

# Stability Analysis

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## Floquet Theory

A Poincarè Map  $P : \Sigma \rightarrow \Sigma$  is the Mapping of the initial values on a trajectory on any orbit of an  $N$  dimensional equation , it tells where the initial point will land on a transverse section which is  $(n-1)$  dimensional. If  $P(x_0) = x_0$  it means the orbit is a periodic orbit. Now here comes a question , if I did a slight perturbation on a periodic let's say  $y$  on the poincarè section itself , how will it evolve in time , will it diverge , will it come back to the orbit , we solve this problem using classical Floquet Theory.

We say if  $P(x_p) = x_p$  is a fixed point on the poincarè section then along the periodic trajectory the slight perturbation  $y$  is given as

$$y(t) = \Phi(t)y(0)$$

$\dot{x} = f(x, t)$  is the system of differential equations evolving the system. Now  $x = x_p + y$  plugging it

$$\dot{x}_p + \dot{y} = f(x_p + y, t)$$

, if we do the taylor series expansion

$$\dot{x}_p + \dot{y} = f(x_p, t) + Df(x_p, t)y + O(|y|^2)$$

As we know that  $f(x_p, t) = \dot{x}_p$  the equation becomes

$$\dot{y} = Df(x_p, t)y$$

And we call the jacobian matrix as  $A(t)$  and  $x_p(t + T) = x_p(t)$  is periodic with period  $T$  ,  $A(t + T) = A(t)$  The relation between the state transition matrix  $\Phi$  is given also we have all the conditions which will help us solve for the state transition matrix

$$\dot{\Phi}(t) = A(t)\Phi(t)$$

$$\Phi(0) = I_{n \times n}$$

We use all these conditions to solve for the state transition matrix , now  $\Phi(T) = M$  , where  $M$  is called the monodromy matrix , it tells where the perturbation we did will land on the poincarè section

$$y(T) = My(0)$$

The eigen values of  $M$  matrix give us the stability analysis of the periodic orbit ,  $\lambda_i \in \mathcal{C}$  so We check the eigen values which are called floquet multipliers if they are inside the unit circle the eigen value corresponds to a stable direction of the perturbation , if it's on the unit circle , it's neutral stable , if it's outside the unit circle , the perturbation corresponds to an unstable direction. The Floquet multipliers are written as

$$\lambda_i = e^{\mu T}$$

where

$$\mu_i = \frac{\log(\lambda_i)}{T}$$

are called the floquet exponent. Where the logarithm is multivalued in  $\mathcal{C}$ . We also need to note a point that , One Floquet exponent is always zero, corresponding to perturbations tangent to the periodic orbit. This reflects the time-translation invariance of the system. But we have a problem , this floquet method is always applicable in the smooth systems means the parameters must be atleast  $\mathcal{C}^1$  that means the application of the floquet theory is very difficult in the case of non smooth systems. Let's show some examples that we can use averaging Theory for this.

# Averaging vs Floquet Stability of a Weakly Forced Damped Harmonic Oscillator

## 0.1 Harmonic equation

We consider the weakly damped forced oscillator

$$\ddot{x} + 2\varepsilon\gamma\dot{x} + x = \varepsilon F \cos t, \quad 0 < \varepsilon \ll 1.$$

This system admits a periodic steady solution. We study its stability using both averaging and Floquet theory.

### Averaging method

We introduce the harmonic ansatz

$$x(t) = a(t) \cos t + b(t) \sin t,$$

where  $a(t), b(t)$  vary slowly on the time scale  $O(\varepsilon^{-1})$ .

Substituting into the equation and projecting onto  $\cos t$  and  $\sin t$ , then averaging over one period, yields the slow amplitude system

$$\dot{a} = -\varepsilon\gamma a, \quad \dot{b} = -\varepsilon\gamma b + \frac{\varepsilon F}{2}.$$

The equilibrium of the averaged system is

$$a_* = 0, \quad b_* = \frac{F}{2\gamma}.$$

Linearization about  $(a_*, b_*)$  gives the Jacobian

$$J_{\text{avg}} = \varepsilon \begin{pmatrix} -\gamma & 0 \\ 0 & -\gamma \end{pmatrix}.$$

Hence the averaged eigenvalues are

$$\lambda_{\text{avg}} = -\varepsilon\gamma.$$

This predicts exponential stability with decay rate  $\varepsilon\gamma$ .

### Floquet analysis

We rewrite the system as a first-order equation

$$\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & -2\varepsilon\gamma \end{pmatrix} y + \begin{pmatrix} 0 \\ \varepsilon F \cos t \end{pmatrix}, \quad y = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}.$$

Perturbations  $u$  about the periodic orbit satisfy the variational system

$$\dot{u} = Au, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & -2\varepsilon\gamma \end{pmatrix}.$$

Since  $A$  is constant, the fundamental matrix is

$$\Phi(t) = e^{At}.$$

The eigenvalues of  $A$  are

$$\lambda_{\pm} = -\varepsilon\gamma \pm i\omega, \quad \omega = \sqrt{1 - \varepsilon^2\gamma^2}.$$

The forcing period is  $T = 2\pi$ . The monodromy matrix is

$$M = \Phi(2\pi),$$

whose eigenvalues (Floquet multipliers) are

$$\mu_{\pm} = e^{-2\pi\varepsilon\gamma} e^{\pm i2\pi\omega}.$$

The Floquet exponents are therefore

$$\boxed{\lambda_{\text{Floquet}} = -\varepsilon\gamma \pm i\omega.}$$

## Comparison

The real part of the Floquet exponents is

$$\text{Re}(\lambda_{\text{Floquet}}) = -\varepsilon\gamma,$$

which matches exactly the averaged eigenvalue:

$$\text{Re}(\lambda_{\text{Floquet}}) = \lambda_{\text{avg}}.$$

Thus averaging preserves the exponential decay rate and correctly predicts the stability of the periodic orbit to first order in  $\varepsilon$ .

## Duffing Oscillator: Averaging and Floquet Stability

### Model equation

We consider the weakly nonlinear Duffing oscillator

$$\ddot{x} + x = \varepsilon (-2\gamma\dot{x} - \alpha x^3 + F \cos t), \quad 0 < \varepsilon \ll 1.$$

The system admits a periodic steady response. We study its stability using both averaging and Floquet theory.

### Averaging method

We introduce the harmonic ansatz

$$x(t) = a(t) \cos t + b(t) \sin t,$$

where  $a(t), b(t)$  vary slowly.

Averaging over one period gives the slow system

$$\dot{a} = -\varepsilon\gamma a + \varepsilon \frac{3\alpha}{8} (a^2 + b^2)b,$$

$$\dot{b} = -\varepsilon\gamma b - \varepsilon \frac{3\alpha}{8} (a^2 + b^2)a + \frac{\varepsilon F}{2}.$$

Let

$$r^2 = a^2 + b^2.$$

The steady amplitude is

$$r_* = \frac{F}{2\gamma}.$$

## Linearization of the averaged system

Linearizing about the equilibrium gives

$$J_{\text{avg}} = \varepsilon \begin{pmatrix} -\gamma & \frac{3\alpha}{4}r_*^2 \\ -\frac{3\alpha}{4}r_*^2 & -\gamma \end{pmatrix}.$$

The averaged eigenvalues are

$$\lambda_{\text{avg},\pm} = \varepsilon \left( -\gamma \pm i \frac{3\alpha}{4} r_*^2 \right).$$

## Floquet variational system

Let  $x_p(t)$  denote the periodic orbit. A perturbation  $u$  satisfies

$$\ddot{u} + 2\varepsilon\gamma\dot{u} + u + 3\varepsilon\alpha x_p^2(t)u = 0.$$

Introduce the state vector

$$y = \begin{pmatrix} u \\ \dot{u} \end{pmatrix}.$$

The variational system becomes

$$\dot{y} = A(t)y,$$

with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 - 3\varepsilon\alpha x_p^2(t) & -2\varepsilon\gamma \end{pmatrix}, \quad A(t+2\pi) = A(t).$$

## Numerical Floquet analysis

The variational equation was integrated numerically along the periodic orbit over one forcing period  $T = 2\pi$  to obtain the monodromy matrix

$$M = \begin{pmatrix} 1.01109922 & 0.08024137 \\ -0.14581054 & 0.91722089 \end{pmatrix}.$$

The Floquet multipliers are the eigenvalues of  $M$ :

$$\mu_{\pm} = 0.96416005 \pm 0.09745128i.$$

The Floquet exponents are defined by

$$\lambda_{\pm} = \frac{1}{T} \log(\mu_{\pm}),$$

which yields

$$\lambda_{\pm} = -0.005 \pm 0.01603194i.$$

Since the real part is negative,

$$\operatorname{Re}(\lambda_{\pm}) = -0.005 < 0,$$

the periodic orbit is exponentially stable.

## Averaged stability eigenvalue

From the averaged amplitude equations the linearized dynamics has eigenvalue

$$\lambda_{\text{avg}} = -\varepsilon\gamma.$$

For the parameters used in the numerical simulation,

$$\varepsilon\gamma = 0.005,$$

so the averaged prediction is

$$\boxed{\lambda_{\text{avg}} = -0.005.}$$

## Comparison

The numerical Floquet exponents were

$$\lambda_{\text{Floquet},\pm} = -0.005 \pm 0.01603i.$$

Thus

$$\text{Re}(\lambda_{\text{Floquet}}) = \lambda_{\text{avg}},$$

showing that the averaging method exactly reproduces the exponential decay rate of perturbations.