Hamiltonian Truncation 101

We want to study 2D ϕ^4 theory in Hamiltonian truncation. The theory is defined by the Euclidean Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial\phi)^2 + \tfrac{1}{2}m^2\phi^2 + \frac{g}{4!}\phi^4. \tag{1}$$

We want to study the theory on a cylinder of radius R. The metric on the cylinder is

$$ds^2 = d\tau^2 + R^2 d\theta^2, \tag{2}$$

where θ is a 2π periodic angle. We will use units where $R = 1$. We expand the fields as

$$\phi(\tau,\theta) = \frac{1}{\sqrt{2\pi}} \sum_{\ell=-\infty}^{\infty} \phi_\ell(\tau) e^{i\ell\theta}, \tag{3}$$

where $\phi^\dagger = \phi$ implies

$$\phi_\ell(\tau)^\dagger = \phi_{-\ell}(\tau). \tag{4}$$

The free action ($g = 0$) is then

$$S_0 = \tfrac{1}{2}\sum_\ell \int d\tau \left(|\dot{\phi}_\ell|^2 + \omega_\ell^2|\phi_\ell|^2\right), \tag{5}$$

where

$$\omega_\ell = \sqrt{\ell^2 + m^2}. \tag{6}$$

This is a sum of independent harmonic oscillators.

To quantize the free theory, we can write the Heisenberg picture operators as

$$\phi_\ell(\tau) = \frac{1}{\sqrt{2\omega_\ell}}\left(a_\ell e^{-\omega_\ell \tau} + a_{-\ell}^\dagger e^{\omega_\ell \tau}\right). \tag{7}$$

Note that

$$\phi_\ell(\tau)^\dagger = \phi_{-\ell}(-\tau), \tag{8}$$

which implies the correct Hermiticity condition for the field in Euclidean space:

$$\phi(\tau,\theta)^\dagger = \phi(-\tau,\theta). \tag{9}$$

The canonical commutation relations for the fields are satisfied provided that the creation and annihilation operators satisfy the standard commutation relations

$$[a_\ell, a_{\ell'}^\dagger] = \delta_{\ell\ell'}. \tag{10}$$

The Hamiltonian of the free theory is then

$$:H_0: = \sum_\ell \omega_\ell a_\ell^\dagger a_\ell. \tag{11}$$

We can subtract an (infinite) constant to drop the constant term, which is equivalent to normal ordering.

When we consider the interacting theory, it is more convenient to use Schrödinger picture because the Hamiltonian depends explicitly on time. The Schrödinger picture fields are equivalent to the Heisenberg fields at $\tau = 0$:

$$\phi_\ell(\tau) = \frac{1}{\sqrt{2\omega_\ell}}\big(a_\ell + a_{-\ell}^\dagger\big). \tag{12}$$

with the creation and annihilation operators satisfying the commutation relations Eq. (10). This amounts to using a basis of states that diagonalizes the free Hamiltonian H_0. We then write the interaction term in terms of the creation and annihilation operators:

$$\begin{aligned} :H_{\text{int}}: &= \frac{g}{4!}\int_0^{2\pi} d\theta : \big[\phi(\tau=0,\theta)\big]^4 : \\ &= \frac{g}{4!}\int_0^{2\pi} d\theta \sum_{\ell_1,\ldots,\ell_4} \frac{e^{i(\ell_1+\ell_2+\ell_3+\ell_4)\theta}}{4\sqrt{\omega_{\ell_1}\omega_{\ell_2}\omega_{\ell_3}\omega_{\ell_4}}} :(a_{\ell_1} + a_{-\ell_1}^\dagger)\cdots(a_{\ell_4} + a_{-\ell_4}^\dagger): \\ &= \frac{g}{4!}\frac{\pi}{2}\sum_{\ell_1,\ldots,\ell_4} \frac{\delta_{\ell_1+\ell_2+\ell_3+\ell_4,0}}{\sqrt{\omega_{\ell_1}\omega_{\ell_2}\omega_{\ell_3}\omega_{\ell_4}}} :(a_{\ell_1} + a_{-\ell_1}^\dagger)\cdots(a_{\ell_4} + a_{-\ell_4}^\dagger): \end{aligned} \tag{13}$$

A general Fock space state can be written in occupation number representation as $|\{n_\ell\}\rangle$, where $n_\ell = 0, 1, 2, \ldots$ is the occupation number of the simple harmonic oscillator with angular momentum ℓ. The unperturbed energy of this state is given by

$$H_0\big|\{n_\ell\}\big\rangle = \left(\sum_\ell n_\ell \omega_\ell\right)\big|\{n_\ell\}\big\rangle \tag{14}$$

The idea of Hamiltonian truncation is defined by truncating the Hilbert space to a finite-dimensional subspace, and then diagonalizing the Hamiltonian projected onto this space. The simplest choice is that a basis for the truncated Hilbert space are the eigenvectors of H_0 with eigenvalues below some $E_{\max} \gg m$.

Renormalization/Improvement of 2D ϕ^4 Theory

Note that the truncation described above throws away states that have large energy because they contain a large number of soft particles, as well as states that have large energy because they contain highly energetic particles. Rather than doing this "one-step" truncation to a finite-dimensional Hilbert space, we will first implement a local UV cutoff. This leaves the Hilbert space infinite-dimensional, but allows us to take into account the modes above the UV cutoff in a systematic way, using methods of effective field theory and renormalization. We want to use a UV cutoff that is compatible with the eventual truncation to a finite-dimensional Hilbert space. We will therefore impose a cutoff on the spatial momentum of single-particle modes:

$$|\ell| \le \ell_{\max}. \tag{15}$$

This corresponds to a cutoff on physical (dimensionful) momenta of

$$\Lambda = \frac{\ell_{\max}}{R}. \tag{16}$$

The reason for choosing this cutoff is that it is easy to implement in the Hamiltonian truncation. It means that the occupation number states are now given by a finite-dimensional vector of natural numbers:

$$|\{n_\ell\}\rangle = |n_{\ell_{\max}}, n_{-\ell_{\max}}, \ldots, n_1, n_{-1}, n_0\rangle. \tag{17}$$

The cutoff theory still has an infinite-dimensional Fock space of states, since it contains states with arbitrarily many particles with small $|\ell|$. An important conceptual point is that this cutoff eliminates modes with large energy and momentum *density*. That is, it is a local UV cutoff, and so standard ideas of renormalization theory apply.

It is a general property of quantum field theory that low energies ($E \ll \Lambda$) we can take into account the modes above the cutoff Λ by adding local counterterms to the action. These counterterms are determined by physics above the scale Λ. Above the scale Λ, the theory is weakly coupled, so these counterterms can be computed in perturbation theory. The counterterms can be classified by dimensional analysis. The leading terms are

$$\Delta\mathcal{L} = \tfrac{1}{2}\Delta m^2\phi^2 + \frac{\Delta g}{4!}\phi^4 + \cdots . \tag{18}$$

The counterterms are determined order by order in an expansion in g. The dependence on Λ is then dictated by dimensional analysis:

$$\Delta m^2 \sim \frac{g^2}{\Lambda^2} + \frac{g^3}{\Lambda^4} + \cdots\,, \tag{19a}$$

$$\Delta g \sim \frac{g^2}{\Lambda^2} + \frac{g^3}{\Lambda^4} + \cdots\,, \tag{19b}$$

etc. We are normal ordering, otherwise there would be an additional term $\Delta m \sim g \ln \Lambda$. At higher orders, we can have counterterms that violate Lorentz invariance, because our cutoff violates Lorentz invariance. For example, we will have terms like

$$\Delta \mathcal{L} \supset \frac{g^2}{\Lambda^4}(\partial_0 \phi)^2. \tag{20}$$

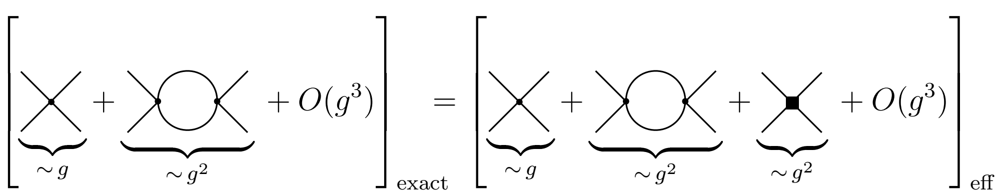
But at order $1/\Lambda^2$, the Lorentz invariant counterterms in Eqs. (19) are all we have.

We want to compute the counterterms. To do this, we move away from the Hamiltonian formulation and consider standard perturbation theory. The theory is defined on a cylinder, so the spatial momentum modes are quantized, but the particle energies are continuous. This means that compared to quantum field theory on $\mathbb{R}^2$ we have

$$\int \frac{d^2k}{(2\pi)^2} f(k) \to \sum_{\ell=-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} f(k^0, \vec{k} = \ell/R). \tag{21}$$

Note we are now labeling the states with plane wave states $\sim e^{i\ell\theta}$ rather than the even/odd circular harmonics. But this does not matter because we are computing counterterms that are independent of how we label the states. Other than this change, we can use the standard Feynman rules. To implement the cutoff, we simply restrict the sum over ℓ to $|\ell| \leq \Lambda$.

For example, at one loop for the 4-point function we have



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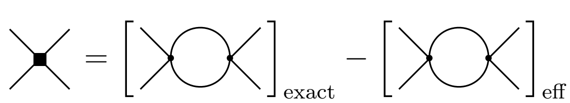
The last term on the right-hand side represents a counterterm in the effective (cutoff) theory. In other words, we have



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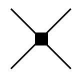
The counterterm in the effective theory is local because it depends only on momenta above the cutoff. This gives (setting $R = 1$ for convenience)



$$= -\Delta g = \tfrac{3}{2} g^2 \sum_{|\ell|\,>\,\Lambda} \int \frac{dk^0}{2\pi} \frac{1}{k^2}, \tag{24}$$

where $k^2 = (k^0)^2 + \ell^2$. Note that we are evaluating the diagram with vanishing external momentum and using the approximation $m^2 = 0$. This is because the counterterm has no derivatives, so we don't care about the momentum. Also, the terms that depend on m^2 are higher order in the $1/\Lambda$ expansion. I tried to get the signs and factors right in Eq. (24). The integral is elementary:

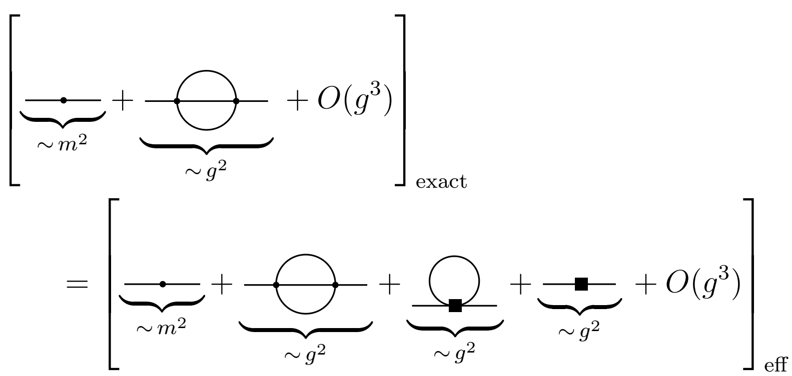
$$-\Delta g = \frac{3g^2}{8} \sum_{|\ell|\,>\,\Lambda} \frac{1}{|\ell|^3}. \tag{25}$$

The sum can be performed exactly in terms of a polygamma function:

$$\sum_{|\ell|\,>\,\Lambda} \frac{1}{|\ell|^3} = 2 \sum_{\ell\,=\,\Lambda+1}^{\infty} = -\psi^{(2)}(\ell_{\max} + 1) = \frac{1}{\Lambda^2} - \frac{1}{\Lambda^3} + O(\Lambda^{-4}). \tag{26}$$

This gives There is no point in keeping the higher order terms in $1/\ell_{\max}$ unless we are also keeping the higher loop diagrams. Since we are interested in the expansion of this result for large Λ, we can also use the Euler-McLauren formula to approximate the sum by an integral with corrections.

For the mass renormalization we have



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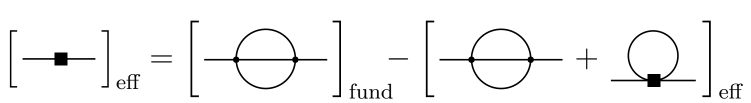
Note that in the effective theory there is an additional diagram that involves the $O(g^2)$ counterterm computed above. We have



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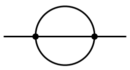
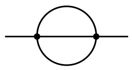
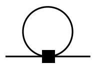
We have





$$\Big[\quad\Big]_{\text{fund}} - \Big[\quad\Big]_{\text{eff}} = \frac{g^2}{6}\left(\sum_{\ell,\ell'} - \sum_{|\ell|,|\ell'|\leq\Lambda}\right)\int \frac{dk^0}{2\pi}\frac{dk'^0}{2\pi}\frac{1}{k^2k'^2(k+\ell)^2}, \tag{29a}$$



$$\Big[\quad\Big]_{\text{eff}} = -\frac{\Delta g}{2}\sum_{|\ell|\leq\Lambda}\int\frac{dk^0}{2\pi}\frac{1}{k^2}. \tag{29b}$$

Note that Eq. (29a) contains terms in the sum with $|\ell| \ll \Lambda$ and $|\ell'| > \Lambda$ (and *vice versa*). These terms are simultaneously sensitive to the UV and IR of the theory; this is called an overlapping divergence. The claim is that this overlapping divergence is canceled by Eq. (29b).

The integral in Eq. (29a) is elementary (meaning `Mathematica` knows it):

$$\int d\omega\, d\omega' \frac{1}{(\omega^2+\ell^2)(\omega'^2+\ell'^2)((\omega+\omega')^2+(\ell+\ell')^2)} = \frac{\pi^2}{2\ell\ell'(\ell+\ell')}. \tag{30}$$

We still have a double sum to perform...