Week - 11 Assignment

Probability and statistics

(1)

Q1 Let $X \sim Bin(n, p)$, where n is known and $0<p<1$, In order to test $H: p = \frac{1}{2}$ vs $K: p = \frac{3}{4}$, a test is "Reject H if $X \geq 2$". Find the power of the test?

Sol: To find the power of the test.

$\boxed{Power = 1-\beta}$ ← power function

We have to first find β.

and β = P(Accepting the H_0 | H_0 is false).

or

P(Accepting H_0 | H_1 true).

$\beta = P(x \in \overline{w} | H_1)$ $H_0 = H: p = \frac{1}{2}$, $H_1 = K: p = 3/2$

Now Here $W : X \geq 2$, $\overline{W} : x < 2$.

or $W = \{x | x \geq 2\}$. $\overline{W} = \{x : x < 2\}$.

Now $\beta = P(x \in \overline{w} | H_1)$

$= P(x < 2 | P = \frac{3}{2})$ [∵ $X \sim Bin(n, p)$ we can use Here Bin thrm. for $x = 0, x = 1$]

$= {}^nC_0 \left(\frac{3}{4}\right)^0 \left(\frac{1}{4}\right)^n + {}^nC_1 \left(\frac{3}{4}\right)^1 \left(\frac{1}{4}\right)^{n-1}$

$= 1 \cdot 1 \cdot \frac{1}{4^n} + n \cdot \frac{3}{\cancel{4}} \times \frac{1}{4^n \cdot \cancel{4^{-1}}}$

$$\boxed{\beta = \frac{1}{4^n} + \frac{3n}{4^n}}$$

Now power of the test $= 1 - \beta$

$= 1 - \left(\frac{1+3n}{4^n}\right)$

Ans (C) $= \boxed{1 - \frac{1+3n}{4^n}}$

Q2 Suppose that X is a random variable with the probability density function (2)

$$f(x,\theta) = \theta x^{\theta-1}, 0<x<1.$$

In order to test the null hypothesis $H_0 : \theta = 2$ against $H_1 : \theta = 3$, the following test is used: "Reject H_0 if $x_1 \geq \frac{1}{2}$", where x_i is a random sample of size 1 drawn from the above distribution. Then the power of the test is-

Sol: Here $H_0 : \theta = 2$. (Null Hyp.) $H_1 : \theta = 3$, H_1 = Alternate Hyp. $W = x \geq \frac{1}{2}$, $\overline{W} = x < \frac{1}{2}$.

Now. $\beta = P(x < \frac{1}{2} \mid \theta = 3)$

$= P(3x^2)$

$= \int_0^{1/2} 3x^2\, dx$

$= 3\left[\frac{x^3}{3}\right]_0^{1/2} = \frac{1}{8}.$

$\left[f(x,\theta) = \theta x^{\theta-1} = 3x^{3-1} = 3x^2 \right]$

$\boxed{\beta = 1/8}$ Now, Power of test $= 1-\beta$

$= 1 - 1/8 = 7/8.$

$= \underline{\underline{0.875}}.$

Q3 A random sample of 500 registered voters in Phoenix is asked if they favor the use of oxygenated fuels year-round to reduce air-pollution. If more than 400 voters respond positively, we will conclude that more than 60% of the voters favor the use of these fuels i.e. we are testing $H_0 : p = 0.6$ vs. $H_1 : p \neq 0.6$. What is the type II error probability if 75% of the voters favor this action? (use normal approximation to the binomial)?

(3)

Sol: Here X = number of voters that respond positively and follows binomial distribution

$X \sim bin(500, 0.6)$.

Now we have to find Type II error means β.

$\beta = P(\text{Accepting } H_0 \mid H_1 \text{ true})$.

$= P(x > 400 \mid p = 0.75)$ ← 75% of voters.

more than 400 voters respond.

$\Rightarrow \beta = P(X > 400 \mid p = 0.75)$

[As mentioned in the quest. use Normal distr $Z = \frac{X - \mu}{\sigma}$]

$= P\left(\frac{X - 375}{9.68245} > \frac{400 - 375}{9.682458}\right)$

$n = 500$

[$\mu = np = 500 \times 0.75 = 375$.

$\sigma = \sqrt{npq}$

$= \sqrt{500 \times (0.75)(0.25)}$

$= \sqrt{93.75}$]

$\boxed{\sigma = 9.68245}$

$= P(Z > 2.582)$

$= 1 - \phi(2.582)$.

$= 1 - 0.9951$

$= 0.00491$

[$\because P(Z > a) = 1 - \phi(a)$]

[$\phi(2.582) = 0.9951$ from S.N.D.T]

Standard Normal distr. table.

Q4 A textile fiber manufacturer is investigating a new drapery yarn, which the company claims follows a normal distribution having mean thread elongation of 12cms. with a standard deviation 0.5 cm. The company wishes to test the hypothesis $H_0: \mu = 12$. against $H_1: \mu \leq 12$, using a random sample of four specimens. What is the type I error probability if the critical region is defined as $\bar{x} < 11.5$ cms.

Sol: Given that the critical region $\bar{x} \leq 11.5$ cms.

$\mu = 12$ cms   Standard deviat. $(\sigma) = 0.5$ cm.

$H_0 : \mu = 12$ and $H_1 : \mu < 12$.
(Null Hyp.)   (Alternate Hyp.)

Here we have to find type I error i.e α

$$\alpha = P(\text{Rejecting } H_0 \mid H_0 \text{ is true})$$

$$\alpha = P(\bar{x} < 11.5 \mid \mu = 12).$$

$$= P\left(\frac{\bar{x} - 12}{0.5/\sqrt{4}} < \frac{11.5 - 12}{0.5/\sqrt{4}}\right)$$

$\frac{x-\mu}{\sigma/\sqrt{n}} \rightarrow$ using this. as n is mentioned for 4 units.

$$= P(z < -2) = \Phi(-2) = \underline{\underline{0.0227}} \text{ Ans.}$$

(A) option

Q5 The probability density function of the random variable X is $f(x) = \frac{1}{\lambda} e^{-x/\lambda}, x > 0, \lambda > 0$. For testing the hypothesis. $H_0 : \lambda = 3$ Vs. $H_A : \lambda = 5$, a test is given as "Reject H_0. if $X \geq 4.5$". The Probability of Type I error and power of this test are respectively:

Sol: Here given $f(x) = \frac{1}{\lambda} e^{-x/\lambda}, x > 0, \lambda > 0.$

$H_0 : \lambda = 3$ $H_A : \lambda = 5$
Null Hyp.   Alternate Hyp.

$x : X \geq 4.5$; $\bar{X} < 4.5$.

Now we have to find. α (Type I error)
and power of the test $(1 - \beta)$

$$\text{Now } \alpha = P(\text{Rejecting } H_0 \mid H_0 \text{ is true})$$

$$= P(X \geq 4.5 \mid \lambda = 3).$$

$$= 1 - P(X < 4.5)$$

$$= 1 - \int_0^{4.5} \frac{1}{3} e^{-x/3}\, dx$$

$\left[\because \lambda = 3, \; f(x) = \frac{1}{\lambda} e^{-x/\lambda} = \frac{1}{3} e^{-x/3}\right]$

$= 1 - \frac{1}{3} \int_0^{4.5} e^{-x/3}.$

$= 1 - \frac{1}{3} \left[\frac{e^{-x/3}}{-1/3} \right]_0^{4.5}$

$= 1 + 1 \left[e^{-45/10 \times 3} - e^0 \right]$

$= 1 + \left[e^{-45/30} - 1 \right]$

$= 1 + \left[e^{-3/2} - 1 \right] = e^{-3/2} = 0.223$

Now, $\boxed{\alpha = 0.223}$

Power of the test $= 1 - \beta = 1 - P(\bar{X} < 4.5 | \lambda = 5)$

$= 1 - \int_0^{4.5} \frac{1}{5} e^{-x/5} \, dx$

$= 1 - \frac{1}{5} \left[\frac{e^{-x/5}}{-1/5} \right]_0^{4.5} = 1 + \frac{1}{5} \left[e^{-x/5} \right]_0^{4.5}$

$= 1 + \left[e^{-9/10} - e^0 \right] = 1 + \left[e^{-9/10} - 1 \right]$

$= e^{-9/10} = 0.407$

$= \boxed{0.407}$

Ans (D) $\cdot 0.223$ and 0.407.

Q6 The proportion of adults living in Tempe, Arizona, who are college graduates is estimated to be $p = 0.4$. To test this hypothesis, a random sample of 20 Tempe adults is selected. If the number of college graduates is between 4 and 8. (endpoints included.), the hypothesis will be accepted; otherwise we will conclude that $p \neq 0.4$. Find the type I error probability for this procedure assuming $p = 0.4$

Sol: Here we have $p = 0.4$.

(6) and the hypothesis given is b/w the end points 4 & 8.

Type I error $(\alpha) = P(X < 4 | p = 0.4) + P(X > 8 | p = 0.4)$

$= P\left(Z < \frac{X-\mu}{\sigma}\right) + P\left(Z > \frac{X-\mu}{\sigma}\right)$

$= P\left(Z < \frac{4-8}{2.190}\right) + P\left(Z > \frac{8-8}{2.190}\right)$

$= P(Z < -1.8257) + P(Z > 0)$

$= \Phi(-1.8257) + 1 - \Phi(0)$

$= 0.0344 + 1 - 0.5000$

$\boxed{\alpha = 0.5339.}$ (C) Ans.

$\mu = np = 20 \times 0.4 = 8$

$\sigma = \sqrt{npq} = \sqrt{20 \times (0.4) \times 0.6} = \sqrt{4.8} = 2.190$

Q7 Let X be a single observation from the population.

$$f(x,\theta) = \theta e^{-\theta x}, x > 0, \theta > 0.$$

If $X > 1$, is a critical region for testing $H: \theta = 1$ vs $K: \theta = 2$. find the type I error and power of the test.

Sol: Here we have to calculate α and power of the test.

$X : X > 1$ $\quad \bar{X} : \bar{X} \leq 1$.

$H: \theta = 1$. Null Hypothesis. $\quad K: \theta = 2$ Alternate hypothesis

Type I error $(\alpha) = P(x \geq 1 | \theta = 1)$

$1 - P(x < 1)$ $= 1 - \int_0^1 e^{-x}$

$= 1 - \int_0^1 e^{-x}$

$= 1 - \left[\frac{e^{-x}}{-1}\right]_0^1 = 1 + [e^{-1} - e^0] = 1 + e^{-1} - 1 = \frac{1}{e}$.

$\left[f(x,\theta) = \theta e^{-\theta x} \quad \theta = 1 \Rightarrow 1 \cdot e^{-1 \cdot x} = e^{-x} \right]$

$\therefore \boxed{\alpha = 1/e}$

Power of test $= 1 - \beta = 1 - P(\bar{X} \leq 1 | \theta = 2)$

$= 1 - \int_0^1 2 e^{-2x} dx$.

$= 1 - 2\left[\frac{e^{-2x}}{-2}\right]_0^1 = 1 + [e^{-2 \times 1} - e^0] = 1 + e^{-2} - 1 = e^{-2} = \frac{1}{e^2}$

$\therefore \boxed{1 - \beta = \frac{1}{e^2}}$

Q8 A manufacturer is interested in the output voltage of a power supply used in a PC. Output voltage is assumed to be normally distributed with standard deviation 0.2 volt and the manufacturer wishes to test $H_0: \mu = 5$ volts against $H_1: \mu \neq 5$ volts using $n = 8$ units. If the acceptance region is $4.85 \le \bar{x} \le 5.15$. find the power of the test for detecting a true mean output voltage of 5.1 volts? (7)

Sol:- Given mean $(\mu) = 5.1$ volts.

and the region of acceptance $\bar{X} = 4.85 \le \bar{x} \le 5.15$.

To find power of test, we have to find 'β.

$$\beta = P(4.85 \le \bar{x} \le 5.15 \mid \mu = 5.1)$$

$$= P\left(\frac{4.85 - 5.1}{0.2/\sqrt{8}} \le \frac{\bar{X} - 5.1}{0.2/\sqrt{8}} \le \frac{5.15 - 5.1}{0.2/\sqrt{8}}\right)$$

$$= P(-3.5335 \le Z \le 0.70711)$$

$$= \phi(0.70711) - \phi(-3.5335)$$

using $\frac{X - \mu}{\sigma/\sqrt{n}}$ for confidence interval.

$$\boxed{\beta = 0.76005.}$$

$$\text{Power} = 1 - \beta = 1 - 0.76005 = 0.23995.$$

Q9 Suppose X is a random variable with $P(X = k) = p(1-p)^k$, $k = 0, 1, 2 \ldots$ and $p \in (0,1)$. For the hypothesis testing problem $H_0: p = 0.5$ vs. $H_1: p \neq 0.5$, consider the test "Reject H_0 if $X \le A$ or $X \ge B$", where $A < B$ are given positive integers. The type I error for this test is –

Sol: Here given that

$$P(X = k) = p(1-p)^k \;; k = 0, 1, 2 \ldots \text{ and } p \in (0,1).$$

and $H_0: p = 0.5$. $\quad H_1: p \neq 0.5$

Null Hypothesis. $\quad$ Alternate hyp.

Now Type I error. $\alpha = P(X \le A \text{ or } X \ge B \mid p = 0.5)$ (8)

$= P(X \le A) + P(X \ge B)$

$= 1 - P(X > A) + P(X \ge B)$

$= 1 - [P(X = A+1) + P(X = A+2) \cdots] + [P(X = B) + P(X = B+1) + \cdots]$

$= 1 - \left[\frac{1}{2^{A+2}} + \frac{1}{2^{A+3}} + \cdots\right] + \left[\frac{1}{2^{B+1}} + \frac{1}{2^{B+2}} + \cdots\right]$

$= 1 - \frac{1}{2^{A+1}} + \frac{1}{2^{B}}$

$\because P(X=k) = p(1-p)^k = \frac{1}{2}\left(\frac{1}{2}\right)^k = (1/2)(1/2)^k = (1/2)^{k+1}$

Q10 Let $X_1, X_2 \ldots X_n$ be a random sample from a $N(\mu, 1)$ population. Consider the hypothesis $H_0: \mu \le 0$ vs. $H_1: \mu > 0$. A random sample of size five from this population is 1.4, 2.4, 4.2, −3.4 and −1.2. Based on this sample which of the following statements is valid for a uniformly most powerful test of size 0.05?

Sol: Using $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim N(0,1)$

$= Z = \frac{\bar{X} - 0}{1/\sqrt{n}} = \sqrt{n}(\bar{X} - 0)$.

Here we have given the no. of observations

1.4, 2.4, 4.2, −3.4 and −1.2

$\bar{X} = \frac{\text{Sum of Obser.}}{\text{Total no. of obser}} = \frac{1.4 + 2.4 + 4.2 + (-3.4) + (-1.2)}{5}$

$= 3.4$

As $n = 5$

Now $Z = \sqrt{5}\cdot(3.4)$

$= 1.52$

Given that $Z_\alpha = Z_{0.05} = 1.645 \ge Z$

Therefore we Accept H_0 (Null Hypothesis)