Random Variable

A random variable is any numerical quantity whose value will be determined by the outcomes of a random experiment & which have a specific range & a definable prob. associated with each value.

Defn:- A real valued fn. X, defined on a sample space S, of a random experiment E, is called a random variable which assigns to each sample point (or to every outcome of random experiment E) $s \in S$ one and only one real number $X(s) = x$

* Random variables are denoted by capital letters, usually from the last part of the alphabet, for instance X, Y, Z etc.

* Random variable is also called chance variable.

Types of Random Variable

- Discrete random variable (Finite set of values)
- Continuous random variable (Infinite set of values)

A discrete random variable is one which can assumes only isolated values.

E.g. The no. of heads in 4 tosses of a coin is a discrete random variable as it cannot assume ~~assume~~ values other than 0, 1, 2, 3, 4.

A continuous random variable is one which can assume any value within an interval i.e. all values of a continuous scale.

E.g. (i) The weights (in kg) of a group of individuals
(ii) the height " " " " "

Discrete Probability Distribution :-

Let a random variable X assume values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ resp., where $P(X=x_i) = p_i \geq 0$ for each x_i & $p_1 + p_2 + \ldots + p_n = \sum_{i=1}^{n} p_i = 1$. Then

$$X: \quad x_1, \quad x_2, \quad x_3, \ldots, x_n$$
$$P(X): \quad p_1, \quad p_2, \quad p_3, \ldots, p_n$$

is called discrete prob. distribution.

Example:- 5 defective bulbs are accidently mixed with 20 good ones bulbs. It is not possible to just look at a bulb & tell whether or not it is defective. Find the prob. distr. of the no. of defective bulbs, if (four) 4 bulbs are drawn at random from this lot.

Soln: Let X: no. of defective bulbs in 4. Clearly X can take the values 0, 1, 2, 3 or 4.

No. of defective bulbs = 5
" " good " = 20
Total no. of bulbs = 25

$$P(X=0) = P(\text{no defective}) = P(\text{all 4 good ones})$$
$$= \frac{{}^{20}C_4}{{}^{25}C_4} = \frac{969}{2530}$$

$P(X=1) = P(\text{one def. \& 3 good ones}) = \frac{{}^5C_1 \times {}^{20}C_3}{{}^{25}C_4} = \frac{1140}{2530}$

$P(X=2) = \frac{{}^5C_2 \times {}^{20}C_2}{{}^{25}C_4} = \frac{380}{2530}$

$P(X=3) = \frac{{}^5C_3 \times {}^{20}C_1}{{}^{25}C_4} = \frac{40}{2530}$

$P(X=4) = \frac{{}^5C_4 \times {}^{20}C_0}{{}^{25}C_4} = \frac{1}{2530}$

$\therefore$ The Prob. distribution of the random variable X is

X:	0	1	2	3	4
$P(X)$:	$\frac{969}{2530}$	$\frac{1140}{2530}$	$\frac{380}{2530}$	$\frac{40}{2530}$	$\frac{1}{2530}$

OR Probability Mass function:-

Let $x_1, x_2, \ldots$ be the values of a discrete random variable X & let $p_1, p_2, p_3, \ldots, p_i \geq 0$, $i = 1, 2, \ldots, n$ be the corresponding probabilities

i.e. $P(X = x_i) = p(x_i)$ or p_i

A function $f(x)$ or $p(x)$ defined by

$$P(X=x) = p(x) = \begin{cases} p(x_i) \text{ or } p_i & \text{if } x = x_i,\ i = 1,2,3,\ldots \\ 0 & \text{, otherwise} \end{cases}$$

is called prob. Mass function of the discrete random variable X.

Example! Suppose a coin is tossed 3 times. Then the distribution of the number of heads is

$X = x$	0	1	2	3
$p(X=x)$ or $p(x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Mean!- Let a random variable X assume values $x_1, x_2, \ldots, x_n$ with prob. $p_1, p_2, \ldots, p_n$. re be discrete prob. distribution.

We denote mean by μ & define

$$\boxed{\mu = \frac{\sum p_i x_i}{\sum p_i}} \quad i = 1, 2, \ldots, n.$$

other names of means are average or expected value $E(x)$.

we denote variance by σ^2 & define

$$\boxed{\sigma^2 = \sum p_i (x_i - \mu)^2}$$

if μ is not a whole no. then

$$\boxed{\sigma^2 = \sum p_i x_i^2 - \mu^2}$$

Standard Deviation. $\boxed{\sigma = +\sqrt{\text{variance}}}$

Example! A R.V. X has following prob. f^n ~~val~~

value of X,

x:	0	1	2	3	4	5	6	7
$P(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

(i) Find k

(ii) Evaluate $P(X<6)$, $P(X\geq 6)$, $P(3<X\leq 6)$

(iii) Find min. value of x so that $P(X\leq x) > \frac{1}{2}$.

Soln Since $\sum_{x=0}^{7} p(x) = 1$, we have

(i) $0+k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

$\Rightarrow (10k-1)(10k+1)=0$

$\Rightarrow k = \frac{1}{10}, -1$

But $p(x)\geq 0$ $\Rightarrow \boxed{k=\frac{1}{10}}$

(ii) (a) $P(X<6) = P(X=0)+P(X=1)+ \cdots + P(X=5)$

$= 0+k+2k+2k+3k+k^2$

$= 8k+k^2 = \frac{8}{10}+\frac{1}{100} = \frac{81}{100}$.

(b) $P(X\geq 6) = 1-P(X<6)$

$= 1-\frac{81}{100} = \frac{19}{100}$.

(c) $P(3<X\leq 6) = P(X=4)+P(X=5)+P(X=6)$

$= 3k+k^2+2k^2 = \frac{33}{100}$.

iii)

$P(X \leq 1) = k = \frac{1}{10} < \frac{1}{2}$;

$P(X \leq 2) = k + 2k = \frac{3}{10} < \frac{1}{2}$.

$P(X \leq 3) = k + 2k + 2k = \frac{5}{10} \leq \frac{1}{2}$.

$P(X \leq 4) = \frac{8}{10} > \frac{1}{2}$.

$\therefore$ The min. value of 'x' so that $P(X \leq x) > \frac{1}{2}$ is 4

Example) A dice is tossed twice getting a number greater than 4 is considered a success. Find the variance of the prob. dist. of the no. of success.

Soln:- Let X: No. of success.

Clearly X can take the values 0, 1, 2.

Prob. of success $= \frac{2}{6} = \frac{1}{3}$. (Getting 5 or 6)

" " failure $= 1 - \frac{1}{3} = \frac{2}{3}$.

$P(X=0) = P(\text{No success}) = \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

$P(X=1) = P(1 \text{ success } \& \text{ } 1 \text{ failure}) = {}^2C_1 \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$

$P(X=2) = P(\text{two successes}) = {}^2C_2 \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

X :	0	1	2
$P(X)$:	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$
$p_i x_i$:	0	$\frac{4}{9}$	$\frac{2}{9}$
$p_i x_i^2$:	0	$\frac{4}{9}$	$\frac{4}{9}$

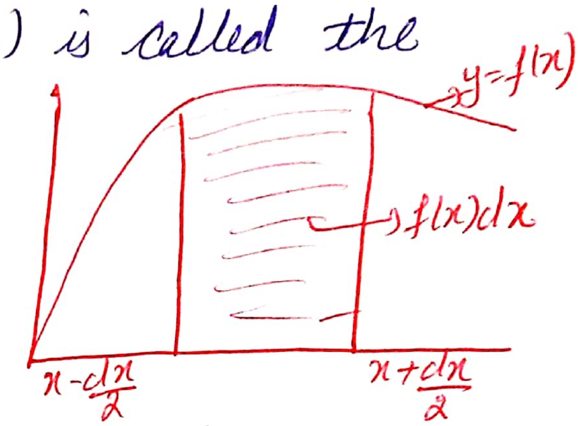
Mean $= \sum p_i x_i = \frac{6}{9} = \frac{2}{3}$

Variance $= \sigma^2 = \sum p_i x_i^2 - \mu^2$

$= \frac{8}{9} - \frac{4}{9} = \frac{4}{9}$.

Probability Density fn:- Let the Prob. of the variate x falling in the infinitesimal interval $(x-\frac{1}{2}dx, x+\frac{1}{2}dx)$ be expressed in the form $f(x)dx$, where $f(x)$ is conti. fn of x. Then $f(x)$ is called the P.d.f.

The continuous curve $y=f(x)$ is called the Prob. density curve.



Symbolically it is expressed as

$$P(x-\tfrac{1}{2}dx \le X \le x+\tfrac{1}{2}dx) = f(x)dx.$$

The interval of the variate may be finite or infinite. A fn. defined only for a finite interval say $f(x)=\phi(x)$, when $a \le x \le b$ can be put in the following form.

$$f(x)=0 \qquad x<a$$
$$f(x)=\phi(x) \qquad a\le x\le b$$
$$f(x)=0 \qquad x>b.$$

The density fn. posses the following two properties:-

(i) $f(x)\geq 0$ for every x, $\because$ -ve Prob. has no meaning.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ this corresponds to the fact that the prob. of an event that is sure to happen is equal to unity.

Ex. If the f^n $f(x)$ is defined by $f(x) = ce^{-x}$, $0 \le x \le \infty$
find the value of c which changes x to a p.d.f.

Sol^n In order that $f(x)$ may p.d.f, we should have

(a) $f(x) \geq 0$, $\forall x$

(b) $\int_{-\infty}^{+\infty} f(x)\,dx = 1$.

Since e^{-x} is always +ve $\forall$ x lying b/w 0 & ∞.
the condition will be satisfied if $c \geq 0$.

The 2nd cond. will be satisfied if

$$\int_0^\infty c \cdot e^{-x}\,dx = 1$$

$\Rightarrow$ i.e. if $\left[-c e^{-x}\right]_0^\infty = 1$

i.e. $\boxed{\text{if } c = 1}$

Eg 2) :- If $f(x)$ has prob. density cx^2, $0<x<1$. Determine c
& find the prob. that $\frac{1}{3} < x < \frac{1}{2}$.

Sol^n :- $f(x)$ will have a prob. density if

$$\int_0^1 cx^2\,dx = 1$$

i.e. if $\left[\frac{1}{3} cx^3\right]_0^1 = 1$

$\Rightarrow \boxed{c = 3}$.

$$P\left(\frac{1}{3} < x < \frac{1}{2}\right) = \int_{1/3}^{1/2} 3x^2\,dx = \frac{19}{216}.$$

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Commulative Distributive fn. [In Case of Continuous]

If X is a conti. random variable with the p.d.f $f(x)$, then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dx \;;\; -\infty < x < \infty$$

is called c.d.f of the R.V. X or simply distributive fn.

Properties of C.d.f :-

(i) $0 \le F(x) \le 1 \;, \quad -\infty < x < \infty$

(2) $F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0$

i.e. $F(x)$ is a non-decreasing fn. of x.

(3) $F(-\infty) = \lim_{x \to -\infty} F(x) = \lim_{x \to -\infty} \int_{-\infty}^{x} f(t)\,dx = \int_{-\infty}^{-\infty} f(t)\,dx = 0$

$F(+\infty) = \int_{-\infty}^{\infty} f(t)\,dx = 1.$

(4) $F(x)$ is a conti. fn. of x on the right.

(5) $P(a \le x \le b) = \int_{a}^{b} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$

$= P(X \le b) - P(X \le a)$

$= F(b) - F(a)$

Ques:- The cd.f of a R.V. X is given by

$$F(x) = \begin{cases} 1-(1+x)e^{-x} & , x \ge 0 \\ 0 & , x < 0 \end{cases}$$

find the corresponding density fn. of R.V. X.

Soln The P.d.f $= f(x) = \frac{d}{dx} F(x).$

$\Rightarrow f(x) = \frac{d}{dx}\left(1-(1+x)e^{-x}\right) = 0 - \frac{d}{dx}(1+x)e^{-x}$

$$= -(1+x)e^{-x}(-1) - e^{-x}(1)$$

$$= x \cdot e^{-x}$$

$$f(x) = \begin{cases} x e^{-x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$$

Ques: A R.v x has the p.d.f. $f(x) = k\frac{1}{1+x^2}$, $-\infty < x < \infty$, determine k & the distⁿ fⁿ.

Solⁿ It will be density fⁿ if

$$\int_{-\infty}^{\infty} k \cdot \frac{1}{1+x^2} = 1 \quad \text{i.e. if} \quad k \tan^{-1}x \Big]_{-\infty}^{\infty} = 1$$

$$\Rightarrow k\pi = 1 \Rightarrow \boxed{k = \frac{1}{\pi}}$$

Now $f(x) = \frac{1}{\pi}\frac{1}{x^2+1}$, $-\infty < x < \infty$

$$f(x) = \frac{d}{dx}F(x) \Rightarrow F(x) = \int f(x)\,dx$$

$$= \frac{1}{\pi}\int \frac{1}{1+x^2}dx$$

$$\boxed{F(x) = \frac{1}{\pi}\tan^{-1}x + C}$$

But $F(-\infty) = 0$ $\Rightarrow$ ~~[illegible]~~

$$\Rightarrow \frac{1}{\pi}\left(\frac{-\pi}{2}\right) + C = 0 \quad \Rightarrow \boxed{C = \frac{1}{2}}$$

$$\boxed{F(x) = \frac{1}{\pi}\tan^{-1}x + \frac{1}{2}} \; ; \; -\infty < x < \infty$$

Mathematical Expectation or Expected value of a random variable :-

It is denoted by $E(x)$ & given by

$$E(x) = \sum_{i=1}^{n} p_i x_i = \sum p x, \quad \sum p = 1.$$

$$\mu_1 = E(x).$$

$$\mu_2 = E(x^2) - [E(x)]^2 \quad \text{(variance)}$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\,dx. \quad \text{[In case of conti.]}$$

Laws of Expectation :-

(1) If C is a constant then $E(C) = C$.

(2) If a is " " then $E(ax) = aE(X)$

(3) $E(X+Y) = E(X) + E(Y)$

(4) $E(ax+b) = E(ax) + E(b) = aE(X) + b$.

(5) $E(XY) = E(X)\,E(Y)$

(6) $E(X-Y) = E(X) - E(Y)$

Ques: what is the expt. value of the no. of points that will be obtained in a single throw with an ordinary die? find variance also.

Sol^n Here the variate is the number of points showing on a die. It assumes the values 1,2,3,4,5,6 with prob. $\frac{1}{6}$ in each case.

$$E(x) = \sum_{i=1}^{6} p_i x_i = \frac{1}{6}.1 + \frac{1}{6}.2 + \frac{1}{6}.3 + \frac{1}{6}.4 + \frac{1}{6}.5 + \frac{1}{6}.6 = 3.5$$

$$Var. = E(x^2) - [E(x)]^2 = \frac{1}{6}[1^2+2^2+3^2+4^2+5^2+6^2] - [\tfrac{7}{2}]^2 = \frac{35}{12}.$$

Ques S.T. if x takes the values $x_n = (-1)^n 2^n n^{-1}$, for $n = 1, 2, ---$ with Prob. $p_n = 2^{-n}$ then $E(x) = -\log 2$.

Sol^n $E(x) = \sum_{n=1}^{\infty} 2^{-n} (-1)^n 2^n n^{-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

$= -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - - - - - -$

$= -\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - - - \right]$

$$\boxed{E(x) = -\log 2}$$

Imp'

Ques:- The diameter say X of an electric ~~bulb~~ Cable, is assumed to be conti. R.V with p.d.f $f(x) = 6x(1-x)$, $0 \le x \le 1$

(i) Check that above is a p.d.f.

(ii) obtain c.d.f of X

(iii) Compute $P\left(X \le \frac{1}{2} \mid \frac{1}{3} \le X \le \frac{2}{3}\right)$

(iv) Determine the no. k st $P(X < k) = P(X > k)$

Sol^n: (i) Since $\int_0^1 f(x)\,dx = \int_0^1 6x(1-x)\,dx = 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$

$\Rightarrow f(x)$ is p.d.f.

$$\Rightarrow f(x) = \begin{cases} 6x(1-x) & ; \ 0 \le x \le 1 \\ 0 & , \ \text{otherwise.} \end{cases}$$

(ii) for any x s.t $-\infty < x \le 0$ s.t

$F(x) = \int_{-\infty}^{x} 0\,dt = 0$

for any x s.t $0 < x \le 1$

$F(x) = \int_{-\infty}^{0} 0\,dt + \int_0^x 6t(1-t)\,dt$.

$$F(x) = 6\left[\frac{t^2}{2} - \frac{t^3}{3}\right]_0^x = 3x^2 - 2x^3$$

for any x s.t $1 \le x < \infty$

$$F(x) = \int_{-\infty}^{0} 0 \cdot dt + \int_0^1 6t(1-t)\,dt + \int_1^x 0 \cdot dt$$

$$= 6\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

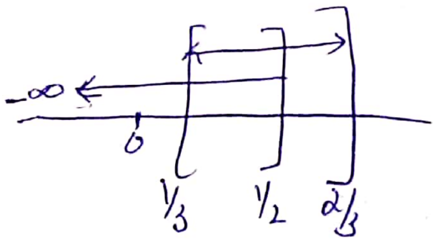
$\Rightarrow$ c.d.f is given by

$$F(x) = \begin{cases} 0 & ; -\infty \le x \le 0 \\ 3x^2 - 2x^3 & ; 0 < x \le 1 \\ 1 & ; 1 < x < \infty \end{cases}$$

(iii) $P\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{3} \le X \le \frac{2}{3}\right) =$

$$\frac{P\left[\left(-\infty < X \le \frac{1}{2}\right) \cap \left(\frac{1}{3} \le X \le \frac{2}{3}\right)\right]}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)}$$

$$= \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)}$$



$$= \frac{\int_{1/3}^{1/2} 6x(1-x)\,dx}{\int_{1/3}^{2/3} 6x(1-x)\,dx} = \frac{\frac{13}{324}}{\frac{13}{162}} = \frac{1}{2}$$

(iv) By given condition $P(x < k) = P(x > k)$

$\Rightarrow P(0 < X < k) = P(k < X < 1)$

$\Rightarrow \int_0^k f(x)\,dx = \int_k^1 f(x)\,dx \Rightarrow 6\int_0^k x(1-x)\,dx = 6\int_k^1 x(1-x)\,dx$

$\Rightarrow \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^k = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_k^1$

$\Rightarrow \frac{k^2}{2} - \frac{k^3}{3} = \left[\frac{1}{2} - \frac{1}{3} - \frac{k^2}{2} + \frac{k^3}{3}\right]$

$\Rightarrow 4k^3 - 6k^2 + 1 = 0 \Rightarrow (2k-1)(2k^2-2k-1) = 0$

$\Rightarrow k = \frac{1}{2}$ or $\frac{1}{2} \pm \frac{\sqrt{3}}{2}$

$\Rightarrow k = \frac{1}{2}$ is the only real value lying b/w 0 & 1 satisfying the given eqn for which $P(X < \frac{1}{2}) = P(X > \frac{1}{2})$

Ques Find the distribution f^n for R.V. X whose p.d.f is

$$f(x) = \begin{cases} \frac{x^2}{9}, & 0 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Sol^n By def^n. $F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt = \int_{-\infty}^{x} 0\,dt = 0$ if $-\infty < x < 0$.

If $0 \le x < 3$, then $F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_0^x f(t)\,dt = \int_0^x \frac{t^2}{9}\,dt = \frac{x^3}{27}$

If $x \geq 3$, then $F(x) = \int_{-\infty}^{0} f(t)\,dt + \int_0^3 f(t)\,dt + \int_3^x f(t)\,dt$

$= \int_0^3 \frac{t^2}{9}\,dt = 1$

$\therefore$ Req. distribution f^n is

$$F(x) = \begin{cases} 0; & x < 0 \\ x^3/27; & 0 \le x < 3 \\ 1; & x \geq 3 \end{cases}$$

Ques] A petrol pump is supplied with petrol once in a day. If its daily volume of sale (X) in 1000 litres is distributed as $f(x) = 5(1-x)^4$; $0 \le x \le 1$

what must be capacity of its tank in order that the prob. that its supply will be exhausted in a given day shall be 0.01.

Soln! Let the capacity of the tank (in 1000 litres) be 'k' s.t

$$P(X \geq k) = 0.01$$

$$\Rightarrow \int_k^1 f(x)\,dx = 0.01 \Rightarrow 5\int_k^1 (1-x)^4 dx = 0.01$$

$$\Rightarrow \left[5\,\frac{(1-x)^5}{-5}\right]_k^1 = 0.01$$

$$\Rightarrow (1-k)^5 = \frac{1}{100} \Rightarrow 1-k = \left[\frac{1}{100}\right]^{1/5}$$

$$\Rightarrow k = 1-0.3981 = 0.6019$$

$\therefore$ Capacity of tank $= 0.6019 \times 1000 = 601.9$ litres