

Note on coherent state

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This note aims to summarize essential properties of the coherent state. I plan to cover these points:

- Basic knowledges on oscillators and coherent states.
- Coherent states under Hamiltonian representation and time.
- Coherent states under coordinate representation.
- Wigner functions for coherent states and time evolution.
- classical propertities of coherent states.

1 Basic knowledges on oscillators

In this section, I'd like to briefly review some basic properties of some operators. The Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, and the commutation relation is $[\hat{x}, \hat{p}] = i\hbar$ (Be careful of the sign!). Then the creation and annihilation operator can be defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}), \quad \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}) \quad (1)$$

And the commutation relation is $[\hat{a}^\dagger, \hat{a}] = -1$ (Be careful of the sign!). Inversely, \hat{x} , \hat{p} and \hat{H} can be described as

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \\ \hat{H} &= (\hat{a}^\dagger\hat{a} + \frac{1}{2})\hbar\omega \end{aligned} \quad (2)$$

Suppose the eigenstate of the Hamiltonian is $|n\rangle$ with eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$ whose wavefunction is

$$\Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n\left(\frac{m\omega}{\hbar}x\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \quad (3)$$

Then the creation and annihilation operators give out

$$\begin{aligned} \hat{a} |n\rangle &= \sqrt{n} |n-1\rangle \\ \hat{a}^\dagger |n\rangle &= \sqrt{n+1} |n+1\rangle \end{aligned} \quad (4)$$

2 Important operator formulas

Then let's review some useful formulas. The first one is Glauber's formula, based on Baker-Hausdorff formula. This formula claims that if operators \hat{A} and \hat{B} satisfies $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, then

$$\exp(\hat{A} + \hat{B}) = \exp(\hat{A}) \exp(\hat{B}) \exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right) \quad (5)$$

This formula tell us the commutation property of $\exp(\hat{A})$ and $\exp(\hat{B})$ which is called Weyl commutation relation:

$$\exp(\hat{A}) \exp(\hat{B}) = \exp(\hat{B}) \exp(\hat{A}) \exp([\hat{A}, \hat{B}]) \quad (6)$$

The 2nd formula is

$$\exp(\lambda \hat{A}) \hat{B} \exp(-\lambda \hat{A}) = \hat{B} + \lambda [\hat{A}, \hat{B}] + \frac{\lambda^2}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{\lambda^3}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots \quad (7)$$

Specially, if $[\hat{A}, \hat{B}] = \text{const} =: C$, then

$$\exp(\lambda \hat{A}) \hat{B} \exp(-\lambda \hat{A}) = \hat{B} + \lambda C \quad (8)$$

which means a translation of operator \hat{B} . More specially, we set $\hat{A} = -\alpha \hat{a}^\dagger + \alpha^* \hat{a}$, $\hat{B} = \hat{a}$ or $\hat{B} = \hat{a}^\dagger$, then we may obtain

$$\exp(-\alpha \hat{a}^\dagger + \alpha^* \hat{a}) \hat{a} \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) = \hat{a} + \alpha \quad (9)$$

$$\exp(-\alpha \hat{a}^\dagger + \alpha^* \hat{a}) \hat{a}^\dagger \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) = \hat{a}^\dagger + \alpha^* \quad (10)$$

Equation (9) is very important. Denote $\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a})$, then (9) presents

$$\hat{D}^\dagger(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha \quad (11)$$

Based on (11), we may prove that all the eigenstates of \hat{a} are

$$\hat{a}(\hat{D}(\alpha)|0\rangle) = \alpha(\hat{D}(\alpha)|0\rangle) \quad (12)$$

Therefore, we may define the coherent states as $|\alpha\rangle = \hat{D}(\alpha)|0\rangle$ with any complex number eigenvalue $\alpha \in \mathbb{C}$, and $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$. Note that all the translation operators forms a group called Heisenberg-Weyl group. It's not difficult to find out that Heisenberg-Weyl group is isomorphic to $(\mathbb{C}, +)$, that is

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha + \beta) \quad (13)$$

Since $\hat{D}(\alpha)$ is an operator on a linear space, the Hilbert space for the oscillator, $\hat{D}(\alpha)$ can be regarded as a group representation of $(\mathbb{C}, +)$.

Now that \hat{a}^\dagger and \hat{a} can be described by \hat{x} and \hat{p} , we can write the translation operator as another form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x} - i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \quad (14)$$

It's not difficult to find that $\hat{D}(\alpha)$ is actually a Weyl translation operator $\hat{W}(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}})$. Based on Glauber's formula, we can express $\hat{D}(\alpha)$ as a separated form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x}\right) \exp\left(-i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \exp(-i\Im(\alpha)\Re(\alpha)) \quad (15)$$

Recall that all $\hat{D}(\alpha)$ forms a group representation of $(\mathbb{C}, +)$, it can also be considered as a group representation of $(\mathbb{R}^2, +)$, where the \mathbb{R}^2 represents the x and p coordinate, which is actually a point in the phase space. So a coherent state is actually a group action by a point $(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}})$ in the phase space, and the evolution of a coherent state is a group action by a trajectory in the phase space. Therefore, we may use points and trajectories to describe coherent states of an oscillator.

In the end of this section, I'd like to mention a further application of (7). Set $\hat{A} = \hat{N} = \hat{a}^\dagger \hat{a}$, and $\hat{B} = \hat{a}^\dagger$ or \hat{a} . Then we may obtain

$$\exp(\lambda \hat{N}) \hat{a} \exp(-\lambda \hat{N}) = \hat{a} e^{-\lambda} \quad (16)$$

$$\exp(\lambda \hat{N}) \hat{a}^\dagger \exp(-\lambda \hat{N}) = \hat{a}^\dagger e^{\lambda} \quad (17)$$

Still pay attention to the sign! This formula will give out the squeezed states.

3 Hamiltonian representation and time evolution

It's not difficult to show that the expansion of a coherent state $|\alpha\rangle$ by the eigenstates of the Hamiltonian eigenstates is

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (18)$$

Therefore the evolution of that state is $|\alpha(t)\rangle = e^{-i\hat{H}t/\hbar} |\alpha\rangle = |\alpha\rangle = e^{-|\alpha|^2/2} \sum \frac{\alpha^n}{\sqrt{n!}} e^{-iE_n t/\hbar} |n\rangle$. Since $E_n = (n + \frac{1}{2})\hbar\omega$, that factor can be written as $e^{-iE_n t/\hbar} = e^{-i\frac{1}{2}\omega t} e^{-in\omega t} = e^{-i\frac{1}{2}\omega t} (e^{-i\omega t})^n$, where the latter phase can be combined with α^n and presents $(\alpha e^{-i\omega t})^n$ and $|\alpha e^{-i\omega t}|^2 = |\alpha|^2$. Therefore,

$$|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t} \exp(-|\alpha e^{-i\omega t}|^2) \sum \frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}} |n\rangle \quad (19)$$

which suggests

$$|\alpha(t)\rangle = e^{-i\frac{1}{2}\omega t} |\alpha e^{-i\omega t}\rangle \quad (20)$$

Now that the translation operator is a group representation of $(\mathbb{R}^2, +)$, where \mathbb{R}^2 is the phase space, the evolution of $|\alpha(t)\rangle$ corresponds to $(\Im(\alpha \exp(-i\omega t))\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha \exp(-i\omega t))\sqrt{\frac{2}{m\hbar\omega}})$. So we can use a circle in \mathbb{C} to described the trajectory of $|\alpha(t)\rangle$.

4 Coherent states under coordinate representation

Now let's calculate the coherent state under coordinate representation, i.e. wavefunction of the coherent state $\langle x|\alpha\rangle$. Notice the expression (14) of $\hat{D}(\alpha)$ by \hat{x} and \hat{p} , we can consider the operation of $\hat{D}(\alpha)$ over coordinate eigenstate $|x\rangle$. Then $\langle x|\alpha\rangle$ can be expressed as $\langle x|\hat{D}(\alpha)|0\rangle$, where the translation operator operates on $\langle x|$. Note that $\hat{D}(\alpha)^\dagger = \hat{D}(-\alpha)$

$$\begin{aligned} \hat{D}(-\alpha) |x\rangle &= \exp\left(i\Im(-\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x}\right) \exp\left(-i\Re(-\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \exp(-i\Im(-\alpha)\Re(-\alpha)) |x\rangle \\ &= \exp\left(-i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\left(x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\right)\right) \exp(-i\Im(\alpha)\Re(\alpha)) |x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\rangle \\ &= \exp\left(-i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}x\right) \exp(i\Im(\alpha)\Re(\alpha)) |x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\rangle \end{aligned} \quad (21)$$

Therefore the wavefunction of a coherent state $|\alpha\rangle$ is

$$\begin{aligned}
\Psi_\alpha(x) &= \langle x | \hat{D}(\alpha) | 0 \rangle = \langle \hat{D}(-\alpha)x | 0 \rangle \\
&= \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}x\right) \exp(-i\Im(\alpha)\Re(\alpha)) \langle x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}} | 0 \rangle \\
&= \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}x\right) \exp(-i\Im(\alpha)\Re(\alpha)) \exp\left(\frac{m\omega}{2\hbar}\left(x - \Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\right)^2\right)
\end{aligned} \tag{22}$$