Note on coherent state

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This note aims to summarize essential properties of the coherent state. I plan to cover these points:

- Basic propertities for creation and annilation operators.
- Coherent states under Hamiltonian representation.
- Coherent states under coordinate representation.
- Wigner functions for coherent states and time evolution.
- classical propertities of coherent states.

1 Properties for operators

In this section, I'd like to briefly review some basic properties of some operators. The Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, and the commute relations is $[\hat{x}, \hat{p}] = i\hbar$(Be careful of the sign!). Then the creation and annilation operator can be defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}),\ \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}) \tag{1}$$

And the commute relations is $[\hat{a}^\dagger, \hat{a}] = -1$(Be careful of the sign!). Inversely, $\hat{x}$, $\hat{p}$ and $\hat{H}$ can be described as

$$\hat{x} =,\ \hat{p} =,\ \hat{H} = (\hat{a}^+\hat{a} + \frac{1}{2})\hbar\omega \tag{2}$$

Suppose the eigenstate of the Hamiltonian is $|n\rangle$ with eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$ whose wavefunction is $\Psi_n(x) =$. Then

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,,\ \hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{3}$$

Then let's review some useful formulas. The first one is Glauber's formula, based on Baker-Hausdorff formula. This formula claims that if operators $\hat{A}$ and $\hat{B}$ satisfies $[\hat{A}, [\hat{A}, \hat{B}]] = [\hat{B}, [\hat{A}, \hat{B}]] = 0$, then

$$\exp(\hat{A} + \hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp\left(-\frac{1}{2}[\hat{A}, \hat{B}]\right) \tag{4}$$

The 2nd formula is

$$\exp(\lambda\hat{A})\hat{B}\exp(-\lambda\hat{A}) = \hat{B} + \lambda[\hat{A}, \hat{B}] + \frac{\lambda^2}{2!}[\hat{A}, [\hat{A}, \hat{B}]] + \frac{\lambda^3}{3!}[\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \ldots \tag{5}$$

Specially, if $[\hat{A}, \hat{B}] = const =: C$, then

$$\exp(\lambda\hat{A})\hat{B}\exp(-\lambda\hat{A}) = \hat{B} + \lambda C \tag{6}$$

which means a translation of operator $\hat{B}$. More specially, we set $\hat{A} = -\alpha\hat{a}^\dagger + \alpha^*\hat{a}$, $\hat{B} = \hat{a}$ or $\hat{B} = \hat{a}^\dagger$, then we may obtain

$$\exp(-\alpha\hat{a}^+ + \alpha^*\hat{a})\hat{a}\exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \hat{a} + \alpha \tag{7}$$

$$\exp(-\alpha\hat{a}^\dagger + \alpha^*\hat{a})\hat{a}^\dagger\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \hat{a}^\dagger + \alpha^* \tag{8}$$

Equation (7) is very important. Denote $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, then (7) presents

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha \tag{9}$$

Based on (9), we may prove that all the eigenstates of $\hat{a}$ are

$$\hat{a}(\hat{D}(\alpha)\left|0\right\rangle) = \alpha(\hat{D}(\alpha)\left|0\right\rangle) \tag{10}$$

Therefore, we may define the coherent states as $\left|\alpha\right\rangle = \hat{D}(\alpha)\left|0\right\rangle$ with ANY complex number eigenvalue $\alpha \in \mathbb{C}$

In the end of this section, I'd like to mention a further appliacation of (5). Set $\hat{A} = \hat{N} = \hat{a}^\dagger\hat{a}$, and $\hat{B} = \hat{a}^\dagger$ or $\hat{a}$. Then we may obtain

$$\exp(\lambda\hat{N})\hat{a}\exp(-\lambda\hat{N}) = \hat{a}e^{-\lambda} \tag{11}$$

$$\exp(\lambda\hat{N})\hat{a}^\dagger\exp(-\lambda\hat{N}) = \hat{a}^\dagger e^{\lambda} \tag{12}$$

Still pay attention to the sign! This formula will give out the squeezed states.

2 Coherent states under Hamiltonian representation

3 Coherent states under coordinate representation