Group Theory in Particle Physics: Classifying the Particle Zoo

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Abstract

In this article we discuss the various groups and it's applications in Particle Physics. Symmetries form a very important part of most physical principles and the study of elementary particles is no different. Groups share a fundamental kinship with the notion of symmetry and as such with every symmetry there is an associated group, which maybe discrete or continuous depending on the nature of the symmetry. Here we concentrate on the continuous lie groups which play an essential role in understanding the framework of the Standard Model of Particle Physics. All the elementary particles can be neatly arranged under some particular representations of these Lie groups which allude to their properties and transformation rules. The forces and interactions between these elementary particles are also governed by their representations under these Lie groups along with their respective quantum numbers.

1 Introduction

In the last century, one of the emerging problems in modern physics has been the constant discovery a new particles. At times seeming overly disorganized and unrelenting in it's lack of order. However as the divisions and spin-statistics of particles gave rise to Bosons and Fermions, newfound clarity in the realm of particle physics pointed towards small finite fundamental set of particles. But unfortunately a lack of relevant mathematical machinery held it back. In 1930, Weyl, Wigner and Van der Waerden bought the theory of group-representations into atomic physics[1] thereby sparking a revolution that continues to affect the field of particle physics till date. Among all the hub-bub of group theory, perhaps the most profound and relevant one was the addition continuous groups and the development of Lie Group theory. This paper seeks to introduce the very same concept and build up the notion of continuous groups to the point of the current Standard Model of Physics.

We begin by introducing the concept of Groups and the the basic axioms, move on to define certain terms that are relevant to this paper. We then develop the notion of group representations and how the abstractness of the group can actually live in Vector spaces. Once a decent base has been set we delve further into Lie Group theory.

Starting with a motivation of Analytical Manifolds and Topological spaces in order to form a rigorous understanding of these groups. We build upon these to define particular Lie Groups heirarchically and land at the relevant groups required to describe the Standard Model of Particle Physics.

In this paper we concentrate more on understanding the representational flexibility of groups and how the various couplings feed into developing notions of classifying our particle zoo so to speak. Concentrating heavily on $SU(2)$ group to show lie group representations and generators as basis vectors. Then we motivate the physical meanings of this group using the $SU(3)$ group and finally give a brief summary of the different sectors within the Standard model gauge group, while cautioning against the fact that Group theory while elegant only forms a part of the puzzle. We end on a note of hope for furthering group theory studies in the pursuit of an all encompassing Standard Model of Physics.

Fair warning, however detailed some portions of this paper, this is atmost aimed to be a primer for more fields related to this topic and a simple place setting for future work.

2 Groups

A Group as mathematical machinery is defined as any set equipped with a binary operation such that it follows certain axioms.

Formally, a Group $(G, *)$, is any set of objects, denoted G with a binary operation $*$ defined on it such that,

1. The operation observes *closure*, $\forall\ g_1, g_2 \in G,\ g_1 * g_2 \in G$

2. The operation is *Associative*, $\forall g_1, g_2, g_3 \in G,$ it must be true that $(g_1 * g_2) * g_3 = g_1 * (g_2 * g_3)$.

3. The group has an identity element, $\exists\ g \in G,$ denoted $e, \ni\ \forall g_i \in G,$ $e * g_i = g_i * e = g_i$

4. There exists an inverse, $\forall\ g \in G, \exists\ g \in G, \ni\ h{*}g = g{*}h = e,\ (\text{so}, h = g^{-1})$

The definition of what an object is quite liberal in this scenario as long as you're able to form a set of those objects attached with it a binary operation (also liberal), that obeys the aforementioned axioms[1].

2.1 Useful Definitions

The goal of this section is to go over some of the basic yet, important concepts[2] within group theory. We will add onto this in another similar section when we come to Lie groups. All these definitions are taken largely from [2],[3]

- **Abelian group**: A group is said to be *Abelian* if all the elements commute with one another, i.e., $\forall g_i, g_j \in (G, *),\ g_i * g_j = g_j * g_i$.

- **Order of the group**: The total number of elements in a group is defined as it's order. Using order as a criterion a group maybe classified as *finite or infinite, countable or non-countable infinite.*

- **Subgroup**: A subset H of the group $(G, *)$, containing elements $g_1', g_2'...,$ that in itself forms a group under the same operation $*$, is called as a *subgroup*[3] H of G. The identity and whole group are always trivial subgroups of any group G.

- **Homomorphism**: A mapping of a group $G \mapsto G'$ is said to be homomorphic if it preserves the products. That is given $h : G \mapsto G',\ \forall g_i, g_j \in G,$ $h(g_i * g_j) = h(g_i) * h(g_j)$.

- **Group Action**: A group action on a space is the homomorphism of the given group into the transformation group of the space. [4]

- **Direct Product**: For two groups G and H, we can form the **Product Group** denoted $K \equiv G \otimes H$, where an arbitrary element of K is (g_i, h_j). This operation is known as the *direct product*.

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[1] Although the axioms themselves do not explicitly state it, it is elementary to note that both the inverse and identity have a uniqueness to them.

[2] While not all definitions will be used explicitly, it is still important to go through them as this paper serves to only be a primer for future work.

[3] $\forall g_i', g_j' \in H$, A necessary and sufficient condition for H to be a subgroup of G, is to have $g_i' * g_j^{'-1}$ also belong to H.

- **Direct Sum**: Take any n-dimensional space V spanned by a basis of n vectors. For the subspaces U and W of V,
  if $\forall\ \bar{v} \in V, \quad \bar{v} = \bar{u} + \bar{w}$, where $\bar{u} \in U$ and $\bar{w} \in W$, and every operator X acting on V can be split into individual parts acting on U and W, we say that V is the **Direct Sum** of U and W denoted, $V = U \oplus W$.

2.2 Representations

Upto this point, we have only talked about groups and it's various aspects in typically abstract terminology. The goal however is to be able to address these abstract objects in terms of numbers and similar entities. We call these 'numbers' as the **Representations** of a group. Please note however that a group can have multiple representations as long as it's inherent structure is preserved. A useful idea that will play a big role later.

For our paper we will choose to form our representations by having our group act on some n-dimensional vector space.
Using formal notation,
For a group $(G, *)$, with element $e, g_1, g_2, g_3, ..$, let's call the Representation of the group $D(G)$ such that the elements of G are represented by $D(e), D(g_1), D(g_2), ...$
Then for some fixed basis on this vector spaces this allows us to say that the Representation $D(g_i)$ is a matrix of some dimension. To go along with this we then choose our $*$ to be matrix multiplication.
Thus, $D(g_i)D(g_j) = D(g_i * g_j)$

While this doesn't seem in anyway profound, the fact remains that by simply choosing a convenient space to act our group on and affixing a basis we have infact created a very familiar lens through which we can see into this abstract world. Further more, by having $D(g_i)D(g_j) = D(g_i * g_j)$, we aren't losing any of the structure of our abstract group by using such a representation.
While there are many more facets to this Representation theory we have introduced here, they unfortunately aren't particularly relevant right now. We will see a more rigorous treatment of how to apply this concept in relation to our Continuous Lie Groups in later sections

3 Lie Groups

Lie Groups refer to a very particular special type of mathematical structures that can be parametrized by one or more continuous variables. They were named after Norwegian mathematician Sophus Lie(1842-1899) who laid the foundations of the theory of continuous groups.[5]. Sophus intended to use it to model the continuous symmetries of differential equations, similar to the usage of finite groups in Galois Theory to model discrete symmetries of algebraic equations.
Lie Groups provide a very intuitive and natural notion of continuous symmetries and as such are very useful to physicists.

3.1 Formal Definition of a Lie Group

Before we get to a formal and rigorous definition of lie groups, let's build upon our previous definitions section here by introducing a couple key terms.[4] All these definitions are referenced from [3].

- **Topological Space**: A non-empty set S for which there is a collection τ of subsets(open sets), is called a topological space with a topology τ defined on it if it satisfies the following conditions:
  1. Both the null set and the whole set S belong to τ
  2. A union of any arbitrary number of sets in τ also belongs to τ
  3. An intersection of any combination of a finite number of sets in τ belongs to τ .
- **Hausdroff Space**: A topological space S with a topology τ which satisfies the *separability axiom*[5] is called a *Hausdroff* space.
- **Metric Space**: A metric space is a particular type of Hausdroff Space on which you define a *distance function* $d(P, P')$ between a set of points $P, P' \in S$. The distance or *metric* is real-valued and is made to satisfy the following axioms.
  1. $d(P, P') = d(P', P)$
  2. $d(P, P) = 0$
  3. $d(P, P') > 0,\ if\ P \neq P'$
  4. $d(P, P') \leq d(P, P'') + d(P'', P'),$ for any three points of S.
- **Second Countable Space**: Given a topological space S with topology τ. If every open set of τ can be expressed as a union of a countable collection of open sets contained in τ, the space is said to be second countable.
- **Homeomorphic Mapping**: Given two topological Spaces (S, τ), and (S', τ').
  For a mapping, $\phi : S \mapsto S'$, if $\forall$ open set $V \in S$, there exists an openset $\phi(V)$ in S', the mapping is said to be *open*
  For such a mapping ϕ if, $\forall$ open set V' of S', the set $\phi^{-1}(V')$ is an open set of S, then the mapping is *continuous.*
  Putting them together, if the mapping $\phi : S \mapsto S'$, is both continuous and open, it then a *homeomorphic* mapping.
- **Locally Euclidean Space**: For a Hausdroff topological space $\mathcal{V}$, if each point of $\mathcal{V}$ is found to belong to an open set homeomorpically mapped to a subset of $\mathbb{R}^n$, the topological space is then called a *locally Euclidean space of dimension n.*

[4] Again, not all of these might be explicitly used within the scope of this paper,but it still helps to garner a broader knowledge of these related topics.

[5] Any two distinct points of S belong to disjoint open subsets of τ.

- **Chart**: Given an open set V of $\mathcal{V}$ and a homeomorphic map $\phi : V \mapsto K$, where $K \subset \mathbb{R}^n$. One can define a set of coordinates $(x_1, x_2, ..., x_n)$ for each point $P \in V$, $\ni$ $\phi(P) = (x_1, x_2, ..., x_n)$; then the touple (V, ϕ) is called a *chart*.

- **Analytic Manifold of Dimension n**: Given a second countable locally Euclidean space $\mathcal{V}$ of dimension n, , and the homeomorphic map $\phi : V \mapsto K$, where $K \subset \mathbb{R}^n$:
Then for ever pair of charts (V_α, ϕ_α) and (V_β, ϕ_β) of $\mathcal{V}$ with a non-empty $V_\alpha \cap V_\beta$, if the mapping $\phi_\beta * \phi_\alpha^{-1}$ becomes an analytic function, then $\mathcal{V}$ is called an analytic manifold of dimension n.[6]

Now that we have our terms in place we can go ahead and give a formal definition of a Lie Group.[3]

A **Lie Group** is a type of mathematical machinery that,

1. Forms a groups as per the group *axioms* mentioned in section 2.

2. The elements of the group constitute a *Topological Space*.[7]

3. The elements of the group form an *analytical manifold* of some arbitrary dimension such that we can define the two analytical mappings,

   (a) $\forall\ g_i, g_j \in (\mathcal{G}, *),\ \phi(g_i, g_j) = g_i * g_j\ (\mathcal{G} \times \mathcal{G} \mapsto \mathcal{G})$
   (b) $\forall\ g_i \in (\mathcal{G}, *),\ \phi(g_i) = g_i^{-1}\ (\mathcal{G} \mapsto \mathcal{G})$

Now that we have our broad, rigorous definition, we can go ahead to look at how we classify these Lie Groups into the particular ones that pique our interest as mentioned in the introduction.

3.2 Classification of Lie Groups

We take a hierarchical approach to the classification of Lie Groups.

To start with, let's consider the most general possible Lie group in an arbitrary number of dimensions, n. All this group does is it takes some random point p in the n-dimensional space, and continuously moves it some other random point within the space. Implying that we can have literally any $n \times n$ matrix relating to a linear transformation, provided it's invertible. We define this group of all $n \times n$ singular matrices, in n dimensions as the **General Linear** group, denoted $GL(n)$.
To tie this space over some basis, we take the elements of these matrices from the Complex field,$\mathbb{C}$. Thus we have a specific version of this general group - $GL(n, \mathbb{C})$, which contains all $n \times n$ non-singular matrices with complex elements, such that all points in $\mathbb{C}^n$ stay in $\mathbb{C}^n$, under the group actions.

[6] A simple example of such a manifold is $\mathbb{R}^n$.
[7] So the group can also be considered as a special case of a topological group

Since we are talking a hierarchical approach, every other group we define is a subgroup of this large overbearing group whose elements are defined over the $\mathbb{C}$ field.

A somewhat subgroup of $GL(n,\mathbb{C})$ is $GL(n,\mathbb{R})$, which would contain all $n \times n$ real non-singular matrices that takes all points in $\mathbb{R}^n$ and leaves it in $\mathbb{R}^n$. To take this a step further let's consider a group such that it preserves the volume within the space. This can be achieved by thinking of preserving the n-parallelotope spanned by our n-vectors in $\mathbb{C}^n$. If we further stipulate that all the general linear transformations transform vectors from the origin, our vectors which span the parellelotope become points. This means that we can only consider all general linear matrices with determinant 1[2]. Luckily, because $det|A.B| = det|A|det|B|$[8], these particular matrices do form a group.We now call this subgroup, the **Special Linear** group denoted, $SL(n,\mathbb{C})$.
The trivial subgroup of this is ofcourse $SL(n,\mathbb{R})$. This group preserves not only the points in the spaces but also the volume.

Now let's consider further, another type of transformation in a n-dimensional vector space using generalized Euler angles. Apart from volume and length, these transformations also leave the square of the radius(r^2) invariant.
Using vector notations,

$$\vec{r}^2 = \vec{r}^T.\vec{r}$$

if we transform this with a rotation matrix R.

$$\begin{aligned} \vec{r} \mapsto \vec{r'} &= R\vec{r} \\ \vec{r}^T \mapsto \vec{r'}^T &= \vec{r}^T R^T \\ \implies \quad \vec{r'}^T.\vec{r'} &= \vec{r}^T R^T.R\vec{r} = \vec{r}^T.\vec{r}, \quad \text{using } r^2 \text{ invariance} \\ \implies \quad R^T.R &= \mathbb{I} \end{aligned}$$

This tells us that the matrices of this particular group are orthogonal. Thus we call this group of generalized rotations as the **Orthogonal** group denoted, $O(n)$.[9]
Also note that since, $\det|R^T \cdot R| = \det|\mathbb{I}| \Rightarrow (\det|R|)^2 = 1 \Rightarrow \det|R| = \pm 1$. We choose the subgroup with $\det|R| = +1$, as the **Special Orthogonal** group denoted, $SO(n)$. Choosing $\det|R| = -1$, simply chooses a particular handedness of the system that doesn't preserve parity, thus by choosing $+1$, we preserve space,radius,volume and parity(handedness) of the space.

In terms of vecotrs in the $\mathbb{C}$ space, it's more relevant to discuss a complex representation of the invariant squared radius. Thus we write, $\vec{r^2} = \vec{r}^\dagger \cdot \vec{r}$, where the dagger denotes the Hermitian conjugate, $\bar{r}^\dagger = (\bar{r}^\star)^T$, where $\star$ denotes complex conjugate.

[8]So, if $det|A| = 1$ and $det|B| = 1$, then $det|AB| = 1$.
[9]There is no need to denote the field here as it is always defined over $\mathbb{R}$.

Using a complex Rotation matrix, we get, $\bar{r} \to R\bar{r}$, and $\bar{r}^\dagger \to \bar{r}^\dagger R^\dagger$.
Thus, $\bar{r}^\dagger \cdot \bar{r} \to \bar{r}^\dagger R^\dagger \cdot R\bar{r}$, similar to the orthgonal matrice treatment above this implies that, $R^\dagger \cdot R = \mathbb{I}$, or $R^\dagger = R^{-1}$. We denote this particular group containing such $n \times n$ invertible unitary matrices as the **Unitary** group denoted, $U(n)$.
And as above, we end up with a special sub group with, $\det |R| = +1$ called **Special Unitary** groups denoted, $SU(n)$.

Although we've finished all the major groups of interest for us under the scope of this paper, let us do one more for posterities sake. We noted that the group $SO(n)$ preserves the radius squared in $\mathbb{R}^n$. In coordinates, this means that $\bar{r}^2 = x_1^2 + x_2^2 + \cdots + x_n^2$, or rather the dot product $\bar{x} \cdot \bar{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$ is preserved.
Now if we generalize this to form a group action that preserves the value of the radius squared,using indicial notation,this gives us, $x^a y_a = -x_1 y_1 - x_2 y_2 - \cdots - x_m y_m + x_{m+1} y_{m+1} + \cdots + x_{m+n} y_{m+n}$. We denote the group that preserves this quantity $SO(m,n)$. The space of interest is still $\mathbb{R}^{m+n}$, but we are making transformations such that we preserve something other than the radius.
This allows $SO(m,n)$ to have an $SO(m)$ subgroup and an $SO(n)$ subgroup, consisting of rotations in the first m and last n components individually.
Now if we take a particular example of this type, $SO(1,3)$, we get a group that preserves the value $s^2 = -x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4$, which can be re-written as, $s^2 = -c^2 t^2 + x^2 + y^2 + z^2$. Thus we call this group as the **Lorentz** Group denoted, $SO(1,3)$.[10]

3.3 Generators-Representation Revisited

Unlike, finite discrete groups, continuous groups cannot be explicitly represented by some set of matrices. Instead we will take the representation theory we introduced in section 2 and build upon it using our notion of continuous parameters mentioned in earlier parts of this section.

Let our Lie group, be parametertized by a set of continuous parameters, called $\alpha_i,\ i = 1, ..., n$, where n is the number of parameters the group is dependant on. The elements of the group then become, $g(\alpha_i)$.

To make our lives easier, we will start with a known element of our group and scale from our known element to traverse the entire group. It's easy to understand that the element we will choose is the identity.(since all groups have a singular unique identity).
This gives us, $g(\alpha_i)\big|_{\alpha_i=0} = e$. Using the matrix representation form from section 2 this will lead us to $D_n(g(\alpha_i))\big|_{\alpha_i=0} = \mathbb{I}_n$.

Let's take α_i to be very small with $\delta\alpha_i << 1$. So, $D_n(g(0 + \delta\alpha_i))$ under a

[10] Any action that is invariant under $SO(1,3)$ is said to be a Lorentz Invariant theory.

Taylor expansion:

$$D_n(g(\delta\alpha_i)) = \mathbb{I} + \delta\alpha_i \frac{\partial D_n(g(\alpha_i))}{\partial \alpha_i}\Big|_{\alpha_i=0} + \cdots \tag{1}$$

The terms $\frac{\partial D_n}{\partial \alpha_i}\Big|_{\alpha_i=0}$ are of importance and so we re-write them using a new variable:

$$X_i \equiv -i\frac{\partial D_n}{\partial \alpha_i}\Big|_{\alpha_i=0}$$

The $-i$ is included here in order to make X_i Hermitian, which will aid us later on.

Rewriting the representation for $\delta\alpha_i$ we get,

$$D_n(\delta\alpha_i) = \mathbb{I} + i\delta\alpha_i X_i + \cdots$$ [11]

Now, that we have an idea of what the representation will look like for an infinitesimal value of α_i, we can repeat it to scale it back to a finite value.Thus by having, $\alpha_i = N\delta\alpha_i$ as $N \to \infty$. So, $\delta\alpha_i = \frac{\alpha_i}{N}$, and performing an infinite number of infinitesimal transformations,

$$\lim_{N\to\infty}(1 + i\delta\alpha_i X_i)^N = \lim_{N\to\infty}\left(1 + i\frac{\alpha_i}{N}X_i\right)^N$$

Which on further expansion of N, give us

$$\lim_{N\to\infty}\left(1 + i\frac{\alpha_i}{N}X_i\right)^N = e^{i\alpha_i X_i}$$

We call these X_i's the **Generators** of the group and there is typically atleast one for each parameter required to describe a particular element of the group.

The most appropriate way to think about these generators is to look at is as though we have mapped our Lie group elements onto an arbitrary parameter space, where the generators of our group form the basis that help us span our space. Which means in general there will be an infinite number of sets of generators for a particular group, and it's the fluidity in choosing some particular ones along with it's attached representations to help describe our natural world that makes Lie groups an unbelievably useful tool in Physics.

We call the number of generators of a group, the **Dimension** of the group. Since the generators of the group from something akin to a basis, the number of unique generator sets you can find will always be the minimum amount that is required. Anything beyond that is quite pointless and would probably boil down to some linear combination or similarity transform of your base generators.

3.4 Lie Algebra

An algebra is a space spanned by elements of a group with $\mathbb{C}$ coefficients parameterizing a Euclidean Space. While this is something quite straightforward for

[11] We have switched our notation from $D_n(g(\alpha))$ to $D_n(\alpha)$ to make things easier, mostly from a typing standpoint.

a finite group, we turn to the generators to form the algebras for the Lie Groups.

Let's take two elements of the same group with generators X_i, parameter values α_i and the other with parameter values β_i. Their product will then be $e^{i\alpha_i X_i}e^{i\beta_j X_j}$. Since this is a group we know that the product must be an element of the group (due to closure), and therefore will be specified by some set of parameters δ_k, so $e^{i\alpha_i X_i}e^{i\beta_j X_j} = e^{i\delta_k X_k}$.[12]
We need to work out what δ_i will be in terms of α_i and β_i.

$$i\delta_i X_k = \ln(e^{i\delta_k X_k}) = \ln(e^{i\alpha_i X_i}e^{\beta_j X_j}) = \ln(1 + e^{i\alpha_i X_i}e^{\beta_j X_j} - 1) \equiv \ln(1+x) \quad (2)$$

Here let, $x \equiv e^{i\alpha_i X_i}e^{\beta_j X_j} - 1$. By expanding only till second order in α_i and β_j,

$$\begin{aligned}
e^{i\alpha_i X_i}e^{\beta_j X_j} - 1 &= (1 + i\alpha_i X_i + \frac{1}{2}(i\alpha_i X_i)^2 + \cdots)(1 + i\beta_j X_j + \frac{1}{2}(i\beta_j X_j)^2 + \cdots) - 1 \\
&= 1 + i\beta_j X_j - \frac{1}{2}(\beta_j X_j)^2 + i\alpha_i X_i - \alpha_i X_i \beta_j X_j - \frac{1}{2}(\alpha_i X_i)^2 - 1 \\
&= i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}\big((\alpha_i X_i)^2 + (\beta_j X_j)^2\big)
\end{aligned}$$

Then, using another Taylor expansion for $\ln(1+x)$, only till the second order again

$$\begin{aligned}
x - \frac{x^2}{2} &= \Big[i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}[(\alpha_i X_i)^2 + (\beta_j X_j)^2]\Big] \\
&\quad -\frac{1}{2}\Big[i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}[(\alpha_i X_i)^2 + (\beta_j X_j)^2]\Big]^2 \\
&= i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}\Big[(\alpha_i X_i)^2 + (\beta_j X_j)^2\Big] \\
&\quad -\frac{1}{2}\Big[-(\alpha_i X_i + \beta_j X_j)(\alpha_i X_i + \beta_j X_j)\Big] \\
&= i(\alpha_i X_i + \beta_j X_j) - \alpha_i X_i \beta_j X_j - \frac{1}{2}\Big[(\alpha_i X_i)^2 + (\beta_j X_j)^2\Big] \\
&\quad +\frac{1}{2}\Big[(\alpha_i X_i)^2 + (\beta_j X_j)^2 + \alpha_i\beta_j(X_i X_j + X_j X_i)\Big] \\
&= i(\alpha_i X_i + \beta_j X_j) + \frac{1}{2}\alpha_i\beta_j(X_j X_i - X_i X_j) \\
&= i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}\alpha_i\beta_j[X_i, X_j] \\
&= i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j X_j]
\end{aligned}$$

This gives us,

$$i\delta_k X_k = i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i, \beta_j, X_j]$$

[12] the product won't tivially be $e^{i\alpha_i X_i}e^{i\beta_j X_j} = e^{i(\alpha_i X_i + \beta_j X_j)}$ because the generators being matrices don't typically commute.

which can be written as

$$e^{i\alpha_i X_i} e^{i\beta_j X_j} = e^{i(\alpha_i X_i + \beta_j X_j) - \frac{1}{2}[\alpha_i X_i,\, \beta_j X_j]} \tag{3}$$

Here, equation(3) is known as the **Baker-Campbell-Hausdroff** formula and it is one of the more profound relations in group theory and in physics. To preserve it's dimensionality, it is quite clear that the commutator $[X_i, X_j]$ must be proportional to some linear combination of the generators. With this we can write,

$$[X_i, X_j] = if_{ijk} X_k \tag{4}$$

for some set of constants f_{ijk}.
These are known as the **Structure Constants** of the group and the specific commutation relations defined by these form the **Lie Algebra** of the group. That is to say, the commutation relations, enables us to determine the group in any representation that we want and thus forms the inherent structure of the Lie Group itself.

3.4.1 Adjoint Representation

Armed with the notion of structure constants and algebra, we can form a particular representation of the group.
By using the Jacobi identity,

$$\big[X_i, [X_j, X_k]\big] + \big[X_j, [X_k, X_i]\big] + \big[X_k, [X_i, X_j]\big] = 0 \tag{5}$$

But, from equation(4), we can write

$$\big[X_i, [X_j, X_k]\big] = if_{jka}[X_i, X_a] = if_{jka} f_{iab} X_b$$

Plugging this into (5) we get

$$\begin{aligned} & if_{jka} f_{iab} X_b + if_{kia} f_{jab} X_b + if_{ija} f_{kab} X_b = 0 && (6)\\ \Rightarrow\quad & (f_{jka} f_{iab} + f_{kia} f_{jab} + f_{ija} f_{kab}) iX_b = 0 && (7)\\ \Rightarrow\quad & f_{jka} f_{iab} + f_{kia} f_{jab} + f_{ija} f_{kab} = 0 && (8) \end{aligned}$$

Thus, if we define our matrices

$$[T^a]_{bc} \equiv -if_{abc} \tag{9}$$

then it trivial to see that equation(8) leads to

$$[T^a, T^b] = if_{abc} T^c$$

This particular representation, as defined by the structure constants themselves is known as the adjoin representation.

3.4.2 Fundamental and Anti-Fundamental Representations

This section while being important also requires notion of topics related to what is known as a Root Space and the concepts that are tied in with it such as weights, classes and Cartan Algebras. While it is something that definitely needs to be learnt,it is frankly beyond the scope of this paper.
All that needs to be garnered is that,
Given a set of structure constants f_{abc}, that define the Lie algebra of some arbitrary lie group, we can form a representation of that group, which we denote R. This will comprise of a set of $D(R)\times D(R)$ matrices, where D is the dimension of the representation R.
We then call the generators of the group(in the representation R) T^a_R, and the obey the commutator relation,[2]

$$[T^a_R, T^b_R] = if_{abc}T^c_R$$

One such representation is of course when we use $n\times n$ matrices to represent $SO(n)$ and $SU(n)$ groups. We call this the **Fundamental Representation** and is often denoted by writing the number in bold. For example, the fundamental representation of the $SU(2)$ group will be denoted **2**, and as such it's generators will be denoted by $T^a_{\mathbf{2}}$. Similarly, $SU(3)$ will correspond to generators $T^a_{\mathbf{3}}$.

Let's further this argument by saying we have some arbitrary representation generated by T^a_R, obeying $[T^a_R, T^b_R] = if_{abc}T^c_R$. If we take the conjugate the commutators we get, $[T^{\star a}_R, T^{\star b}_R] = -if_{abc}T^{\star c}_R$.
Using this, a *new* set of generators $T'^a_R \equiv -T^{\star a}_R$, can be defined such that T'^a_R will also obey the commutation relations, thereby also forming a representation of the group.
Say it turns out that $T'^a_R = -(T^a_R)^\star = T^a_R$, or perhaps a series of some unitary similarity transformation give us $T^a_R \to U^{-1}T^a_R U,\ \ni\ T'^a_R = -(T^a_R)^\star = T^a_R$, then we call this representation **Real**, and it's complex conjugate, T^a_R's is also the same representation.
However, in absence of such a transformation, then we have created a *new* representation, denoted as the **Complex Conjugate** representation to R, or the **Anti-R** representation, which we denote $\bar{R}$.

For example, given a fundamental representation of $SU(3)$, denoted **3**, generated by $T^a_{\mathbf{3}}$, there exists the **Anti-Fundamental representation $\bar{\mathbf{3}}$**, generated by $T^a_{\bar{\mathbf{3}}}$.[2]

These fundamental, anti-fundamental, and adjoin representations of our Lie groups are of utmost importance for our particle classification.

4 SU(2)

Perhaps the most intuitive way to understand Lie groups, is to study the space created by it's generators. The easiest way to do this would be through it's eigenspace. The eigen vectors of the generators of our chosen representation will definitely form a basis for this eigenspace of the physical space of the group action.

Now, let's quantify this approach and try to find representations of the SU(2) group.

4.1 Generator relations

We know that the $SU(2)$ group requires 3 generators similar to the $SO(3)$ group and also obeys the same commutator relationship[13] with ϵ_{ijk} as it's structure constants[9]

Now, using similarity transformations we can diagonalize one of these arbitrary generators for our Lie Group[10].

Combining these two aforementioned statements, let the generators of $SU(2)$ be J^1, J^2, and J^3, and let J^3 to be the diagonal one. This makes the eigenvectors of J^3 to be the basis vectors of the physical vector space upon which $SU(2)$ acts.

Let's denote the eigenvectors of J^3 as $|j;m\rangle$, with j as the highest eigenvalue of the eigenvector. Then we use these notations to create scaling functions to traverse the space, similar to how we first developed generators, like so.

$$J^{\pm} \equiv \frac{1}{\sqrt{2}}(J^1 \pm iJ^2) \tag{10}$$

Using the commutator relations and these J^+(**Raising**) and J^-(**Lowering**) operators and some mathematics, we can get these following relations and properties of the $SU(2)$ group, for some arbitrary maximum value of j. A decent calculation can be found here [2].

For a general representation of $SU(2)$, we have $2j+1$ states:

$$\{j, j-1, j-2, \ldots, -j+2, -j+1, -j\}$$

Which then demands that $j = \frac{n}{2}$ for some integer n. Thus imposing bounds on the highest eigenvalue of an $SU(2)$ eigenvector: $0, \frac{1}{2}, 1, \frac{3}{2}, 2$, etc.

Furthermore, using these states, we see that

$$\langle j;m'|J^3|j;m\rangle = m\delta_{m',m} \tag{11}$$

$$\langle j;m'|J^+|j;m\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m+1)(j-m)}\delta_{m',m+1} \tag{12}$$

$$\langle j;m'|J^-|j;m\rangle = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}\delta_{m',m-1} \tag{13}$$

4.2 Representations for different j values

Using our above developed formalism we can now see how representations of j work out for some important and familiar values of j.

[13] $[J_i, J_j] = i\epsilon_{ijk}J_k$, where ϵ_{ijk} is the rank 1 anti-symmetric tensor

- $\mathbf{j = \frac{1}{2}}$,
  We find the generators to be[2]:

$$J^1_{1/2} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\sigma^1}{2}, \quad J^2_{1/2} = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\sigma^2}{2}, \quad J^3_{1/2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\sigma^3}{2} \quad (14)$$

  Where the σ^i are the familiar **Pauli Spin Matrices**

- $\mathbf{j = 1}$,
  This value yields the generators[2],

$$J^1_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad J^2_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J^3_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad (15)$$

  Going from our notion of the adjoint representation that for a $n-dimensional$ group, the adjoint would have $n \times n$ matrices, we see now that the $j = 1$ generators indeed form the adjoint representation of the $SU(2)$ group.

4.3 Further Discussions

If we notice from the above section, it is no mere co-incidence that the Pauli spin matrices popped out for a $j = \frac{1}{2}$ value representation of the $SU(2)$ group. After all the $SU(2)$ group represents quantum mechanical spin. It is important to note however, that the quantum notion of spin does not imply rotation through space time,(that is described by the SO(3) group, hence a shared kinship between the two) but rather through the spinor space. Both the spinor space value and the space-time angular momentum are measurable and conserved quantities and contribute equally to the angular momentum of a particle.

However it is important to note that while spin does feature an important role in the use of the $SU(2)$ group, it actually has much more profound effects when it describes the Weak force and it's mediators. The adjoint representation plays a substantially large roll and is something that will be covered in later sections.

5 SU(3)

All our discussions till now have largely been relying on the mathematical side of things and in that we have then since established enough rigour that we can from now on skip most of it in favour of the discussion of physics and how groups and it's notion of symmetries find uses in Particle Physics. The $SU(3)$ group is a brilliant place to start.

5.1 Representations

The generators of $SU(3)$, can be written in multiple bases, but unlike it's $SU(2)$ counterpart, it is quite tedious to deal with in the Adjoint space as it consists of 8×8 matrices.[14]

Instead we stick to the fundamental representation of this group, which yield 3×3 matrices, given that these share the same space as the adjoint representations of $SU(2)$ alludes to some greater implications.

The **Fundamental Representations** of the $SU(3)$ group are of the form, $T^a = \frac{1}{2}\lambda^a$ for $a = 1, \ldots, 8$, where[2]

$$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

These are called the Gell-Mann λ matrices. Since only two of these are diagonal, λ^3 and λ^8, $SU(3)$ is a rank 2 group.

Expanding the basis from the two diagonal matrices we get the eigenvectors to be:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{16}$$

Now, we find the corresponding eigen-vectors under the new basis $|T_3, T_8\rangle$ The new vectors(called weight vectors), are

$$\vec{t}^1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \end{pmatrix}^T, \quad \vec{t}^2 = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2\sqrt{3}} \end{pmatrix}^T, \quad \vec{t}^3 = \begin{pmatrix} 0 & -\frac{1}{\sqrt{3}} \end{pmatrix}^T \tag{17}$$

By plotting these vectors on the graph we get an equilateral triangle. This is the point where things really leave the boundary of abstractions and get rooted in the real world, cause as it turns out, these three points on the equilateral triangle, with the abscissa as the z-component of isospin, I_3 and the ordinate as a value, $\frac{\sqrt{3}Y}{2}$, for Hyper charge Y[15] corresponds to the up, down and strange quarks. More over, by varying the 3 values to from various other points, we form the meson octet - giving birth to the eight fold way of quark treatment.

5.2 Approximate and Exact Symmetries

Similar to the Meson octet, it is possible to get similar pictures for Baryons as well, although a bit more complex owing to it's three quark structure. The

[14]An arbitrary $SU(n)$ group will always have $n^2 - 1$ generators, and will be rank $n - 1$. An arbitrary $SO(n)$ group (for n even) will always have $\frac{n(n-1)}{2}$ generators. We won't worry about the rank of the orthogonal groups.[2]

[15]Defined by the sum of Baryon and strangeness number

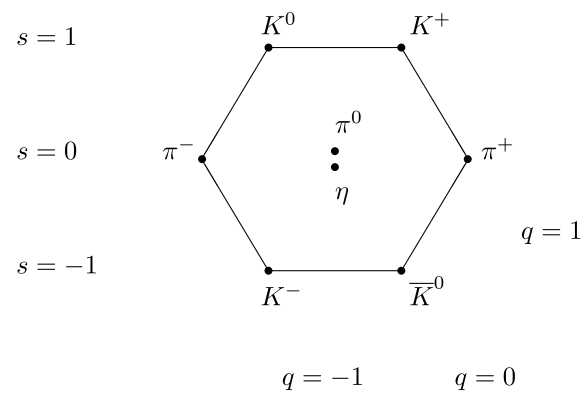

Figure 1: Meson Octet drawn using LaTex code[6]

group involved adopts a $3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1$ coupling, there by giving us 27 elements in total. But thanks to reducibility, it is changed into 4 irreducible representations, each of which form a basis group in it's own right, similar to our meson octet. They are the baryonic decouplet, two octets, and a singlet.

While organization using the Isospin and HyperCharge quantum number is indeed exact, there are some approximation in terms of mass within this model. Due to the eight-fold Octet way arising because of the **approximate symmetries** of the $SU(3)$ group between the three light quarks, up, down and strange. There are a lot of similar particles in the rows of the arrangements. Thus when extending the model to include the other generations of the quarks, it tends to be tedious and often times not effective at all.

However, while $SU(3)$ is inexact under flavour symmetries, it is exact for color. The 8 matrix generator basis of this particular group forms the very basis of Quantum Chromo Dyanamics, and is reflected in the structure of gluons. While all combinations of red, green and blue with their anticolors should give nine states, one of them is a color singlet, and would have some non-gluon like properties such as a long range, and thus it is believed that this does not exist (at least as a gluon)[8]

$\lvert 1\rangle = \frac{(r\bar{b}+b\bar{r})}{\sqrt{2}}$	$\lvert 2\rangle = \frac{-i(r\bar{b}-b\bar{r})}{\sqrt{2}}$
$\lvert 3\rangle = \frac{(r\bar{r}-b\bar{b})}{\sqrt{2}}$	$\lvert 4\rangle = \frac{(r\bar{g}+g\bar{r})}{\sqrt{2}}$
$\lvert 5\rangle = \frac{-i(r\bar{g}+g\bar{r})}{\sqrt{2}}$	$\lvert 6\rangle = \frac{(b\bar{g}+g\bar{b})}{\sqrt{2}}$
$\lvert 7\rangle = \frac{-i(b\bar{g}-g\bar{b})}{\sqrt{2}}$	$\lvert 8\rangle = \frac{(r\bar{r}+b\bar{b}-2g\bar{g})}{\sqrt{6}}$

Table 1: The gluon color octet under $SU(3)$

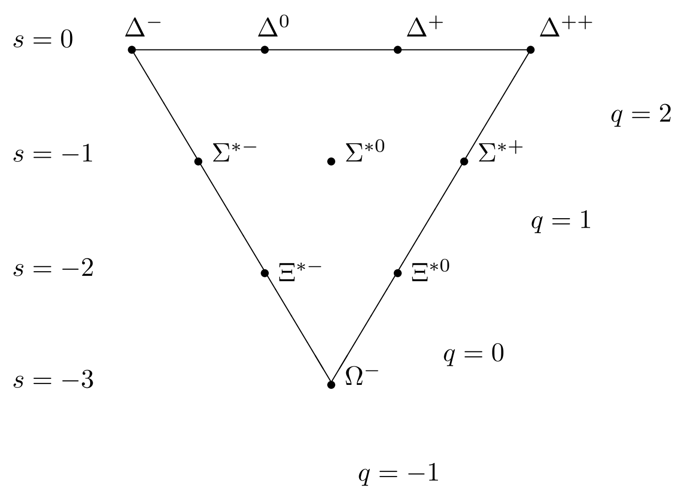

Figure 2: Baryonic Decouplet drawn using LaTeX code[7].
Especially notable is the Ω^-, which was predicted using the Eight-fold Way by Gell-Mann before it was discovered[8].

Thus the Gluons, who are carriers of the strong force are described by the **Exact symmetry** of the $SU(3)$ group.

6 Particle Content of the Standard Model

This section is more or less glorified labelling. This is simply because, while Group theory and it's representations do form an intrinsic part of the Standard Model of Particle physics and aid in assigning quantum numbers to the particles in it, it is not the whole puzzle. We are still sorely lacking in terms of Relativistic Quantum Mechanics, Quantum Field Theory, Lagrangian Dynamics and other topics that required to give a clear picture. Group theory simply forms the canvas for all of these things to shine and interact beautifully.

With that in mind, we formally write the Standard Model[2] as, A Yang-Mills (Gauge) Theory with Gauge Group

$$SU(3) \otimes SU(2) \otimes U(1)$$

with left-handed Weyl fields fields in three copies of the representation

$$(1, \mathbf{2}, -1/2) \oplus (1, 1, 1) \oplus (\mathbf{3}, \mathbf{2}, 1/6) \oplus (\bar{\mathbf{3}}, 1, -2, 3) \oplus (\bar{\mathbf{3}}, 1, 1/3)$$

(where the last entry specifies the value of the $U(1)$ hypercharge),

and a single copy of a complex scalar field in the representation

$$(1, \mathbf{2}, -1/2)$$

Which in summation gives the gauge representation,
$SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$

Here, the c stands for the color charge, L stands for left-chiral and Y is hypercharge.
Now we will go through the three major sectors and observe only the influence of Lie group theory and it's representations in them. Anything apart from that is unfortunately beyond the scope of this paper.

6.1 Gauge and Higgs sector

This sector deals with the gauge and Higgs bosons that feature within the Standard model. The Gauge bosons are - $W^{\pm}, Z, g$(gluons), and γ Since we have already covered gluons under $SU(3)$, let's concentrate on the **Electroweak** part of the Standard model gauge group - $SU(2)_L \otimes U(1)_Y$ and in tandem with it the Higgs.

- **Higgs Boson**: The Higgs field is a scalar field in the $(\mathbf{2}, -1/2)$ representation of $SU(2) \otimes U(1)$. The Higgs-Boson, is the associated particle with this field. It interacts as part of a symmetry-breaking mechanism and gives mass to the standard model particles. That is under the $SU(2) \otimes U(1)$ group, certain actions of this group leaves the ground state representations changed, and the strength of this interaction directly correlates to the mass of the particle interacting with it.
- **W and Z**: The vector fields of these bosons break symmetry at low energy and thus have mass. They are three in number($W^{\pm}, Z$). As such they are represented by the Adjoint representations of the $SU(2)_L$ group which is responsible for the weak force and thus these particles govern the **Weak force**.

- **Photon(γ)**: Within all the interacting fields of the Electro-weak nature, there is one with an unbroken symmetry and therefore remains massless. The $U(1)$ group containing a singular generator. This $U(1)$ forms the gauge group and field of **Electro-Magnetism** and it's associated particle is the **photon**.

6.2 Lepton sector

This sector deals with the leptons, which are fermions. But we are still in the $SU(2) \otimes U(1)$ part of the gauge group. This is where our notion of differing representations for groups come into the picture. A **Lepton** is a spin-$\frac{1}{2}$ particle that is not charged under color. That is to say it has no number representation under the $SU(3)_c$ color group. It is grouped in terms of six **Flavors** arranged into three **Generations**, much like quarks. The first generation consists of the electron (e) and the electron neutrino (ν_e), the second generation the muon (μ) and the muon neutrino (ν_μ), and the third the tau (τ) and tau neutrino (ν_τ). Each family behaves exactly the same way and as such it is enough to consider just one of them.

The biggest anomaly in this group are the neutrino's. They're stubborn particles that don't really interact with anything on their own, but however does interact as part of the $SU(2)_L$ doublet.

Before we get to that let's get the small notion of the charge away, since the leptons do interact with $SU(2)_L$(Isospin) and $U(1)_Y$(HyperCharge), they do have an intrinsic electric charge - whose value is given by the representation numbers in both those groups.

$$Q = I_{3L} + Y/2$$

Now that that part is over, let us take a look at the interactions of the leptons as doublets in the $SU(2)_L$ group. Here they interact with the gauge bosons of the weak force. The $W^{\pm}$ acts as raising and lowering operators similar to the linear combination we saw in the $SU(2)$ group generators. While the $Z's\ and\ \gamma$ don't affect the charge. There is also a notion of chirality involved in it. Rigorous treatments aren't particularly relevant to this paper but a somewhat cursory idea may be given as follows.

The primary idea is that the electrons,positrons[16] and neutrinos all interact with the $SU(2)\otimes U(1)$ gauge group and the particles that arise from their various representations(i.e., the $W^{\pm}$, Z_μ, and $A_\mu(photon)$). Their intrinsic chirality ensures that the W^+ interacts with a left-handed electron such that raises it's electric charge from minus one to zero, producing a neutrino. The W^+ however does not interact with left-handed neutrinos. Similarly, the W^- will lower the electric charge of a neutrino, producing an electron. But W^- will not interact with an electron.[2]

[16]Leptons much like other elementary particles have anti-particles which are given by their anti-fundamental or anti-R representations of their respective Lie Groups

6.3 Quark Sector

We covered quarks as part of the approximate/inexact symmetries of the $SU(3)$ group.

A quark is a fermion, spin-$\frac{1}{2}$ particle that interacts with the entire gauge group as a whole. As we saw in the $SU(3)$, section it has representations within the color force group as a triplet and as such interacts with gluons to be charged under color. It is quite similar to Leptons in terms of it's arrangement in that it is a 6 flavored, 3 generational $SU(2)_L$ doublet and takes a non-adjoint representation under the group. This also gives rise to idea, that the $SU(2)$ doublet behaves quite similar to the lepton doublet. This notion is infact correct, under the $SU(2)$ interactions, the quark doublet couples with the W bosons and has it's state/charge lowered and raised.

Although, the quarks and their formations themselves were detailed in section 5 quite well, here we can take a look at the interactions that arise out of the group algebra between the gluons and quarks.

The $SU(3)$ index runs from 1 to 3, and the values are conventionally denoted *red, green, and blue*(r, g, b).[17]

The eight gauge fields correlated with the eight $SU(3)$ generators are called gluons , We label them as follows:

$$g_\alpha^\beta \doteq \begin{pmatrix} r\bar{r} & r\bar{g} & r\bar{b} \\ g\bar{r} & g\bar{g} & g\bar{b} \\ b\bar{r} & b\bar{g} & b\bar{b} \end{pmatrix} \tag{18}$$

Note that the upper index is the anti-color index, and denotes the column, and the lower index is the color index denoting the row. Now consider the gluon as follows,

$$g_r^{\bar{g}} \propto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{19}$$

and the quarks

$$q_r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad q_g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad q_b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{20}$$

This gives us the interactions,

$$g_r^{\bar{g}} q_r = 0 \qquad g_r^{\bar{g}} q_g = q_r \qquad g_r^{\bar{g}} q_b = 0 \tag{21}$$

By repeating this for the other generators of the SU(3) group and looking at it's respective eigen state you can in essence work out all of the interaction rules between quark and gluons as such.

[17] These are merely labels and have nothing to do with the colors in any type of visible spectrum

6.4 Tabulated Summary

In general with this we have seen how the standard model particles and their respective quantum numbers are arranged in order to classify this particle zoo(arranged in a table on the last page). By using Group theory, with its organisation of like elements into flexible recognizable patterns we were able to easily achieve this goal. There are still many more facets of group theory that are yet to be observed and as such may still hold the key toward unifying all the forces and really giving us a complete standard model of physics itself.

7 References

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	Leptons			Hadrons				Higgs
	$(1, \mathbf{2}, -1/2)$	$(1, 1, 1)$		$(\mathbf{3}, \mathbf{2}, 1/6)$	$(\mathbf{3}, 1, -2/3)$	$(\mathbf{3}, 1, 1/3)$		$(1, \mathbf{2}, -1/2)$
Generation 1	$\begin{pmatrix}\text{electron neutrino}\\\text{electron}\end{pmatrix}$	electron		$\begin{pmatrix}\text{up}\\\text{down}\end{pmatrix}$	up	down		1 Generation Only
Generation 2	$\begin{pmatrix}\text{muon neutrino}\\\text{muon}\end{pmatrix}$	muon		$\begin{pmatrix}\text{charm}\\\text{strange}\end{pmatrix}$	charm	strange		
Generation 3	$\begin{pmatrix}\text{tau neutrino}\\\text{tau}\end{pmatrix}$	tau		$\begin{pmatrix}\text{top}\\\text{bottom}\end{pmatrix}$	top	bottom		

Table 2: Summary of Standard Model Particles[2]