# Note on Wigner Function

Mengzhi Wu

March 18, 2020

This note aims to summarize the main idea of the Wigner function and the Weyl transformation. It is mainly based on DOI 10.1007/978-1-4419-8840-9 chap 5 and my own calculation.

Let's begin with the statiscal average value of an operator $\hat{A}$ in a quantum system described by a density operator $\rho$:

$$\begin{aligned} < A > = Tr(\hat{A}\hat{\rho}) &= \int dx \left\langle x\right| \hat{A}\hat{\rho} \left|x\right\rangle \\ &= \int dx dx' \left\langle x\right| \hat{A} \left|x'\right\rangle \left\langle x'\right| \hat{\rho} \left|x\right\rangle \end{aligned} \tag{1}$$

If we want to know the time evolution of $< \hat{A} >$, since $\hat{A}$, $|x\rangle$ and $|x'\rangle$ are time-independent, the unique term that influence the evolution of $< \hat{A} >$ is $\frac{d}{dt}\hat{\rho} = \frac{1}{i\hbar}[\hat{H}, \hat{\rho}]$, according to Liouville's equation. Therefore,

$$\begin{aligned} \frac{d}{dt} < A > &= \int dx dx' \left\langle x\right| \hat{A} \left|x'\right\rangle \left\langle x'\right| \frac{d}{dt}\hat{\rho} \left|x\right\rangle \\ &= \int dx dx' \left\langle x\right| \hat{A} \left|x'\right\rangle \left\langle x'\right| \frac{1}{i\hbar}[\hat{H}, \hat{\rho}] \left|x\right\rangle \\ &= \frac{1}{i\hbar} \int dx dx' \left\langle x\right| \hat{A} \left|x'\right\rangle \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial x'^2} \right) + (V(x) - V(x')) \right] \left\langle x'\right| \hat{\rho} \left|x\right\rangle \end{aligned} \tag{2}$$

According to these 2 equations above, we find out that $\langle x'| \hat{\rho} |x\rangle$ plays a central role to determine the behavior of an operator $\hat{A}$. The method to deal with $\langle x'| \hat{\rho} |x\rangle$ is double Fourier transformation which is called Weyl transformation.