Note on coherent state

Mengzhi Wu

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This note aims to summarize essential properties of the coherent state. I plan to cover these points:

- Basic knowledges on oscillators and coherent states.
- Coherent states under Hamiltonian representation and time.
- Coherent states under coordinate representation.
- Wigner functions for coherent states and time evolution.
- classical propertities of coherent states.

1 Basic knowledges on oscillators

In this section, I'd like to briefly review some basic properties of some operators. The Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, and the commutation relation is $[\hat{x}, \hat{p}] = i\hbar$(Be careful of the sign!). Then the creation and annihilation operator can be defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}),\ \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}) \tag{1}$$

And the commutation relation is $[\hat{a}^\dagger, \hat{a}] = -1$(Be careful of the sign!). Inversely, $\hat{x}$, $\hat{p}$ and $\hat{H}$ can be described as

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \\ \hat{H} &= (\hat{a}^+\hat{a} + \frac{1}{2})\hbar\omega \end{aligned} \tag{2}$$

Suppose the eigenstate of the Hamiltonian is $|n\rangle$ with eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$ whose wavefunction is

$$\Psi_n(x) = \big(\frac{m\omega}{\pi\hbar}\big)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n(\frac{m\omega}{\hbar}x)\exp\big(-\frac{m\omega}{2\hbar}x^2\big) \tag{3}$$

Then the creation and annihilation operators give out

$$\begin{aligned} \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{4}$$

2 Important opertor formulas

Then let's review some useful formulas. The first one is Glauber's formula, based on Baker-Hausdorff formula. This formula claims that if operators $\hat{A}$ and $\hat{B}$ satisfies $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$, then

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp\left(-\frac{1}{2}[\hat{A},\hat{B}]\right) \tag{5}$$

This formula tell us the commutation property of $\exp(\hat{A})$ and $\exp(\hat{B})$ which is called Weyl commutation relation:

$$\exp(\hat{A})\exp(\hat{B}) = \exp(\hat{B})\exp(\hat{A})\exp([\hat{A},\hat{B}]) \tag{6}$$

The 2nd formula is

$$\exp(\lambda\hat{A})\hat{B}\exp(-\lambda\hat{A}) = \hat{B} + \lambda[\hat{A},\hat{B}] + \frac{\lambda^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{\lambda^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots \tag{7}$$

Specially, if $[\hat{A},\hat{B}] = const =: C$, then

$$\exp(\lambda\hat{A})\hat{B}\exp(-\lambda\hat{A}) = \hat{B} + \lambda C \tag{8}$$

which means a translation of operator $\hat{B}$. More specially, we set $\hat{A} = -\alpha\hat{a}^\dagger + \alpha^*\hat{a}$, $\hat{B} = \hat{a}$ or $\hat{B} = \hat{a}^\dagger$, then we may obtain

$$\exp(-\alpha\hat{a}^+ + \alpha^*\hat{a})\hat{a}\exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \hat{a} + \alpha \tag{9}$$

$$\exp(-\alpha\hat{a}^\dagger + \alpha^*\hat{a})\hat{a}^\dagger\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \hat{a}^\dagger + \alpha^* \tag{10}$$

Equation (9) is very important. Denote $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, then (9) presents

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha \tag{11}$$

Based on (11), we may prove that all the eigenstates of $\hat{a}$ are

$$\hat{a}\big(\hat{D}(\alpha)\left|0\right\rangle\big) = \alpha\big(\hat{D}(\alpha)\left|0\right\rangle\big) \tag{12}$$

Therefore, we may define the coherent states as $\left|\alpha\right\rangle = \hat{D}(\alpha)\left|0\right\rangle$ with any complex number eigenvalue $\alpha \in \mathbb{C}$, and $\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$. Note that all the translation opertors forms a group called Heisenberg-Weyl group. It's not difficult to find out that Heisenberg-Weyl group is isomorphic to $(\mathbb{C},+)$, that is

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta) \tag{13}$$

Since $\hat{D}(\alpha)$ is an operator on a linear space ,the Hilbert space for the oscillator, $\hat{D}(\alpha)$ can be regarded as a group representation of $(\mathbb{C},+)$.

Now that $\hat{a}^\dagger$ and $\hat{a}$ can be described by $\hat{x}$ and $\hat{p}$, we can write the translation operator as another form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x} - i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \tag{14}$$

It's not difficult to find that $\hat{D}(\alpha)$ is actually a Weyl translation operator $\hat{W}(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}})$. Based on Glauber's formula, we can express $\hat{D}(\alpha)$ as a separated form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\alpha}{\hbar}}\hat{x}\right)\exp\left(-i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right)\exp(-i\Im(\alpha)\Re(\alpha)) \tag{15}$$

Recall that all $\hat{D}(\alpha)$ forms a group representation of $(\mathbb{C},+)$, it can also be considered as a group representation of $(\mathbb{R}^2,+)$, where the $\mathbb{R}^2$ represnts the x and p coordinate, which is actually a point in the phase space. So a coherent state is actually a group action by a point $(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}})$ in the phase space, and the evolution of a coherent state is a group action by a trajectory in the phase space. Therefore, we may use points and trajectories to describe coherent states of an oscillator.

In the end of this section, I'd like to mention a further appliacation of (7). Set $\hat{A} = \hat{N} = \hat{a}^\dagger\hat{a}$, and $\hat{B} = \hat{a}^\dagger$ or $\hat{a}$. Then we may obtain

$$\exp(\lambda\hat{N})\hat{a}\exp(-\lambda\hat{N}) = \hat{a}e^{-\lambda} \tag{16}$$

$$\exp(\lambda\hat{N})\hat{a}^\dagger\exp(-\lambda\hat{N}) = \hat{a}^\dagger e^{\lambda} \tag{17}$$

Still pay attention to the sign! This formula will give out the squeezed states.

3 Hamiltonian representation and time evolution

It's not difficult to show that the expansion of a coherent state $|\alpha\rangle$ by the eigenstates of the Hamiltonian eigenstates is

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{18}$$

Therefore the evolution of that state is $|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar}|\alpha\rangle = |\alpha\rangle = e^{-|\alpha|^2/2}\sum\frac{\alpha^n}{\sqrt{n!}}e^{-iE_nt/\hbar}|n\rangle$. Since $E_n = (n+\frac{1}{2})\hbar\omega$, that factor can be written as $e^{-iE_nt/\hbar} = e^{-i\frac{1}{2}\omega t}e^{-in\omega t} = e^{-i\frac{1}{2}\omega t}\big(e^{-i\omega t}\big)^n$, where the latter phase can be combined with α^n and presents $(\alpha e^{-i\omega t})^n$ and $|\alpha e^{-i\omega t}|^2 = |\alpha|^2$. Therefore,

$$|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t}\exp\big(-|\alpha e^{-i\omega t}|^2\big)\sum\frac{(\alpha e^{-i\omega})^n}{\sqrt{n!}}|n\rangle \tag{19}$$

which suggests

$$|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t}|\alpha e^{-i\omega t}\rangle \tag{20}$$

Now that the translation operator is a group representation of $(\mathbb{R}^2,+)$, where $\mathbb{R}^2$ is the phase space, the evolution of $|\alpha(t)\rangle$ corresponds to $(\Im(\alpha\exp(-i\omega t))\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha\exp(-i\omega t))\sqrt{\frac{2}{m\hbar\omega}})$. So we can use a circle in $\mathbb{C}$ to described the trajectory of $|\alpha(t)\rangle$.

4 Coherent states under coordinate representation

Now let's calculate the coherent state under coordinate representation, i.e. wavefunction of the coherent state $\langle x|\alpha\rangle$. Notice the expression (14) of $\hat{D}(\alpha)$ by $\hat{x}$ and $\hat{p}$, we can consider the operation of $\hat{D}(\alpha)$ over coordinate eigenstate $|x\rangle$. Then $\langle x|\alpha\rangle$ can be expressed as $\langle x|\hat{D}(\alpha)|0\rangle$, where the translation operator operates on $\langle x|$. Note that $\hat{D}(\alpha)^\dagger = \hat{D}(-\alpha)$

$$\begin{aligned}
\hat{D}(-\alpha)|x\rangle &= \exp\left(i\Im(-\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x}\right)\exp\left(-i\Re(-\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right)\exp(-i\Im(-\alpha)\Re(-\alpha))|x\rangle \\
&= \exp\left(-i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}})\right)\exp(-i\Im(\alpha)\Re(\alpha))|x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\rangle \\
&= \exp\left(-i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}x\right)\exp(i\Im(\alpha)\Re(\alpha))|x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\rangle
\end{aligned} \tag{21}$$

Therefore the wavefunction of a coherent state $|\alpha\rangle$ is

$$\begin{aligned}\Psi_\alpha(x) &= \langle x|\,\hat{D}(\alpha)\,|0\rangle = \langle \hat{D}(-\alpha)x|0\rangle \\ &= \exp\Big(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}x\Big)\exp(-i\Im(\alpha)\Re(\alpha))\,\langle x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}|0\rangle \\ &= N\exp\Big(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}x\Big)\exp(-i\Im(\alpha)\Re(\alpha))\exp\Big(-\frac{m\omega}{2\hbar}\big(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)\end{aligned} \tag{22}$$

where N is the normalization constant.
Since $|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t}\,|\alpha e^{-i\omega t}\rangle$, the wavefunction at time t is

$$\begin{aligned}\Psi_\alpha(x,t) &= N\exp\Big(i\Im(\alpha e^{-i\omega t})\sqrt{\frac{2m\omega}{\hbar}}x\Big)\exp(-i\Im(\alpha e^{-i\omega t})\Re(\alpha e^{-i\omega t}))\exp\Big(-\frac{m\omega}{2\hbar}\big(x-\Re(\alpha e^{-i\omega t})\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)\exp(-i\frac{1}{2}\omega t) \\ &= N\exp\Big(i\big(\Im(\alpha)\cos(\omega t)-\Re(\alpha)\sin(\omega t)\big)\sqrt{\frac{2m\omega}{\hbar}}x\Big)\exp\Big(-i\big(\Re(\alpha)\Im(\alpha)\cos(2\omega t)-\frac{(\Re(\alpha)^2-\Im(\alpha)^2)}{2}\sin(2\omega t)\big)\Big) \\ &\times\exp\Big(-\frac{m\omega}{2\hbar}\big(x-\big(\Re(\alpha)\cos(\omega t)+\Im(\alpha)\sin(\omega t)\big)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)\exp(-i\frac{1}{2}\omega t)\end{aligned} \tag{23}$$

5 Wigner function for coherent states

The Wigner function is defined as

$$W(x,p) = \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\,\langle x+\frac{1}{2}\xi|\,\hat{\rho}\,|x-\frac{1}{2}\xi\rangle\,e^{-\frac{i}{\hbar}p\xi} \tag{24}$$

where the density matrix $\hat{\rho} = |\alpha\rangle\langle\alpha|$. Therefore,

$$\begin{aligned}W(x,p) &= \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\,\langle x+\frac{1}{2}\xi|\alpha\rangle\,\langle\alpha|x-\frac{1}{2}\xi\rangle\,e^{-\frac{i}{\hbar}p\xi} \\ &= N^2\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}e^{i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\left(x+\frac{1}{2}\xi\right)}e^{-i\Im(\alpha)\Re(\alpha)}e^{-\frac{m\omega}{2\hbar}\left(x+\frac{1}{2}\xi-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\right)^2} \\ &\times e^{-i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\left(x-\frac{1}{2}\xi\right)}e^{i\Im(\alpha)\Re(\alpha)}e^{-\frac{m\omega}{2\hbar}\left(x-\frac{1}{2}\xi-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\right)^2}\times e^{-\frac{i}{\hbar}p\xi} \\ &= N^2\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\exp\Big[i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}-\frac{m\omega}{2\hbar}\Big(\big(x+\frac{1}{2}\xi-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2+\big(x-\frac{1}{2}\xi-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)-\frac{i}{\hbar}p\xi\Big] \\ &= N^2\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\exp\Big[i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}-\frac{m\omega}{2\hbar}\Big(\frac{1}{2}\xi^2+2\big(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)-\frac{i}{\hbar}p\xi\Big] \\ &= N^2\exp\Big(-\frac{m\omega}{\hbar}\big(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)\int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\exp\Big[-\frac{m\omega}{2\hbar}\xi^2+\Big(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}-\frac{p}{\hbar}\Big)\xi\Big] \\ &= N^2\exp\Big(-\frac{m\omega}{\hbar}\big(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)\sqrt{\frac{\hbar}{2\pi m\omega}}\exp\Big[-\frac{\hbar}{2m\omega}\Big(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}-\frac{p}{\hbar}\Big)^2\Big] \\ &= N^2\sqrt{\frac{\hbar}{2\pi m\omega}}\exp\Big(-\frac{m\omega}{\hbar}\big(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\big)^2\Big)\exp\Big(-\frac{1}{2m\hbar\omega}\big(p-\Im(\alpha)\sqrt{2m\hbar\omega}\big)^2\Big)\end{aligned} \tag{25}$$

In the 6th "=", we use the formula $\int \frac{dx}{2\pi} e^{-ax^2+ibx} = \frac{e^{-b^2/4a}}{2\sqrt{\pi a}}$.

It's not difficult to check the property of the Wigner function that the x and p probability distributions are given by the marginals:

$$\int dpW(x,p) = \langle x| \hat{\rho} |x\rangle \simeq |\Psi(x)|^2 \tag{26}$$

$$\int dxW(x,p) = \langle p| \hat{\rho} |p\rangle \simeq |\phi(p)|^2 \tag{27}$$