1.8.5 Theorem of Total Probability

Let $E_1, E_2 \cdots E_n$ be the partition of sample sapce S Such that $P(E_i) \neq 0 \; \forall i$ then for any event E in S.

$$P(E) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)$$

or $$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right) + \ldots + P(E_n) \cdot P\left(\frac{E}{E_n}\right)$$

Proof. Since $E_1, E_2 \cdots E_n$ be the partition of sample space S.

then we have

$$E_i \cap E_j = \phi \quad \forall \; i \neq j$$

and $E_1 \cup E_2 \cup E_3 \ldots \cup E_n = S$, therefore

$$E = E \cap S$$

$$= E \cap (E_1 \cup E_2 \cup E_3 \ldots \cup E_n)$$

$\Rightarrow \quad E = (E \cap E_1) \cup (E \cap E_2) \cup (E \cap E_3) \ldots \cup (E \cap E_n)$ (By distributive property)

$\Rightarrow \quad E = (E \cap E_1) \cup (E \cap E_2) \cup (E \cap E_3) \ldots \cup (E \cap E_n)$ (By addition law of probability)

$$\Rightarrow \quad P(E) = P(E \cap E_1) + P(E \cap E_2) + P(E \cap E_3) + \ldots\ldots\ldots + P(E \cap E_n)$$

by multiplication theorem of probability

$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right) + \ldots + P(E_n) \cdot P\left(\frac{E}{E_n}\right)$$

Hence, $$\boxed{P(E) = \sum_{i=1}^{n} P(E_i) \cdot P\left(\frac{E}{E_i}\right)}$$

Remark :

1. *For two events.*

$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right)$$

2. *For three events*

$$P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right)$$

$2 \cdot 6 \quad 2 \quad 36 \quad - \quad 12 \quad + \quad 72 \quad = \quad 72$

Example 1.45 *A bag contains 10 red and 3 blue balls. Another bag contains 3 red and 5 blue balls. Two balls are drawn randomly from the first bag and placed in the second bag and then 1 ball is taken randomly from the latter. What is the probability that it is a red ball?*

Solution : Let events be

$E_1 \rightarrow$ drawing red ball from first bag

$E_2 \rightarrow$ drawing blue ball from first bag

$E_3 \rightarrow$ drawing balls one red and one blue

$E \rightarrow$ drawing a red ball from the second bag.

it is given, $$P(E_1) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{10\times 9}{1\times 2}\times\frac{1\times 2}{13\times 12} = \frac{90}{12\times 13} = \frac{15}{26}$$

then $$P(E_2) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{1}{26} \quad \text{and} \quad P(E_3) = \frac{{}^{10}C_1 \cdot {}^{3}C_1}{{}^{13}C_2} = \frac{5}{13}$$

Now $$P\left(\frac{E}{E_1}\right) = P\text{ (drawing a red ball from 2nd bag)}$$

$$= \frac{{}^{5}C_1}{{}^{10}C_1} = \frac{1}{2}$$

Similarly, $$P\left(\frac{E}{E_2}\right) = \frac{3}{10} \quad \text{and} \quad P\left(\frac{E}{E_3}\right) = \frac{4}{10}$$

by total law of probability

$$P(E) = P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right) + P(E_3)\cdot P\left(\frac{E}{E_3}\right)$$

$$= \frac{15}{26}\times\frac{5}{10} + \frac{1}{26}\times\frac{3}{10} + \frac{10}{26}\times\frac{4}{10}$$

$$= \frac{59}{130}$$

Example 1.46 *A bolt is made by 3 machines M_1, M_2 and M_3. M_1 turns out twice as many items as M_2 and machine M_2 and M_3 produce equal number of items, 2% of bolts produced by M_1 and M_2 are defective and 0.4% of bolts produced by M_3 are defective. All bolts are put into 1 stock pile and 1 is chosen from this pile. What is the probability that it is defective.*

Solution : Let events be

$E_1 \rightarrow$ item has been produced by machine M_1

$E_2 \rightarrow$ item has been produced by machine M_2

$E_3 \rightarrow$ item has been produced by machine M_3

$D \rightarrow$ item is defective

it is given that $P(E_1)=\frac{1}{2}, P(E_2)=\frac{1}{4}$

$$P(E_3)=\frac{1}{4} \text{ and } P\left(\frac{D}{E_1}\right)=P\text{ (an item in } M_1 \text{ is defective)}=0.02$$

also $$P\left(\frac{D}{E_2}\right)=0.02 \text{ and } P\left(\frac{D}{E_3}\right)=0.04$$

$\therefore$ by theorem of total probability

$$P(D)=P(E_1)\cdot P\left(\frac{D}{E_1}\right)+P(E_2)\cdot P\left(\frac{D}{E_2}\right)+P(E_3)\cdot P\left(\frac{D}{E_3}\right)$$

$$=\frac{1}{2}(0.02)+\frac{1}{4}(0.04)=\frac{1}{40}$$

> **Note :** *If we have to find the probability of the defective bolt manufactured by M_1 or M_2 or M_3, then we will use Baye's theorem.*

1.8.6 Baye's Theorem

In many cases outcomes depend on intermediate stages. We use the Baye's theorem for solving such type of problems.

1.8.7 Proof of Baye's theorem

[Raj. Univ. B.E. 1995, 2003, 2006, RTU IVth Sem. CS, 2008, 2009]

Statement 1 : If $E_1, E_2 \cdots E_n$ be n mutually exclusive and exhaustive set of events of a sample sapce S and E is an event which occurs together with either of $E_1, E_2 \cdots E_n$ (form a partition of S) and let E be any event then

$$\boxed{P\left(\frac{E_i}{E}\right)=\frac{P(E_i)\cdot P\left(\frac{E}{E_i}\right)}{\sum_{j=1}^{n}P(E_j)\cdot P\left(\frac{E}{E_j}\right)}} \quad ; i = 1, 2, \ldots\ldots n$$

or

$$P\left(\frac{E_i}{E}\right)=\frac{P(E_i)\cdot P\left(\frac{E}{E_i}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+\cdots+P(E_j)P\left(\frac{E}{E_j}\right)} \quad ; i = 1, 2, \ldots\ldots n$$

Example 1.50 *Three bags B_1, B_2, B_3 contains 6 red and 4 black balls, 2 red and 6 black balls and 1 red and 8 black balls respectively. A bag is chosen and a ball is drawn from the bags. If the drawn ball is red, find the probability that the ball was drawn from bag B_1.*

Solution : Let events be

$E_1 \rightarrow$ Choosing bag B_1, $E_2 \rightarrow$ Choosing bag B_2,
$E_3 \rightarrow$ Choosing bag B_3, $E \rightarrow$ Drawn ball is red.

According to question $P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$

Now $P\left(\frac{E}{E_1}\right) = \frac{6}{10}$, $P\left(\frac{E}{E_2}\right) = \frac{7}{8}$, $P\left(\frac{E}{E_3}\right) = \frac{1}{9}$.

Now $P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right)$

$$P(E) = \frac{1}{3} \times \frac{3}{5} + \frac{1}{3} \times \frac{1}{4} + \frac{1}{3} \times \frac{1}{9} = \frac{1}{5} + \frac{1}{12} + \frac{1}{27} = \frac{173}{540}$$

Now required probability, by Baye's theorem

$$P\left(\frac{E_1}{E}\right)=\frac{P(E\cap E_1)}{P(E)}$$

$$=\frac{\frac{1}{3}\times\frac{3}{5}}{\frac{173}{540}}=\frac{108}{173}$$

Example 1.51. *Three factories A, B, C does 30%, 50% and 20% production of certain item. Out of their production 8%, 5% and 10% of the items produced are defective respectively. An item is purchased and is found to be defective. Find the probability that it was a product of factory A.* **[Raj. Univ. B.E. IVth SEM. CS, 1997, 2002, 2007]**

Solution : Let events be

$E_1 \to$ Item is produced by machine A, $E_2 \to$ Item is produced by machine B.

$E_3 \to$ Item is produced by machine C,

Let $D \to$ Item pruchased is defective.

According to question, it is given that

$P\left(\frac{D}{E_1}\right)=\frac{8}{100}$	$P(E_1)=\frac{30}{100}=\frac{3}{10}$
$P\left(\frac{D}{E_2}\right)=\frac{5}{100}$	$P(E_2)=\frac{50}{100}=\frac{5}{10}$
$P\left(\frac{D}{E_3}\right)=\frac{10}{100}$	$P(E_3)=\frac{20}{100}=\frac{2}{10}$

Now by Baye's theorem

$$P\left(\frac{E_1}{D}\right)=\frac{P\left(\frac{D}{E_1}\right)\cdot P(E_1)}{P\left(\frac{D}{E_1}\right)\cdot P(E_1)+P\left(\frac{D}{E_2}\right)\cdot P(E_2)+P\left(\frac{D}{E_3}\right)\cdot P(E_3)}$$

$$=\frac{\frac{8}{100}\times\frac{3}{10}}{\frac{8}{100}\times\frac{3}{10}+\frac{5}{100}\times\frac{5}{10}+\frac{10}{100}\times\frac{2}{10}}=\frac{3\times 8}{3\times 8+25+20}=\frac{24}{69}$$

Similarly, we can find

$$P\left(\frac{E_2}{D}\right)=\frac{25}{69} \text{ and } P\left(\frac{E_3}{D}\right)=\frac{20}{69}$$

Example 1.48 *In an examination with multiple choice answer, each question has four choice answers, out of which, one is correct. A candidate ticks his answer either by his skill or by guess or by copying from his neighbours. The probability of guess is $\frac{1}{3}$ and that of copying is $\frac{1}{6}$. The probability of correct answer by copying is $\frac{1}{8}$. If a candidate answers a question correctly, find the probability that he knew the answer.*

Solution : Let events be

$E_1 \rightarrow$ answering by skill, $E_2 \rightarrow$ answering by guessing, $E_3 \rightarrow$ answering by copying,
$E \rightarrow$ answering correct

it is given that $P(E_2)=\frac{1}{3},\ P(E_3)=\frac{1}{6}$

$$\because \quad P(E_1)+P(E_2)+P(E_3)=1$$

$$P(E_1)=1-P(E_2)-P(E_3) \ =1-\frac{1}{3}-\frac{1}{6}=\frac{3}{6}$$

Also given that

$$P\left(\frac{E}{E_3}\right)=\frac{1}{8},\ P\left(\frac{E}{E_1}\right)=1,\ P\left(\frac{E}{E_2}\right)=\frac{1}{4}$$

Now we have to find $P\left(\frac{E_1}{E}\right)$.

Then by Baye's theorem

$$P\left(\frac{E_1}{E}\right)=\frac{P(E/E_1)\cdot P(E_1)}{P(E_1)P(E/E_1)+P(E_2)+(E/E_2)+P(E_3)\cdot P(E/E_3)}$$

$$=\frac{1.3/6}{1\times 3/6+1/4\times 2/6\times 1/8\times 1/6}=\frac{24}{29}$$

$$\boxed{P\left(\frac{E_i}{E}\right)=\frac{P(E_i)\cdot P\left(\frac{E}{E_i}\right)}{\sum_{j=1}^{n} P(E_j)\cdot P\left(\frac{E}{E_j}\right)}} \quad ; i = 1, 2, \ldots.. n$$

or $$P\left(\frac{E_i}{E}\right)=\frac{P(E_i)\cdot P\left(\frac{E}{E_i}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right)+P(E_2)\cdot P\left(\frac{E}{E_2}\right)+\cdots+P(E_j)P\left(\frac{E}{E_j}\right)} \quad ; i = 1, 2, \ldots.. n$$

Proof :

Since $E_1, E_2 \cdots E_n$ be the partition of sample sapce S.

So $S = E_1 \cup E_2 \cup E_3 \cdots \cup E_n$ ...(1)

Now, $E = E \cap S$

$$E = E \cap (E_1 \cup E_2 \cup E_3 \cdots \cup E_n) \quad \text{...(2)}$$

$$E = (E \cap E_1) \cup (E \cap E_2) \cup (E \cap E_3) \ldots \cup (E \cap E_n)$$ (By distributive properly)

$$P(E) = P(E \cap E_1) + P(E \cap E_2) + P(E \cap E_3) \ldots P(E \cap E_n) \quad \text{...(3)}$$

(By addition rule of probability)

Now $P\left(\frac{E_i}{E}\right) = \frac{P(E \cap E_i)}{P(E)}$ (by multiplication rule)

$$= \frac{P(E \cap E_i)}{P(E \cap E_1) + P(E \cap E_2) + P(E \cap E_3) + \ldots P(E \cap E_n)}$$

$$P(E_i)\cdot P\left(\frac{E}{\ }\right.$$