Undergraduate Mathematical Analysis

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Chapter 1

In the Beginning

Before we begin studying anything about real numbers, we need a couple of ground rules to start our work. And everything else about the real numbers needs to be proven from these ground rules. Now we will attempt to formalize a fundamental structure that will help us to define what the real numbers are, and what makes the real numbers different than other number systems, such as the natural numbers and rational numbers.

It is important to note that historically, we just assumed the definition and properties of real numbers intuitively, and only about 120 years ago, the mathematics community actually constructed real numbers formally from the very fundamental ideas of natural numbers. These fundamental ideas regarding the natural numbers are called the ***Peano Axioms*** as they were put forth by the Italian mathematician Giuseppe Peano (1858-1932). There are a couple of ways to construct the reals from the naturals, but these methods are lengthy, and the steps are just some 'unwieldy' manipulations and repeated use of the principle of induction. Furthermore, in my personal experience, most first-time learners are unable to appreciate the beauty behind such constructions. But the importance of this matter cannot be ignored at all, and hence, one such construction is given in the appendix section of the text.

The objectives of this chapter are-

- Establishing the Fundamentals.
- The Least Upper Bound and the Greatest Lower Bound.
- The Completeness Property of the Real Numbers and its Consequences.
- The Real Numbers.
- The Complex Numbers.
- The Metric Spaces.
- The Cardinality of Sets.

1.1 The Structure of Real Numbers

1.1.1 Fields

Before we dive into the heart of real numbers or any other numbers, we want to ensure that the number we are studying obeys the usual algebra laws of addition and multiplication. If our number type does not follow the usual law of algebra, then it will become difficult for us to study and analyze it. So, in order to study nice numbers that obey the algebra rules we are familiar with, we need to formalize the algebra rules, and this gives rise to a fundamental notion in mathematics, termed 'field'.

Definition 1.1.1: Field, F

Fields are an algebraic structure equipped with the operations of addition($+$) and multiplication($\cdot$) such that the following properties hold:

1. **Closure property of addition:**
For all $a, b \in F$, then $(a + b) \in F$.

2. **Commutativity property of addition:**
For all $a, b \in F$, then $a + b = b + a$.

3. **Associativity property of addition:**
For all $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.

4. **Existence of additive entity element:**
There exists $0 \in F$, such that for all $a \in F,\ a + 0 = a$.

5. **Existence of additive inverse element:**
For each and every individual $a \in F$, there exists a corresponding individual element $-a \in F$, such that $a + (-a) = 0$.

6. **Closure property of multiplication:**
For all $a, b \in F$, then $(a \cdot b) \in F$.

7. **Commutativity property of multiplication:**
For all $a, b \in F$, then $a \cdot b = b \cdot a$.

8. **Associativity property of multiplication:**
For all $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

9. **Existence of multiplicative entity element:**
There exists $1 \in F$ & $1 \neq 0$, such that for all $a \in F,\ a \cdot 1 = a$.

10. **Existence of multiplicative inverse element:**
For each and every individual element $a \in F$ & $a \neq 0$, there exists a corresponding individual element $a^{-1} \in F$, such that $a \cdot a^{-1} = 1$.

11. **Distributivity of multiplication over addition:**
If $a, b, c \in F$, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

Disclaimer: New readers of mathematics textbooks are advised to notice that the usage of 'for all' and 'for each and every' are different, for these terms will accompany you for the rest of your mathematics career.

1. 'For all' is the real-life equivalent to 'one size fits all', i.e., the same element acts the same for other members. Like 0 is the additive identity, so does not matter our choice of number, $1 + 0 = 1, 2 + 0 = 2, \ldots$. So, 0 is the additive identity for $1, 2, 3, \ldots$.

2. 'For each' means that all elements will have an element that acts on it, but unlike the 'for all' case, the acting element will be dependent on the choice of the original element. In the case of additive inverse, 1 has -1, 2 has -2, and so on and so forth. -1 is the additive inverse of 1 only, not 2 or other members. This sense of partnership is represented by 'for each'.

Although we have used addition and multiplication in the definition of fields, the two operations can be anything else as long as they satisfy the *properties* similar to addition and multiplication as per the definition, not necessarily *be* addition or multiplication themselves. One such example has been given as one of the exercises.
Most of the times, when checking if a set equipped with the operations of addition and multiplication is a field or not, the main property to investigate is the closure under addition and multiplication. As we shall see in an example later in this chapter, sometimes, investigating this property is also one of the most difficult tasks ever.
Now, let us see what number types we encounter primarily are fields and what are not fields in the following table:

Number Types	**Is it a Field?**
Natural Numbers, $\mathbb{N}$	No
Integer Numbers, $\mathbb{Z}$	No
Rational Numbers, $\mathbb{Q}$	Yes
Irrational Numbers, $\mathbb{I}$	No
Real Numbers, $\mathbb{R}$	Yes
Complex Numbers, $\mathbb{C}$[1]	Yes

Table 1.1: An Illustration of Number Types and Fields.

The verification of the statements in this table is left as an exercise to the readers. (Not joking, it actually is the last exercise of this section! So, the best course of action for the reader is to solve it right now.)
This investigation tells us that $\mathbb{Q}, \mathbb{R}, \mathbb{C}$ are quite similar in structure and also presents us with some questions to ponder over if we *combine* the sets of numbers mentioned above. Such as -

Illustrative Example 1.1.1

A. What type of number is the sum & product of a rational and an irrational number?

B. What type of number is the sum & product of a real and a complex number?

C. As the set of irrationals is not closed under addition & multiplication, what type of numbers are $\pi + e,\ \pi \cdot e,\ \pi^e,\ e^\pi$? Are they rational or irrational?

In addition to these questions, we can also ask about the nature of the sum and product of other number types. Now, we will attempt to answer these questions one by one.

Nature of sum of a rational and an irrational number:
Proof. Before we try to answer in general settings, it is wise to gather as much information as possible in the context of some particular setting. Doing so will give us an idea about how to proceed to answer, and it is also instructive for our *mathematical intuition*.
We can clearly see that $\pi + 0 = \pi$, so the sum of an irrational number with a rational number cannot be a rational number. We

[1] Just as a reminder, we define a complex number, z, to be $z = a + ib$ where $a, b \in \mathbb{R}$ and $i^2 = -1$.

know that a number must be either rational or irrational. With these pieces of information, it is time to introduce a wonderful *trick* that we will use in our journey of mathematical analysis, and the name of that trick is ***proof by contradiction***.

Trick 1: Proof by Contradiction

When to use proof by contradiction?

- Impossible to directly prove a statement.
- The possible answers to the question are exhaustive, i.e., at least one out of the possible outcomes must be true. Normally, the options are binary, i.e., mutually exclusive and exhaustive.

Where is proof by contradiction used?
In binary outcome scenarios such as rational or irrational, countable or uncountable, exist or does not exist, true or false, etc.

Structure of the proof by contradiction:-

1. Check if the possible outcomes are exhaustive or not. Then check if the options are mutually exclusive and exhaustive.
2. From studying specific cases, identify the outcome that is blatantly true but not provable directly.
3. Assume the ***"opposite of the true outcome"*** to be true.
4. Continue calculation until a statement is found that is absolutely against previously learned knowledge, such as going against definitions or definitive properties.
5. Conclude that the assumption is wrong and as the options are mutually exhaustive and exclusive, the opposite of the assumption, i.e. ***the original statement we wanted to prove*** is right.
6. Bask in the aftermath of the glory of proof by contradiction.

Now, by definition, a rational number is a number $x = \frac{p}{q}$ where p, q are co-prime numbers and $q \neq 0, 1$. We can see from $\pi + 0 = \pi$ that the sum of a rational and an irrational number is an irrational number. But we cannot prove it directly. So, we will assume that the sum of a rational and an irrational number is the *opposite* of an irrational number, i.e., a rational number.
As per our assumption, $a + b = c$ where a is an irrational number, b is a rational number, and c is a rational number. Then by definition of the rational numbers, we can express $b = \frac{p_1}{q_1}$ and $c = \frac{p_2}{q_2}$ where both q_1 and q_2 are not equal to 0 and the numerators and denominators are co-prime numbers. Now,

$$\begin{aligned} & a + b = c \\ \implies & a = c - b \\ \implies & a = \frac{p_2}{q_2} - \frac{p_1}{q_1} \\ \implies & a = \frac{p_2 q_1 - p_1 q_2}{q_1 q_2} \end{aligned}$$

But a is an irrational number by construction. So, it should not be possible to express a as a fraction of integers. Therefore, our initial assumption is wrong.
$\therefore$ The sum of an irrational number and a rational number is an irrational number. ■

Nature of the product of an irrational and a rational number:

Proof. Similar to the proof above, we can see that $\pi \cdot 1 = \pi$. So the product is an irrational number. However, for proof by contradiction, we will assume that the product is a rational number.
According to our assumption, $a \cdot b = c$ where a is an irrational number, b is a rational number, and c is a rational number. Then by definition of the rational numbers, we can express $b = \frac{p_1}{q_1}$ and $c = \frac{p_2}{q_2}$ where both q_1 and q_2 are not equal to 0 and the

numerators and denominators are co-prime numbers. Now,

$$\begin{aligned} a \cdot b &= c \\ \implies a &= \frac{c}{b} \\ \implies a &= \frac{\frac{p_2}{q_2}}{\frac{p_1}{q_1}} \\ \implies a &= \frac{p_2 q_1}{p_1 q_2} \end{aligned}$$

To be well-defined, we also have to impose the extra condition $p_1 \neq 0$. But a is an irrational number by construction. So, it should not be possible to express a as a fraction of integers. Therefore, our initial assumption is wrong.
$\therefore$ The product of an irrational number and a rational number is an irrational number. ■

Nature of the sum and product of a real and a complex number:
Proof. We know that a complex number $z = a+ib$ where $a, b \in \mathbb{R}$ and $i^2 = -1$. When we set $b = 0$, then we get $z = a \implies z \in \mathbb{R}$. Again, setting $a = 0$ gives $z = ib \implies z \in \mathbb{C}, z \notin \mathbb{R}$. So, $\mathbb{R} \subset \mathbb{C}$, that is, the real numbers are a proper subset of the complex numbers. This implies that $\mathbb{C} \cap \mathbb{R} = \mathbb{R}$ and $\mathbb{C} \cup \mathbb{R} = \mathbb{C}$. So the set of real numbers and complex numbers is not a disjoint set, and o any attempts to use proof by contradiction will fail here.
Let, $r \in \mathbb{R}$, then, $r + z = r + a + ib = (r + a) + ib = a' + ib \in \mathbb{C}$ where $a' = r + a$.
Again, $rz = r(a + ib) = (ra) + i(rb) = c + id \in \mathbb{C}$ where $c = ra, d = rb$.
$\therefore$The sum and product of a real number and a complex number is a complex number. ■

☠ **Nature of the combination of π and e:**
Proof. We know that both π and e are irrational numbers; not only that, but they are also **transcendental numbers**. The proofs of their irrationality & transcendentality are complicated and hence, we will take them for granted. Recall that transcendental numbers are real or complex numbers that are not the roots of a non-zero polynomial with rational coefficients. Now, let us construct a quadratic equation with the roots being π and e.

$$\begin{aligned} (x - \pi)(x - e) &= 0 \\ \implies x^2 - (\pi + e)x + \pi e &= 0 \end{aligned}$$

By definition of transcendental numbers, the quadratic equation is not a polynomial with rational coefficients. As such, we can infer that at least one of the expressions $\pi + e$ or πe is irrational.
The number e^π is called **Gelfond's constant** and is an irrational number. The proof is quite complicated and beyond the scope of this text.
As for π^e, we do not know if it is rational or irrational. So, as of writing this, we have no answer about the irrationality (or rationality?) of $\pi + e,\ \pi e,\ \pi^e$. ■

The reason for mentioning these simple but unsolved problems this early into the text is to demonstrate that despite the extensive research into mathematics for quite a long time, we have yet to tackle the more fundamental questions. It also reaffirms the fact that sometimes, the simplest question has the hardest answer. Oftentimes, mathematics textbooks present the topics in such a way that the students are lulled into thinking that everything is solved sleekly and systematically, while the reality is far from it. These unsolved problems are my way of showing the 'reality' of mathematical research.

Exercise 1A

1. **Field without Addition and Multiplication as Operations?** All of us are familiar with division. Let $a, b \in \mathbb{N}_0$, that is a, b are non-negative integers, then if a is divided by b, we can write it as follows, $a = qb + r$ where $q = quotient$ and $r = remainder$. We also know that the remainder, r, has the further restriction $0 \leq r \leq (b - 1)$. The number of possible values taken by r is b as r can be $0, 1, 2, \ldots, (b - 2), (b - 1)$.
   If we fix b but keep a as a variable, then we will find that some numbers will have remainder 0; we group these numbers under a new set denoted as $\{\bar{0}\}$. In the same way, for numbers with $r = 1$, we construct the set $\{\bar{1}\}$, for numbers with $r = 2$, we construct the set $\{\bar{2}\}$, so on and so forth. We will end up with b number of new sets, with the sets being $\{\bar{0}\}, \{\bar{1}\}, \{\bar{2}\}, \ldots, \{\overline{(b-2)}\}, \{\overline{(b-1)}\}$. In this way, we have managed to categorize the numbers a with respect to the

remainder r, when we divide a by b.
Now, we will impose further restrictions on b by taking b to be a prime number, i.e., $b = 2, 3, 5, \ldots$. We now introduce the two operations of remainder addition and remainder multiplication to the sets we constructed above, and keep in mind that the sum and product of the sets have to be expressed in the form of the sets constructed above too, i.e. the sum and product also have to be grouped as different remainder sets.

a. Are the operations of addition and multiplication introduced in the question in the setting of the remainder, the same as usual addition and usual multiplication we are familiar with? Explain why they are the same or different. [Hint: Start with $b = 2$ first and see where it takes you.]
b. Does this construction of sets with respect to the remainder, equipped with the operations introduced in the question, form a field? Explain your answer fully by examining a particular choice of b. [Hint: Same as the previous question.]
c. The restriction of b to prime numbers may seem a little bit out of the box. So, check if the same construction of the question forms a field or not, but this time take b to be a composite number. [Hint: Check what happens if $b = 4$.]

For further investigation, look up 'Modular Arithmetic and Fields' on the internet. (After solving this exercise, of course!)

2. **Consequences of the Axioms of Addition of Field:** For any $x, y, z \in \mathbb{F}$, prove that

   a. If $x + y = x + z$, then $y = z$.
   b. If $x + y = x$, then $y = 0$.
   c. If $x + y = 0$, then $x = -y$.
   d. $-(-x) = x$.
   e. The Additive Identity is Unique[2]. [Hint: Either there is another additive inverse or there are no different additive inverses, so binary outcome scenario. Wish the text introduced some *trick* about such things!]
   f. The Additive Inverse is Unique.

3. **Consequences of the Axioms of Multiplication of Field:** For any $x, y, z \in \mathbb{F}$, prove that

   a. If $x \neq 0$ and $xy = xz$, then $y = z$.
   b. If $x \neq 0$ and $xy = x$, then $y = 1$.
   c. If $x \neq 0$ and $xy = 1$, then $y = \frac{1}{x}$.
   d. If $x \neq 0$, then $\frac{1}{\frac{1}{x}} = x$.
   e. The Multiplicative Identity is Unique.
   f. The Multiplicative Inverse, when it exists, is Unique.

4. **Consequences of the Field Axioms:** For any $x, y \in F$, prove that

   a. $0x = 0$.
   b. If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$.
   c. $(-x)y = x(-y) = -(xy)$.
   d. $(-x)(-y) = xy$.

5. **The numbers that do not form a field in the table (1.1), elaborate why.**

1.1.2 Ordered Sets and Fields

Now that we have defined a field, we want to sort out how the members are arranged in said field. This necessity for sorting provides us with the need for the notion of ordering.

[2] In mathematics, the uniqueness of something means that there is only one of that thing. This is one of the wonderful times when English and Mathematics agree on their meaning.

Definition 1.1.2: Totally Ordered Set

Let there exist a set X and assume there exists a homogeneous binary relation[3] $\leq$, which will be called order if it satisfies the following:

1. Reflexivity: For all $a \in X$, $a \leq a$.
2. Connectedness: For all $a, b \in X$, either $a \leq b$ or $b \leq a$.
3. Anti-symmetry: For all $a, b \in X$, if $a \leq b$ and $b \leq a$, then $a = b$.
4. Transitivity: For all $a, b, c \in X$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

The set X will then be called a totally ordered set.

If we take a close look at the definition, it is quite apparent that from connectedness, we get reflexivity. While reflexivity, anti-symmetry, and transitivity are familiar to most, the connectedness property may be new to some, and it is also quite consequential. Connectedness implies that we can establish the binary relation between any two elements of the set; that is, every element of the set can be taken to establish the binary relation. As this binary relation can be applied to the entirety of the set, this is defined as 'total ordering'. Now, one thing the reader can ask is how the notion of ordering is relevant to the notion of field we have defined before. We will soon introduce a concept that combines the ideas of both field and order. Without even defining that concept, some questions may arise in the mind of the interested readers after some deliberation, such as whether it is always possible to equip any field with an order, whether we can equip a field with an order, and whether the ordering is unique or not. These questions will be answered gradually, and for now, let us keep these questions on the sidelines so that we can finally tackle the crux of this section, an ordered field, and define it.

Definition 1.1.3: Ordered Field

Let F be a field. A field is said to be ordered if equipped with the total order $\leq$, it satisfies the following properties:

1. For all $a, b, c \in F$, if $a \leq b$ then $a + c \leq b + c$.
2. For all $a, b \in F$, if $a \geq 0, b \geq 0$ then $ab \geq 0$ with $ab = 0$ being true when at least one of a, b is 0.

We can also define an ordered field in another way, which utilizes the concept of a positive cone. This alternative approach and its equivalence with the totally ordered field will be handled as one of the exercises.

Number Types	**Is it an Ordered Field?**
Rational Numbers, $\mathbb{Q}$	Yes
Real Numbers, $\mathbb{R}$	Yes
Complex Numbers, $\mathbb{C}$	No

Table 1.2: An Illustration of Number Types and Ordering.

While it is apparent that the fields $\mathbb{Q}, \mathbb{R}$ are ordered fields, the field $\mathbb{C}$ being unordered may come across as quite surprising to the learners. The proof of $\mathbb{C}$ being an unordered field will be tackled here.

[3]**Homogeneous Binary Relation:** Subset of the Cartesian product of $X \times X$. The symbol $\leq$ is used for convenience as it looks natural in the setting of the real numbers; the order symbol can also be $\sim$ or any other symbol of choice.

Illustrative Example 1.1.2

The Field of Complex Numbers, $\mathbb{C}$, is NOT Totally Ordered.

Proof. The number that defines $\mathbb{C}$ is i, which has the property $i^2 = -1$. Keeping this in mind, let us assume that $\mathbb{C}$ is totally ordered, then by the connectedness property of a totally ordered set, we can say that,

$$\text{For all } a, b \in \mathbb{C}, \text{ either } a \leq b \text{ or } b \leq a$$

Setting $a = i, b = 0$ gives,

$$\text{Either } i \leq 0, \text{ or } i \geq 0$$

Case I: Let $i \leq 0 \implies -i \geq 0 \implies (-i)(-i) \geq 0 \implies i^2 \geq 0$. However, $i^2 = -1 \implies i^2 \ngeq 0$.
So, $i \leq 0$ is not possible.
Case II: Let $i \geq 0 \implies i.i \geq 0 \implies i^2 \geq 0$. Again, $i^2 = -1 \implies i^2 \ngeq 0$.
So, $i \geq 0$ is also not possible.
This means that our assumption of i being totally ordered is false. Then we deduce that i cannot be totally ordered and this, in turn, implies that $\mathbb{C}$ cannot be totally ordered[4]. ■

Naturally, we have known $\mathbb{R}$ far longer than $\mathbb{C}$, and the reals being a totally ordered set was also known. But as example 1.1.2 showed us, the complex numbers are not totally ordered; rather, only a particular subset of $\mathbb{C}$ (i.e. $\mathbb{R}$) is totally ordered. This mathematical fact actually gave rise to the concepts of 'partially ordered set' and 'chains'.

Definition 1.1.4: Partially Ordered Set

Let there exist a set X and assume there exists a homogeneous binary relation $\preceq$, which will be called order if it satisfies the following:

1. Reflexivity: For all $a \in X$, $a \preceq a$.
2. Anti-symmetry: For $a, b \in X$, if $a \preceq b$ and $b \preceq a$, then $a = b$.
3. Transitivity: For $a, b, c \in X$, if $a \preceq b$ and $b \preceq c$, then $a \preceq c$.

The set X will then be called a partially ordered set.

An important distinction is that connectedness is not required in a partially ordered set. This means that not every element of the set can be subject to the relation defined. As we can invoke the relation with partial members of the set, this ordering is called 'partial ordering'. This also implies that **every totally ordered set is a partially ordered set, but the opposite is not true**.

Illustrative Example 1.1.3

Some examples of partial order relations are:

1. Set Inclusion: $A \subseteq B$. For some A, A might partially overlap B that makes $A \nsubseteq B$, or A, B are completely different sets, making $A \nsubseteq B$.
2. Number Divisibility: For two numbers $a, b \in \mathbb{N}$, $a|b \iff$ there exists some $k \in \mathbb{N}$ such that $b = k \cdot a$. So, b is divisible by a without any remainder. Such a relation is an example of partial order relation as not all numbers are divisible with remainder zero.

The verification of the reflexivity, anti-symmetry, and transitivity property is left to the readers.

[4] Notice that we have used proof by contradiction here without stating it explicitly. In the upcoming proofs, when they are quite straightforward, the text will not mention the method of proof used. It is beneficial for the readers to figure out these short proofs so that they can learn to handle the more sophisticated ones on their own.

Definition 1.1.5: Chains

Let X be a partially ordered set and $E \subset X$, i.e., E is a proper subset of X. If the set E itself is a totally ordered set with respect to the same homogeneous binary relation $\preceq$, then E is called a Chain.
In other words, a chain is a totally ordered subset of a partially ordered set.

Now, we will go back to discussing totally ordered sets. The rationale for using the $\leq$ symbol in the definition of the totally ordered set will be justified now. The real numbers equipped with the binary relation of the usual less than or equal to ($\leq$) form a totally ordered set.

One thing to note is that total ordering is shown by the $\leq$ relation, but it is also possible to define total ordering using just the $<$ relation. This requires modifying the definition of a totally ordered set a little bit.

Definition 1.1.6: Strict Totally Ordered Set

Let there exist a set X and assume there exists a homogeneous binary relation $<$, which will be called order if it satisfies the following:

1. Irreflexivity: For all $a \in X$, $a \nless a$.
2. Connectedness: For all $a, b \in X$, if $a \neq b$, then either $a < b$, or $b < a$.
3. Asymmetry[5]: For all $a, b \in X$, if $a < b$ then $b \nless a$.
4. Transitivity: For all $a, b, c \in X$, if $a < b$ and $b < c$, then $a < c$.

The set X will then be called a strict totally ordered set.

As the new order ($<$) is stricter than the old order ($\leq$), the term 'strict' is incorporated. Again, it is apparent that the real numbers equipped with the $<$ order also form a strict totally ordered set. In a similar fashion, the definition of the strict partially ordered set can also be constructed.

Definition 1.1.7: Strict Partially Ordered Set

Let there exist a set X and assume there exists a homogeneous binary relation $\prec$, which will be called order if it satisfies the following:

1. Irreflexivity: For all $a \in X$, $a \nprec a$.
2. Asymmetry: For $a, b \in X$, when $a \neq b$, if $a \prec b$ then $b \nprec a$.
3. Transitivity: For $a, b, c \in X$, if $a \prec b$ and $b \prec c$, then $a \prec c$.

The set X will then be called a strict partially ordered set.

Mathematicians make their bread and butter by extending definitions and it is no surprise to attempt to do the same thing for the concept of ordering. If we look at the definitions of the totally and partially ordered set, we will see that both of the definitions have two properties in common, namely reflexivity and transitivity. Even the definition of equivalence class has these two properties as well. This gives us sufficient motivation to study a relation having only these two properties. Furthermore, if we are confining ourselves to the strict ordering, and desire a similar sort of generalization, then we get a relation that only satisfies irreflexivity and transitivity. And to that end, let us take advantage of this moment to introduce two new and more general ideas as follows -

[5]**Anti-symmetry vs Asymmetry:** Note that anti-symmetry and asymmetry are different. An asymmetric relation is where (a, b) is in the relation, but (b, a) is not. An anti-symmetric relation is where (a, b) and (b, a) are in the relation if and only if $a = b$.

Definition 1.1.8: Preordered Set

Let there exist a set X and assume there exists a homogeneous binary relation $\lesssim$, which will be called a preorder if it satisfies the following:

1. Reflexivity: For all $a \in X, a \lesssim a$.
2. Transitivity: For $a, b, c \in X$, if $a \lesssim b$ and $b \lesssim c$, then $a \lesssim c$.

The set X will then be called a preordered set.

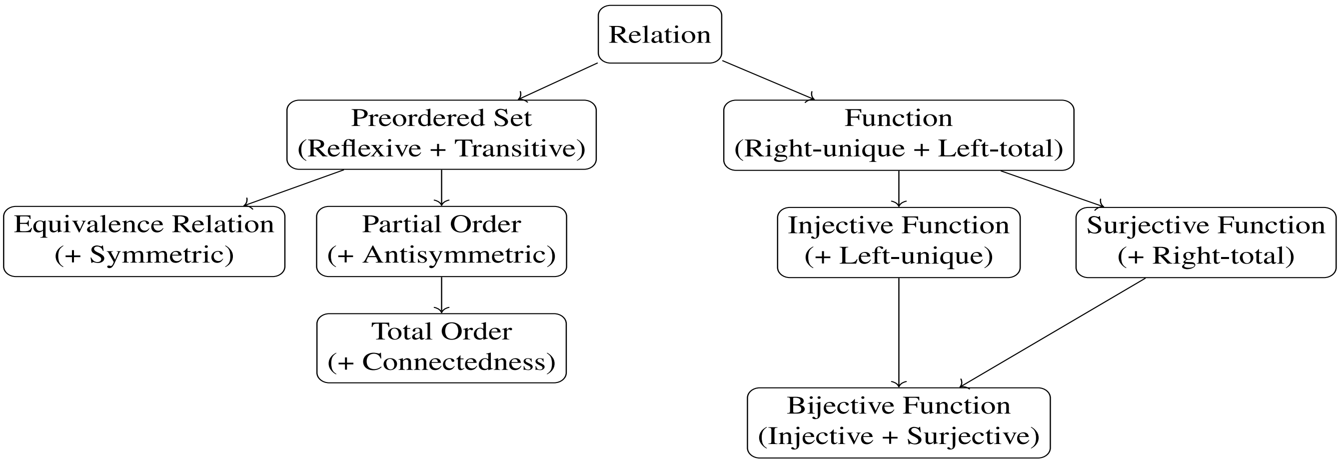
Definition 1.1.9: Strict Preordered Set

Let there exist a set X and assume there exists a homogeneous binary relation R_{spreo}, which will be called a strict preorder if it satisfies the following:

1. Irreflexivity: For all $a \in X, aR_{spreo}a$.
2. Transitivity: For $a, b, c \in X$, if $aR_{spreo}b$ and $bR_{spreo}c$, then $aR_{spreo}c$.

The set X will then be called a strictly preordered set.

Below is a diagram summarizing all the important relations we are going to encounter in this text :



1.1.2.1 Ordering and Elements

With all this talk about the ordering, it is natural to ask which elements are the smallest and which are the greatest. This gives us the following -

Definition 1.1.10: Greatest Element of a Set

Let $(X, \lesssim)$ be a preordered set. For the set $Y \subseteq X$, an element $g \in X$ is defined to be a greatest element of the set Y if:

1. $g \in Y$.
2. $a \lesssim g$ for all $a \in Y$.

Definition 1.1.11: Least Element of a Set

Let $(X, \lesssim)$ be a preordered set. For the set $Y \subseteq X$, an element $l \in X$ is defined to be a least element of the set Y if:

1. $l \in Y$.
2. $l \lesssim a$ for all $a \in Y$.

We have discussed elements that are bigger or smaller than all other elements, now let us put a spin to this idea. Let us look for elements that are not smaller (or larger) than any other elements.

Definition 1.1.12: Maximal Element of a Set

Let $(X, \lesssim)$ be a preordered set. For the set $Y \subseteq X$, an element $m \in Y$ is defined to be a maximal element of the set Y if $s \in Y$ satisfies $m \lesssim s$, then $s \lesssim m$ must be true.

Definition 1.1.13: Minimal Element of a Set

Let $(X, \lesssim)$ be a preordered set. For the set $Y \subseteq X$, an element $m \in Y$ is defined to be a minimal element of the set Y if $s \in Y$ satisfies $s \lesssim m$, then $m \lesssim s$ must be true.

A question might come to the mind of the readers, are these elements unique? The answer to this question is left as an interesting exercise for the readers. With that out of the way, we have one final concept to acquaint ourselves with, and that is as follows -

Definition 1.1.14: Well-ordered Set

A totally ordered set $(X, \leq)$ is defined to be well-ordered if every non-empty subset $Y \subseteq X$ has a least element in the ordering.

While we have no use of well-ordered set for now, it will come handy in the future, especially when dealing with the foundational matters of the natural numbers.

Exercise 1B

1. **An Alternate Way of Defining Ordered Field:** A prepositive cone or preordering of a field F is a subset $P \subseteq F$ that abides by following rules:

   a. For all $x, y \in P$ both $x + y,\ x \cdot y \in P$. An immediate consequence of this is that if $x \in P$ and $x = y$, then $x^2 \in P$.

   b. For all $x \in F$, $x^2 \in P$. Notice here that x belongs to the field F, not the subset P.

   c. The element $-1 \notin P$. This means that negative numbers are excluded from P by definition.

   A preordered field is a field F equipped with the preordering P.
   If the field F is the union of P and $-P$, i.e. $P \cup -P = F$, then P is called a positive cone of F. The non-zero members of P are called the positive elements of F.
   Prove the equivalence between the definitions of ordered field and positive cone[6].

2. **Consequences of an Ordered Field:** For $x, y, z \in F$, where F is a strict totally ordered field, prove that

   a. If $x > 0$, then $-x < 0$ and vice versa.

   b. If $x > 0$ and $y > z$, then $xy > xz$.

   c. If $x < 0$ and $y > z$, then $xy < xz$.

   d. If $x \neq 0$, then $x^2 > 0$.

[6] Two statements P, Q are equivalent if $P \implies Q$ and $Q \implies P$. That is, $P \iff Q$. These sorts of statements are also called **if and only if statements, necessary and sufficient conditions**.

e. If $y > x > 0$, then $\frac{1}{x} > \frac{1}{y} > 0$.

3. **Multiple Ordering on the Same Set:** Consider the particular subset of a real number $A = \{t = r + s\sqrt{2} : r, s \in \mathbb{Q}\}$. Notice that $A \subset \mathbb{R}$. So, the usual notion of ordering in real numbers works for the set A. But an interesting thing is that the set A can be ordered in a different way. For that, we will make use of the positive cone definition of an ordered field. Prove that the function $f : r + s\sqrt{2} \to r - s\sqrt{2}$ produces a positive cone[7]. More specifically, prove that for $r - s\sqrt{2} \in P \subseteq A$ and P satisfies all the conditions of a positive cone. This is the second way of ordering for the set A.

4. **Some uniqueness checks:** Prove the following -

   a. Greatest and least element of a set are unique.

   b. Maximal and minimal elements of a set are not unique. Prove it by constructing an example.

1.1.3 A Glimpse into Completeness

In the previous parts, using the concept of fields and orders, we can distinguish $\mathbb{R}, \mathbb{Q}$ from $\mathbb{N}, \mathbb{Z}, \mathbb{I}, \mathbb{C}$. Now, what can we do to distinguish between $\mathbb{R}, \mathbb{Q}$? $\mathbb{Q}$ obviously does not contain any irrational numbers, which $\mathbb{R}$ has. So, let us start from there. We first show the irrationality of a particular number in $\mathbb{R}$, namely $\sqrt{2}$.

Illustrative Example 1.1.4

$\sqrt{2}$ is an Irrational Number.

Proof. Recall that $\mathbb{Q} \cup \mathbb{I} = \mathbb{R}, \mathbb{Q} \cap \mathbb{I} = \varnothing$. So we have a binary scenario, and proof by contradiction is our friend in this case. So, let us assume that $\sqrt{2}$ is a rational number.
At first, we want to show that $\sqrt{2}$ is not an integer. So,

$$\begin{aligned} & 1 < 2 < 4 \\ \implies & 1^2 < (\sqrt{2})^2 < 2^2 \\ \implies & 1 < \sqrt{2} < 2 \end{aligned}$$

With this out of the way, by our assumption of $\sqrt{2}$ being a rational number, we can define $\sqrt{2}$ as follows:

$$\sqrt{2} = \frac{p}{q}, \text{ where } p, q \text{ are relatively prime or co-prime integers and } q \neq 0, 1$$

Now,

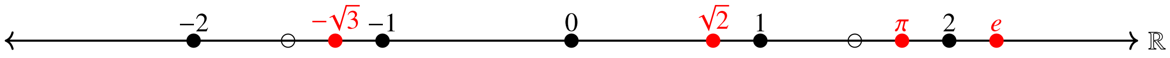
$$\begin{aligned} & \sqrt{2} = \frac{p}{q} \\ \implies & 2 = \frac{p^2}{q^2} \\ \implies & 2q^2 = p^2 \end{aligned}$$

This means that p^2 is divisible by 2 and is an even number. Again, we see that $q^2 = \frac{p^2}{2}$ and so q^2 is also an even number. This means that both p, q are even numbers too! (When a^2 is even, a is also even.[8])
But if p, q are both even numbers, then they have a common factor of 2, and this means that p, q are not relatively prime numbers. This means our assumption of defining $\sqrt{2} = \frac{p}{q}$, i.e., taking $\sqrt{2}$ to be a rational number, is incorrect. So, $\sqrt{2}$ is an irrational number. ∎

[7] The reason why $r - s\sqrt{2}$ is selected is due to **automorphism**. Automorphism is generally covered in an abstract algebra course, so you will learn about it there. In short automorphism is a bijective function f defined on the set A such that $f : A \to A$ and the function f preserves all the algebraic structures of the set A.

[8] If a^2 is even, then for some positive (square number, that is why positive) integer k, $a^2 = 2k \implies a^2 - 2k = 0$. Now, $a = a + o = a + a^2 - 2k = a(a+1) - 2k$. Now, even and odd numbers alternate, so either a or $a + 1$ is an even number. Which makes $a(a + 1)$ an even number, i.e., for some non-negative integer j, $a(a + 1) = 2j$. Now, $a = a(a + 1) - 2k = 2j - 2k = 2(j - k)$. So a is an even number.

With this fact established, if we visualize the real numbers as a straight line, we see that the real numbers, $\mathbb{R}$, do not have any sort of 'gaps' while in the same line if we consider the rational numbers only, we will find some gaps. This property of not having any gaps is so important and consequential that it has its own name - completeness.



Completeness: Gaps of $\mathbb{Q}$ are filled in $\mathbb{R}$ by $\mathbb{I}$.

This visualization is intuitive, but we still need a rigorous proof of the existence of gaps in the rational number line.

Illustrative Example 1.1.5

Gaps Exist in the Rational Numbers.

Proof. Let us define the following two sets:

$$A = \{m \in \mathbb{Q} : m^2 < 2\},\ B = \{p \in \mathbb{Q} : p^2 > 2\}$$

If we can always find a rational number, n, that has the properties $n^2 < 2$ and $n > m$ (i.e. $\sqrt{2} > n > m$) for any choice of $m \in A$, then it proves the existence of gaps in the rational numbers.
Now, any choice of m adheres by $m^2 < 2 \implies 2 - m^2 > 0$. However, we need to show that n is larger than m. So, we need $n = m + f(2 - m^2)$ where $f(2 - m^2) > 0$.[9]
Again,

$$\begin{aligned}
&2 > n^2 > m^2 \\
\implies &2 - m^2 > n^2 - m^2 > 0 \\
\implies &2 - m^2 > (n + m)(n - m) > 0 \\
\implies &2 - m^2 > \{2m + f(2 - m^2)\}f(2 - m^2) > 0 \\
\implies &\frac{2 - m^2}{f(2 - m^2)} > 2m + f(2 - m^2) > 0 \\
\implies &\frac{2 - m^2}{f(2 - m^2)} > 2m + f(2 - m^2) > 2m > 0 \\
\implies &\boxed{\frac{2 - m^2}{2m} > f(2 - m^2) > 0}
\end{aligned}$$

Another piece of information we have is that,

$$2 > \sqrt{2} > m \implies 2 > m \implies m + 2 > 2m \implies \frac{1}{2m} > \frac{1}{m + 2} \implies \boxed{\frac{2 - m^2}{2m} > \frac{2 - m^2}{m + 2}}$$

So, combining both boxed equations, we can define[10]

$$\boxed{f(2 - m^2) := \frac{2 - m^2}{m + 2};\ n = m + \frac{2 - m^2}{m + 2}}$$

Now we will verify the properties we are seeking.

$$n = m + f(2 - m^2) \implies n - m = f(2 - m^2) = \frac{2 - m^2}{m + 2} > 0 \implies n > m$$

[9]The reason for writing $f(2 - m^2)$ instead of $f(m)$ is just my own choice; the readers can denote the function as the function of $2 - m^2$ or m as they please.
[10]The symbol := is used when defining something in mathematics.

$$
\begin{aligned}
2-n^2 &= (\sqrt{2})^2 - n^2 = (\sqrt{2}+n)(\sqrt{2}-n) = \Big(\sqrt{2}+m+\frac{2-m^2}{m+2}\Big)\Big(\sqrt{2}-m-\frac{2-m^2}{m+2}\Big) \\
&= \Big(\sqrt{2}+\frac{2(m+1)}{m+2}\Big)\Big(\sqrt{2}-\frac{2(m+1)}{m+2}\Big) = 2-\frac{4(m+1)^2}{(m+2)^2} = \frac{2}{(m+2)^2}\big((m+2)^2-2(m+1)^2\big) \\
&= \frac{2}{(m+2)^2}(2-m^2) > 0 \\
\implies 2-n^2 > 0
\end{aligned}
$$

As we can always construct n from any arbitrary choice of m (It is the arbitrary choice that gives generality to the statement we are trying to prove), there will always exist a rational number $\sqrt{2} > n > m$ and $n \in A$. This proves the existence of gaps in the rational numbers utilizing the set A. What can be done for the set B? ∎

Exercise 1C

1. **Prove that $\sqrt{3}$ is an Irrational Number.**
2. **Prove that $\sqrt{6}$ is an Irrational Number.**
3. **If a^2 is an odd number, then prove that a is also an odd number. Prove it using the method used in the footnote 8 and then prove it again by using the method of contradiction.**
4. **What happens if one tries proving $\sqrt{4}$ as an irrational number in the same way as $\sqrt{2}$? Explain what goes wrong, if there is any!**
5. **Prove the existence of gaps in the rational numbers using set B by constructing a rational number q such that $p > q > \sqrt{2}$ in the same way we have shown for set A in example (1.1.5).**

1.2 Completeness

In this section, we will investigate the completeness property more deeply. For that, we at first have to go back and see what type of set makes completeness apparent, just like example (1.1.5). Now we want to study a broader class of such examples first, so that we can be precise about any assertions regarding completeness.

1.2.1 Bounded Sets, Supremum & Infimum of Sets

A general prescription of the type of set in example (1.1.5) is given below:

Definition 1.2.1: Bounded Sets

Let A be a non-empty subset of $\mathbb{R}$, then the set A will be called bounded if there exist two real numbers a, b[11] and $a \le b$ such that for all $x \in A$, we get $a \le x \le b$.
If the set A satisfies the condition $a \le x$ only, for all $x \in A$, then the set A is called a **bounded below** set and the number a is called a **lower bound**.
If the set A satisfies the condition $x \le b$ only, for all $x \in A$, then the set A is called a **bounded above** set and the number b is called an **upper bound**.

Now, we want to find the absolute number (or point in the real line) from where the upper bounds and lower bounds start to emerge. A simple solution is to take that upper bound below which no numbers are upper bounds, that is, taking the smallest of the upper bounds. This motivates us to define the following -

[11] Very importantly, notice that the numbers a, b do not necessarily have to be members of the reference set A. Also, note that we have not yet introduced the concept of infinity.

Definition 1.2.2: Supremum of a Set

An upper bound of a set $A \subset \mathbb{R}$ and $A \neq \emptyset$, denoted by b, will be called the supremum of the set A if for all other upper bounds v, we get $b \leq v$.
The supremum is denoted by $\sup A = b$ or simply $\sup A$.

Thinking similarly for the lower bounds gives us the infimum.

Definition 1.2.3: Infimum of a Set

A lower bound of a set $A \subset \mathbb{R}$ and $A \neq \emptyset$, denoted by a, will be called the infimum of the set A if for all other lower bounds u, we get $u \leq a$.
The infimum is denoted by $\inf A = a$ or simply $\inf A$.

One can remember the definitions more easily by the following lines:

1. A supremum is the smallest of the upper bounds.
2. An infimum is the largest of the lower bounds.

Another important thing to consider in the definitions is that the bounds, supremum, and infimum do not have to be members of the set itself.
Although the concepts of upper and lower bounds are introduced here in the context of a totally ordered set, they can be defined exactly the same in the context of a preordered set.

1.2.1.1 Properties of the Supremum

Before we investigate the properties of the supremum, let us establish some notations.

$$\begin{aligned}
&A = \{a : a \in A, A \subset \mathbb{R}\},\ cA = \{x : x = ca,\ \text{for all } x \in A \text{ and } c \in \mathbb{R}\} \\
&B = \{b : b \in B, B \subset \mathbb{R}\},\ A + B = \{p : p = a + b,\ \text{where } a \in A, b \in B\} \\
&A \cdot B = \{q : q = a \cdot b,\ \text{where } a \in A, b \in B\} \\
&A \leq B \implies a \leq b,\ \text{for all } a \in A, b \in B
\end{aligned}$$

Theorem 1.2.1: Uniqueness of Supremum

The supremum of a set, if it exists, is unique.

Proof. Let s_1, s_2 be two suprema of a non-empty subset of real numbers, A. Then, both s_1, s_2 are upper bounds of the set A.
Now, s_1 by definition of supremum, $\boxed{s_1 \leq s_2}$.
Similarly, s_2 by definition of supremum, $\boxed{s_2 \leq s_1}$.
Combining both boxed equations, we get $s_1 = s_2$. So, the supremum of the set A is unique. As A was an arbitrary set, we can say, the supremum is unique. ■

Theorem 1.2.2: Necessary and Sufficient Conditions of Supremum

Let $A \subset \mathbb{R}$ and $A \neq \emptyset$, then following are the necessary and sufficient conditions for $\sup A = s$:

1. For all $x \in A$, $x \leq s$.
2. For any $\epsilon > 0, \epsilon \in \mathbb{R}$, there exists at least one $x \in A$ such that $s - \epsilon < x$.[12]

[Check **footnote** 6 for the meaning of necessary and sufficient conditions.]

Proof. For the forward direction ($\implies$), we have to start by assuming that $\sup A = s$ and then prove the two conditions. The supremum is an upper bound by definition, and therefore, for all $x \in A, x \leq s$ is true by the definition of an upper bound. Now, for the second condition, for the sake of contradiction, let us assume there exists no such $x \in A$ for which $s - \epsilon < x$. That is, we consider for any $\epsilon > 0$ such that $x \leq s - \epsilon$ for all $x \in A$. So, $s - \epsilon$ is an upper bound. As s is the supremum, then

$$\begin{aligned} &s \leq s - \epsilon \\ \implies &0 \leq -\epsilon \\ \implies &\epsilon \leq 0 \end{aligned}$$

This is clearly not true, and so we must have at least one $x \in A$ such that $s - \epsilon < x$. This proves the second condition. So, we have proved the forward direction ($\implies$).
As for the backward direction ($\impliedby$), we have to start by assuming the two conditions to be true and then prove that $\sup A = s$. By the first condition, we know that s is an upper bound by definition.
As for the second condition, for the sake of contradiction, let us assume that s is not the supremum, and then, there exists another upper bound $s' < s$. Then, $s - s' > 0$, and let us set $\epsilon = s - s'$. Then, by the second condition, $s - \epsilon = s - s + s' < x \implies s' < x$. This means that s' is not an upper bound. So, no such $s' < s$ exists, that means all upper bounds follow $s \leq s'$. Thus, the second condition along with the first condition proves that $\sup A = s$.
This completes the backwards ($\impliedby$) proof. ■

Theorem 1.2.3: Positive Scalar Product of Supremum[13]

Let A be a non-empty proper subset of $\mathbb{R}$, with supremum $\sup A = s$, then for $c > 0,\ c \in \mathbb{R}$,

$$\sup(cA) = c \sup A = cs$$

Proof. By definition of the supremum, for all $a \in A$,

$$\begin{aligned} &a \leq \sup A \\ \implies &ca \leq c \sup A \end{aligned}$$

So, $c \sup A$ is an upper bound for the set cA.
Again, by definition of the upper bound, for all upper bounds M of the set cA, for all $a \in A$,

$$\begin{aligned} &ca \leq M \\ \implies &a \leq \frac{M}{c} \end{aligned}$$

As $\sup A$ is the least upper bound, by definition we get,

$$\begin{aligned} &\sup A \leq \frac{M}{c} \\ \implies &c \sup A \leq M \end{aligned}$$

By definition of the supremum, we get $\sup(cA) = c \sup A$. ■

Theorem 1.2.4: Addition of Supremum

Let A, B be non-empty proper subsets of $\mathbb{R}$, with their supremum $\sup A = s, \sup B = t$. Then,

$$\sup(A + B) = \sup A + \sup B = s + t$$

[12] Congratulations! This is your first exposure to the famous epsilon (ϵ) in analysis. Remember this number very vividly, because we will have numerous encounters with this throughout our analysis journey.
[13] In mathematics, product by a constant number is called *scalar product*.

Proof. By definition of the supremum,

$$\begin{aligned} &a \leq \sup A, \text{ for all } a \in A \ \& \ b \leq \sup B, \text{ for all } b \in B \\ \Longrightarrow\ &a + b \leq \sup A + \sup B, \text{ for all } a \in A, b \in B \end{aligned}$$

So, $\sup A + \sup B$ is an upper bound of the set $(A + B)$.
Let M be an arbitrary upper bound of $(A + B)$. Then,

$$\begin{aligned} &a + b \leq M, \text{ for all } a \in A, b \in B \\ \Longrightarrow\ &b \leq M - a, \text{ for all } a \in A, b \in B \end{aligned}$$

So, $M - a$ is an upper bound of the set B, then by definition of the supremum,

$$\begin{aligned} &\sup B \leq M - a, \text{ for all } a \in A \\ \Longrightarrow\ &a \leq M - \sup B, \text{ for all } a \in A \end{aligned}$$

Again, $M - \sup B$ is an upper bound of the set A, then by definition of the supremum,

$$\begin{aligned} &\sup A \leq M - \sup B \\ \Longrightarrow\ &\sup A + \sup B \leq M \end{aligned}$$

As M was an arbitrary upper bound of $(A + B)$, by definition of the supremum, we can say that,

$$\sup(A + B) = \sup A + \sup B$$

This concludes the proof. ■

Theorem 1.2.5: Product of Supremum

Let A, B be non-empty proper subsets of $\mathbb{R}$, with their supremum $\sup A = s, \sup B = t$. Then,

$$\sup(A \cdot B) = \sup A \cdot \sup B = s \cdot t$$

Proof. By definition of the supremum,

$$\begin{aligned} &a \leq \sup A, \text{ for all } a \in A \ \& \ b \leq \sup B, \text{ for all } b \in B \\ \Longrightarrow\ &a \cdot b \leq \sup A \cdot \sup B, \text{ for all } a \in A, b \in B \end{aligned}$$

So, $\sup A \cdot \sup B$ is an upper bound of the set $(A \cdot B)$.
Let M be an arbitrary upper bound of $(A \cdot B)$. Then,

$$\begin{aligned} &a \cdot b \leq M, \text{ for all } a \in A, b \in B \\ \Longrightarrow\ &b \leq \frac{M}{a}, \text{ for all } a \neq 0, a \in A, b \in B \end{aligned}$$

So, $\frac{M}{a}$ is an upper bound of the set B, then by definition of the supremum,

$$\begin{aligned} &\sup B \leq \frac{M}{a}, \text{ for all } a \neq 0, a \in A \\ \Longrightarrow\ &a \leq \frac{M}{\sup B}, \text{ for all } a \in A, \sup B \neq 0 \end{aligned}$$

Again, $\frac{M}{\sup B}$ is an upper bound of the set A, then by definition of the supremum,

$$\begin{aligned} &\sup A \leq \frac{M}{\sup B} \\ \Longrightarrow\ &\sup A \cdot \sup B \leq M \end{aligned}$$

As M was an arbitrary upper bound of $(A \cdot B)$, by definition of the supremum, we can say that,

$$\sup(A \cdot B) = \sup A \cdot \sup B$$

This concludes the proof. ■

Theorem 1.2.6: Supremum and Order

Let A, B be non-empty proper subsets of $\mathbb{R}$, with their supremum $\sup A = s, \sup B = t$. Then,

$$A \leq B \implies \sup A \leq \sup B \implies s \leq t$$

Proof. By definition of the supremum, we have,

$$a \leq \sup A, \text{ for all } a \in A, \; b \leq \sup B, \text{ for all } b \in B$$

Now, as $a \leq b$, for all $a \in A, b \in B$, we have $a \leq b \leq \sup B \implies a \leq \sup B$ for all $a \in A$. So, $\sup B$ is an upper bound of A. By definition of the supremum, $\sup A \leq M$, where M is any upper bound of A. Setting $M = \sup B$, we get $\sup A \leq \sup B$. This concludes the proof. ■

1.2.1.2 Properties of the Infimum

Theorem 1.2.7: Uniqueness of Infimum

The infimum of a set, if it exists, is unique.

Proof. Let i_1, i_2 be two infima of a non-empty subset of real numbers, A. Then, both i_1, i_2 are lower bounds of the set A. Now, i_1 by definition of infimum, $\boxed{i_1 \geq i_2}$.
Similarly, i_2 by definition of infimum, $\boxed{i_2 \geq i_1}$.
Combining both boxed equations, we get $i_1 = i_2$. So, the infimum of the set A is unique. As A was an arbitrary set, we can say, the infimum is unique. ■

Theorem 1.2.8: Necessary and Sufficient Conditions of Infimum

Let $A \subset \mathbb{R}$ and $A \neq \emptyset$, then following are the necessary and sufficient conditions for $\inf A = i$:

1. For all $x \in A, x \geq i$.
2. For any $\epsilon > 0, \epsilon \in \mathbb{R}$, there exists at least one $x \in A$ such that $x < i + \epsilon$.

Proof. For the forward direction ($\implies$), assume $\inf A = i$.
The infimum is a lower bound by definition, so for all $x \in A, x \geq i$ holds.
For the second condition, suppose for contradiction that there exists $\epsilon > 0$ such that $x \geq i + \epsilon$ for all $x \in A$. Then $i + \epsilon$ is a lower bound. Since i is the infimum:

$$\begin{aligned} & i \geq i + \epsilon \\ \implies & 0 \geq \epsilon \end{aligned}$$

This is clearly not true, and so we must have at least one $x \in A$ such that $x < i + \epsilon$. This proves the second condition.
For the backward direction ($\impliedby$), assume the two conditions hold. By (1), i is a lower bound. Suppose there exists another lower bound $i' > i$. Let $\epsilon = i' - i > 0$. By (2), there exists $x \in A$ with $x < i + \epsilon = i'$, contradicting that i' is a lower bound. Thus i is the greatest lower bound. ■

Theorem 1.2.9: Positive Scalar Product of Infimum

Let A be a non-empty proper subset of $\mathbb{R}$, with infimum $\inf A = i$, then for $c > 0,\ c \in \mathbb{R}$,

$$\inf(cA) = c \inf A = ci$$

Proof. By definition of the infimum, for all $a \in A$,

$$\begin{aligned} a &\geq \inf A \\ \implies ca &\geq c \inf A \end{aligned}$$

So, $c \inf A$ is a lower bound for the set cA.
For any lower bound m of cA, for all $a \in A$,

$$\begin{aligned} ca &\geq m \\ \implies a &\geq \frac{m}{c} \end{aligned}$$

As $\inf A$ is the greatest lower bound, by definition we get,

$$\begin{aligned} \inf A &\geq \frac{m}{c} \\ \implies c \inf A &\geq m \end{aligned}$$

By definition of the infimum, we get $\inf(cA) = c \inf A$. ∎

Theorem 1.2.10: Addition of Infimum

Let A, B be non-empty proper subsets of $\mathbb{R}$, with their infimum $\inf A = i, \inf B = j$. Then,

$$\inf(A + B) = \inf A + \inf B = i + j$$

Proof. By definition of the infimum,

$$\begin{aligned} a &\geq \inf A,\ \text{for all } a \in A \ \&\ b \geq \inf B,\ \text{for all } b \in B \\ \implies a + b &\geq \inf A + \inf B,\ \text{for all } a \in A, b \in B \end{aligned}$$

So, $\inf A + \inf B$ is a lower bound of the set $(A + B)$.
Let m be an arbitrary lower bound of $(A + B)$. Then,

$$\begin{aligned} a + b &\geq m,\ \text{for all } a \in A, b \in B \\ \implies b &\geq m - a,\ \text{for all } a \in A, b \in B \end{aligned}$$

So, $m - a$ is a lower bound of the set B, then by definition of the infimum,

$$\begin{aligned} \inf B &\geq m - a,\ \text{for all } a \in A \\ \implies a &\geq m - \inf B,\ \text{for all } a \in A \end{aligned}$$

Again, $m - \inf B$ is a lower bound of the set A, then by definition of the infimum,

$$\begin{aligned} \inf A &\geq m - \inf B \\ \implies \inf A + \inf B &\geq m \end{aligned}$$

As m was an arbitrary lower bound of $(A + B)$, by definition of the infimum, we can say that,

$$\inf(A + B) = \inf A + \inf B$$

This concludes the proof. ∎

Theorem 1.2.11: Product of Infimum

Let A, B be non-empty proper subsets of $\mathbb{R}$, with their infimum $\inf A = i, \inf B = j$. Then,

$$\inf(A \cdot B) = \inf A \cdot \inf B = i \cdot j$$

Proof. By definition of the infimum,

$$\begin{aligned} & a \geq \inf A, \text{ for all } a \in A \text{ \& } b \geq \inf B, \text{ for all } b \in B \\ \implies & a \cdot b \geq \inf A \cdot \inf B, \text{ for all } a \in A, b \in B \end{aligned}$$

So, $\inf A \cdot \inf B$ is a lower bound of the set $(A \cdot B)$.
Let m be an arbitrary lower bound of $(A \cdot B)$. Then,

$$\begin{aligned} & a \cdot b \geq m, \text{ for all } a \in A, b \in B \\ \implies & b \geq \frac{m}{a}, \text{ for all } a \neq 0, a \in A, b \in B \end{aligned}$$

So, $\frac{m}{a}$ is a lower bound of the set B, then by definition of the infimum,

$$\begin{aligned} & \inf B \geq \frac{m}{a}, \text{ for all } a \neq 0, a \in A \\ \implies & a \geq \frac{m}{\inf B}, \text{ for all } a \in A, \inf B \neq 0 \end{aligned}$$

Again, $\frac{m}{\inf B}$ is a lower bound of the set A, then by definition of the infimum,

$$\begin{aligned} & \inf A \geq \frac{m}{\inf B} \\ \implies & \inf A \cdot \inf B \geq m \end{aligned}$$

As m was an arbitrary lower bound of $(A \cdot B)$, by definition of the infimum, we can say that,

$$\inf(A \cdot B) = \inf A \cdot \inf B$$

This concludes the proof. ■

Theorem 1.2.12: Infimum and Order

Let A, B be non-empty proper subsets of $\mathbb{R}$, with their infimum $\inf A = i, \inf B = j$. Then,

$$A \leq B \implies \inf A \leq \inf B \implies i \leq j$$

Proof. By definition of the infimum, we have,

$$a \geq \inf A, \text{ for all } a \in A, \ b \geq \inf B, \text{ for all } b \in B$$

Now, as $a \leq b$ for all $a \in A, b \in B$, we have $\inf A \leq a \leq b \implies \inf A \leq b$ for all $b \in B$. So, $\inf A$ is a lower bound of B.
By definition of the infimum, $m \leq \inf B$, where m is any lower bound of B. Setting $m = \inf A$, we get $\inf A \leq \inf B$.
This concludes the proof. ■

1.2.1.3 Connection between Supremum and Infimum

Theorem 1.2.13: Supremum and Infimum Inequality

Let $A \subseteq \mathbb{R}$ and $A \neq \varnothing$. Then, $\inf A \leq \sup A$.

Proof. By definition of the supremum and infimum,

$$\begin{aligned} & \inf A \le a \ \& \ a \le \sup A \text{ for all } a \in A \\ \implies & \inf A \le \sup A \end{aligned}$$

This concludes the proof. ■

Theorem 1.2.14: Infimum in terms of Supremum

Let $A \subseteq \mathbb{R}$ and $A \neq \emptyset$. Then, $\inf A = -\sup(-A)$.

Proof. Let $\sup(-A) = s \implies -a \le s$ for all $a \in A \implies -s \le a$ for all $a \in A$.
So, $-\sup(-A) = -s$ is a lower bound of A.
Again, as $\sup(-A) = s$ implies that for any $\epsilon > 0$, there exists at least one $a \in A$ such that $s - \epsilon < -a \implies a < -s + \epsilon$.
Now, by theorem (1.2.8), we see that $-\sup(-A) = -s$ satisfies the necessary and sufficient condition for the infimum of the set A.
Therefore, $\inf A = -\sup(-A)$. ■

Theorem 1.2.15: Negative Scalar Product of the Supremum and Infimum

Let A be a non-empty proper subset of $\mathbb{R}$, with supremum $\sup A = s$, then for $c < 0,\ c \in \mathbb{R}$,

$$\sup(cA) = c \inf(A)$$

$$\inf(cA) = c \sup(A)$$

Proof. As $c < 0$, we can write c as $c = -p$ where $p > 0$. Then, we need to prove $\sup(-pA) = -p\inf(A)$ and $\inf(-pA) = -p\sup(A)$.
Now, by theorem (1.2.14), we know that $\inf A = -\sup(-A)$. From the positive scalar product of both supremum and infimum, we know that $\sup(pA) = p \sup A$ and $\inf(pA) = p \inf A$. Now,

$$\begin{aligned} & \inf A = -\sup(-A) \\ \implies & -\inf A = \sup(-A) \\ \implies & -p\inf(A) = p\sup(-A) \\ \implies & \sup(-pA) = -p\inf(A) \\ \implies & \boxed{\sup(cA) = c\inf(A)} \end{aligned}$$

Again, $-(-A) = A$, so setting $A = -A$ (This is just a substitution, not an equation!) gives $\inf(-A) = -\sup(A)$. Like before,

$$\begin{aligned} & \inf(-A) = -\sup\big(-(-A)\big) \\ \implies & \inf(-A) = -\sup(A) \\ \implies & p\inf(-A) = -p\sup(A) \\ \implies & \inf(-pA) = -p\sup(A) \\ \implies & \boxed{\inf(cA) = c\sup(A)} \end{aligned}$$

This concludes the proof. ■

Exercise 1D

1. **Supremum, Infimum and Preordering:** Let $(K, \lesssim)$ be a preordered set. Then define the following in the context of $(K, \lesssim)$:

   a. Bounded set, upper bound and lower bound.

b. Supremum and infimum of the set.

[Hint: Read the last line of the part discussing supremum and infimum.]

2. Supremum, Infimum and Subsets: Let A, B be non-empty bounded subsets of $\mathbb{R}$ and $B \subseteq A$. Then prove the following:

a. $\sup(B) \leq \sup(A)$.

b. $\inf(A) \leq \inf(B)$.

c. $\inf(A) \leq \inf(B) \leq \sup(B) \leq \sup(A)$.

3. Supremum, Infimum, Unions and Intersections: Let A, B be non-empty bounded subsets of $\mathbb{R}$. Then prove the following:

a. $\inf(A \cup B) \leq \inf(A)$ and $\inf(A \cup B) \leq \inf(B)$.

b. $\sup(A \cup B) \geq \sup(A)$ and $\sup(A \cup B) \geq \sup(B)$.

c. $\inf(A \cap B) \geq \inf(A)$ and $\inf(A \cap B) \geq \inf(B)$.

d. $\sup(A \cap B) \leq \sup(A)$ and $\sup(A \cap B) \leq \sup(B)$.

4. Simple Computation: Compute the supremum and infimum (if they exist) of the following sets:

a. $\frac{1}{n}$ where $n \in \mathbb{N}$.

b. $\frac{n}{m+n}$ where $n, m \in \mathbb{N}$.

c. $\frac{an}{bn+c}$ where $a, b, c \in \mathbb{R}$, $a, b, c > 0$, and $n \in \mathbb{N}$.

d. $(-1)^n \frac{an}{bn+c}$ where $a, b, c \in \mathbb{R}$, $a, b, c > 0$, and $n \in \mathbb{N}$.

5. Supremum and Infimum of the Empty Set: In our discussion of the supremum and infimum, we have always assumed our reference set $A \neq \emptyset$, but what will be the case when $A = \emptyset$? In this exercise, we will work through to find the solution step by step.

a. What are the upper bounds and lower bounds of the empty set?
[Hint: It is time to introduce another trick, vacuous truth, into our analysis repertoire. The trick is given below.]

Trick 2: Vacuous Truth

When and where to use this?

- Mainly when proving something for the empty sets, and the property that we wish to prove makes no sense for empty sets.

- Most of the time, when a statement for no member or every member of the set cannot be verified exactly.

Structure of this proof:
A statement of the form: for all $x \in S$ such that $P(x) \implies Q(x)$, is said to be vacuously true if there are no elements in S for which $P(x)$ is true. That is, for all $x \in S, \neg P(x)$ is true, and this implies that $P(x)$ is never true. Then, we cannot say anything about $Q(x)$ because, as $P(x)$ is always false, $Q(x)$ can be either true or false. Thus, the statement $P(x) \implies Q(x)$ is taken to be true.
This is the mathematician's way of taking things as granted. Who said mathematicians are skeptical and snobby?

A simple example:
There are no students in the classroom. So, let us take a look at the following statements -

I. All students were present.

II. No student was present.

Both of them make no sense as there are no students to begin with, and neither statement can be proved nor disproved. Here, x denotes the students, $P(x)$ denotes the number of students, and $Q(x)$ denotes the presence of the students. So, vacuously, $P(x) \implies Q(x)$ is true for all $x \in S$, meaning that a class having 0 students implies that all students were present is a vacuously true statement.

b. Now, if you have managed to solve the first step, then we need to find such numbers that are less than the upper bounds and greater than the lower bounds for determining the supremum and infimum, respectively. Do such numbers exist?

c. If such numbers do exist, then our work is done, but if they do not exist, then what concept do we need to introduce? After introducing that concept, can it be regarded as a member of the field of real numbers?

This whole exercise is a foreshadowing of a very important concept that we will see in this chapter.

6. **An off-beat way of identifying the empty set:** Let $A \subset \mathbb{R}$ be a bounded set. Then $\inf(A) \leq \sup(A)$ will be true if and only if $A \neq \emptyset$.

1.2.2 The Maximum and Minimum of Sets

1.2.3 The Axiom of Completeness and its Consequences

1.3 Real Numbers and the Extended Real Numbers

1.3.1 Real Numbers

Trick 3: Contrapositive Statements

When and where to use this?
Mainly when the opposite direction of a proof is significantly easier than the original direction. In other terms, if we wish to prove $P \implies Q$, but we know very little or can infer little from P, but we know a lot of things about Q, then the contrapositive is the way to go.
Structure of this proof:
If we have a statement $P \implies Q$, then its contrapositive statement is $\boxed{\neg Q \implies \neg P}$, where $\neg$ means not or negation. The contrapositive statement is equivalent to the original statement. This is proved via the truth table below.

Lemma 1.3.1: Original Statement and Contrapositive Statement

The original statement and the contrapositive statement are the same in logic.

Proof of Lemma. Some things to learn at first:

1. In $P \implies Q$, when P is false, regardless of the value of Q, we have $P \implies Q$ being true. This is because P is the precedent, and if the precedent is not valid (that is, false), then Q is true vacuously. See trick (2) for a refresher. The same applies to the rest of the statements in this proof.
2. As both $A \implies B$ and $B \implies A$ are true regardless of the value of the inputs, we can conclude that $A \iff B$.

The truth table is as follows:

P	Q	$\neg P$	$\neg Q$	$A := (P \implies Q)$	$B := (\neg Q \implies \neg P)$	$A \implies B$	$B \implies A$
T	T	F	F	T	T	T	T
T	F	F	T	F	F	T	T
F	T	T	F	T	T	T	T
F	F	T	T	T	T	T	T

This concludes the proof of the original statement, and the contrapositive statement is logically equivalent. □

1.3.2 Extended Real Numbers

1.4 Complex Numbers and the Extended Complex Numbers

1.4.1 Complex Numbers

1.4.2 Extended Complex Numbers

1.5 Metric Spaces

We have seen the real numbers, complex numbers, and we can observe that these numbers have a notion of distance defined on them. We wish to create a framework that will encapsulate all things distance, this very attempt motivates us to define the notion of a metric space where the notion of distance is measured by the 'metric' function.

1.5.1 Definition and Examples of Metric Spaces

Now, we will need two points to measure distance, so the metric function must have two inputs and give one non-negative real number (distance cannot have negative value!) as output. So, if we denote the metric function by d and X can be any set:

$$d : X \times X \to \mathbb{R}^+ \cup \{0\}$$

$$\text{For all } x, y \in X, \; \boxed{d(x, y) \geq 0}$$

As d spits out the distance between any two points, say x, y of the set X, it should not make any difference if we measure the distance from x to y, or from y to x. This can be written as $\boxed{d(x, y) = d(y, x)}$.
Now, any distance from a point to the same point is 0, so our distance function must follow that $d(x, x) = 0$, or written otherwise, $\boxed{d(x, y) = 0 \text{ if and only if } x = y}$.
Now, when going from one point, x, to another point, y, if we take any detour or stoppage on the point z, then we get two cases:

1. The point of detour or stoppage z is not on the direct path of x and y. So, the distance traversed by going from x to y is strictly shorter than going from x to z and then going from z to y. So, in terms of the metric function $d(x, y) < d(x, z) + d(z, y)$.

2. Now we consider the same case as before, but z is a point on the direct path of x and y. Then, $d(x, y) = d(x, z) + d(z, y)$.

Combining both cases, we get the classic triangle inequality, but in terms of the metric function as $\boxed{d(x, y) \leq d(x, z) + d(z, y)}$ for all $x, y, z \in X$.
Incorporating all the boxed equation gives us the following definition of the metric space :

Definition 1.5.1: Metric Space

A space (X, d) is defined as a metric space if there exists a set X and a function $d : X \times X \to \mathbb{R}$ such that it possess the following properties for all $x, y, z \in X$,:

1. Non-negativity: $d(x, y) \geq 0$.

2. Symmetry: $d(x, y) = d(y, x)$.

3. Indiscernibility: $d(x, y) = 0$ if and only if $x = y$.

4. Triangle Inequality: $d(x, y) \leq d(x, z) + d(z, y)$.

The function d is called the metric function. When there is no scope of confusion, we omit the mention of d.

1.5.2 More about Metric Spaces

1.6 Cardinality of Sets

In our mathematics journey, we use sets quite extensively, we work with sets of different sizes, some sets have no members like the empty set, some have 3 members like $\{1, 2, 3\}$, some have infinite members like $\mathbb{N} = \{1, 2, 3, \dots\}$. This notion of the number of members of the set is called the cardinality of the set.

1.6.1 The Basics

Definition 1.6.1: Cardinality of a Set

The number of elements of a set is defined as the cardinality of that set. For a reference set A, the cardinality of the set A is denoted by $|A|$.

Now, based on counting the number of elements of a set, sets are classified into two types:

1. Finite Set.
2. Infinite Set.

Definition 1.6.2: Finite Set

A set is called finite if there is a finite number of elements, that is, the counting of the number of elements of the set stops after a certain time. The cardinality of the finite set can be expressed as a natural number. That is if A is finite, then $|A| = n$ where $n \in \mathbb{N}$.
Some classic examples of finite sets are sets like $\{1, 2\}$, $\{1, 2, 3, \dots, 420\}$, $\{x, y, z, p, q, r, s, t\}$ which have cardinality 2, 420, and 8, respectively.

Definition 1.6.3: Infinite Set

A set is called infinite if there are an infinite number of elements, that is, the counting of the number of elements of the set never stops. The cardinality of the infinite set cannot be expressed as a natural number. That is if A is infinite, then $|A| \neq n$ where $n \in \mathbb{N}$.
Some classic examples of infinite sets are sets like $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{I}$, $\mathbb{R}$, $\mathbb{C}$. The cardinality of the naturals has a special symbol denoted by $|\mathbb{N}| = \aleph_0$ (Pronounced as 'aleph zero').

1.6.1.1 Countable Sets: Bigger & Smaller Infinite Sets?!

Mathematicians are a funny group of people; they always try to find new methods to extend the domain of certain ideas. By directly counting the number of elements, one can easily differentiate between the finite and infinite sets. However, what happens if we try to count the sizes of two infinite sets? It seems that our method of direct counting will fail, as for both infinite sets, the counting will never stop. So, we need a new (and indirect!) method of counting. One of the most common method of indirect test we see in our daily lives is the test by comparison. But this test, by comparison, presents us with two new questions, namely

1. What will be the standard for comparison?
2. What will be the process of comparison?

Let us try to answer the second question first! As we are working in the confines of set theory, it is no surprise that our method of comparison will involve a function, but what type of function is the million-dollar question! When we try to compare two infinite sets, say A and B, using a function f, to ensure that they are of the same size (the comparison will hold when they are of same size, if they are not of the same size the comparison will not hold and so we cannot infer any new information from that case), we must check two things first:

1. Both A, B are of the same size, so if we remove one member of the set A as the input of the function, then it should also mean that we will be able to remove the corresponding member of the set B. This means that the function must be one-one or injective.
2. $f(A) = B$, the function must be onto or surjective because, as they are of the same size, we cannot have the image of the function include more or less members than the set B.

This means that **our function for size comparison has to be bijective or one-to-one correspondence**. A wonderful detail about bijective functions is stated without proof (the readers are implored to work it out by themselves) as follows -

Theorem 1.6.1: Bijective Functions and Inverse Functions

The inverse of a function exists, if and only if the function is bijective.

So, when comparing the sizes of two sets A, B, if we can find a one-to-one correspondence or a bijective function $f : A \longrightarrow B$, then we can say that the sets have the same cardinality. Now that we have managed to find the process of comparison for infinite sets, we must also check if it holds for finite sets. If it does not, then all our work attempting to generalize the counting process will fail. But fortunately, the readers can verify (quite easily too, but it is still kept as one of the exercises) that this comparison holds true for finite sets too!
Now we try to set the standard of comparison for infinite sets. In our journey, we have encountered some seriously important numbers such as $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}, \mathbb{C}$. But infinities are not fun to work with, as you have seen before in the consequences of the axiom of completeness section. This gives us the incentive to work with the smallest possible infinite set we can find and by membership alone, we see that $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$. So, we set our **standard for infinite sets as the set of natural numbers $\mathbb{N}$**.
This wonderful method of generalizing cardinality to infinite sets was put forth by a wonderful Russian-German mathematician named Georg Cantor (1845-1918).

Definition 1.6.4: Countably Infinite Set

A set A is defined to be countably infinite if its cardinality is exactly equal to the cardinality of the natural numbers, that is, $|A| = |\mathbb{N}| = \aleph_0$.

Now, let us explore some instances to apply our newfound method of counting.

Illustrative Example 1.6.1

The sets of Even and Odd numbers are countably infinite.

Proof. Let the set of even numbers be denoted by $E = \{2, 4, 6, 8, \dots\}$.
Let the set of odd numbers be denoted by $O = \{1, 3, 5, 7, \dots\}$.
Recall that we have considered $\mathbb{N} = \{1, 2, 3, \dots\}$.
To prove that the sets $E,\ O$ are countably infinite, we need to show a one-to-one correspondence or bijective function from $\mathbb{N}$ to E and O. By theorem (1.6.1), the determination of the inverse alone should suffice!
Now, $E = \{2, 4, 6, \dots\}$ can be expressed in terms of $\mathbb{N}$ by the function $f_1 : \mathbb{N} \longrightarrow E$ where

$$f_1(n) = 2n \implies n = \frac{f_1(n)}{2} \implies f_1^{-1}(f_1(n)) = \frac{f_1(n)}{2} \implies f_1^{-1}(y) = \frac{y}{2}$$

Similarly, $O = \{1, 3, 5, \dots\}$ can be expressed in terms of $\mathbb{N}$ by the function $f_2 : \mathbb{N} \longrightarrow O$ where

$$f_2(n) = 2n - 1 \implies n = \frac{f_2(n) + 1}{2} \implies f_2^{-1}(f_2(n)) = \frac{f_2(n) + 1}{2} \implies f_2^{-1}(y) = \frac{y + 1}{2}$$

Therefore, both $E,\ O$ are countably infinite sets. ∎

Illustrative Example 1.6.2

The set of Integers is countably infinite.

Proof. Like the previous illustration, we need a bijective function from $\mathbb{N}$ to $\mathbb{Z}$. One way we can accomplish that is to make a function that assigns the even numbers to the positive integers and the odd numbers to the non-positive integers. So, we establish $f : \mathbb{N} \to \mathbb{Z}$ where

$$f(n) = \begin{cases} \frac{n}{2}, & \text{where } n \text{ is even} \\ -(\frac{n-1}{2}), & \text{where } n \text{ is odd} \end{cases}$$

Just like before, we can construct the inverse function $f^{-1} : \mathbb{Z} \to \mathbb{N}$ where

$$f^{-1}(y) = \begin{cases} 2y, & \text{where } y \text{ is positive integer number} \\ 1 - 2y, & \text{where } y \text{ is non-positive integer number} \end{cases}$$

This concludes the proof that the set of integers is also countably infinite. ■

A very interesting (and bizarre) thing to notice is that $E, O \subset \mathbb{N} \subset \mathbb{Z}$, but $|E| = |O| = |\mathbb{N}| = |\mathbb{Z}| = \aleph_0$. Upon inspection, one might think that this kind of bizarre characteristic occurs because all involved sets are infinite, and that is actually quite the correct observation. This behavior will motivate us to define an alternate type of finite and infinite sets, which are named after the German mathematician Richard Dedekind (1831-1916).

Definition 1.6.5: Dedekind Finite Set

A set is defined as Dedekind finite if the cardinality of the set itself is strictly greater than the cardinality of all of its proper subsets.

Similarly, we define a Dedekind infinite set.

Definition 1.6.6: Dedekind Infinite Set

A set is defined to be Dedekind infinite if the cardinality of the set itself is equal to the cardinality of at least one of its proper subsets.

One might wonder what the relationship is between the two types of finite and infinite sets. Are they equivalent or not? This gives us the next theorem.

Theorem 1.6.2: Dedekind Finite Sets & Finite Sets, Dedekind Infinite Sets & Infinite Sets

1. All finite sets are Dedekind finite sets.
2. All Dedekind infinite sets are infinite sets.

Proof. (1) By definition of finite set, $|A| = n$ where $n \in \mathbb{N}$. If $B \subset A$ and A is finite, then there exists at least one element x such that $x \in A, x \notin B$. So, $B \subseteq A \setminus \{x\} \implies |B| \leq n - 1 \implies |B| \leq n - 1 < n = |A| \implies |B| < |A|$. So, we have got for all $B \subset A$, $|B| < |A|$, from A being finite set. So, this concludes the proof that all finite sets are Dedekind finite sets.
(2) This is just the contrapositive statement of (1), and as the original statement & the contrapositive statement are logically the same, following the proof of (1), (2) is also proved. This concludes the proof of Dedekind infinite sets being infinite sets. ■

An interesting thing to note is that it is not as straight forward to prove the opposite direction, that is, proving all Dedekind finite sets are finite sets. This is not as easy as it looks and requires the axiom of choice. As such, it is left for section (1.7).
Now we want to create a new idea of the size of sets that will allow us to combine the usual finite sets with the countably infinite sets. Now, as the cardinality of the naturals is defined as countably infinite, then any other arbitrary set with cardinality less than the naturals should also be countable in some sense, and it also agrees with the finite sets then being countable as well. With that out of the way, we now define a set with a more generalized notion of finiteness.

Definition 1.6.7: Countable Sets

Let A be an arbitrary set, and it will be defined as countable if the cardinality of A is equal to or less than the cardinality of the naturals. Symbolically, set A is countable if $|A| \leq \aleph_0$.

Some Other Equivalent Definitions of Countable Sets: The following are some of the alternate definitions of countable sets..

Definition 1.6.8: Alternate Definitions of Countable Sets

1. An arbitrary set A is countable if there exists a bijective function or map from the set to a subset of the naturals. Symbolically $f : A \xrightarrow{\text{bij}} B \subseteq \mathbb{N}$.

2. An arbitrary set A is countable if there exists an injective function or map from the set to the naturals, i.e., $f : A \xrightarrow{\text{inj}} \mathbb{N}$.

3. An arbitrary set A is countable if there exists a surjective function or map from the naturals to the set, i.e., $f : \mathbb{N} \xrightarrow{\text{sur}} A$.

Proof. Let the original definition be definition 1, with the other alternate definitions being 2, 3, and 4 in the order of their presentation in definition (1.6.8).

1. **1 $\iff$ 3**
   Let us assume the original definition, then for countable sets, $|A| \leq |\mathbb{N}|$. Then by the definition of injective function, there exists an injective function, $f : A \xrightarrow{\text{inj}} \mathbb{N}$. So, $1 \implies 3$.
   Using the same logic in the opposite direction, we can show that $3 \implies 1$.

2. **1 $\iff$ 2**
   Let us start from the second definition. Then, $B \subseteq \mathbb{N} \implies |B| \leq |\mathbb{N}| = \aleph_0$. By definition $f : A \xrightarrow{\text{bij}} B \subseteq \mathbb{N}$ and we know that a bijection preserves the cardinality, i.e., $|A| = |B|$. So, $|B| \leq \aleph_0 \implies |A| \leq \aleph_0$. So, $2 \implies 1$.
   Now let us assume the original definition. As definition $1 \iff 3$, by definition 3, we already have an $f : A \xrightarrow{\text{inj}} \mathbb{N}$. By definition, $f : A \to f(A)$ is surjective, so let us set $f(A) = B$ and then we get $f : A \xrightarrow{\text{bij}} B \subseteq \mathbb{N}$. This proves $1 \implies 2$.

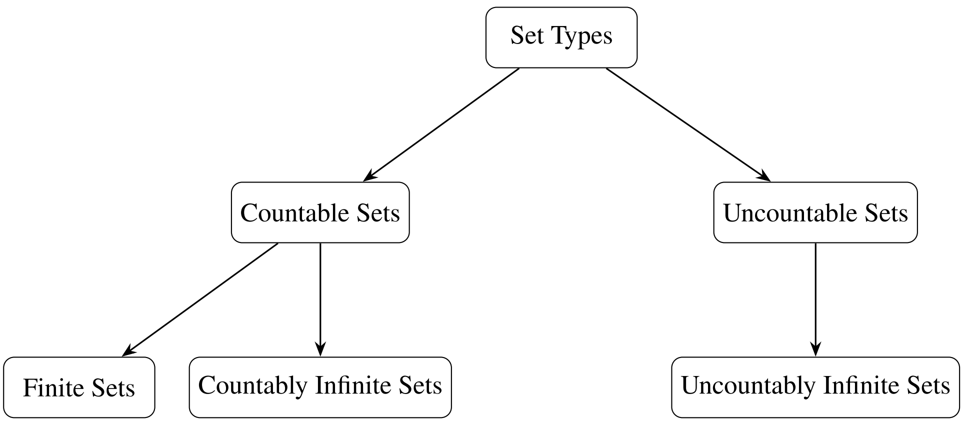
3. **1 $\iff$ 4**
   Left for section (1.7) as its proof requires the axiom of choice.

This concludes the proof. ■

Definition 1.6.9: Uncountable Sets

Let A be an arbitrary set, and it will be defined as uncountable if the cardinality of A is greater than the cardinality of the naturals. Symbolically, set A is uncountable if $|A| > \aleph_0$.
Quite simply, a set that is not countable, is defined as uncountable.

As a small refresher of what we learned in this section, the diagram below is provided:



1.6.1.2 Arbitrary Unions, Intersections: A Fun Excursion

Exercise 1E

1. **Prove that bijection preserves the cardinality for finite sets.**
2. **Prove the following theorem -**

   Theorem 1.6.3: Cantor's Theorem about Power Sets

   The cardinality of the power set of a reference set is always strictly greater than the cardinality of the reference set itself.

1.6.2 Properties of Finite and Countable Sets

With the basics out of the way, we now focus on the properties of finite sets, countable sets, and uncountable sets.

1.6.2.1 Properties of Finite Sets

Theorem 1.6.4: Cardinality of Subsets of Finite Sets

The subsets of a finite set are finite.

Proof. ■

Theorem 1.6.5: Finite Union of Finite Sets

The finite union of finite sets is finite.

Proof. ■

Theorem 1.6.6: Finite Cartesian Product of Finite Sets

The finite cartesian product of finite sets is finite.

Proof. ■

Theorem 1.6.7: Arbitrary Intersection with at least one Finite Set

In an arbitrary intersections of sets, if at least one of the set is finite, then the entire intersection will also be a finite set.

Proof. ■

Theorem 1.6.8: Image of a Finite Set is Finite

The image of a finite set under any function is finite.

Proof. ■

Theorem 1.6.9: Cardinality of the Power Set of Finite Set

The power set of a finite set is finite. More specifically, if the cardinality of an arbitrary finite set $|A| = n$ where $n \in \mathbb{N}$, then the cardinality of the power set $|\mathbb{P}(A)| = 2^{|A|} = 2^n$.

Proof.

■

1.6.2.2 Properties of Countable Sets

Theorem 1.6.10: Cardinality of Subsets of Countable Sets

The subsets of a countable set are countable.

Proof.

■

Theorem 1.6.11: Finite Union of Countable Sets

The finite union of countable sets is countable.

Proof.

■

Theorem 1.6.12: Countable Union of Countable Sets

The countable union of countable sets is countable.

Proof. **Requires the axiom of choice in its proof and hence, left for section (1.7).** ■

Theorem 1.6.13: Finite Cartesian Product of Countable Sets

The finite cartesian product of countable sets is countable.

Proof.

■

Theorem 1.6.14: Arbitrary Intersection with at least one Countable Set

In an arbitrary intersections of sets, if at least one of the set is countable, then the entire intersection will also be a countable set.

Proof.

■

Theorem 1.6.15: Image of a Countable Set is Countable

The image of a countable set under any function is countable.

Proof.

■

Theorem 1.6.16: Set of Finite Subsets and n-tuples of Natural Numbers and Countable Sets

The following sets of natural numbers are countable:

1. Set of all finite subsets of $\mathbb{N}$.
2. Set of all finite tuples or n-tuples of $\mathbb{N}$.

Proof.

■

Exercise 1F

1. **Investigate the following query -**

   Query 1.6.1: Countable Cartesian Product of Countable Sets

   Is the countable cartesian products of countable sets countable? If yes, then provide a proof, and if no, then provide a counterexample.

2. **Investigate the following query -**

   Query 1.6.2: Power Set of Countable Sets

   Is the power set of countable sets countable? If yes, then provide a proof, and if no, then provide a counterexample.

1.6.3 Different Representations of Numbers

1.7 Things involving the Axiom of Choice

1.7.1 Order Theory and the Axiom of Choice

1.7.2 Cardinality of Sets and the Axiom of Choice

1.8 Supplementary Material: Construction of the Natural Numbers from ZF

In this section, the natural numbers will be established from the the Peano axioms and the Peano axioms will be established from the ZF (that is right, AoC not required!) axioms. Usually, one should start from the ZF axioms, then the Peano axioms, and then the natural numbers. But note that this will require us to know what the Peano axioms are in the first place. To get rid of this issue, we will first construct the natural numbers from the Peano axioms, and then construct the Peano axioms from the axioms of ZF. But the more observant readers may note that this presents us with a new problem, what if the Peano axioms use something defined in the ZF axioms? We are truly lucky in the sense that **the Peano axioms are arithmetic in nature while the ZF axioms are set theoretic in nature**. While the Peano axioms do use the concepts of set and function, they do not define set and function, so there is no fear of circular arguments.

1.8.1 From the Peano Axioms to the Natural Numbers

We will attempt to construct the natural numbers from the Peano axioms, but to do that, first we need to start with the Peano axioms. Speaking of which, we first need to discuss from which number the natural numbers start. Do the natural numbers start with 0 or 1? Many contemporary textbooks consider that the natural numbers start with 0 while the older textbooks consider that the natural numbers start with 1. To appease both parties, some of the textbooks have adopted a very ingenious scheme. They take the natural numbers to be $\mathbb{N} := \{1, 2, 3, \dots\}$, but they define a new set of numbers $\mathbb{N}_0 := \{0, 1, 2, 3, \dots\}$. As $\mathbb{N} \subset \mathbb{N}_0$, if we construct $\mathbb{N}_0$ from the Peano axioms, then we will get the natural numbers for free as well. That is the scheme this text will be following. So, let us start with the Peano axioms.

Definition 1.8.1: Peano Axioms

(PA1) 0 is a natural[14] number.

(PA2) There is a 'successor' operation, $S(n)$, on the natural numbers, such that,

a. If n is a natural number, then $S(n)$ is also a natural number.

b. If for two natural numbers n, m, and we have $S(n) = S(m)$, then $n = m$.

c. There is no natural number whose successor is 0, that is, there is no n such that $S(n) = 0$.

(PA3) If $p(n)$ expresses a property of natural numbers such that

a. $p(0)$ is true.

b. For every natural number n, if $p(n)$ being true implies that $p(S(n))$ is true, then $p(n)$ is true for all natural numbers.

(PA3) is also called the induction axiom.

As the name 'successor operation' suggests, it is simply a rule that takes one natural number as input and spits out its next natural number as output. Most of the readers will immediately think of the successor operation as adding 1 to the natural number n, but there is a catch. We have yet to define addition and 1. So, we cannot express the successor operation with anything else yet. With that in mind, let us define the rest of the modified natural numbers as follows:

$$\begin{aligned}
1 &:= S(0) \\
2 &:= S(1) = S(S(0)) \\
3 &:= S(2) = S(S(S(0))) \\
4 &:= S(3) = S(S(S(S(0)))) \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}$$

Now, we will try to prove the following two model examples.

Illustrative Example 1.8.1

1. Prove that 6 is a natural number.
2. Prove that $3 \neq 1$.

Proof. (1) We know that 4 is a natural number and $S(4) := 5, S(5) := 6$. So, $6 = S(S(4))$. By axiom (PA2)(a), the successor operation gives back a natural number, so repeated application of this axiom gives us that 6 is a natural number. This concludes that 6 is a natural number.
(2) We know that $3 := S(S(S(0)))$ and $1 := S(0)$. For the sake of contradiction, let us assume that $3 = 1$. Then, by (PA2)(b), $S(2) = S(0) \implies 2 = 0$. Again, $0 = 2 = S(1) \implies 0 = S(1)$. But, by (PA2)(c), we know there exists no natural number whose successor is 0. So, our initial assumption is false. This proves that $3 \neq 1$. ■

Now, let us introduce the binary operation of addition using the successor function.

[14] Modified natural number, $\mathbb{N}_0$, in the context of this text.

Definition 1.8.2: Addition

Addition is a binary operation $+ : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$, such that

1. $n + 0 = n$.
2. $n + S(m) = S(n + m)$.

Theorem 1.8.1: Addition is Commutative

Addition obeys the following rules:

1. $0 + n = n$.
2. $S(m) + n = S(m + n)$.
3. $m + n = n + m$.

Proof. For this proof, our main weapon will be (PA3) or the axiom of Induction.
(1) We want to apply (PA3) on $0 + n = n$ by inducting on n. For that, we first need to determine whether the property holds for $n = 0$. So, by using 1st point of the definition of addition, we get for $n = 0$, $0 + n = 0 + 0 = 0$. So the base case for $0 + n = n$ is indeed true. Now, by the induction axiom, we assume that $0 + p = p$ is also true for some natural number p. Now we have to show that this implies that $0 + S(p) = S(p)$. Now, applying the 2nd point of the definition of addition, we get, $0 + S(p) = S(0 + p)$, and we already assumed $0 + p = p$ by induction hypothesis. This implies that, $0 + S(p) = S(0 + p) = S(p)$. So, we have proved that $0 + p = p \implies 0 + S(p) = S(p)$. This concludes the proof that $0 + n = n$.
The rest of the two proofs are done similarly to the proof of (1), induction on n, and as such, are left as exercises to the readers. ■

Now, we go on to define the operation of multiplication.

Definition 1.8.3: Multiplication

Multiplication is a binary operation $\cdot : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$, such that

1. $n \cdot 0 = 0$.
2. $n \cdot S(m) = n + (n \cdot m)$.

Theorem 1.8.2: Multiplication is Commutative

Multiplication obeys the following rules:

1. $0 \cdot n = 0$.
2. $S(m) \cdot n = (m \cdot n) + n$.
3. $m \cdot n = n \cdot m$.

Proof. Similar to the proof of theorem (1.8.1), we have to use induction on n. As such, it is left as an exercise to the readers. ■

Theorem 1.8.3: Associativity

Prove that both addition and multiplication are associative. In other words, prove the following

1. $a + (b + c) = (a + b) + c$.
2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

Proof. (1) This can be proven by inducting on a. For that, we first check if this holds for $a = 0$.

$$\begin{aligned} 0 + (b + c) &= b + c \\ &= (0 + b) + c \\ &= (a + b) + c \end{aligned}$$

So, this property holds for $a = 0$. Then, by the induction hypothesis, $n + (b + c) = (n + b) + c$ is true for all natural numbers up to n.

$$\begin{aligned} S(n) + (b + c) &= S(n + (b + c)) \\ &= S((n + b) + c) \quad \text{[By induction hypothesis.]} \\ &= S(n + b) + c \\ &= (S(n) + b) + c \end{aligned}$$

Thus, we have proved that addition is associative.
(2) This proof is similar to (1), and hence, is left as an exercise to the readers. ■

Theorem 1.8.4: Distributive Law

Multiplication distributes over addition, that is for all $a, b, c \in \mathbb{N}_0$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

Proof. This can be proven by inducting on a. For that, we first check if this holds for $a = 0$.

$$\begin{aligned} 0 \cdot (b + c) &= 0 \\ &= 0 + 0 \\ &= 0 \cdot b + 0 \cdot c \end{aligned}$$

So, this property holds for $a = 0$. Then, by the induction hypothesis, $n \cdot (b + c) = n \cdot b + n \cdot c$ is true for all natural numbers up to n.

$$\begin{aligned} S(n) \cdot (b + c) &= (n \cdot (b + c)) + (b + c) \quad \text{[By property 2 of theorem (1.8.2).]} \\ &= n \cdot b + n \cdot c + b + c \\ &= n \cdot b + b + n \cdot c + c \\ &= S(n) \cdot b + S(n) \cdot c \end{aligned}$$

Thus, we have proved the distributive law holds for $S(n)$ as well. This concludes the proof of the distributive law of multiplication over addition. ■

Now, we introduce the concept of inequality in the realm of natural numbers.

Definition 1.8.4: Inequality

Two natural numbers a, b are defined to be $a \leq b$ if there exists a natural number c such that $a + c = b$.

Theorem 1.8.5: Inequality and Natural Numbers

If $a, b, c \in \mathbb{N}_0$ and $a \leq b$, then

1. $a + c \leq b + c$.
2. $a \cdot c \leq b \cdot c$.

Proof. (1) This proof can also be done by inducting on c. For $c = 0$, we have $a \leq b \implies a + d = b$, where $d \in \mathbb{N}_0$, which is valid. So, let us assume that for all natural numbers up to n, we have

$$b + n = (a + d) + n \implies a + n + d = b + n \implies a + n \leq b + n$$

Then, we need to prove that,

$$b + S(n) = (a + d) + S(n) \implies a + d + S(n) = b + S(n) \implies a + S(n) \leq b + S(n)$$

Now,

$$\begin{aligned} a + S(n) + d &= a + S(n + 0) + d \\ &= a + n + S(0) + d \\ &= a + n + d + S(0) \\ &= b + n + S(0) \\ &= b + S(n + 0) \\ &= b + S(n) \end{aligned}$$

This concludes the proof.
(2) Similar to the proof of (1) and left as an exercise to the readers. ■

Notice that the modified natural numbers start with 0 by construction and we have $0 \leq n$ for all $n \in \mathbb{N}_0$. So, 0 is the least element of the natural numbers, and this ensures that **the natural numbers are well-ordered**. The proof is given below.
Disclaimer: Previously, we have seen that the axiom of choice is equivalent to the well-ordering theorem that states that every set can be well-ordered. This might make some infer that the proof of well-ordering of natural numbers also require the axiom of choice. However, the well-ordering of natural numbers can be proven from the Peano axioms, and the Peano axioms themselves can be constructed from the ZF axioms. So, the well-ordering of natural numbers is inherent by construction and independent of AoC. AoC or equivalently the well-ordering theorem is the more generalized and powerful version that we use to ensure that all sets other than the natural numbers are well-ordered.

Theorem 1.8.6: The Subsets of Natural Numbers and Well-Ordering

The natural numbers are well-ordered.

Proof. Let $A \subseteq \mathbb{N}$ be any non-empty subset of the natural numbers. Assume for the sake of contradiction that A is not well-ordered, i.e., has no least element.
As the natural numbers start with 0, it is the least element of the natural numbers and that means $0 \notin A$.
By our assumption of A having no least element, for every $n \in A \subseteq \mathbb{N}$, we cannot have $k \leq n$ where $k \in \mathbb{N}$, ensuring that $k \notin A$.
As we are working with the natural numbers, and see that the property of not having least element is true for when $n = 0$ (base case), and for any n, by induction axiom, we have that for $S(k) \notin A$.
This means for all $k \in \mathbb{N}$, that is for all natural numbers $n \in \mathbb{N}$, we have $n \notin A$.
Now, $n \in \mathbb{N}, n \notin A \implies A \cap \mathbb{N} = \varnothing$. As $A \subseteq \mathbb{N}$, this implies that $A = \varnothing$. But by our construction of the question, A is non-empty.
This means our assumption must be wrong, and so, every non-empty subset of natural numbers has a least element and the natural numbers are well-ordered. ■

Exercise 1G

1. **Exercises related to the successor operation:**

   a. Investigate what will happen if the following Peano axioms are dropped:

   I. (PA2)(b).

   II. (PA2)(c).

   The answer(s) to this question will shed light on the motivation behind the axioms.

b. What characteristics does the 'successor function' have?

2. **Exercises related to Addition, Multiplication, Inequality, and Induction Axiom:**
   Finish all the incomplete proofs from theorem (1.8.1) to theorem (1.8.5).

3. **Some more properties related to inequality:** Prove the following theorem -

Theorem 1.8.7: Transitivity of Inequality and Law of Trichotomy

The set of natural numbers with 0 equipped with the operation of $\leq$ and $<$ has the following properties:

a. **Transitivity:** For all $a, b, c \in \mathbb{N}_0$, equipped with $\leq$, if $a \leq b$ and $b \leq c$, then $a \leq c$.

b. **Law of Trichotomy:** For all $a, b \in \mathbb{N}_0$ equipped with $<$, only one of the following three must be true:

$$a < b, \; a = b, \; b < a$$

1.8.2 From ZF to the Peano Axioms

1.9 Supplementary Material: From the Natural Numbers to the Rational Numbers

1.9.1 Constructing the Integers

1.9.2 Constructing the Rationals

We keep our discussions restricted up to here, in the supplementary material section of the next chapter, we construct the real numbers from the rational numbers by filling up the gaps in the rationals through a process called 'completion'.

Chapter 2

Series and Sequences

Chapter 3

Limits and Continuity

Chapter 4

Topology: What & Why?

Continuity is of special importance in mathematical analysis, as it allows us to study 'nice' functions and their behavior. However, rather than studying individual continuous functions, it is better to study a space of continuous functions. Furthermore, it is natural to attempt to generalize the tools we are studying further in the space of continuous functions. The most general structure that we have worked with up to now is metric spaces, which gives us a notion of distance in our space. So, it is quite natural to try to look beyond such constructions, and as all our analysis has been developed on set theory, we look to establish continuity with as little input as possible from anything more than that. The need to go beyond the notion of distance and study the space of continuous functions with nothing but just the basics of set theory actually gave rise to the popularity of 'topology'.

4.1 Open Sets, Closed Sets and Topology

Open sets are the main instrument of topology, but before we tackle topology itself, we need to understand open sets first. Due to the very abstract nature of open sets, one feels that open sets should be motivated in the simplest setting available to us, the real line. That is why, in this instance, we are breaking from the usual norm of using metric spaces to introduce concepts, and are sticking to the ever-reliable real line. But to ensure complete understanding, the formulation of open sets in terms of metric spaces has been left as an exercise.

4.1.1 Motivation behind Open Sets in $\mathbb{R}$

Recall the definition of continuity at a point in $\mathbb{R}$ equipped with the usual Euclidean metric is:
A function $f : \mathbb{R} \to \mathbb{R}$ is said to be continuous at some point $x_0 \in \mathbb{R}$ if for each $\epsilon > 0$, there exists some $\delta(\epsilon) > 0$ such that $|f(x) - f(x_0)| < \epsilon$ is true whenever $|x - x_0| < \delta(\epsilon)$ is true.
Unpacking the modulus, we can get from the definition,

$$
\begin{aligned}
&|f(x) - f(x_0)| < \epsilon && |x - x_0| < \delta(\epsilon) \\
&-\epsilon < f(x) - f(x_0) < \epsilon && -\delta(\epsilon) < x - x_0 < \delta(\epsilon) \\
&\boxed{f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon} && \boxed{x_0 - \delta(\epsilon) < x < x_0 + \delta(\epsilon)}
\end{aligned}
$$

So, now we can formulate the definition in terms of open intervals.
$f(x)$ is contained in an open interval of center $f(x_0)$ and radius ϵ, let us define this open interval as U. x is similarly contained in an open interval of center x_0 and radius $\delta(\epsilon)$, and let us define this open interval as V. Now, $f(x) \in U$ and $x \in V$, combining these two we get, $\boxed{f(V) \subseteq U}$.
Things would be nice if we could keep things like this, but we have dependencies on two sets, namely U, V. Keeping true to the original definition of $\epsilon, \delta(\epsilon)$, we wish to make the sets in the form $U, V(U)$. We can easily achieve that in the following manner: $f(V) \subseteq U \implies \boxed{V \subseteq f^{-1}(U)}$. That is, we have taken the pre-image of V.
A common mistake is to confuse the pre-image with the inverse of a function. Here, we are not talking about the inverse of the function. If it were to be true, then all continuous functions would be bijective, as inverse functions exist when it is bijective. A quick refutation to this is the function $f(x) = x^2$, which is continuous but not bijective.
Up to now, we are only considering continuity at a particular point x_0, but we want to include functions that are just continuous in

the whole $\mathbb{R}$. To do that we just add the condition $\boxed{U \subseteq \mathbb{R}}$.
Now we look at the characteristics of V:

a. For all x, $x \in V$.

b. V is an open interval with center x_0 and radius $\delta(\epsilon)$.

c. $V \subseteq f^{-1}(U)$.

Combining all three together, we get a peculiar set where for all x, $x \in V \subseteq f^{-1}(U)$. That is, every point of the set forms an open interval that is contained in the set itself. Let us define a new type of set having this property, and as this new type of set utilizes the concept of open interval, we term it an open set.

Definition 4.1.1: Open Sets in $\mathbb{R}$

A subset $O \subseteq \mathbb{R}$ is said to be an open subset if for each $x \in O$, there is some open interval centered at x with appropriate radius $\epsilon > 0$, $N_\epsilon(x)$, such that $x \in N_\epsilon(x) \subseteq O$.

But what of U? It is an open interval with center $f(x_0)$ and radius ϵ. Is it an open set? Well, this gives us the next theorem.

Theorem 4.1.1: Open Intervals and Open Sets in $\mathbb{R}$

Every Open Interval in $\mathbb{R}$ is an Open Set.

Proof. Let us consider an open interval $(c, d) = \{x \in \mathbb{R} : c < x < d\}$.
Let $a \in (c, d)$ and let us construct an open interval of center a and radius ϵ and call it $N_\epsilon(a)$. So,

$$N_\epsilon(a) = \{x \in \mathbb{R} : |x - a| < \epsilon\}$$

To satisfy the definition of open sets, we want to choose the smallest possible radius we can,n, and to do that, we choose $\epsilon = \min\{a - c, d - a\}$ [Remember that $c < a < d$].

$$\begin{aligned} & |x - a| < \epsilon \\ \implies & -\epsilon < x - a < \epsilon \\ \implies & \boxed{a - \epsilon < x < a + \epsilon} \end{aligned} \tag{4.1.1}$$

Now, as $\epsilon = \min\{a - c, d - a\}$, this means that,

$$\begin{aligned} & \epsilon \leq a - c \text{ and } \epsilon \leq d - a \\ \implies & \boxed{c \leq a - \epsilon \text{ and } a + \epsilon \leq d} \end{aligned} \tag{4.1.2}$$

Combining both (4.1.1) and (4.1.2) gives,

$$c \leq a - \epsilon < x < a + \epsilon \leq d$$

[Someone new might be confused by this statement, as it seems that c, d are included in this open interval, but notice that if we write it like this $c \leq a - \epsilon$, $a + \epsilon \leq d$, and $a - \epsilon < x < a + \epsilon$, then we get the bounds of the open interval for any arbitrary x to be $c < x < d$ which exactly is the open interval (c, d) that we started with.]
So, the open interval for any arbitrary $x \in (c, d)$ is always contained in the open interval (c, d) itself, and hence, the open interval (c, d) is an open set. As (c, d) is an arbitrary open interval, we have proven that every open interval in $\mathbb{R}$ is an open set. ■

By theorem (4.1.1) we can conclude that U is also an open set. So, combining everything, we get our new definition of continuity to be -

Theorem 4.1.2: Continuity using Open Sets in $\mathbb{R}$

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous if and only if the pre-image of each open set is an open set[1]. (i.e. for each open set $U \subseteq \mathbb{R}$, $f^{-1}(U) \subseteq \mathbb{R}$ is open)

Proof. Formalize the argument described before the statement of this theorem in this section. ■

Now, a question might arise in the mind of the readers that everything we did in this section regarding open sets is open intervals. What about closed intervals? Are they open sets in $\mathbb{R}$? This brings us to the following -

Theorem 4.1.3: Closed Intervals and Open Sets in $\mathbb{R}$

Every Closed Interval in $\mathbb{R}$ is **NOT** an Open Set.

Proof. Let $x \in [c, d]$ where $[c, d] = \{x \in \mathbb{R} : c \leq x \leq d\}$.
By definition of open sets, if $[c, d]$ is open then we should be able to construct an open interval of center x and radius ϵ, $N_\epsilon(x)$, for any $x \in [c, d]$ and $N_\epsilon(x) \subseteq [c, d]$.
We already know that (c, d) is an open set by theorem (4.1.1), and that leaves us with the two terminal points, namely c, d.
Let us construct an open interval with center c and radius $\epsilon > 0$, i.e.

$$N_\epsilon(c) = \{x \in \mathbb{R} : |x - c| < \epsilon\}$$

Unpacking modulus gives us,

$$c - \epsilon < x < c + \epsilon$$

As $\epsilon > 0$, $c - \epsilon < c$. So, we get,

$$c - \epsilon < c < x < c + \epsilon$$

However, the set $[c, d]$ obviously does not contain any $x < c$, so $(c - \epsilon, c) \notin [c, d]$.
Therefore, $[c, d]$ is not an open set. As our choice of c, d is arbitrary, every closed interval in $\mathbb{R}$ is not open. ■

4.1.1.1 Properties of Open Sets

Now, we want to investigate the properties of open sets to understand them better. From here on, we will again revert to using metric spaces to discuss things.

Theorem 4.1.4: Arbitrary Unions of Open Sets

Arbitrary unions (can be infinite too!) of open sets are an open set. That is if $\{U_i\}_{i \in I}$ is any collection of open subsets of $\mathbb{R}$, then $\bigcup_{i \in I} U_i$ is open.

Proof. Let $\{U_i\}_{i \in I}$ be open sets. We want to prove that $\bigcup_{i \in I} U_i$ is open.
Now, if we take any arbitrary $x \in \left(\bigcup_{i \in I} U_i\right)$, then that means for some $i \in I$, $x \in U_i$, and $U_i \subseteq \left(\bigcup_{i \in I} U_i\right)$ we have

$$x \in U_i \subseteq \Big(\bigcup_{i \in I} U_i\Big)$$

By construction, U_i is open. As the choice of x is arbitrary, this is always true for $\bigcup_{i \in I} U_i$. So, $\bigcup_{i \in I} U_i$ is open. ■

Theorem 4.1.5: Finite Intersections of Open Sets

Finite intersections of open sets are an open set. That is if $\{U_n\}_{n \in \mathbb{N}}$ is any collection of open subsets of $\mathbb{R}$, then $\bigcap_{n=1}^{n} U_n$ is open.

[1] Remember that $f(V) \subseteq U$, so U actually is the 'output' of function f. For clarity, let us denote $f : \mathbb{R}_{in} \to \mathbb{R}_{out}$ where in, out stands for input and output respectively, then $U \subseteq \mathbb{R}_{out}$ and $f^{-1}(U) \subseteq \mathbb{R}_{in}$.

Proof. Let $\{U_n\}_{n\in\mathbb{N}}$ be open sets. We want to prove that $\bigcap_{n=1}^{n} U_n$ is open.
Now if $\bigcap_{n=1}^{n} U_n = \emptyset$, then by **exercise 4 of 4A**, we know that the $\emptyset$ is open.
Again, if $\bigcap_{n=1}^{n} U_n \neq \emptyset$, then, let us choose any arbitrary $x \in \big(\bigcap_{n=1}^{n} U_n\big)$. For some n, we will have $x \in U_n$ and $U_n \subseteq \big(\bigcap_{n=1}^{n} U_n\big)$.
We have

$$x \in U_n \subseteq \big(\bigcap_{n=1}^{n} U_n\big)$$

By construction, U_n is open. As the choice of x is arbitrary, this is always true for $\bigcap_{n=1}^{n} U_n$.
So, $\bigcap_{n=1}^{n} U_n$ is open. ■

Query 4.1.1: Infinite Intersections of Open Sets

Are infinite intersections of open sets open?

Investigation. From previously learned knowledge, we know that open intervals are open sets and closed intervals are not open sets. A singleton set is a closed interval, and that makes a singleton set not an open set. So, what we will try is to construct an example of open sets whose infinite intersection results in a singleton set. The most straightforward way of producing such a construction is by taking an open interval and slowly decreasing its radius until it becomes just a point. As we also have to incorporate the intersection of sets, what we can do is to intersect many open intervals with decreasing radius. The most common decreasing function we know is the 'Harmonic series'. So let us construct an open interval with center at 0 of the real line and radius $\frac{1}{n}$, i.e. $\big(-\frac{1}{n}, \frac{1}{n}\big)$. Now,

$$\bigcap_{n=1}^{\infty}\Big(-\frac{1}{n}, \frac{1}{n}\Big) = \{0\}$$

This construction is complete, and as we can see, the infinite intersections of open sets can be not open.
Again, if we are only taking a random open interval (c, d) and infinitely intersecting the open interval with itself, then we shall get (c, d) itself, which is an open set. That is

$$\bigcap_{n=1}^{\infty}(c, d) = (c, d)$$

So, infinite intersections of open sets can be open too.
So, it turns out that the infinite intersection of open sets is **not necessarily open**. That is why the number of intersections is taken to be finite in theorem (4.1.5). ■

Theorem 4.1.6: The Empty Set, $\emptyset$, and the Whole Space, X (for real numbers $\mathbb{R}$), and Open Sets

The empty set, $\emptyset$, and the whole space, X (for real numbers $\mathbb{R}$), is open.

Proof. Let us have two disjoint open sets, O_1, O_2. Then, $O_1 \cap O_2 = \emptyset$. By theorem (4.1.5), finite intersections of open sets are open. So the empty set, $\emptyset$, is open.
Again, let us take any $x \in X$ and construct an open ball of center x and radius ϵ, and denote it by $B_\epsilon(x)$. Obviously, as X is the whole space, $B_\epsilon(x) \subseteq X$. As our choice of x is arbitrary, we can construct an open ball contained in X for any $x \in X$. So, $\cup_{x\in X} B_\epsilon(x) = X$ and by theorem (4.1.4), arbitrary unions of open sets are open. Therefore, the whole space X is also open.
For $\mathbb{R}$, we can do the proof in the same way, or we can have an alternate way. We can find two open sets, namely $(-\infty, a)$ and (b, ∞) where $a, b \in \mathbb{R}$, $a > b$ such that $(-\infty, a) \cup (b, \infty) = \mathbb{R}$. The reason for such a construction is that we wish to apply the knowledge of theorem (4.1.4), which states that arbitrary unions of open sets are open. As the union is $\mathbb{R}$ itself, we conclude that $\mathbb{R}$ is open. ■

Exercise 4A

1. **Open Sets in the Setting of Metric Space:** Using the metric space structure, from the definition of continuity, motivate and prove the following definition and theorem -

Definition 4.1.2: Open Sets in Metric Spaces

Let U be a subset of a metric space (M, d). U is called an open set if for every $x \in U$ there exists an open ball $B_\epsilon(x)$ such that $B_\epsilon(x) \subseteq U$.

Theorem 4.1.7: Open Balls and Open Sets

Every Open Ball is an Open Set.

2. **Going down the metric space route, do we get something the same or similar to the theorem (4.1.2)?** In other words, prove the following theorem -

Theorem 4.1.8: Continuity, Metric Spaces and Open Sets

Let (X, d_1) and (Y, d_2) be two metric spaces and then the function $f : (X, d_1) \to (Y, d_2)$ is called continuous if and only if for every open set $U \subseteq Y$ in (Y, d_2), there exists some open set $f^{-1}(U) \subseteq X$ in (X, d_1).

3. **Prove the following theorem in metric spaces -**

Theorem 4.1.9: Closed Balls and Open Sets

Every Closed Ball is **NOT** an Open Set.

4. **★★★ Prove that ∅ is an Open Set. [To the unfamiliar, you will need the concept of 'vacuously true' to answer this problem when dealing with the empty set.]**

4.1.2 Closed Sets, Clopen Sets

Now that we know open sets as a generalization of open intervals in $\mathbb{R}$, we want to define something similar for closed intervals too. Keeping true to our set-theoretic theme, any arbitrary closed interval in $\mathbb{R}$, such as $[c, d]$, can be expressed in terms of open intervals as follows -

$$[c, d] = \mathbb{R} \setminus (-\infty, c) \cup (d, \infty) = \big((-\infty, c) \cup (d, \infty)\big)^c$$

So, closed intervals can be expressed as complements of open intervals quite easily, and as we want to formulate a generalization of closed intervals like we formulated open sets for open intervals, let us call it 'closed set'.

Definition 4.1.3: Closed Sets

A set is closed if its complement is open.

With the information we learned in the open sets section, we now formulate the following theorems:

Theorem 4.1.10: Closed Intervals and Closed Sets

Every closed interval in $\mathbb{R}$ is a closed set.

Proof. Let, $[c, d]$ be any closed interval in $\mathbb{R}$, then $[c, d]^c = (-\infty, c) \bigcup (d, \infty)$. We know from theorems (4.1.1) that $(-\infty, c)$ and (d, ∞) are open sets. Again, we know by theorem (4.1.4) that their union, i.e. $[c, d]^c$, is open. This means that, by definition of closed sets, $[c, d]$ is closed. ■

Theorem 4.1.11: Closed Ball and Closed Sets

Every closed ball is a closed set.

Proof. Let $C_r(x)$ be a closed ball of center x and radius $r > 0$.
Then, $\left(C_r(x)\right)^c = \{x, y \in M : d(x, y) > r\}$.
For the purpose of proving the theorem, we now need to show that any open ball, $B_\epsilon(y)$, constructed in $\left(C_r(x)\right)^c$ belongs to $\left(C_r(x)\right)^c$ itself.
At first, we need to relate ϵ with r. So, in that regard, as $d(x, y) > r$, let us take $\boxed{\epsilon = d(x, y) - r > 0}$.
Now, let us consider an arbitrary open ball $B_\epsilon(y)$ [Remember that $y \in \left(C_r(x)\right)^c$] and $z \in B_\epsilon(y)$, then $d(y, z) < \epsilon$.
Now, by the triangle law of metric spaces,

$$\begin{aligned} & d(x,z) + d(z,y) \geq d(x,y) \\ \Longrightarrow\ & d(x,z) \geq d(x,y) - d(z,y) > d(x,y) - \epsilon = r \\ \Longrightarrow\ & d(x,z) > r \end{aligned}$$

So, $B_\epsilon(y) \subseteq \left(C_r(x)\right)^c$. As our choice of y was arbitrary, it is true for all the complements. So, every closed ball is a closed set.
■

Theorem 4.1.12: The Empty Set, $\varnothing$, and the whole space, X, and Closed Sets

The empty set, $\varnothing$, and the whole space, X, are closed.

Proof. $\varnothing^C = X$, and by definition of open sets, X is open. So, $\varnothing$ is closed.
Again, $X^c = \varnothing$, and by definition of open sets, $\varnothing$ is open. So, X is closed. ■

Now, it seems that we have reached a peculiar conclusion, by the theorems (4.1.6) and (4.1.12), the empty set and the whole space are simultaneously open and closed. This might seem perplexing to the readers initially, but if you take a close look at the definition of a closed set, you will realize that there is nothing stopping both a set and its complement from being open. Again, nothing is stopping a set from being neither open nor closed. This brings us to define the following set -

Definition 4.1.4: Clopen Sets

If a set is both open and closed, then it is called a clopen set.
The empty set $\varnothing$ and the whole space X are always clopen.

Curious minds may ask if there are any more clopen sets like the empty set, the whole space? Well, we will answer this question in full generality a bit later, but with what we have already learnt, we can assert a certain theorem about clopen sets in $\mathbb{R}$.

Theorem 4.1.13: Other Clopen Sets in $\mathbb{R}$

Excluding the empty set, $\varnothing$, and the real line, $\mathbb{R}$, there is no other clopen set in $\mathbb{R}$.

Proof. Let $A \subseteq \mathbb{R}$ and $A \neq \varnothing$. For the sake of contradiction, let us assume A is a clopen set.
Then, by definition of open sets in $\mathbb{R}$, for all $x \in A$, we can find an open interval of radius ϵ with center at x, $N_\epsilon(x)$, such that $x \in N_\epsilon(x) \subseteq A$.
Again, as A is closed, the complement of A, i.e. $A^c = \mathbb{R} \setminus A$, is also open. Then for any $y \in A^c$, we can similarly find an open interval $y \in N_\epsilon(y) \subseteq A^c$.
Now, as we are working in $\mathbb{R}$, we will recall the absolute fundamentals of $\mathbb{R}$, i.e., **completeness axiom**.
Let $S = \{y \in S, x \in A : x < y \text{ for some } y \in \mathbb{R}\}$. Then by completeness axiom, $\sup S = s$ exists.
<u>Case I:</u> If $s \in A$, then A being an open set, $(s - \epsilon, s + \epsilon) \subseteq A \implies (s, s + \epsilon) \subseteq A$. That means some $p \in A$ where $s < p < s + \epsilon$. So, p can be the supremum and $\sup S \neq s$.
<u>Case II:</u> If $s \in A^c$. Then again, by similar logic from the definition of open sets, we can find some $q \in A^c$ where $s - \epsilon < q < s$. So, q can be $\sup S$. This means, $\sup S \neq s$.
Therefore, we arrive at a contradiction, and that means that our initial assumption of A being clopen is incorrect.
$\therefore$ Clopen non-empty proper subset of $\mathbb{R}$ does not exist. ■

Exercise 4B

1. **Properties of Closed Sets:** Verify the following properties of closed sets:

   Theorem 4.1.14: Properties of Closed Sets

   A closed set satisfies the following properties -

   a. The empty set and the whole space are closed.

   b. The arbitrary intersection of closed sets is closed.

   c. The finite union of closed sets is closed.

   [Hint: These properties follow quite naturally from the properties of open sets, and if this is not enough, as the closed set is the complement of open sets, think of some common law about complements, unions, and intersections! (De Morgan's law perhaps?)]

2. **Set that is neither open nor closed?** Prove that the semi-open (or semi-closed) intervals such as $(c, d]$, $[c, d)$ are neither open nor closed.

3. **Formulate a definition of continuous functions in terms of closed sets similar to theorem (4.1.8).**
   [Hint: Prove that $f^{-1}(A^c) = \left(f^{-1}(A)\right)^c$. Then the rest should be easy sailing.]

4. **Investigate the following query -**

   Query 4.1.2: Infinite Unions of Closed Sets

   Are infinite unions of closed sets closed? If not, then provide a counterexample to hammer home your claim.

4.1.3 Topology, Open Sets, and Closed Sets

In the introduction to this chapter, we said that topology wishes to go beyond the metric spaces, i.e., a notion of distance. However, up to now we have defined open sets in terms of open balls (in metric space), open intervals (in $\mathbb{R}$), where some notion of distance is used as evidenced by ϵ in their definition. How do we go beyond it? Well, we also said that we wanted to bring the basics of set theory in our discussions, too. We have established three very powerful theorems on open sets, namely theorems (4.1.4), (4.1.5), and (4.1.6), where just the basic union, intersection of set theory are used. So instead of using the usual definition of open sets in $\mathbb{R}$ and metric space (M, d), we redefine open sets in the form of these theorems. But before we do that, for union and intersection to make sense, we need some collection of sets, the whole space X and the empty set $\varnothing$ being the bare minimum. So let us take $\mathcal{T}$ to be a collection of subsets of the whole space X. That is,

$$\mathcal{T} = \{\varnothing, U_1, U_2, \ldots, X\}$$

With this we finally proceed to define a topology $\mathcal{T}$ on set X -

Definition 4.1.5: Topology, Open Sets, and Closed Sets

Let X be any arbitrary set, and then take $\mathcal{T}$ to be a collection of subsets of X. If the following properties are satisfied -

1. $\varnothing$ and X in $\mathcal{T}$. [Theorem (4.1.6)]
2. Arbitrary unions of any sub-collections of $\mathcal{T}$ is in $\mathcal{T}$. [Theorem (4.1.4)]
3. Finite intersection of any sub-collection of $\mathcal{T}$ is in $\mathcal{T}$. [Theorem (4.1.5)]

Then the collection $\mathcal{T}$ defines a **topology** on the set X. In other words, a topological space is an ordered pair of $(X, \mathcal{T})$ where X is the set and $\mathcal{T}$ is the topology, but when no scope of confusion is possible, we omit specific mention of $\mathcal{T}$.
A set U is called an **open set** if $U \subseteq X$ and $U \in \mathcal{T}$.
A set V is called a **closed set** if its complement V^c is an open set.

It is better to further acquaint the readers with topology by discussing some examples.

Illustrative Example 4.1.1

Standard Topology

Proof. Standard topology is simply the topology that is induced from the Euclidean metric $d_{euc} = |x - y|$ (or the Unitary metric $d_{uni} = \sqrt{|x-y|^2}$) $(\mathbb{R}, d_{euc})$ (or $(\mathbb{C}, d_{uni})$). It can also be $(\mathbb{R}^n, d_{euc})$ or $(\mathbb{C}^n, d_{uni})$ where $d_{euc} = \sum_{i=0}^{i=n} \sqrt{(x_i - y_i)^2}$ and $d_{uni} = \sum_{i=0}^{i=n} \sqrt{|x_i - y_i|^2}$. The verification of this as a topology has already been given before. ■

Illustrative Example 4.1.2

Metric Topology

Proof. The metric topology is defined as the topology generated by utilizing the concept of open balls in the metric space. This has also been covered as exercises to part 4A. ■

Illustrative Example 4.1.3

From Discrete Metric to Discrete Topology.

Proof. Recall that the discrete metric is (X, d) where

$$d(x, y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

Let us define an open ball $B_\epsilon(x) = \{x, y \in X : d(x, y) < \epsilon\}$.
Case I: If $\epsilon \leq 1$, then only $d(x, y) = 0$ is possible in the metric space, meaning that $x = y$ and so $B_\epsilon(x) = \{x\}$.
Case II: If $\epsilon > 1$, then for any $x, y \in X$, $d(x, y) < \epsilon$, so $B_\epsilon(x) = X$.
We know by theorem (4.1.7) that open balls are open sets, and by the properties of open sets, we see that the open ball forms a topology.
Some unique features of this topology: Notice that in this topology, all the subsets of X are clopen, and quite interestingly, the singleton sets are also clopen. ■

Theorem 4.1.15: Discrete Topology and Topology Size

The discrete topology is the largest possible topology.

Proof. Let the discrete topology be denoted as τ_{dis}. To prove that τ_{dis} is the largest possible topology, we need to show that for any topology τ: $\tau \subseteq \tau_{dis}$.
Now, we know from example (4.1.3) that all the subsets of τ_{dis} are clopen. We will use that as follows:

1. The singleton sets are open as seen in example (4.1.3). So, $\{a\}, \{b\}$ are open.
2. Arbitrary unions of open sets are open in topology, so arbitrary unions of singleton sets are also open. So, $\{a\} \cup \{b\} = \big\{\{a\}, \{b\}\big\}$ is open. In the same way we can see that $\big\{\{a\}, \{b\}, \ldots, \{z\}\big\}$, $\big\{\{\{a\}, \{b\} \ \ldots\}, \ldots, \{z\}\big\}$ also is open.
3. The empty set and the whole space, by definition, are open.

Then we can write $\tau_{dis} = \mathcal{P}(X)$. So, τ_{dis} contains all subsets of X. So for any τ we will get $\tau \subseteq \tau_{dis}$.
This proves that the discrete topology is the largest possible topology. ■

Illustrative Example 4.1.4

Trivial Topology

Proof. The trivial topology is the topology constructed using only the sets $\varnothing, X$, where X is the whole space. It is quite easy to verify that the trivial topology forms a topology. ∎

Theorem 4.1.16: Trivial Topology and Topology Size

The trivial topology is the smallest possible topology.

Proof. Let us denote the trivial topology as $\tau_{tri} = \{\varnothing, X\}$. To prove that τ_{tri} is the smallest possible topology, we need to show that for any other topology τ, $\tau_{tri} \subseteq \tau$.
By definition of topology, $\varnothing, X \in \tau$. So, $\tau_{tri} = \{\varnothing, X\} \subseteq \tau$.
This proves that the trivial topology is the smallest possible topology. ∎

4.1.3.1 Properties of Topology

Now, sharp and trained minds may notice an important thing, in mathematics, not everything is a two-way street (i.e., if and only if statements, bijective functions). We have simply constructed a one-way road (from distance to topology), but does the other way (from topology to distance) always exist? This question gives us the next example.

Illustrative Example 4.1.5

The Particular Point Topology does not have a Metric.

Proof. **Construction of Particular Point Topology:** Let X be a non-empty set and $p \in X$ be any point. The topology, $\mathcal{T} = \{U \subseteq X : p \in U\} \bigcup \varnothing$. In other words, a set is **open** if:

1. U contains a particular point p.

2. $U = \varnothing$.

By utilizing the definition of closed sets, a set is **closed** if:

1. U does not contain a particular point p.

2. $U = X$.

Verification of Topology:

1. $\varnothing$ is open and $p \in X$ as X is the whole space, so X is open too.

2. Let $\{U_i\}_{i\in I}$ be open sets. By construction either $p \in U$ or $U = \varnothing$.
Case I: If all $U_i = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ which is open.
Case II: If for at least one U_i, we have $p \in U_i$, then we have $p \in U_i \subseteq \bigcup_{i\in I} U_i$. This means that $\bigcup_{i\in I} U_i$ is open too.
So, an arbitrary union of open sets is open.

3. Let $\{U_n\}_{n\in\mathbb{N}}$ be open sets. By construction either $p \in U$ or $U = \varnothing$.
Case I: If at least one $U_n = \varnothing$ then, $\bigcap_{n=1}^{n} U_n = \varnothing$ which is open.
Case II: If none of the open sets are empty, then for all U_n, by construction, we have $p \in U_n$ for all U_n. This means that $\bigcap_{n=1}^{n} U_n = \{p\}$ at the bare minimum. Then, as $p \in \bigcap_{n=1}^{n} U_n$, $\bigcap_{n=1}^{n} U_n$ is open too.
So, the finite intersection of open sets is open.
Indeed, the particular point topology forms a topology!

Absence of Metric Space Structure: By construction of the particular point topology, we see that the singleton set $\{p\}$ is an open set. Any other singleton set $\{a\}$ where $a \neq p$ is not open. Before proving the absence of a metric space explicitly first, notice that in any metric space, all sets of a particular type, including singleton sets, will be either open or not open. A metric space does not distinguish a particular singular set from other singleton sets. So, intuitively, we see that metric space structure is not present here. Now, recall that from metric spaces, we know that singleton sets are:

1. Always closed, because $\{a\}^c = X \setminus \{a\}$ is open [Basically any open ball centered at a with radius $\epsilon > 0$.] where X is the whole space of the metric space (X, d).
2. Most of the time, it is not open except when metric induces the discrete topology, where every subset of the whole space X in metric space (X, d) is open.

Now, comparing these two points, we clearly see that the particular point topology disagrees with the properties of a metric space and therefore, the particular point topology does not have any metric space structure. ■

Illustrative Example 4.1.6

The Finite Complement Topology or Cofinite Topology does not have a Metric.

Proof. **Construction of Finite Complement Topology:** Let X be an **infinite** set.
The set U is **open** (i.e. $U \subseteq X$ and $U \in \mathcal{T}$) if either:

1. $U = \varnothing$.
2. $U^c = X \setminus U$ is finite.

Utilizing the definition of closed set, the set V is **closed** if either:

1. $V = X$.
2. $\left(V^c\right)^c = V$ is finite.

Verification of Topology:

1. By definition, $\varnothing,\ X$ is open.
2. Arbitrary unions of open sets are an open set.
   Case I: If all $U = \varnothing$, then $\bigcup_{i\in I} U_i = \varnothing$ which is open.
   Case II: If at least one $U \neq \varnothing$, then we need to show that $\bigcup_{i\in I} U_i$ is also open. Then
   $$\{\bigcup_{i\in I} U_i\}^c = X \setminus \bigcup_{i\in I} U_i = \bigcap_{i\in I} (X \setminus U_i) = \bigcap_{i\in I} (U_i)^c$$
   Now, with U_i being open sets, U_i^c is finite, and infinite intersections of finite sets result in a finite set or the empty set. This means that $\bigcup_{i\in I} U_i$ is also open.
3. Finite intersections of open sets are an open set.
   Case I: If all $U = \varnothing$ then $\bigcap_{n=1}^{n} U_n = \varnothing$ which is open.
   Case II: If not all $U \neq \varnothing$, then we need to show that $\bigcap_{n=1}^{n} U_n$ is open. Then
   $$\{\bigcap_{n=1}^{n} U_n\}^c = X \setminus \bigcap_{n=1}^{n} U_n = \bigcup_{n=1}^{n} (X \setminus U_n) = \bigcup_{n=1}^{n} (U_n)^c$$
   Now, with U_n being open sets, U_n^c is finite, and finite unions of finite sets result in a finite set. This means that $\bigcap_{n=1}^{n} U_n$ is also open.

So, the finite complement topology actually forms a topology!
Absence of Metric Space Structure: Observe that in the finite complement set, the closed set can only be finite sets or the whole space X where X is an infinite set. So, excluding the whole space, no other closed set is infinite.
However, in metric spaces (excluding the discrete metric space) where the whole space is infinite, some closed sets (not being the whole space itself) can exist that are infinite. A short proof is given below.

Lemma 4.1.1: Infinite Closed Sets in Infinite Metric Spaces

Excluding the discrete metric space, in an infinite metric space (X, d) (i.e., X is infinite), there exist closed sets $C \subset X$ that are infinite too.

Proof of Lemma. Since X is infinite, there exists an infinite sequence of distinct points $\{x_n\}_{n\in\mathbb{N}} \subset X$. Let us define $A = \{x_n : n \in \mathbb{N}\}$. We want to show that A is closed. Let $p \in X \setminus A$. Then, for each $x_n \in A$, the distance $d(p, x_n) > 0$. Now, if we define

$$\epsilon = \frac{1}{2} \min_{n\in\mathbb{N}} d(p, x_n).$$

This is positive due to the distinctness of the points in A. The open ball $B_\epsilon(p)$ does not intersect A, as for any $y \in B_\epsilon(p)$, $d(p, y) < \epsilon$, which is strictly smaller than $d(p, x_n)$ for all $x_n \in A$. Thus, $B_\epsilon(p) \subseteq X \setminus A$, proving that $X \setminus A$ is open. Therefore, A is closed.
Since A is infinite and $A \subsetneq X$, it is an infinite closed proper subset. □

Lemma 4.1.2: Co-existence of the Discrete and Cofinite Topology

The Discrete Topology and Cofinite Topology cannot exist simultaneously if the whole space X is infinite.

Proof of Lemma. Now, what about the case of the discrete metric space? As seen in example (4.1.3), the discrete metric induces the discrete topology, which is different from the finite complement topology. Because the finite complement topology only allows infinite sets to be open, the discrete topology allows finite sets, such as singleton sets, to be open too! They can only be the same if X were finite, but by construction, X is infinite. So, they cannot exist simultaneously. □

The finite complement topology does not have a metric. ■

So, we have found two such examples of topologies that do not admit of any metric. This gives us the basis for the following important theorem -

Theorem 4.1.17: Topology and Metric

Every metric space induces a topology, but not every topology induces a metric.

Proof. To prove that every metric space induces a topology, just use the concept of open balls as open sets.
To prove that not all topologies admit of a metric, one can simply refer to examples (4.1.5) and (4.1.6). There are many more topologies that do not admit of a metric, these two are just 2 simple examples. ■

Curious readers may think, what sort of restriction do we need to impose on topological spaces so that they have a metric? We will slowly but definitely come to answer this question in the future when our knowledge about topological space will be more bolstered. With that established, we now need to define continuity in topological spaces. Following theorems (4.1.2) and (4.1.8), we define continuity in topological spaces as below:

Definition 4.1.6: Continuity in Topological Spaces

Let X, Y be topological spaces, then a function $f : X \longrightarrow Y$ is continuous if for every open set $U \subseteq Y$, we can always find the preimage to be an open set in X, that is $f^{-1}(U) \subseteq X$ is open.

Theorem 4.1.18: Intersections of Topologies

The arbitrary intersection of topologies, in the same whole space, is a topology.

Proof. Let $\{\tau_i\}_{i\in I}$ be a collection of topologies on the set X. We want to prove that $\bigcap_{i\in I} \tau_i$ is also a topology.

1. Since for each τ_i, $\tau_{tri} \subseteq \tau_i$ by theorem (4.1.16), we know that $\{\varnothing, X\} = \tau_{tri} \subseteq \bigcap_{i\in I} \tau_i$.

2. Let $\{U_j\}_{j\in J} \in \bigcap_{i\in I} \tau_i$. This means that for every $j \in J, U_j \in \tau_i$ for all $i \in I$. Since each τ_i is a topology, $\bigcup_{j\in J} U_j \in \tau_i$ for all $i \in I$. Therefore, $\bigcup_{j\in J} U_j \in \bigcap_{i\in I} \tau_i$

3. Let $U_1, U_2, \ldots, U_n \in \bigcap_{i\in I} \tau_i$. This means that for every $k = 1, 2, \ldots, n,\ U_k \in \tau_i$ for all $i \in I$. Since each τ_i is a topology, $\bigcap_{k=1}^{n} U_k \in \tau_i$ for all $i \in I$. Therefore, $\bigcap_{k=1}^{n} U_k \in \bigcap_{i\in I} \tau_i$.

Now, we see that $\bigcap_{i\in I} \tau_i$ satisfies every axiom of topology and hence the arbitrary intersection of topologies is also a topology. ■

Now, what about the union of topologies? Is it a topology? Due to the interesting nature of this question (no trick needed here, just an interesting exercise genuinely!), it is left as an exercise to the readers. However, I have provided hints to this question in the exercises so that the readers do not get lost. Below is the list of 22 distinct topologies possible from the set $X = \{a, b, c\}$, which is part of a hint to the union of topologies question.

1. $\{\emptyset, X\}$
2. $\{\emptyset, \{a\}, X\}$
3. $\{\emptyset, \{b\}, X\}$
4. $\{\emptyset, \{c\}, X\}$
5. $\{\emptyset, \{a\}, \{b\}, X\}$
6. $\{\emptyset, \{a\}, \{c\}, X\}$
7. $\{\emptyset, \{b\}, \{c\}, X\}$
8. $\{\emptyset, \{a\}, \{b\}, \{c\}, X\}$
9. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$
10. $\{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$
11. $\{\emptyset, \{b\}, \{c\}, \{b, c\}, X\}$
12. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, X\}$
13. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, X\}$
14. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{b, c\}, X\}$
15. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$
16. $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$
17. $\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{b, c\}, X\}$
18. $\{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$
19. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, X\}$
20. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, X\}$
21. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, c\}, \{b, c\}, X\}$
22. $\{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$

4.1.4 Supplementary Material: Basis for the Topology

As one can see, explicitly writing out the possible topologies is very difficult and cumbersome. So, it is quite natural to search for something that simplifies this. Basically, we want something smaller that produces or generates the topology!
Characteristics desired: We want a sub-collection to have the following characteristics:

1. Be smaller than the topology itself.
2. Only the sets, their unions, and intersections are allowed to be used to keep the sub-collection foundational and simple.

Restrictions imposed: This sub-collection has to generate members that basically satisfy the definition of topology. That is,

1. The empty set and the whole space must be among the generated members.
2. Generated members must satisfy closure under arbitrary unions.
3. Generated members must satisfy closure under finite intersections.

Now, we have to answer two questions: what are the defining characteristics of this sub-collection, and through what process does this sub-collection generate the topology? But before that, we need a name and symbol for this sub-collection. As it is kinda like the foundation for the topology, let us term it the basis for the topology and denote it by $\mathcal{B}$. Now, let us try to answer the questions with as much flexibility as we can.
Following desired characteristics (1), we know that $\boxed{\mathcal{B} \subseteq \mathcal{T}}$. Now, we have a smaller object, and from that smaller object, we have to generate something larger, so we need to combine the smaller objects. So, this gives us an idea about the method of generating members from the basis. Using this idea and desired characteristics (2), we set our generating method to be unions of the members of the basis. But what type of union? To keep things flexible, let us set the union as arbitrary unions (Similar to the definition of

topology). This generating process must generate all the members of the topology. So, we have $\boxed{\cup_{i\in I} B_i = \mathcal{T}_j}$ for each $B_i \in \mathcal{B}$ where $\mathcal{T}_j \in \mathcal{T}$ for some index $j \in J$. To get the whole space, we require $\mathcal{T}_j = X$. In fact, we can do a more direct setup for the whole space, we can write $\boxed{\cup_{x\in X} B_x = X}$. In general we can get, $\cup_{j\in J}\mathcal{T}_j = \mathcal{T} \implies \cup_{j\in J}(\cup_{i\in I} B_i) = \mathcal{T} \implies \cup_{k\in K} B_k = \mathcal{T}$. This means that we also get the arbitrary union of the topology property! What about the empty set? According to the convention of set theory, the arbitrary union of the empty family is the empty set, i.e. $\cup_{B\in\varnothing} B = \varnothing$. Now, from the finite intersection property of topology, we must have $\boxed{B_p \subseteq B_m \cap B_n}$ as every $\mathcal{B} \subseteq \mathcal{T}$. Now, we have finished our synthesis for the basis.

Notice that in the four boxed equations, the first two set the domain and the generating process, respectively. The fourth box preserves the finite intersection property. But why the third box? Well, when working with topological spaces, the same topology can be applied to different whole spaces. So, we need something to capture the whole space, and that is why we have boxed the third one. An alternate formulation of the third box is that for every $x \in X$, there exists a B_x such that $x \in B_x$. With this out of the way, let us define the following concepts.

Definition 4.1.7: Basis for the Topology

The basis for the topology, denoted by $\mathcal{B}$, is a sub-collection of sets $\mathcal{B} \subseteq \mathcal{T}$ (or if one wishes to circumvent the mention of the topology itself, can quite simply write it as a collection of subsets of X) such that :

1. For every $x \in X$, there exists a B_x such that $x \in B_x \subseteq \mathcal{B}$. This implies that $\cup_{x\in X} B_x = X$.
2. For every B_i, B_j there exists at least one B_k such that $B_k \subseteq (B_i \cap B_j)$.

The basis for the topology is also called the **base** for the topology.

Definition 4.1.8: Generated Topology

The generated topology is the topology that we get from the union of the arbitrary family of basis for the topology. Denoted by $\mathcal{T}_{gen}$, symbolically,

$$\mathcal{T}_{gen} = \{U \subseteq X : U = \cup_{k\in K} B_k, B_k \in \mathcal{B}\}$$

Now, we have thematically went from the topology to the basis and generated topology definition. But we need to check if the other way, that is from the basis and generated topology to the topology, checks out as well.

Theorem 4.1.19: Generated Topology forms a Topology!

The generated topology indeed forms a topology.

Proof. ■

Definition 4.1.9: Basic Open Sets

The members of the basis for the topology, when equipped with the generated topology, are open, and thus they are called basic open sets.

Now, one question may arise to the mind of the readers, is the generated topology unique or not? The following theorem tells us exactly that -

Theorem 4.1.20: The Uniqueness of Bases and Generated Topologies

1. Given a basis, the generated topology is unique.
2. Given a topology, the basis is not unique.

Proof. ■

With that out of the way, let us look at some examples.

Illustrative Example 4.1.7

Some examples of bases for different topologies:

1. For metric spaces, the collection of open balls forms a basis for the topology.
2. For the discrete topology, the set of all singleton sets forms a basis for the topology.

Our discussion on the topic of basis for the topology is confined up to this much for now. Hopefully, the readers will get to pursue more in the topology course.

Exercise 4C

1. **Investigate the claims for the following sets in the table.** Consider the **standard topology** when dealing with the numbers $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}, \mathbb{C}$. For the rest, consider the usual topology generated by open sets in a metric space.

Sets	Is it Open?	Is it Closed?	Is it Clopen?
Empty Set	Yes	Yes	Yes
Non-empty Finite Sets	No	Yes	No
Open Balls	Yes	No	No
Closed Balls	No	Yes	No
Whole Space, X	Yes	Yes	Yes
Natural Numbers, $\mathbb{N}$	No	No	No
Integer Numbers, $\mathbb{Z}$	No	No	No
Irrational Numbers, $\mathbb{I}$	No	No	No
Rational Numbers, $\mathbb{Q}$	No	No	No
Real Numbers, $\mathbb{R}$	Yes	Yes	Yes
Complex Numbers, $\mathbb{C}$	Yes	Yes	Yes

Table 4.1: An Illustration of Sets and Their Topological Properties.

2. **Let $A \subseteq X$ and for every $x \in A$, there exists an open set U such that $x \in U \subseteq A$. Prove the set A is open.**

3. **Investigate the following query -**

Query 4.1.3: Union of Topologies

Is the union of two topologies a topology? If not, then under what condition (if and only if statement) will the union be a topology? If one can find such a condition, can it be generalized to arbitrary unions?
[Hint: (I) Think of some of the topologies from the set $X = \{a, b, c\}$. (II) You have two bags but can only carry one. But you want to keep both bags, so what would you do in that case? You simply put the smaller bag into the larger one!]

4. **Investigate the following query -**

Query 4.1.4: Topology where Sets are either Clopen, or not open and not closed in $\mathbb{R}$

Excluding the τ_{dis}, τ_{tri}, which topology ensures that every open set in $\mathbb{R}$ is closed and vice versa?
[Hint: Look up **the partition topology**. Is it a generalization of some other topology we covered?]

5. **Supplementary Material Exercise: Can you formulate the definition of basis for the topology in terms of closed sets?**

4.2 Same Space, Different Topologies

4.2.1 Subspace Topology

4.2.2 Homeomorphism

4.2.3 Order Topology

4.2.4 Product Topology

4.2.5 Box Topology

4.3 Interior, and Closure of a Set

Now that we have familiarized ourselves with the basics of topology, we will move on to some new terminology.

4.3.1 Interior, and Closure of a Set

We know that sets can be open, closed, both, or none. Open sets are our main target here for studying continuous spaces. So with a particular emphasis on open sets, it is quite natural to ask **if it is always possible to find an open set inside an arbitrary set**. In $\mathbb{R}$, we can always find an open interval (c, d) inside a closed interval $[c, d]$. Notice that here we have looked 'inside' a closed set to search for an open set. In this instance, we have found the largest possible open set (c, d) in $[c, d]$. It comes quite naturally that the union of all open sets is the largest possible open set. This looking inside an arbitrary set to find the largest open set is internalized in the following definition -

Definition 4.3.1: Interior of a Set

In a topological space X, let $A \subseteq X$. Then, the interior of the set A is a subset that contains the union of all open sets in A. The interior of set A is denoted by $\text{Int}_X(A)$ or A^o.
If the set A itself is open, then $A^o = A$.

Again looking at the example of (c, d), $[c, d]$, we see that $[c, d]^o = (c, d) \subset [c, d]$. So in general, $\boxed{A^o \subseteq A}$. We will come back to discuss the properties of the interior of a set soon. For now, we will tackle another concept.
Now we will reverse the script! We see that the smallest possible closed set that contains the open interval (c, d) is the closed interval $[c, d]$. And it is easy to verify that the smallest closed set is the intersection of all the possible closed sets.

Definition 4.3.2: Closure of a Set

In a topological space X, let $A \subseteq X$. Then, a closure of a set A is a superset that contains the intersection of all the closed sets containing A. The closure of a set A is denoted by $\mathrm{Cl}_X(A)$ or $\overline{A}$.
If the set A itself is closed, then $\overline{A} = A$.

Just like the previous case, looking at the example of (c, d), $[c, d]$ gives $\overline{(c, d)} = [c, d]$ and $(c, d) \subset [c, d]$. So, $\boxed{A \subseteq \overline{A}}$. Combining both the interior and closure of a set gives us -

$$\boxed{A^o \subseteq A \subseteq \overline{A}}$$

Now, notice here that if the set A is **clopen**, then $A^o = \overline{A} = A$. Now, we want to formalize a concept for clopen sets. In fact, one can easily introduce this new concept with the help of complements. As $A^o \subseteq \overline{A}$ in general, we take $\overline{A} \setminus A^o$, not $A^o \setminus \overline{A}$. Now, we can define the following -

Definition 4.3.3: Boundary of a Set

In a topological space X, let $A \subseteq X$. Then, a boundary of a set A is the subtraction of the interior of set A from the closure of set A. The boundary of a set is also called the Frontier of a set, and denoted by $\mathrm{Bd}_X(A)$, $\mathrm{Fr}_X(A)$, or ∂A. Symbolically, $\partial A = \overline{A} \setminus A^o$.
For a clopen set A, $\partial A = \varnothing$.

With that out of the way, we can finally study the properties of these sets.

4.3.1.1 Properties of the Interior of a Set

Theorem 4.3.1: Interior of Sets and Open Sets

For any set in a topological space, the interior of a set is always open.

Proof. For a set $A \subseteq X$ in the topological space X, by definition, the interior of set A is the union of all open sets contained in A. We know from the topological definition of open sets that an arbitrary union of open sets is open. Hence, A^o is open. ■

Theorem 4.3.2: Interior of a Set and the Original Set

If A is any set in the topological space X, then $A^o \subseteq A$.

Proof. By definition, the interior of a set is the union of all of its open subsets. For a set $A \subseteq X$ in a topological space X, let us denote the open subsets of A by U. Then, by construction, $U \subseteq A$. Let $x \in \bigcup U$, then there exists a U for which $x \in U \subseteq A \implies x \in A$. So, $\bigcup U = A^o \subseteq A$. ■

Theorem 4.3.3: Largest Open Set and Interior Sets

The interior of the set is the largest open subset of the original set.

Proof. Let $U \subseteq A$ be any open set in the topological space X where $A \subseteq X$. Now, for any U, by definition of the interior of a set, $U \subseteq A^o$. So, even if U is the largest open set, it will be contained in A^o. From theorem (4.3.1), the interior is always open. So, A^o is the largest open set in A. ■

Theorem 4.3.4: Open Set in terms of Interior Sets

A set A in a topological space X is open if and only if $A^o = A$.

Proof. If A is open, then $A = A^o$:
Combining from the theorems (4.3.2) and (4.3.3), we get, $U \subseteq A^o \subseteq A$. However, A is open, so $U = A$. Substituting this gives, $A \subseteq A^o \subseteq A \implies A^o = A$.
If $A = A^o$, then A is open:
By theorem (4.3.1), the set A^o is open, and as $A^o = A$, the set A is open too. ■

Theorem 4.3.5: Idempotence of the Interior Sets

The interior of a set is idempotent. That is $(A^o)^o = A^o$.

Proof. Let $A^o = M$. Now by theorems (4.3.1) and (4.3.4), $M^o = M \implies (A^o)^o = A^o$. ■

Theorem 4.3.6: Intersection of Interior of Sets

Let U, V be two sets in the topological space X. Then, $(U \cap V)^o = U^o \cap V^o$.

Proof. Let,

$$\begin{aligned} & x \in (U \cap V)^o \\ \implies & x \in (U \cap V)^o \subseteq (U \cap V) \\ \implies & x \in (U \cap V)^o \subseteq U \;\&\; x \in (U \cap V)^o \subseteq V \end{aligned}$$

Now, $(U \cap V)^o$ is an open set as it is an interior set, and every open subset of U, V is contained in their interiors U^o, V^o respectively. So,

$$\begin{aligned} \implies & x \in (U \cap V)^o \subseteq U^o \;\&\; x \in (U \cap V)^o \subseteq V^o \\ \implies & x \in U^o \;\&\; x \in V^o \\ \implies & x \in (U^o \cap V^o) \end{aligned}$$

So, $\boxed{(U^o \cap V^o) \subseteq U^o \cap V^o}$.
Again, let,

$$\begin{aligned} & x \in U^o \cap V^o \\ & x \in U^o \cap V^o \subseteq U \cap V \quad [\text{As } A^o \subseteq A] \end{aligned}$$

Now, U^o, V^o are open sets, and their finite intersection is also an open set. As $(U \cap V)^o$ is the union of all open sets in $(U \cap V)$, we get,

$$\begin{aligned} & x \in U^o \cap V^o \subseteq (U \cap V)^o \\ & x \in (U \cap V)^o \end{aligned}$$

So, $\boxed{U^o \cap V^o \subseteq (U \cap V)^o}$.
Combining both boxed equations, we get $(U \cap V)^o = U^o \cap V^o$. ■

Some interesting ideas:

1. By using the method of induction, one can also prove that this equality holds for finite intersections.
2. What about infinite intersections? Does the equality hold in that case? This is handled as one of the exercises.

Theorem 4.3.7: Union of Interior of Sets

Let U, V be two sets in the topological space X. Then, $U^o \cup V^o \subseteq (U \cup V)^o$.

Proof. Let,

$$\begin{aligned}
&x \in U^o \cup V^o \\
\implies &x \in U^o \text{ or } x \in V^o \\
\implies &x \in U^o \subseteq U \text{ or } x \in V^o \subseteq V \text{ [As } A^o \subseteq A] \\
\implies &x \in U \cup V
\end{aligned}$$

Now U^o, V^o are open sets, and their arbitrary union is also an open set. So, x is a member of an open set. Combining this with $x \in U \cup V$, we can conclude that $x \in (U \cup V)^o$ as $(U \cup V)^o$ contains all other open sets in $U \cup V$.
So, $U^o \cup V^o \subseteq (U \cup V)^o$. ■

Concrete example where the equality does not hold:
Let $A = (a, b]$ and $B = [b, c)$ where $a, b, c \in \mathbb{R}$ and $a < b < c$. Then $A \cup B = (a, c)$.
Now, $A^o = (a, b)$, $B^o = (b, c)$, and $(A \cup B)^o = (a, c)$. But $A^o \cup B^o = (a, c) - \{b\} \neq (A \cup B)^o$.

4.3.1.2 Properties of the Closure of a Set

Theorem 4.3.8: Closure of Sets and Closed Sets

For any set in a topological space, the closure of a set is always closed.

Proof. For a set $A \subseteq X$ in the topological space X, by definition, the closure of set A is the intersection of all closed sets contained in A. We know from the topological definition of closed sets that an arbitrary intersection of closed sets is closed. Hence, $\overline{A}$ is closed. ■

Theorem 4.3.9: Closure of a Set and the Original Set

If A is any set in the topological space X, then $A \subseteq \overline{A}$.

Proof. By definition, the closure of a set is the intersection of all of its closed supersets. For a set $A \subseteq X$ in a topological space X, let us denote the closed supersets of A by U. Then, by construction, $A \subseteq U$. Let $x \in A$, then there exists a U for which $x \in A \subseteq U \implies x \in U$. So, $\bigcap U = \overline{A} \supseteq A$. ■

Theorem 4.3.10: Smallest Closed Set and Closure of a Set

The closure of the set is the smallest closed superset of the original set.

Proof. Let $U \supseteq A$ be any closed set in the topological space X where $A \subseteq X$. Now, for any U, by definition of the closure of a set, $U \supseteq \overline{A}$. So, even if U is the smallest closed set, it will be containing $\overline{A}$. From theorem (4.3.8), the closure is always closed. So, $\overline{A}$ is the smallest closed set in A. ■

Theorem 4.3.11: Closed Sets in terms of Closure of a Set

A set A in a topological space X is closed if and only if $A = \overline{A}$.

Proof. If A is closed, then $A = \overline{A}$:
Combining from the theorems (4.3.9) and (4.3.10), we get, $A \subseteq \overline{A} \subseteq U$. However, A is closed, so $U = A$. Substituting this gives, $A \subseteq \overline{A} \subseteq A \implies \overline{A} = A$.
If $A = \overline{A}$, then A is closed:
By theorem (4.3.8), the set $\overline{A}$ is closed, and as $\overline{A} = A$, the set A is closed too. ■

Theorem 4.3.12: Idempotence of the Closure

The closure of a set is idempotent. That is $\overline{(\overline{A})} = \overline{A}$.

Proof. Let $\overline{A} = M$. Now by theorems (4.3.8) and (4.3.11), $\overline{M} = M \implies \overline{(\overline{A})} = \overline{A}$. ■

Theorem 4.3.13: Union of Closures of Sets

Let U, V be two sets in the topological space X. Then, $\overline{(U \cup V)} = \overline{U} \cup \overline{V}$.

Proof. $\overline{U}, \overline{V}$ are closed set and their finite union, $\overline{U} \cup \overline{V}$, is also closed. Utilizing theorem (4.3.9), we can say that, $U \cup V \subseteq \overline{U} \cup \overline{V}$. Then, by definition of closure of $U \cup V$, $\boxed{\overline{U \cup V} \subseteq \overline{U} \cup \overline{V}}$.
Now, $U \subseteq U \cup V \subseteq \overline{U \cup V}$. Recall that $\overline{U}$ is the smallest closed set containing U, so $\overline{U} \subseteq \overline{U \cup V}$. Similarly, $\overline{V} \subseteq \overline{U \cup V}$. Then combining both, we get, $\boxed{\overline{U} \cup \overline{V} \subseteq \overline{U \cup V}}$.
Again, combining the boxed equations we conclude that, $\overline{U} \cup \overline{V} = \overline{U \cup V}$. ■

Some interesting ideas:

1. By using the method of induction, one can also prove that this equality holds for finite unions.

2. What about infinite unions? Does the equality hold in that case? This is handled as one of the exercises.

Theorem 4.3.14: Intersection of Closures of Sets

Let U, V be two sets in the topological space X. Then, $\overline{(U \cap V)} \subseteq \overline{U} \cap \overline{V}$.

Proof. We know that $U \cap V \subseteq U \subseteq \overline{U}$. Now, in $U \cap V$, by theorem (4.3.10), the smallest closed superset is $\overline{U \cap V}$. So, $\overline{U \cap V} \subseteq \overline{U}$. Similarly, $\overline{U \cap V} \subseteq \overline{V}$. Combining both, we get $\overline{(U \cap V)} \subseteq \overline{U} \cap \overline{V}$. ■

Concrete example where the equality does not hold:

Let $A = (a, b)$ and $B = (b, c)$ where $a, b, c \in \mathbb{R}$ and $a < b < c$. Then $A \cap B = \varnothing \implies \overline{A \cap B} = \varnothing$.
However, $\overline{A} = [a, b], \overline{B} = [b, c]$. So, $\overline{A} \cap \overline{B} = \{b\} \neq \overline{A \cap B}$.

Theorem 4.3.15: Closure of Sets and Open Sets

Let $A \subseteq X$ in a topological space X. Then $x \in \overline{A}$ if and only if for every open set U containing x (i.e. $x \in U$) we have $U \cap A \neq \varnothing$.

Proof. In the case of this proof, proving the contrapositive is much easier than proving the statement directly. **Just as a refresher if $P \implies Q$ is true, then its contrapositive statement is $\neg Q \implies \neg P$ is also true.**
As the original statement is a bi-conditional statement (if and only if statement), both $P \implies Q$ and $Q \implies P$ are true. Taking each of their contrapositive give us $\neg Q \implies \neg P$ and $\neg P \implies \neg Q$. Combining both gives us $\neg P \iff \neg Q$. **So, the contrapositive statement of a bi-conditional statement $P \iff Q$ is $\neg P \iff \neg Q$.**
So, the contrapositive statement that we have to prove is $x \notin \overline{A} \iff U \cap A = \varnothing$ where U is an open set containing x.
$x \notin \overline{A} \implies x \notin A$ as $A \subseteq \overline{A}$. By theorem (4.3.8), we know closure is a closed set. So, $x \in \overline{A}^c = U$, which by definition of a closed set is open. It is quite apparent from here that $U \cap A = \overline{A}^c \cap A = \varnothing$.
So, $\boxed{x \notin \overline{A} \implies U \cap A = \varnothing}$.
Now let us assume that we have an open set U containing x such that $U \cap A = \varnothing \implies U^c \cap A \neq \varnothing$. Now, the set U^c is a closed set by definition of a closed set. By theorem (4.3.10), $\overline{A} \subseteq U^c$ and $x \notin U^c$. This implies that $x \notin \overline{A}$.
So, $U \cap A = \varnothing \implies x \notin \overline{A}$.
Combining the two boxed equations gives $x \notin \overline{A} \iff U \cap A = \varnothing$. ■

4.3.1.3 Properties of the Boundary of a Set

An alternate definition of the boundary of a set can be formulated as follows -

Definition 4.3.4: Alternate Definition of Boundary of a Set

For the topological space X, the boundary of a set A, denoted by ∂A, is defined by $\partial A = \overline{A} \cap \overline{X \setminus A}$.

Proof. As per our first definition of the boundary of a set, $\partial A = \overline{A} \setminus A^o$. This means that,

$$x \in \overline{A} \text{ and } x \notin A^o$$
$$\implies x \in \overline{A} \text{ and } x \in \left(A^o\right)^c$$

Now, by definition of the interior of a set, $A^o = \cup U$ where U denotes open sets in A. Then,

$$\left(A^o\right)^c = \left(\cup U\right)^c = \cap U^c = \cap (X \setminus U)$$

Again, by definition of a closed set, $(X \setminus U)$ is closed.
We also know that $A^o \subseteq A$. So $U \subseteq A$. This gives us,

$$(A^o)^c = \bigcap (X \setminus U) \supseteq \bigcap (X \setminus A)$$
$$\implies \bigcap (X \setminus U) \supseteq (X \setminus A)$$

Now, the intersection of all closed supersets containing the set is called the closure of that set. So,

$$\overline{(X \setminus A)} = \bigcap (X \setminus U) = (A^o)^c$$

Then,

$$x \in \overline{A} \text{ and } x \in \left(A^o\right)^c$$
$$\implies x \in \overline{A} \text{ and } x \in \overline{(X \setminus A)}$$
$$\implies x \in \overline{A} \cap \overline{(X \setminus A)}$$

So, we have shown that $\boxed{\overline{A} \setminus A^o \subseteq \overline{A} \cap \overline{(X \setminus A)}}$.

Reversing the logic of the same steps, we can easily show that $\boxed{\overline{A} \cap \overline{(X \setminus A)} \subseteq \overline{A} \setminus A^o}$.
This means that $\partial A = \overline{A} \setminus A^o = \overline{A} \cap \overline{(X \setminus A)}$. ■

There are **three immediate consequences** of this definition. They are:

1. The boundary of a set is always closed, as it is the intersection of two closures. (i.e. intersection of two closed sets.)
2. The boundary of a set and the boundary of the complement of the same set are the same. (i.e. $\partial A = \partial(A^c)$.)
3. The closure of a boundary of a set is the boundary of a set itself. (i.e. $\overline{\partial A} = \partial A$.)

Exercise 4D

1. **Prove that a set A is clopen if and only if $\partial A = \varnothing$.**
2. **Is the equality in theorem (4.3.6) valid for infinite intersections? If yes, then provide proof, and if no, then provide a counterexample.**
   [Hint: Follow the proof of the finite case closely. Can you find any argument that breaks down if the intersection is infinite?]
3. **Is the equality in theorem (4.3.13) valid for infinite unions? If yes, then provide proof, and if no, then provide a counterexample.**
   [Hint: Same as the previous question.]
4. **Prove the following theorem -**

Theorem 4.3.16: Monotonic Property of Closure of Sets

Let $A \subset B$ be two subsets in the topological space X. Then $\overline{A} \subseteq \overline{B}$.

Find an example where $A \subset B$ but $\overline{A} = \overline{B}$. When $A = B$, then it is trivial that $\overline{A} = \overline{B}$. That is why the theorem says $A \subset B$, not $A \subseteq B$.

5. **Investigate the following query -**

Query 4.3.1: Idempotence of Boundary of a Set

Check the validity of the following statement -

$$\partial A \supseteq \partial(\partial A)$$

Is there any if and only if condition imposed on A or ∂A such that $\partial(\partial A) = \partial A$?

[Hint: From the 3rd consequence of the alternate definition of the boundary of a set, recall that $\overline{\partial A} = \partial A$. From here on, I believe you should be able to do the rest.]

6. Prove the following theorem -

Theorem 4.3.17: Interesting Connection Between the Interior and Closure

a. $\mathrm{Int}_X(A) = X \setminus \mathrm{Cl}_X(X \setminus A)$.

b. $\mathrm{Cl}_X(A) = X \setminus \mathrm{Int}_X(X \setminus A)$.

[Hint: Look at the proof of the definition (4.3.4) closely.]

4.4 Neighborhoods, Limit Points, Dense Sets, and Separable Space

Now we will develop another way of defining the closure of a set, and this way utilizes the concept of limit points.

4.4.1 Neighborhoods, Limit Points, Derived Sets

In metric spaces, we knew points were close by utilizing the notion of open balls. However, a topological space is more general than a metric space. So, a notion of 'closeness' needs to be defined in the topological space. Now, to capture this we have to create a more flexible framework than open sets that encompasses the outer area of open sets. To do just that, we introduce the following definition -

Definition 4.4.1: Neighborhood of a Set

Let X be a topological space and $S \subseteq X$, then the subset $N \subseteq X$ is called a neighborhood of S if there exists an open set U such that $S \subseteq U \subseteq N \subseteq X$.
In other words, N contains an open set U that includes S. A neighborhood does not need to be open itself, but must contain at least one open set containing the point.
If the set in question is a singleton, then the neighborhood is determined with respect to a point, and such a neighborhood is defined as the neighborhood of a point.

There is another type of neighborhood set that some authors prefer to introduce, and just to be exhaustive about the materials, I have included it below as well.

Definition 4.4.2: Deleted Neighborhood of a Set

Let N be a neighborhood of a set $S \subseteq X$ in the topological space X. Then a deleted neighborhood is a set $D = N \setminus S$. If the set in question is a singleton, then the deleted neighborhood is called the deleted neighborhood of a point.

Now, one might ask the question, how do these definitions imply that two sets are close? If two sets are close to each other, then one of their neighborhood must overlap with the other set and thus, have a non-empty intersection! Another important distinction we have to make is that we want sets (or points) that are strictly 'close' to the set, not identical to it. Because otherwise, we will get each individual point as its own neighbor, which is trivial. We are mainly interested in finding the non-trivial sets (or points) that our reference set can reach out to, and hence, we get the following definition -

Definition 4.4.3: Limit Points of a Set

A point $x \in X$ is called the limit point of the set $A \subseteq X$ in a topological space X if the neighborhood of x, denoted by N_x, intersects the set A in other points excluding x itself. Symbolically, $N_x \cap (A \setminus \{x\}) \neq \varnothing$.
Now, $N_x \cap (A \setminus \{x\}) = (N \cap A) \setminus \{x\} = (N \setminus \{x\}) \cap A$. But we know that the deleted neighborhood of x, $D_x = N_x \setminus \{x\}$. So, limit points are the non-empty intersection of the deleted neighborhood and the reference set. Symbolically, $D_x \cap A \neq \varnothing$.

Other names of limit points are **accumulation points, cluster points**.

Definition 4.4.4: Derived Set

The set of all limit points of $A \subseteq X$ in a topological space X is called the derived set. It is denoted by A' or $D(A)$.

Illustrative Example 4.4.1

1. The derived set of (a, b) where $a, b \in \mathbb{R}$ and $a < b$ is $[a, b]$.
2. The derived set of $[a, b]$ where $a, b \in \mathbb{R}$ and $a < b$ is $[a, b]$.
3. The derived set of $[a, b)$ where $a, b \in \mathbb{R}$ and $a < b$ is $[a, b]$.
4. The derived set of $(a, b]$ where $a, b \in \mathbb{R}$ and $a < b$ is $[a, b]$.
5. The derived set of $\{a\}$ where $a \in X$ and X is a topological space is $\varnothing$.
6. The derived set of $\mathbb{N}, \mathbb{Z}$ is $\varnothing$.
7. The derived set of $\mathbb{Q}, \mathbb{I}, \mathbb{R}$ is $\mathbb{R}$.
8. The derived set of $\mathbb{C}$ is $\mathbb{C}$.

With that out of the way, we define a few more closely related terms too.

Definition 4.4.5: Isolated Point of a Set

A point x is called an isolated point of the set $A \subseteq X$ in a topological space X if the neighborhood of x, N_x, does not intersect the set A in other points excluding x itself.
Symbolically, $N_x \cap (A \setminus \{x\}) = \varnothing$ or $N_x \cap A = \{x\}$.
In terms of deleted neighborhoods, $D_x \cap A = \varnothing$.
In other words, any point that is not a limit point for the set A, is called an isolated point for the set A.

Definition 4.4.6: Discrete Set

A set $A \subseteq X$ is called discrete in a topological space X if all of its points are isolated points.

Illustrative Example 4.4.2

The set $A = \{\frac{1}{n} : n \in \mathbb{N}\}$ in $\mathbb{R}$ equipped with the standard topology is a common example of a discrete set. If written explicitly, then $A = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$.

Definition 4.4.7: Adherent Point of a Set

A point x is called an adherent point of the set $A \subseteq X$ in a topological space X if the neighborhood of x, N_x, intersects the set A at some point(s). Symbolically, $N_x \cap A \neq \varnothing$.

Adherent points are also known as **contact point**. Some textbooks introduce a special type of adherent point, and just to be exhaustive, I have included it below.

Definition 4.4.8: Condensation Point of a Set

A point x is called a condensation point of the set $A \subseteq X$ in a topological space X if the neighborhood of x, N_x, intersects the set A at uncountable points. Symbolically, $|N_x \cap A| > \aleph_0$ where $|\cdot|, \aleph_0$ denote cardinality and cardinality of the natural numbers(countably infinite), respectively.

Some interesting facts:

1. Limit points and isolated points are mutually exclusive and exhaustive.
2. Adherent point is the union of limit points and isolated points. So, all limit points or isolated points are adherent points, but the converse is not true.
3. Condensation point is a special type of adherent point where the cardinality of the intersection of the neighborhood of a point and the reference set is uncountably infinite.

Armed with all these definitions, we can now shift our focus to prove the following theorems regarding derived sets, closed sets, and closure of sets.

4.4.1.1 Properties of Derived Sets

Theorem 4.4.1: Derived Set of an Empty Set

The derived set of an empty set is the empty set itself. That is $\varnothing' = \varnothing$.

Proof. As the reference set $S = \varnothing$ is empty, for any $x \in X$ if we find a neighborhood N_x, then we will always get

$$N_x \cap (S \setminus x) = N_x \cap (\varnothing \setminus x) = N_x \cap \varnothing = \varnothing$$

So, there exists no $x \in X$ that satisfies the definition of the limit point, and the limit points of $\varnothing$ are precisely $\varnothing$. Hence, $\varnothing' = \varnothing$. ∎

Theorem 4.4.2: Monotonic Property of Derived Set

Let $A, B \subseteq X$ in a topological space X. Then, $A \subset B \implies A' \subseteq B'$.

Proof. Let $x \in A'$. Then by definition of limit points, we can have a point $x \in X$ (Here X is the topological space) where we can get a neighborhood N_x such that $N_x \cap (A \setminus \{x\}) \neq \varnothing$.
As $A \subset B \implies (A \setminus \{x\}) \subseteq B \setminus \{x\}$. So,

$$\begin{aligned} &\varnothing \neq N_x \cap (A \setminus \{x\}) \subseteq N_x \cap (B \setminus \{x\}) \\ \implies &\varnothing \neq N_x \cap (B \setminus \{x\}) \end{aligned}$$

So, $x \in B'$. Therefore, $A' \subseteq B'$. ■

Theorem 4.4.3: Derived Set of Union of Sets

In a topological space X, let $A, B \subseteq X$. Then, $(A \cup B)' = A' \cup B'$.

Proof. Let $x \in (A \cup B)'$ and to exclude trivial cases take $A, B \neq \varnothing$. Then, by definition

$$
\begin{aligned}
& N_x \cap \big((A \cup B) \setminus \{x\}\big) \neq \varnothing \\
\implies & N_x \cap \big((A \setminus \{x\}) \cup (B \setminus \{x\})\big) \neq \varnothing \\
\implies & \big(N_x \cap (A \setminus \{x\})\big) \cup \big(N_x \cap (B \setminus \{x\})\big) \neq \varnothing \\
\implies & \text{Either } N_x \cap (A \setminus \{x\}) \text{ or } N_x \cap (B \setminus \{x\}) \text{ is non-empty} \\
\implies & x \in A' \text{ or } x \in B' \\
\implies & x \in A' \cup B'
\end{aligned}
$$

So, $\boxed{(A \cup B)' \subseteq (A' \cup B')}$.
Reversing the same steps we get, $\boxed{(A' \cup B') \subseteq (A \cup B)'}$.
Combining both boxed equations gives $(A \cup B)' = A' \cup B'$. ■

<u>Some interesting ideas:</u>

1. By using the method of induction, one can also prove that this equality holds for finite unions.
2. What about infinite unions? Does the equality hold in that case? This is handled as one of the exercises.

Theorem 4.4.4: Closure of a Set & Limit Points

In a topological space X, for any set $A \subseteq X$, $\mathrm{Cl}_X(A) = A \cup D(A)$.
In other words, $\overline{A} = A \cup A'$.

Proof. If the set is the empty set, then it is trivial. So, let us assume $A \neq \varnothing$.
By theorem (4.3.9), we get $A \subseteq \overline{A}$.
Now we have to show that $A' \subseteq \overline{A}$. Let us assume for the sake of contradiction that there exists a $x \in A'$ and $x \notin \overline{A} \implies x \in (\overline{A})^c$. By theorem (4.3.8), we know $\overline{A}$ is closed. So, $(\overline{A})^c$ is open by definition of closed sets. So, we can easily set our neighborhood to be $(\overline{A})^c$. But $(\overline{A})^c \cap A = \varnothing$. This means that x is not a limit point and $x \notin A'$, which is clearly a contradiction. So, there exists no $x \in A'$ and $x \notin \overline{A}$. So, $x \in A' \implies x \in \overline{A}$.
Therefore, $\boxed{A \cup A' \subseteq \overline{A}}$.
Let $x \in \overline{A}$. If $x \in A$, then $\overline{A} \subseteq A \cup A'$.
Now $x \notin A \implies (A \setminus \{x\}) = A$. By theorem (4.3.15), as $x \in \overline{A}$, we know that we can find an open set $x \in U$ such that $U \cap A \neq \varnothing \implies U \cap (A \setminus \{x\}) \neq \varnothing \implies x \in A'$.
Therefore, $\boxed{\overline{A} \subseteq A \cup A'}$. Combining both boxed equations, we get $\overline{A} = A \cap A'$. ■

Theorem 4.4.5: Closed Sets & Limit Points

A set is closed if and only if it contains all of its limit points.
Worded differently, a set is closed if and only if it contains its derived set.

Proof. We know from theorem (4.3.11) that a set $A \subseteq X$ is closed in a topological space X if and only if $\overline{A} = A$. We also know from theorem (4.4.4) that $\overline{A} = A \cup A'$.
If A is closed then combining both gives us, $A = A \cup A' \implies A' \subseteq A$.
This shows us that, $\boxed{\text{Set } A \text{ is closed} \implies A' \subseteq A}$.
Reversing the same steps, we get, $\boxed{A' \subseteq A \implies \text{Set } A \text{ is closed}}$.
Combining both boxed equations gives that a set A is closed if and only if $A' \subseteq A$. ■

4.4.1.2 Closedness of the Derived Set

From the properties of the derived set, we see that it has a close relation with the closure of a set, closed sets. So, it is quite natural to ask if the derived set itself is closed or not. We will investigate exactly that.

Theorem 4.4.6: The Derived Set of a Closed Set

The derived set of a closed set is closed.

Proof. By theorem (4.4.5), we know that a set is closed if and only if it contains its derived set. So for any set $A \subseteq X$ in a topological space X, A is closed if and only if $A' \subseteq A$. By theorem (4.4.2), we know that derived sets are monotonic. So, $A' \subseteq A \implies (A')' \subseteq A'$. So, A' contains its own derived set $(A')'$. Then we can conclude that the derived set of a closed set is closed. ■

Now, what about open sets? Is the derived set of an open set closed? As we are looking for a general thread for all topologies, it is best to use such a topology as an example that can be found inside all topologies, that is, the trivial topology.

Illustrative Example 4.4.3

Topology where the derived set is not closed.

Proof. Let us consider the trivial topology where $X = \{a, b\}$. Here

1. $\varnothing' = \varnothing$, which is closed.
2. $\{a\}' = \{b\}$, which is not closed as we cannot get $\{b\}$ as an intersection or union of X and $\varnothing$.
3. $\{b\}' = \{a\}$, which is not closed as we cannot get $\{a\}$ as an intersection or union of X and $\varnothing$.
4. $X' = X$, which is closed.

In this example, we can easily find two derived sets $\{a\}', \{b\}'$ which are not closed. Interestingly, they are not open either. ■

This example shows that in order for the derived set to be closed in a topology, the singleton sets need to have some sort of restriction imposed on them. So, let us investigate what restriction may be required to be imposed on the singleton sets if the derived set is assumed to always be a closed set.

Query 4.4.1: Derived Sets being Closed

Is there any condition required to be imposed on singleton sets so that all derived sets are closed?

Investigation. Being inspired by the example (4.4.3), we proceed by assuming all derived sets are closed in the topological space X. We focus on the derived set of singletons specifically. So, for any $x \in X$, let us assume that $\{x\}'$ is closed. Before we progress any further, it would be better to gather as much information as we can about the derived set of singleton sets we can.

1. By definition of limit points for a singleton set $\{x\}$ has to satisfy for any $p \in X$, $N_p \cap (\{x\} \setminus \{p\}) \neq \varnothing$. When $p = x$, then $N_x \cap (\{x\} \setminus \{x\} = N_x \cap \varnothing = \varnothing$. So, in any topology, $x \notin \{x\}'$. But when $p \neq x$, then $N_p \cap (\{x\} \setminus \{p\}) = N_p \cap \{x\}$. But we already know that $x \notin \{x\}'$, so $N_p \cap \{x\} \neq \{x\}$. Therefore, the only remaining option is $\boxed{\{x\}' = \varnothing}$.
2. We know from theorem (4.4.4) that, $\overline{\{x\}} = \{x\} \cup \{x\}' \implies \overline{\{x\}} = \{x\}$ as $\{x\}' = \varnothing$. By theorem (4.3.11) we can then say that $\boxed{\text{all singleton sets have to be closed}}$.

So, we have found that in any topology, for all derived sets to be closed, we need the singleton sets to be closed as well.
Now, let us check if the converse is true or not. Reversing the steps will only take us as far as showing that the derived set of singleton sets is closed, and if we use theorem (4.4.3), we can assert that the derived set of all the finite sets is closed. So we need to take a different approach.
The next direct way is to go by the original definition of a closed set introduced in the text. So, we aim to show that $(A')^c$ is open so that A' is closed for any $A \subseteq X$.
Let $x \in (A')^c$, then for a $x \in X$, we have a neighborhood N_x (Remember that by definition of neighborhood $U \subseteq N_x$) such that $N_x \cap (A \setminus \{x\}) = \varnothing \implies U \cap (A \setminus \{x\}) = \varnothing$. We get two specific cases.

1. If $x \notin A$, then $U \cap A = \varnothing$. Now, by the contrapositive statement of the theorem (4.3.15), we know that, $x \in U$, $U \cap A = \varnothing \iff x \notin \overline{A}$. By theorem (4.3.8), we know that $\overline{A}$ is closed. So, $x \in (\overline{A})^c$ is open. Thus, $x \in (A')^c \cap A^c$ is open.
2. If $x \in A$, then $U \cap A = \{x\}$. Now, $\{x\}$ is closed, and $\{x\}^c$ is open. So, $U \cap \{x\}^c = U \setminus \{x\}$ is open too, as it is the intersection of two open sets. So, $x \in (A')^c \cap A$ is open.

The union of two open sets is an open set. So, from both cases,

$$\left((A')^c \cap A^c\right) \cup \left((A')^c \cap A\right) = (A')^c \cap (A^c \cup A) = (A')^c \cap X = (A')^c$$

So, $(A')^c$ is open. Then, A' is closed.
With that, we have also proved the converse statement. We can finally conclude that, in a topology, the derived sets are always closed if and only if the singleton sets are closed. ■

Theorem 4.4.7: Derived Sets being Closed

In a topology, the derived set will be closed if and only if all the singleton sets are closed in that topology.

Proof. See the investigation before. ■

Exercise 4E

1. **Does the equality in theorem (4.4.3) hold for infinite unions? If yes, then provide proof, and if no, provide a counterexample.**
   [Hint: Think about sets with pathological behavior. The answer is quite *discreet*, and yes, the pun is intended.]
2. **Rigorously verify each of the examples (4.4.1) and (4.4.2).** When dealing with $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{I}, \mathbb{R}$, and the discrete set, assume the topology is the standard topology.
3. **Continuity via Neighborhoods:** Prove the following theorem -

   Theorem 4.4.8: Continuity & Neighborhoods

   Let X, Y be topological spaces, then a function $f : X \to Y$ is continuous if and only if for every neighborhood of $f(x)$, $N_{f(x)} \in Y$ there exists some neighborhood of x, $N_x \subseteq f^{-1}(N_{f(x)}) \subseteq X$.
4. **Prove the following things:**
   a. Limit points and Isolated points are mutually exclusive and exhaustive.
   [Hint: The deleted neighborhood definition becomes handy in this case!]
   b. The set of all adherent points (or contact points) of a set is equivalent to its closure. Let the set of adherent points of the set A be denoted by A_{Ad}, then show that $A_{Ad} = \overline{A}$.
5. **Working in the Opposite Order:** From definition (4.1.3), we arrived at theorem (4.4.5). Now, dear reader, you have to go from theorem (4.4.5) to definition (4.1.3). That is, prove that,

$$\boxed{\text{A set is closed if it contains all of its limit points} \implies \text{The complement of a closed set is open}}$$

4.4.2 Separation Axioms and T_1 Space

4.4.2.1 Separation Axioms

4.4.2.2 T_1 Space

In the proof of the derived sets being closed, only having a topology was not enough. We had to introduce a further restriction on the topology by requiring singleton sets to be closed. Now, one can find a surprising connection between singleton sets being always closed with the neighborhood of such a topology. This is illustrated here.

Theorem 4.4.9: Singleton Sets being Closed and Neighborhoods

In a topological space X, if $x, y \in X$ are distinct points, then all the singleton sets are always closed if and only if there exist neighborhoods N_x, N_y such that $y \notin N_x, x \notin N_y$.

Proof. Let us assume that all singleton sets are closed. Then $\{x\}, \{y\}$ are closed. By definition of closed sets, we get $\{x\}^c, \{y\}^c$ to be open sets. It is apparent that $y \in \{x\}^c, x \notin \{x\}^c$ and $x \in \{y\}^c, y \notin \{y\}^c$. We can set $N_y = \{x\}^c, N_x = \{y\}^c$.
So, $\boxed{\text{singleton sets being always closed} \implies y \notin N_x, x \notin N_y}$.
For the converse statement, let us assume there exists $y \notin N_x, x \notin N_y$. If $y \notin N_x \implies y \notin U_x$. We know arbitrary unions of open sets are open, so $\bigcup_{x \in X, x \neq y} U_x = X \setminus \{y\} = \{y\}^c$ is open. Then by definition of closed sets, $\{y\}$ is closed. Similarly, from $x \notin N_y$ it can be shown that $\{x\}$ is closed. As x, y are arbitrary distinct points, all closed sets are closed.
So, $\boxed{y \notin N_x, x \notin N_y \implies \text{singleton sets being always closed}}$.
Therefore, all the singleton sets are always closed if and only if there exist neighborhoods N_x, N_y such that $y \notin N_x, x \notin N_y$. ■

So, we have found that the two seemingly different restrictions are actually the same restriction on the topology. A topology with such a restriction is special. This special type of topology is defined below.

Definition 4.4.9: T_1 Space

A topological space X is called a T_1 space if any one of the following conditions is met -

1. All singleton sets are always closed.
2. Distinct points $x, y \in X$ guarantees that $y \notin N_x, x \notin N_y$.
3. The set of the limit points, i.e. the derived set, is always closed.

Now, it is natural to look for examples of T_1 space.

Illustrative Example 4.4.4

All metric spaces are examples of T_1 space.
[N.B. But not all T_1 space has a metric. An example of such a space is the cofinite or the finite complement topology. This is a T_1 space but not a metric space.]

Exercise 4F

1.

4.4.3 Dense Sets, Perfect Sets & Separable Spaces

From the previous section, we learned that in a set, the limit points are the points which the set itself 'reaches out to'. We also found the utility of limit points in an alternate definition of the closure (theorem (4.4.4)). Now, it is natural to look for sets which reaches out to points in itself! Like these sets do not venture outside of themselves.

Definition 4.4.10: Dense Set

A subset $A \subseteq X$ in a topological space X, is said to be dense in X if $\mathrm{Cl}_X(A) = X$ or $\overline{A} = X$.

Now this is a callback to what we learned in the very first chapter. Let us verify if our new notion of density does follow our old.

1. $\overline{\mathbb{N}} = \mathbb{N} \cup \mathbb{N}' = \mathbb{N} \cup \varnothing = \mathbb{N} \neq \mathbb{R}$. Similarly, $\overline{\mathbb{Z}} = \mathbb{Z} \neq \mathbb{R}$. So, $\mathbb{N}, \mathbb{Z}$ are not dense in $\mathbb{R}$.
2. $\overline{\mathbb{Q}} = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$. Similarly, $\overline{\mathbb{I}} = \mathbb{R}$. So, $\mathbb{Q}, \mathbb{I}, \mathbb{R}$ are dense in $\mathbb{R}$.

3. Another thing that is trivial, in every topological space X, $\overline{X} = X$, so every whole space is dense in itself.

We have established an even more powerful tool to find dense sets. This will be extremely important in further studies of analysis. Now, we want to find some correlation between the derived set and the reference set itself. Most notably when $A' = A$.

Definition 4.4.11: Perfect Sets

A subset $A \subseteq X$ in a topological space X, is called perfect if the derived set is equal to the set itself. Symbolically, $A' = A$.

Are perfect sets closed or open? To answer this question, we invoke theorem (4.4.4) and the definition of the perfect set.

$$\begin{aligned} &\overline{A} = A \cup A' \\ \implies &\overline{A} = A \cup A \\ \implies &\overline{A} = A \end{aligned}$$

So, **a perfect set is always closed**. A classic example of a perfect set in $\mathbb{R}$ is closed intervals.
Now, we again go back to our roots in chapter 1, we know that $\mathbb{Q}$ is countably infinite and $\mathbb{I}$ is uncountably infinite. But both of them are dense in $\mathbb{R}$. We want to work with topological spaces that have dense subsets AND are manageable in size, so let us define a new topological space where such interests are reflected.

Definition 4.4.12: Separable Space

A topological space $(X, \mathcal{T})$ is called a separable space if there exists a dense and countable subset $A \subseteq X$.

Illustrative Example 4.4.5

1. If the topological space itself is finite or infinitely countable, then it will be a separable topological space.
2. The real numbers $\mathbb{R}$ with the usual topology is a separable space as it contains a dense and countable subset, the set of rational numbers $\mathbb{Q}$.

Now, we can see from $\mathbb{Q}' = \mathbb{R}$, that there exists a countable dense subset for an uncountable whole space. Now, let us try to push things further in terms of manageable size. Can there exist a finite dense subset for an arbitrarily sized whole space? Common sense tells us a finite dense set can exist for a finite whole space, but can we always find a finite dense subset when the whole space is infinite? This gives us the following investigation.

Query 4.4.2: Existence of Finite Dense Subset in an Infinite Whole Space

Can there exist a finite dense subset in an infinite whole space?

Investigation. When investigating topological spaces, it is always a good habit to check using pathological examples, such as the indiscrete topology. So, let us consider $\mathcal{T} = \{\varnothing, X\}$. Here, X is an infinite set, and recall that $\varnothing, X$ are clopen sets. Let $A \subset X$ be a finite proper subset. Then, the only closed superset containing A is X, so $\overline{A} = X$. Thus, we have found a finite dense subset for an infinite whole space. So, **it is possible**. ■

As we are working with the closure of sets, the derived set becomes a focal point. This, in turn, makes T_1 space (Just need to look up definition (4.4.9) only, no need to see anything else!) an important distinguishing factor. So, let us investigate if we can find a finite dense subset in an infinite whole space.

Theorem 4.4.10: Finite Dense Subsets in Infinite Whole Space and T_1 Space

In a T_1 space, if the whole space is infinite, then NO finite dense subset can exist.

Proof. Let us assume, for the sake of contradiction, that there exists a finite dense subset $D \subset X$, where X is an infinite T_1 space. By the definition of T_1 spaces, all singleton sets are closed, and we know that finite unions of closed sets are closed. All finite

sets can be expressed as finite unions of singleton sets. So, all finite sets are closed in T_1 space. Therefore, in T_1 space, $\boxed{\overline{D} = D}$. By assumption, we know D is dense in X. So, $\boxed{\overline{D} = X}$. Combining both boxed equations gives us $D = X$. But D is also finite and X is infinite by construction. So, $D = X$ definitely is not possible. Hence, we have reached a contradiction.
This proves that there exists no finite dense subset if the whole space is infinite. ■

As most of the spaces we are interested in, such as $\mathbb{Q}, \mathbb{R}, \mathbb{C}$, are T_1 spaces, we can only find countable dense subsets, and that is what makes separable space so important.
Another important thing to note about separable spaces is that, as the dense subset is countable, we can find a way to relate the dense set to the natural numbers, and that means we can construct the dense set explicitly. This is extremely helpful when proving the existence of certain mathematical objects by explicitly constructing an example. The branch of mathematics that deals with this sort of proof of existence by construction of examples is called 'constructive mathematics'.
We keep our discussion on separable spaces restricted to here for now; the next section will provide more ammunition to study such spaces further.

Exercise 4G

1.

4.5 Connectedness and Compactness

In this section, we will discuss two specific types of sets and their impressive impact on continuous functions.

4.5.1 Basic Intuition and Definitions

Here, we are introduced to two major types of topological spaces, connected and compact spaces.

4.5.1.1 Connectedness or Connected Space

In theorem (4.1.13), we see that no non-trivial clopen sets exist in $\mathbb{R}$. We want to find a certain restriction on the topological space $(X, \mathcal{T})$ such that no non-trivial clopen sets exist in that space.
So, let us assume we are working in a topological space where non-trivial clopen sets exist. Let the non-trivial clopen set be $A \subset X$ and $A \neq \emptyset$. We know that then, the boundary of A will be empty, i.e., $\partial A = \emptyset$. Again, by the second consequence of the alternate definition of the boundary of the set, we know that $\partial A = \partial(A)^c$. So, the complement of A is also a clopen set as $\partial(A)^c = \partial A = \emptyset$. Now, we observe that,

$$A \cap A^c = \emptyset, A \cup A^c = X$$

As open sets are the main method of analysis, we consider A, A^c as open sets. Then, we term it as follows, a topological space will have a non-trivial clopen set if the whole space can be expressed as the union of two disjoint open sets. Now, we want to verify whether the other way around is true or not.
Let A, B be two non-empty open sets such that,

$$A \cap B = \emptyset, A \cup B = X$$

Now, it is obvious that $B = A^c$. Then, by construction, A, A^c are open, and that makes both A, A^c to be clopen. This implies that there exists at least a non-trivial clopen set A. So, the reverse also holds.
Now, let us look at the condition closely.

$$A \cap B = \emptyset \text{ and } A \cup B = X$$

This condition basically means that the topological space can be split apart into two smaller pieces. So, we can disconnect the two parts forming the whole space. Obviously, in $\mathbb{R}$, we cannot find two disjoint open sets that give back $\mathbb{R}$ under union. So, this gives us a solid reason to define this condition as a concept entirely. As analysis starts at $\mathbb{R}$, which does not possess these disconnected parts, we call $\mathbb{R}$ a connected space.

Definition 4.5.1: Connectedness

A topological space X is defined as connected if there exists NO such open subsets $A, B \subset X$ such that

$$A \cap B = \varnothing \text{ and } A \cup B = X$$

If there exist such A, B, then the topological space is defined to be disconnected.

Definition 4.5.2: Alternate Definition of Connectedness

A topological space X is defined as connected if there exists NO non-trivial clopen sets.

Proof. Formalize the arguments before in this section. ■

Illustrative Example 4.5.1

1. $\mathbb{R}, \mathbb{C}$ are connected topological spaces.
2. $\mathbb{R} \setminus \{x\}$ where $x \in \mathbb{R}$ is an example of a disconnected space, because we can find two open sets $A = (-\infty, x), B = (x, \infty)$ such that $A \cap B = \varnothing, A \cup B = \mathbb{R} \setminus \{x\}$.

4.5.1.2 Compactness or Compact Space

It is quite natural to want to work with as little number of sets as possible in a topological space X. We have formulated the basis for topology (do not worry if you have skipped it!) and separable spaces following this demand. Now, we look at something similar of sorts taking inspiration from both of these concepts.
First things first, we want to preserve the whole space X. Open sets are the modus operandi in dealing with topological spaces. So, let us imagine a collection of open subsets of X, denote it by C, so that

$$C = \{\text{Collection of open subsets of } X\}$$

Now, we want this collection to give us the whole space. Also, X is a topological space, so arbitrary unions of open sets are valid in this space. So, we can encode X in terms of C quite easily by setting

$$X = \bigcup_{U \in C} U$$

This property is quite important as it preserves the whole space, and this is why it has its own name, the covering property. Now, we have considered C as a collection of *open* subsets, but as you have seen already, mathematicians love generalization, so we want to drop the requirement of the collection containing only open sets. This gives us the following concept.

Definition 4.5.3: Covers and Open Covers

Let X be a topological space and C be a collection of subsets of X such that

$$X = \bigcup_{U \in C} U$$

Then C is defined as the cover of X.
If all the members of the cover are open, then the cover is defined as an **open cover**.

But wait a minute! The cover C can be quite large, and that is no good. So, we need to find a smaller sub-collection that still 'covers' X.

Definition 4.5.4: Sub-cover

Let C be a cover of a topological space X and $\mathcal{S} \subseteq C$ such that $\mathcal{S}$ also covers X, that is

$$X = \bigcup_{U \in \mathcal{S}} U$$

Then $\mathcal{S}$ is called a sub-cover of X.
If all the members of the sub-cover are open, then it is called an **open sub-cover** of X.

In separable spaces, we took countable sets, so let us do exactly that. This idea was put forth by a Finnish mathematician named Ernst Leonard Lindelöf, and so the following concept is named after him.

Definition 4.5.5: Countable Sub-cover and Lindelöf Space

If every open cover of X has a countable sub-cover, then the topological space is defined as Lindelöf.

Now, let us make the restriction on the size of the sub-cover even stricter and require the subset to be finite. The space having such a property is of massive interest, and we call it a compact space.

Definition 4.5.6: Finite Sub-cover and Compact Space

If every open cover of X has a finite sub-cover, then the topological space is defined as compact.

Definition 4.5.7: Compact Subset

A subset A of a topological space X is itself compact if for every open cover C of X, there exists a finite open sub-cover $\mathcal{F} \subseteq C$ of X such that $A \subseteq \bigcup_{U \in \mathcal{F}} U$.

From the definitions, it is quite clear that **every compact space is Lindelöf, but every Lindelöf space is not compact**. Now, we can immediately come to the following conclusions based on the cardinality of the whole space X:

1. If X itself is countable, then it is Lindelöf.

2. If X itself is finite, then it is compact.

Now, how do you get examples of compact sets? This is a task that is not as easy as connected spaces, aside from the two mentioned above. So, this means that we probably should explore a bit more about compact spaces. However, we can prove two certain facts about closed sets in a compact space.

Theorem 4.5.1: Closed Sets and Compact Space

In a compact space X, a closed subset $A \subseteq X$ is compact.

Proof. Let A have an arbitrary open cover C in a compact space X. We need to prove that there exists a finite sub-cover $\mathcal{F}$ for the arbitrary open cover

$$C = \cup_{a \in \alpha} U_\alpha$$

By definition of closed sets, A is closed implies $X \setminus A$ is open.
Now, let us construct a new open cover denoted by $C_1 := C \cup (X \setminus A)$. This new open cover C_1, covers the whole space X. Now, by construction of the theorem, X is compact, so every open cover of it admits of a finite sub-cover, by the definition of a compact space. Let us denote this finite open sub-cover by

$$\mathcal{F}_1 = \cup_{i=1}^{n} U_i \cup (X \setminus A)$$

This means that,

$$
\begin{aligned}
& X \subseteq \cup_{i=1}^{n} U_i \cup (X \setminus A) \\
\implies & X \cap A \subseteq \{\cup_{i=1}^{n} U_i \cup (X \setminus A)\} \cap A \\
\implies & A \subseteq \cup_{i=1}^{n} (U_i \cap A) \\
\implies & A \subseteq \cup_{i=1}^{n} U_i
\end{aligned}
$$

Therefore, $\cup_{i=1}^{n} U_i$ is a finite open sub-cover of A and so define it as $\mathcal{F}$, i.e., $\mathcal{F} := \cup_{i=1}^{n} U_i$. This concludes our proof as we can always find a finite open sub-cover $\mathcal{F}$ for an arbitrary open cover C of the closed set A in the compact space X. ■

Theorem 4.5.2: Finite Union of Compact Sets

Finite unions of compact sets are compact.

Proof. Let $A = A_1 \cup A_2 \cup \cdots \cup A_n$ where $A_1, A_2, \ldots, A_n$ are compact. Let us consider an arbitrary open cover C. We are required to show that A is also compact.
Each individual A_i has a finite sub-cover $F_i \subseteq C \implies \bigcup_{i=1}^{n} F_i \subseteq C$. Let $F = \bigcup_{i=1}^{n} F_i \implies F \subseteq C$. We know the finite union of finite sets is finite. So, F is also finite.
Now, by definition of compact set,

$$
A_i \subseteq \cup_{U \in F_i} U \implies \cup_{i=1}^{n} A_i \subseteq \cup_{i=1}^{n} (\cup_{U \in F_i} U) = \cup_{U \in F_1 \cup F_2 \cup \cdots \cup F_n} U = \cup_{U \in F} U \implies A \subseteq \cup_{U \in F} U
$$

So, F is a finite open sub-cover of A, and hence, A is compact. ■

The properties of compact sets exhibited by theorems (4.5.1) and (4.5.2) implies that **compact sets are generalizations of finite sets in a topological sense**, because every finite set is compact[2] and the finite union of finite sets is finite.
Utilizing these two theorems, we can come up with two test cases that maybe, just maybe, are examples of compact subsets. The examples are quite simply the open interval and closed interval in the real line with the usual topology in $\mathbb{R}$, that is, $(a, b),\ [a, b]$ where $a \le b$ & $a, b \in \mathbb{R}$.

Illustrative Example 4.5.2

The closed interval $[a, b]$ is a compact set in $\mathbb{R}$ equipped with the usual topology, while the open interval (a, b) is not.

Proof.

■

4.5.2 Connectedness and Continuity

4.5.3 The Big Three of Compactness

Previously, we have familiarized ourselves with the definition of compact spaces without going into much detail. Now, we want to explore this concept more deeply.

4.5.3.1 Finite Intersection Property

4.5.3.2 Bolzano-Weierstrass Theorem

4.5.3.3 Heine-Borel Theorem

4.5.4 Compactness and Continuity

4.6 Supplementary Material: Metrizable and Baire Spaces

[2] Think about it for a minute, this is also one of the exercises!

4.6.1 Metrizable Space

4.6.2 Baire Space

4.6.2.1 Some Preliminaries and Definitions

4.6.2.2 The 1st Baire Category Theorem

4.6.2.3 The 2nd Baire Category Theorem

Chapter 5

Differentiation in $\mathbb{R}$

Chapter 6

Optimization in $\mathbb{R}$ and Indeterminate Forms

Chapter 7

Recap of Algebra & Topology in $\mathbb{R}^n$

7.1

Chapter 8

Differentiation in $\mathbb{R}^n$

Chapter 9

Optimization in $\mathbb{R}^n$

Chapter 10

Integration in $\mathbb{R}$: The Riemann Integral

Chapter 11

Function Spaces, Uniform Convergence, and Power Series

Chapter 12

Beyond the Riemann Integral: Measure Spaces

Chapter 13

Beyond the Riemann Integral: Lebesgue Integral

Chapter 14

Solutions to the Exercises of Part A

Chapter 1

1A

1B

1C

Chapter 2

Chapter 3

Chapter 4

4A

4B

4C

4D

4E

Part B: Topology

Part C: Measure Theory & Integration

Part D: Differential Geometry

Chapter 15

Differential Geometry: The Basics

15.1 Building Blocks of Manifolds

15.1.1 Locally Euclidean Spaces and Topological Manifolds

15.1.1.1 The Definitions

If we want to do any calculus in any weird or arbitrary space, first we have to convert it to a space in which we already know how to do calculus well. Also as we have learned before, the precondition of doing calculus is ensuring the space is continuous first, so we need the concept of the space of continuous functions, that is, topology. The most general setting in which we have learned differential calculus so far is in the Euclidean spaces, so we need a topological comparison (homeomorphism) from the arbitrary space to the Euclidean spaces. Now, such topological comparisons may not exist always for the whole weird or arbitrary space itself; but we can 'zoom in' to all parts of this weird space until it resembles a certain known Euclidean space. An example of this 'zoom in' mantra is the Earth, which is roughly a sphere, but zoom in enough on the surface of the Earth, it seems that the land is flat. How can we mathematically capture this notion of 'zooming in' in the setting of topology? Well we are truly looking for is a sense of 'closeness' to the weird space, and whenever closeness and topology comes up, the concept of a neighborhood becomes very useful. So we formulate 'zooming in' via open neighborhoods of the weird space.

Definition 15.1.1: Locally Euclidean Space

A topological space $(X, \mathcal{T})$ is defined to be a locally Euclidean space of dimension n, where n is a non-negative integer, if for every point in X there exists an open neighborhood U such that a homeomorphism ϕ from U to an open subset $\phi(U)$ of $\mathbb{R}^n$.

This means that the following topological properties will be inherited by the open neighborhood U of the new space $(X, \mathcal{T})$ from $\mathbb{R}^n$:

1. Locally compact.
2. Locally path connected.
3. Locally metrizable.

An Important Detail: Some may think that what if the locally Euclidean space has two homeomorphisms, one at an open subset of $\mathbb{R}^n$ and another at an open subset of $\mathbb{R}^m$, where $n \neq m$! There is a wonderful but difficult theorem from algebraic topology called 'Brouwer's Invariance of Dimension' that states that for the locally Euclidean space both homeomorphisms must be of dimension $m = n$. The proof of this theorem is quite non-trivial and requires sophisticated machineries which are beyond the scope of our syllabus. So, for now, we take the theorem to be true by faith and conclude that $n = m$ is true.
Now, let us investigate the homeomorphism ϕ very closely, as understanding it better will help us develop our intuition more. With this homeomorphism what we are essentially doing is correlating a local part of the weird space with the geometry of a locally Euclidean space. Now, geometry is often understood best by the means of a coordinate system. So, in that small neighborhood, we are basically bestowing the coordinate system of the Euclidean space. Within this neighborhood, with this coordinate system,

we can traverse the unknown space as if it is a common place we have already visited before; so we have essentially found a map (Geographical meaning implied here) of the unknown place. Now, in mathematics, the word 'map' is already taken, so the next best name we can use is 'chart', which in my personal opinion is a cooler name than 'map'.

Definition 15.1.2: Chart, Coordinate Neighborhood, and Coordinate System

In a locally Euclidean space, the homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$ is characterized by the pair (U, ϕ) and this pair is defined as the **chart**.
The neighborhood U is defined as the **coordinate neighborhood**.
The homeomorphism ϕ is defined as the **coordinate system** on U.
The chart (U, ϕ) is defined to be centered at $p \in U$ if $\phi(p) = \vec{0}$.

If we try to work with these concepts, a few practical problems come up. Such as:

1. What to do if two coordinate neighborhoods overlap? Which coordinate system do we apply for the common domain of the two coordinate neighborhoods?
2. In which process do we 'patch up' the coordinate neighborhood and how many numbers of coordinate neighborhoods do we need to get a global behavior of the new arbitrary space?

Let us attempt to answer these questions one by one. For the first question, we have two open neighborhoods U, V that overlap, making $U \cap V \neq \emptyset$. Suppose with U there is an associated coordinate system ϕ, and with V there is an associated coordinate system ψ. So, we essentially have the charts $(U, \phi), (V, \psi)$ with $U \cap V \neq \emptyset$. We wish to create a system in which we can easily interchange the coordinate systems of the domain $U \cap V$, and for that we need to find the following two functions

$$a : \phi(U \cap V) \to \psi(U \cap V),\ b : \psi(U \cap V) \to \phi(U \cap V)$$

Notice that both ϕ, ψ are homeomorphisms, and by definition of homeomorphism, are bijective functions, and so their inverse functions ϕ^{-1}, ψ^{-1} exist. It takes little effort to verify that the functions a, b above are defined as

$$a := \psi \circ \phi^{-1},\ b := \phi \circ \psi^{-1}$$

The functions a, b are essentially functions that 'transitions' from one coordinate system to another coordinate system. Therefore, we define the following concept below -

Definition 15.1.3: Transition Functions between Charts

Let $(U, \phi), (V, \psi)$ be two charts, then the transition functions between the charts are defined as the following two functions

1. $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$.
2. $\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$.

If we look closely into the transition functions, then we see that we have essentially found a way to go from one arbitrary Euclidean space to another arbitrary Euclidean space. So, we have found our gateway to do calculus in these small overlapping regions. All that is required to specify is what sort of characteristics do we wish to impose on these transition functions in terms of differentiability. With that in mind, we can impose the following restrictions on the transition functions themselves based on the number of times they are continuously differentiable as follows:

1. $\boldsymbol{C^0}$: If both transition functions are C^0, that is just continuous.
2. $\boldsymbol{C^{0,\alpha}}$: If both transition functions are $C^{0,\alpha}$ where $\alpha \in (0, 1)$, that is α-Hölder continuous.
3. $\boldsymbol{C^{0,1}}$: If both transition functions are $C^{0,1}$, that is Lipschitz continuous.
4. $\boldsymbol{C^r}$: If both transition functions are C^r where $r = 1, 2, 3, \ldots$, that is $r \in \mathbb{N}$ times continuously differentiable.
5. $\boldsymbol{C^\infty}$: If both transition functions are C^∞, that is infinitely differentiable.
6. $\boldsymbol{C^\omega}$: If both transition functions are analytic.

Notation Alert: Throughout this section whenever we use the variable k in C^k, it will refer to these 6 options before. This means that the transition functions are classified into 6 types, and to specify which one we are working with, we define the following concept -

Definition 15.1.4: C^k Compatible Charts

Let (U,ϕ) and (V,ψ) be two arbitrary charts of a topological manifold $(M,\mathcal{T})$. Then based on the transitions functions

$$\psi\circ\phi^{-1}:\phi(U\cap V)\to\psi(U\cap V),\ \phi\circ\psi^{-1}:\psi(U\cap V)\to\phi(U\cap V)$$

we call them C^k compatible if both transition functions are C^k.

Just like in topology, we tried to encode and define everything via open sets, in differential geometry, we would like to explain everything via C^k compatible charts. Why not only charts? Because we have a further choice on what sort of charts we are working with, and so we require C^k compatible charts. With that outlook, we also wish to express the whole locally Euclidean space as some collection of C^k compatible charts, so we require a sort of covering property. This gives us the idea of 'Atlas'.

Definition 15.1.5: Atlas

A C^k atlas on a locally Euclidean space $(M,\mathcal{T})$ of dimension n is a collection of pairwise C^k compatible charts that covers M, that is $\mathcal{U}=\{(U_\alpha,\phi_\alpha)\}_{\alpha\in A}$ and $M=\cup_{\alpha\in A}U_\alpha$.

It is easy to check that if the transition functions do not overlap, i.e. if $U\cap V=\varnothing$, then both transition functions reduces to the empty function, i.e. $\psi\circ\phi^{-1}=\phi\circ\psi^{-1}:\varnothing\to\varnothing$. Furthermore, the empty function is analytic (C^ω) vacuously. The hierarchy between all sorts of continuity is given below where $0<\beta<\alpha<1$:

$$\boxed{C^\omega\subset C^\infty\subset\cdots\subset C^{r+1}\subset C^r\subset\cdots\subset C^2\subset C^1\subset C^{0,1}\subset C^{0,\alpha}\subset C^{0,\beta}\subset C^0}$$

When doing calculations, it is quite natural to want as large an atlas as possible so that we can do calculus in as many coordinates as possible. This brings us to define an atlas that must be larger under comparison with other atlases. There comes two methods of comparison, by membership and by cardinality. As membership is comparatively easier to grasp, we proceed using comparison of atlases through membership. Membership means that we require $\subseteq$, which is a reflexive and transitive. So a preorder relation, and the idea of the largest element under preorder relation is given by the maximal element. So, we need to define a maximal atlas.

Definition 15.1.6: Maximal Atlas

A C^k atlas, $\mathcal{U}$ is defined as a maximal atlas if for every other C^k atlas $\mathcal{P}$, whenever we have $\mathcal{U}\subseteq\mathcal{P}$, we must also have $\mathcal{P}\subseteq\mathcal{U}\implies\mathcal{U}=\mathcal{P}$.

Now let us try to answer the question about global behavior. An important thing to notice is that for the non-overlapping regions, such as $(U\setminus V),(V\setminus U)$, we cannot have transition functions. So, the region $(U\setminus V)$ only has coordinates of ϕ and the region $(V\setminus U)$ only has the coordinates of ψ. So, it is impossible to 'patch up' local open neighborhoods to establish a global property of the new arbitrary space we wish to do calculus on. This means that we need to add further global restrictions or conditions to the definition of a locally Euclidean space. Now there are two types of global restrictions one can impose on an arbitrary topological space, namely -

1. Separation axioms. Some prominent examples are $\mathrm{T}_1,\mathrm{T}_2,\mathrm{T}_3,\mathrm{R}_0,\mathrm{R}_1$, Regular.
2. Countability axioms. These are first countable, second countable and separable. The second countability condition is the strongest.

Now the question is which out of the many global axioms do we pick! This is a question that is still to be answered definitively and many mathematicians are working with different sets of these axioms to figure it out. As this is intended to be a first course in differential geometry, it will be wise to keep things as simple as possible and stick with already well established concepts. In that vein, we pick the **second countable** condition as it is the most restrictive one in the countability axioms. About the separation axioms, we can sort of work backwards about what we need from what we desire. To do calculus in our locally Euclidean space, it is very useful if the limits are unique. From topology we know that in Hausdorff spaces the limit of a convergent sequence is unique. So, we also include the **Hausdorff** condition as another of the global restrictions. Again, as a reminder, the choice of the global restrictions are currently undergoing research, and so many mathematicians are working on locally Euclidean spaces with weaker separation and countability axioms; some of them even drop the requirement of countability axioms all together. But for the purposes of a first course, the Hausdorff and second countable space will do just fine. With these two global restrictions, we finally define a topological manifold.

Definition 15.1.7: Topological Manifold

A topological manifold of dimension n is defined to be a Hausdorff, second countable, locally Euclidean space of dimension n.

When working with this definition, we should keep the following things in mind -

1. **Whenever we are proving any theorem using topological manifolds, we should take note of which global restrictive property we are using in the proof. Doing so will give us a clear idea about which theorems or properties would fail should we weaken the definition of a topological manifold.**
2. Although it might be obvious to some, it might still not dawn on the minds of other readers that these global restrictive conditions such as the separation axioms and countability axioms are 'hereditary'. Hereditary means that if an arbitrary space has some separation and countability axiom added to it, then its subspace will also inherit the same separation and countability axiom. So, the subspace of any topological manifold (which itself is Hausdorff and second countable), will also be Hausdorff and second countable.
3. It is very easy to check that all topological manifolds have transition functions that are minimum C^0. So, topological manifolds can also be labeled as C^0 manifold.

Now, to incorporate the differentiability structure into the definition of manifolds, we need to include C^k atlas. For versatility of calculations, we require maximal atlas. So, we build our definition as follows.

Definition 15.1.8: C^k Manifold

A C^k manifold is defined as an ordered tuple $(M, \mathcal{T}, \mathcal{M})$, where M is the whole space, $\mathcal{T}$ is the topology, and $\mathcal{M}$ is the maximal C^k atlas.
When there is no scope of confusion for the topology, a C^k manifold is denoted by an ordered pair $(M, \mathcal{M})$.
The maximal C^k atlas is also called the differentiable structure on $(M, \mathcal{T})$.

As we are working with differentiability, it is an appropriate time to introduce the following isomorphism (or equivalency) concept.

Definition 15.1.9: C^k Diffeomorphism

Let $(M_1, \mathcal{M}_1)$ be a C^{k_1} manifold, $(M_2, \mathcal{M}_2)$ be C^{k_2} manifold and $k = \min(k_1, k_2)$. A diffeomorphism is a bijective C^k function, F, from M_1 to M_2, that is $F : M_1 \to M_2$, whose inverse, that is $F^{-1} : M_2 \to M_1$ is also a C^k function.

Here, we used $k = \min(k_1, k_2)$ as such C^k functions will always exist in the two manifolds with differing differentiability structure. So, our comparison function will always be valid for both manifolds.

15.1.1.2 Some Examples and Non-examples

In the previous subsection, we noticeably did not provide any examples of the concepts, to fill in this gap, many examples and non-examples are given a separate subsection all-together. Each example is accompanied by its construction and its characteristics with proof (some of the constructions and proofs are omitted as they require substantial amount of sophisticated mathematics), with some of the characteristics not mentioned as we have yet to define and/or understand said characteristics. It is quite practical for the readers to have a repertoire of these examples for these examples come up often in discussion and acts as a handy reference.

Illustrative Example 15.1.1

The Euclidean space.

Illustrative Example 15.1.2

The cusp.

Illustrative Example 15.1.3

The cross.

Characteristics: It is Hausdorff, second countable, but NOT LOCALLY EUCLIDEAN. So, it is not a topological manifold.

Illustrative Example 15.1.4

The line with two origins.

Characteristics: It is locally Euclidean, second countable, but NOT HAUSDORFF. So, it is not a topological manifold, rather it is a **non-Hausdorff topological manifold**.

Illustrative Example 15.1.5

The long line or Alexandroff line.

Characteristics: It is locally Euclidean, Hausdorff, but NOT SECOND COUNTABLE. So, it is not a topological manifold, rather it is a **non-second countable topological manifold**.

Illustrative Example 15.1.6

E_8 Manifold.

Construction: Requires extremely specialized knowledge of algebraic topology and differential geometry, which is beyond the scope of introductory differential geometry, hence not provided in this text.
Characteristics: It is a topological manifold, but NOT A SMOOTH TOPOLOGICAL MANIFOLD.

Proof. For the same reasons mentioned in the construction section, the proof is not provided here. ∎

Illustrative Example 15.1.7

Product of an uncountable set with the indiscrete topology with an Euclidean space of finite dimension.

Construction:
Characteristics: It is NOT A TOPOLOGICAL MANIFOLD as it does not satisfy locally Euclidean, Hausdorff and second countable properties.

15.1.1.3 Some Properties of Compatible Charts and Atlases

Theorem 15.1.1: Existence of C^0 and C^∞ Atlas

There always exists a C^0 atlas in a topological manifold, but a C^∞ atlas may not exist in a topological manifold.

Proof. By definition of a topological manifold, it is also locally Euclidean. And by definition of locally Euclidean, every point of it can be equipped with a chart. By taking union of the charts of all the points, we can construct a C^0 atlas. This completes our

first part of the proof.
For the second part of the proof, we only present a counterexample, namely the E_8 manifold. Proving this, as mentioned before in the examples and non-examples section, requires extremely advanced tools. So, for now we take it by faith. ■

15.1.2 Smooth Manifolds

Part E: Complex Analysis

Chapter 16

Recap of $\mathbb{C}$

16.1

Chapter 17

Solutions to the Exercises of Part D

Part F: Functional Analysis

Chapter 18

Inner Products, Norms and Metrics

18.1 The Basics

18.1.1 Inner Product

18.1.2 Norm

18.1.3 Some Recapitulation on Metric and Topology

18.2 Continuity of the Operations

18.2.1 Continuity of Inner Product

18.2.2 Continuity of Norm

18.2.3 Continuity of Metric

18.3 Continuity and Some Interesting Concepts

18.3.1 Semi-Inner Product

18.3.2 Semi-Norm

18.3.3 Pseudometric

Chapter 19

Normed and Topological Vector Spaces

19.1 Normed Vector Spaces

Definition 19.1.1: Normed Vector Space

A normed vector space is defined as a vector space V over the field K that is equipped with a norm,

$$\|\cdot\| : V \to \mathbb{R}$$

Such that the following properties hold:

1. Non-negativity: For all $x \in V,\ ||x|| \geq 0$.
2. Positive Definiteness: For every $x \in V,\ ||x|| = 0$ if and only if x is the zero vector.
3. Absolute Homogeneity: For all $\lambda \in K,\ x \in V,\ ||\lambda x|| = |\lambda|||x||$.
4. Triangle Inequality: For all $x, y \in V$, $||x + y|| \leq ||x|| + ||y||$.

19.2 Topological Vector Spaces

Definition 19.2.1: Topological Vector Space

A topological vector space is defined to be a vector space V over a topological field K (Generally $\mathbb{R}$ or $\mathbb{C}$ equipped with the standard topology), and is endowed with a topology such that the following operations are continuous:

1. Vector Addition, $+ : V \times V \to V$.
2. Scalar Multiplication, $\cdot : K \times V \to V$.

The domain associated with these operations are the product topology.

Theorem 19.2.1: Relation between Normed and Topological Vector Spaces

Every normed vector space is a topological vector space.

Proof. Let V be a normed vector space. ∎

Chapter 20

Banach Spaces

Definition 20.0.1: Banach Space

A complete normed space is defined as a Banach space.

Chapter 21

Hilbert Spaces

Part F: Special Functions

Chapter 22

Special Functions: An Introduction

There are some functions in mathematics that arise frequently when dealing with mathematical analysis, functional analysis, and as solutions to various equations concerning application in physics, engineering, statistics, economics etc. Such functions are more often than not attributed with special names, and these types of functions are called 'special functions'. In this chapter, we will be learning about the fundamental types of these special functions that will serve as a blueprint of sorts and allow us to study even more advanced special functions in the following chapters.

22.1 Exploring a Non-elementary Integral: The Error Function and the Gaussian Integral

22.1.1 The Context

In real life applications, we often encounter the following probability distribution model -

$$P(x) = C_N \int_b^a e^{-x^2}\,\mathrm{d}x$$

Here, $P(x)$ is the probability, and C_N is called the 'normalization constant'. In probability, we know that the sum of all probabilities should be equal to 1. This condition is 'normal' (the usual English meaning, not the one in mathematics that is associated with being perpendicular to the tangent) in the sense that it aligns with our usual notion of discrete probability we learned throughout high-school. This fact is written as $1 = C_N \int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$. From here, we can get

$$C_n = \frac{1}{\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x}$$

Hence the name of normalization constant.

22.1.2 Some Basics on Elementary and Non-Elementary Integral

Definition 22.1.1: Elementary Integral

Let $F(x) = \int f(x)\,\mathrm{d}x$. We define the integration to be elementary if $F(x)$ can be constructed from polynomials, exponentials, logarithm, hyperbolic functions, trigonometric functions, and its inverses, using finite number of algebraic operations and compositions.

22.1.2.1 The Gaussian Integral

When we try to evaluate the normalization constant above, we see that we require to evaluate the integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,\mathrm{d}x$$

This integral is the famous 'Gaussian Integral'.

22.2 The Gamma and Pi Function: The Functions that Generalize the Factorial

22.2.1 Finding the Pi and Gamma Function

We are all familiar with the factorial function, but just a refresher, the factorial function is

$$f(n) = n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot (n-1) \cdot n$$

where $n \in \mathbb{N}_0$. An important property of the factorial function is that $(n!) = n \cdot (n-1)!$, in which if we substitute $f(n) = n!$ we get the functional equation $f(n) = n \cdot f(n-1)$. Now, we arrive at a slight problem if we leave the functional equation as it is because it becomes $f(0) = 0 \cdot f(-1)$ for $n = 0$. We can sidestep this issue quite easily by taking the functional equation as $f(n+1) = (n+1) \cdot f(n)$ instead of what we took earlier. Now, we wish to extend the domain of the factorial function to more than the non-negative integers only. To what extend it is possible we do not know yet, but whatever the extension may be, it must follow these:

1. $f(x+1) = (x+1) \cdot f(x)$, where $x \in X$ and X is the extended domain (which we are yet to determine).
2. When $x \in \mathbb{N}_0 \subseteq X$, it must reduce to the familiar factorial function that we know of.

Now comes the tricky part, we have to guess what the solution to this functional equation might look like. For that reason, it is better to take a look at the common functions that we already know a lot about.

$$\begin{aligned} e^x &= \sum_{k=0}^{k=\infty} \frac{x^k}{k!} \\ \sin(x) &= \sum_{k=0}^{k=\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\ \cos(x) &= \sum_{k=0}^{k=\infty} (-1)^k \frac{x^{2k}}{(2k)!} \end{aligned}$$

Now, from Euler's formula ($e^{ix} = \cos(x) + i\sin(x)$), we know that both sine and cosine functions can be encoded via the exponential function, and the exponential function also contains the factorial function in it. So, the solution to the functional equation must be closely related to the exponential function. So, let us investigate this function very carefully.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left(e^{kx}\right) &= ke^{kx} \\ \int e^{kx}\,\mathrm{d}x &= \frac{e^{kx}}{k} + c \quad \text{Here } k \neq 0, c \in \mathbb{R}. \end{aligned}$$

As we have our original factorial defined on the non-negative integers, let us evaluate the following definite integral of the exponential function -

$$\int_b^a e^{kx}\,\mathrm{d}x = \left[\frac{e^{kx}}{k}\right]_b^a = \frac{1}{k}\left[e^{ka} - e^{kb}\right]$$

Normally we would like to be able to integrate over the whole real line (that is from $-\infty$ to $+\infty$), but if we do that we can clearly see that the integral would diverge. So, the next best option would be to integrate from 0 to ∞ or from $-\infty$ to 0. We integrate from 0 to ∞ first and after doing it, then we will see what happens when we integrate from $-\infty$ to 0. So, we have

$$\int_0^\infty e^{kx}\,\mathrm{d}x = \left[\frac{e^{kx}}{k}\right]_0^\infty = \frac{1}{k}\left[e^{k\cdot\infty} - 1\right]$$

Here we have two options, for positive value of k, the integral diverges, which is of no good to us. But for negative values of k, we can get a finite value. So setting $k = -p$ we get -

$$\int_0^\infty e^{-px}\,\mathrm{d}x = \frac{1}{p}$$

But we are yet to introduce the variable n that corresponds to the factorial function. So for that let us suppose we have the following function that is a solution to the functional equation -

$$\int_0^\infty g(x,n)e^{-px}\ \mathrm{d}x$$

As it is our proposed solution, we should have $\int_0^\infty g(x,0)e^{-px}\ \mathrm{d}x = 0! = 1$. Now we can make an interesting observation in the equation $\int_0^\infty e^{-px}\ \mathrm{d}x = \frac{1}{p}$, when $p = 1$, we have precisely $\int_0^\infty e^{-x}\ \mathrm{d}x = 1 = 0!$. Written more elaborately, we can see that $\int_0^\infty g(x,0)e^{-x}\ \mathrm{d}x = 0! = 1$. From here, we can say that $g(x,0) = 1$. The simplest and classic example of such behavior is seen for $g(x,n) = x^n$. So, we have a candidate for the solution to the function equation -

$$f_{can}(n) = \int_0^\infty x^n e^{-x}\ \mathrm{d}x$$

Now we need to check if it verifies our original two constraints.

1. The functional equation:

$$\begin{aligned}
f_{can}(n+1) &= \int_0^\infty x^{n+1}e^{-x}\ \mathrm{d}x \\
&= \left[x^{n+1}\int e^{-x}\ \mathrm{d}x\right]_0^\infty - \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}x}\left(x^{n+1}\right)\left(\int e^{-x}\ \mathrm{d}x\right)\mathrm{d}x \\
&= \left[-x^{n+1}e^{-x}\right]_0^\infty + (n+1)\int_0^\infty x^n e^{-x}\ \mathrm{d}x \\
&= 0 + (n+1)\int_0^\infty x^n e^{-x}\ \mathrm{d}x \\
&= (n+1)\int_0^\infty x^n e^{-x}\ \mathrm{d}x \\
&= (n+1)f_{can}(n)
\end{aligned}$$

2. Coinciding with the factorial function for non-negative integer values of n:
   For $n = 0$, we already have $f_{can}(n) = 0!$, and right before, we saw that the function satisfies the functional equation. So, by the method of induction, we can quite easily prove that $f_{can}(n) = n!$ for all non-negative integers n.

Now, we go back and try to do the same thing for the option we left out before, that is we integrate from $-\infty$ to 0. Following the same ideas for this would yield another candidate

$$h_{can}(n) = \int_{-\infty}^0 x^n e^x\ \mathrm{d}x$$

Now, if we substitute $x = -m$ we have

$$\begin{aligned}
h_{can}(n) &= \int_{-\infty}^0 x^n e^x\ \mathrm{d}x \\
&= (-1)^{n+1}\int_\infty^0 m^n e^{-m}\ \mathrm{d}m \\
&= (-1)^n\int_0^\infty m^n e^{-m}\ \mathrm{d}m \\
&= (-1)^n \cdot f_{can}(n)
\end{aligned}$$

So, the function $h_{can}(n)$ is just a transformed version of $f_{can}(n)$, and thus, they are essentially the same. The function $f_{can}(n)$ is defined to be the Euler's factorial function or the Pi function.

Definition 22.2.1: Euler's Factorial Function or Pi Function

Euler's factorial function or Pi function, denoted by $\Pi(n)$, is defined as follows -

$$\Pi(n) := \int_0^\infty z^n e^{-z} \, \mathrm{d}z$$

When $n \in \mathbb{N}_0$, $\Pi(n) = n!$.

Now, another French mathematician named Legendre used the same function but defined it a bit differently than Euler and that function is called the Gamma function, denoted by $\Gamma(n)$, has the property $\Gamma(n+1) := \Pi(n)$. If one wishes to write it in terms of $\Gamma(n)$, then the function is defined as $\Gamma(n) = \int_0^\infty z^{n-1} e^{-x} \, \mathrm{d}z$.

Definition 22.2.2: Gamma Function

The gamma function, denoted by $\Gamma(n)$, is defined as follows -

$$\Gamma(n) := \int_0^\infty z^{n-1} e^{-z} \, \mathrm{d}z$$

When $n \in \mathbb{N}_0$, $\Gamma(n+1) = n!$.

The readers may ask that why is the same function defined in two ways? Well, historically speaking, Euler's Pi function came before Legendre's Gamma function, but Legendre's Gamma function got more popularity for some reason. Furthermore, the rationale behind the usage of z^{n-1} instead of z^n in the definition of the Gamma function by Legendre was also never disclosed, so we genuinely do not know what made Legendre define the Gamma function in the manner he did, just that it stuck with most of the mathematicians. In the context of generalizing the factorial function, the Pi function may seem to be more natural than the Gamma function, but in other contexts, the Gamma function seems to be more natural. So, both functions have their respective merits in terms of usage. From both definitions, we can quite clearly observe that

$$\boxed{\Gamma(n+1) = \Pi(n) = n!}$$

Now, we can also get another equivalent definition for the Pi function. For that, let us consider the substitution $z = -\ln(t)$. Then, $\mathrm{d}z = -\frac{1}{t}\,\mathrm{d}t$ and $e^{-z} = e^{\ln(t)} = e^{\ln(t)} = t$. Furthermore, as $z \to 0 \implies t \to 1, z \to \infty \implies t \to 0$. So after substitution, we have the Pi function as -

$$\begin{aligned}
\Pi(n) &= \int_0^\infty z^n e^{-z} \, \mathrm{d}z \\
&= \int_1^0 (-\ln(t))^n (t)(-\frac{1}{t}) \, \mathrm{d}t \\
&= \int_0^1 (-\ln(t))^n \, \mathrm{d}t
\end{aligned}$$

So, we can have the Pi and Gamma functions defined as follows -

$$\boxed{\Pi(n) = \int_0^1 (-\ln(t))^n \, \mathrm{d}t, \; \Gamma(n+1) = \int_0^1 (-\ln(t))^n \, \mathrm{d}t}$$

Now, all the things that we have proved about the Pi and Gamma function is the modern approach. Historically Euler actually discovered something different at first. We will discuss that historical approach by Euler in detail now.

For any $p, q \in \mathbb{N}_0$, we have that

$$\begin{aligned}
&(p+q)! = (p!)(p+1)(p+2)\dots(p+q) \\
\implies &\frac{p!}{(p+q)!} = \frac{1}{(p+1)(p+2)\dots(p+q)} \\
\implies &\frac{(p!)(p+1)^q}{(p+q)!} = \frac{(p+1)^q}{(p+1)(p+2)\dots(p+q)} \\
\implies &\lim_{p\to\infty} \frac{(p!)(p+1)^q}{(p+q)!} = \lim_{p\to\infty} \frac{(p+1)^q}{(p+1)(p+2)\dots(p+q)} {}^{1} \\
\implies &\boxed{\lim_{p\to\infty} \frac{(p!)(p+1)^q}{(p+q)!} = 1}
\end{aligned}$$

The main objective for finding this limit is to find a property that is true for the factorial of large numbers. In order to define an extension of the factorial function, we must have some property of the factorial that is preserved even in the extended definition. Another important thing to notice is that when we discuss something about large numbers, more often than not, we have to use limits. This also helps us to introduce the methods of calculus into our new extended functions. This is the genius of Euler that is still relevant today when doing mathematical research!
Now, with that, let us assume the extended definition of the factorial function also obeys the limit above. Then we have for two numbers p, q (Reminder: we are yet to discover the domain!) -

$$\begin{aligned}
1 &= \lim_{p\to\infty} \frac{(p!)(p+1)^q}{(p+q)!} \\
\implies (q-1)! &= (q-1)! \lim_{p\to\infty} \frac{(p!)(p+1)^q}{(p+q)!} \\
&= \frac{1}{q} \lim_{p\to\infty} \frac{(p!)(q!)(p+1)^q}{(p+q)!} \\
&= \frac{1}{q} \lim_{p\to\infty} (p!) \frac{q!}{(p+q)!} (p+1)^q \\
&= \frac{1}{q} \lim_{p\to\infty} \left(\prod_{i=1}^{i=p} i\right) \frac{1}{(q+1)(q+2)\dots(q+p)} \left(\frac{2}{1}\cdot\frac{3}{2}\dots\frac{p+1}{p}\right)^q \\
&= \frac{1}{q} \lim_{p\to\infty} \left(\prod_{i=1}^{i=p} i\right)\left(\prod_{i=1}^{i=p} \frac{1}{q+i}\right)\left(\frac{2}{1}\cdot\frac{3}{2}\dots\frac{p+1}{p}\right)^q \\
&= \frac{1}{q} \lim_{p\to\infty} \left(\prod_{i=1}^{i=p} i\right)\left(\prod_{i=1}^{i=p} \frac{1}{q+i}\right)\left(\prod_{i=1}^{i=p} \frac{i+1}{i}\right)^q \\
&= \frac{1}{q} \lim_{p\to\infty} \left(\prod_{i=1}^{i=p} i\right)\left(\prod_{i=1}^{i=p} \frac{1}{q+i}\right)\left(\prod_{i=1}^{i=p} (\frac{i+1}{i})^q\right) \\
&= \frac{1}{q} \lim_{p\to\infty} \prod_{i=1}^{i=p} \left(\frac{i}{q+i}(\frac{i+1}{i})^q\right) \\
&= \frac{1}{q} \lim_{p\to\infty} \prod_{i=1}^{i=p} \left(\frac{1}{1+\frac{q}{i}}(1+\frac{1}{i})^q\right) \\
&= \frac{1}{q} \prod_{p=1}^{\infty} \left[\frac{1}{1+\frac{q}{p}}(1+\frac{1}{p})^q\right] \quad \text{The index } i \text{ can be easily replaced by } p \text{ as } i \text{ is a dummy (or placeholder) index.}
\end{aligned}$$

[1] The limit is evaluated as follows: $\lim_{p\to\infty} \frac{(p+1)^q}{(p+1)(p+2)\dots(p+q)} = \lim_{p\to\infty} \frac{p^q(1+\frac{1}{p})^q}{p^q(1+\frac{1}{p})(1+\frac{2}{p})\dots(1+\frac{q}{p})} = \lim_{p\to\infty} \frac{(1+\frac{1}{p})^q}{(1+\frac{1}{p})(1+\frac{2}{p})\dots(1+\frac{q}{p})} = 1$

In the recent calculations, we should take notice of two things in particular:

$$(q-1)! = \lim_{p\to\infty} \frac{(p!)(p+1)^q}{q(q+1)(q+2)\dots(q+p)}, \quad (q-1)! = \frac{1}{q}\prod_{p=1}^{\infty} \frac{1}{1+\frac{q}{p}}(1+\frac{1}{p})^q$$

The first one expresses the extended factorial function as limit, and the second one expresses the extended factorial function as an infinite product. So we want to focus on both of them. Before we make our notations consistent, we have to make just one tiny modification to the limit expression.

$$\begin{aligned}
(q-1)! &= \lim_{p\to\infty} \frac{(p!)(p+1)^q}{q(q+1)(q+2)\dots(q+p)} \\
&= \lim_{p\to\infty} \frac{(p!)(p)^q}{q(q+1)(q+2)\dots(q+p)}\left(\frac{p+1}{p}\right)^q \\
&= \lim_{p\to\infty} \frac{(p!)(p)^q}{q(q+1)(q+2)\dots(q+p)} \lim_{p\to\infty}\left(\frac{p+1}{p}\right)^q \\
&= \lim_{p\to\infty} \frac{(p!)(p)^q}{q(q+1)(q+2)\dots(q+p)}
\end{aligned}$$

Now, we fix our notations by taking $q = n, p = z$ to have the Euler's limit definition of the extended factorial function as -

$$\boxed{\Gamma(n) = (n-1)! = \lim_{z\to\infty} \frac{(z!)z^n}{n(n+1)(n+2)\dots(n+z)}}$$

Doing the same notation change gives us Euler's infinite product definition as -

$$\boxed{\Gamma(n) = (n-1)! = \frac{1}{n}\prod_{z=1}^{\infty}\left[\frac{1}{(1+\frac{n}{z})}\left(1+\frac{1}{z}\right)^n\right]}$$

If we want to go from the infinite product definition to the limit definition, we just reverse the same steps we took to go from the limit definition to the infinite product definition. This means that both definitions are equivalent as well. Also notice that in this context, the notation of the Gamma function makes more sense!

But are these definitions equivalent to the integral definition of the Gamma function? Interestingly enough, they are!

Definition 22.2.3: Alternate Definitions of the Pi and Gamma Functions

The followings definitions of the Pi and Gamma Functions are equivalent to our original definitions (22.2.1) and (22.2.2):

1. Logarithmic Definition:
$$\Gamma(n+1) = \Pi(n) = \int_0^1 (-\ln(t))^n \, \mathrm{d}t$$
2. Euler's Limit Definition:
$$\Gamma(n) = \Pi(n-1) = \lim_{z\to\infty} \frac{(z!)z^n}{n(n+1)\dots(n+z)}$$
3. Euler's Infinite Product Definition:
$$\Gamma(n) = \Pi(n-1) = \frac{1}{n}\prod_{z=1}^{\infty}\left[\frac{1}{(1+\frac{n}{z})}\left(1+\frac{1}{z}\right)^n\right]$$
4. Weierstrass's Definition:
$$\Gamma(n) = \Pi(n-1) = \frac{e^{-\gamma n}}{n}\prod_{z=1}^{\infty}\left[\frac{1}{(1+\frac{n}{z})}\left(e^{\frac{n}{z}}\right)\right]$$

Here $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$ is the Euler-Mascheroni constant.

Proof. **(22.2.1), (22.2.2) $\iff$ (1)**
(1) can be easily shown as an equivalent definition by substituting and back-substituting $z = -\ln(t)$ as done before. This proves (1) is an equivalent definition.
(22.2.1), (22.2.2) $\iff$ (2)
Left as an exercise for the readers.
(22.2.1), (22.2.2) $\iff$ (3)
We know that the 2nd definition is equivalent to our original definitions, and the 2nd definition itself is equivalent to the 3rd definition. So, this concludes that definition (3) is equivalent to our original definitions.
(22.2.1), (22.2.2) $\iff$ (4)

■

22.2.2 The Domain of the Pi and Gamma Function

Now we will tackle the important issue of the domain of the Pi (and Gamma) functions.

22.3 The Riemann-Zeta Function

Part G: Miscellaneous Topics

Chapter 23

List of Open Problems

Chapter 24

List of Interesting Examples

24.1 Classification of Special Functions

The usage of special functions are inevitable when analyzing real life equations in physics, chemistry, biology, statistics & economics. It will be of very useful for the readers to at least know the names of such special functions and what sort of classification they belong to. To that end, a comprehensive classification of special functions is given below. The classification contains some duplicate entries as some special functions pop up under different considerations and these duplicate entries also serve to highlight the cross-connection between the many types of special functions themselves. Furthermore, there are still many special functions that are being discovered and being researched on currently, so this classification is not definitive. This classification is only meant to act as an introduction for the readers to the wonderful world of special functions.

1. **Foundational Functions**
   - **a)** Gamma Function $\Gamma(z)$ and its Classification
     - **i)** Complete Gamma Function $\Gamma(z)$
     - **ii)** Incomplete Gamma Functions $\gamma(s,z), \Gamma(s,z)$
     - **iii)** Generalized Gamma Functions
     - **iv)** Beta Function $B(x,y)$
     - **v)** Digamma Function $\psi(z)$
     - **vi)** Polygamma Functions $\psi^{(m)}(z)$
     - **vii)** Log Gamma Function $\ln\Gamma(z)$
     - **viii)** Pochhammer Symbol $(a)_n$
     - **ix)** q-Gamma Function $\Gamma_q(z)$
     - **x)** Multivariate Gamma Functions
   - **b)** Riemann Zeta Function $\zeta(s)$
2. **Hypergeometric Universe**
   - **a)** Generalized Hypergeometric Functions ${}_pF_q$
     - **i)** Gauss Hypergeometric Function ${}_2F_1(a,b;c;z)$
       - **1)** Elementary Functions
       - **2)** Complete Elliptic Integrals $K(k), E(k)$
       - **3)** Legendre Functions P_ν^μ, Q_ν^μ
       - **4)** Chebyshev Polynomials T_n, U_n
     - **ii)** Confluent Hypergeometric Family ${}_1F_1$
       - **1)** Laguerre Polynomials $L_n^{(\alpha)}$
       - **2)** Hermite Polynomials H_n
       - **3)** Parabolic Cylinder Functions D_ν

4) Error Function $\mathrm{erf}(z)$
5) Incomplete Gamma Functions $\gamma(s,z), \Gamma(s,z)$
6) Coulomb Wave Functions $F_L(\eta,\rho), G_L(\eta,\rho)$
7) Whittaker Functions $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$

iii) Bessel Family (from ${}_0F_1$)
1) Bessel Functions J_ν, Y_ν
2) Modified Bessel Functions I_ν, K_ν
3) Spherical Bessel Functions j_n, y_n
4) Hankel Functions $H_\nu^{(1)}, H_\nu^{(2)}$
5) Struve Functions $\mathbf{H}_\nu, \mathbf{L}_\nu$
6) Kelvin Functions $\mathrm{ber}_\nu, \mathrm{bei}_\nu$
7) Lommel Functions
8) Tomson Functions
9) Anger-Weber Functions

b) Heun Functions (4 singular points)
i) General Heun
ii) Confluent Heun
iii) Doubly Confluent Heun
iv) Biconfluent Heun
v) Triconfluent Heun

c) Multivariate Hypergeometric Functions
i) Appell Functions F_1, F_2, F_3, F_4
ii) Lauricella Functions
iii) Kampé de Fériet Function
iv) MacRobert E-Function

3. q-Special Functions & Basic Hypergeometric Series

a) Basic Hypergeometric Series ${}_p\phi_q$
b) q-Pochhammer Symbol $(a;q)_n$
c) q-Gamma Function $\Gamma_q(z)$
d) q-Beta Function
e) q-Orthogonal Polynomials
i) Askey-Wilson Polynomials
ii) q-Racah Polynomials
iii) q-Hahn Polynomials
iv) q-Jacobi Polynomials
v) q-Laguerre Polynomials
vi) q-Hermite Polynomials
vii) Continuous q-Hermite Polynomials

f) q-Bessel Functions
g) q-Exponential Functions

4. Orthogonal Polynomials (Askey Scheme)

a) Continuous Orthogonal Polynomials
i) Wilson Polynomials (most general continuous)

1) Continuous Dual Hahn Polynomials
2) Continuous Hahn Polynomials
3) Jacobi Polynomials $P_n^{(\alpha,\beta)}$
A) Gegenbauer/Ultraspherical Polynomials $C_n^{(\lambda)}$
B) Legendre Polynomials P_n
C) Chebyshev Polynomials T_n, U_n
ii) Other Continuous Polynomials
1) Hermite Polynomials H_n
2) Laguerre Polynomials $L_n^{(\alpha)}$
3) Bessel Polynomials

b) Discrete Orthogonal Polynomials

i) Racah Polynomials (most general discrete)
1) Hahn Polynomials
2) Dual Hahn Polynomials
3) Krawtchouk Polynomials
4) Meixner Polynomials
5) Charlier Polynomials

c) Multiple Orthogonal Polynomials

i) Hermite-Padé Approximants
ii) Angelesco Systems
iii) AT Systems

d) Matrix Orthogonal Polynomials

i) Matrix-valued Orthogonal Polynomials

5. Elliptic Functions & Integrals

a) Elliptic Integrals
i) Legendre's Complete: $K(k), E(k)$
ii) Jacobi's Incomplete: $F(\varphi|k), E(\varphi|k), \Pi(\varphi, n|k)$

b) Elliptic Functions
i) Jacobi Elliptic: $\mathrm{sn}(u,k), \mathrm{cn}(u,k), \mathrm{dn}(u,k)$
ii) Weierstrass Elliptic: $\wp(z; g_2, g_3)$
iii) Theta Functions $\theta_1, \theta_2, \theta_3, \theta_4$
iv) Neville Theta Functions

6. Exponential, Logarithmic & Trigonometric Integrals

a) Exponential Integral $\mathrm{Ei}(z)$
b) Logarithmic Integral $\mathrm{li}(z)$
c) Sine Integral $\mathrm{Si}(z)$
d) Cosine Integral $\mathrm{Ci}(z)$
e) Hyperbolic Sine Integral $\mathrm{Shi}(z)$
f) Hyperbolic Cosine Integral $\mathrm{Chi}(z)$
g) Fresnel Integrals $S(z), C(z)$
h) Dawson's Integral $F(z)$
i) Generalized Exponential Integrals $E_n(z)$

j) Incomplete Gamma Functions $\gamma(s, z), \Gamma(s, z)$

7. Angular & Radial Functions

a) Angular Functions

i) Spherical Harmonics $Y_l^m(\theta, \varphi)$

ii) Wigner D-matrices $D^j_{mm'}$

iii) Wigner 3-j, 6-j, 9-j Symbols

iv) Clebsch-Gordan Coefficients

v) Zernike Polynomials

b) Radial Functions

i) Coulomb Wave Functions $F_L(\eta, \rho), G_L(\eta, \rho)$

ii) Whittaker Functions $M_{\kappa,\mu}(z), W_{\kappa,\mu}(z)$

iii) Bateman Functions $k_\nu(x)$

iv) Miller Functions

8. Painlevé Transcendents

a) Painlevé I

b) Painlevé II

c) Painlevé III

d) Painlevé IV

e) Painlevé V

f) Painlevé VI

9. Number-Theoretic Functions

a) Partition Function $p(n)$

b) Divisor Functions $\sigma_k(n)$

c) Euler Totient Function $\varphi(n)$

d) Möbius Function $\mu(n)$

e) Liouville Function $\lambda(n)$

f) Von Mangoldt Function $\Lambda(n)$

g) Ramanujan Tau Function $\tau(n)$

h) Dedekind Eta Function $\eta(\tau)$

10. Statistical Functions

a) Error Function $\text{erf}(z)$

b) Complementary Error Function $\text{erfc}(z)$

c) Owen's T-function $T(h, a)$

d) Marcum Q-function $Q_M(a, b)$

e) Rice Ie-function $I_e(k, x)$

f) Non-central Distributions (χ^2, t, F)

11. Advanced & Specialized Functions

a) Wave Equation Solutions

i) Mathieu Functions $\text{ce}_n(z, q), \text{se}_n(z, q)$

ii) Modified Mathieu Functions

- **iii)** Lamé Functions $E_n^m(z)$
- **iv)** Spheroidal Wave Functions $\mathrm{PS}_n^m, \mathrm{QS}_n^m$
- **v)** Cylindrical Wave Functions

b) Other Transcendental Functions

- **i)** Airy Functions $\mathrm{Ai}(z), \mathrm{Bi}(z)$
- **ii)** Scorer's Functions $\mathrm{Gi}(z), \mathrm{Hi}(z)$
- **iii)** Goodman Functions
- **iv)** Lerch Transcendent $\Phi(z, s, a)$
- **v)** Polylogarithm $\mathrm{Li}_s(z)$
- **vi)** Dilogarithm $\mathrm{Li}_2(z)$
- **vii)** Clausen Functions $\mathrm{Cl}_2(\theta), \mathrm{Sl}_2(\theta)$
- **viii)** Lambert W Function $W(z)$

c) Miscellaneous Special Functions

- **i)** Struve Functions $\mathbf{H}_\nu(z), \mathbf{L}_\nu(z)$
- **ii)** Weber Functions $\mathbf{E}_\nu(z)$
- **iii)** Lommel Functions $s_{\mu,\nu}(z), S_{\mu,\nu}(z)$
- **iv)** Bateman Functions $k_\nu(x)$

12. Transform-Related Functions

a) Fourier Transform Kernels

b) Laplace Transform Kernels

c) Hankel Transform Kernels

d) Mellin Transform Kernels

e) Hilbert Transform Kernels

24.2 The Cantor Set

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