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1. Stochastic G.W. Background (SGWB)

- Linearized Einstein Equation:

$$\Box h_{\mu\nu} = \frac{-16\pi G}{c^4} T_{\mu\nu} \quad \rightarrow (1)$$

$\partial^\mu \partial_\mu$

-> Expand GR through small perturbations in flat background

-> 2 dof propagating: 2 polarization of spin(2) -> G. Wave.

-> T.T. Gauge: $h^{TT}_{0\mu} = 0$, $h^{TT\,i}_{i} = 0$, $\partial^i h^{TT}_{ij} = 0$

$$\equiv \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_\times & 0 \\ 0 & h_\times & h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{(for } 3\times3\text{)}$$

- Einstein Eqn: $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \rightarrow (1^*)$

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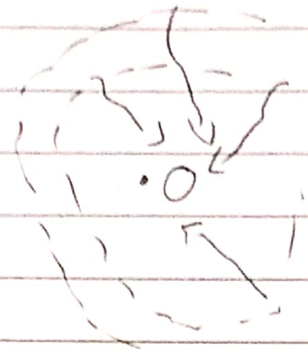
SGWB



→ Observer looking at G wave

- Detector: Super position of GW's with $\vec{k}_i$

- Sources: Both Astro-physical and 'Cosmological'

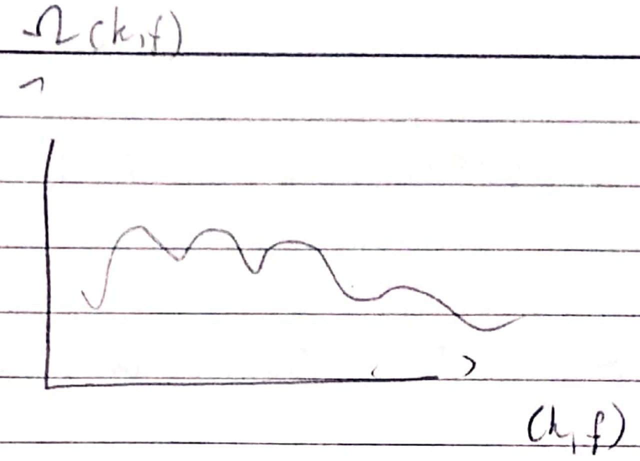
- Detectors have viewing volume.

- Events outside this volume will appear as noise.

- typically: isotropic, unpolarized and gaussian

Spectral Shape: $\Omega_{GW} = \underbrace{\left(\frac{d\,\rho_{gw}}{d\ln(k)}\right)\frac{1}{\rho_c}}$

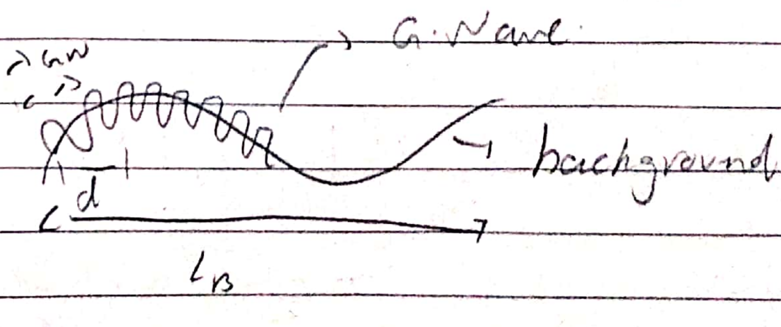
How much energy in G Waves per 'ln k' interval.



- Different volumes <-> simply correspond to different times <-> cosmological history book
- "To look for additional noise"
  <-> analogous to CMB measurements.

☆☆ Energy Momentum Tensor of G. Waves.

- G.Wave Radiation in curved space time!



Separation only possible if $L_B \neq \lambda_{GW}$

[Scale separation

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- Let's look at $\lambda_{GW} \ll d \ll L_B$, $[f_{GW} \gg f_B]$

- Average over $d \rightarrow$ Only obtain background contribution
- Subtract from total $\rightarrow$ G. wave (λ_{GW})

- Expand $G_{\mu\nu} \rightarrow$ in powers of h

$$G_{\mu\nu} = \underbrace{G_{\mu\nu}^{(B)}}_{O(h^0)} + \underbrace{G_{\mu\nu}^{(1)}}_{} + \underbrace{G_{\mu\nu}^{(2)}}_{O(h^2)} + \cdots \rightarrow (2)$$

$O(h^0)$
long length
small k
(L_B)

large k
(λ_{GW})

$O(h^2)$
[can have
both]
↓
No longer
linearized.
$(k_{small}$ & $k_{large})$
$\Rightarrow |\vec{k}_1 + \vec{k}_2| \simeq 0$, $k_1 \simeq -k_2$

Sub (2) in (1^*) $\rightarrow$ focus on small k

$$G_{\mu\nu} = G_{\mu\nu}^{(B)} + [G_{\mu\nu}^{(2)}]$$

$$\Rightarrow \quad G_{\mu\nu}^{(B)} = -[G_{\mu\nu}^{(2)}] + \frac{8\pi G}{c^4}[T_{\mu\nu}]^{\bar{h}\text{ small}}$$

$$= \underbrace{-\langle G_{\mu\nu}^{(2)}\rangle_d}_{\text{Extra term from G-wave superposition.}} + \frac{8\pi G}{c^4}\underbrace{\langle T_{\mu\nu}\rangle_d}_{\text{E-P tensor of background}}$$

$$= \frac{8\pi G}{c^4} t_{\mu\nu}$$

Energy density curves the background.

☆ Explicit computation of $G_{\mu\nu}$ at 2^{nd} Order

[Maggiore : Vol 1.]

$$\langle R_{\mu\nu}^{(2)}\rangle_d = -\frac{1}{4}\langle \partial_\mu h^{TT}_{\alpha\beta}\, \partial_\nu h^{TT\,\alpha\beta}\rangle_d ,$$

$$\langle R^{(2)}\rangle = 0 \, , \quad \langle R^{(1)}\rangle = 0$$

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$$:) \; t_{\mu\nu} = \frac{c^4}{32\pi G} \langle \partial_\mu h_{\alpha\beta}^{TT} \partial_\nu h^{\alpha\beta \, TT} \rangle_d$$

$$☆ \quad \rho_{GW}(t_{00}) = \underbrace{\frac{c^4}{32\pi G} \left(\langle \dot{h}_{ij}^{TT} \dot{h}^{ij\,TT} \rangle \right)}$$

2-point function (Stochastic Background)

(3☆)

☆☆ GWs in FRW universe [$c=1$, $\hbar=1$]

$$ds^2 = -dt^2 + a^2(t)\, dx^i dx_i = a^2(\tau)(d\tau^2 - dx^2)$$

↳ (4)

→ Impose on (1) :

$$\Box \bar{h}_{\mu\nu}(\vec{x},\tau) - \underbrace{2\frac{\dot{a}}{a}\bar{h}'_{\mu\nu}(\vec{x},\tau)}_{\substack{=0 \text{ for flat} \\ \text{static} \\ \text{universe}}} = -16\pi G T_{\mu\nu}$$

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* F.T (for k space) $\longrightarrow$

notation: $\tilde{h} \equiv a h$, $\lambda = +, \times$

$$\Rightarrow \tilde{h}''_\lambda(\vec{k}, \tau) + \Big(k + \overbrace{\frac{a''}{a}}^{\equiv (aH)^2}\Big) \tilde{h}_\lambda(\vec{k}, \tau)$$

$$= 16\pi G \underbrace{T_\lambda(\vec{k}, \tau)}_{\Lambda_{\lambda,\mu\nu} T^{\mu\nu}}$$

} E-p tensor projected onto the +, × polarization

Case 1: $k >> aH \rightarrow \frac{\dot{a}}{a}$ [hubble parameter] [sub-horizon scales]

$\hookrightarrow$ $\tilde{h}''_\lambda + k \tilde{h}_\lambda = 0$ [wave eqn]

$\hookrightarrow$ $h = \frac{A}{a} \cos(k\tau + \phi)$ { $\frac{1}{a}$ scale factor amplitude switch

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Case 2: $k << aH$ (Super-Horizon)

$$2a'h_\lambda' + a h_\lambda'' = 0 \qquad [T_{\mu\nu} = 0 \text{ in vacuum}]$$

$$\hookrightarrow h_\lambda = A_\lambda + B_\lambda \underbrace{\int_0^{\tau} \frac{d\tau'}{a^2(\tau')}}_{=0 \text{ for evolving } a^2(\tau)}$$

$\Rightarrow$ $h_\lambda = A_\lambda \simeq$ const.

$\Rightarrow$ "GWs are frozen outside the Hubble horizon."

A useful parametrization:

$$h_{ij}^{TT}(\vec{x},\tau) = \sum_{\lambda=+,\times} \int d^3k \; h_\lambda(\vec{k}) \, T_k(\tau) \times e^\lambda_{ij}(\hat{k}) \exp[(-i(k\tau - \vec{k}\vec{x})) + h.c.]$$

$\hookrightarrow$ 5

here, $e_{ij}^{\vec{}}(\hat{k}) \longleftrightarrow$ polarization tensor

$\underbrace{T_k(\tau)\, e_{ij}^{\vec{}}(\hat{k})}_{\text{expansion of the universe,}} \longleftrightarrow$ Transfer function

- $T_k(\tau) = \dfrac{a(\tau_*)}{a(\tau)}$ $\rightarrow$ time of formation / enters into horizon.

- $h_\lambda(\vec{k}) \longleftrightarrow$ Primordial Fourier co-efficient

- This splitting is due to $a(t)$ scale evolution.

☆ Sub ⑤ in (3^*) while applying homogeniety and isotropy.

- $$\langle h_\lambda(\vec{k})\, h_{\lambda'}(\vec{k}')\rangle = 2\pi^3\, \delta_{\lambda\lambda'}\, \underbrace{\delta(\vec{k}-\vec{k}')}_{\text{homogeniety}}\, \underbrace{P_\lambda(|\vec{k}|)}_{\text{isotropy}}$$

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$\Rightarrow$

$$\rho_{GW}(\tau_0) = \frac{1}{32\pi G} \frac{1}{\pi^2 a^2(\tau_0)} \int k^2 dk \underbrace{\sum_\lambda P_\lambda(k)}_{\text{Source information. [primordial power spectrum]}} \underbrace{\frac{a^2(\tau_x)}{a^2(\tau_0)}}_{\text{transfer function (history)}}$$

$$= \rho_c \int d\ln k \underbrace{\frac{1}{\rho_c}\left(\frac{\partial \rho_{GW}}{\partial \ln k}\right)}_{\Omega_{GW}(k,\tau_0)}$$

$\Omega_{GW}(k,\tau_0)$ → model prediction

GW spectrum.

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Eg: Single field slow-roll inflation: (10^{-35} s after bang)

$$\Delta_t^2 = \frac{k^3}{2\pi^2} \sum_\lambda P_\lambda \sim \frac{H_{inf}^2}{M_{planck}^2} \sim \text{const.}$$

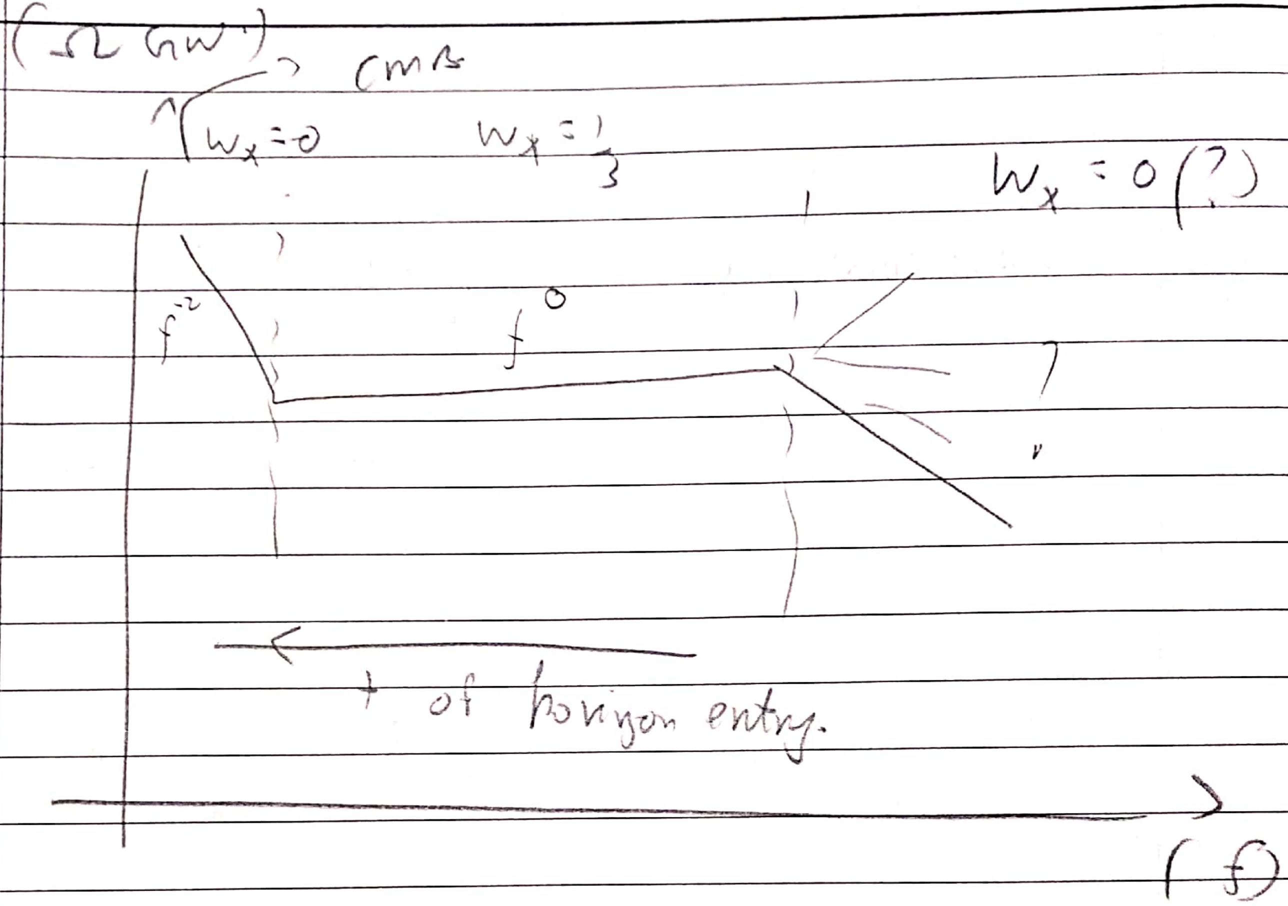
$$\Omega_{gw}^0 = \frac{\Delta_t^2}{12} \frac{k^2}{a_0^2 (H_0^2)} \left(\frac{a(\tau_{*,k})}{a(\tau_0)} \right)^2$$

↳ · diff $k \leftrightarrow$ diff τ_*

· $a_* H_* = k$

$$\Rightarrow \Omega_{GW}^0 = \frac{\Delta_t^2}{12} \underbrace{\frac{k^2}{(a_* H_*)^2}}_{1} \underbrace{\frac{a_*^4 H_*^2}{a_0^4 H_0^2}}_{\text{only variation.}}$$

but $H = \frac{\dot{a}}{a}$, evaluate in Λ-CDM



$f \sim 10^{-16}$ Hz

Cosmological Sources

Characteristics of Relic G.Ws

- Redshift : $f_0 = f_* \dfrac{a(\tau_*)}{a(\tau_0)}$, $\rightarrow$ Sub

$f_* = (\epsilon_* H_*^{-1})^{-1}$ $\mapsto$ Impose for relevant cosmic energy scales

$\epsilon_* < 1$ for causality

- Radiation Era: $H_*^2 = T_*^4 / m_*^2$ $\leftarrow$ $\left\{ \begin{array}{l} 10^{17}\ \text{GeV} \\ a \sim \frac{1}{T} \end{array} \right.$

- $\Rightarrow f_0 \simeq 10^{-8} \epsilon_*^{-1} \left(\dfrac{T_*}{\text{GeV}} \right) \text{Hz}$

- $\Rightarrow t_* = 10^{-22}\,\text{s}\ \epsilon_*^{-1} \left(\dfrac{1\,\text{Hz}}{f_0} \right)$

★ $\epsilon_* = 1$ | $\epsilon_* = \rho_* c^2$

$f_0\ \text{Hz}$

- $\epsilon_* = 1$

	f_0 [Hz]	T_* [GeV]
PTA	10^{-8}	1 Gev
LISA	10^{-2}	10^5 GeV
LIGO	10^2	10^9 Gev

☆ Caveat: Higher Energy => decayed amplitude of gravitational wave <-> harder to see

☆☆ Constraints: BBN & CMB

- "extra radiation" <-> anything BSM

☆ after e^- decoupling: $\rho_{rad} = \frac{\pi^2}{30} (\underbrace{**}_{dof}) T^4$

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$\Rightarrow$ $** = \underbrace{2}_{\gamma} + \frac{7}{4} N_{eff} \left(\frac{4}{11}\right)^{4/3})$

$\left(\frac{4}{11}\right)$ → e^+e^- only heat γ plasma

$\frac{7}{4}$ ⟷ fermionic dof becomes exotic

N_{eff} ⟷ eff. dof

SM; $N_{eff}^{sm} = 3.046$

☆ Extra Radiation; $N_{eff} = N_{eff}^{sm} + \Delta N_{eff}$ (model dependent)

$$\Rightarrow \frac{\rho_{GW}(T)}{\rho_\gamma(T)} < \frac{\rho_{rad}^{obs} - \rho_{rad}^{sm}}{\rho_\gamma(T)} << \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} \Delta N_{eff}$$

at BBN, CMB decoupling: $\Delta N_{eff} \lesssim 0.2$

$$\Rightarrow \left.\frac{\rho_{GW}}{\rho_\gamma c}\right|_{T_{BBN/CMB}} \leq 10\%$$

Since both radiations redshift the same way ⟷ the bound is unchanged.

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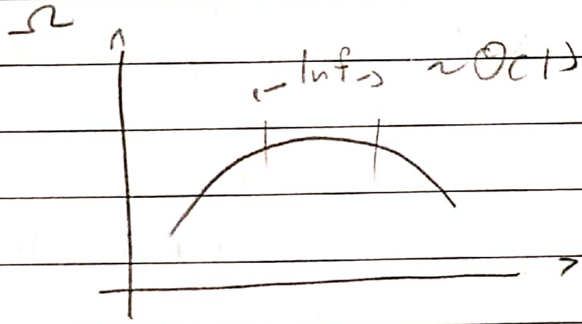
• today,

$$\Omega_\gamma = \frac{\rho_\gamma}{\rho_{tot}} \sim 10^{-5} \Rightarrow \rho_{GW} \lesssim \Omega_\gamma \Delta N_{eff} \rho_{tot}$$

☆ $$\Rightarrow \boxed{\frac{\rho_{GW}}{\rho_{to}} \lesssim 10^{-6}}$$

☆ Ω_{GW} [broad spectra]



$$\Rightarrow \boxed{\Omega_{GW} \lesssim 10^{-6}}$$ [for GWs inside horizon at $T_{BBN/CMB}$]

Serious constraint on Early Universe Models

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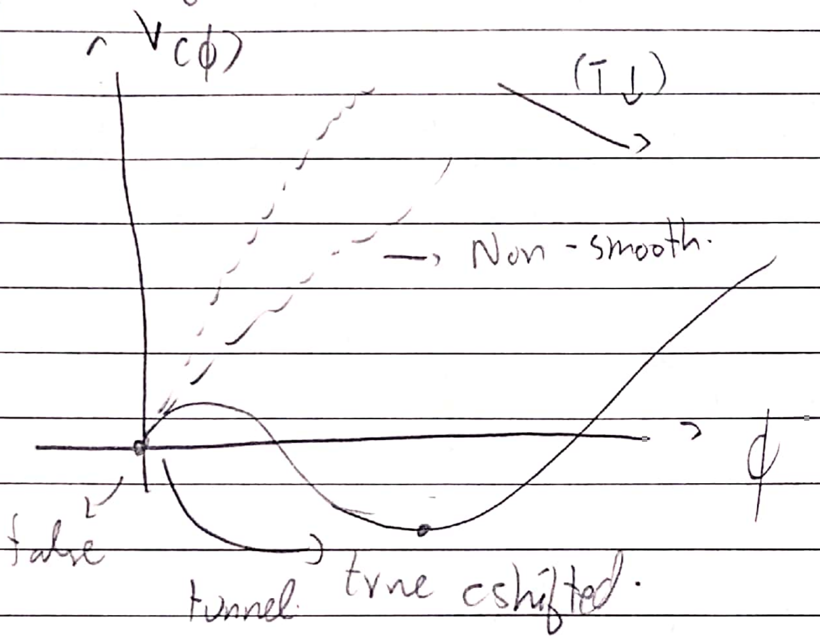
☆ This bound is not frequency dependant

☆☆☆ Cosmological Sources: 1st Order Phase transitions



SM ⟷ has no quantum tunnelling.

↳ bubbles of true vacuum form in the false vacuum phase

("violent interactions")

☆☆ Sources:

- Bubble Collisions
- MHD turbulence
- SOUND WAVES

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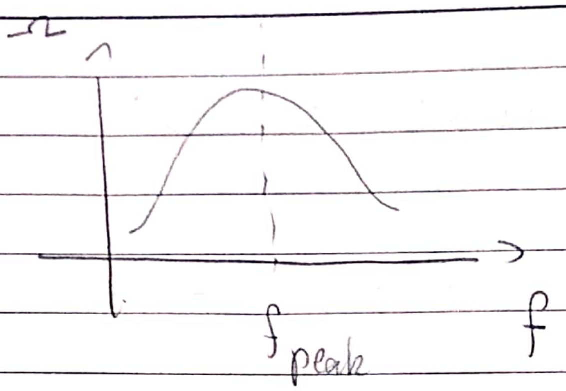
☆ Sketch:



- $f_{peak} \sim 10^{-3}\ Hz \left(\frac{T}{100\ GeV}\right)$ <--> LISA limit.
  will probe EWPT

- $\Omega_{GW}^{MAX} \gtrsim 10^{-6}$ (depending on PT)

☆ Shape of the potential:

- To have a violent enough phase transition will imply,

That there needs to be a potential barrier big enough to not allow any classical tunnelling.

$V(\phi) \equiv \mu^2 |\phi|^2 + \lambda |\phi|^4$, but

μ^2 & λ need to be large enough to measure enough value to fit our bounds. ($\Omega \leq 10^{-6}$)

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- Order parameter: VEV(Φ)

i.e., VEV(ϕ) = 0 $\longleftrightarrow$ VEV(ϕ) $\neq$ 0

Ordered — Disordered ("Violent") phase

II. Gravitational Waves from 1st order Phase transitions

☆ There are no calculable P.T. which generate GWs in the standard model. But, BSM models can be defined to allow for this to happen.

☆☆ Relevant Parameters of the SGWB

- We require a scalar potential, $V(\phi, T)$ at a finite temperature.

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(a) Rate of variation of Bubble nucleation: β

β → size ($R_b \simeq v_b \beta^{-1}$)

→ duration (β^{-1})

(b) G.W Spectrum: $\alpha = \frac{\epsilon}{\rho_{rad}} = \frac{\rho_{vac}}{\rho_{rad}}$, $\epsilon \leftrightarrow$ latent heat

Derived

(c)* Nucleation Rate: $\Gamma(t) = A(t)\, e^{-S_c(t)}$

$S \rightarrow$ euclidean action (S_4 or S_3/T)

(d)* $\beta \equiv \left. \frac{-dS}{dt} \right|_{t^*} \simeq \left. \frac{\dot{\Gamma}}{\Gamma} \right|_{t_*} \equiv \frac{d}{dt} \ln(\Gamma(t))$

⊕ For EW scales, $S_c \simeq 140$

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S_c (const) : $141 + \log(A/T^4) - 4\log\left(\frac{T}{100\,GeV}\right) - \log\left(\frac{\beta/H}{100}\right)$

S_c (percolation) : $131 + \log(A/T^4) - 4\log\left(\frac{T}{100\,GeV}\right) - 4\log\left(\frac{\beta/H}{100}\right) + 3\log(v_w)$

EWPT for Baryogenesis [radiation dominated]

x.(c) $\frac{\beta}{H_*} = T_* \frac{dS}{dT}\Big|_{T_*}$

f. Determining these parameters help set specific bounds for the model.

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★★ **Model Specific Overview**

(a) **Singlet extension of the SM**

★ $$V(H,S) = -\mu^2 |H|^2 + \lambda |H|^4 + \frac{a_1}{2} |H|^2 S + \frac{a_2}{2} |H|^2 S^2$$

$$+ b_1 S + \frac{1}{2} b_2 S^2 + \frac{1}{3} b_3 S^3 + \frac{1}{4} b_4 S^4$$

At EWSSB, $H = \left[0, \frac{(h+v)}{\sqrt{2}}\right]^T$, $v = 246$ GeV

★ $$\left.\begin{aligned} h_1 &= h\cos\theta - s\sin\theta \\ h_2 &= h\sin\theta + s\cos\theta \end{aligned}\right\} \text{ mixing.}$$

★ $|\sin\theta| \leq 0.2 - 0.3$ [EW mixing angle]

(assuming, $m_1 = 125$ GeV.

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