Derivatives: A Comprehensive Guide

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1 Introduction

The derivative is one of the most important concepts in calculus. It measures how a function changes as its input changes. In other words, it represents the rate of change or the slope of a function at a particular point.

The derivative has numerous applications in physics, engineering, economics, and many other fields.

2 Definition of Derivative

2.1 Formal Definition

The derivative of a function $f(x)$ at a point $x = a$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{1}$$

Alternatively, using the limit definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{2}$$

2.2 Notation

The derivative of $f(x)$ can be denoted as:

- $f'(x)$ (Lagrange notation)
- $\frac{df}{dx}$ (Leibniz notation)
- $\frac{d}{dx} f(x)$ (operator notation)
- $D_x f(x)$ (operator notation)

2.3 Geometric Interpretation

The derivative represents the slope of the tangent line to the curve $y = f(x)$ at a given point. A positive derivative means the function is increasing, while a negative derivative means it's decreasing.

3 Rules of Differentiation

3.1 Power Rule

For $f(x) = x^n$ where n is any real number:

$$\frac{d}{dx}(x^n) = nx^{n-1} \tag{3}$$

Example: $\frac{d}{dx}(x^3) = 3x^2$

3.2 Sum and Difference Rule

For functions $u(x)$ and $v(x)$:

$$\frac{d}{dx}[u(x) + v(x)] = \frac{du}{dx} + \frac{dv}{dx} \tag{4}$$

$$\frac{d}{dx}[u(x) - v(x)] = \frac{du}{dx} - \frac{dv}{dx} \tag{5}$$

3.3 Product Rule

For functions $u(x)$ and $v(x)$:

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x) \cdot v(x) + u(x) \cdot v'(x) \tag{6}$$

Or in short form: $(uv)' = u'v + uv'$

3.4 Quotient Rule

For functions $u(x)$ and $v(x)$ where $v(x) \neq 0$:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{[v(x)]^2} \tag{7}$$

Or in short form: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

3.5 Chain Rule

For a composite function $y = f(g(x))$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} \tag{8}$$

Where $u = g(x)$.

4 Common Derivatives

This section provides a reference table of derivatives for common functions, along with brief explanations of each.

4.1 Basic Derivatives

Function	Derivative	Notes
c (constant)	0	The derivative of any constant is zero
x	1	The derivative of x is always 1
x^n	nx^{n-1}	Power rule applies for any real n

Table 1: Basic polynomial derivatives

Function	Derivative	Notes
e^x	e^x	The exponential function is its own derivative
a^x	$a^x \ln(a)$	For any base $a > 0$, $a \neq 1$
$\ln(x)$	$\frac{1}{x}$	Natural logarithm; domain: $x > 0$
$\log_a(x)$	$\frac{1}{x \ln(a)}$	Logarithm with base a

Table 2: Exponential and logarithmic derivatives

4.2 Exponential and Logarithmic Derivatives

Examples:

- $\frac{d}{dx}(e^x) = e^x$
- $\frac{d}{dx}(2^x) = 2^x \ln(2) \approx 0.693 \cdot 2^x$
- $\frac{d}{dx}(\ln(x)) = \frac{1}{x}$ for $x > 0$

4.3 Trigonometric Derivatives

Function	Derivative
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

Table 3: Trigonometric derivatives

Key observations:

- The derivative of $\sin(x)$ is $\cos(x)$ and vice versa (with a sign change)
- Derivatives of reciprocal trig functions include the original function in the result
- All trig derivatives involve other trig functions

4.4 Inverse Trigonometric Derivatives

4.5 Hyperbolic Derivatives

4.6 Absolute Value and Piecewise Derivatives

Explanation:

Function	Derivative	Domain
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$	$\|x\| < 1$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$	$\|x\| < 1$
$\arctan(x)$	$\frac{1}{1+x^2}$	All real x

Table 4: Inverse trigonometric derivatives

Function	Derivative
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\mathrm{sech}^2(x)$

Table 5: Hyperbolic derivatives

Function	Derivative	Notes
$\|x\|$	$\mathrm{sgn}(x)$	Undefined at $x = 0$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$	Domain: $x > 0$
$\sqrt[n]{x}$	$\frac{1}{n\sqrt[n]{x^{n-1}}}$	For positive integer n

Table 6: Absolute value and root derivatives

- The derivative of $|x|$ is the sign function $\mathrm{sgn}(x) = \begin{cases} -1 & x < 0 \\ 0 & x = 0 \\ 1 & x > 0 \end{cases}$
- For square root: $\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$ (using power rule with $n = 1/2$)
- For n-th roots, use the power rule: $\frac{d}{dx}(x^{1/n}) = \frac{1}{n}x^{(1/n)-1}$

4.7 Composition Rules for Derivatives

When derivatives are combined, we use the following composition rules:

4.8 Practical Tips for Computing Derivatives

1. **Identify the structure:** Determine if the function is a sum, product, quotient, or composition.
2. **Apply the appropriate rule:** Use sum, product, quotient, or chain rule as needed.
3. **Use tables:** Refer to the common derivatives table for basic functions.

Rule	Formula	Example
Sum Rule	$(u+v)' = u' + v'$	$\frac{d}{dx}(x^2 + \sin x) = 2x + \cos x$
Difference Rule	$(u-v)' = u' - v'$	$\frac{d}{dx}(e^x - x^3) = e^x - 3x^2$
Constant Multiple	$(cu)' = cu'$	$\frac{d}{dx}(5\cos x) = -5\sin x$
Product Rule	$(uv)' = u'v + uv'$	$\frac{d}{dx}(x \cdot e^x) = e^x + xe^x$
Quotient Rule	$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$	$\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \frac{x\cos x - \sin x}{x^2}$
Chain Rule	$(f \circ g)' = f'(g) \cdot g'$	$\frac{d}{dx}(\sin(x^2)) = 2x\cos(x^2)$

Table 7: Composition and combination rules for derivatives

4. **Simplify:** Always simplify the final answer by combining like terms and factoring when possible.
5. **Check domain restrictions:** Remember where derivatives may be undefined (e.g., $\ln(x)$ only for $x > 0$).

Common mistakes to avoid:

- Forgetting to apply the chain rule to composite functions
- Misapplying the product rule (it's $u'v + uv'$, not $(u'v)'$ or $(uv')'$)
- Ignoring domain restrictions of the original function
- Forgetting constant factors in the numerator of quotient rule results

4.9 Summary Table: All Common Derivatives

5 Applications of Derivatives

5.1 Rate of Change

Derivatives measure how quantities change over time. For example, velocity is the derivative of position with respect to time.

5.2 Optimization

Finding maximum and minimum values of functions. This is crucial in business (maximizing profit), physics (finding equilibrium points), and engineering.

5.3 Curve Sketching

The first derivative tells us where the function is increasing or decreasing. The second derivative tells us about concavity.

Function	Derivative
c (constant)	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x \ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$
$\arcsin(x)$	$\frac{1}{\sqrt{1-x^2}}$
$\arccos(x)$	$-\frac{1}{\sqrt{1-x^2}}$
$\arctan(x)$	$\frac{1}{1+x^2}$
$\sinh(x)$	$\cosh(x)$
$\cosh(x)$	$\sinh(x)$
$\tanh(x)$	$\operatorname{sech}^2(x)$

Table 8: Comprehensive list of common derivatives

5.4 Related Rates

Problems where multiple quantities are related and change with respect to time.

5.5 Physics

- **Velocity:** $v = \frac{ds}{dt}$ (derivative of position)
- **Acceleration:** $a = \frac{dv}{dt}$ (derivative of velocity)
- **Force:** $F = \frac{dp}{dt}$ (derivative of momentum)

6 Examples

6.1 Example 1: Power Rule

Find the derivative of $f(x) = 5x^4 - 3x^2 + 7$.

Solution:

We apply the power rule to each term separately:

- For $5x^4$: The exponent is 4, so $\frac{d}{dx}(5x^4) = 5 \cdot 4x^{4-1} = 20x^3$
- For $-3x^2$: The exponent is 2, so $\frac{d}{dx}(-3x^2) = -3 \cdot 2x^{2-1} = -6x$
- For the constant 7: The derivative of any constant is 0

Therefore:

$$f'(x) = 20x^3 - 6x \tag{9}$$

This tells us the instantaneous rate of change of $f(x)$ at any point x. For example, at $x = 1$, the slope is $f'(1) = 20(1)^3 - 6(1) = 14$.

6.2 Example 2: Product Rule

Find the derivative of $f(x) = (2x + 1)(x^2 - 3)$.

Solution:

We have a product of two functions, so we use the product rule: $(uv)' = u'v + uv'$.

Let:

- $u = 2x + 1$, so $u' = 2$
- $v = x^2 - 3$, so $v' = 2x$

Applying the product rule:

$$\begin{aligned} f'(x) &= u'v + uv' && (10)\\ &= 2(x^2 - 3) + (2x + 1)(2x) && (11)\\ &= 2x^2 - 6 + 4x^2 + 2x && (12)\\ &= 6x^2 + 2x - 6 && (13) \end{aligned}$$

Note: We could have also expanded the original function first as $f(x) = 2x^3 + x^2 - 6x - 3$, then applied the power rule directly to get the same result.

6.3 Example 3: Chain Rule

Find the derivative of $f(x) = (3x^2 + 2)^5$.

Solution:

This is a composite function (a function inside another function), so we use the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Let:

- Inner function: $u = 3x^2 + 2$, with $\frac{du}{dx} = 6x$
- Outer function: $y = u^5$, with $\frac{dy}{du} = 5u^4$

Using the chain rule:

$$f'(x) = \frac{dy}{du} \cdot \frac{du}{dx} \tag{14}$$
$$= 5u^4 \cdot 6x \tag{15}$$
$$= 5(3x^2 + 2)^4 \cdot 6x \tag{16}$$
$$= 30x(3x^2 + 2)^4 \tag{17}$$

At $x = 0$, we get $f'(0) = 30(0)(2)^4 = 0$, which means the tangent line is horizontal at this point.

6.4 Example 4: Quotient Rule

Find the derivative of $f(x) = \frac{x^2 + 1}{x - 1}$.

Solution:

We have a quotient of two functions, so we use the quotient rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$.

Let:

- $u = x^2 + 1$ (numerator), so $u' = 2x$
- $v = x - 1$ (denominator), so $v' = 1$

Applying the quotient rule:

$$f'(x) = \frac{u'v - uv'}{v^2} \tag{18}$$
$$= \frac{2x(x - 1) - (x^2 + 1)(1)}{(x - 1)^2} \tag{19}$$
$$= \frac{2x^2 - 2x - x^2 - 1}{(x - 1)^2} \tag{20}$$
$$= \frac{x^2 - 2x - 1}{(x - 1)^2} \tag{21}$$

Note: The domain of the derivative excludes $x = 1$ (where the denominator equals zero). Also, we can find critical points by setting $f'(x) = 0$, which occurs when $x^2 - 2x - 1 = 0$, giving $x = 1 \pm \sqrt{2}$.

7 Conclusion

Understanding derivatives is fundamental to calculus and has applications throughout science, engineering, and economics. By mastering the rules and concepts presented here, you'll be well-equipped to solve a wide variety of problems involving rates of change and optimization.