(2) Let X and Y be two r.v with the joint density function given by

$$f(x,y) = \begin{cases} ax+1 & x,y>0, \quad x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(Y<X)$.

Ans:

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} ax+1 \, dy\,dx$$

$$= \int_0^1 (ax+1)(1-x)\,dx$$

$$= \frac{1}{2} + \frac{a}{6}$$

$$\Rightarrow \quad a = 3.$$

$$\therefore f(x,y) = \begin{cases} 3x+1 & x,y>0 \quad x+y<1 \\ 0 & \text{otherwise} \end{cases}$$

Now $P(Y<X) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{XY}(x,y)\,dy\,dx$

$$= \int_0^1 \int_0^{\min(x,1-x)} (3x+1)\,dy\,dx$$

$$= \int_0^{1/2} x(3x+1)\,dx + \int_{1/2}^{1} (3x+1)(1-x)\,dx$$

$$= \frac{3x^3}{3} + \frac{x^2}{2}\Big|_0^{1/2} + \left[\frac{3x^2}{2} - 3\cdot\frac{x^3}{3} + x - \frac{x^2}{2}\right]_{1/2}^{1}$$

$$= \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^3 + \left[\tfrac{3}{2} - \cancel{1} + \cancel{1} - \tfrac{1}{2} - 3\left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 - \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^3\right]$$

$$= 2\left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^3[-3+1+1] + \left[\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{1}{2}\right]$$

$$= 2\left(\tfrac{1}{2}\right)^3 - \left(\tfrac{1}{2}\right)^3 + \tfrac{1}{2}$$

$$= \left(\tfrac{1}{2}\right)^3 + \tfrac{1}{2} = \tfrac{1}{8} + \tfrac{1}{2} = \frac{1+4}{8} = \frac{5}{8}$$

Ans (A)

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(3) Suppose that X and Y are two r.v. with joint pmf

| | $Y=0$ | $Y=1$ | $Y=2$ | $Y=3$ | |
|---|---|---|---|---|---|
| $X=0$ | $a$ | $a$ | $2a$ | $3a$ | $7a$ |
| $X=1$ | $3a$ | $a$ | $a$ | $2a$ | $7a$ |
| $X=2$ | $5a$ | $3a$ | $2a$ | $4a$ | $14a$ |
| | $9a$ | $5a$ | $5a$ | $9a$ | $28a$ |

Find $Var(X)$

Ans: $28a = 1 \Rightarrow a = \frac{1}{28} = 0.0357142$

$P_X(x=0) = 7a = \frac{7}{28} = \frac{1}{4}$ 0.3571

$P_X(x=1) = 7a = \frac{7}{28} = \frac{1}{4}$ 0.3571

$P_X(x=2) = 14a = \frac{14}{28} = \frac{1}{2}$

$\therefore Var(X) = \sum x^2 P_X(x) - (E(X))^2$

$= \sum x^2 P_X(x) - \left[\sum x P_X(x)\right]^2$

$= \left[0^2 . P(0) + 1^2 P(1) + 2^2 P(2)\right] - \left[0.P(0) + 1 P(1) + 2 P(2)\right]^2$

$= \left[\frac{1}{4} + \frac{4}{2}\right] - \left[\frac{1}{4} + \frac{2}{2}\right]^2$

$= \left(\frac{1}{4} + 2\right) - \left(\frac{1}{4} + 1\right)^2 =$

$= 2.25 - (1.25)^2 = 2.25 - 1.5625$

$= 0.6875$

Ans. (D)

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(4) Qu: Let X and Y be two r.v. with the joint density function

$$f(x,y) = \begin{cases} 6xy & 0 \le x \le 1, \quad 0 \le y \le \sqrt{x} \\ 0 & \text{otherwise.} \end{cases}$$

Calculate $E(Y|X=x)$

Ans: $f_{Y|X}(y|x) = \dfrac{f(x,y)}{f_X(x)}$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \int_0^{\sqrt{x}} 6xy\,dy$$

$$= 6x \left.\frac{y^2}{2}\right|_0^{\sqrt{x}} = 6x \cdot \frac{x}{2} = 3x^2$$

$$\therefore f_{Y|X}(y|x) = \frac{6xy}{3x^2} = \frac{2y}{x}$$

$$\therefore E(Y|X=x) = \int_{-\infty}^{\infty} y\, f_{Y|X}(y|x)\,dy$$

$$= \int_0^{\sqrt{x}} y \cdot \frac{2y}{x}\,dy$$

$$= \left.\frac{2y^3}{3x}\right|_0^{\sqrt{x}} = \frac{2}{3x}\left[x^{3/2}\right]$$

$$= \frac{2}{3}\sqrt{x}$$ (Ans)

Ans - (A)

(5) $$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_0^1 6xy\,dx$$

$$= 6y \cdot \left.\frac{x^2}{2}\right|_0^1 = 6y \cdot \frac{1}{2} = 3y$$

$$\therefore f_X(x) \cdot f_Y(y) = 3x^2 \cdot 3y = 9x^2y$$

$$f_1(x,y) = \begin{cases} 6\,e^{-(2x_1+3y_1)} & x_1, y_1 > 0 \\ 0 & \text{otherw} \end{cases}$$

An (A) $(X_1, Y_1)$ independent but $(X_2, Y_2)$ is not independent.

⑥ Consider X and Y be two r.v. with joint mass function

| | Y=1 | Y=2 | Y=3 | |
|---|---|---|---|---|
| X=1 | 0 | 1/2 | 1/8 | 0.625 |
| X=2 | 0 | 0 | 1/8 | 0.125 |
| X=3 | 1/8 | 0 | 1/8 | ~~0.125~~ 0.25 |
| | 0.125 | 0.5 | 0.375 | |

Find Cov (X, Y).

$$Cov(X\,Y) = E(XY) - E(X)\cdot E(Y)$$

$$E(XY) = \sum xy\, P(x,y)$$

$$= 1\cdot 2\cdot \frac{1}{2} + 1\cdot 3\cdot \frac{1}{8} + 2\cdot 3\cdot \frac{1}{8} + 3\cdot 1\cdot \frac{1}{8} + 3\cdot 3\cdot \frac{1}{8}$$

$$= 1 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} + \frac{9}{8}$$

$$= \frac{8+3+6+3+9}{8} = \frac{29}{8}$$

$$E(X) = \sum x\, P_x(x) = 1\cdot\left(\frac{1}{2}+\frac{1}{8}\right) + 2\left(\frac{1}{8}\right) + 3\left(\frac{1}{8}+\frac{1}{8}\right)$$

$$= \frac{1}{2} + \frac{1}{8} + \frac{2}{8} + \frac{6}{8}$$

$$= \frac{4+1+2+6}{8} = \frac{13}{8}$$

$$E(Y) = \sum y\, P_y(y) = 1\left(\frac{1}{8}\right) + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)$$

$$= \frac{1}{8} + 1 + \frac{9}{8} = \frac{1+8+9}{8} = \frac{18}{8}$$

$$\therefore Cov(X\,Y) = \frac{29}{8} - \left(\frac{13}{8}\cdot\frac{18}{8}\right)$$

$$= \frac{29\times 8 - 13\times 18}{64} = \frac{232-234}{64}$$

$$= \frac{-2}{64} = -0.03125$$

Ans (D)

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(7) Suppose a fair die rolled $n$ times.
Let $X$ and $Y$ be the r.v. denoting the number 1's and number 2's respectively. Find $n$ such that
$Cov(X, Y) = \frac{-1}{4}$.

Ans: $Cov(X, Y) = -\frac{n}{36}$

$\therefore -\frac{n}{36} = \frac{-1}{4} \Rightarrow n = 9$

Ans. (C)