

# Note on coherent state

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This note aims to summarize essential properties of the coherent state. I plan to cover these points:

- Basic properties for creation and annihilation operators.
- Coherent states under Hamiltonian representation.
- Coherent states under coordinate representation.
- Wigner functions for coherent states and time evolution.
- classical properties of coherent states.

## 1 Basic knowledges on oscillators

In this section, I'd like to briefly review some basic properties of some operators. The Hamiltonian is  $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$ , and the commutation relation is  $[\hat{x}, \hat{p}] = i\hbar$  (Be careful of the sign!). Then the creation and annihilation operator can be defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}) \quad (1)$$

And the commutation relation is  $[\hat{a}^\dagger, \hat{a}] = -1$  (Be careful of the sign!). Inversely,  $\hat{x}$ ,  $\hat{p}$  and  $\hat{H}$  can be described as

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \\ \hat{H} &= (\hat{a}^\dagger\hat{a} + \frac{1}{2})\hbar\omega \end{aligned} \quad (2)$$

Suppose the eigenstate of the Hamiltonian is  $|n\rangle$  with eigenvalue  $E_n = (n + \frac{1}{2})\hbar\omega$  whose wavefunction is

$$\Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{m\omega}{\hbar}x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad (3)$$

Then the creation and annihilation operators give out

$$\begin{aligned} \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \\ \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \end{aligned} \quad (4)$$

## 2 Important operator formulas

Then let's review some useful formulas. The first one is Glauber's formula, based on Baker-Hausdorff formula. This formula claims that if operators  $\hat{A}$  and  $\hat{B}$  satisfies  $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$ , then

$$\exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \exp(\hat{B}) \exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right) \quad (5)$$

The 2nd formula is

$$\exp(\lambda \hat{A}) \hat{B} \exp(-\lambda \hat{A}) = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{\lambda^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots \quad (6)$$

Specially, if  $[\hat{A}, \hat{B}] = \text{const} =: C$ , then

$$\exp(\lambda \hat{A}) \hat{B} \exp(-\lambda \hat{A}) = \hat{B} + \lambda C \quad (7)$$

which means a translation of operator  $\hat{B}$ . More specially, we set  $\hat{A} = -\alpha \hat{a}^\dagger + \alpha^* \hat{a}$ ,  $\hat{B} = \hat{a}$  or  $\hat{B} = \hat{a}^\dagger$ , then we may obtain

$$\exp(-\alpha \hat{a}^\dagger + \alpha^* \hat{a}) \hat{a} \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) = \hat{a} + \alpha \quad (8)$$

$$\exp(-\alpha \hat{a}^\dagger + \alpha^* \hat{a}) \hat{a}^\dagger \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) = \hat{a}^\dagger + \alpha^* \quad (9)$$

Equation (8) is very important. Denote  $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$ , then (8) presents

$$\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha \quad (10)$$

Based on (10), we may prove that all the eigenstates of  $\hat{a}$  are

$$\hat{a}(\hat{D}(\alpha) |0\rangle) = \alpha(\hat{D}(\alpha) |0\rangle) \quad (11)$$

Therefore, we may define the coherent states as  $|\alpha\rangle = \hat{D}(\alpha) |0\rangle$  with any complex number eigenvalue  $\alpha \in \mathbb{C}$ , and  $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$ . Note that all the translation operators forms a group called Heisenberg-Weyl group. It's not difficult to find out that Heisenberg-Weyl group is isomorphic to  $(\mathbb{C}, +)$ , that is

$$\hat{D}(\alpha) \hat{D}(\beta) = \hat{D}(\alpha + \beta) \quad (12)$$

Since  $\hat{D}(\alpha)$  is an operator on a linear space, the Hilbert space for the oscillator,  $\hat{D}(\alpha)$  can be regarded as a group representation of  $(\mathbb{C}, +)$ .

Now that  $\hat{a}^\dagger$  and  $\hat{a}$  can be described by  $\hat{x}$  and  $\hat{p}$ , we can write the translation operator as another form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x} - i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \quad (13)$$

It's not difficult to find that  $\hat{D}(\alpha)$  is actually a Weyl translation operator  $\hat{W}(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}})$ . Based on Glauber's formula, we can express  $\hat{D}(\alpha)$  as a separated form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x}\right) \exp\left(-i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \exp(-i\Im(\alpha)\Re(\alpha)) \quad (14)$$

Recall that all  $\hat{D}(\alpha)$  forms a group representation of  $(\mathbb{C}, +)$ , it can also be considered as a group representation of  $(\mathbb{R}^2, +)$ , where the  $\mathbb{R}^2$  represents the  $x$  and  $p$  coordinate, which is actually a point in the phase space. So a coherent state is actually a group action by a point in the phase space, and the evolution of a coherent state is a group action by a trajectory in the phase space.

In the end of this section, I'd like to mention a further application of (6). Set  $\hat{A} = \hat{N} = \hat{a}^\dagger \hat{a}$ , and  $\hat{B} = \hat{a}^\dagger$  or  $\hat{a}$ . Then we may obtain

$$\exp(\lambda \hat{N}) \hat{a} \exp(-\lambda \hat{N}) = \hat{a} e^{-\lambda} \quad (15)$$

$$\exp(\lambda \hat{N}) \hat{a}^\dagger \exp(-\lambda \hat{N}) = \hat{a}^\dagger e^\lambda \quad (16)$$

Still pay attention to the sign! This formula will give out the squeezed states.

### 3 Hamiltonian representation and time evolution

It's not difficult to show that the expansion of a coherent state  $|\alpha\rangle$  by the eigenstates of the Hamiltonian eigenstates is

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (17)$$

Therefore the evolution of that state is  $|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar} |\alpha\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$ . Since  $E_n = (n + \frac{1}{2})\hbar\omega$ , that factor can be written as  $e^{-iE_n t/\hbar} = e^{-i\frac{1}{2}\omega t} e^{-in\omega t} = e^{-i\frac{1}{2}\omega t} (e^{-i\omega t})^n$ , where the latter phase can be combined with  $\alpha^n$  and presents  $(\alpha e^{-i\omega t})^n$  and  $|\alpha e^{-i\omega t}|^2 = |\alpha|^2$ . Therefore,

$$|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t} \exp(-|\alpha e^{-i\omega t}|^2) \sum \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle \quad (18)$$

which suggests

$$|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t} |\alpha e^{-i\omega t}\rangle \quad (19)$$

### 4 Coherent states under coordinate representation

Now let's calculate the coherent state under coordinate representation, i.e. wavefunction of the coherent state  $\langle x|\alpha\rangle$ . Notice the expression (13) of  $\hat{D}(\alpha)$  by  $\hat{x}$  and  $\hat{p}$ , we can consider the operation of  $\hat{D}(\alpha)$  over coordinate eigenstate  $|x\rangle$ . Then  $\langle x|\alpha\rangle$  can be expressed as  $\langle x|\hat{D}(\alpha)|0\rangle$ , where the translation operator operates on  $\langle x|$ . Note that  $\hat{D}(\alpha)^\dagger = \hat{D}(-\alpha)$

$$\begin{aligned} \hat{D}(\alpha) |x\rangle &= \exp\left(i\Im(\alpha)\sqrt{\frac{2m\alpha}{\hbar}}\hat{x}\right) \exp\left(-i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \exp(-i\Im(\alpha)\Re(\alpha)) |x\rangle \\ &= \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\left(x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\right)\right) \exp(-i\Im(\alpha)\Re(\alpha)) |x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\rangle \end{aligned} \quad (20)$$