Unit 16

THEOREMS RELATED WITH AREA

Area of a Figure

The region enclosed by the bounding lines of a closed figure is called the area of the figure.

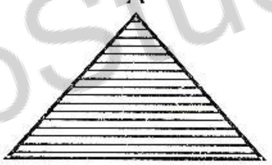
The area of a closed region is expressed in square units (say, sq. m or m^2) i.e., a positive real number.

Triangular region

The interior of a triangle is the part of the plane enclosed by the triangle.

A triangular region is the union of a triangle and its interior i.e., the three line segments forming the triangle and its interior.

By area of a triangle, we mean the area of its triangular region.



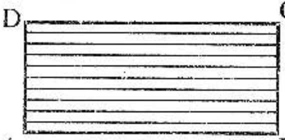
Congruent Area Axiom

If $\Delta ABC \cong \Delta PQR$, then area of (region ΔABC) = area of (region ΔPQR)

Define Rectangular Region

The interior of a rectangle is the part of the plane enclosed by the rectangle.

A rectangular region is the union of a rectangle and its interior.



A rectangular region can be divided into two or more than two triangular regions in many ways.

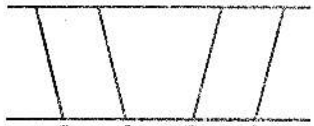
Note

If the length and width of a rectangle are a units and b units respectively, then the area of the rectangle is equal to $a \times b$ square units.

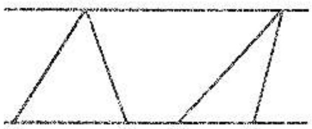
If a is the side of a square, its area $= a^2$, square units.

Between the same Parallels

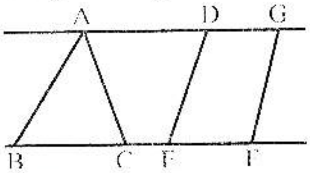
(i) Two parallelograms are said to be between the same parallels, when their bases are in the same straight line and their sides opposite to these bases are also in a straight line; as the parallelograms ABCD, EFGH in the given figure.



(ii) Two triangles are said to be between the same parallels, when their bases are in the same straight line and the line joining their vertices is parallel to their bases; as the Δ ABC, Δ DEF in the given figure.



(iii) A triangle and a parallelogram are said to be between the same parallels, when their bases are in the same straight line, and the side of the parallelogram opposite the base, produced if necessary, passes through the vertex of the triangle as are the ΔABC and the parallelogram DEFG in the given figure.



Altitude of Parallelogram

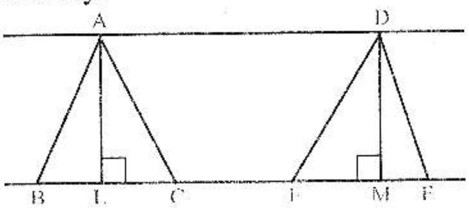
If one side of a parallelogram is taken as its base, the perpendicular distance between that side and the side parallel to it, is called the Altitude or Height of the parallelogram.

Altitude of the triangle

If one side of a triangle is taken as its base, the perpendicular to that side, from the opposite vertex is called the Altitude or Height of the triangle.

Example

"Triangles or parallelograms having the same or equal altitudes can be placed between the same parallels and conversely."



Place the triangles ABC, DEF so that their bases $\overline{BC}$, $\overline{EF}$ are in the same straight line and the vertices on the same side of it and suppose $\overline{AL}$, $\overline{DM}$ are the equal altitudes. We have to show that $\overline{AD}$ is parallel to BCEF.

Proof

$\overline{AL}$ and $\overline{DM}$ are parallel, for they are both perpendicular to $\overline{BF}$. Also $m\overline{AL} = m\overline{DM}$. (given)

$\therefore$ $\overline{AD}$ is parallel to $\overline{LM}$. A similar proof may be given in the case of parallelograms.

Note:

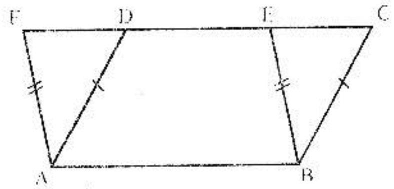
A diagonal of a parallelogram divides it into two congruent triangles (SSS) and hence of equal area.

Theorem

Parallelograms on the same base and between the same parallel lines (or of the same altitude) are equal in area.

Given

Two parallelograms ABCD and ABEF having the same base $\overline{AB}$ and $\overline{DE}$ between the same parallel lines $\overline{AB}$ and $\overline{DE}$.



To Prove

Area of parallelogram ABCD = area of parallelogram ABEF

Proof

Statements	Reasons
Area of (parallelogram ABCD) = Area of (quad. ABED) + area of (ΔCBE)..(i)	[Area addition axiom]

Area of (parallelogram ABEF)	
= area of (quad. ABED) + area of (ΔDAF)..(ii)	[Area addition axiom]
In Δs CBE and DAF	
$m\overline{CB} = m\overline{DA}$	[opposite sides of a parallelogram]
$m\overline{BE} = m\overline{AF}$	[opposite sides of a parallelogram]
$\angle CBE = \angle DAF$	[$\because \overline{BC} \parallel \overline{AD}, \overline{BE} \parallel \overline{AF}$]
$\therefore \Delta CBE \cong \Delta DAF$	[S.A.S. cong. Axiom]
$\therefore$ area of (ΔCBE) = area of (ΔDAF).........(iii)	[cong. Area axiom]
Hence area of (parallelogram ABCD) = area of (parallelogram ABEF)	From (i), (ii) and (iii)

Example

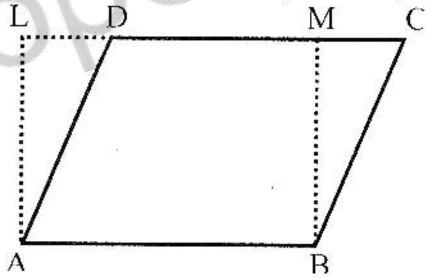
(i) The area of a parallelogram is equal to that of a rectangle on the same base and having the same altitude.

(ii) Hence area of parallelogram = base $\times$ altitude

Proof

Let ABCD be a parallelogram. $\overline{AL}$ is an altitude corresponding to side $\overline{AB}$.

(i) Since parallelogram ABCD and rectangle ALMB are on the same base $\overline{AB}$ and between the same parallels,



$\therefore$ by above theorem it follows that area of (parallelogram ABCD) = area of (rect. ALMB)

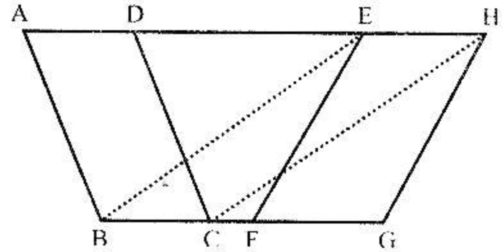
(ii)But area of (rect. ALMB) = $\overline{AB} \times \overline{AL}$

Hence

Area of (parallelogram ABCD) = $\overline{AB} \times \overline{AL}$

Theorem

Parallelograms on equal bases and having the same (or equal) altitude are equal in area.



Given

Parallelograms ABCD, EFGH are on the equal bases $\overline{BC}$, $\overline{FG}$, having equal altitudes.

To Prove

Area of (parallelogram ABCD) = area of (parallelogram EFGH)

Construction

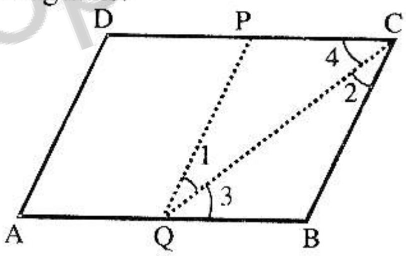
Place the parallelograms ABCD and EFGH so that their equal bases $\overline{BC}$, $\overline{FG}$ are in the straight line BCFG. Join $\overline{BE}$ and $\overline{CH}$.

Proof

Statements	Reasons
The given $\parallel^{gms}$ ABCD and EFGH are between the same parallels	Their altitudes are equal (given)
Hence ADEH is a straight line $\parallel$ to $\overline{BC}$	
$\therefore\ m\overline{BC} = m\overline{FG}$	Given
$= m\overline{EH}$	EFGH is a parallelogram
Now $m\overline{BC} = m\overline{EH}$ and they are $\parallel$	
$\therefore\ \overline{BE}$ and $\overline{CH}$ are both equal and $\parallel$ Hence EBCH is a parallelogram	A quadrilateral with two opposite sides congruent and parallel is a parallelogram
Now $\parallel^{gm}$ ABCD $= \parallel^{gm}$ EBCH ...(i)	Being on the same base $\overline{BC}$ and between the same parallels
But $\parallel^{gm}$ EBCH $= \parallel^{gm}$ EFGH ...(ii)	Being on the same base $\overline{EH}$ and between the same parallels
Hence area ($\parallel^{gm}$ ABCD) = area ($\parallel^{gm}$ EFGH)	From (i) and (ii)

Exercise 16.1

(1) Show that the line segment joining the mid-points of opposite sides of a parallelogram, divides it into two equal parallelograms.



Given ABCD is parallelogram. point p is midpoint of side $\overline{DC}$ i.e. $\overline{DP} \cong \overline{PC}$ and point Q is midpoint of side $\overline{AB}$ i.e. $\overline{AQ} \cong \overline{QB}$.

To Prove

Parallelogram AQPD $\cong$ parallelogram QBCP

Construction

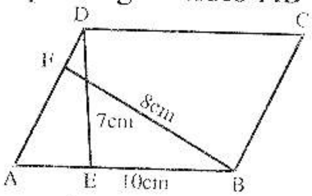
Join P to Q and Q to C.

Proof

Statements	Reasons
$m\overline{AB} = m\overline{DC}$	
$\frac{1}{2} m\overline{AB} = \frac{1}{2} m\overline{DC}$	Dividing by 2
$m\overline{QB} = m\overline{PC}$	

Now	
$\Delta PQC \leftrightarrow \Delta QBC$	
$\overline{QC} \cong \overline{QC}$	Common
$\overline{QB} \cong \overline{PC}$	Proved
$\angle 3 \cong \angle 4$	Alt. Angles $\overline{AB} \parallel \overline{DC}$
$\Delta PQC \cong \Delta QBC$	S.A.S = S.A.S
$\overline{PQ} \cong \overline{CB}$ ……………………(i)	Corresponding sides of congruent triangles
$\overline{AD} \cong \overline{CB}$ …………………(ii)	
$\overline{PQ} \cong \overline{AD} \cong \overline{CB}$	
$\angle 1 \cong \angle 2$	Corresponding angles of congruent triangles
$m\angle 1 + m\angle 3 = m\angle 2 + m\angle 4$	Corresponding angles of $\parallel$gm
$\angle PQB \cong \angle PCB$	
$\angle A \cong \angle PCB$	
$\angle A \cong \angle PQB$	
Now	
$\parallel$gm AQPD and $\parallel$gm QBCP	
$\overline{AQ} \cong \overline{QB}$	Given
$\overline{AD} \cong \overline{PQ}$	Proved
$\angle A \cong \angle PQB$	
Thus $\parallel$gm AQPD $\cong$ $\parallel$gm QBCP	

(2) In a parallelogram ABCD, $m\overline{AB} = 10$cm. The altitudes corresponding to sides AB and AD are respectively 7 cm and 8 cm. Find $\overline{AD}$.



Given Parallelogram ABCD, $m\overline{AB} = 10$cm altitudes. Corresponding to the sides $\overline{AB}$ and $\overline{AD}$ are 7cm and 8cm.

To Prove: $m\overline{AD} = ?$

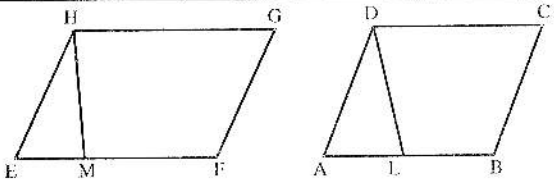
Construction Make $\parallel$gm ABCD and show the given altitudes $\overline{DE} = 7$cm, $\overline{BF} = 8$cm.

Proof The area of parallelogram = base x altitude

Statements	Reasons
$\therefore$ Area of parallelogram ABCD = 10 x 7 ………(i)	
Also area of the $\parallel$gm ABCD = $\overline{AD}$ x 8 ……..(ii)	
$\therefore$ $m\overline{AD}$ x 8 = 10 x 7	

$m\overline{AD} = \frac{10 \times 7}{8}$ $m\overline{AD} = \frac{35}{4} = 8\frac{3}{4}\text{cm}$	

(3) If two parallelograms of equal areas have the same or equal bases, their altitudes are equal.



Given Two parallelograms of same or equal bases and same areas.

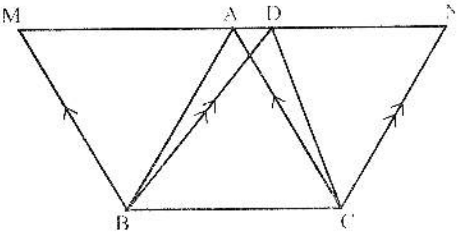
To Prove Their altitudes are equal.

Construction Make the ||gm ABCD and EFGH. Draw $\overline{DL} \perp \overline{AB}$ and $\overline{HM} \perp \overline{EF}$

Proof

Statements	Reasons
Area of the ||gm ABCD = area of the ||gm EFGH	
base x altitude = base x altitude	
$m\overline{AB} \times m\overline{DL} = m\overline{EF} \times m\overline{HM}$	Area = base x altitude
But $m\overline{AB} = m\overline{EF}$	
$\therefore$ $m\overline{EF} \times m\overline{DL} = m\overline{EF} \times m\overline{HM}$	Dividing by $m\overline{EF}$ we get
$m\overline{DL} = m\overline{HM}$ so altitudes are equal	

Theorem Triangles on the same base and of the same (i.e., equal) altitudes are equal in area.



Given Δs ABC, DBC on the same base $\overline{BC}$ and having equal altitudes.

To Prove Area of (ΔABC) = area of (ΔDBC)

Construction Draw $\overline{BM}$ || to $\overline{CA}$, $\overline{CN}$ || to $\overline{BD}$ meeting $\overline{AD}$ produced in M, N.

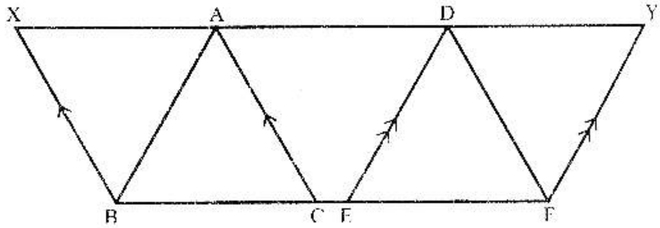
Proof

Statements	Reasons
Δ ABC and Δ DBC are between the same $\|^s$	Their altitudes are equal
Hence MADN is parallel to $\overline{BC}$	
$\therefore$ Area ($\|^{gm}$ BCAM)=Area ($\|^{gm}$ BCND)....(i)	These $\|^{gms}$ are on the same base $\overline{BC}$ and between the same $\|^s$
But $\Delta ABC = \frac{1}{2}(\|^{gm}\text{ BCAM})$ .....(ii)	Each diagonal of a $\|^{gm}$ bisects it into two congruent triangles

and $\quad \Delta DBC = \frac{1}{2}(\|_{gm}\ BCND) \quad .....(iii)$ Hence area (Δ ABC) = Area (Δ DBC)	From (i), (ii) and (iii)

Theorem **Triangles on equal bases and of equal altitudes are equal in area.**

Given



Δs ABC, DEF on equal bases $\overline{BC}$, $\overline{EF}$ and having altitudes equal.

To Prove

Area (Δ ABC) = Area (Δ DEF)

Construction

Place the Δs ABC and DEF so that their equal bases $\overline{BC}$ and $\overline{EF}$ are in the same straight line BCEF and their vertices on the same side of it. Draw BX ‖ to CA and FY ‖ to ED meeting AD produced in X, Y respectively

Proof

Statements	Reasons
Δ ABC and Δ DEF are between the same parallels	Their altitudes are equal (given)
$\therefore$ XADY is ‖ to BCEF	
$\therefore$ Area ($\|^{gm}$ BCAX) = Area ($\|^{gm}$ EFYD) .....(i)	These $\|^{gms}$ are on equal bases and between the same parallels
But $\quad \Delta ABC = \frac{1}{2}(\|^{gm}\ BCAX) \quad .....(ii)$	Diagonal of a $\|^{gm}$ bisects it
and $\quad \Delta DEF = \frac{1}{2}(\|_{gm}\ EFYD) \quad .....(iii)$	
$\therefore$ area (ΔABC) = area (ΔDEF)	From (i), (ii) and (iii)

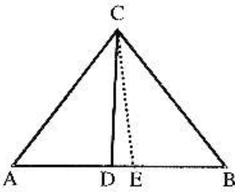
Corollaries

(1) Triangles on equal bases and between the same parallels are equal in area.

(2) Triangles having a common vertex and equal bases in the same straight line, are equal in area.

Exercise 16.2

(1) Show that a median of a triangle divides it into two triangles of equal area.



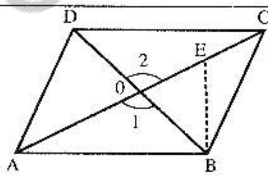
Given Median of the triangle

To Prove: Median divides the triangle into two triangles of equal area.

Proof Make Δ ABC, with $\overline{CD}$ as median and $\overline{CE}$ as altitude

Statements	Reasons
$m\overline{AD} = m\overline{DB}$ ……..(i)	D is midpoint of $m\overline{AB}$
Area of the $\Delta ACD = \frac{1}{2} \cdot m\overline{AD} . m\overline{CE}$ …(ii)	
Area of the $\Delta BCD = \frac{1}{2} \cdot m\overline{BD} . m\overline{CE}$	
$= \frac{1}{2} . m\overline{AD} . m\overline{CE}$ …(iii)	By (i)
$\Delta ACD = \Delta\ BCD$	By (ii) and (iii)

(2) Prove that a parallelogram is divided by its diagonals into four triangles of equal area.



Given

||gm divided by its diagonals into four triangles

To Prove

Areas of the four triangles arc equal

Construction Make the ||gm ABCD with diagonals $m\overline{AC}, m\overline{BD}$ intersecting each other at O. Draw $\overline{BE} \perp \overline{AC}$.

Proof

Statements	Reasons
Area of $\Delta OBC = \frac{1}{2}\ m\overline{OA} . m\overline{BE}$	
$= \frac{1}{2}\ m\overline{OC} . m\overline{BE}$ …………(i)	
The diagonals of the ||gm bisect each other	
$\therefore\ m\overline{OA} \cong m\overline{OC}$	
In $\Delta OAB \leftrightarrow \Delta OCD$	
$m\overline{OB} \cong m\overline{OD}$	
$m\overline{OA} \cong m\overline{OC}$	
$<1 \cong\ <2$	opposite angles
$\Delta\ OAB \cong \Delta OCD$ ………… (ii)	
$\Delta\ OAD \cong \Delta\ OBC$ …………(iii)	
$\therefore$ Area Δ OAB = Area Δ OBC = Area ΔOCD = Area Δ ODA	By (i), (ii) ,(iii)

(3) Which of the following are true and which are false?

(i) Area of a figure means region enclosed by bounding lines of closed figure. TRUE

(ii) Similar figures have same area. FALSE

(iii) Congruent figures have same area. TRUE

(iv) A diagonal of a parallelogram divides it into two non-congruent triangles. FALSE

(v) Altitude of a triangle means perpendicular from vertex to the opposite side (base). TRUE

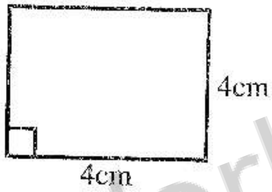
(vi) Area of a parallelogram is equal to the product of base and height. TRUE

Q.4 Find the area of the following.

(i)

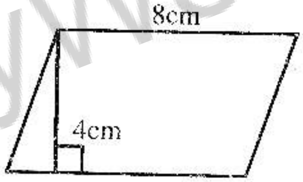
Area $= 6 \times 3 = 18\text{cm}^2$

(ii)



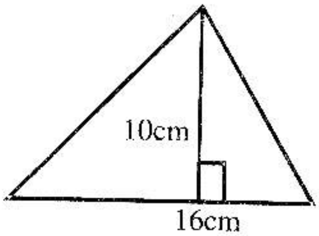
Area $= 4 \times 4 = 16\text{cm}^2$

(iii)



Area $= 8 \times 4 = 32\text{cm}^2$

(iv)



Area $= \frac{1}{2} \times 16 \times 10 = 80\text{cm}^2$

OBJECTIVE

1. The region enclosed by the bounding lines of a closed figure is called the __ of the figure:
   (a) Area (b) Circle
   (c) Boundary (d) None
2. Base x altitude =
   (a) Area of parallelogram
   (b) Area of square
   (c) Area of Rectangular
   (d) None
3. The union of a rectangular and its interior is called:
   (a) Circle region
   (b) Rectangular region
   (c) Triangle region
   (d) None
4. If a is the side of a square, its area =
   (a) a square unit
   (b) a^2 square units
   (c) a^3 square units
   (d) a^4 square units
5. The union of a triangle and its interior is called as:
   (a) Triangular region
   (b) Rectangular region
   (c) Circle region
   (d) None of these
6. Altitude of a triangle means perpendicular distance to base from its opposite:___
   (a) Vertex (b) Side
   (c) Midpoint (d) None

ANSWER KEY

1.	a	**2.**	a	**3.**	b	**4.**	b	**5.**	a	**6.**	a