Chaos and Analysis in Classical Kicked Rotors

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Introduction

In the 1600s, Newton developed the laws of motion and universal gravitation, enabling accurate calculation of the equations of motion for two-body systems. However, he encountered significant difficulty when attempting to understand the gravitational interaction among three bodies — now known as the Three-Body Problem. In a letter to his friend Edmond Halley, Newton expressed his frustration: “No problem has ever made my head ache like the problem of the Earth, Moon, and Sun.”This problem remained unsolved for centuries until the 1890s, when Henri Poincaré made groundbreaking progress. He proved that no general algebraic solution exists for the three-body problem, and that its long-term behavior is highly sensitive to initial conditions — meaning even tiny differences in the starting state can lead to drastically different outcomes. This insight laid the foundation for what we now call chaos theory.Chaos theory gained wider attention in the 1960s when Edward Lorenz, while running weather simulations, discovered what is now known as the Butterfly Effect: the idea that small changes in initial conditions can lead to vastly different outcomes. Since then, chaotic systems have been observed across numerous fields, including physics, chemistry, biology, and economics.One particularly useful system for studying classical and quantum chaos is the kicked rotor. Despite having a simple Hamiltonian and only one degree of freedom, the kicked rotor is nonlinear and explicitly time-dependent, which allows it to exhibit rich and complex chaotic behavior. As a Hamiltonian system, it also connects naturally to quantum mechanics, where the Hamiltonian becomes an operator central to the Schrödinger equation.In this work, we derive the equations governing the kicked rotor and explore its behavior in both the classical and quantum regimes. *Classical Kicked Rotors

Theory

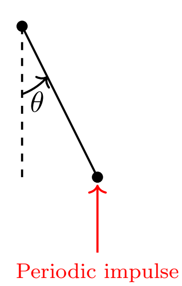

Figure 1: The kicked rotor: no gravity, frictionless pivot, periodic impulse applied at intervals.

In the classical prototype of the kicked rotor it is a bar of length l and the moment of inertia associated with it is I.The one end is pivoted without friction and the other end is subjected to a impulse and fixed intervels t = 0 , T , 2T , 3T..... with impulse of strength K/l (see fig 1). The motion of the system is governed by the following equation:

$$H(p_\theta, \theta, t) = \frac{p_\theta^2}{2I} + K cos\theta \sum_n \delta(t - nT) \qquad (1)$$

Now we should derrive this equation with full explanation for better understanding.

Derrivation

The angular impulse J = $\vec{r}$ × impulse.

$$J = l(sin\theta\hat{i} + cos\theta\hat{j}) \times impluse$$
$$= l(sin\theta\hat{i} + cos\theta\hat{j}) \times \frac{K}{l}\hat{j}$$
$$= Ksin\theta\hat{k}$$

But this angular impulse will not be acting on the bar for forever, it will be acting on it just at t = nT. which means that that that the torque acting on the bar would just be for t = nT which will be same for every other mT because the impulse is not related with any other variable and will be zero for other t which means we can express the torque with a dirc delta notation with some constant C.

$$\tau(t) = C\sum_n \delta(t - nT)$$

The next part would be to calculate the value of C using the info we have. Stating the relation of angular impulse and torque for t ϵ (nT - ϵ , nT + ϵ),[$\epsilon << 1$]:

$$\frac{dL}{dt} = \tau(t)$$
$$dL = \tau(t)dt$$
$$\int dL = \int_{nT-\epsilon}^{nT+\epsilon} \tau(t)dt$$
$$\Delta L = \int_{nT-\epsilon}^{nT+\epsilon} C\sum_n \delta(t - nT)dt$$
$$J = C\int_{nT-\epsilon}^{nT+\epsilon} \delta(t - nT)dt$$
$$J = C$$
$$Ksin\theta = C$$
$$\tau(t) = Ksin\theta\sum_n \delta(t - nT)$$

Here we have derived the equation of torque now we can find the energy of the system.The Hamiltonian will have two parts , one: the free motion in between the kicks and other one would be the energy it will get after kicks.

$$H = H_{\text{free}} + H_{\text{kick}}$$

So in between the kicks the only energy present in the system would be due to rotation of the bar , hence the rotational kinetic energy.

$$= \frac{1}{2}I\omega^2$$
$$Note : L = p$$
$$p = I\omega$$
$$\omega = \frac{p}{I}$$
$$H_{\text{free}} = \frac{p^2}{2I}$$

So we got the expression of H_{free} now for the H_{kick} we have to use the expression the torque we got just back cause that's what is imparting torque and cause of which the change is state occurs which results in energy formation due to external interaction(the energy needs to be conserved).

$$-\frac{dV}{d\theta} = \tau(t)$$

$$-\frac{dV}{d\theta} = Ksin\theta \sum_n \delta(t - nT)$$

$$-\int dV = \int Ksin\theta \sum_n \delta(t - nT) d\theta$$

$$-\Delta V = K \sum_n \delta(t - nT) \int sin\theta d\theta$$

$$H_{\text{kick}} = Kcos\theta \sum_n \delta(t - nT)$$

$$H(p_\theta, \theta, t) = H_{\text{free}} + H_{\text{kick}}$$

$$H(p_\theta, \theta, t) = \frac{p_\theta^2}{2I} + Kcos\theta \sum_n \delta(t - nT)$$

Equations of Motion

We have formulated the hamiltonian of the system now it's time to actually solve it and get the equations governing motion of the system.

Using the Hamiltonian equations:

$$\frac{\partial H}{\partial p_\theta} = \frac{d\theta}{dt} = \frac{p_\theta}{I}$$

$$\frac{\partial H}{\partial \theta} = -\frac{dp_\theta}{dt} = -Ksin\theta \sum_n \delta(t - nT)$$

$$\frac{dp_\theta}{dt} = Ksin\theta \sum_n \delta(t - nT)$$

Now at time t = nT we can do a very small changes ϵ around it and there we can formulate the results. So at time t = nT - ϵ we know there was no torque to change the angular momentum so it was conserved from the earlier kick let's call it p_n and at t = nT + ϵ it changes and the angular momentum becomes conserved for the next interval , let's call it $p_{\text{n+1}}$.

Now using the equation obtained from the equation of motion:

$$\int_{p_n}^{p_{\text{n+1}}} dp_\theta = \int_{nT-\epsilon}^{(n+1)T+\epsilon} Ksin\theta \sum_n \delta(t - nT) dt$$

$$p_{\text{n+1}} - p_n = \int_{nT-\epsilon}^{nT+\epsilon} Ksin\theta_n \delta(t) dt$$

$$p_{\text{n+1}} - p_n = Ksin\theta_n$$

In the other equation obtained we know that between the kicks p_θ is constant so theta grows linearly in the interval but changes discontinously after each kick.

Now solving this equation would yield:

$$\int_{\theta_n}^{\theta_{\mathrm{n+1}}} d\theta = \int_{nT-\epsilon}^{(n+1)T+\epsilon} \frac{p_\theta}{I} dt$$

$$\theta_{\mathrm{n+1}} - \theta_n = \int_{nT-\epsilon}^{(n+1)T+\epsilon} \frac{p_\theta}{I} dt$$

$$\theta_{\mathrm{n+1}} - \theta_n = \int_{nT-\epsilon}^{(n+1)T+\epsilon} \frac{p_{\mathrm{n+1}}}{I} dt$$

As after the kick at t = nT the momentum becomes constant which is equal to $p_{\mathrm{n+1}}$

$$\theta_{\mathrm{n+1}} - \theta_n = \frac{p_{\mathrm{n+1}}}{I}\tau$$

We can simplify the equation by $\frac{\tau}{I} = 1$ and no generality will be lost. We obtained the Standard map of the classical kicked rotor:

$$p_{\mathrm{n+1}} = p_n + Ksin\theta_n$$

$$\theta_{\mathrm{n+1}} = \theta_n + p_{\mathrm{n+1}} \mod 2\pi$$

Phase Space Analysis

Phase space is a mathematical representation of all the physical possible states.The plot is done by the momentum vs the position of the system.Each point in the phase space gives us a state of the system (all different points are unique).When these points are joined together they forms what is called trajectory which gives the time evolution of the system.

When the phase space is plotted it can be of three different types:

Regular:

In these plots the trajectory forms a closed,smooth loops like ellipse , circles. They are well behaved and predictable.These systems are integrable.

KAM Systems:

These systems are the intermediate between the Regular and the chaotic.These systems are nearly integrable.They are formed when the regular systems are pertubed ($X_o + \epsilon X_1$) some regions remaining in regular and some in chaotic. The regular one's are represented by still surviving closed deformed loops and the chaotic one's represented by fuzzy cloudy region.

Chaotic system:

Chaotic systems exhibit extreme sensitivity to initial conditions, meaning even tiny differences in starting points can lead to vastly different outcomes over time. Phase space trajectories in chaotic systems are highly complex and unpredictable. In some cases, chaotic systems can exhibit strange attractors, which are sets of points in phase space that attract nearby trajectories.

Now let's study the Phase Space of the standard map (θ, p_θ)

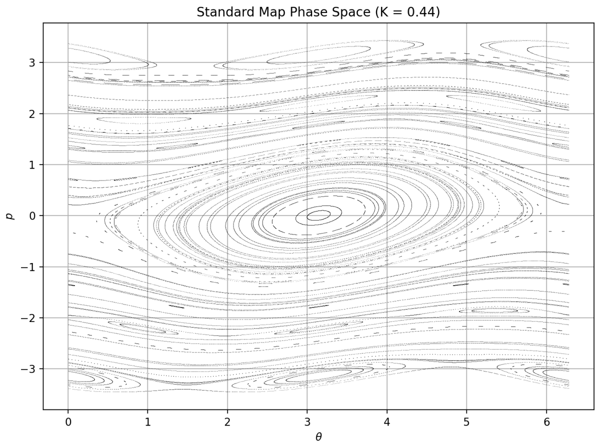

Figure 2: K = 0.44

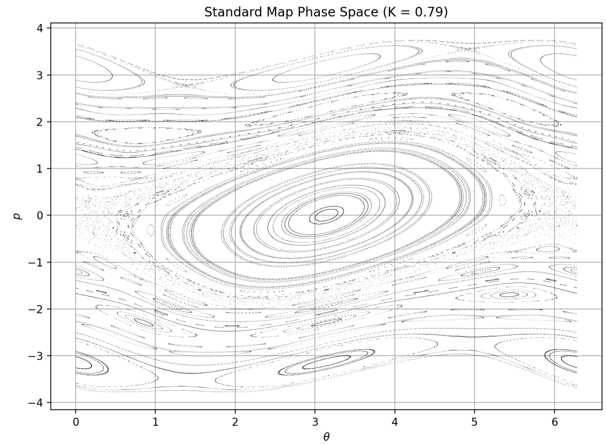

Figure 3: K = 0.79

For $K = 0.44$, the phase space exhibits a largely **regular structure**. The trajectories form smooth, continuous, closed curves, which represent invariant tori—indicative of quasi-periodic motion. The system behaves in an integrable fashion with very little to no visible chaos. This suggests that the nonlinear effects introduced by the kick strength are too weak to break the underlying integrability of the system.

On the other hand, for $K = 0.79$, the structure is still predominantly regular, but subtle signs of **incipient chaos** begin to emerge. While many closed curves persist, some regions—especially near the separatrix layers and resonant islands—start to show deformation. Small chaotic layers or fuzzy areas begin to form around resonances, hinting at the onset of instability. These are precursors to more dramatic chaos at higher K values.

Together, these plots illustrate the transition from an almost fully integrable regime at low K values to the early stages of chaos as K increases. As the kicking strength increases, the influence of nonlinearity becomes stronger, leading to the gradual breakdown of invariant tori and the formation of chaotic seas in the phase space.

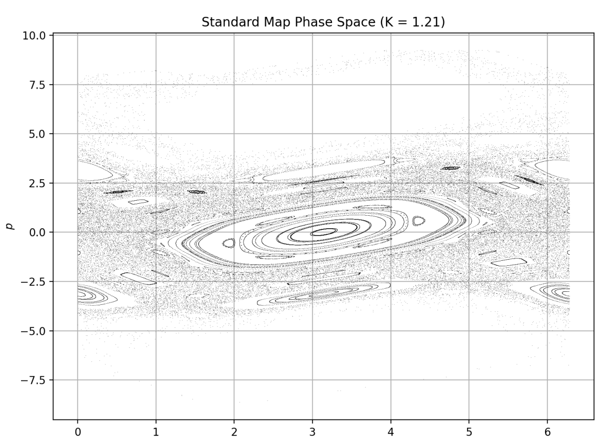

Figure 4: K = 1.21

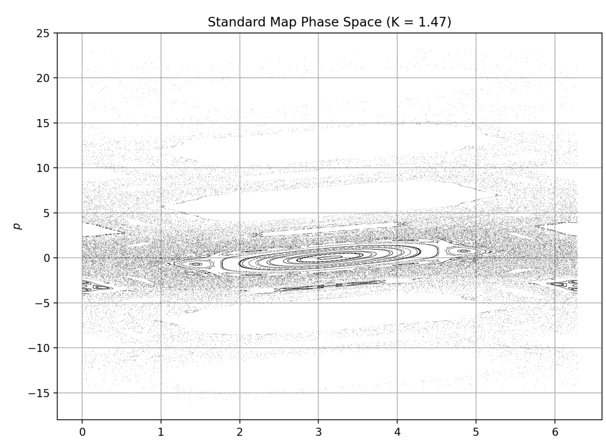

Figure 5: K = 1.47

For $K = 1.21$, the phase space shows a **mixed structure**—a combination of regular and chaotic behavior. Some invariant tori still survive, appearing as smooth, closed curves. However, around these regions, we observe significant chaotic layers and broken tori, particularly near resonant zones. This indicates that nonlinearity is strong enough to cause the breakdown of integrable motion in parts of the system, resulting in a coexistence of ordered islands and chaotic seas.

When $K = 1.47$, the phase space becomes even more **dominated by chaos**. The majority of the invariant tori have been destroyed, and the motion in a large part of phase space is irregular and highly sensitive to initial conditions. Only a few small islands of stability persist, surrounded by vast chaotic regions. The contours are much more densely scattered and less structured compared to lower K values, reflecting the increased unpredictability of the system.

These two plots clearly illustrate how increasing K leads to the progressive destruction of regular motion and the expansion of chaotic dynamics. This transition embodies the typical route to chaos in Hamiltonian systems, where nonlinearity acts to erode the stability of phase space trajectories.

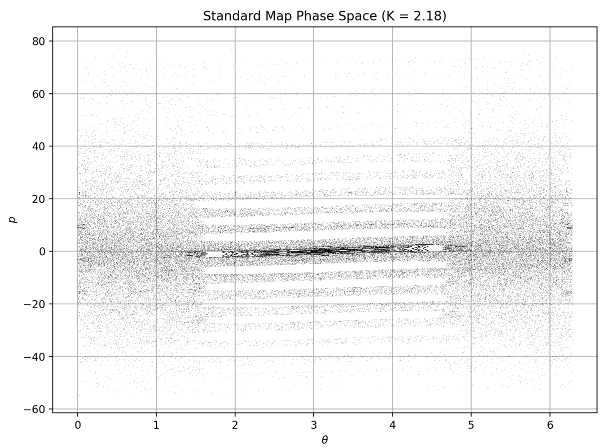

Figure 6: K = 2.18

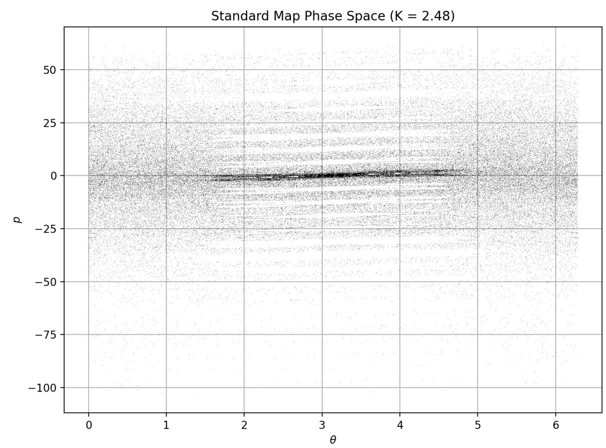

Figure 7: K = 2.48

For $K = 2.18$ and $K = 2.48$, the phase space plots of the standard map show a striking dominance of chaotic behavior. The momentum p exhibits significant diffusion over a wide range of values, indicating that most trajectories are no longer confined to invariant curves but instead wander irregularly through phase space. Although some tiny remnants of regular structures—such as small elliptic islands—are still visible, particularly around $p = 0$, they are barely noticeable compared to the vast sea of chaotic trajectories. This widespread chaos is characteristic of the standard map for large K, where the nonlinearity introduced by the kick term is strong enough to break up most of the tori, allowing the dynamics to become highly sensitive to initial conditions and enabling diffusion in the momentum direction.