Derivatives: A Comprehensive Guide

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1. Introduction

The derivative is one of the most important concepts in calculus. It measures how a function changes as its input changes. In other words, it represents the rate of change or the slope of a function at a particular point.

The derivative has numerous applications in physics, engineering, economics, and many other fields.

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2. Definition of Derivative

Formal Definition

The derivative of a function $f(x)$ at a point $x = a$ is defined as:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Alternatively, using the limit definition:

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Notation

The derivative of $f(x)$ can be denoted as: - $f'(x)$ (Lagrange notation) - $\frac{df}{dx}$ (Leibniz notation) - $\frac{d}{dx}f(x)$ (operator notation) - $D_x f(x)$ (operator notation)

Geometric Interpretation

The derivative represents the slope of the tangent line to the curve $y = f(x)$ at a given point. A positive derivative means the function is increasing, while a negative derivative means it's decreasing.

3. Rules of Differentiation

Power Rule

For $f(x) = x^n$ where n is any real number:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Example: $\frac{d}{dx}(x^3) = 3x^2$

Sum and Difference Rule

For functions $u(x)$ and $v(x)$:

$$\frac{d}{dx}[u(x) + v(x)] = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}[u(x) - v(x)] = \frac{du}{dx} - \frac{dv}{dx}$$

Product Rule

For functions $u(x)$ and $v(x)$:

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x) \cdot v(x) + u(x) \cdot v'(x)$$

Or in short form: $(uv)' = u'v + uv'$

Quotient Rule

For functions $u(x)$ and $v(x)$ where $v(x) \neq 0$:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x) \cdot v(x) - u(x) \cdot v'(x)}{[v(x)]^2}$$

Or in short form: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Chain Rule

For a composite function $y = f(g(x))$:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Where $u = g(x)$.

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4. Common Derivatives

Function	Derivative
c (constant)	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln(a)$
$\ln(x)$	$\frac{1}{x}$
$\log_a(x)$	$\frac{1}{x \ln(a)}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$
$\csc(x)$	$-\csc(x)\cot(x)$

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5. Applications of Derivatives

Rate of Change

Derivatives measure how quantities change over time. For example, velocity is the derivative of position with respect to time.

Optimization

Finding maximum and minimum values of functions. This is crucial in business (maximizing profit), physics (finding equilibrium points), and engineering.

Curve Sketching

The first derivative tells us where the function is increasing or decreasing. The second derivative tells us about concavity.

Related Rates

Problems where multiple quantities are related and change with respect to time.

Physics

- **Velocity:** $v = \frac{ds}{dt}$ (derivative of position)
- **Acceleration:** $a = \frac{dv}{dt}$ (derivative of velocity)
- **Force:** $F = \frac{dp}{dt}$ (derivative of momentum)

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6. Examples

Example 1: Power Rule

Find the derivative of $f(x) = 5x^4 - 3x^2 + 7$

Solution:

$$f'(x) = 5 \cdot 4x^3 - 3 \cdot 2x + 0 = 20x^3 - 6x$$

Example 2: Product Rule

Find the derivative of $f(x) = (2x + 1)(x^2 - 3)$

Solution:

Let $u = 2x + 1$ and $v = x^2 - 3$

Then $u' = 2$ and $v' = 2x$

Using the product rule:

$$f'(x) = 2(x^2 - 3) + (2x + 1)(2x)$$
$$= 2x^2 - 6 + 4x^2 + 2x$$
$$= 6x^2 + 2x - 6$$

Example 3: Chain Rule

Find the derivative of $f(x) = (3x^2 + 2)^5$

Solution:

Let $u = 3x^2 + 2$, then $\frac{du}{dx} = 6x$

Using the chain rule:

$$f'(x) = 5(3x^2 + 2)^4 \cdot 6x = 30x(3x^2 + 2)^4$$

Example 4: Quotient Rule

Find the derivative of $f(x) = \frac{x^2+1}{x-1}$

Solution:

Let $u = x^2 + 1$ and $v = x - 1$

Then $u' = 2x$ and $v' = 1$

Using the quotient rule:

$$f'(x) = \frac{2x(x-1) - (x^2+1)(1)}{(x-1)^2}$$

$$= \frac{2x^2 - 2x - x^2 - 1}{(x-1)^2}$$

$$= \frac{x^2 - 2x - 1}{(x-1)^2}$$

Conclusion

Understanding derivatives is fundamental to calculus and has applications throughout science, engineering, and economics. By mastering the rules and concepts presented here, you'll be well-equipped to solve a wide variety of problems involving rates of change and optimization.