

ALGEBRAIC EXPRESSIONS AND ALGEBRAIC FORMULAS

Define the following terms

Algebraic Expressions

When operations of addition and subtraction are applied to algebraic terms we obtain an algebraic expression. For

Polynomials

A polynomial in the variable x is an algebraic expression of the form

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0, a_n \neq 0, \dots (i)$$

Where n , the highest power of x , is a non-negative integer called the degree of the polynomial and each coefficient a_n is a real number. The coefficient a_n of the highest power of x is called the leading coefficient of the polynomial. $2x^4 y^3 + x^2 y^2 + 8x$ is a polynomial in two variables x and y and has degree 7.

Rational Expression

The quotient $\frac{p(x)}{q(x)}$ of two polynomials $p(x)$ and $q(x)$, where $q(x)$ is a non-zero polynomial, is called a rational expression.

For example, $\frac{2x+1}{3x+8}$, $3x+8 \neq 0$ is a rational expression.

In the rational expression $\frac{p(x)}{q(x)}$, $p(x)$ is called the numerator and $q(x)$ is known as the denominator of the rational expression $\frac{p(x)}{q(x)}$. The rational expression

$\frac{p(x)}{q(x)}$ need not be a polynomial.

instance, $5x^2 - 3x + \frac{2}{\sqrt{x}}$ and

$3xy + \frac{3}{x} (x \neq 0)$ are algebraic expressions.

Example

Reduce the following algebraic fractions to their lowest forms.

$$(i) \quad \frac{lx+mx-ly-my}{3x^2-3y^2} \quad (ii)$$

$$\frac{3x^2+18x+27}{5x^2-45}$$

Solution

$$\begin{aligned} (i) \quad & \frac{lx+mx-ly-my}{3x^2-3y^2} \\ &= \frac{x(l+m)-y(l+m)}{3(x^2-y^2)} \\ &= \frac{(l+m)(x-y)}{3(x+y)(x-y)} \\ &= \frac{l+m}{3(x+y)} \end{aligned}$$

Which is in the lowest forms.

$$(ii) \quad \frac{3x^2+18x+27}{5x^2-45} = \frac{3(x^2+6x+9)}{5(x^2-9)}$$

$$\frac{3(x+3)(x+3)}{5(x+3)(x-3)}$$

$$\frac{3(x+3)}{5(x-3)}$$

Which is in the lowest forms

Example

Simplify (i) $\frac{1}{x-y} - \frac{1}{x+y} + \frac{2x}{x^2-y^2}$

(ii) $\frac{2x^2}{x^4-16} - \frac{x}{x^2-4} + \frac{1}{x+2}$

Solution

$$(i) \quad \frac{1}{x-y} - \frac{1}{x+y} + \frac{2x}{x^2-y^2}$$

$$= \frac{1}{x-y} - \frac{1}{x+y} + \frac{2x}{(x+y)(x-y)}$$

$$= \frac{x+y-(x-y)+2x}{(x+y)(x-y)}$$

(L.C.M of denominators)

$$= \frac{\cancel{x}+y-\cancel{x}+y+2x}{(x+y)(x-y)}$$

$$= \frac{2x+2y}{(x+y)(x-y)}$$

$$= \frac{2(x+y)}{(x+y)(x-y)} = \frac{2}{x-y}$$

$$(ii) \quad \frac{2x^2}{x^4-16} - \frac{x}{x^2-4} + \frac{1}{x+2}$$

$$= \frac{2x^2}{(x^2+4)(x^2-4)} - \frac{x}{x^2-4} + \frac{1}{x+2}$$

$$= \frac{2x^2}{(x^2+4)(x+2)(x-2)} - \frac{x}{(x+2)(x-2)} + \frac{1}{x+2}$$

$$= \frac{2x^2 - x(x^2+4) + (x^2+4)(x-2)}{(x^2+4)(x+2)(x-2)} = \frac{\cancel{2x^2} - \cancel{x^3} - \cancel{4x} + \cancel{x^3} + \cancel{4x} - \cancel{2x^2} - 8}{(x^2+4)(x+2)(x-2)}$$

$$= \frac{-8}{(x^2+4)(x+2)(x-2)}$$

$$= \frac{-8}{(x^2+4)(x^2-4)} = \frac{-8}{x^4-16}$$

Example

Find the product $\frac{x+2}{2x-3y} \cdot \frac{4x^2-9y^2}{xy+2y}$

Solution

$$\frac{x+2}{2x-3y} \cdot \frac{4x^2-9y^2}{xy+2y} = \frac{(x+2)[(2x)^2-(3y)^2]}{(2x-3y)(x+2)y}$$

$$= \frac{(x+2)(2x+3y)(2x-3y)}{y(x+2)(2x-3y)}$$

$$= \frac{2x+3y}{y}$$

Example

Simplify $\frac{7xy}{x^2-4x+4} \div \frac{14y}{x^2-4}$

Solution

$$\frac{7xy}{x^2-4x+4} \div \frac{14y}{x^2-4}$$

$$= \frac{7xy}{x^2-4x+4} \times \frac{x^2-4}{14y}$$

$$= \frac{7xy}{(x-2)(x-2)} \times \frac{(x+2)(x-2)}{14y}$$

$$= \frac{x(x+2)}{2(x-2)}$$

Example

Evaluate $\frac{3x^2\sqrt{y}+6}{5(x+y)}$ if $x = -4$ and $y = 9$

Solution

We have, by putting $x = -4$ and $y = 9$,

$$\frac{3x^2\sqrt{y}+6}{5(x+y)} = \frac{3(-4)^2\sqrt{9}+6}{5(-4+9)} = \frac{3(16)(3)+6}{5(5)} = \frac{150}{25} = 6$$

Exercise 4.1

1. Identify whether the following algebraic expression are polynomials (yes or no).

(i) $3x^2 + \frac{1}{x} - 5$ No

(ii) $3x^3 - 4x^2 - x\sqrt{x} + 3$ No

(iii) $x^2 - 3x + \sqrt{2}$ Yes

(iv) $\frac{3x}{2x-1} + 8$ No

2. State whether each of the following expression is a rational expression or not.

(i) $\frac{3\sqrt{x}}{3\sqrt{x}+5}$ No

(ii) $\frac{x^3 - 2x^2 + \sqrt{3}}{2+3x-x^2}$ Yes

(iii) $\frac{x^2+6x+9}{x^2-9}$ Yes

$$(iv) \frac{2\sqrt{x}+3}{2\sqrt{x}-3}$$

No

3. Reduce the following rational expression to the lowest forms.

$$(i) \frac{120x^2y^3z^5}{30x^3yz^2}$$

$$= 4x^{2-3}y^{3-1}z^{5-2}$$

$$= 4x^{-1}y^2z^3$$

$$= \frac{4y^2z^3}{x}$$

$$(ii) \frac{8a(x+1)}{2(x^2-1)} = \frac{4a(\cancel{x+1})}{(x-1)(\cancel{x+1})} = \frac{4a}{x-1}$$

$$(iii) \frac{(x+y)^2-4xy}{(x-y)^2} = \frac{x^2+y^2+2xy-4xy}{(x-y)(x-y)}$$

$$= \frac{x^2+y^2-2xy}{(x-y)(x-y)}$$

$$= \frac{(x-y)^2}{(x-y)(x-y)}$$

$$= \frac{(\cancel{x-y})^2}{(\cancel{x-y})(\cancel{x-y})} = 1$$

$$(iv) \frac{(x^3-y^3)(x^2-2xy+y^2)}{(x-y)(x^2+xy+y^2)}$$

$$= \frac{(\cancel{x^3-y^3})(x-y)^2}{\cancel{x^3-y^3}} = (x-y)^2$$

$$(v) \frac{(x+2)(x^2-1)}{(x+1)(x^2-4)}$$

$$= \frac{(\cancel{x+2})(x-1)(\cancel{x+1})}{(\cancel{x+1})(x-2)(\cancel{x+2})} = \frac{x-1}{x-2}$$

$$(vi) \frac{x^2-4x+4}{2x^2-8} = \frac{(x-2)^2}{2(x^2-4)}$$

$$= \frac{(x-2)^2}{2(x-2)(x+2)}$$

$$= \frac{(\cancel{x-2})(x-2)}{2(\cancel{x-2})(x+2)}$$

$$= \frac{x-2}{2(x+2)}$$

$$(vii) \frac{64x^5-64x}{(8x^2+8)(2x+2)}$$

$$= \frac{64x(x^4-1)}{(8x^2+8) \cdot 2(x+1)}$$

$$= \frac{64x(x^4-1)}{16(x^2+1)(x+1)}$$

$$= \frac{4x(x^2+1)(x^2-1)}{(x^2+1)(x+1)}$$

$$= \frac{4x(\cancel{x^2+1})(x-1)(\cancel{x+1})}{(\cancel{x^2+1})(\cancel{x+1})}$$

$$= 4x(x-1)$$

$$\frac{9x^2-(x^2-4)^2}{4+3x-x^2} = \frac{(3x)^2-(x^2-4)^2}{4+3x-x^2}$$

$$= \frac{(3x+x^2-4)(\cancel{3x-x^2+4})}{(4+3x-x^2)}$$

$$= 3x+x^2-4$$

$$= x^2+3x-4$$

4. Evaluate (a) $\frac{x^3y-2z}{xz}$ for (i) $x = 3$

$y = -1, z = -2$.

$$(a) \frac{(3)^3(-1)-2(-2)}{3(-2)} = \frac{-27+4}{-6}$$

$$= \frac{-23}{-6} = \frac{23}{6} = 3\frac{5}{6}$$

$$\begin{aligned}
 \text{(b)} \quad & \frac{x^2y^3 - 5z^4}{xyz} \text{ for } x = 4, y = -2, z = -1 \\
 &= \frac{(4)^2(-2)^3 - 5(-1)^4}{(4)(-2)(-1)} = \frac{-16(8) - 5}{8} \\
 &= \frac{-128 - 5}{8} = \frac{-133}{8} = -16\frac{5}{8}
 \end{aligned}$$

5. Perform the indicated operation and simplify

$$\begin{aligned}
 \text{(i)} \quad & \frac{15}{2x-3y} - \frac{4}{3y-2x} \\
 &= \frac{15(3y-2x) - 4(2x-3y)}{(2x-3y)(3y-2x)} \\
 &= \frac{45y - 30x - 8x + 12y}{(2x-3y)(3y-2x)} \\
 &= \frac{57y - 38x}{(2x-3y)(3y-2x)} \\
 &= \frac{19(3y-2x)}{(2x-3y)(3y-2x)} = \frac{19}{2x-3y}
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad & \frac{1+2x}{1-2x} - \frac{1-2x}{1+2x} \\
 &= \frac{(1+2x)^2 - (1-2x)^2}{(1-2x)(1+2x)} \\
 &= \frac{(1+4x^2+4x) - (1+4x^2-4x)}{(1-2x)(1+2x)} \\
 &= \frac{\cancel{1} + \cancel{4x^2} + 4x - \cancel{1} - \cancel{4x^2} + 4x}{(1-2x)(1+2x)} \\
 &= \frac{8x}{(1-2x)(1+2x)} = \frac{8x}{1-4x^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{(iii)} \quad & \frac{x^2-25}{x^2-36} - \frac{x+5}{x+6} \\
 &= \frac{(x-5)(x+5)}{(x-6)(x+6)} - \frac{x+5}{x+6} \\
 &= \frac{(x-5)(x+5) - (x+5)(x-6)}{(x+6)(x-6)}
 \end{aligned}$$

$$= \frac{(x-5)(x+5) - (x+5)(x-6)}{(x+6)(x-6)}$$

$$= \frac{(x+5)[(x-5) - (x-6)]}{(x+6)(x-6)}$$

$$= \frac{(x+5)(\cancel{x} - \cancel{5} + x + 6)}{(x+6)(x-6)}$$

$$= \frac{(x+5)(1)}{(x+6)(x-6)} = \frac{x+5}{x^2-36}$$

$$\begin{aligned}
 \text{(iv)} \quad & \frac{x}{x-y} - \frac{y}{x+y} - \frac{2xy}{x^2-y^2} \\
 &= \frac{x(x+y) - y(x-y)}{(x-y)(x+y)} - \frac{2xy}{x^2-y^2} \\
 &= \frac{x^2 + \cancel{xy} - \cancel{xy} + y^2}{x^2-y^2} - \frac{2xy}{x^2-y^2}
 \end{aligned}$$

$$= \frac{x^2+y^2}{x^2-y^2} - \frac{2xy}{x^2-y^2}$$

$$= \frac{x^2+y^2-2xy}{(x^2-y^2)}$$

$$= \frac{(x-y)^2}{(\cancel{x-y})(x+y)} = \frac{x-y}{x+y}$$

$$\begin{aligned}
 \text{(v)} \quad & \frac{x-2}{x^2+6x+9} - \frac{x+2}{2x^2-18} \\
 &= \frac{x-2}{x^2+3x+3x+9} - \frac{x+2}{2(x^2-9)} \\
 &= \frac{x-2}{x(x+3)+3(x+3)} - \frac{x+2}{2(x-3)(x+3)} \\
 &= \frac{x-2}{(x+3)(x+3)} - \frac{x+2}{2(x-3)(x+3)} \\
 &= \frac{2(x-3)(x-2) - (x+3)(x+2)}{2(x-3)(x+3)(x+3)} \\
 &= \frac{2(x^2-2x-3x+6) - (x^2+2x+3x+6)}{2(x-3)(x+3)^2}
 \end{aligned}$$

$$= \frac{2(x^2 - 5x + 6) - (x^2 + 5x + 6)}{2(x-3)(x+3)^2}$$

$$= \frac{2x^2 - 10x + 12 - x^2 - 5x - 6}{2(x-3)(x+3)^2}$$

$$= \frac{x^2 - 15x + 6}{2(x-3)(x+3)^2}$$

$$(vi) \quad \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4-1}$$

$$= \frac{x+1-(x-1)}{(x-1)(x+1)} - \frac{2}{x^2+1} - \frac{4}{x^4-1}$$

$$= \frac{\cancel{x+1} - \cancel{x-1}}{x^2-1} - \frac{2}{x^2+1} - \frac{4}{x^4-1}$$

$$= \frac{2}{x^2-1} - \frac{2}{x^2+1} - \frac{4}{x^4-1}$$

$$= \frac{2(x^2+1) - 2(x^2-1)}{(x^2-1)(x^2+1)} - \frac{4}{x^4-1}$$

$$= \frac{\cancel{2x^2} + 2 - \cancel{2x^2} + 2}{x^4-1} - \frac{4}{x^4-1}$$

$$= \frac{4}{x^4-1} - \frac{4}{x^4-1}$$

$$= \frac{4-4}{x^4-1}$$

$$= \frac{0}{x^4-1}$$

$$= 0$$

6. Perform the indicated operation and simplify:

$$(i) \quad (x^2 - 49) \frac{5x+2}{x+7}$$

$$= (x-7)(\cancel{x+7}) \frac{5x+2}{\cancel{x+7}}$$

$$= (x-7)(5x+2)$$

$$(ii) \quad \frac{4x-12}{x^2-9} \div \frac{18-2x^2}{x^2+6x+9}$$

$$= \frac{4(x-3)}{(x-3)(x+3)} \div \frac{2(9-x^2)}{x^2+3x+3x+9}$$

$$= \frac{4(x-3)}{(x-3)(x+3)} \div \frac{2(3-x)(3+x)}{x(x+3)+3(x+3)}$$

$$= \frac{4(x-3)}{(x-3)(x+3)} \div \frac{2(3-x)(3+x)}{(x+3)(x+3)}$$

$$= \frac{4(x-3)}{(x+3)(x-3)} \times \frac{(x+3)(x+3)}{2(3+x)(3-x)}$$

$$= \frac{2}{3-x}$$

$$(iii) \quad \frac{x^6-y^6}{x^2-y^2} \div (x^4+x^2y^2+y^4)$$

$$= \frac{(x^3)^2 - (y^3)^2}{x^2-y^2} \div (x^4+x^2y^2+y^4)$$

$$= \frac{(x^3-y^3)(x^3+y^3)}{x^2-y^2} \div (x^4+x^2y^2+y^4)$$

$$= \frac{(x-y)(x^2+xy+y^2)(x+y)(x^2-xy+y^2)}{x^2-y^2}$$

$$\times \frac{1}{x^4+x^2y^2+y^4}$$

$$= \frac{(\cancel{x^2-y^2})(x^2+xy+y^2)(x^2-xy+y^2)}{\cancel{x^2-y^2}}$$

$$\times \frac{1}{x^4+x^2y^2+y^4}$$

$$= \frac{x^4+x^2y^2+y^4}{x^4+x^2y^2+y^4} = 1$$

$$(iv) \quad \frac{x^2-1}{x^2+2x+1} \cdot \frac{x+5}{1-x}$$

$$= \frac{-(\cancel{x-1})(x+1)}{x^2+x+x+1} \cdot \frac{x+5}{(x-1)}$$

$$= \frac{-(x+1)(x+5)}{x(x+1)+1(x+1)}$$

$$= \frac{\cancel{-(x+1)}(x+5)}{(x+1)\cancel{(x+1)}} = -\frac{x+5}{x+1}$$

$$(v) \quad \frac{x^2+xy}{y(x+y)} \cdot \frac{x^2+xy}{y(x+y)} + \frac{x^2-x}{xy-2y}$$

$$= \frac{x\cancel{(x+y)}}{y\cancel{(x+y)}} \cdot \frac{\cancel{x}(x+y)}{\cancel{y}(x+y)} + \frac{\cancel{x}(x-2)}{\cancel{y}(x-1)}$$

$$= \frac{x(x-2)}{y(x-1)}$$

Example

If $a+b=7$ and $a-b=3$, then find the value of (a) a^2+b^2 (b) ab

Solution

We are given that $a+b=7$ and $a-b=3$

(a) To find the value of (a^2+b^2) , we use the formula

$$(a+b)^2 + (a-b)^2 = 2(a^2+b^2)$$

Substituting the values $a+b=7$ and $a-b=3$, we get

$$(7)^2 + (3)^2 = 2(a^2+b^2)$$

$$\Rightarrow 49+9 = 2(a^2+b^2)$$

$$\Rightarrow 58 = 2(a^2+b^2)$$

$$\Rightarrow 29 = a^2+b^2,$$

(b) To find the value of ab , we make use of the formula

$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\Rightarrow (7)^2 - (3)^2 = 4ab,$$

$$\Rightarrow 49-9 = 4ab$$

$$\Rightarrow 40 = 4ab,$$

$$\Rightarrow 10 = ab,$$

Hence $a^2+b^2=29$ and $ab=10$.

Example

If $a^2+b^2+c^2=43$ and $ab+bc+ca=3$, then find the value of $a+b+c$.

Solution

We know that

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$(a+b+c)^2 = a^2+b^2+c^2+2(ab+bc+ca)$$

$$\Rightarrow (a+b+c)^2 = 43+2 \times 3$$

(Putting $a^2+b^2+c^2=43$ and $ab+bc+ca=3$)

$$\Rightarrow (a+b+c)^2 = 49$$

$$\Rightarrow a+b+c = \pm\sqrt{49}$$

$$\text{Hence } a+b+c = \pm 7$$

Example

If $a+b+c=6$ and $a^2+b^2+c^2=24$ then find the value of $ab+bc+ca$.

Solution

We have

$$(a+b+c)^2 = a^2+b^2+c^2+2ab+2bc+2ca$$

$$(6)^2 = 24+2(ab+bc+ca)$$

$$\Rightarrow 36 = 24+2(ab+bc+ca)$$

$$\Rightarrow 12 = 2(ab+bc+ca)$$

$$\text{Hence } ab+bc+ca=6$$

Example

If $a+b+c=7$ and $ab+bc+ca=9$, then find the value of $a^2+b^2+c^2$.

Solution

We know that

$$\begin{aligned}(a+b+c)^2 &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\ \Rightarrow (a+b+c)^2 &= a^2 + b^2 + c^2 + 2(ab+bc+ca) \\ \Rightarrow (7)^2 &= a^2 + b^2 + c^2 + 2(9) \\ \Rightarrow 49 &= a^2 + b^2 + c^2 + 18 \\ \Rightarrow 31 &= a^2 + b^2 + c^2\end{aligned}$$

Hence $a^2 + b^2 + c^2 = 31$

Example

If $2x - 3y = 10$ and $xy = 2$, then find the value of $8x^3 - 27y^3$.

Solution

$$\begin{aligned}\text{We are given that } 2x - 3y &= 10 \\ \Rightarrow (2x - 3y)^3 &= (10)^3 \\ \Rightarrow 8x^3 - 27y^3 - 3 \times 2x \times 3y(2x - 3y) &= 1000 \\ \Rightarrow 8x^3 - 27y^3 - 18xy(2x - 3y) &= 1000 \\ \Rightarrow 8x^3 - 27y^3 - 18 \times 2 \times 10 &= 1000 \\ \Rightarrow 8x^3 - 27y^3 - 360 &= 1000\end{aligned}$$

Hence

$$8x^3 - 27y^3 = 1360$$

Example

If $x + \frac{1}{x} = 8$, then find the value of $x^3 + \frac{1}{x^3}$

Solution

$$\begin{aligned}\text{We have been given } x + \frac{1}{x} &= 8 \\ \Rightarrow \left(x + \frac{1}{x}\right)^3 &= (8)^3 \\ \Rightarrow x^3 + \frac{1}{x^3} + 3 \times x \times \frac{1}{x} \left(x + \frac{1}{x}\right) &= 512\end{aligned}$$

$$\begin{aligned}\Rightarrow x^3 + \frac{1}{x^3} + 3 \left(x + \frac{1}{x}\right) &= 512 \\ \Rightarrow x^3 + \frac{1}{x^3} + 3 \times 8 &= 512 \\ \Rightarrow x^3 + \frac{1}{x^3} + 24 &= 512 \\ \Rightarrow x^3 + \frac{1}{x^3} &= 512 - 24 \\ \Rightarrow x^3 + \frac{1}{x^3} &= 488\end{aligned}$$

Example

If $x - \frac{1}{x} = 4$, then find $x^3 - \frac{1}{x^3}$

Solution

$$\begin{aligned}\text{We have } x - \frac{1}{x} &= 4 \\ \Rightarrow \left(x - \frac{1}{x}\right)^3 &= (4)^3 \\ \Rightarrow x^3 - \frac{1}{x^3} - 3x \times \frac{1}{x} \left(x - \frac{1}{x}\right) &= 64 \\ \Rightarrow x^3 - \frac{1}{x^3} - 3(4) &= 64 \\ \Rightarrow x^3 - \frac{1}{x^3} - 12 &= 64 \\ \Rightarrow x^3 - \frac{1}{x^3} &= 64 + 12 \\ \Rightarrow x^3 - \frac{1}{x^3} &= 76\end{aligned}$$

Example

Factorize $64x^3 + 343y^3$

Solution

$$\begin{aligned}\text{We have} \\ 64x^3 + 343y^3 &= (4x)^3 + (7y)^3\end{aligned}$$

$$\begin{aligned}
 &= (4x+7y)[(4x)^2 - (4x)(7y) + (7y)^2] \\
 &= (4x+7y)(16x^2 - 28xy + 49y^2)
 \end{aligned}$$

Example

Factorize $125x^3 - 1331y^3$

Solution

We have

$$\begin{aligned}
 125x^3 - 1331y^3 &= (5x)^3 - (11y)^3 \\
 &= (5x-11y)[(5x)^2 + (5x)(11y) + (11y)^2] \\
 &= (5x-11y)(25x^2 + 55xy + 121y^2)
 \end{aligned}$$

Example

Factorize

$$\left(\frac{2}{3}x + \frac{3}{2x}\right)\left(\frac{4}{9}x^2 - 1 + \frac{9}{4x^2}\right)$$

Solution

$$\begin{aligned}
 &\left(\frac{2}{3}x + \frac{3}{2x}\right)\left(\frac{4}{9}x^2 - 1 + \frac{9}{4x^2}\right) \\
 &= \left(\frac{2}{3}x + \frac{3}{2x}\right)\left[\left(\frac{2}{3}x\right)^2 - \left(\frac{2}{3}x\right)\left(\frac{3}{2x}\right) + \left(\frac{3}{2x}\right)^2\right] \\
 &= \left(\frac{2}{3}x\right)^3 + \left(\frac{3}{2x}\right)^3 \\
 &= \frac{8}{27}x^3 + \frac{27}{8x^3}
 \end{aligned}$$

Example

Find the product $\left(\frac{4}{5}x - \frac{5}{4x}\right)$

$$\left(\frac{16}{25}x^2 + \frac{25}{16x^2} + 1\right)$$

Solution

$$\begin{aligned}
 &\left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16}{25}x^2 + \frac{25}{16x^2} + 1\right) \\
 &= \left(\frac{4}{5}x - \frac{5}{4x}\right)\left(\frac{16x^2}{25} + 1 + \frac{25}{16x^2}\right) \\
 &\text{(rearranging)} \\
 &= \left(\frac{4}{5}x - \frac{5}{4x}\right)\left[\left(\frac{4}{5}x\right)^2 + \left(\frac{4}{5}x\right)\left(\frac{5}{4x}\right) + \left(\frac{5}{4x}\right)^2\right] \\
 &= \left(\frac{4}{5}x\right)^3 - \left(\frac{5}{4x}\right)^3 = \frac{64}{125}x^3 - \frac{125}{64x^3}
 \end{aligned}$$

Example

Find the continued product of $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$

Solution

$$\begin{aligned}
 &(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2) \\
 &= (x+y)(x^2-xy+y^2)(x-y)(x^2+xy+y^2) \\
 &= (x^3+y^3)(x^3-y^3) = (x^3)^2 - (y^3)^2 = x^6 - y^6
 \end{aligned}$$

Exercise 4.2

1.(i) If $a + b = 10$ and $a - b = 6$ then find value of $a^2 + b^2$.

Solution:

$$\begin{aligned}
 2(a^2 + b^2) &= (a+b)^2 + (a-b)^2 \\
 2(a^2 + b^2) &= (10)^2 + (6)^2 \\
 2(a^2 + b^2) &= 100 + 36 \\
 a^2 + b^2 &= \frac{136}{2} = 68
 \end{aligned}$$

(ii) If $a + b = 5$, $a - b = \sqrt{17}$ then find value of ab .

Solution:

$$\begin{aligned}
 4ab &= (a+b)^2 - (a-b)^2 \\
 4ab &= (5)^2 - (\sqrt{17})^2 \\
 4ab &= 25 - 17 \\
 4ab &= 8
 \end{aligned}$$

$$ab = \frac{8}{4} = 2$$

2. If $a^2 + b^2 + c^2 = 45$ and $a + b + c = -1$ find value of $ab + bc + ca$.

Solution:

$$a+b+c = -1$$

Squaring

$$(a+b+c)^2 = (-1)^2$$

$$a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 1$$

$$a^2 + b^2 + c^2 + 2(ab + bc + ca) = 1$$

$$45 + 2(ab + bc + ca) = 1$$

$$2(ab + bc + ca) = 1 - 45$$

$$2(ab + bc + ca) = -44$$

$$ab + bc + ca = \frac{-44}{2} = -22$$

3. If $m+n+p = 10$, $mn + np + pm = 27$ find value of $m^2 + n^2 + p^2$.

Solution:

$$m+n+p = 10$$

Squaring both sides

$$(m+n+p)^2 = (10)^2$$

$$m^2 + n^2 + p^2 + 2mn + 2np + 2mp = 100$$

$$m^2 + n^2 + p^2 + 2(mn + np + mp) = 100$$

$$m^2 + n^2 + p^2 + 2(27) = 100$$

$$m^2 + n^2 + p^2 + 54 = 100$$

$$m^2 + n^2 + p^2 = 100 - 54$$

$$m^2 + n^2 + p^2 = 46$$

4. If $x^2 + y^2 + z^2 = 78$ and $y + yz + zx = 59$ find $x + y + z$.

Solution:

$$(x+y+z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$$

$$= x^2 + y^2 + z^2 + 2(xy + yz + zx)$$

$$= 78 + 2(59)$$

$$= 78 + 118$$

$$= 196$$

$$\sqrt{(x+y+z)^2} = \sqrt{196} = \sqrt{(\pm 14)^2}$$

$$x + y + z = \pm 14$$

5. If $x + y + z = 12$ and $x^2 + y^2 + z^2 = 64$ find value of $xy + yz + zx$.

Solution:

$$x + y + z = 12$$

Squaring both sides

$$(x + y + z)^2 = (12)^2$$

$$x^2 + y^2 + z^2 + 2xy + 2yz + 2zx = 144$$

$$x^2 + y^2 + z^2 + 2(xy + yz + zx) = 144$$

$$64 + 2(xy + yz + zx) = 144$$

$$2(xy + yz + zx) = 144 - 64$$

$$2(xy + yz + zx) = 80$$

$$xy + yz + zx = \frac{80}{2} = 40.$$

6. If $x + y = 7$ and $xy = 12$ then find value of $x^3 + y^3$.

Solution:

$$x + y = 7$$

$$(x + y)^3 = (7)^3$$

$$x^3 + y^3 + 3xy(x + y) = 343$$

$$x^3 + y^3 + 3(12)(7) = 343$$

$$x^3 + y^3 + 252 = 343$$

$$x^3 + y^3 = 343 - 252$$

$$x^3 + y^3 = 91$$

7. If $3x + 4y = 11$ and $xy = 12$ then find value of $27x^3 + 64y^3$.

Solution:

$$3x + 4y = 11$$

$$(3x + 4y)^3 = (11)^3$$

$$(3x)^3 + (4y)^3 + 3(3x)(4y)(3x + 4y) = 1331$$

$$27x^3 + 64y^3 + 36xy(3x + 4y) = 1331$$

$$27x^3 + 64y^3 + 36(12)(11) = 1331$$

$$27x^3 + 64y^3 + 4752 = 1331$$

$$27x^3 + 64y^3 = 1331 - 4752 = -3421$$

8. If $x - y = 4$ and $xy = 21$ then find value of $x^3 - y^3$.

Solution:

$$x - y = 4$$

$$(x - y)^3 = (4)^3$$

$$x^3 - y^3 - 3xy(x - y) = 64$$

$$x^3 - y^3 - 3(21)(4) = 64$$

$$x^3 - y^3 - 252 = 64$$

$$x^3 - y^3 = 64 + 252$$

$$x^3 - y^3 = 316$$

9. If $5x - 6y = 13$ and $xy = 6$ then find value of $125x^3 - 216y^3$.

Solution:

$$5x - 6y = 13$$

$$\Rightarrow (5x - 6y)^3 = (13)^3$$

$$\Rightarrow (5x)^3 - (6y)^3 - 3(5x)(6y)(5x - 6y) = 2197$$

$$125x^3 - 216y^3 - 90xy(5x - 6y) = 2197$$

$$125x^3 - 216y^3 - 90(6)(13) = 2197$$

$$125x^3 - 216y^3 - 7020 = 2197$$

$$125x^3 - 216y^3 = 2197 + 7020$$

$$125x^3 - 216y^3 = 9217$$

10. If $x + \frac{1}{x} = 3$ then find $x^3 + \frac{1}{x^3}$.

$$x + \frac{1}{x} = 3 \text{ Cubing both sides}$$

$$\left(x + \frac{1}{x}\right)^3 = (3)^3$$

$$x^3 + \frac{1}{x^3} + 3(x)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) = 27$$

$$x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = 27$$

$$x^3 + \frac{1}{x^3} + 3(3) = 27$$

$$x^3 + \frac{1}{x^3} = 27 - 9$$

$$x^3 + \frac{1}{x^3} = 18$$

11. If $x - \frac{1}{x} = 7$, then find value of

$$x^3 - \frac{1}{x^3}$$

$$x - \frac{1}{x} = 7 \text{ Taking cube of both sides}$$

$$\left(x - \frac{1}{x}\right)^3 = (7)^3$$

$$x^3 - \frac{1}{x^3} - 3(x)\left(\frac{1}{x}\right)\left(x - \frac{1}{x}\right) = 343$$

$$x^3 - \frac{1}{x^3} - 3\left(x - \frac{1}{x}\right) = 343$$

$$x^3 - \frac{1}{x^3} - 3(7) = 343$$

$$x^3 - \frac{1}{x^3} - 21 = 343$$

$$x^3 - \frac{1}{x^3} = 343 + 21$$

$$x^3 - \frac{1}{x^3} = 364$$

12. If $3x + \frac{1}{3x} = 5$, then find value of

$$27x^3 + \frac{1}{27x^3}$$

$$\left(3x + \frac{1}{3x}\right)^3 = (5)^3$$

$$(3x)^3 + \left(\frac{1}{3x}\right)^3 + 3(3x)\left(\frac{1}{3x}\right)\left(3x + \frac{1}{3x}\right) = 125$$

$$27x^3 + \frac{1}{27x^3} + 3\left(3x + \frac{1}{3x}\right) = 125$$

$$27x^3 + \frac{1}{27x^3} + 3(5) = 125$$

$$27x^3 + \frac{1}{27x^3} + 15 = 125$$

$$27x^3 + \frac{1}{27x^3} = 125 - 15$$

$$27x^3 + \frac{1}{27x^3} = 110$$

13. If $\left(5x - \frac{1}{5x}\right) = 6$, then find value of

$$125x^3 - \frac{1}{25x^3}$$

$$\left(5x - \frac{1}{5x}\right) = 6$$

Taking cube of both sides

$$\left(5x - \frac{1}{5x}\right)^3 = (6)^3$$

$$(5x)^3 - \left(\frac{1}{5x}\right)^3 - 3\left(5x\right)\left(\frac{1}{5x}\right)\left(5x - \frac{1}{5x}\right) = 216$$

$$125x^3 - \frac{1}{125x^3} - 3\left(5x - \frac{1}{5x}\right) = 216$$

$$125x^3 - \frac{1}{125x^3} - 3(6) = 216$$

$$125x^3 - \frac{1}{25x^3} - 18 = 216$$

$$125x^3 - \frac{1}{125x^3} = 216 + 18$$

$$125x^3 - \frac{1}{125x^3} = 234$$

14. Factorize (i) $x^3 - y^3 - x + y$

$$(i) \quad x^3 - y^3 - x + y$$

$$= (x - y)(x^2 + xy + y^2) - 1(x - y)$$

$$= (x - y)[x^2 + xy + y^2 - 1]$$

$$(ii) \quad 8x^3 - \frac{1}{27y^3}$$

$$= (2x)^3 - \left(\frac{1}{3y}\right)^3$$

$$= \left(2x - \frac{1}{3y}\right) \left[(2x)^2 + (2x)\left(\frac{1}{3y}\right) + \left(\frac{1}{3y}\right)^2 \right]$$

$$= \left(2x - \frac{1}{3y}\right) \left[4x^2 + \frac{2x}{3y} + \frac{1}{9y^2} \right]$$

15. Find products, using formulae

$$(i) \quad (x^2 + y^2)(x^4 - x^2y^2 + y^4)$$

$$= (x^2)^3 + (y^2)^3$$

$$\text{Ref. } (a + b)(a^2 - ab + b^2) = a^3 + b^3$$

$$= x^6 + y^6$$

$$(ii) \quad (x^3 - y^3)(x^6 + x^3y^3 + y^6)$$

$$= (x^3)^3 - (y^3)^3$$

$$\text{Ref. } (a - b)(a^2 + ab + b^2) = a^3 - b^3$$

$$= x^9 - y^9$$

$$(iii) \quad (x - y)(x + y)(x^2 + y^2)(x^2 + xy + y^2)$$

$$(x^2 - xy + y^2)(x^4 - x^2y^2 + y^4)$$

$$= (x - y)(x^2 + xy + y^2)(x + y)(x^2 - xy + y^2)$$

$$(x^2 + y^2)(x^4 - x^2y^2 + y^4)$$

$$= (x^3 - y^3)(x^3 + y^3) \left[(x^2)^3 + (y^2)^3 \right]$$

$$= \left[(x^3)^2 - (y^3)^2 \right] (x^6 + y^6)$$

$$= (x^6 - y^6)(x^6 + y^6)$$

$$= (x^6)^2 - (y^6)^2$$

$$= x^{12} - y^{12}$$

$$16. \quad (2x^2 - 1)(2x^2 + 1)(4x^4 + 2x^2 + 1)$$

$$(4x^4 - 2x^2 + 1)$$

$$= (2x^2 - 1)(4x^4 + 2x^2 + 1)(2x^2 + 1)$$

$$(4x^4 - 2x^2 + 1)$$

$$= ((2x^2)^3 - (1)^3)((2x^2)^3 + (1)^3)$$

$$= (8x^6 - 1)(8x^6 + 1)$$

$$= (8x^6)^2 - (1)^2$$

$$= 64x^{12} - 1$$

Define Surd

An irrational radical with rational radicand is called a surd.

Hence the radical $\sqrt[n]{a}$ is a surd if

- (i) a is rational
- (ii) the result $\sqrt[n]{a}$ is irrational.

e.g., $\sqrt{3}, \sqrt{2/5}, \sqrt[3]{7}, \sqrt[4]{10}$ are surds.

But $\sqrt{\pi}$ is not surd because π is not rational.

Note: Every surd is an irrational number but every irrational number is not surd

Example

Simplify by combining similar terms.

$$(i) \quad 4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$$

$$(ii) \quad \sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$$

Solution

$$(i) \quad 4\sqrt{3} - 3\sqrt{27} + 2\sqrt{75}$$

$$= 4\sqrt{3} - 3\sqrt{9 \times 3} + 2\sqrt{25 \times 3} = 4\sqrt{3} - 3\sqrt{9} \sqrt{3} + 2\sqrt{25} \times \sqrt{3}$$

$$= 4\sqrt{3} - 9\sqrt{3} + 10\sqrt{3} = (4 - 9 + 10)\sqrt{3} = 5\sqrt{3}$$

$$(ii) \quad \sqrt[3]{128} - \sqrt[3]{250} + \sqrt[3]{432}$$

$$= \sqrt[3]{64 \times 2} - \sqrt[3]{125 \times 2} + \sqrt[3]{216 \times 2}$$

$$= \sqrt[3]{(4)^3 \times 2} - \sqrt[3]{(5)^3 \times 2} + \sqrt[3]{(6)^3 \times 2}$$

$$= \sqrt[3]{(4)^3} \sqrt[3]{2} - \sqrt[3]{(5)^3} \sqrt[3]{2} + \sqrt[3]{(6)^3} \sqrt[3]{2}$$

$$= 4\sqrt[3]{2} - 5\sqrt[3]{2} + 6\sqrt[3]{2} = (4 - 5 + 6)\sqrt[3]{2} = 5\sqrt[3]{2}$$

Example

Simplify and express the answer in the simplest form.

$$(i) \quad \sqrt{14}\sqrt{35}$$

$$(ii) \quad \frac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}}$$

Solution

$$(i) \quad \sqrt{14}\sqrt{35} = \sqrt{14 \times 35} = \sqrt{7 \times 2 \times 7 \times 5} = \sqrt{(7)^2 \times 2 \times 5}$$

(ii) We have $\frac{\sqrt[6]{12}}{\sqrt{3} \sqrt[3]{2}}$

Thus $\sqrt{3} = (3)^{1/2} = (3)^{3/6} = \sqrt[6]{3^3} = \sqrt[6]{27}$

Hence $\frac{\sqrt[6]{12}}{\sqrt{3}\sqrt[3]{2}} = \frac{\sqrt[6]{12}}{\sqrt[6]{27}\sqrt[6]{4}} = \frac{\sqrt[6]{12}}{\sqrt[6]{108}} = \sqrt[6]{\frac{12}{108}} = \sqrt[6]{\frac{1}{9}}$

$$\sqrt[6]{\left(\frac{1}{3}\right)^2} = \left(\frac{1}{3}\right)^{2/6} = \left(\frac{1}{3}\right)^{1/3} = \sqrt[3]{\frac{1}{3}}$$

1. Express each of the following surd in the simplest form.

$$\begin{aligned} \text{(ii)} \quad & 3\sqrt{162} \\ &= 3\sqrt{2 \times 3 \times 3 \times 3 \times 3} \\ &= 3(3 \times 3)\sqrt{2} \\ &= 27\sqrt{2} \end{aligned}$$

$$= \frac{3}{4} (2^3 \times 2^3 \times 2)^{\frac{1}{3}}$$

$$\begin{aligned} &= \frac{3}{4} (2^3)^{\frac{1}{3}} \times (2^3)^{\frac{1}{3}} \times 2^{\frac{1}{3}} \\ &= \frac{3}{4} (2)(2) \times \sqrt[3]{2} \\ &= 3\sqrt[3]{2} \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad & \sqrt[5]{96x^6y^7z^8} \\
 &= \sqrt[5]{2x2x2x2x2x3x^6y^7z^8} \\
 &= \left(2^5 x^3 x^5 \cdot x \cdot y^5 \cdot y^2 \cdot z^5 \cdot z^3 \right)^{\frac{1}{5}} \\
 &= (2^5)^{\frac{1}{5}} (3)^{\frac{1}{5}} (x^5)^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot (y^5)^{\frac{1}{5}} \cdot (y^2)^{\frac{1}{5}} \cdot (z^5)^{\frac{1}{5}} (z^3)^{\frac{1}{5}} \\
 &= 2 \cdot 3^{\frac{1}{5}} \cdot x \cdot x^{\frac{1}{5}} y \cdot y^{\frac{2}{5}} z \cdot z^{\frac{3}{5}} \\
 &= 2xyz \cdot 3^{\frac{1}{5}} \cdot x^{\frac{1}{5}} \cdot y^{\frac{2}{5}} \cdot z^{\frac{3}{5}}
 \end{aligned}$$

$$= 2xyz\sqrt[5]{3xy^2z^3}$$

2. Simplify

$$(i) \quad \frac{\sqrt{18}}{\sqrt{3} \cdot \sqrt{2}} = \frac{\sqrt{3 \cdot 3 \cdot 2}}{\sqrt{3} \cdot \sqrt{2}} = \frac{3\cancel{\sqrt{2}}}{\sqrt{3} \cdot \cancel{\sqrt{2}}} \\ = \frac{3}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\cancel{3}\sqrt{3}}{\cancel{3}} = \sqrt{3}$$

$$(ii) \quad \frac{\sqrt{21} \times \sqrt{9}}{\sqrt{63}} = \frac{\sqrt{3 \times 7} \times \sqrt{3 \times 3}}{\sqrt{3 \times 3 \times 7}} \\ = \frac{\sqrt{3 \times 7 \times 3 \times 3}}{\sqrt{3 \times 3 \times 7}} \\ = \frac{\cancel{3}\sqrt{21}}{\cancel{3}\sqrt{7}} = \sqrt{\frac{21}{7}} \\ = \sqrt{3}$$

$$(iii) \quad \sqrt[5]{243x^5y^{10}z^{15}} \\ = (3^5 \cdot x^5 y^{10} z^{15})^{\frac{1}{5}} \\ = (3^5)^{\frac{1}{5}} (x^5)^{\frac{1}{5}} (y^{10})^{\frac{1}{5}} (z^{15})^{\frac{1}{5}} \\ = 3xy^2z^3$$

$$(iv) \quad \frac{4}{5}\sqrt[3]{125} \\ = \frac{4}{5} \left(\cancel{5}^3 \right)^{\frac{1}{3}} \\ = 4$$

$$(v) \quad \sqrt{21} \times \sqrt{7} \times \sqrt{3} \\ = \sqrt{3 \times 7} \times \sqrt{7} \times \sqrt{3} \\ = \sqrt{3 \times 7 \times 7 \times 3} = (3^2 \times 7^2)^{\frac{1}{2}} \\ = (3^2)^{\frac{1}{2}} \times (7^2)^{\frac{1}{2}} \\ = 3 \times 7$$

$$= 21$$

3. Simplify by combining similar terms:

$$(i) \quad \sqrt{45} - 3\sqrt{20} + 4\sqrt{5} \\ = \sqrt{9 \times 5} - 3\sqrt{4 \times 5} + 4\sqrt{5} \\ = 3\sqrt{5} - 6\sqrt{5} + 4\sqrt{5} \\ = (3 - 6 + 4)\sqrt{5} \\ = (-3 + 4)\sqrt{5} \\ = \sqrt{5}$$

$$(ii) \quad 4\sqrt{12} + 5\sqrt{27} - 3\sqrt{75} + \sqrt{300} \\ = 4\sqrt{3 \times 4} + 5\sqrt{3 \times 3 \times 3} - 3\sqrt{3 \times 5 \times 5} \\ + \sqrt{3 \times 2 \times 5 \times 2 \times 5} \\ = 8\sqrt{3} + 15\sqrt{3} - 15\sqrt{3} + 10\sqrt{3} \\ = (8 + 15 - 15 + 10)\sqrt{3} \\ = 18\sqrt{3}$$

$$(iii) \quad \sqrt{3}(2\sqrt{3} + 3\sqrt{3}) \\ = \sqrt{3}((2 + 3)\sqrt{3}) \\ = \sqrt{3}(5\sqrt{3}) \\ = 5\sqrt{3} \times \sqrt{3} \\ = 5(\sqrt{3} \times 3) \\ = 5(3) \\ = 15$$

$$(iv) \quad 2(6\sqrt{5} - 3\sqrt{5}) \\ = 2((6 - 3)\sqrt{5}) \\ = 2(3\sqrt{5}) \\ = 6\sqrt{5}$$

4. Simplify:

$$(i) \quad (3 + \sqrt{3})(3 - \sqrt{3}) \\ = (3)^2 - (\sqrt{3})^2$$

$$= 9 - 3$$

$$= 6$$

$$\begin{aligned} \text{(ii)} \quad & (\sqrt{5} + \sqrt{3})^2 \\ &= (\sqrt{5})^2 + (\sqrt{3})^2 + 2\sqrt{5}\sqrt{3} \\ &= 5 + 3 + 2\sqrt{15} \\ &= 8 + 2\sqrt{15} \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad & (\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) \\ &= (\sqrt{5})^2 - (\sqrt{3})^2 \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad & \left(\sqrt{2} + \frac{1}{\sqrt{3}}\right)\left(\sqrt{2} - \frac{1}{\sqrt{3}}\right) \\ &= (\sqrt{2})^2 - \left(\frac{1}{\sqrt{3}}\right)^2 \end{aligned}$$

$$= 2 - \frac{1}{3}$$

$$= \frac{6-1}{3} = \frac{5}{3}$$

$$\begin{aligned} \text{(v)} \quad & (\sqrt{x} + \sqrt{y})(\sqrt{x} - \sqrt{y})(x + y) \\ & (x^2 + y^2) \\ &= \left((\sqrt{x})^2 - (\sqrt{y})^2\right)(x + y)(x^2 + y^2) \\ &= (x - y)(x + y)(x^2 + y^2) \\ &= (x^2 - y^2)(x^2 + y^2) \\ &= (x^2)^2 - (y^2)^2 \\ &= x^4 - y^4 \end{aligned}$$

Define monomial surd

- (i) A surd which contains a single term is called a monomial surd. e.g., $\sqrt{2}, \sqrt{3}$ etc.
- (ii) A surd which contains sum of two monomial surds or sum of a monomial surd and a rational number is called a binomial surd.
e.g., $\sqrt{3} + \sqrt{7}$ or $\sqrt{2} + 5$ $\sqrt{11} - 8$ etc.
- (iii) If the product of two surds is a rational number, then each surd is called the rationalizing factor of the other.

Example

$$\text{Rationalize the denominator } \frac{58}{7 - 2\sqrt{5}}$$

Solution

To rationalize the denominator, we multiply both the numerator and denominator by the conjugate $(7 + 2\sqrt{5})$ of $(7 - 2\sqrt{5})$, i.e.

$$\frac{58}{7 - 2\sqrt{5}} = \frac{58}{7 - 2\sqrt{5}} \times \frac{7 + 2\sqrt{5}}{7 + 2\sqrt{5}} = \frac{58(7 + 2\sqrt{5})}{(7)^2 - (2\sqrt{5})^2}$$

$$\begin{aligned}
 &= \frac{58(7+2\sqrt{5})}{49-20}; \text{ (radical is eliminated in the denominator)} \\
 &= \frac{58(7+2\sqrt{5})}{29} = 2(7+2\sqrt{5})
 \end{aligned}$$

Example

Rationalize the denominator $\frac{2}{\sqrt{5}+\sqrt{2}}$

Solution

Multiplying both the numerator and denominator by the conjugate $(\sqrt{5}-\sqrt{2})$ of $(\sqrt{5}+\sqrt{2})$, to get

$$\begin{aligned}
 \frac{2}{\sqrt{5}+\sqrt{2}} &= \frac{2}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}} = \frac{2(\sqrt{5}-\sqrt{2})}{(\sqrt{5})^2-(\sqrt{2})^2} = \frac{2(\sqrt{5}-\sqrt{2})}{5-2} \\
 &= \frac{2(\sqrt{5}-\sqrt{2})}{3} = \frac{2(\sqrt{5}-\sqrt{2})}{3}
 \end{aligned}$$

Example

Simplify $\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}}$

Solution

First we shall rationalize the denominators and then simplify. We have

$$\begin{aligned}
 &\frac{6}{2\sqrt{3}-\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \\
 &= \frac{6}{2\sqrt{3}-\sqrt{6}} \times \frac{2\sqrt{3}+\sqrt{6}}{2\sqrt{3}+\sqrt{6}} + \frac{\sqrt{6}}{\sqrt{3}+\sqrt{2}} \times \frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} - \frac{4\sqrt{3}}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}} \\
 &= \frac{6(2\sqrt{3}+\sqrt{6})}{(2\sqrt{3})^2-(\sqrt{6})^2} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{(\sqrt{3})^2-(\sqrt{2})^2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{(\sqrt{6})^2-(\sqrt{2})^2} \\
 &= \frac{6(2\sqrt{3}+\sqrt{6})}{12-6} + \frac{\sqrt{6}(\sqrt{3}-\sqrt{2})}{3-2} - \frac{4\sqrt{3}(\sqrt{6}+\sqrt{2})}{6-2} \\
 &= \frac{12\sqrt{3}+6\sqrt{6}}{6} + \frac{\sqrt{6}\sqrt{3}-\sqrt{6}\sqrt{2}}{1} - \frac{4\sqrt{3}\sqrt{6}+4\sqrt{3}\sqrt{2}}{4} \\
 &= 2\sqrt{3}+\sqrt{6}+3\sqrt{2}-2\sqrt{3}-3\sqrt{2}-\sqrt{6} = 0
 \end{aligned}$$

Example

Find rational numbers x and y such that $\frac{4+3\sqrt{5}}{4-3\sqrt{5}} = x + y\sqrt{5}$

Solution

$$\begin{aligned}\frac{4+3\sqrt{5}}{4-3\sqrt{5}} &= \frac{4+3\sqrt{5}}{4-3\sqrt{5}} \times \frac{4+3\sqrt{5}}{4+3\sqrt{5}} = \frac{(4+3\sqrt{5})^2}{(4)^2 - (3\sqrt{5})^2} \\ &= \frac{16+24\sqrt{5}+45}{16-45} = \frac{61+24\sqrt{5}}{-29}\end{aligned}$$

$$\Rightarrow \frac{-61}{29} - \frac{24}{29}\sqrt{5} = x + y\sqrt{5} \quad (\text{given})$$

Hence, on comparing the two sides, we get

$$x = \frac{-61}{29}, \quad y = \frac{-24}{29}$$

Example

If $x = 3 + \sqrt{8}$, then evaluate

(i) $x + \frac{1}{x}$ and (ii) $x^2 + \frac{1}{x^2}$

Solution

Since $x = 3 + \sqrt{8}$, therefore,

$$\begin{aligned}\frac{1}{x} &= \frac{1}{3+\sqrt{8}} = \frac{1}{3+\sqrt{8}} \times \frac{3-\sqrt{8}}{3-\sqrt{8}} = \frac{3-\sqrt{8}}{(3)^2 - (\sqrt{8})^2} \\ &= \frac{3-\sqrt{8}}{9-8} = 3-\sqrt{8}\end{aligned}$$

(i) $x + \frac{1}{x} = 3 + \sqrt{8} + 3 - \sqrt{8} = 6$

(ii) $\left(x + \frac{1}{x}\right)^2 = 36$

or $x^2 + 2x \times \frac{1}{x} + \frac{1}{x^2} = 36$

or $x^2 + \frac{1}{x^2} = 34$

Exercise 4.4

1. Rationalize the denominator

$$(i) \quad \frac{3}{4\sqrt{3}} = \frac{3}{4\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{3\sqrt{3}}{4\sqrt{3} \times 3}$$

$$= \frac{3\sqrt{3}}{4(3)} = \frac{\sqrt{3}}{4}$$

$$(ii) \quad \frac{14}{\sqrt{98}} = \frac{14}{7\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$$

$$= \frac{14\sqrt{2}}{14} = \sqrt{2}$$

$$(iii) \quad \frac{6}{\sqrt{8} \cdot \sqrt{27}} = \frac{6}{2\sqrt{2} \cdot 3\sqrt{3}}$$

$$= \frac{\cancel{6}}{\cancel{6}\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}} \times \frac{\sqrt{6}}{\sqrt{6}}$$

$$= \frac{\sqrt{6}}{6}$$

$$(iv) \quad \frac{1}{3+2\sqrt{5}} = \frac{1}{3+2\sqrt{5}} \times \frac{3-2\sqrt{5}}{3-2\sqrt{5}}$$

$$= \frac{3-2\sqrt{5}}{(3)^2 - (2\sqrt{5})^2} = \frac{3-2\sqrt{5}}{9-20}$$

$$= \frac{3-2\sqrt{5}}{-11}$$

$$(v) \quad \frac{15}{\sqrt{31}-4}$$

$$= \frac{15}{\sqrt{31}-4} \times \frac{\sqrt{31}+4}{\sqrt{31}+4}$$

$$= \frac{15(\sqrt{31}+4)}{(\sqrt{31})^2 - (4)^2}$$

$$= \frac{15(\sqrt{31}+4)}{31-16}$$

$$= \frac{15(\sqrt{31}+4)}{15}$$

$$= \sqrt{31}+4$$

$$(vi) \quad \frac{2}{\sqrt{5}-\sqrt{3}} = \frac{2}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$= \frac{2(\sqrt{5}+\sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2}$$

$$= \frac{2(\sqrt{5}+\sqrt{3})}{5-3}$$

$$= \frac{\cancel{2}(\sqrt{5}+\sqrt{3})}{\cancel{2}}$$

$$= \sqrt{5}+\sqrt{3}$$

$$(vii) \quad \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}-1)(\sqrt{3}-1)}{(\sqrt{3})^2 - (1)^2}$$

$$= \frac{(\sqrt{3}-1)^2}{3-1}$$

$$= \frac{(\sqrt{3})^2 + 1^2 - 2(1)\sqrt{3}}{2}$$

$$= \frac{3+1-2\sqrt{3}}{2}$$

$$= \frac{4-2\sqrt{3}}{2}$$

$$= \frac{2(2-\sqrt{3})}{2}$$

$$= 2-\sqrt{3}$$

$$(viii) \quad \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} \times \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$= \frac{(\sqrt{5}+\sqrt{3})^2}{(\sqrt{5})^2 - (\sqrt{3})^2}$$

$$= \frac{(\sqrt{5}+\sqrt{3})^2}{5-3}$$

$$= \frac{(\sqrt{5})^2 + (\sqrt{3})^2 + 2(\sqrt{5})(\sqrt{3})}{2}$$

$$= \frac{5+3+2\sqrt{15}}{2}$$

$$= \frac{8+2\sqrt{15}}{2}$$

$$= \frac{2(4+\sqrt{15})}{2}$$

$$= 4+\sqrt{15}$$

(2) Find conjugate of $x+\sqrt{y}$:

(i) $3+\sqrt{7}$

Conjugate of $3+\sqrt{7}$ is $3-\sqrt{7}$

(ii) $4-\sqrt{5}$

Conjugate of $4-\sqrt{5}$ is $4+\sqrt{5}$

(iii) $2+\sqrt{3}$

Conjugate of $2+\sqrt{3}$ is $2-\sqrt{3}$

(iv) $2+\sqrt{5}$

Conjugate of $2+\sqrt{5}$ is $2-\sqrt{5}$

(v) $5+\sqrt{7}$

Conjugate of $5+\sqrt{7}$ is $5-\sqrt{7}$

(vi) $4-\sqrt{15}$

Conjugate of $4-\sqrt{15}$ is $4+\sqrt{15}$

(vii) $7-\sqrt{6}$

Conjugate of $7-\sqrt{6}$ is $7+\sqrt{6}$

(viii) $9+\sqrt{2}$

Conjugate of $9+\sqrt{2}$ is $9-\sqrt{2}$

Q.3 If $x=2-\sqrt{3}$ find $\frac{1}{x}$

(i) $x = 2-\sqrt{3}$

$$\frac{1}{x} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}}$$

$$\frac{1}{x} = \frac{2+\sqrt{3}}{(2)^2 - (\sqrt{3})^2}$$

$$\frac{1}{x} = \frac{2+\sqrt{3}}{4-3}$$

$$\frac{1}{x} = 2+\sqrt{3}$$

(ii) $x=4-\sqrt{17}$ find $\frac{1}{x}$

$$\frac{1}{x} = \frac{1}{4-\sqrt{17}} \times \frac{4+\sqrt{17}}{4+\sqrt{17}}$$

$$\frac{1}{x} = \frac{4+\sqrt{17}}{(4)^2 - (\sqrt{17})^2}$$

$$= \frac{4+\sqrt{17}}{16-17}$$

$$= \frac{4+\sqrt{17}}{-1}$$

$$= -(4+\sqrt{17})$$

$$= -4-\sqrt{17}$$

(iii) If $x = \sqrt{3} + 2$, find $x + \frac{1}{x}$

$$x = \sqrt{3} + 2$$

$$\frac{1}{x} = \frac{1}{\sqrt{3} + 2} \times \frac{\sqrt{3} - 2}{\sqrt{3} - 2}$$

$$\frac{1}{x} = \frac{\sqrt{3} - 2}{(\sqrt{3})^2 - (2)^2}$$

$$\frac{1}{x} = \frac{\sqrt{3} - 2}{3 - 4}$$

$$\frac{1}{x} = \frac{\sqrt{3} - 2}{-1}$$

$$\frac{1}{x} = -\sqrt{3} + 2 = 2 - \sqrt{3}$$

$$x + \frac{1}{x} = \sqrt{3} + 2 - \sqrt{3} + 2$$

$$x + \frac{1}{x} = 4$$

Q4. Simplify

(i) $\frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} + \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}}$

$$\frac{1 + \sqrt{2}}{\sqrt{5} + \sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}} + \frac{1 - \sqrt{2}}{\sqrt{5} - \sqrt{3}} \times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}}$$

$$= \frac{(1 + \sqrt{2})(\sqrt{5} - \sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} + \frac{(1 - \sqrt{2})(\sqrt{5} + \sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2}$$

$$= \frac{(1 + \sqrt{2})(\sqrt{5} - \sqrt{3})}{5 - 3} + \frac{(1 - \sqrt{2})(\sqrt{5} + \sqrt{3})}{5 - 3}$$

$$= \frac{\sqrt{5} - \sqrt{3} + \sqrt{2}\sqrt{5} - \sqrt{2}\sqrt{3}}{2} + \frac{\sqrt{5} + \sqrt{3} - \sqrt{2}\sqrt{5} - \sqrt{2}\sqrt{3}}{2}$$

$$= \frac{\sqrt{5} - \sqrt{3} + \sqrt{10} - \sqrt{6}}{2} + \frac{\sqrt{5} + \sqrt{3} - \sqrt{10} - \sqrt{6}}{2}$$

$$= \frac{\sqrt{5} - \sqrt{3} + \sqrt{10} - \sqrt{6} + \sqrt{5} + \sqrt{3} - \sqrt{10} - \sqrt{6}}{2}$$

$$= \frac{2\sqrt{5} - 2\sqrt{6}}{2}$$

$$= \cancel{2}(\sqrt{5} - \sqrt{6})$$

$$= \sqrt{5} - \sqrt{6}$$

(ii) $\frac{1}{2 + \sqrt{3}} + \frac{2}{\sqrt{5} - \sqrt{3}} + \frac{1}{2 + \sqrt{5}}$

$$= \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}} + \frac{2}{\sqrt{5} - \sqrt{3}}$$

$$\times \frac{\sqrt{5} + \sqrt{3}}{\sqrt{5} + \sqrt{3}} + \frac{1}{2 + \sqrt{5}} \times \frac{2 - \sqrt{5}}{2 - \sqrt{5}}$$

$$= \frac{2 - \sqrt{3}}{(2)^2 - (\sqrt{3})^2} + \frac{2(\sqrt{5} + \sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} + \frac{2 - \sqrt{5}}{(2)^2 - (\sqrt{5})^2}$$

$$= \frac{2 - \sqrt{3}}{4 - 3} + \frac{2(\sqrt{5} + \sqrt{3})}{5 - 3} + \frac{2 - \sqrt{5}}{4 - 5}$$

$$= 2 - \sqrt{3} + \frac{2(\sqrt{5} + \sqrt{3})}{2} + \frac{2 - \sqrt{5}}{-1}$$

$$= \cancel{2} - \sqrt{3} + \sqrt{5} + \sqrt{3} - \cancel{2} + \sqrt{5} = 2\sqrt{5}$$

(iii) $\frac{2}{\sqrt{5} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{2}} - \frac{3}{\sqrt{5} + \sqrt{2}}$

$$= \frac{2}{\sqrt{5} + \sqrt{3}} \times \frac{\sqrt{5} - \sqrt{3}}{\sqrt{5} - \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{2}}$$

$$\times \frac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} - \frac{3}{\sqrt{5} + \sqrt{2}} \times \frac{\sqrt{5} - \sqrt{2}}{\sqrt{5} - \sqrt{2}}$$

$$= \frac{2(\sqrt{5} - \sqrt{3})}{(\sqrt{5})^2 - (\sqrt{3})^2} + \frac{\sqrt{3} - \sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} - \frac{3(\sqrt{5} - \sqrt{2})}{(\sqrt{5})^2 - (\sqrt{2})^2}$$

$$= \frac{2(\sqrt{5} - \sqrt{3})}{5 - 3} + \frac{\sqrt{3} - \sqrt{2}}{3 - 2} - \frac{3(\sqrt{5} - \sqrt{2})}{5 - 2}$$

$$\begin{aligned}
 &= \frac{\cancel{2}(\sqrt{5}-\sqrt{3})}{\cancel{2}} + \frac{\sqrt{3}-\sqrt{2}}{1} - \frac{\cancel{2}(\sqrt{5}-\sqrt{2})}{\cancel{2}} \\
 &= \cancel{\sqrt{5}} - \cancel{\sqrt{3}} + \sqrt{3} - \sqrt{2} - \cancel{\sqrt{5}} + \cancel{\sqrt{2}} \\
 &= 0
 \end{aligned}$$

Q5(i) If $x = 2 + \sqrt{3}$, find value of $x - \frac{1}{x}$

and $\left(x - \frac{1}{x}\right)^2$

$$x = 2 + \sqrt{3}$$

$$\frac{1}{x} = \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$$

$$\frac{1}{x} = \frac{2 - \sqrt{3}}{(2)^2 - (\sqrt{3})^2}$$

$$\frac{1}{x} = 2 - \sqrt{3}$$

$$\begin{aligned}
 x - \frac{1}{x} &= 2 + \sqrt{3} - (2 - \sqrt{3}) \\
 &= \cancel{2} + \sqrt{3} - \cancel{2} + \sqrt{3} \\
 &= 2\sqrt{3}
 \end{aligned}$$

$$\left(x - \frac{1}{x}\right)^2 = (2\sqrt{3})^2$$

$$\left(x - \frac{1}{x}\right)^2 = 12$$

(ii) If $x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$ find the value of

$$x + \frac{1}{x}, x^2 + \frac{1}{x^2} \text{ and } x^3 + \frac{1}{x^3}$$

$$x = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}} \times \frac{\sqrt{5}-\sqrt{2}}{\sqrt{5}-\sqrt{2}}$$

$$x = \frac{(\sqrt{5}-\sqrt{2})^2}{(\sqrt{5})^2 - (\sqrt{2})^2}$$

$$x = \frac{(\sqrt{5})^2 + (\sqrt{2})^2 - 2(\sqrt{5})(\sqrt{2})}{5 - 2}$$

$$x = \frac{5 + 2 - 2\sqrt{10}}{3}$$

$$x = \frac{7 - 2\sqrt{10}}{3}$$

$$\frac{1}{x} = \frac{3}{7 - 2\sqrt{10}} \times \frac{7 + 2\sqrt{10}}{7 + 2\sqrt{10}}$$

$$\frac{1}{x} = \frac{3(7 + 2\sqrt{10})}{(7)^2 - (2\sqrt{10})^2}$$

$$\frac{1}{x} = \frac{3(7 + 2\sqrt{10})}{49 - 40}$$

$$\frac{1}{x} = \frac{3(7 + 2\sqrt{10})}{9}$$

$$\frac{1}{x} = \frac{7 + 2\sqrt{10}}{3}$$

$$\begin{aligned}
 x + \frac{1}{x} &= \frac{7 - 2\sqrt{10}}{3} + \frac{7 + 2\sqrt{10}}{3} \\
 &= \frac{7 - 2\sqrt{10} + 7 + 2\sqrt{10}}{3} = \frac{14}{3}
 \end{aligned}$$

Now

$$x + \frac{1}{x} = \frac{14}{3}$$

Squaring

$$\left(x + \frac{1}{x}\right)^2 = \left(\frac{14}{3}\right)^2$$

$$x^2 + \frac{1}{x^2} + 2 = \frac{196}{9}$$

$$x^2 + \frac{1}{x^2} = \frac{196}{9} - 2$$

$$x^2 + \frac{1}{x^2} = \frac{196 - 18}{9} = \frac{178}{9}$$

Also

$$x^3 + \frac{1}{x^3} = ?$$

$$x + \frac{1}{x} = \frac{14}{3}$$

$$\left(x + \frac{1}{x}\right)^3 = \left(\frac{14}{3}\right)^3$$

$$x^3 + \frac{1}{x^3} + 3\left(x\right)\left(\frac{1}{x}\right)\left(x + \frac{1}{x}\right) = \frac{2744}{27}$$

$$x^3 + \frac{1}{x^3} + 3\left(x + \frac{1}{x}\right) = \frac{2744}{27}$$

$$x^3 + \frac{1}{x^3} + 3\left(\frac{14}{3}\right) = \frac{2744}{27}$$

$$x^3 + \frac{1}{x^3} = \frac{2744}{27} - 14$$

$$= \frac{2366}{27}$$

Q6. Determine the rational numbers a and b. If

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} = a + b\sqrt{3}$$

Given

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} = a + b\sqrt{3}$$

$$\frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} + \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = a + b\sqrt{3}$$

$$\frac{(\sqrt{3}-1)^2}{(\sqrt{3})^2 - (1)^2} + \frac{(\sqrt{3}+1)^2}{(\sqrt{3})^2 - (1)^2} = a + b\sqrt{3}$$

$$\frac{(\sqrt{3})^2 + (1)^2 - 2(\sqrt{3})(1)}{3-1} + \frac{(\sqrt{3})^2 + (1)^2 + 2\sqrt{3}}{3-1} = a + b\sqrt{3}$$

$$\frac{3+1-2\sqrt{3}}{2} + \frac{3+1+2\sqrt{3}}{2} = a + b\sqrt{3}$$

$$\frac{4-2\sqrt{3}}{2} + \frac{4+2\sqrt{3}}{2} = a + b\sqrt{3}$$

$$\frac{\cancel{2}(2-\sqrt{3})}{\cancel{2}} + \frac{\cancel{2}(2+\sqrt{3})}{\cancel{2}} = a + b\sqrt{3}$$

$$2 - \sqrt{3} + 2 + \sqrt{3} = a + b\sqrt{3}$$

$$4 = a + b\sqrt{3}$$

$$\Rightarrow a + b\sqrt{3} = 4$$

Hence on comparing the two sides, we get

$$\Rightarrow a = 4 \text{ and } b = 0$$

Exercise

Q1. If $x + \frac{1}{x} = 3$ find

(i) $x^2 + \frac{1}{x^2}$

(ii) $x^4 + \frac{1}{x^4}$

(i) $x + \frac{1}{x} = 3$

$$\left(x + \frac{1}{x}\right)^2 = (3)^2$$

$$x^2 + \frac{1}{x^2} + 2 = 9$$

$$x^2 + \frac{1}{x^2} = 9 - 2$$

$$x^2 + \frac{1}{x^2} = 7$$

(ii) $x^4 + \frac{1}{x^4}$

$$\left(x^2 + \frac{1}{x^2}\right)^2 = (7)^2$$

$$x^4 + \frac{1}{x^4} + 2 = 49$$

$$x^4 + \frac{1}{x^4} = 49 - 2$$

$$x^4 + \frac{1}{x^4} = 47$$

Q2. If $x - \frac{1}{x} = 2$ find

(i) $x^2 + \frac{1}{x^2}$

(ii) $x^4 + \frac{1}{x^4}$

(i) $x - \frac{1}{x} = 2$

Squaring

$$\left(x - \frac{1}{x}\right)^2 = (2)^2$$

$$x^2 + \frac{1}{x^2} - 2 = 4$$

$$x^2 + \frac{1}{x^2} = 4 + 2$$

$$x^2 + \frac{1}{x^2} = 6$$

(ii) $\left(x^2 + \frac{1}{x^2}\right) = (6)^2$

$$x^4 + \frac{1}{x^4} + 2 = 36$$

$$x^4 + \frac{1}{x^4} = 36 - 2$$

$$x^4 + \frac{1}{x^4} = 34$$

Q3. Find value of $x^3 + y^3$ and xy if

$$x + y = 5 \text{ and } x - y = 3$$

$$4xy = (x + y)^2 - (x - y)^2$$

$$4xy = (5)^2 - (3)^2$$

Now

$$4xy = 25 - 9 = 16$$

$$xy = \frac{16}{4} = 4$$

$$x + y = 5$$

taking cube both sides

$$(x + y)^3 = (5)^3$$

$$x^3 + y^3 + 3xy(x + y) = 125$$

$$x^3 + y^3 + 3(4)(5) = 125$$

$$x^3 + y^3 + 60 = 125$$

$$x^3 + y^3 = 125 - 60$$

$$x^3 + y^3 = 65$$

Q4. If $P = 2 + \sqrt{3}$ find (i) $P + \frac{1}{P}$

(ii) $P - \frac{1}{P}$ (iii) $P^2 + \frac{1}{P^2}$ (iv) $P^2 - \frac{1}{P^2}$

$$P = 2 + \sqrt{3}$$

$$\frac{1}{P} = \frac{1}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$$

$$\frac{1}{P} = \frac{2 - \sqrt{3}}{(2)^2 - (\sqrt{3})^2} = \frac{2 - \sqrt{3}}{4 - 3} = 2 - \sqrt{3}$$

i) $P + \frac{1}{P} = 2 + \cancel{\sqrt{3}} + 2 - \cancel{\sqrt{3}} = 4$

ii) $P - \frac{1}{P} = \cancel{2} + \sqrt{3} - \cancel{2} + \sqrt{3} = 2\sqrt{3}$

iii) $P^2 + \frac{1}{P^2} = ?$

$$\left(P + \frac{1}{P}\right)^2 = (4)^2$$

$$P^2 + \frac{1}{P^2} + 2 = 16$$

$$P^2 + \frac{1}{P^2} = 16 - 2$$

$$P^2 + \frac{1}{P^2} = 14$$

iv) $P^2 - \frac{1}{P^2} = ?$

$$\begin{aligned}
 P^2 - \frac{1}{P^2} &= \left(P + \frac{1}{P}\right) \left(P - \frac{1}{P}\right) \\
 &= (4)(\sqrt{3}) \\
 &= 8\sqrt{3}
 \end{aligned}$$

Q5. If $q = \sqrt{5} + 2$ Find (i) $q + \frac{1}{q}$

(ii) $q - \frac{1}{q}$ (iii) $q^2 + \frac{1}{q^2}$ (iv) $q^2 - \frac{1}{q^2}$

Solution: $q = \sqrt{5} + 2$

$$\frac{1}{q} = \frac{1}{\sqrt{5}+2} \times \frac{\sqrt{5}-2}{\sqrt{5}-2}$$

$$\frac{1}{q} = \frac{\sqrt{5}-2}{(\sqrt{5})^2 - (2)^2}$$

$$\frac{1}{q} = \frac{\sqrt{5}-2}{1} = \sqrt{5}-2$$

(i) $q + \frac{1}{q} = \sqrt{5} + 2 + \sqrt{5} - 2$
 $= 2\sqrt{5}$

(ii) $q - \frac{1}{q} = \sqrt{5} + 2 - \sqrt{5} + 2$
 $= 4$

(iii) $q^2 + \frac{1}{q^2}$

$$\left(q + \frac{1}{q}\right)^2 = (2\sqrt{5})^2$$

$$q^2 + \frac{1}{q^2} + 2 = 20$$

$$q^2 + \frac{1}{q^2} = 20 - 2$$

$$q^2 + \frac{1}{q^2} = 18$$

(iv) $q^2 - \frac{1}{q^2} = \left(q + \frac{1}{q}\right) \left(q - \frac{1}{q}\right)$

$$\begin{aligned}
 &= (2\sqrt{5})(4) \\
 &= 8\sqrt{5}
 \end{aligned}$$

Q6. Simplify

i) $\frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} - \sqrt{a^2-2}}$

$$\begin{aligned}
 &= \frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} - \sqrt{a^2-2}} \times \frac{\sqrt{a^2+2} + \sqrt{a^2-2}}{\sqrt{a^2+2} + \sqrt{a^2-2}} \\
 &= \frac{(\sqrt{a^2+2} + \sqrt{a^2-2})^2}{(\sqrt{a^2+2})^2 - (\sqrt{a^2-2})^2} \\
 &= \frac{(\sqrt{a^2+2})^2 + (\sqrt{a^2-2})^2 + 2(\sqrt{a^2+2})(\sqrt{a^2-2})}{a^2 + 2 - a^2 + 2} \\
 &= \frac{a^2 + 2 + a^2 - 2 + 2\sqrt{a^4-4}}{4} \\
 &= \frac{2a^2 + 2\sqrt{a^4-4}}{4} \\
 &= \frac{\cancel{2}(a^2 + \sqrt{a^4-4})}{\cancel{2}} \\
 &= \frac{a^2 + \sqrt{a^4-4}}{2}
 \end{aligned}$$

(ii) $\frac{1}{a - \sqrt{a^2 - x^2}} - \frac{1}{a + \sqrt{a^2 - x^2}}$

$$\begin{aligned}
 &= \frac{1}{a - \sqrt{a^2 - x^2}} \times \frac{a + \sqrt{a^2 - x^2}}{a + \sqrt{a^2 - x^2}} \\
 &\quad - \frac{1}{a + \sqrt{a^2 - x^2}} \times \frac{a - \sqrt{a^2 - x^2}}{a - \sqrt{a^2 - x^2}} \\
 &= \frac{a + \sqrt{a^2 - x^2}}{(a)^2 - (\sqrt{a^2 - x^2})^2} - \frac{a - \sqrt{a^2 - x^2}}{(a)^2 - (\sqrt{a^2 - x^2})^2}
 \end{aligned}$$

$$= \frac{a + \sqrt{a^2 - x^2}}{a^2 - a^2 + x^2} - \frac{a - \sqrt{a^2 - x^2}}{a^2 - a^2 + x^2}$$

$$= \frac{a + \sqrt{a^2 - x^2}}{x^2} - \frac{a - \sqrt{a^2 - x^2}}{x^2}$$

$$= \frac{a + \sqrt{a^2 - x^2} - a + \sqrt{a^2 - x^2}}{x^2}$$

$$= \frac{2\sqrt{a^2 - x^2}}{x^2}$$

Objective

- $4x + 3y - 2$ is an algebraic _____.
 (a) Expression
 (b) Sentence
 (c) Equation
 (d) In equation
- The degree of polynomial $4x^4 + 2x^2y$ is _____.
 (a) 1 (b) 2
 (c) 3 (d) 4
- $a^3 + b^3$ is equal to _____.
 (a) $(a-b)(a^2 + ab + b^2)$
 (b) $(a+b)(a^2 - ab + b^2)$
 (c) $(a-b)(a^2 - ab + b^2)$
 (d) $(a-b)(a^2 + ab - b^2)$
- $(3 + \sqrt{2})(3 - \sqrt{2})$ is equal to: _____.
 (a) 7 (b) -7
 (c) -1 (d) 1
- Conjugate of Surd $a + \sqrt{b}$ is _____.
 (a) $-a + \sqrt{b}$ (b) $a - \sqrt{b}$
 (c) $\sqrt{a} + \sqrt{b}$ (d) $\sqrt{a} - \sqrt{b}$
- $\frac{1}{a-b} - \frac{1}{a+b}$ is equal to
 (a) $\frac{2a}{a^2 - b^2}$ (b) $\frac{2b}{a^2 - b^2}$
 (c) $\frac{-2a}{a^2 - b^2}$ (d) $\frac{-2b}{a^2 - b^2}$
- $\frac{a^2 - b^2}{a + b}$ is equal to:
 (a) $(a-b)^2$ (b) $(a+b)^2$
 (c) $a+b$ (d) $a-b$
- $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$ is equal to: _____.
 (a) $a^2 + b^2$ (b) $a^2 - b^2$
 (c) $a - b$ (d) $a + b$
- The degree of the polynomial $x^2y^2 + 3xy + y^3$ is _____.
 (a) 4 (b) 5
 (c) 6 (d) 2
- $x^2 - 4 =$ _____.
 (a) $(x-2)(x+2)$ (b) $(x-2)(x-2)$
 (c) $(x+2)(x+2)$ (d) None
- $x^3 + \frac{1}{x^3} = \left(x + \frac{1}{x}\right)(\dots\dots\dots)$
 (a) $x^2 - 1 + \frac{1}{x^2}$ (b) $x^2 + 1 + \frac{1}{x^2}$
 (c) $x^2 + 1 - \frac{1}{x^2}$ (d) $x^2 - 1 - \frac{1}{x^2}$
- $2(a^2 + b^2) =$ _____.
 (a) $(a+b)^2 + (a-b)^2$
 (b) $(a+b)^2 - (a-b)^2$
 (c) $(a+b)^2$ (d) $4ab$
- Order of surd $\sqrt[3]{x}$ is _____.
 (a) 3 (b) $\frac{1}{3}$
 (c) 0 (d) 1

14. $\frac{1}{2-\sqrt{3}} = \underline{\hspace{2cm}}$

- (a) $2+\sqrt{3}$ (b) $2-\sqrt{3}$
 (c) $-2+\sqrt{3}$ (d) $-2-\sqrt{3}$

15. $(a+b)^2 - (a-b)^2 = \underline{\hspace{2cm}}$

- (a) $2(a^2 + b^2)$ (b) $4ab$
 (c) $2ab$ (d) $3ab$

16. $\sqrt{14} \cdot \sqrt{35} = \underline{\hspace{2cm}}$

- (a) $\sqrt[4]{10}$ (b) $\sqrt[5]{10}$
 (c) $7\sqrt{10}$ (d) $8\sqrt{10}$

17. A surd which contains a single term is called surd.

- (a) Monomial
 (b) Binomial
 (c) Trinomial
 (d) None

ANSWER KEY

1.	a	2.	d	3.	b	4.	a	5.	b
6.	b	7.	d	8.	c	9.	a	10.	a
11.	a	12.	a	13.	a	14.	a	15.	b
16.	c	17.	a						