Introduction to Statistical Learning with Applications

CM3: Cross-validation, model selection, and bias-variance

Pedro L. C. Rodrigues

Questions from the final exam from 2018

Consider the following Python script:

```python
import numpy as np
import pandas as pd
import statsmodels.api as sm
np.random.seed(0)
# number of variables
pt = 201
# number of predictors
p = pt - 1
# sample size
n = 30 * p
# generate data
D = np.random.randn(n, pt)
df = pd.DataFrame(data=D)
df = df.rename(columns={0:'Y'})
# do multiple linear regression
df['intercept'] = 1
model = sm.OLS(df['Y'], df.drop(columns='Y'))
results = model.fit()
print(results.summary())
```

- What does the script do?
- What is the true distribution of the random variable **Y** given the first 200 columns of **D** ($X_1, X_2, \ldots, X_{200}$)?
- Write an equation defining the model estimated by `model.fit()`. What is the difference between this model and the one defined above?
- Print `results.params` what's going on?

IN OUR PREVIOUS EPISODES…

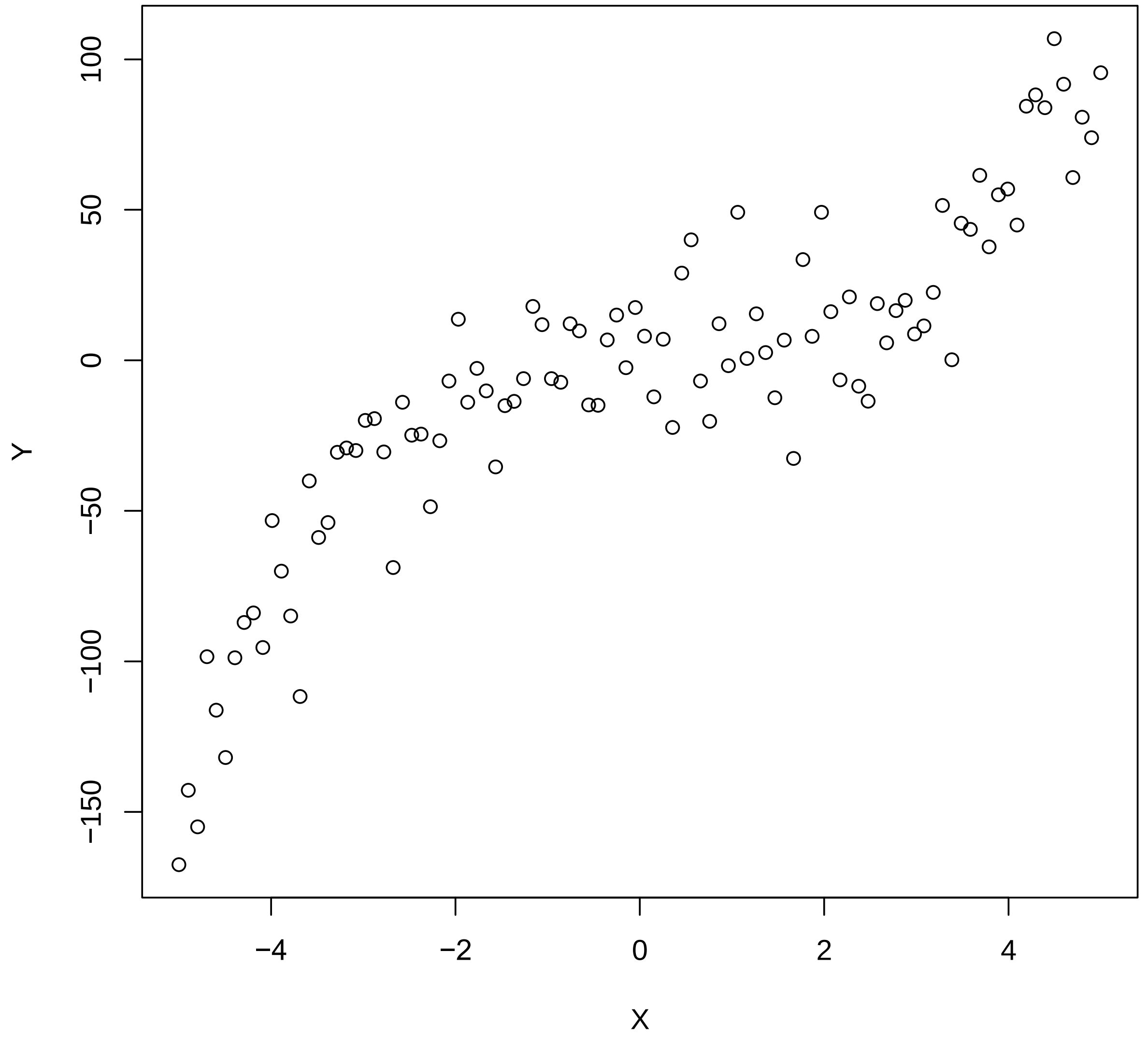
Our current worflow

1. We want to estimate the values of Y based on predictors $X_1, \ldots, X_p$
2. We are given a set of N examples of y_i with corresponding $x_{i1}, \ldots, x_{ip}$
3. We estimate parameters $\hat{\beta}$ that minimize $\frac{1}{N}\sum_{i=1}^{N}(y_i - \beta^T x_i)^2$
4. We do statistical inference on the values of the $\hat{\beta}_j$
5. How can we **quantify the quality** of a model?

$$x_i = \begin{bmatrix} x_{i1} \\ \vdots \\ x_{ip} \end{bmatrix}$$

- What makes a model **good**?
- Estimating the quality of a model
- Comparing and selecting models

What makes a model good?



Suppose our data is generated as

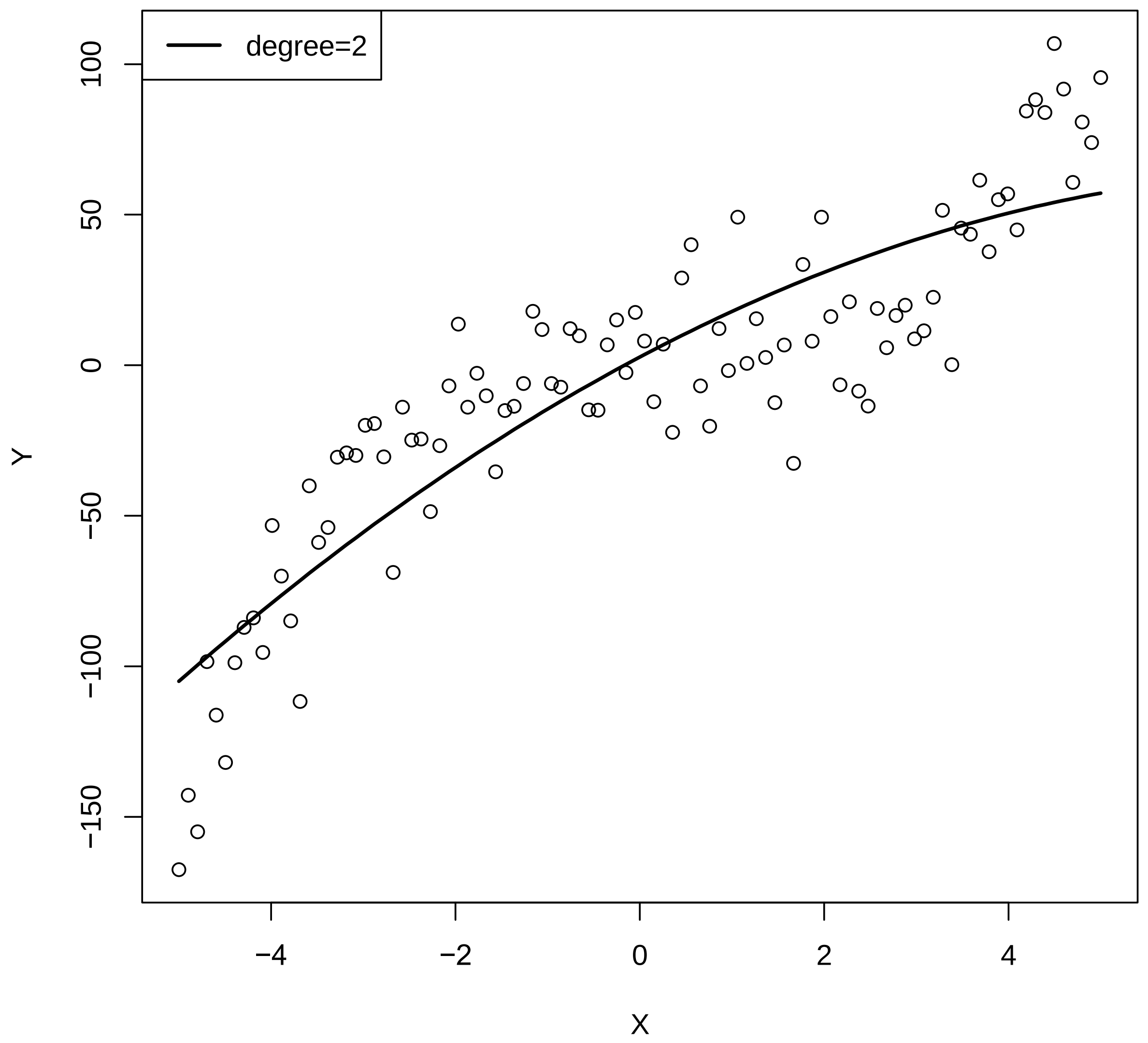
$$Y = \beta_0 + \sum_{i=1}^{d} \beta_i X^i + \varepsilon$$

we can run multiple linear regression with

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & x_N^2 & \dots & x_N^d \end{bmatrix}$$

What's the best choice for *d* ?

What makes a model good?



Suppose our data is generated as

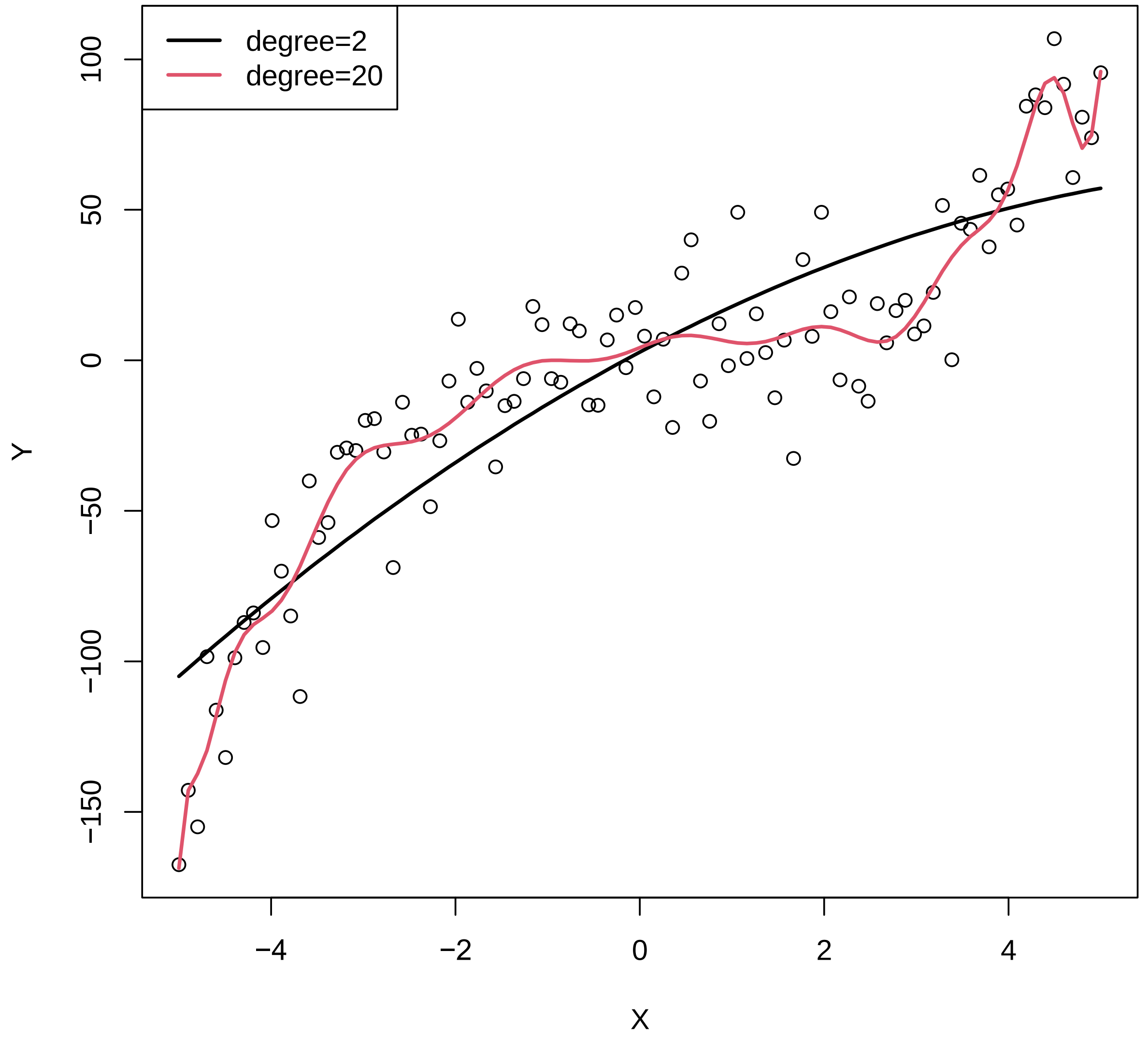
$$Y = \beta_0 + \sum_{i=1}^{d} \beta_i X^i + \varepsilon$$

we can run multiple linear regression with

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & x_N^2 & \dots & x_N^d \end{bmatrix}$$

What's the best choice for *d* ?

What makes a model good?



Suppose our data is generated as

$$Y = \beta_0 + \sum_{i=1}^{d} \beta_i X^i + \varepsilon$$

we can run multiple linear regression with

$$\mathbf{X} = \begin{bmatrix} 1 & x_1 & x_1^2 & \dots & x_1^d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_1 & x_N^2 & \dots & x_N^d \end{bmatrix}$$

What's the best choice for *d* ?

What makes a model good? – The generalization error

Remember that our goal in regression was to minimize the generalization error:

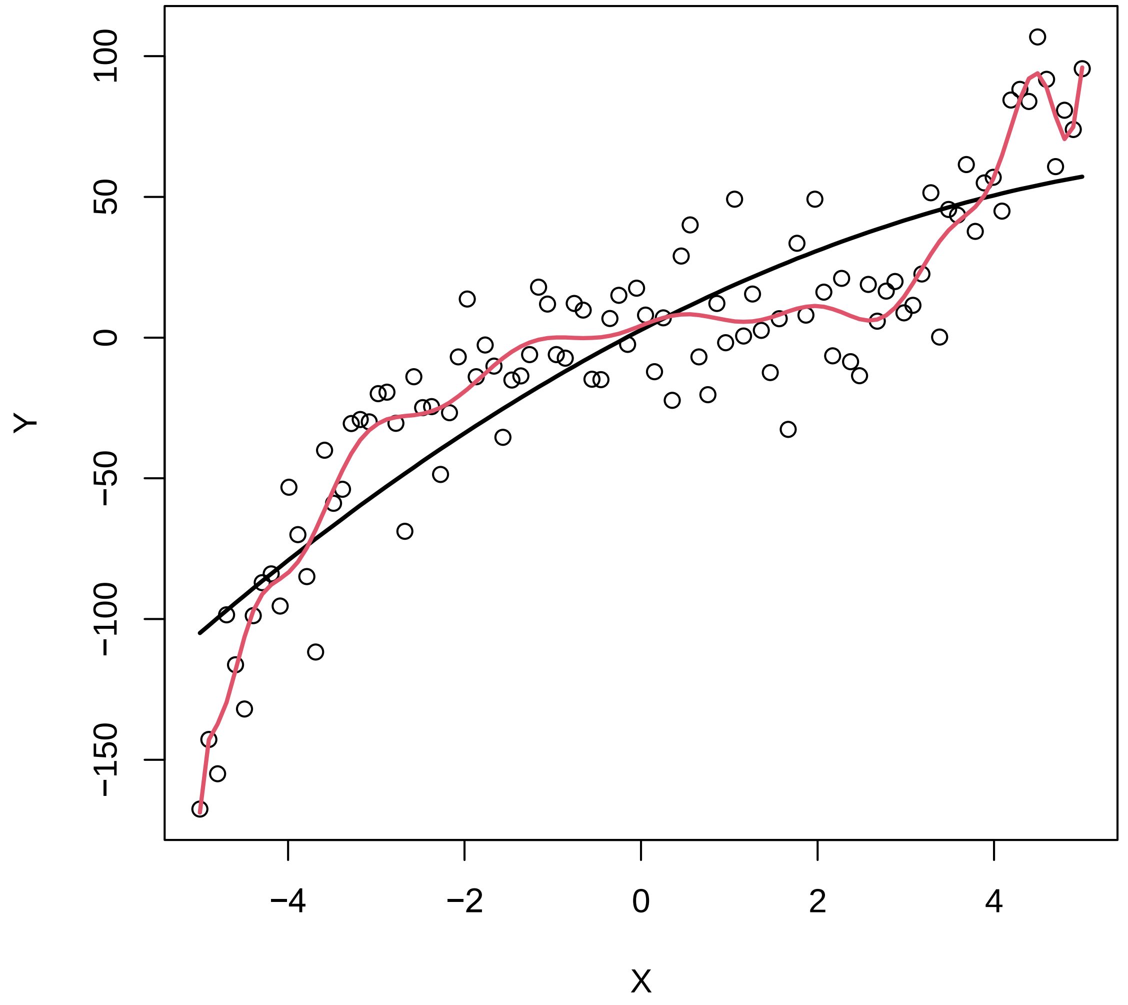
$$\boxed{\mathcal{L}(r) = \mathbb{E}_{Y,X}\left[\left(Y - r(X)\right)^2\right]}$$

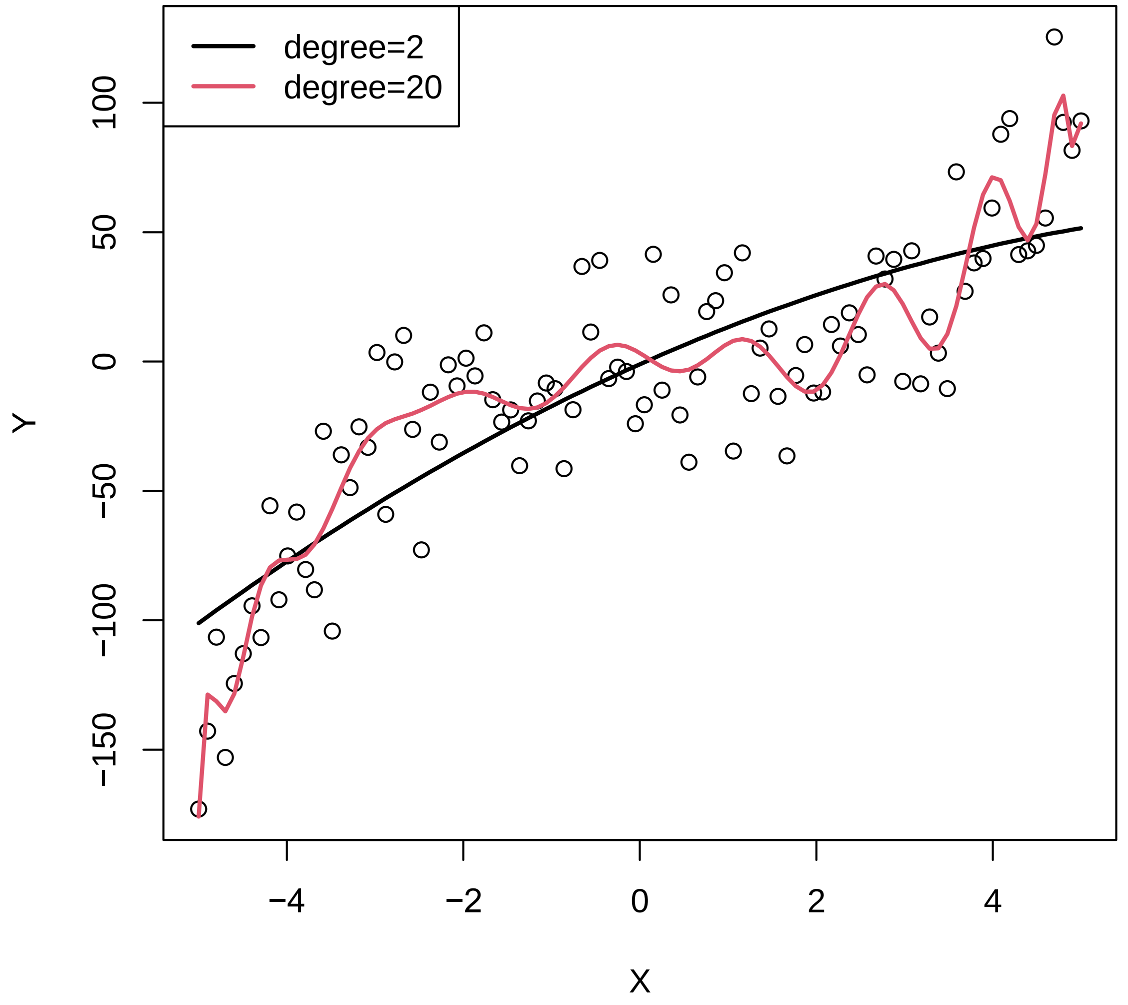
Intuitively, we can expect this error to be decomposed into a few informative parts:

- The **irreducible error**: can we ever predict Y from X with zero prediction error? Probably not, since we always assume having some observation error ε in the data model.
- The **estimation bias**: when estimating the conditional expectation, we always have to choose a family of approximators, which may not always be sufficiently flexible.
- The **estimation variance**: for families of approximators that are too flexible, it might happen that for every small change in the observed data points, the estimated model changes.

What makes a model good?

Estimation variance. The same model fit on two slightly different datasets.





What makes a model good?

What makes a model **good**? – The bias-variance tradeoff

Remember that the data model is assumed to be $Y = r(X) + \varepsilon$ with $\mathrm{Var}(\varepsilon) = \sigma^2$

We use a training dataset $\mathcal{D} = \left\{(y_i, x_i)\right\}_{i=1}^{i=N}$ to estimate $\hat{r}_{\mathcal{D}}$

For a given new observation x of interest, we can write the bias-variance decomposition

$$
\begin{aligned}
\mathcal{L}\big(\hat{r}_{\mathcal{D}}(x)\big) &= \mathbb{E}_{Y,X}\left[\big(Y - \hat{r}_{\mathcal{D}}(X)\big)^2 \,\middle|\, X = x\right] \\
&= \mathbb{E}_{Y,X}\left[\big(Y - r(X) + r(X) - \hat{r}_{\mathcal{D}}(X)\big)^2 \,\middle|\, X = x\right] \\
&= \mathbb{E}_{Y,X}\left[\big(Y - r(X)\big)^2 \,\middle|\, X = x\right] + \mathbb{E}_{Y,X}\left[\big(r(X) - \hat{r}_{\mathcal{D}}(X)\big)^2 \,\middle|\, X = x\right] \\
&= \underbrace{\mathbb{E}_{Y,X}\left[\big(Y - r(X)\big)^2 \,\middle|\, X = x\right]}_{\text{Irreducible error } \sigma^2} + \underbrace{\Big(\mathbb{E}_{Y,X}\left[r(X) - \hat{r}_{\mathcal{D}}(X) \,\middle|\, X = x\right]\Big)^2}_{\text{Squared estimation bias}} + \underbrace{\mathrm{Var}\Big(\hat{r}_{\mathcal{D}}(X) \,\Big|\, X = x\Big)}_{\text{Estimation variance}}
\end{aligned}
$$

- What makes a model **good**?
- Estimating the quality of a model
- Comparing and selecting models

Estimating the quality of a model

How can we estimate the generalization error of an estimated model in practice?

$$\mathcal{L}(\hat{r}_{\mathcal{D}}) = \mathbb{E}_{Y,X}\left[\left(Y - \hat{r}_{\mathcal{D}}(X)\right)^2\right]$$

We don't know the $p(Y, X)$

Estimating the quality of a model

How can we estimate the generalization error of an estimated model in practice?

$$\mathcal{L}(\hat{r}_{\mathcal{D}}) = \mathbb{E}_{Y,X}\left[\Big(Y - \hat{r}_{\mathcal{D}}(X)\Big)^2\right] \approx \underbrace{\frac{1}{M}\sum_{i=1}^{M}\Big(y_i - \hat{r}_{\mathcal{D}}(x_i)\Big)^2}_{} = L(\hat{r}_{\mathcal{D}}, \mathcal{X}) \quad \text{with } \mathcal{X} = \Big\{(y_i, x_i)\Big\}_{i=1}^{i=M}$$

We don't know the $p(Y, X)$

So we approximate it with M data points

Note that for $\mathcal{D} = \Big\{(y_i, x_i)\Big\}_{i=1}^{i=N}$ we have $\hat{r}_{\mathcal{D}} = \underset{r \in \mathcal{F}}{\operatorname{argmin}}\ L(r, \mathcal{D})$

We want to minimize one quantity but can only estimate a proxy

Estimating the quality of a model

How can we estimate the generalization error of an estimated model in practice?

$$\mathcal{L}(\hat{r}_{\mathcal{D}}) = \mathbb{E}_{Y,X}\left[\left(Y - \hat{r}_{\mathcal{D}}(X)\right)^2\right] \approx \underline{\frac{1}{M}\sum_{i=1}^{M}\left(y_i - \hat{r}_{\mathcal{D}}(x_i)\right)^2} = L(\hat{r}_{\mathcal{D}}, \mathcal{X}) \quad \text{with} \quad \mathcal{X} = \left\{(y_i, x_i)\right\}_{i=1}^{i=M}$$

We don't know the $p(Y,X)$

So we approximate it with M data points

Note that for $\mathcal{D} = \left\{(y_i, x_i)\right\}_{i=1}^{i=N}$ we have $\hat{r}_{\mathcal{D}} = \underset{r\in\mathcal{F}}{\operatorname{argmin}}\ L(r, \mathcal{D})$

We want to minimize one quantity but can only estimate a proxy

Question: Can we say that $\mathcal{L}(\hat{r}_{\mathcal{D}}) \approx L(\hat{r}_{\mathcal{D}}, \mathcal{D})$?

ABSOLUTELY NOT!

In fact $\mathcal{L}(\hat{r}_{\mathcal{D}}) \geq L(\hat{r}_{\mathcal{D}}, \mathcal{D})$ i.e. the true generalization error is larger than the estimated one

The optimism of the training error

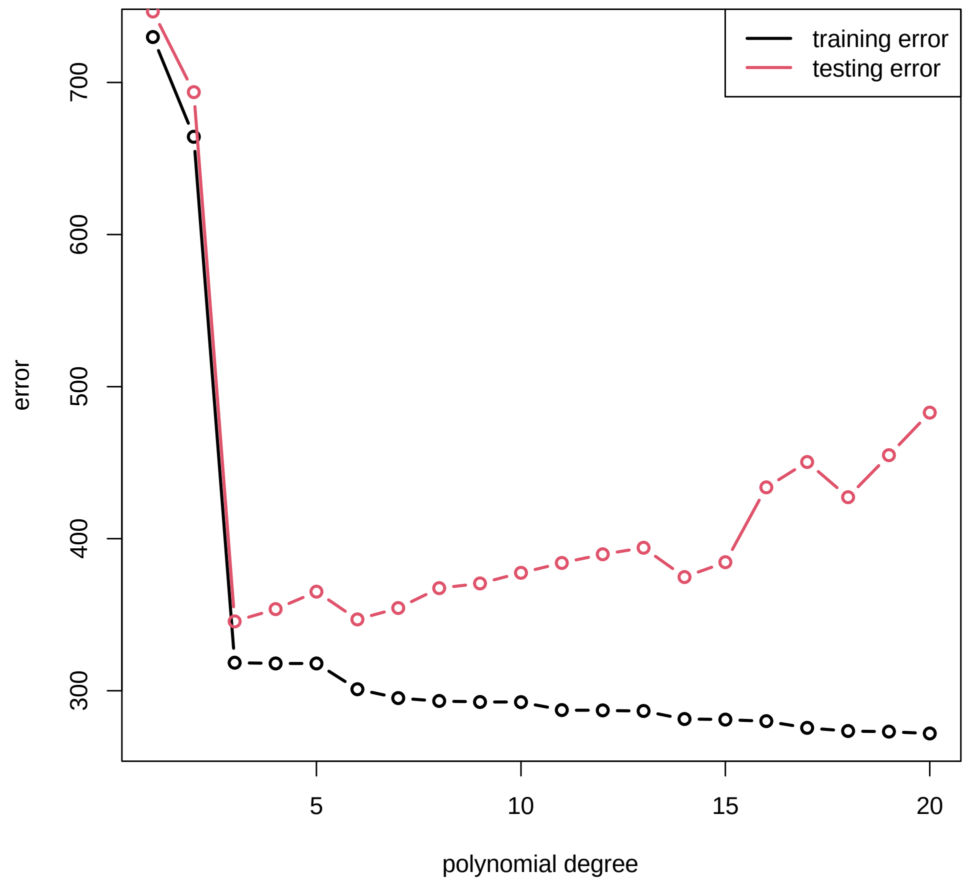
Different ways of seeing the problem:

- Intuitively, since $\hat{r}_{\mathcal{D}} = \underset{r \in \mathcal{F}}{\operatorname{argmin}}\ L(r, \mathcal{D})$ then $L(\hat{r}_{\mathcal{D}}, \mathcal{D})$ will always look small…
- Consider our example with polynomials
  - Increasing the degree improves training error
  - But the testing error grows!

$L(\hat{r}_{\mathcal{D}}, \mathcal{D})$ is the **training** error

$L(\hat{r}_{\mathcal{D}}, \mathcal{X})$ with $\mathcal{X} \neq \mathcal{D}$ is the **testing** error

Error on unseen data



The optimism of the training error

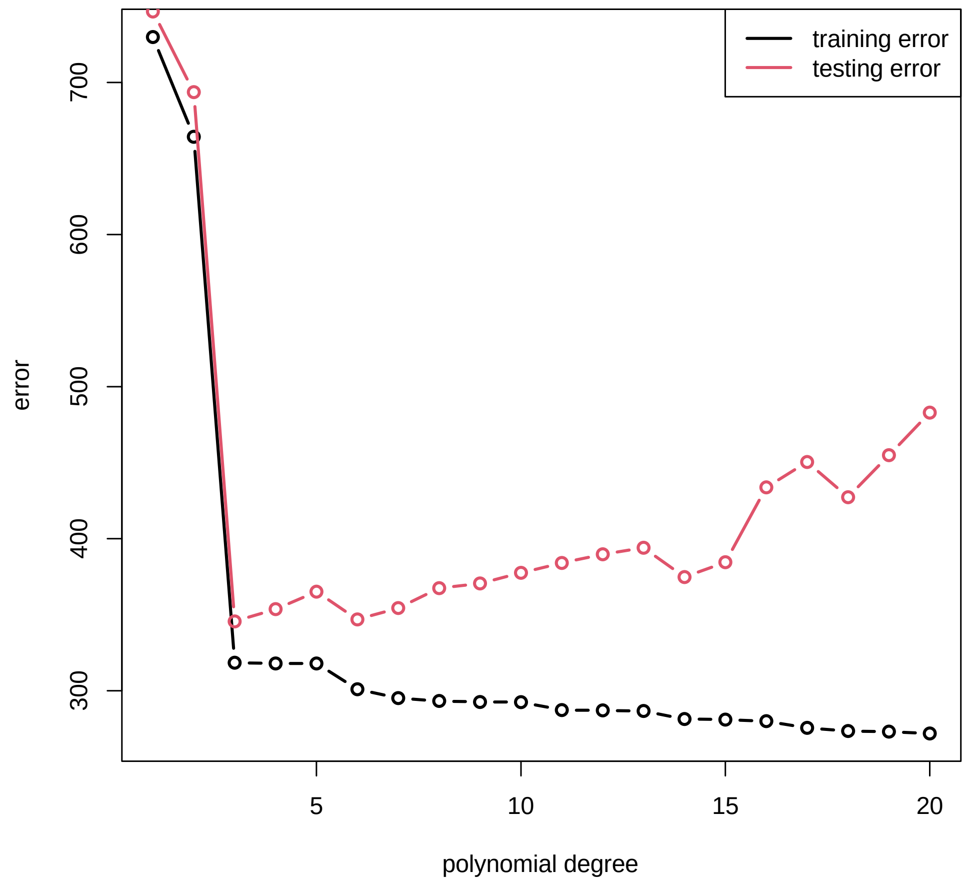
Different ways of seeing the problem:

- Intuitively, since $\hat{r}_{\mathcal{D}} = \underset{r \in \mathcal{F}}{\operatorname{argmin}}\, L(r, \mathcal{D})$ then $L(\hat{r}_{\mathcal{D}}, \mathcal{D})$ will always look small…
- Consider our example with polynomials
  - Increasing the degree improves training error
  - But the testing error grows!
- Mathematical illustration on linear regression

$$\mathcal{D} = \Big\{(x_i, y_i)\Big\}_{i=1}^{i=N} \quad \mathcal{D}' = \Big\{(x_i, y_i')\Big\}_{i=1}^{i=N} \quad Y = \beta_0 + \sum_{k=1}^{p} \beta_k X_k + \varepsilon$$

(same predictors but different observations)

$$\mathbb{E}\Big[L(\hat{r}_{\mathcal{D}}, \mathcal{D}')\Big] = \mathbb{E}\Big[L(\hat{r}_{\mathcal{D}}, \mathcal{D})\Big] + \frac{2}{N}\sigma^2(p+1)$$



The optimism of the training error

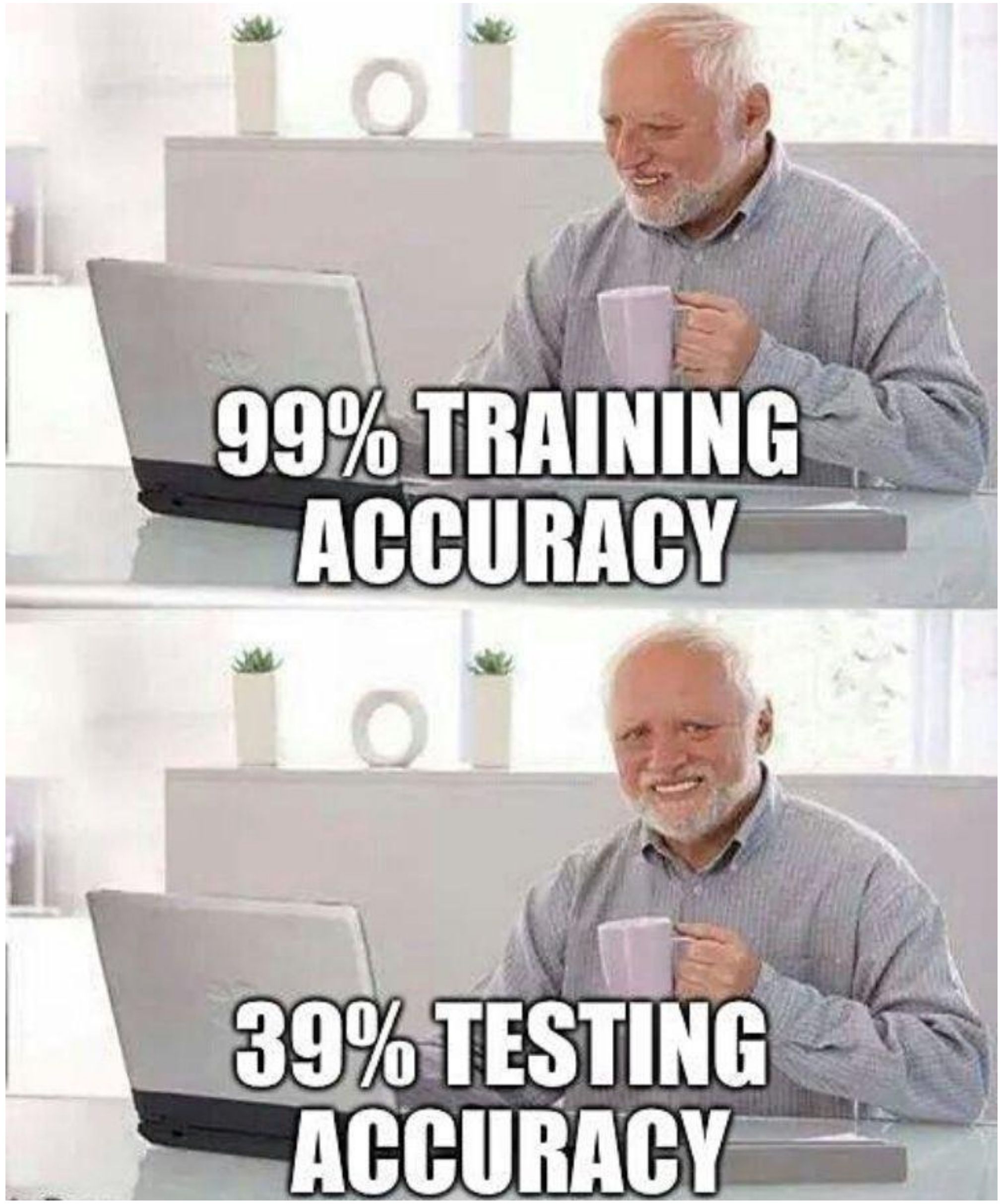
Conclusion:

- We can only know if a model is good or not if we **correctly evaluate** its performance

- The objective function that we minimize when training a model (**training error**) is never the same as the one we are actually interested in minimizing (**test error**)

$$\hat{r}_{\mathcal{D}} = \underset{r \in \mathcal{F}}{\operatorname{argmin}} \frac{1}{M} \sum_{i=1}^{M} \Big(y_i - r(x_i)\Big)^2$$

$$\mathcal{L}(\hat{r}_{\mathcal{D}}) = \mathbb{E}_{(X,Y)}\left[\Big(Y - \hat{r}_{\mathcal{D}}(X)\Big)^2\right]$$



Cross-validation

We are given a dataset with N datapoints $\mathcal{D} = \left\{(y_i, x_i)\right\}_{i=1}^{i=N}$

How can we estimate the **generalization error** of a model we train on this dataset?

Remember: We should evaluate the model on samples that were not used for training

Cross-validation

We are given a dataset with N datapoints $\mathcal{D} = \left\{(y_i, x_i)\right\}_{i=1}^{i=N}$

How can we estimate the **generalization error** of a model we train on this dataset?

Remember: We should evaluate the model on samples that were not used for training

○ **Strategy 1 : Single hold-out test point**

We fit a model on the first N-1 training samples, calling it $\hat{r}^{(-N)}$

Treat the last sample (X_N, Y_N) as a test sample and estimate generalization error as

$$\mathcal{L}(\hat{r}) \approx \left(y_N - \hat{r}^{(-N)}(x_N)\right)^2$$

The estimator is easy to calculate but we can expect a rather large variance.

Cross-validation

We are given a dataset with N datapoints $\mathcal{D} = \left\{(y_i, x_i)\right\}_{i=1}^{i=N}$

How can we estimate the **generalization error** of a model we train on this dataset?

Remember: We should evaluate the model on samples that were not used for training

- **Strategy 2 : Leave one out cross-validation (LOOCV)**

We fit N models on N-1 training samples, holding out x_i at each time to get $\hat{r}^{(-i)}$

The test error is approximate as

$$\mathcal{L}(\hat{r}) \approx \frac{1}{N} \sum_{i=1}^{N} \left(y_i - \hat{r}^{(-i)}(x_i) \right)^2$$

The variance decreases, but the computational burden is much higher

Cross-validation

We are given a dataset with N datapoints $\mathcal{D} = \Big\{(y_i, x_i)\Big\}_{i=1}^{i=N}$

How can we estimate the **generalization error** of a model we train on this dataset?

Remember: We should evaluate the model on samples that were not used for training

- **Strategy 3 : K-Fold cross-validation**

Split the training dataset randomly into K folds so to have $\mathcal{D}_1 \cup \cdots \cup \mathcal{D}_K = \mathcal{D}$

For k=1,...,K fit a model $\hat{r}^{(-k)}$ on a training set except that excludes $\mathcal{D}_k$

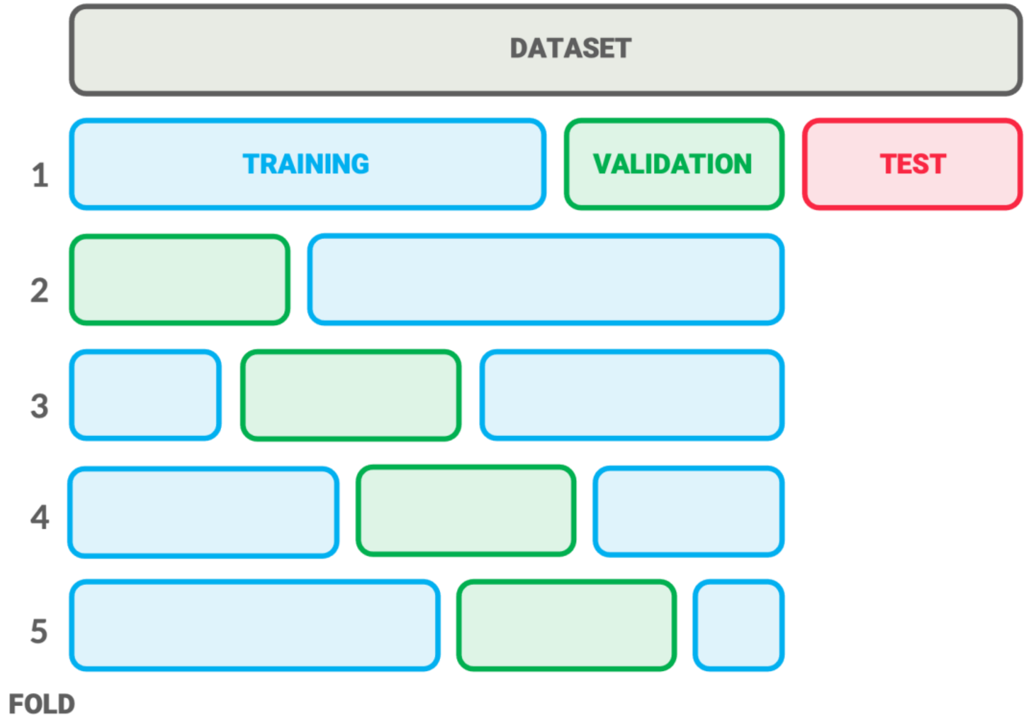
$$\mathcal{L}(\hat{r}) \approx \underset{\text{Take average over folds}}{\frac{1}{K}\sum_{k=1}^{K}} \left(\underbrace{\frac{1}{N_k} \sum_{i \in \mathcal{D}_k} \Big(y_i - \hat{r}^{(-k)}(x_i)\Big)^2}_{\text{Estimate test error for each fold}} \right)$$

Cross-validation

Strategy 3 : K-Fold cross-validation

Using scikit learn

Note that the errors calculated in each fold are not IID random variables, since the models share some of their data points.
↳ **Question:** What does this imply?

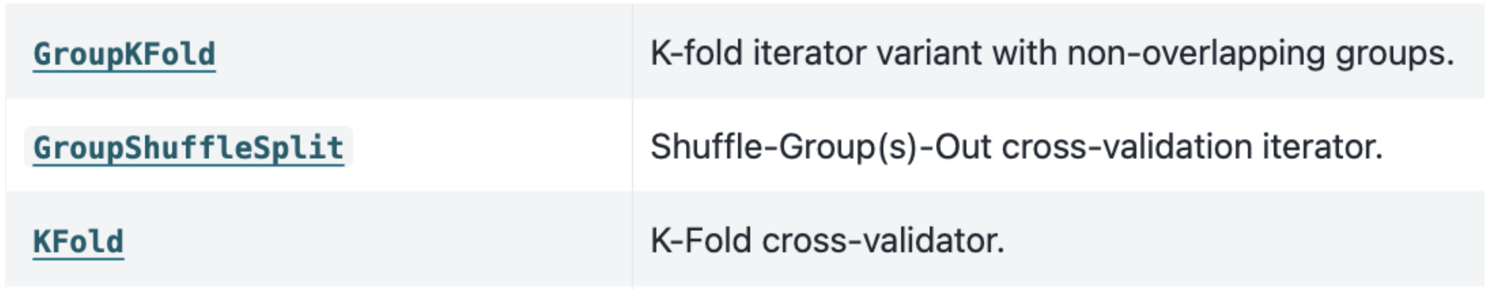


```python
from sklearn.model_selection import KFold
from sklearn.datasets import fetch_california_housing
from sklearn.linear_model import LinearRegression
kf = KFold(n_splits=5)
dataset = fetch_california_housing()
lm = LinearRegression()
X = dataset.data
y = dataset.target
scores = []
for idx_train, idx_test in kf.split(X):
    X_train, y_train = X[idx_train], y[idx_train]
    X_test, y_test = X[idx_test], y[idx_test]
    lm.fit(X_train, y_train)
    scores.append(lm.score(X_test, y_test))
print(np.mean(scores))
```

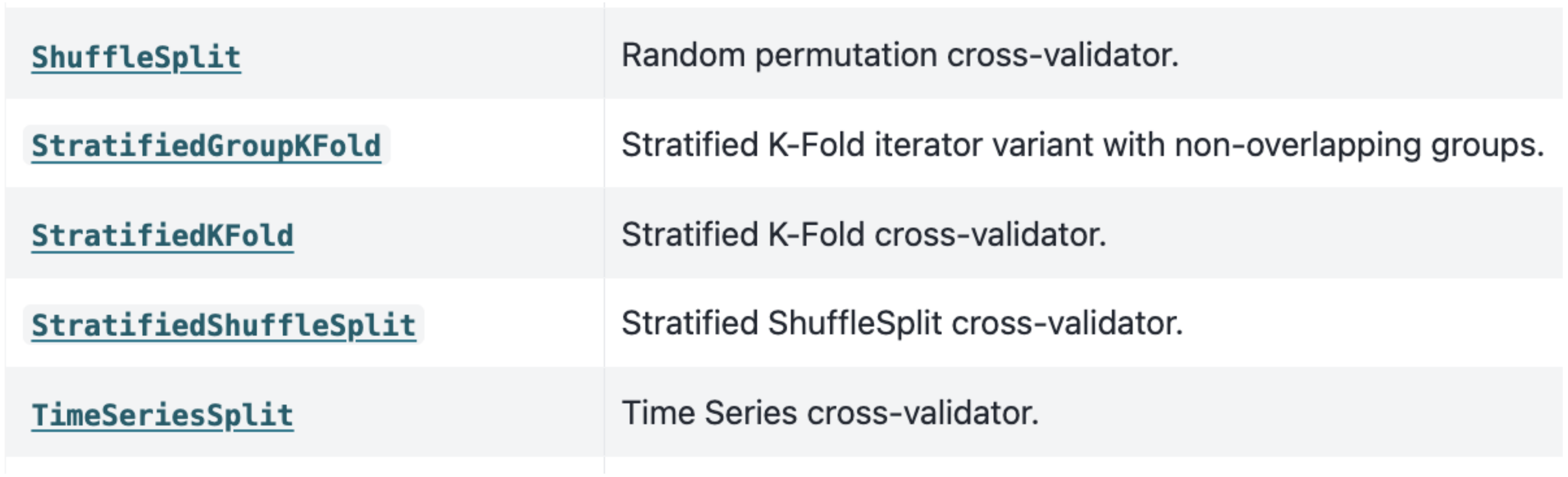
See the documentation for more details

Cross-validation

But there are **several** other strategies!



GroupKFold	K-fold iterator variant with non-overlapping groups.
GroupShuffleSplit	Shuffle-Group(s)-Out cross-validation iterator.
KFold	K-Fold cross-validator.

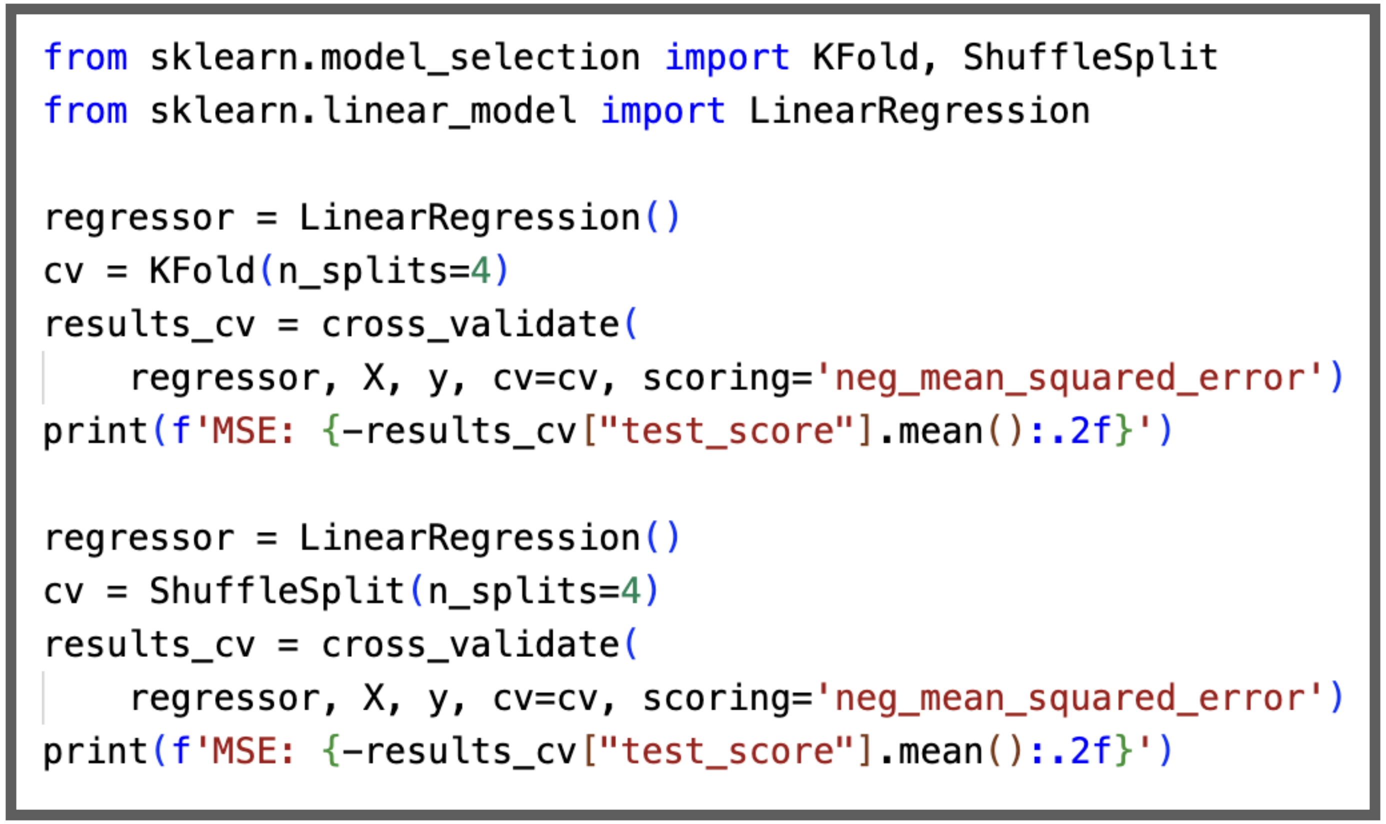


ShuffleSplit	Random permutation cross-validator.
StratifiedGroupKFold	Stratified K-Fold iterator variant with non-overlapping groups.
StratifiedKFold	Stratified K-Fold cross-validator.
StratifiedShuffleSplit	Stratified ShuffleSplit cross-validator.
TimeSeriesSplit	Time Series cross-validator.

and more…

Cross-validation

(1) Example with the **categorical variables** from last week



```
from sklearn.model_selection import KFold, ShuffleSplit
from sklearn.linear_model import LinearRegression

regressor = LinearRegression()
cv = KFold(n_splits=4)
results_cv = cross_validate(
    regressor, X, y, cv=cv, scoring='neg_mean_squared_error')
print(f'MSE: {-results_cv["test_score"].mean():.2f}')

regressor = LinearRegression()
cv = ShuffleSplit(n_splits=4)
results_cv = cross_validate(
    regressor, X, y, cv=cv, scoring='neg_mean_squared_error')
print(f'MSE: {-results_cv["test_score"].mean():.2f}')
```

17.50

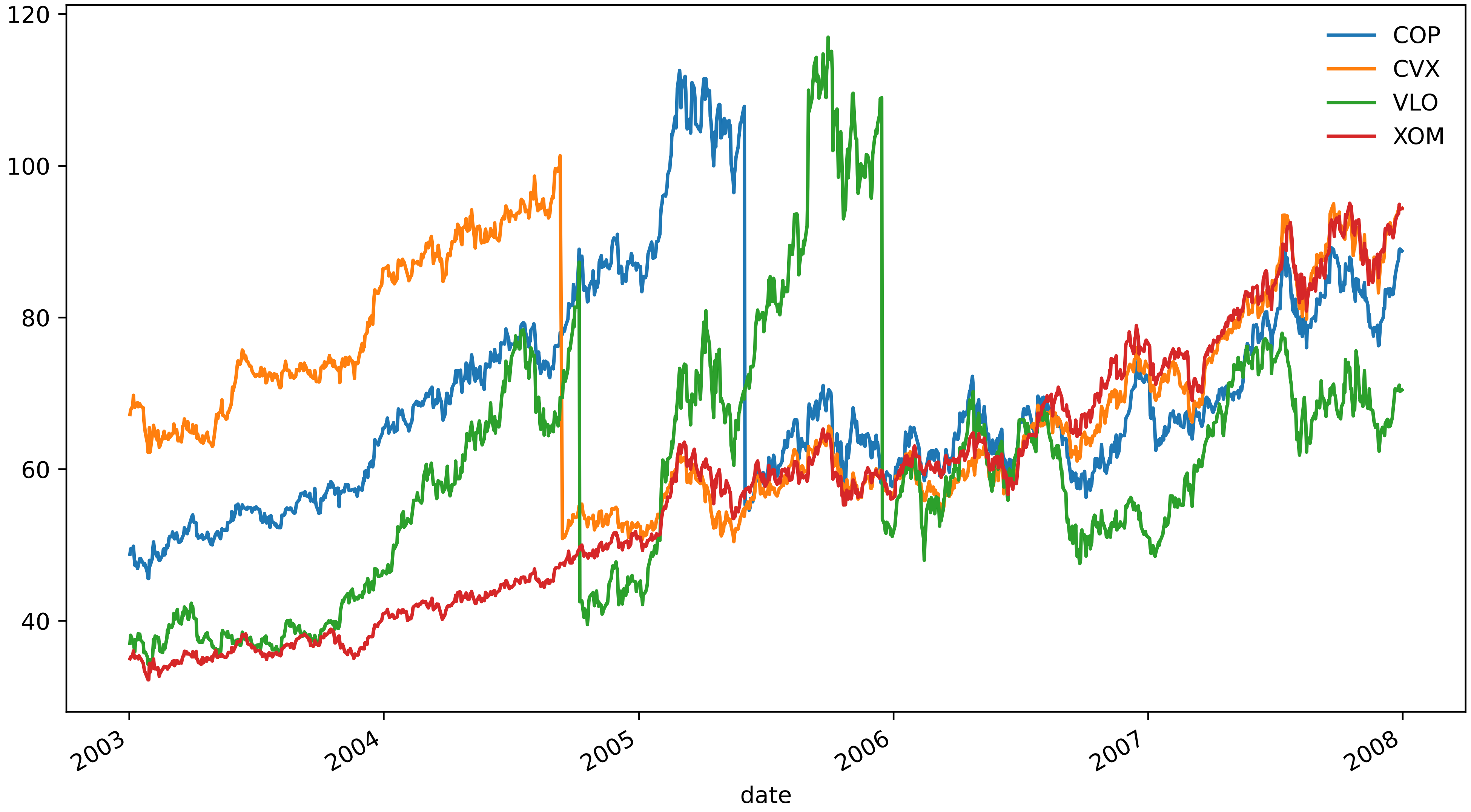
4.60

```
In [85]: df
Out[85]:
     life   rpm brand
0   18.73   610     A
1   14.52   950     A
2   17.43   720     A
3   14.54   840     A
4   13.44   980     A
5   24.39   530     A
6   13.34   680     A
7   22.71   540     A
8   12.68   890     A
9   19.32   730     A
10  30.16   670     B
11  27.09   770     B
12  25.40   880     B
13  26.05  1000     B
14  33.49   760     B
15  35.62   590     B
16  26.07   910     B
17  36.78   650     B
18  34.95   810     B
19  43.67   500     B
```

What is going on? 😅

Cross-validation

(2) Example with **financial** time series: predict CVX's quotes based on other quotes



Cross-validation

(2) Example with **financial** time series: predict CVX’s quotes based on other quotes

```python
from sklearn.ensemble import GradientBoostingRegressor
from sklearn.model_selection import ShuffleSplit, cross_validate
X = quotes.drop(columns=["CVX"])
y = quotes["CVX"]
regressor = GradientBoostingRegressor()
cv = ShuffleSplit(n_splits=10)
results_cv = cross_validate(regressor, X, y, cv=cv)
print(f'Mean R2: {results_cv["test_score"].mean():.2f}')
```

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y}_i)^2}$$

0.95

This looks like an almost perfect prediction! Does it sound right to you?

Cross-validation

(2) Example with **financial** time series: predict CVX's quotes based on other quotes

```
from sklearn.ensemble import GradientBoostingRegressor
from sklearn.model_selection import ShuffleSplit, cross_validate
X = quotes.drop(columns=["CVX"])
y = quotes["CVX"]
regressor = GradientBoostingRegressor()
cv = ShuffleSplit(n_splits=10)
results_cv = cross_validate(regressor, X, y, cv=cv)
print(f'Mean R2: {results_cv["test_score"].mean():.2f}')
```

0.95

$$R^2 = 1 - \frac{\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}{\sum_{i=1}^{n}(y_i - \bar{y}_i)^2}$$

This looks like an almost perfect prediction! 🎊 Does it sound right to you?

```
from sklearn.model_selection import TimeSeriesSplit
cv = TimeSeriesSplit(n_splits=10)
results_cv = cross_validate(regressor, X, y, cv=cv)
print(f'Mean R2: {results_cv["test_score"].mean():.2f}')
```

-3.10

Disappointing, but closer to reality…

- What makes a model **good**?
- Estimating the quality of a model
- Comparing and selecting models

Comparing and selecting models

Suppose we are given a dataset with *p* predictors.

We want to estimate a linear model with only a subset of them.

There are mainly **three strategies** for doing this properly:

- **Shrinkage** fits a model with all p predictors but using a modified loss function that drives some parameters to zero automatically. (Razan will talk about this)

- We can use **dimensionality reduction** techniques to project the *p* predictors to a lower-dimensional subspace. (We will see how to do this in CM4 and TP2)

- In **subset selection** we identify a subset of the predictors that seems the most adequate and then fit a model with them.

Comparing and selecting models

When p is large, testing all 2^p possible models can be very time consuming…

So we prefer to proceed greedily with e.g. **forward stepwise selection**

(1) Let $\mathcal{M}_0$ denote a model with no predictors (i.e. just the intercept)

(2) For $k = 0, \ldots, p-1$

- **(a)** Consider all $p - k$ models that augment the predictors in $\mathcal{M}_k$ by one extra predictor
- **(b)** Choose the **best** among these $p - k$ models and call it $\mathcal{M}_{k+1}$

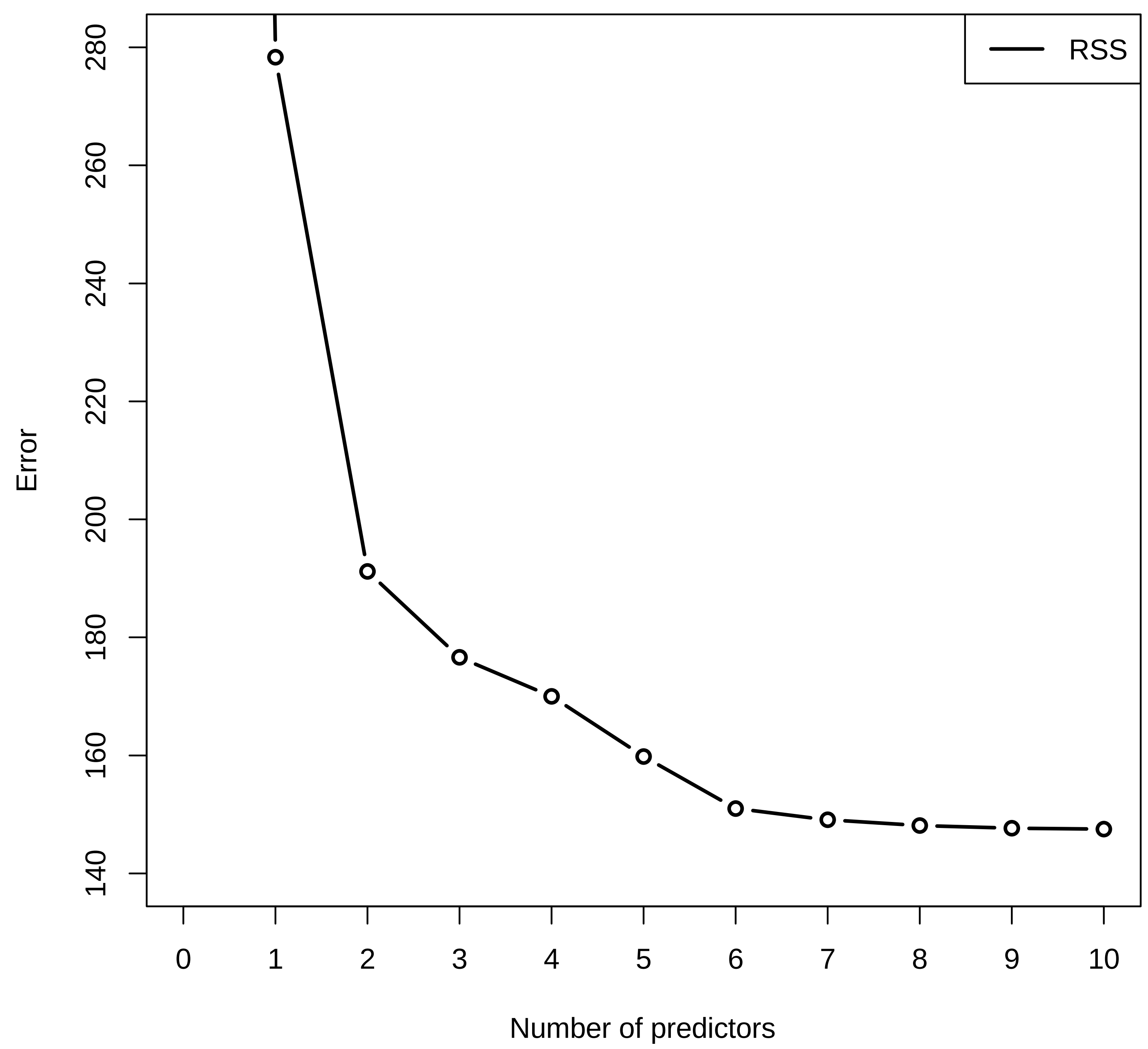
(3) Select single **best** model among $\mathcal{M}_0, \ldots, \mathcal{M}_p$ using cross-validated prediction error

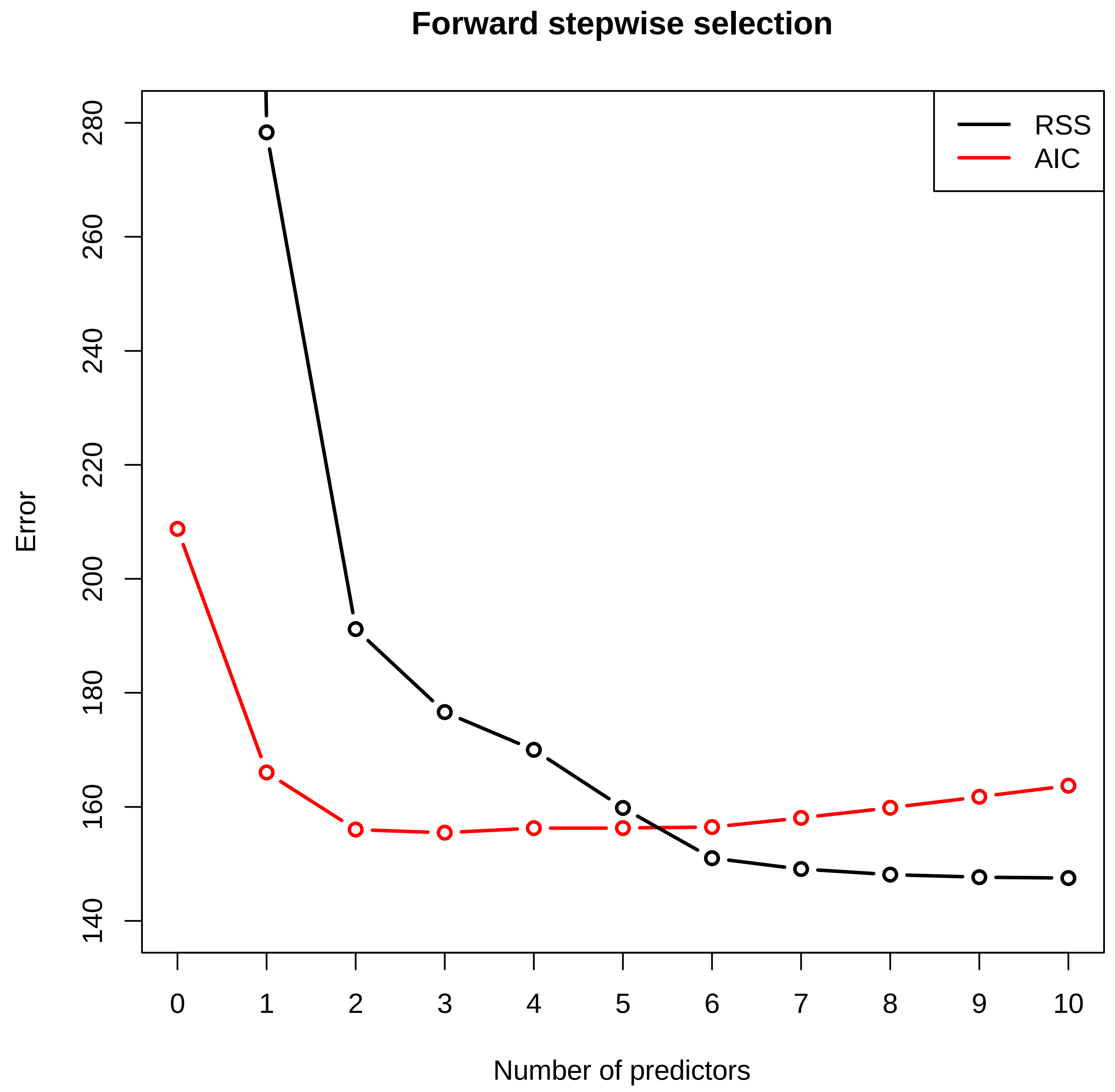
Example: Consider the mtcars dataset – we want to predict **mpg**

```
Description:

     The data was extracted from the 1974 _Motor Trend_ US magazine,
     and comprises fuel consumption and 10 aspects of automobile design
     and performance for 32 automobiles (1973-74 models).

Format:

     A data frame with 32 observations on 11 variables.

       [, 1]  mpg   Miles/(US) gallon
       [, 2]  cyl   Number of cylinders
       [, 3]  disp  Displacement (cu.in.)
       [, 4]  hp    Gross horsepower
       [, 5]  drat  Rear axle ratio
       [, 6]  wt    Weight (lb/1000)
       [, 7]  qsec  1/4 mile time
       [, 8]  vs    V/S
       [, 9]  am    Transmission (0 = automatic, 1 = manual)
       [,10]  gear  Number of forward gears
       [,11]  carb  Number of carburetors
```

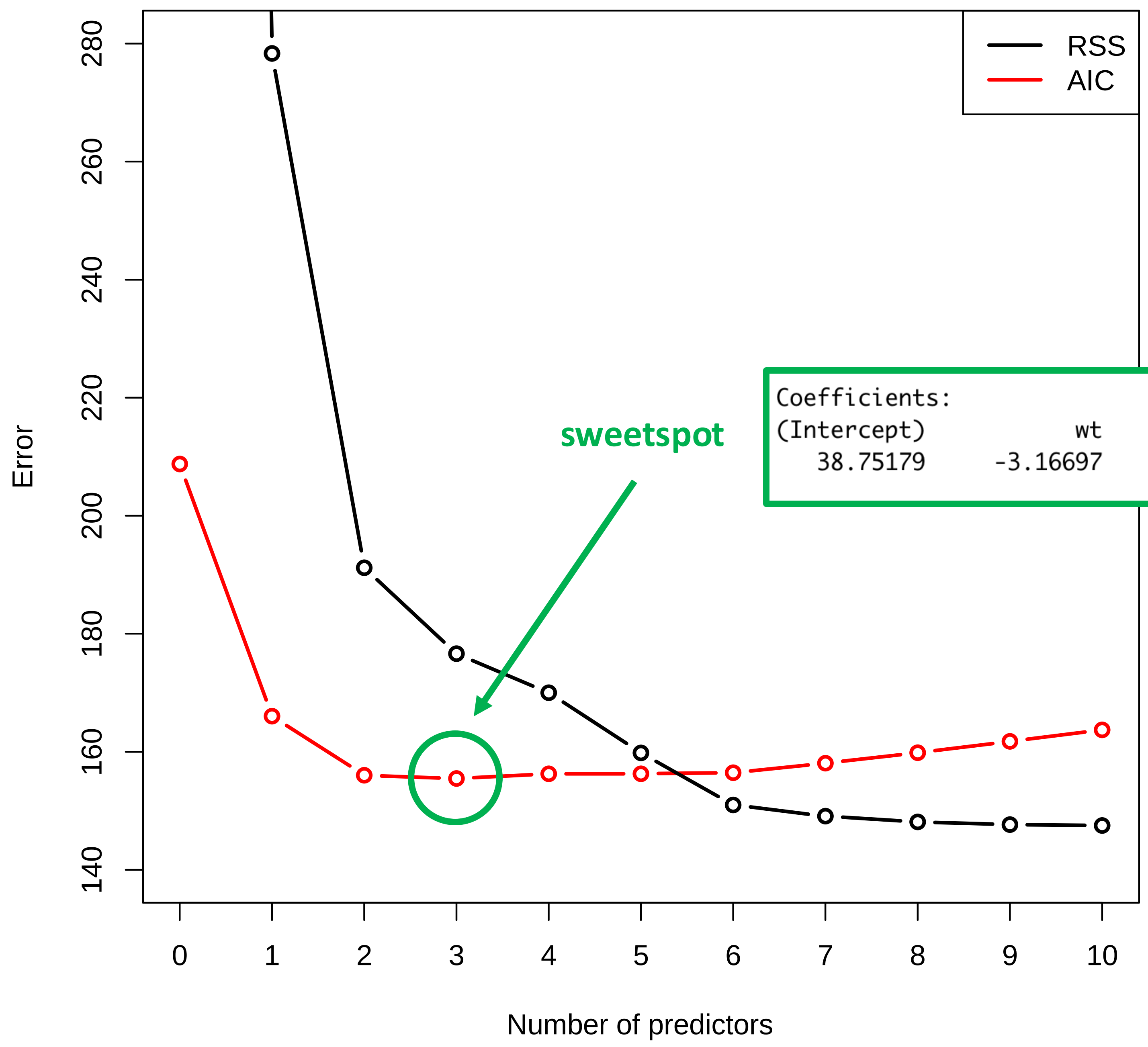
```
                            OLS Regression Results
==============================================================================
Dep. Variable:                    mpg   R-squared:                       0.869
Model:                            OLS   Adj. R-squared:                  0.807
Method:                 Least Squares   F-statistic:                     13.93
Date:                Fri, 27 Dec 2024   Prob (F-statistic):           3.79e-07
Time:                        15:09:11   Log-Likelihood:                -69.855
No. Observations:                  32   AIC:                             161.7
Df Residuals:                      21   BIC:                             177.8
Df Model:                          10
Covariance Type:            nonrobust
==============================================================================
                 coef    std err          t      P>|t|      [0.025      0.975]
------------------------------------------------------------------------------
cyl           -0.1114      1.045     -0.107      0.916      -2.285       2.062
disp           0.0133      0.018      0.747      0.463      -0.024       0.050
hp            -0.0215      0.022     -0.987      0.335      -0.067       0.024
drat           0.7871      1.635      0.481      0.635      -2.614       4.188
wt            -3.7153      1.894     -1.961      0.063      -7.655       0.224
qsec           0.8210      0.731      1.123      0.274      -0.699       2.341
vs             0.3178      2.105      0.151      0.881      -4.059       4.694
am             2.5202      2.057      1.225      0.234      -1.757       6.797
gear           0.6554      1.493      0.439      0.665      -2.450       3.761
carb          -0.1994      0.829     -0.241      0.812      -1.923       1.524
intercept     12.3034     18.718      0.657      0.518     -26.623      51.229
==============================================================================
```

Forward stepwise selection




Forward stepwise selection
RSS
AIC
Error
280
260
240
220
200
180
160
140
0
1
2
3
4
5
6
7
8
9
10
Number of predictors

Forward stepwise selection



```
Coefficients:
(Intercept)           wt          cyl           hp
   38.75179     -3.16697     -0.94162     -0.01804
```

– **Question 10: (credits to Berkeley CS-189)**

In the following statements, the word “bias” is referring to the bias-variance decomposition.

(A) A model trained with N training points is likely to have lower variance than a model trained with $2N$ training points.

(B) If my model is underfitting, it is more likely to have high bias than high variance.

(C) Increasing the number of parameters (weights) in a model usually improves the test set accuracy.

(D) None of the above.

– **Question 19: Bias-variance decomposition (credits to EPFL CS-433)**

Consider a regression model where data (x, y) is generated by input $x \in \mathbf{R}$ uniformly sampled between $[0,1]$ and $y = x + \varepsilon$, where ε is random noise with mean 0 and variance 1. Two models are carried out for regression: model $\mathcal{A}$ is a trained quadratic function $g_{\mathcal{A}}(x, \boldsymbol{\beta}) = \beta_0 + \beta_1 x + \beta_2 x^2$ and model $\mathcal{B}$ is a constant function $g_{\mathcal{B}}(x) = \frac{1}{2}$. Compared to model $\mathcal{B}$, model $\mathcal{A}$ has

(A) Higher bias, higher variance.

(B) Lower bias, higher variance.

(C) Higher bias, lower variance.

(D) Lower bias, lower variance.

– **Question 22: Linear regression (credits to Berkeley CS-189)**

In linear regression, we model $p(y \mid \mathbf{x}) \sim \mathcal{N}(\beta^\top \mathbf{x} + \beta_0, \sigma^2)$. The irreducible error in this model is

(A) σ^2

(B) $\mathbb{E}[y \mid \mathbf{x}]$

(C) $\mathbb{E}[(y - \mathbb{E}[y \mid \mathbf{x}])^2 \mid \mathbf{x}]$

(D) None of the above.