Negative Binomial Dist.

A random variable X is said to follow a negative binomial dist. if its p.m.f is given by

$$P(X) = P(X=n) = (n+k-1)C_{k-1}\, p^k q^n \qquad n = 0,1,2 ---$$

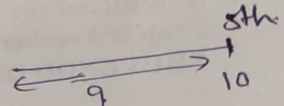
If $k=1$, the prob. fun. of negative binomial dist. reduces to the prob fun. of geometric dist.

$$\text{Mean} = \frac{kq}{p}, \quad \text{Var.} = \frac{kq}{p^2}$$

$$M_X(t) = \left(\frac{p}{1-qe^t}\right)^k$$

e.g. If a boy is throwing stones at a target, what is the prob. that his 10th throw is 5th hit, if the prob. of hitting the target at any trial is 1/2

Sol.



$$9c_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^5 \cdot \tfrac{1}{2}$$

Means. p = Prob. of hitting the target $= \frac{1}{2}$

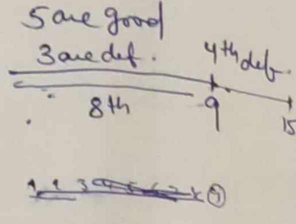
$q = 1-p = \frac{1}{2}$

If k denotes the no. of failure. In the first nine throw 4 hits the target and 4 failures and 10th throw is also a hit. Then

$n+k-1 = 9, \; k=5 \; \therefore n=5$

$$P(X=5) = 9c_4\, q^5 p^4 \cdot p = 9c_4 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^4 \cdot \tfrac{1}{2} = 0.123$$

e.g. A consignment of 15 tubes contains 4 def. tubes and the tubes are selected at random one by one and examined. Assuming that the tubes tested are not put back, what is the prob. that the ninth one examined is the last def. tube.



sol. Out of 15 tubes, 4 are def.

If the ninth one examined is the 4th def. tube, then in the first eight tubes examined 3 are def. and 5 are good and the tested tube are not put back.

The prob of selecting 3 def. tube and 5 goods out of 15 tubes (in which 4 are def. and 11 are good)

$$= \frac{4C_3 \times 11C_5}{15C_8} = \frac{56}{195}$$

After selecting 8 tubes from 15 tubes the ~~remaining~~ remaining tubes are 7 which contain only one def.

The prob. of selecting 1 def. tube (ninth one is def) out of 7 tubes $= \frac{1}{7}$

$\therefore$ The required prob. $= \frac{56}{195} \times \frac{1}{7} = 0.0410$

Gamma Dist.

A continuous random variable X is said to follow general gamma dist. or Erlang dist. with two parameters $\lambda>0$ and $k>0$, if its prob density fun. (PDF) is given by

$$f(x)=\begin{cases}\dfrac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma k}, & x\geq 0\\ 0, & \text{otherwise}\end{cases}$$

Note (i) when $\lambda=1$, the Erlang dist. is called simple gamma dis.

$$f(x)=\frac{1}{\Gamma k}x^{k-1}e^{-x}, \quad x\geq 0,\ k>0$$ with one parameter $k>0$

(ii) $\displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \frac{\lambda^k}{\Gamma k}\int_0^{\infty} x^{k-1}e^{-\lambda x}\,dx = \frac{\lambda^k}{\Gamma k}\cdot\frac{\Gamma k}{\lambda^k}=1$

$\left[\because \displaystyle\int_0^{\infty} x^{k-1}e^{-ax}\,dx = \frac{\Gamma k}{a^k}\right.$

(iii) when $k=1$, the Erlang dist. reduces to exponential distribution.

Moment Generating function of Gamma Dist.

$$M_X(t)= E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty}\frac{e^{tx}\lambda^k x^{k-1}e^{-\lambda x}}{\Gamma k}\,dx = \frac{\lambda^k}{\Gamma k}\int_0^{\infty} e^{tx}x^{k-1}e^{-\lambda x}\,dx$$

$$= \frac{\lambda^k}{\Gamma k}\int_0^{\infty} x^{k-1}e^{-x(\lambda-t)}\,dx$$

$$= \frac{\lambda^k}{\Gamma k}\cdot\frac{\Gamma k}{(\lambda-t)^k} = \left(\frac{\lambda}{\lambda-t}\right)^k$$

$$\text{Mean} = \mu_1' = \left[\frac{d}{dt}\left(\frac{\lambda}{\lambda-t}\right)^k\right]_{t=0}$$

$$= \left[k\left(\frac{\lambda}{\lambda-t}\right)^{k-1}\cdot\frac{\lambda}{(\lambda-t)^2}\right]_{t=0} = \frac{k}{\lambda}$$

$$\mu_2' = \left[\frac{d^2}{dt^2}\left(\frac{\lambda}{\lambda-t}\right)^k\right]_{t=0} = \lambda^k\left\{\frac{d}{dt}\left(k(\lambda-t)^{-k-1}\right)\right\}_{t=0}$$

$$= k\lambda^k\left[(-k-1)(\lambda-t)^{-k-2}(-1)\right]_{t=0}$$

$$= k\lambda^k\left[(k+1)(\lambda-t)^{-k-2}\right]_{t=0} = k\lambda^k(k+1)\lambda^{-k-2}$$

$$= \frac{k(k+1)}{\lambda^2} = \frac{k^2+k}{\lambda^2}$$

$$\text{Var.} = E(x^2) - [E(x)]^2 = \frac{k^2+k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}$$

Weibull distribution

$$f(x)=\begin{cases}\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}} & , x>0\\ 0 & , \text{otherwise}\end{cases}$$

where $\alpha>0$, $\beta>0$ are the two parameters of the Weibull dist. When $\beta=1$, the weibull dist. reduces to the exponential dist. with dist. α.

$$\text{Mean}= E(x)=\mu_1'=\int_0^\infty x\,\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}\,dx$$

$$=\alpha\beta\int_0^\infty x^{\beta}e^{-\alpha x^{\beta}}\,dx$$

Put $\alpha x^{\beta}=y$

$$\Rightarrow x=\left(\frac{y}{\alpha}\right)^{1/\beta}\Rightarrow dx=\frac{1}{\beta}\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1}\cdot\frac{1}{\alpha}dy$$

when $x=0$, $y=0$ and $x=\infty$, $y=\infty$

$$E(x)=\alpha\beta\int_0^\infty\left(\frac{y}{\alpha}\right)^{\beta/\beta}e^{-y}\cdot\frac{1}{\beta}\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1}\cdot\frac{1}{\alpha}\,dy$$

$$=\frac{\alpha\beta}{\alpha\beta}\int_0^\infty\left(\frac{y}{\alpha}\right)e^{-y}\cdot\left(\frac{y}{\alpha}\right)^{\frac{1}{\beta}-1}dy$$

$$=\left(\frac{1}{\alpha}\right)^{1/\beta}\int_0^\infty e^{-y}y^{\frac{1}{\beta}+1-1}dy=\left(\frac{1}{\alpha}\right)^{1/\beta}\int_0^\infty e^{-y}y^{\frac{1}{\beta}+1-1}dy$$

$$=\frac{1}{\alpha^{1/\beta}}\Gamma\left(\frac{1}{\beta}+1\right)$$

$$E(x^2)=\mu_2'=\int_0^\infty x^2\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}dx$$

$$=\alpha\beta\int_0^\infty x^{\beta+1}e^{-\alpha x^{\beta}}dx$$

$$\alpha x^{\beta} = y$$

$$\mu_2' = \int_0^\infty e^{-y}\left(\frac{y}{\alpha}\right)^{2/\beta} dy = \frac{1}{\alpha^{2/\beta}}\int_0^\infty e^{-y} y^{2/\beta}\, dy$$

$$= \frac{1}{\alpha^{2/\beta}}\int_0^\infty e^{-y} y^{\left(\frac{2}{\beta}+1-1\right)} dy$$

$$= \frac{1}{\alpha^{2/\beta}}\Gamma\left(\frac{2}{\beta}+1\right)$$

$$\text{Var.} = \mu_2' - (\mu_1')^2$$

$$= \frac{1}{\alpha^{2/\beta}}\Gamma\left(\frac{2}{\beta}+1\right) - \frac{1}{\alpha^{2/\beta}}\left[\Gamma\left(\frac{1}{\beta}+1\right)\right]^2$$

$$= \frac{1}{\alpha^{2/\beta}}\left[\Gamma\left(\frac{2}{\beta}+1\right) - \left\{\Gamma\left(\frac{1}{\beta}+1\right)\right\}^2\right]$$

In general, the rth moment about the origin is

$$\mu_r' = E(x^r) = \frac{1}{\alpha^{(r/\beta)}}\Gamma\left(\frac{r}{\beta}+1\right)$$

For a weibull dist.

$$P(x \leq a) = 1 - e^{-\alpha x^{\beta}} \; ; \; P(x > a) = e^{-\alpha x^{\beta}}.$$

eg. Each of the 6 tubes of a radio set has a life length (in years) which may be considered as a random variable that follows a Weibull dist. with parameter $\alpha = 25$ and $\beta = 2$. If these tubes fun. independently of one another, what is the prob. that no tube will have to be replaced during the first two months of service?

sol. If X represents the life length of each tube, then its density fun. $f(x)$ is given by

$$f(x) = \alpha\beta x^{\beta-1} e^{-\alpha x^{\beta}}, \quad x > 0$$

i.e. $$f(x) = 50 x e^{-25x^2}, \quad x > 0$$

$2 \text{ months} = \frac{2}{12} = \frac{1}{6}$ years.

P (a tube is not to be replaced during the first two months)

$$= P\left(X > \tfrac{1}{6}\right) = \int_{1/6}^{\infty} 50 x e^{-25x^2}\, dx$$

Put $X = 25x^2$

$dX = 50x\, dx$

$$= \int_{1/6}^{\infty} e^{-X}\, dX = \left[-e^{-X}\right]_{1/6}^{\infty} = \left[-e^{-25x^2}\right]_{1/6}^{\infty}$$

$$= e^{-25/36} = 0.4993.$$

$\therefore$ P(all the 6 tubes are not to be replaced during the first two months)

$$= e^{(-25/36)^6} = e^{-25/6} = 0.0155 \quad \text{(by independence)}$$

Normal Distribution

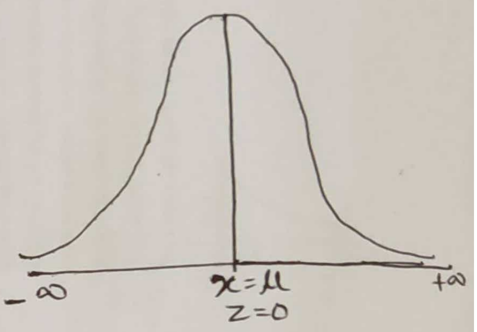
This is a continuous distribution.

- limiting case of binomial dist. i.e. no. of trials $n \to \infty$ with no restriction on p or q. (neither $p \to 0$ nor $q \to 0$)

Def: A continuous random variable X is said to be normal variate if it has prob. density function of the following form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty.$$

μ (Mean) and σ (S.D.)



Properties:-

- It is bell shaped curve.
- It is symmetrical about $z=0$ i.e. $x=\mu$
  area under normal curve is defined as

$$z = \frac{X-\mu}{\sigma}$$

- In this dis. Mean = mode = Median.
- Area lying under the normal prob. curve is 1

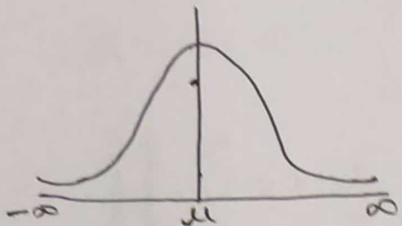
To convert the normal variate to standard normal standard variate:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty.$$

As $z = \dfrac{x-\mu}{\sigma}$

$$f(z) = \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \qquad -\infty < z < \infty$$

Area under the standard prob. curve



$-\infty$ to $\mu = 0.5$

μ to $\infty = 0.5$

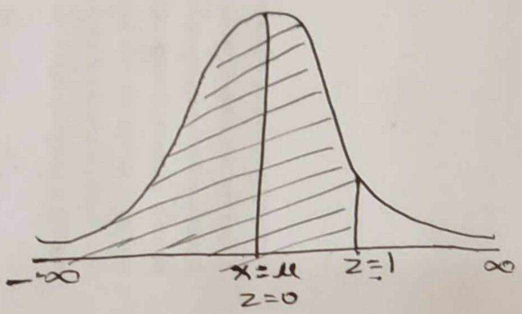
Eg. Evaluate the prob. at $\mu = 10$ and $\sigma = 5$

(i) $P(X \le 15)$ (ii) $P(X \ge 15)$ (iii) $P(10 \le X \le 15)$

Sol.

$\mu = 10$ and $\sigma = 5$

$z = \frac{x-\mu}{\sigma} = \frac{15-10}{5} = 1$



$P(X \le 15) = P(Z \le 1)$

$= P(-\infty \le Z \le 0) + P(0 \le Z \le 1)$

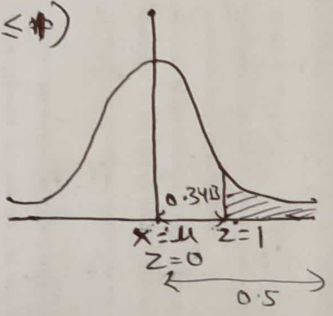
$= 0.5 + 0.3413 = 0.8413$

(ii) $P(X \ge 15) = P(Z \ge 1)$

$= P(0 \le Z \le \infty) - P(0 \le Z \le 1)$

$= 0.5 - 0.3413$

$= 0.1587$



(iii) $P(10 \le X \le 15)$

when $x = 10$

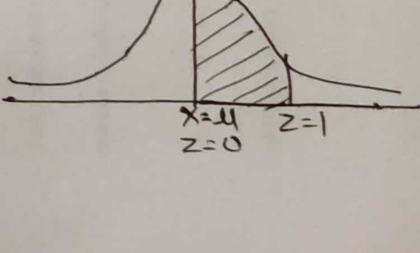
$z = \frac{10-10}{\sigma} = \frac{0}{5} = 0$

when $x = 15$

$z = 1$

$P(10 \le X \le 15) = P(0 \le Z \le 1)$

$= 0.3413$



Mean of Normal Distribution

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad -\infty < x < \infty$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

Let $z = \frac{x-\mu}{\sigma}$ $\qquad x-\mu = \sigma z$

$x = \sigma z + \mu$

$dz = \frac{1}{\sigma} dx$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z + \mu) e^{-\frac{1}{2}(z^2)} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-z^2/2} dz$$

$$= \frac{2}{\sqrt{2\pi}} \left[\int_0^{\infty} \mu e^{-z^2/2} dz + \int_0^{\infty} \sigma z e^{-z^2/2} dz \right.$$

$$= \frac{2}{\sqrt{2\pi}} \left[\mu \int_0^{\infty} e^{-z^2/2} dz + \sigma \int_0^{\infty} \underbrace{z e^{-z^2/2} dz}_{\text{odd fun}} \right.$$

$$= \frac{2}{\sqrt{2\pi}} \left[\mu \cdot \sqrt{\frac{\pi}{2}} + \sigma(0) \right]$$

$$= \underline{\underline{\mu}}$$

And Variance $= \underline{\sigma^2}$

Karl Pearson coeff. β & γ for normal dist.

$$\beta_1 = \frac{\mu_3^2}{\mu_2^2} = 0 \quad , \beta_1 = 0 \quad , \gamma_1 = \sqrt{\beta_1} = 0$$

Therefore curve is symmetric

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3$$

$$\beta_2 = 3$$

Hence curve is normal curve.

e.g. The diameter of a dot produced by a printer is normally distributed with a mean diameter of 0.005 cm and S.D. of 0.001 cm. What is the prob. that a diameter is between 0.0035 and 0.0065 cm.

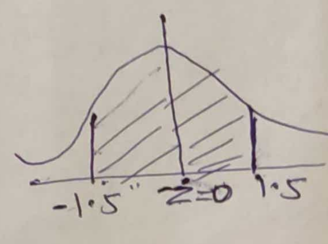
sol.

$$Z = \frac{x-\mu}{\sigma}$$

$$= \frac{0.0035 - 0.005}{0.001} = \frac{-0.0015}{0.001} = -1.5$$

$$Z = \frac{0.0065 - 0.005}{0.001}$$

$$= \frac{0.0015}{0.001} = 1.5$$



According to table

$z(0 \text{ to } 1.5) = .4332$

$.4332 + .4332 = .8664$

$$P(0.0035 \le x \le 0.0065)$$

$$= P(-1.5 \le z \le 1.5)$$

$$= P(-1.5 \le z \le 0) + P(0 \le z \le 1.5)$$

$$= P(0 \le z \le 1.5) + P(0 \le z \le 1.5)$$

$$= .4332 + .4332$$

$$= .8664$$