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Some remarks on the extended Galilean transformation

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The Galilean transformation carries one inertial frame into another with a different velocity. The extended Galilean transformation carries an inertial frame into a rigid frame with uniform spatial, but arbitrary translational acceleration $\mathbf{x}' = \mathbf{x} + \xi(t)$. Besides being useful in discussing the equivalence principle, this transformation provides a physical interpretation for the theorem in nonrelativistic quantum mechanics, that we cannot coherently combine states of different mass. This restriction comes about because we cannot distinguish between coordinate and proper time in nonrelativistic physics.

The Galilean transformation carries one inertial frame into another one, moving with constant velocity, $\mathbf{v}$, with respect to the first. The law of transformation,

$$\mathbf{x}' = \mathbf{x} + \mathbf{v}t, \quad t' = t, \tag{1}$$

is consistent with the laws of three-dimensional Euclidean geometry. (It preserves the magnitude and direction of the vector $\mathbf{x}_1 - \mathbf{x}_2$ joining two points.) Galilean invariance refers to the fact that the laws of classical nonrelativistic mechanics are invariant with respect to such transformations. Galilean invariance was first discussed in quantum mechanics by Bargmann,[1] and the topic has found its way into the textbooks.[2]

The extension of Galilean invariance to cover the invariance of all the laws of physics with respect to transformations between inertial frames, is the principle of special relativity. In this case the relevant transformation is no longer given by Eq. (1), but by the (non-Euclidean) Lorentz transformation. The final extension, to arbitrary local coordinate systems, rather than rigid inertial frames, states that the laws of physics take the same form in all coordinate systems, and is called the principle of general covariance, or sometimes the principle of general relativity. In this most general form its physical content is much more restricted, as has been especially stressed by Fock.[3]

However, there is an intermediate transformation, sometimes called the extended Galilean transformation, which can shed light on both the transition to special and general relativity. Specifically, the transformation is from an inertial system to another extended, rigid reference system moving with an arbitrary translational, but nonrotating motion, namely,

$$\mathbf{x}' = \mathbf{x} + \xi(t), \; t' = t.$$

The origin of the $\mathbf{x}'$ system, $\mathbf{x}' = \mathbf{0}$, is located at $\mathbf{x} = -\xi$, and the entire system has the spatially uniform (but time-varying) acceleration $(-\ddot{\xi})$. This transformation has been noted many times,[4] but its capacity to provide insight into the connection between the various transformations mentioned above has been explored only to a very limited extent.

In this paper we are not going to give an extensive discussion of this transformation, but rather, we will merely point out one topic on which it casts a special light. This concerns a theorem, due to Bargmann, which states that in nonrelativistic quantum mechanics, one cannot have a coherent superposition of states of different mass. The standard proof of this theorem[2] uses a series of Galilean transformations, but gives no insight as to why the theorem is true. If one considers that a series of Galilean transformations is nothing but an extended Galilean transformation, then one sees that it is really the extended transformation that is relevant to the problem. And as soon as one discusses it in this light, it immediately becomes clear that the theorem is true because of the residual effects of the difference between proper and coordinate time that persist in the nonrelativistic limit.

Consider the Schrödinger equation,

$$mc^2\psi - (\hbar^2/2m)\nabla^2\psi + V\psi = i\hbar\partial\psi/\partial t. \tag{3}$$

We have added a term mc^2, corresponding to the rest mass, to the Hamiltonian. Nonrelativistically, this has no observable effect on the equation, since one can add any constant to the energy, without physically affecting the result. In fact the only effect it produces is to multiply all solutions to the equation by the phase factor exp $(-imc^2t/\hbar)$. Nonetheless, the term is there relativistically, and it will simplify the task of interpreting the results.

In this equation, the potential V is of the form $V(\mathbf{x} - \mathbf{x}_0)$, where $\mathbf{x}_0$ is a point of origin, either distinguished physically by the form of the force, or else arbitrarily chosen. Under the transformation (2), not only is the point $\mathbf{x}$ transformed, but so is the origin $\mathbf{x}_0$. Thus the force V becomes

$$V(\mathbf{x}' - \mathbf{x}_0') = V(\mathbf{x} - \mathbf{x}_0). \tag{4}$$

For example, if the potential is that of a harmonic oscillator, $V = \frac{1}{2}kx^2$, this expression implies that there is a force center at $x_0 = 0$. When the whole reference frame is accelerated, as in Eq. (2), the force center is then located at $x_0' = \xi(t)$, and Eq. (4) says that the attraction is to the force center, and not to the new origin, which is moving arbitrarily. So, under the transformation (2), we have

$$\nabla = \nabla', \quad \frac{\partial}{\partial t} = \dot{\xi}\cdot\nabla' + \frac{\partial}{\partial t'}. \tag{5}$$

Then the Schrödinger equation becomes

$$mc^2\psi - \left(\frac{\hbar^2}{2m}\right)\nabla'^2\psi - i\hbar\dot{\xi}\cdot\nabla'\psi + V\psi - \frac{i\hbar\partial\psi}{\partial t} = 0, \tag{6}$$

where we have replaced t' by t $(= t')$.

We now write

$$\psi(x,t) = u(x',t)e^{-imc^2t/\hbar}\, e^{if(x',t)} \equiv ue^{ik}. \tag{7}$$

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This gives for the derivatives of ψ,

$$\nabla'\psi = e^{ik}(\nabla' u + iu\nabla' f),$$
$$\nabla'^2\psi = e^{ik}(\nabla'^2 u + 2i\nabla' u\cdot\nabla' f + iu\nabla'^2 f - u\nabla' f\cdot\nabla' f),$$
$$\dot\psi = e^{ik}[\dot u + iu\dot f - i(mc^2/\hbar)u]. \quad (8)$$

If we choose for f,

$$f = \left(\frac{m}{\hbar}\right)\left(-\dot{\xi}\cdot x' + \frac{1}{2}\int^t \dot{\xi}^2 dt\right), \quad (9)$$

then the Schrödinger equation becomes

$$-(\hbar^2/2m)\,\nabla'^2 u + Vu - m\ddot{\xi}\cdot\mathbf{x}'\,u = i\hbar\dot u, \quad (10)$$

where

$$\psi = u(x',t)\exp\left\{\left(\frac{-im}{\hbar}\right)\left[\dot{\xi}\cdot\mathbf{x}' + c^2\int\left(1-\frac{\dot{\xi}^2}{2c^2}\right)dt\right]\right\}. \quad (11)$$

The first thing to note is that the new Schrödinger equation, Eq. (10), contains the extra potential term $(-m\ddot{\xi}\cdot\mathbf{x}')$, representing the effective gravitational potential, which enters via the equivalence principle. While this effect could be predicted classically, it has certain nonclassical features as well.[5] The phase factor proportional to $\dot{\xi}\cdot\mathbf{x}'$ is necessary in Eq. (11) in order to transform the momentum properly,

$$(\hbar/i)\nabla'\psi = e^{ik}[(\hbar/i)\nabla' u - m\dot{\xi}u], \quad (12)$$

which gives the extra factor $(-m\dot{\xi})$, which is the result of transforming to the new system. But the second phase factor in Eq. (11), which also corresponds to what happens classically to the Lagrangian, can now be seen to represent the proper time, to order $(v/c)^2$, in the accelerated frame,

$$\left(\frac{mc^2}{\hbar}\right)\int d\tau = \left(\frac{mc^2}{\hbar}\right)\int\left(1-\frac{\dot{\xi}^2}{c^2}\right)^{1/2}dt \approx \left(\frac{mc^2}{\hbar}\right)\int\left(\frac{1-\dot{\xi}^2}{2c^2}\right)dt. \quad (13)$$

Unlike the first term, which depends on the instantaneous value of $\dot{\xi}$, this term contains the accumulated effects of the entire, integrated history of the transformation to the accelerated frame.

We can now discuss the Bargmann theorem, on the inconsistency of combining different masses in nonrelativistic quantum theory. Consider the transformation to a system, given by Eq. (2), with the following properties:

$$\xi(0) = \dot{\xi}(0) = 0, \qquad \xi(T) = \dot{\xi}(T) = 0. \quad (14)$$

In other words, we are transforming to an accelerated system which takes off from the original system, travels around for a while, and then returns, with zero velocity, at time T. Now nonrelativistically, this transformation has no physical consequences. Once the coordinate system has returned to where it started, it is not possible to tell when the system wandered away, or in fact, whether it has. The actual physics that has transpired does not depend on the machinations of the coordinate system in which we choose to describe it.

Since $\xi = \mathbf{0}$ in the original, unaccelerated system, the wave function at times 0, and T, will be given by some function independent of ξ. Introducing $\varphi(\mathbf{x},t) = u(\mathbf{x},t)\exp(-imc^2t/\hbar)$, then

$$\psi(\mathbf{x},0) = u_0(\mathbf{x},0) = \varphi_0(\mathbf{x},0),$$
$$\psi(\mathbf{x},T) = u_0(\mathbf{x},T)e^{-imc^2T/\hbar} = \varphi_0(\mathbf{x},T). \quad (15)$$

In the accelerated system, since $\mathbf{x}'(0) = \mathbf{x}$, we have

$$\psi(\mathbf{x},0) = \varphi_0(\mathbf{x},0),$$
$$\psi(\mathbf{x},T) = u(\mathbf{x},\mathrm{T})\exp\left[\left(\frac{-imc^2}{\hbar}\right)\int_0^T\left(1-\frac{\dot{\xi}^2}{2c^2}\right)dt\right]$$
$$= \varphi(\mathbf{x},T)\exp\left(im\int\dot{\xi}^2\frac{dt}{2\hbar}\right)$$
$$\equiv \varphi(\mathbf{x},T)\,e^{im\eta(T)}. \quad (16)$$

Now, since at time T the accelerated system has returned and is indistinguishable from the original system, its wave function $u(\mathbf{x}',T) = u(\mathbf{x},T)$ can differ from that in the original system $u_0(\mathbf{x},T)$ only by a phase factor. And that phase factor is in fact precisely that given above by Eq. (16),

$$u(x,T) = u_0(x,T)e^{-im\eta(T)},$$
$$\varphi(x,T) = \varphi_0(x,T)e^{-im\eta(T)}. \quad (17)$$

So the "memory" of the trip is completely contained in the phase factor $\eta(T)$.

If now we have two coherent wave functions, both of different masses, m_1 and m_2, then

$$\varphi(\mathbf{x},0) = \varphi_0^{(1)}(\mathbf{x},0) + \varphi_0^{(2)}(\mathbf{x},0), \quad (18)$$

and

$$\varphi(\mathbf{x},T) = \varphi_0^{(1)}(\mathbf{x},T)e^{-im_1\eta(T)} + \varphi_0^{(2)}(\mathbf{x},T)e^{-im_2\eta(T)}$$
$$= e^{-im_1\eta}[\varphi_0^{(1)}(\mathbf{x},T) + \varphi_0^{(2)}(\mathbf{x},T)e^{-i\Delta m\eta}], \quad (19)$$

where

$$\Delta m = m_2 - m_1. \quad (20)$$

This is the wave function in the frame which has been accelerated and returned, and is to be compared with the wave function in the original unaccelerated frame

$$\psi(\mathbf{x},T) = \varphi_0^{(1)}(\mathbf{x},T) + \varphi_0^{(2)}(\mathbf{x},T). \quad (21)$$

We therefore have the situation that the wave functions calculated in these two coordinate systems, which are in fact the very same system for times $t \geq T$, differ in an observable way, since the two components $\varphi_0^{(1)}$ and $\varphi_0^{(2)}$ have different relative phases.

The last phase factor, $\exp(-i\Delta m\eta) = \exp(-i\Delta m\int\times\dot{\xi}^2 dt/2\hbar)$, depends explicitly on the integrated transformation. This means that if you made the thought experiment of calculating the experimental quantities from the accelerated system, then when the system ultimately returned to coincide with the original one, there would actually be an observable phase factor generated between the wave functions $u^{(1)}$ and $u^{(2)}$, which would cause interference effects. And yet there cannot be actual physical effects that depend on which coordinate system one chooses to calculate from. So this phase factor cannot possibly be real. But there is only one way to eliminate this phase factor, and that is to arbitrarily decree that $\Delta m = 0$, or that we cannot superpose these two different mass solutions. Such a restriction, imposed from outside the theory, is called a superselection rule.

However, in relativistic quantum mechanics, the theorem breaks down. For example, since mass and energy are equivalent, an atom at rest but in a superposition of excited states, actually is in a superposition of different mass states. Nonrelativistically, of course, energy changes do not affect the mass. An even clearer example is the superposition of states of the two K_0 mesons, K_S and K_L, which have slightly different masses.

We can also see the reason why the theorem breaks down relativistically. It is because the effect of going into an accelerated frame is no longer unobservable. If one has spent time in an accelerated frame and returned, there is a definite residual effect, and that is the twin paradox effect, namely, that the accelerated observer will find that his clocks read less elapsed time than the inertial observer's. And this will produce the same phase factor which nonrelativistically we said could not be real, because there, there is no difference between coordinate and proper time. We will analyze this in more detail when we discuss Kaempffer's proof.

And so, the Bargmann theorem is true because this v^2/c^2 effect manages to slip into the nonrelativistic formalism. The way this comes about, although the phase factor $\exp[i(\Delta m/2\hbar)\int \dot{\xi}^2 dt]$ seems to be independent of c, is that classically Δm is arbitrary, but relativistically, $E = mc^2$. If $\Delta m \sim m$, then there will be no effective coherence relativistically, either. But if the mass difference is caused by a small nonrelativistic binding energy correction to the relativistic Hamiltonian, we will have $\Delta m = \Delta E/c^2$. Then the phase factor becomes

$$i\left(\frac{\Delta m}{2\hbar}\right)\int \dot{\xi}^2 dt = i\left(\frac{\Delta E}{2\hbar c^2}\right) \times \int \dot{\xi}^2 dt = i\omega \int \left(\frac{\dot{\xi}^2}{2c^2}\right) dt, \quad (22)$$

where ω is the frequency corresponding to transitions between the states, $\omega = \Delta E/\hbar$. This result is down by $(\dot{\xi}^2/c^2)$ from $\exp(i\omega t)$, and we see that relativistically the term $\exp(i\Delta mc^2\Delta\tau/\hbar)$ is of order (v^4/c^4). But classically, where Δm is an independent parameter, it must be considered of order (v^2/c^2), which gives the Bargmann theorem.

We might also note that the same effects must be operating in the "standard proof," as presented by Kaempffer,[2] although there they are less apparent. Because he is restricted to constant velocity transformations, he generates the inconsistent phase difference between $u^{(1)}$ and $u^{(2)}$ in the following manner. The system is first translated by $\mathbf{a}$, then transformed (nonrelativistically) to velocity $\mathbf{V}$, after which it is translated back through $-\mathbf{a}$, and finally transformed back through $-\mathbf{V}$, to velocity zero. After this sequence of operations the phase difference between $u^{(1)}$ and $u^{(2)}$ is $\exp(i\Delta m\mathbf{V}\cdot\mathbf{a}/\hbar)$, which must be unobservable, as in our case, and so implies the rule $\Delta m = 0$.

Unfortunately, Kaempffer's discussion does not indicate the physical basis for this phase difference which is, as in our case, precisely due to the difference between coordinate and proper time. He points out that relativistically (for a scalar field)

$$\psi'(r') = \psi(r), \quad r' = Lr, \quad (23)$$

where L is an inhomogeneous Lorentz transformation, and that there is no phase factor in this equation, which of course is true. This equation indicates that the new value of the field, as measured at the transformed point, numerically equals the old value at the original point. This guarantees that all matrix elements and expectation values of scalar observables will remain unchanged, so that physical measurements will yield the same results, as interpreted by the two observers.

Equation (23) represents a numerical equality between two different functions of two different arguments.[6] Mathematically, if we ask "How does the function ψ' actually differ from ψ?," we want to know how $\psi'(r)$ differs from $\psi(r)$. This refers to a change in the function at a fixed point, and it can be represented by a unitary operator. From Eq. (23) we have

$$\psi'(r) = U_L\psi(r) = \psi(L^{-1}r). \quad (24)$$

Now the relativistic analog of the sequence of operations that correspond to the four nonrelativistic operations of Kaempffer are (restricting ourselves for simplicity to the special case where the translation and "boost" are both parallel to the x axis): (i) a translation by a, so $x_1 = x + a$, $t_1 = t$; (ii) a "boost" by velocity V (a Lorentz transformation to a system moving with velocity V), so $x_2 = \gamma(x_1 - Vt_1)$, $t_2 = \gamma(t_1 - Vx_1/c^2)$; (iii) a translation back by a/γ, so $x_3 = x_2 - a/\gamma$, $t_3 = t_2$; and (iv), the inverse boost, by $-V$, so that $x_4 = \gamma(x_3 + Vt_3) = x$, and $t_4 = \gamma(t_3 + Vx_3/c^2) = t - Va/c^2$. The two important points to note here are that the translation back, in (iii), is not the same as the first one, in (i), because there is a Lorentz contraction in the moving frame, and much more importantly, the time t_4 is not equal to t, because in the moving system the translation back is made instantaneously everywhere, but the two ends of the vector $\mathbf{a}/\gamma$ are not translated back simultaneously in the original rest system, and this leaves the residue $\Delta t = t_4 - t = -\mathbf{V}\cdot\mathbf{a}/c^2$. (In this form it is valid even if $\mathbf{a}$ and $\mathbf{V}$ are not parallel.) This implies that the series of four transformations do not form a closed cycle relativistically, and this is the physical reason why $V^{-1}T^{-1}VT \neq 1$, (where V is equal to the boost, T is equal to the translation). This time difference between the frame that has been accelerated and returned, and the original system, is the spacelike analog to the twin paradox.

Now because $(r^\mu)' \equiv r_4^\mu \neq r^\mu$, it follows from Eq. (24) that the transformed field, $\xi'(r^\mu)$, i.e., the field as seen by the observer who has been accelerated and returned, is not the same as the field seen by the inertial observer at the same point $\psi(r^\mu)$. In fact the operator U of Eq. (24) is just given by

$$\psi'(r^\mu) = U\psi(r^\mu) = \psi(L^{-1}r^\mu) = \psi(\mathbf{x}, t - \Delta t) = e^{imc^2\Delta t/\hbar}\psi(r^\mu), \quad (25)$$

or

$$\psi(t) = e^{imV\cdot a/\hbar}\psi'(t). \quad (26)$$

This equation is to be interpreted as follows: when the accelerated system returns to the same spatial point $\mathbf{x}$, as the original, it will be true that all physical quantities measured within it must give the same values as in the original system $\psi'(r') = \psi(r)$; however this system has used up less time than the original system, and so an observer in the original system must multiply the accelerated wave functions by a phase factor, at any given time, in order to agree with his own wave functions $\psi(r) = U^{-1}\psi'(r)$.

If there are two different coherent masses, this procedure shows that there will be a phase difference $\exp(i\Delta m\mathbf{V}\cdot\mathbf{a}/\hbar)$ between them, from Eq. (26), when the accelerated wave function is compared with the inertial one. And this is just the phase difference given by Kaempffer. So one sees that in the relativistic case the exact same phase difference shows up as in the nonrelativistic one. Only in the relativistic case it has a physical interpretation, exactly the same one as in our original derivation. It is the residue of an acceleration around a nonclosed path, which leaves a time shift between the systems when the accelerated system is brought back, because its proper time differs from the coordinate time in the original inertial system.

The problem is that in the nonrelativistic limit this proves rather embarrassing, because in this limit

$$\tau = t + \mathcal{O}(v^2/c^2) \approx t, \tag{27}$$

and there is no distinction between coordinate and proper time. Thus, nonrelativistically, the accelerated observer comes back with a phase-shifted wave function, but once he has returned he is the same physical system as the inertial one (**x** is the same, and t is the same), and there is no possible way to interpret the phase difference. Therefore one arbitrarily rules it out, by legislating that $\Delta m = 0$, nonrelativistically.

In other words, relativistically, one inertial observer can ascertain a physical difference between two other nearby observers, one of whom has always remained nearby, and the other of whom has sneaked off somewhere and returned. But nonrelativistically, the wandering observer leaves no trace on returning, and if his trip causes a phase shift, there will be a physical ambiguity in interpreting the theory, which is why the phase shift must be removed in this limit.

The final comment on the theorem is that if one considers a two-particle system, rather than just a single particle, so that

$$\begin{aligned} H\psi &= [m_1c^2 + m_2c^2 - (\hbar^2/2m_1)\nabla_1^2 \\ &\quad - (\hbar^2/2m_2)\nabla_2^2 + V(x_1 - x_2)]\,\psi, \\ &= [Mc^2 - (\hbar^2/2M)\nabla_R^2 \\ &\quad - (\hbar^2/2\mu)\nabla_r^2 + V(r)]\psi = i\hbar\dot{\psi} \end{aligned} \tag{28}$$

(where $\mu = m_1m_2/M$, $M = m_1 + m_2$, R is the center of mass coordinate, and r is the relative coordinate), then the transformation (2) yields

$$\begin{aligned} \psi &= u_R(R',t)u_r(r',t) \\ &\times \exp\left\{\left(\frac{-iM}{\hbar}\right)\left[(\dot{\xi}\cdot R' + c^2\int\left(1 - \frac{\dot{\xi}^2}{2c^2}\right)dt\right]\right\}, \end{aligned} \tag{29}$$

where the Hamiltonian for **r**′ is the same as that for **r**,

$$H_{r'} = -(\hbar^2/2\mu)\nabla_{r'}^2 + V(r'), \tag{30}$$

and the gravitational force appears in the Hamiltonian for **R**′,

$$H_{R'} = Mc^2 - (\hbar^2/2M)\nabla_{R'}^2 - M\,\ddot{\xi}\cdot R'. \tag{31}$$

It follows from the form of the phase factor in Eq. (19), that if one has two sets of nonrelativistic particle superpositions, then this is permissible provided they both have the same value of M. (We believe that this theorem, which is generalizable to n particles, has not been stated before.) Research supported in part by a grant from Professional Staff Congress-Board of Higher Education Research Award Program of the City University of New York.

[1] V. Bargmann, Ann. Math. **59**, 1 (1954).

[2] The subject is at least mentioned in most modern texts. There is an extended discussion in an Appendix of F. A. Kaempffer, *Concepts in Quantum Mechanics* (Academic, New York, 1965).

[3] V. Fock, *The Theory of Space, Time, and Gravitation*, 2nd ed. (Pergamon, London, 1964).

[4] Three recent papers referring to this transformation in different contexts are G. Rosen, Am. J. Phys. **40**, 683 (1972); D. M. Greenberger, J. Math. Phys. **15**, 406 (1974); R. Colella, A. W. Overhauser, and S. A. Werner, Phys. Rev. Lett. **34**, 1472 (1975); see also the appendix to D. M. Greenberger and A. W. Overhauser, Rev. Mod. Phys. (to be published).

[5] The classical and quantum mechanical implications of the equivalence principle are compared in D. Greenberger, Ann. Phys. (NY) **47**, 116 (1968).

[6] This is clearly explained in E. Wigner, *Group Theory* (Academic, New York, 1959), p. 106. Transformations of relativistic fields are explained in N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields* (Interscience, New York, 1959), for classical fields in Chap. 2.