Journal of Mathematical Physics

An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field

Cite as: Journal of Mathematical Physics **10**, 1458 (1969); https://doi.org/10.1063/1.1664991
Submitted: 08 July 1968 . Published Online: 04 November 2003

H. R. Lewis, and W. B. Riesenfeld

zero. However, given a conjectured exact quantization rule, i.e., given $S(x)z'^2$, it is not a simple matter to determine precisely what are the higher-order integrals, since this requires knowing $S(x)$ and z'^2 separately. In fact, given a choice for $S(x)z'^2$, we may obtain z' by solving a second-order nonlinear differential equation obtained by multiplying Eq. (6) by z'^2. We have not yet been successful in finding the appropriate z' for Cases IV, V, and VI, and thus cannot explicitly demonstrate that the higher-order integrals are zero in these cases also.

V. DISCUSSION

Using the method of coordinate transformations, we have derived a generalization of the usual WKB quantization condition through the third-order integral. Since a necessary condition for the first-order integral to give an exact quantization condition is the vanishing of the higher-order integrals, this method leads to the possibility of finding exact quantization conditions in those cases where the usual first-order integral in the original coordinate system is not sufficient. Furthermore, even in those cases where the transformed higher-order integrals are not zero, the technique is still useful in providing a means of reducing the size of these terms and hence increasing the accuracy of eigenvalues computed from the first-order integral alone.

ACKNOWLEDGMENTS

We are indebted to Professor G. Wannier for his stimulating correspondence. We would also like to express our gratitude to C. Rosenzweig and J. Bekenstein for several helpful discussions.

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 10, NUMBER 8 AUGUST 1969

An Exact Quantum Theory of the Time-Dependent Harmonic Oscillator and of a Charged Particle in a Time-Dependent Electromagnetic Field*

H. R. Lewis, Jr., and W. B. Riesenfeld
Los Alamos Scientific Laboratory, University of California, Los Alamos, New Mexico

(Received 8 July 1968)

The theory of explicitly time-dependent invariants is developed for quantum systems whose Hamiltonians are explicitly time dependent. The central feature of the discussion is the derivation of a simple relation between eigenstates of such an invariant and solutions of the Schrödinger equation. As a specific well-posed application of the general theory, the case of a general Hamiltonian which settles into constant operators in the sufficiently remote past and future is treated and, in particular, the transition amplitude connecting any initial state in the remote past to any final state in the remote future is calculated in terms of eigenstates of the invariant. Two special physical systems are treated in detail: an arbitrarily time-dependent harmonic oscillator and a charged particle moving in the classical, axially symmetric electromagnetic field consisting of an arbitrarily time-dependent, uniform magnetic field, the associated induced electric field, and the electric field due to an arbitrarily time-dependent uniform charge distribution. A class of explicitly time-dependent invariants is derived for both of these systems, and the eigenvalues and eigenstates of the invariants are calculated explicitly by operator methods. The explicit connection between these eigenstates and solutions of the Schrödinger equation is also calculated. The results for the oscillator are used to obtain explicit formulas for the transition amplitude. The usual sudden and adiabatic approximations are deduced as limiting cases of the exact formulas.

I. INTRODUCTION

The use of explicitly time-dependent invariants in applications of quantum theory has received little attention, if any. Presumably, the reason for this lack of attention has been the dearth of examples in which the use of such quantities was both possible and fruitful. Recently, a class of exact invariants for time-dependent harmonic oscillators, both classical and quantum, was reported.[1] The simplicity of the rules for constructing these invariants and the instructive relation of the invariants to the asymptotic expansion of adiabatic invariant theory have stimulated an interest in using the invariants for solving some explicit quantum-mechanical problems. We discuss

* Work performed under the auspices of the U.S. Atomic Energy Commission.

[1] H. R. Lewis, Jr., J. Math. Phys. **9**, 1976 (1968); also, H. R. Lewis, Jr., Phys. Rev. Letters **18**, 510, 636 (1967).

two systems in detail: the time-dependent harmonic oscillator and a charged particle in a particular type of time-dependent, classical electromagnetic field.

In Sec. II we consider the theory of explicitly time-dependent invariants for a general quantum system whose Hamiltonian operator $H(t)$ is explicitly time-dependent. Of course, such a system is not closed, in the sense that some external influence, which need not be specified, may change the parameters of the system, alter its total energy or angular momentum, etc. The semiclassical theory of radiation provides a well-known example. In that case the quantum system is taken to be an atom or molecule which undergoes radiative transitions, and the explicitly time-dependent term in the Hamiltonian operator is the interaction with the classical radiation field. The usual approximation techniques for treating such a system are time-dependent perturbation theory (in which the time-dependent term is considered small), the adiabatic approximation (in which the time scale of variation of the time-dependent term is long compared to all of the characteristic periods of the system), and the "sudden" approximation (in which the external changes are fast compared to the shortest characteristic period). The results of the adiabatic and "sudden" approximations will be deduced as limiting cases of rigorous results that are presented in this article for the time-dependent harmonic oscillator.

The central feature of our discussion of general systems is the derivation of the relation between eigenstates of an explicitly time-dependent invariant and solutions of the Schrödinger equation. A time-dependent phase transformation can be found for each eigenstate of an invariant such that the eigenstate becomes a solution of the Schrödinger equation, and the phase is determined by solving a simple, first-order differential equation. Later in the article, for the two special systems that we discuss in detail, we derive explicit formulas for the eigenstates and eigenvalues of the invariants and for the phases. Also, in these examples, we evaluate all physically relevant matrix elements elegantly by operator techniques.

To provide a specific well-posed application of these ideas, we consider a Hamiltonian which settles into constant operators in the sufficiently remote past and future, and we assume that each of these two limiting operators has a known complete set of eigenstates and eigenvalues. The time dependence of $H(t)$ for intermediate times is to be at least piecewise continuous, but otherwise arbitrary, and we calculate the transition amplitude connecting any initial state in the remote past to any final state in the remote future.

The first special physical system to which we apply the general results, in Sec. III, is that of a time-dependent harmonic oscillator, that is, a system whose Hamiltonian has the form of the Hamiltonian of a simple harmonic oscillator, but for which the frequency parameter is allowed to vary with time.[2] To begin with, we derive a class of exact invariants for this system by means of a method different than that used previously.[1] Then we calculate the eigenvalues and eigenstates of these invariants, and we also calculate the appropriate time-dependent phase factors that make the eigenstates solutions of the Schrödinger equation. Finally, as in Sec. II, we specialize to the case that the Hamiltonian is a constant operator in the remote past and future and calculate explicit formulas for the transition amplitude between arbitrary states at these times. Using these exact formulas, we discuss the adiabatic and "sudden" approximations and deduce the usual formulas for those limiting cases.

In Sec. IV we consider a charged particle in the classical, axially symmetric electromagnetic field consisting of an arbitrarily time-dependent, uniform magnetic field, the associated induced electric field, and the electric field due to an arbitrarily time-dependent, uniform charge distribution. The dynamical variables of this system are simply related to those of the time-dependent harmonic oscillator by a noncanonical transformation. We use this noncanonical transformation to derive a class of invariants for the particle system from the invariants for the oscillator. These invariants for the particle system are not Hermitian. However, it turns out to be possible to derive from them a class of Hermitian invariants that are formally identical to the Hamiltonian for a particle in a uniform, time-independent magnetic field. Using operator techniques, we derive the eigenstates and eigenvalues of this class of Hermitian invariants, and we find the phases for which the eigenstates are solutions of the Schrödinger equation. The results are a generalization of the solution[3,4,5] for a particle in a uniform, time-independent magnetic field.

---

[2] The two special systems that we consider in Secs. III and IV have been treated along different lines by M. Kolsrud: (a) "Exact Quantum Dynamical Solutions for Oscillator-Like Systems," Institute for Theoretical Physics, University of Oslo (Norway), Institute Report No. 28 (1965); (b) Kgl. Norske Videnskab. Selskabs Forh. **31**, No. 5 (1958); (c) Phys. Rev. **104**, 1186 (1956).

[3] L. Landau, Z. Physik **64**, 629 (1930).

[4] R. B. Dingle, Proc. Roy. Soc. (London) **A211**, 500 (1952).

[5] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics: Nonrelativistic Theory* (Addison-Wesley Publ. Co., Inc., Reading, Mass., 1965), 2nd. ed., p. 426. There is an error in this derivation: the wavefunction is assumed proportional to $e^{im\phi}$, but the subsequent formulas are derived for $e^{-im\phi}$.

II. EXPLICITLY TIME-DEPENDENT INVARIANTS AND THEIR RELATION TO SOLUTIONS OF THE SCHRÖDINGER EQUATION

We consider a system whose Hamiltonian operator $H(t)$ is an explicit function of time, and we assume the existence of another explicitly time-dependent nontrivial Hermitian operator $I(t)$, which is an invariant. That is, $I(t)$ satisfies the conditions

$$\frac{dI}{dt} \equiv \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0 \tag{1}$$

and

$$I^\dagger = I. \tag{2}$$

The equation determining the time-dependent Schrödinger state vector $|\ \rangle$ is

$$i\hbar \frac{\partial}{\partial t} |\ \rangle = H(t)\,|\ \rangle. \tag{3}$$

By operating with the left-hand side of Eq. (1) on $|\ \rangle$ and using Eq. (3), we obtain the relation

$$i\hbar \frac{\partial}{\partial t}(I\,|\ \rangle) = H(I\,|\ \rangle), \tag{4}$$

which implies that the action of the invariant operator on a Schrödinger state vector produces another solution of the Schrödinger equation. This result is valid for any invariant, even if the latter involves the operation of time differentiation. If the invariant does *not* involve time differentiation, then we are able to derive a simple and explicit rule for choosing the phases of the eigenstates of $I(t)$ such that these states themselves satisfy the Schrödinger equation. In what follows, we assume that $I(t)$ does not involve time differentiation. The invariants with which we treat the time-dependent harmonic oscillator, described in Sec. III, and the motion of a charged particle, described in Sec. IV, satisfy this requirement.

We assume that the invariant operator is one of a complete set of commuting observables, so that there is a complete set of eigenstates of I. We denote the eigenvalues of I by λ, and the orthonormal eigenstates associated with a given λ by $|\lambda, \kappa\rangle$, where the label κ represents all of the quantum numbers other than λ that are necessary for specifying the eigenstates:

$$I(t)\,|\lambda, \kappa\rangle = \lambda\,|\lambda, \kappa\rangle, \tag{5a}$$

$$\langle \lambda', \kappa' \mid \lambda, \kappa\rangle = \delta_{\lambda'\lambda}\delta_{\kappa'\kappa}. \tag{5b}$$

The eigenvalues λ are real by virtue of Eq. (2). They are also time independent, as we can deduce in the following simple way. By differentiating Eq. (5a) with respect to time, we obtain

$$\frac{\partial I}{\partial t}|\lambda, \kappa\rangle + I\frac{\partial}{\partial t}|\lambda, \kappa\rangle = \frac{\partial \lambda}{\partial t}|\lambda, \kappa\rangle + \lambda\frac{\partial}{\partial t}|\lambda, \kappa\rangle. \tag{6}$$

We also operate with the left-hand side of Eq. (1) on $|\lambda, \kappa\rangle$ to obtain

$$i\hbar\frac{\partial I}{\partial t}|\lambda, \kappa\rangle + IH\,|\lambda, \kappa\rangle - \lambda H\,|\lambda, \kappa\rangle = 0. \tag{7}$$

The scalar product of Eq. (7) with a state $|\lambda', \kappa'\rangle$ is

$$i\hbar\,\langle \lambda', \kappa'|\frac{\partial I}{\partial t}|\lambda, \kappa\rangle + (\lambda' - \lambda)\langle \lambda', \kappa'|\,H\,|\lambda, \kappa\rangle = 0, \tag{8}$$

implying

$$\langle \lambda, \kappa'|\frac{\partial I}{\partial t}|\lambda, \kappa\rangle = 0. \tag{9}$$

Now taking the scalar product of Eq. (6) with $|\lambda, \kappa\rangle$, we obtain

$$\frac{\partial \lambda}{\partial t} = \langle \lambda, \kappa|\frac{\partial I}{\partial t}|\lambda, \kappa\rangle = 0. \tag{10}$$

Since the eigenvalues are time independent, it is clear that the eigenstates must be time dependent.

In order to investigate the connection between eigenstates of I and solutions of the Schrödinger equation, we first write the equation of motion of $|\lambda, \kappa\rangle$, starting from Eq. (6) and using Eq. (10):

$$(\lambda - I)\frac{\partial}{\partial t}|\lambda, \kappa\rangle = \frac{\partial I}{\partial t}|\lambda, \kappa\rangle. \tag{11}$$

By taking the scalar product with $|\lambda', \kappa'\rangle$ and using Eq. (8) to eliminate

$$\langle \lambda', \kappa'|\frac{\partial I}{\partial t}|\lambda, \kappa\rangle,$$

we get

$$i\hbar(\lambda - \lambda')\langle \lambda', \kappa'|\frac{\partial}{\partial t}|\lambda, \kappa\rangle = (\lambda - \lambda')\langle \lambda', \kappa'|\,H\,|\lambda, \kappa\rangle. \tag{12}$$

From this, for $\lambda' \neq \lambda$, we infer

$$i\hbar\,\langle \lambda', \kappa'|\frac{\partial}{\partial t}|\lambda, \kappa\rangle = \langle \lambda', \kappa'|\,H\,|\lambda, \kappa\rangle. \tag{13}$$

Equation (12) does not imply

$$i\hbar\,\langle \lambda, \kappa'|\frac{\partial}{\partial t}|\lambda, \kappa\rangle = \langle \lambda, \kappa'|\,H\,|\lambda, \kappa\rangle.$$

If Eq. (13) held for $\lambda' = \lambda$ as well as for $\lambda' \neq \lambda$, then we would immediately deduce that $|\lambda, \kappa\rangle$ satisfies the Schrödinger equation, i.e., is a special solution for $|\ \rangle$.

The phase of $|\lambda, \kappa\rangle$ has not been fixed by our definitions. We assume that some definite phase has been chosen, but we are still free to multiply $|\lambda, \kappa\rangle$ by an arbitrarily time-dependent phase factor. That is, we can define a new set of eigenvectors of $I(t)$ related to our initial set by a time-dependent gauge transformation

$$|\lambda, \kappa\rangle_\alpha = e^{i\alpha_{\lambda\kappa}(t)}|\lambda, \kappa\rangle, \tag{14}$$

where the $\alpha_{\lambda\kappa}(t)$ are arbitrary real functions of time. Because $I(t)$ is assumed not to contain time-derivative operators, the $|\lambda, \kappa\rangle_\alpha$ are orthonormal eigenstates of $I(t)$ just as are the $|\lambda, \kappa\rangle$. For $\lambda' \neq \lambda$, Eq. (13) also holds for matrix elements taken with respect to the new eigenstates. Each of the new eigenstates will satisfy the Schrödinger equation if we choose the phases $\alpha_{\lambda\kappa}(t)$ such that Eq. (13) holds for $\lambda' = \lambda$. This requirement is equivalent to the following first-order differential equation for the $\alpha_{\lambda\kappa}(t)$:

$$\hbar\delta_{\kappa\kappa'}\frac{d\alpha_{\lambda\kappa}}{dt} = \langle\lambda, \kappa'|\, i\hbar\frac{\partial}{\partial t} - H\,|\lambda, \kappa\rangle.$$

In order to satisfy this equation, the states $|\lambda, \kappa\rangle$ must be chosen such that the right-hand side vanishes for $\kappa' \neq \kappa$. This diagonalization is always possible because the operator $i\hbar(\partial/\partial t) - H$ is Hermitian. Once the states have been so chosen, the phases $\alpha_{\lambda\kappa}(t)$ are chosen to satisfy the simple equation

$$\hbar\frac{d\alpha_{\lambda\kappa}}{dt} = \langle\lambda, \kappa|\, i\hbar\frac{\partial}{\partial t} - H\,|\lambda, \kappa\rangle. \tag{15}$$

Since each of the new set of eigenstates of $I(t)$, $|\lambda, \kappa\rangle_\alpha$, satisfies the Schrödinger equation, the general solution is

$$|t\rangle = \sum_{\lambda,\kappa} c_{\lambda\kappa}e^{i\alpha_{\lambda\kappa}(t)}|\lambda, \kappa; t\rangle, \tag{16}$$

where the $c_{\lambda\kappa}$ are time-independent coefficients. All of the state vectors with which we have dealt so far are time dependent, and we have revised the notation in Eq. (16) slightly by indicating the dependences on time explicitly. Now the Schrödinger state vector is denoted by $|t\rangle$ and the eigenstates of the invariant by $|\lambda, \kappa; t\rangle$.

We now assume that in the remote past the Hamiltonian $H(t)$ settled into a constant operator $H(-\infty)$ having a complete, orthonormal set of time-independent eigenstates $|n; i\rangle$, n being a label for all relevant quantum numbers including the energy eigenvalue and i standing for "initial state." Similarly, we assume that the Hamiltonian settles into a constant operator $H(\infty)$ in the distant future and that it possesses time-independent eigenstates $|m;f\rangle$, m labeling the quantum numbers and f standing for "final state." The explicit time variation of $H(t)$ for intermediate times is arbitrary except for piecewise continuity; in particular, we do not exclude the possibility of variations rapid enough to render an analysis in terms of quasi-stationary states of $H(t)$ impossible. Our aim is to calculate the transition amplitude $T(n \to m)$ connecting an initial state $|n; i\rangle$ to a final state $|m;f\rangle$. Thus we consider the case in which the Schrödinger state vector $|-\infty\rangle$ in the remote past corresponds to an eigenstate $|n; i\rangle$ and, after tracing the exact time evolution of $|t\rangle$ into the distant future, we compute the overlap of $|\infty\rangle$ with the desired final state $|m;f\rangle$ to obtain the exact transition amplitude. The superposition coefficients of Eq. (16) for this problem are given by

$$c_{\lambda\kappa} = e^{-i\alpha_{\lambda\kappa}(-\infty)}\langle\lambda, \kappa; -\infty \mid n; i\rangle, \tag{17}$$

from which we obtain

$$|t\rangle = \sum_{\lambda,\kappa}\exp\{i[\alpha_{\lambda\kappa}(t) - \alpha_{\lambda\kappa}(-\infty)]\}\,|\lambda, \kappa; t\rangle \times \langle\lambda, \kappa; -\infty \mid n; i\rangle. \tag{18}$$

The transition amplitude is therefore given by

$$\begin{aligned} T(n \to m) &= \langle m;f \mid \infty\rangle \\ &= \sum_{\lambda,\kappa}\exp\{i[\alpha_{\lambda\kappa}(\infty) - \alpha_{\lambda\kappa}(-\infty)]\} \\ &\quad \times \langle m;f \mid \lambda, \kappa; \infty\rangle\langle\lambda, \kappa; -\infty \mid n; i\rangle. \end{aligned} \tag{19a}$$

Our discussion of the properties of $I(t)$ applies equally well to any operator that is an invariant corresponding to a given $H(t)$. In general, for a system of f degrees of freedom, there is an infinite family of such invariants, the members of which are functions of a set of f independent invariants. Two invariants $I_1(t)$ and $I_2(t)$ will, in general, have different eigenstates, different time derivatives, and different commutators with the Hamiltonian. In Secs. III and IV we give examples of this by constructing families of invariants for our two special systems in detail. Of course, we must obtain the same physical results no matter what invariant we use and, therefore, the choice of which particular invariant to use may be made on the basis of mathematical convenience. In order to illustrate explicitly that the physical results do not depend on our choice of invariant, we give a simple and direct proof that a transition amplitude like that of Eq. (19a) is indeed independent of our choice of invariant.

Suppose that we have two complete orthonormal sets of states, $|v; t\rangle$ and $|w; t\rangle$, all of which satisfy the time-dependent Schrödinger equation; and suppose that the states $|v; t\rangle$ are eigenstates of one set of operators, whose eigenvalues are labeled by v, and that the states $|w; t\rangle$ are eigenstates of a different set of operators, whose eigenvalues are labeled by w. The transition amplitude $T(n \to m)$ can be expressed as

$$T(n \to m) = \sum_v \langle m;f \mid v; \infty\rangle\langle v; -\infty \mid n; i\rangle \tag{19b}$$

or as

$$T(n \to m) = \sum_w \langle m; f \mid w; \infty \rangle \langle w; -\infty \mid n; i \rangle. \quad \text{(19c)}$$

We want to show directly that these two expressions are the same. The completeness of the states $|w; t\rangle$ requires

$$|v; t\rangle = \sum_w |w; t\rangle \langle w; t \mid v; t\rangle. \quad (20)$$

Operating on this equation with $(i\hbar(\partial/\partial t) - H)$, and using the facts that all of the states satisfy the Schrödinger equation and that the states $|w; t\rangle$ are orthogonal, we obtain

$$\frac{\partial}{\partial t}\langle w; t \mid v; t\rangle = 0. \quad (21)$$

Thus the quantity $\langle w; t \mid v; t\rangle$ is independent of time. We now use the completeness of the states $|v; t\rangle$ and $|w; t\rangle$, Eq. (21), and the orthonormality of the states $|w; t\rangle$ to rewrite Eq. (19b) as

$$\begin{aligned} T(n \to m) &= \sum_v \sum_w \sum_{w'} \langle m; f \mid w; \infty\rangle \langle w; \infty \mid v; \infty\rangle \\ &\quad \times \langle v; -\infty \mid w'; -\infty\rangle \langle w'; -\infty \mid n; i\rangle \\ &= \sum_v \sum_w \sum_{w'} \langle m; f \mid w; \infty\rangle \langle w; -\infty \mid v; -\infty\rangle \\ &\quad \times \langle v; -\infty \mid w'; -\infty\rangle \langle w'; -\infty \mid n; i\rangle \\ &= \sum_w \langle m; f \mid w; \infty\rangle \langle w; -\infty \mid n; i\rangle. \end{aligned} \quad (22)$$

Thus, Eqs. (19b) and (19c) are the same, as asserted.

We have used the $t \to \pm\infty$ limits in the above expressions as if the limits exist. In fact, however, the factors entering Eq. (19a) are generally undamped oscillating quantities for $t \to \pm\infty$ [for example, see Eq. (62), which gives the form of $\alpha_{\lambda\kappa}(t)$ for a time-dependent harmonic oscillator]. Nevertheless, this circumstance generates no difficulties in the calculation of transition probabilities in this limit. If t_1 and t_2 are finite times in the sufficiently remote past and future, respectively, then it is easily shown with the argument leading to Eq. (21) that the dependence of the transition amplitude of Eq. (19) on t_1 and t_2 is only $\exp[i(E_n t_1 - E_m t_2)/\hbar]$, where E_n and E_m are the initial and final state energies, respectively. The transition probability does not involve this phase factor, and therefore we shall continue to use the limits $t \to \pm\infty$.

Suppose for simplicity that the eigenstates of I are nondegenerate, so that the eigenvalue of I is the only quantum number required for describing the system. When this is so, as it is in our discussion of the time-dependent harmonic oscillator, then it is particularly convenient to choose an invariant having the property that it becomes time-independent as $t \to -\infty$ so that the commutator $[I(-\infty), H(-\infty)]$ vanishes. Then the normalized eigenvectors of $H(-\infty)$ and $I(-\infty)$ are identical to within arbitrary constant phase factors. Consequently, we may choose the initial state $|n; i\rangle$ simply to be a given eigenstate of $I(-\infty)$, say $|\lambda_n; -\infty\rangle$. Equation (19a) then reduces to

$$T(n \to m) = \exp\{i[\alpha_n(\infty) - \alpha_n(-\infty)]\}\langle m; f \mid \lambda_n; \infty\rangle, \quad (23)$$

and the transition probability is given by

$$\begin{aligned} P_{nm} &= |T(n \to m)|^2 \\ &= |\langle m; f \mid \lambda_n; \infty\rangle|^2. \end{aligned} \quad (24)$$

As $t \to \infty$, the invariant operator $I(t)$ in general remains time dependent and does not commute with the Hamiltonian. Therefore, the state $|\lambda_n; \infty\rangle$ in Eq. (24) is a superposition of eigenstates of $H(\infty)$; this is another expression of the fact that energy is not conserved in our system.

From the structure of Eq. (19a), it is apparent that we may express the transition amplitude as a matrix element of an S matrix by writing [keeping in mind the comment following Eq. (22)]

$$\begin{aligned} S &= \sum_{\lambda,\kappa} e^{i\alpha_{\lambda\kappa}(\infty)} |\lambda, \kappa; \infty\rangle\langle\lambda, \kappa; -\infty| \, e^{-i\alpha_{\lambda\kappa}(-\infty)}, \\ T(n \to m) &= \langle m; f| \, S \, |n; i\rangle. \end{aligned} \quad (25)$$

It is easily verified that this operator is unitary:

$$S^\dagger S = S S^\dagger = 1. \quad (26)$$

In the special case that the Hamiltonian operators in the remote past and distant future are identical, $H(-\infty) = H(\infty)$, so that the initial and final states are the same set, we may define an elastic scattering operator R in the standard fashion:

$$S = 1 + 2\pi i R. \quad (27)$$

The operator R describes the nondiagonal transitions just as S does, but subtracts a noninteracting part from the diagonal amplitudes so that $\langle n| R |n\rangle$ represents a "forward reaction amplitude" from the state $|n\rangle$ to the same state. The unitarity of the S matrix implies

$$\sum_m |\langle m| R |n\rangle|^2 = \frac{1}{\pi} \operatorname{Im}(\langle n| R |n\rangle), \quad (28)$$

which is a statement of the optical theorem: the total reaction probability is proportional to the imaginary part of the forward reaction amplitude.

III. APPLICATION TO TIME-DEPENDENT HARMONIC OSCILLATORS

A. A Family of Invariant Operators for a Time-Dependent Harmonic Oscillator

A time-dependent, one-dimensional harmonic oscillator is a system whose Hamiltonian operator is of the form

$$H(t) = (1/2M)[p^2 + \Omega^2(t)q^2], \tag{29}$$

where q is a canonical coordinate, p is its conjugate momentum, $\Omega(t)$ is an arbitrary, piecewise-continuous function of time, and M is a real, positive mass parameter. The variables q and p satisfy the canonical commutation relation

$$[q, p] = i\hbar, \tag{30}$$

and the canonical equations of motion are

$$\dot{q} = \frac{1}{i\hbar}[q, H] = \frac{1}{M}p,$$
$$\dot{p} = \frac{1}{i\hbar}[p, H] = -\frac{1}{M}\Omega^2(t)q, \tag{31}$$

where the dots denote time derivative operators. To obtain the simple harmonic oscillator in the limit that $\Omega(t)$ is time independent, we would have to require that Ω be a real function. However, our discussion is equally valid if Ω is imaginary; all that is necessary is that Ω^2 be real, either positive or negative. Therefore we allow Ω^2 to be a positive or negative real function. In order for the usual adiabatic approximation to be applicable, M and $\Omega(t)$ must satisfy the criterion

$$\frac{1}{M} \gg \frac{1}{\Omega^2}\left|\frac{d\Omega}{dt}\right| \tag{32}$$

for all t. However, except where we discuss the adiabatic approximation specifically, we do not impose such a restriction on M and $\Omega(t)$.

For such oscillator systems, a convenient representation has been derived for the class of invariants which are homogeneous, quadratic expressions in the dynamical variables p and q.[1] This representation was constructed as a result of an examination of the classical trajectories, and the invariants were normalized in such a way as to reduce to the usual adiabatic invariant (energy divided by frequency) in the limit that the inequality (32) is satisfied. Here we present a purely quantum-mechanical derivation of this representation of the quadratic invariants.

We assume the existence of a Hermitian invariant of the homogeneous, quadratic form

$$I(t) = \tfrac{1}{2}[\alpha(t)q^2 + \beta(t)p^2 + \gamma(t)\{q, p\}_+], \tag{33}$$

where α, β, and γ are real functions of time, the multiplicative numerical factor has been chosen for convenience, and we have used the conventional anticommutator notation $\{q, p\}_+ \equiv qp + pq$. The time derivative of $I(t)$ is given by

$$\begin{aligned}\dot{I} &= \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H]\\ &= \frac{1}{2}\Big[\Big(\dot{\alpha} - \frac{2\Omega^2}{M}\gamma\Big)q^2 + \Big(\dot{\beta} + \frac{2}{M}\gamma\Big)p^2\\ &\quad + \Big(\dot{\gamma} + \frac{1}{M}\alpha - \frac{\Omega^2}{M}\beta\Big)\{q, p\}_+\Big].\end{aligned} \tag{34}$$

In order to satisfy Eq. (1), we demand

$$\begin{aligned}\dot{\alpha} &= \frac{2\Omega^2}{M}\gamma,\\ \dot{\beta} &= -\frac{2}{M}\gamma,\\ \dot{\gamma} &= -\frac{1}{M}\alpha + \frac{\Omega^2}{M}\beta.\end{aligned} \tag{35}$$

It is convenient to introduce another function $\sigma(t)$, defined by

$$\beta(t) = \sigma^2(t), \tag{36}$$

where $\sigma^2(t)$ is a real function of time. The second of Eqs. (35) then becomes

$$\gamma = -M\sigma\dot{\sigma}, \tag{37}$$

and the third equation yields

$$\alpha = M^2(\dot{\sigma}^2 + \sigma\ddot{\sigma}) + \Omega^2\sigma^2. \tag{38}$$

The first of Eqs. (35) imposes a constraint on $\sigma(t)$ which may be expressed in the form

$$\sigma\frac{d}{dt}(M^2\ddot{\sigma} + \Omega^2\sigma) + 3\dot{\sigma}(M^2\ddot{\sigma} + \Omega^2\sigma) = 0. \tag{39}$$

A first integral of Eq. (39) may immediately be written in the form

$$M^2\ddot{\sigma} + \Omega^2\sigma = c/\sigma^3, \tag{40}$$

where c is an arbitrary real constant of integration. Then Eq. (38) becomes

$$\alpha = M^2\dot{\sigma}^2 + c/\sigma^2. \tag{41}$$

The invariant may therefore be expressed in the form

$$I = \tfrac{1}{2}[(c/\sigma^2)q^2 + (\sigma p - M\dot{\sigma}q)^2], \tag{42}$$

with Eq. (40) as a subsidiary condition. The arbitrariness implied by the presence of the constant c is illusory, as may be verified by making the scale transformation

$$\sigma(t) = c^{\frac{1}{4}}\rho(t), \tag{43}$$

$\rho(t)$ being a new auxiliary function of time. After discarding a constant multiplicative factor $c^{\frac{1}{2}}$, we may write Eq. (42) in the form

$$I = \tfrac{1}{2}[(1/\rho^2)q^2 + (\rho p - M\dot{\rho}q)^2], \quad (44)$$

and the auxiliary condition given by Eq. (40) becomes

$$M^2\ddot{\rho} + \Omega^2(t)\rho - 1/\rho^3 = 0. \quad (45)$$

In order to make $I(t)$ Hermitian, we choose only the real solutions of this equation.

Any particular solution of Eq. (45) may be used to construct an invariant operator of the form given by Eq. (44). We thus have obtained a family of operators which is in one-to-one correspondence with the family of solutions of the nonlinear differential equation (45). Later in this section we shall consider the special case of a system for which Ω is a constant function in the remote past and in the remote future, and we shall calculate transition amplitudes connecting states at these two times. In obtaining these transition amplitudes, we shall need the general solution of Eq. (45) for constant Ω, which we now derive.[6] The problem is not quite so trivial as it appears at first glance because the obvious time-independent, real solution,

$$\rho = \pm\, |\Omega|^{-\frac{1}{2}}, \quad (46)$$

is by no means the most general solution. According to the discussion of Sec. II, we are free to choose the solution given by Eq. (46) for $t \to -\infty$ if we like. But then we shall find that the time dependence of $\Omega(t)$ produces a more general solution for $\rho(t)$ as $t \to \infty$. The above choice for ρ as $t \to -\infty$ leads to the condition $[I(-\infty), H(-\infty)] = 0$, which, according to the discussion preceding Eq. (23), is a particularly convenient choice.

To find the general solution of Eq. (45) for constant Ω, we note that $\dot{\rho}$ is an integrating factor of this equation and immediately obtain the first integral

$$M^2\dot{\rho}^2 + \Omega^2\rho^2 + 1/\rho^2 = 2\,|\Omega| \cosh\delta, \quad (47)$$

where δ is an arbitrary real constant. The right-hand side of Eq. (47) is the integration constant, which, it turns out, must be greater than or equal to $2\,|\Omega|$ if ρ is to be real. The integration constant has been written in this way so that ρ will be real for all values of the real parameter δ. Solution of Eq. (47) is straightforward and leads to the result

$$\rho(t) = \gamma_1\,|\Omega|^{-\frac{1}{2}}\,[\cosh\delta + \gamma_2 \sinh\delta \sin((2\Omega/M)t + \varphi)]^{\frac{1}{2}}, \quad (48)$$

where γ_1 and γ_2 can each independently assume the values ± 1, and φ is a real phase constant. The special solution of Eq. (46) corresponds to the case $\delta = 0$. Whenever $\Omega(t)$ becomes constant, the solution for $\rho(t)$ is necessarily of the form given by Eq. (48). Therefore, the transition amplitudes that we shall calculate are completely determined by the parameters in this expression that are appropriate to the limit $t \to \infty$, no matter how complicated or violent the time dependence of $\Omega(t)$ for earlier times. We shall express the transition amplitudes in terms of these parameters in Part C of this Section.

B. Eigenstates and Eigenvalues of $I(t)$ and the Phases

The eigenstates and eigenvalues of the invariant operator $I(t)$ may be found by an operator technique that is completely analogous to the method introduced by Dirac[7] for diagonalizing the Hamiltonian of a constant-frequency harmonic oscillator. Thus we define time-dependent canonical lowering and raising operators a and $a^\dagger$ by the relations

$$a = (2\hbar)^{-\frac{1}{2}}[(1/\rho)q + i(\rho p - M\dot{\rho}q)],$$
$$a^\dagger = (2\hbar)^{-\frac{1}{2}}[(1/\rho)q - i(\rho p - M\dot{\rho}q)]. \quad (49)$$

These operators satisfy the canonical commutation rule

$$[a, a^\dagger] = 1, \quad (50)$$

so that the operator $a^\dagger a$ is a number operator with nonnegative integer eigenvalues. The invariant operator given by Eq. (44) can be written in terms of a and $a^\dagger$ as

$$I = \hbar(a^\dagger a + \tfrac{1}{2}), \quad (51)$$

from which it follows[8] that the normalized eigenstates $|\lambda\rangle$ of I are the same as the normalized eigenstates $|s\rangle$ of $a^\dagger a$:

$$a^\dagger a\,|s\rangle = s\,|s\rangle, \quad s = 0, 1, 2, \cdots. \quad (52)$$

We specify the relative phases of these normalized eigenstates $|s\rangle$ by requiring the standard lowering and raising relations:

$$a\,|s\rangle = s^{\frac{1}{2}}\,|s-1\rangle,$$
$$a^\dagger\,|s\rangle = (s+1)^{\frac{1}{2}}\,|s+1\rangle. \quad (53)$$

The eigenvalue spectrum of I is given by

$$\lambda_s = (s + \tfrac{1}{2})\hbar, \quad s = 0, 1, 2, \cdots. \quad (54)$$

To effect the transformation of Eqs. (14) and (15) we need to calculate the diagonal matrix elements of the operators H and $\partial/\partial t$. The former are obtained by

[6] A method for expressing the general solution of Eq. (45) for arbitrary $\Omega(t)$ in terms of independent solutions of the equations for a classical oscillator has been described in Ref. 1.

[7] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Clarendon Press, Oxford, 1947), 3rd ed. Also see A. Messiah, *Quantum Mechanics* (Interscience Publishers, New York, 1962), Vol. I.

[8] For the present we are omitting the time label t in our notation for these eigenstates. When it is required for clarity, we shall replace $|s\rangle$ by $|s; t\rangle$ to denote an eigenstate at time t.

using Eqs. (49) to express H in terms of a and $a^\dagger$ and then applying Eqs. (53):

$$\langle s|\, H\, |s\rangle = \frac{\hbar}{4M}\left(M^2\dot\rho^2 + \Omega^2\rho^2 + \frac{1}{\rho^2}\right)\langle s|\, \{a, a^\dagger\}_+\, |s\rangle$$
$$= \frac{1}{2M}\left(M^2\dot\rho^2 + \Omega^2\rho^2 + \frac{1}{\rho^2}\right)(s + \tfrac{1}{2})\hbar. \quad (55)$$

The Hamiltonian, of course, also has nondiagonal matrix elements since the representation defined by Eqs. (52) and (53) does not diagonalize this operator.

To evaluate the diagonal matrix elements of $\partial/\partial t$, we take the partial derivative of the second of Eqs. (53) with respect to time, and then take the appropriate scalar product, obtaining

$$\langle s|\, \frac{\partial}{\partial t}\, |s\rangle = \langle s-1|\, \frac{\partial}{\partial t}\, |s-1\rangle + s^{-\frac{1}{2}}\langle s|\, \frac{\partial a^\dagger}{\partial t}\, |s-1\rangle. \quad (56)$$

The expression for $\partial a^\dagger/\partial t$ in terms of a and $a^\dagger$ is

$$\frac{\partial a^\dagger}{\partial t} = \frac{1}{2}\left\{\left[-\frac{2\dot\rho}{\rho} + iM(\rho\ddot\rho - \dot\rho^2)\right]a + iM(\rho\ddot\rho - \dot\rho^2)a^\dagger\right\}, \quad (57)$$

so that Eq. (56) becomes

$$\langle s|\, \frac{\partial}{\partial t}\, |s\rangle = \langle s-1|\, \frac{\partial}{\partial t}\, |s-1\rangle + i\frac{M}{2}(\rho\ddot\rho - \dot\rho^2)$$
$$= \langle 0|\, \frac{\partial}{\partial t}\, |0\rangle + i\frac{s}{2}M(\rho\ddot\rho - \dot\rho^2). \quad (58)$$

It is clear that the anti-Hermitian character of $\partial/\partial t$ requires all diagonal matrix elements of $\partial/\partial t$ to be purely imaginary. However, no further information about $\langle 0|\, \partial/\partial t\, |0\rangle$ can be determined from Eq. (58); indeed, the choice of relative phases given by Eqs. (53) leaves the phase of a given state, say the state $|0\rangle$, undetermined. This time-dependent state can, in general, have a time-dependent phase factor, the choice of which is arbitrary. A convenient choice, which we now adopt, is one which makes $\langle 0|\, \partial/\partial t\, |0\rangle$ vanish in the limit that ρ becomes a constant, and which makes a "zero-point" contribution to Eq. (58):

$$\langle 0|\, \frac{\partial}{\partial t}\, |0\rangle = i\frac{M}{4}(\rho\ddot\rho - \dot\rho^2). \quad (59)$$

With this convention we can now write the general diagonal matrix element of $\partial/\partial t$ as

$$\langle s|\, \frac{\partial}{\partial t}\, |s\rangle = i\frac{M}{2}(\rho\ddot\rho - \dot\rho^2)(s + \tfrac{1}{2}). \quad (60)$$

The phases required for carrying out the transformation of Eq. (14) may be calculated by substituting Eqs. (55) and (60) into Eq. (15) to give

$$\frac{d\alpha_s}{dt} = -\frac{1}{2M}\left[M^2(\rho\ddot\rho - \dot\rho^2) + M^2\dot\rho^2 + \Omega^2\rho^2 + \frac{1}{\rho^2}\right](s + \tfrac{1}{2})$$
$$= -\frac{1}{M}(s + \tfrac{1}{2})\frac{1}{\rho^2}, \quad (61)$$

where we have made use of the subsidiary condition of Eq. (45). Thus the phase functions may be written in the form

$$\alpha_s(t) = -\frac{1}{M}(s + \tfrac{1}{2})\int^t dt'\, \frac{1}{\rho^2(t')}. \quad (62)$$

It is interesting to note that these phases are closely related to a quantity that occurs in the analysis of classical time-dependent harmonic oscillators.[1] In the classical case, the invariant I can be chosen as a generalized canonical momentum and the corresponding cyclic canonical coordinate is then equal to $-\alpha_s/(s + \frac{1}{2})$.

The off-diagonal matrix elements of H and $\partial/\partial t$, though not required for the purpose of the present discussion, are straightforward to compute and are given for completeness. The expression of H in terms of the raising and lowering operators immediately yields

$$\langle s'|\, H\, |s\rangle = (\hbar/4)\{[M(\dot\rho^2 - \rho\ddot\rho) - 2i(\dot\rho/\rho)][s(s-1)]^{\frac{1}{2}}\delta_{s'+2,s} + [M(\dot\rho^2 - \rho\ddot\rho) + 2i(\dot\rho/\rho)][(s+1)(s+2)]^{\frac{1}{2}}\delta_{s',s+2}\},$$
$$s' \neq s; \quad (63a)$$

and from Eq. (13) we obtain

$$\langle s'|\, \frac{\partial}{\partial t}\, |s\rangle = \frac{1}{i\hbar}\langle s'|\, H\, |s\rangle, \quad s' \neq s. \quad (63b)$$

C. Calculation of the Transition Probability

We assume that the Hamiltonian operator in the remote past, $H(-\infty)$, corresponds to a harmonic oscillator whose frequency parameter Ω_1 is constant and positive, and we choose the convenient form of $I(t)$ that leads to Eq. (23) by taking

$$\rho(-\infty) = \Omega_1^{-\frac{1}{2}}, \quad (64a)$$

from which follows

$$I(-\infty) = (M/\Omega_1)H(-\infty), \quad (64b)$$

so that

$$[I(-\infty), H(-\infty)] = 0. \quad (64c)$$

In the distant future the Hamiltonian is to settle into a harmonic oscillator Hamiltonian $H(\infty)$, with a constant and positive frequency parameter Ω_2. The form of the invariant at any time is given by Eq. (44) or Eq. (51), and, as $t \to \infty$, the auxiliary function ρ necessarily satisfies Eq. (48) with $|\Omega|$ replaced by Ω_2. The detailed dynamics of the time variation determines the parameters δ and φ. We assume these parameters to be known, and we shall express the transition amplitude of Eq. (23) in terms of them. In general, specific numerical values for δ and φ can only be obtained by integrating Eq. (45) numerically.

Let us first suppose that the initial state $|n; i\rangle$ is the ground state of $H(-\infty)$. From Eq. (64b) it is clear that this state, apart from an arbitrary phase factor, is the same as the "ground state" $|0; -\infty\rangle$ of $I(-\infty)$. The Schrödinger state vector of the system at all later times is $e^{i\alpha_0(t)}\,|0; t\rangle$, where $\alpha_0(t)$ is given by Eq. (62), and this state vector is at all times an eigenvector (corresponding to the "ground state") of $I(t) = \hbar[a^\dagger(t)a(t) + \frac{1}{2}]$. We seek the transition amplitude to an eigenstate $|m;f\rangle$ of the final Hamiltonian, and, according to Eq. (23), this transition amplitude is given by

$$T(0 \to m) = \exp\{i[\alpha_0(\infty) - \alpha_0(-\infty)]\}\langle m;f\,|\,0;\infty\rangle. \tag{23'}$$

The final Hamiltonian may be written in the form

$$H(\infty) = (\hbar\Omega_2/M)(b^\dagger b + \tfrac{1}{2}), \tag{65}$$

where

$$b = \left(\frac{\Omega_2}{2\hbar}\right)^{\frac{1}{2}}\left(q + i\frac{1}{\Omega_2}p\right),$$
$$b^\dagger = \left(\frac{\Omega_2}{2\hbar}\right)^{\frac{1}{2}}\left(q - i\frac{1}{\Omega_2}p\right), \quad [b, b^\dagger] = 1. \tag{66}$$

The lowering and raising operators of the invariant, $a(\infty)$ and $a^\dagger(\infty)$, may be expressed in terms of the lowering and raising operators of the final Hamiltonian, b and $b^\dagger$, by use of Eqs. (49):

$$a(\infty) = \eta(\infty)b + \zeta(\infty)b^\dagger,$$
$$a^\dagger(\infty) = \zeta^*(\infty)b + \eta^*(\infty)b^\dagger, \tag{67a}$$

where

$$\eta(t) = (4\Omega_2)^{-\frac{1}{2}}((1/\rho) + \Omega_2\rho - iM\dot\rho),$$
$$\zeta(t) = (4\Omega_2)^{-\frac{1}{2}}((1/\rho) - \Omega_2\rho - iM\dot\rho), \tag{67b}$$

and ρ is given by the final state form of Eq. (48). The condition that the transformation of Eq. (67) satisfy Eq. (50),

$$|\eta|^2 - |\zeta|^2 = 1,$$

clearly is satisfied.

To calculate the matrix element $\langle m;f\,|\,0;\infty\rangle$, we expand the state $|0;\infty\rangle$ in terms of the eigenstates $|n;f\rangle$ of $H(\infty)$:

$$|0;\infty\rangle = \sum_{n=0}^{\infty}|n;f\rangle\langle n;f\,|\,0;\infty\rangle. \tag{68}$$

By applying the lowering operator $a(\infty)$ to the left-hand side of this equation, we obtain zero, while the right-hand side may be transformed by use of Eq. (67). After regrouping terms, the resulting equation may be written as

$$0 = |0;f\rangle\,\eta\langle 1;f\,|\,0;\infty\rangle + \sum_{n=1}^{\infty}|n;f\rangle[\eta(n+1)^{\frac{1}{2}}\langle n+1;f\,|\,0;\infty\rangle + \zeta n^{\frac{1}{2}}\langle n-1;f\,|\,0;\infty\rangle]. \tag{69}$$

In this equation and in what follows, by η and ζ we mean $\eta(\infty)$ and $\zeta(\infty)$. The orthonormality of the eigenvectors $|n;f\rangle$ then yields the recursion relations

$$\langle 1;f\,|\,0;\infty\rangle = 0,$$
$$\langle n+1;f\,|\,0;\infty\rangle = -\frac{\zeta}{\eta}\left(\frac{n}{n+1}\right)^{\frac{1}{2}}\langle n-1;f\,|\,0;\infty\rangle, \tag{70}$$

which have the solution

$$\langle 2r+1;f\,|\,0;\infty\rangle = 0,$$
$$\langle 2r;f\,|\,0;\infty\rangle = \left(-\frac{\zeta}{\eta}\right)^r\frac{[(2r)!]^{\frac{1}{2}}}{2^r r!}\langle 0;f\,|\,0;\infty\rangle, \tag{71}$$

where r is an integer. The first of Eqs. (71) expresses the usual parity selection rule: states of negative parity have vanishing overlap with a state of positive parity. By combining Eqs. (71) with the expansion of Eq. (68) and imposing the normalization requirement on the state $|0;\infty\rangle$, we obtain

$$|\langle 0;f\,|\,0;\infty\rangle|^2 = \left[\sum_{r=0}^{\infty}\left|\frac{\zeta}{\eta}\right|^{2r}\frac{(2r)!}{2^{2r}(r!)^2}\right]^{-1} = \left(1 - \left|\frac{\zeta}{\eta}\right|^2\right)^{\frac{1}{2}} = \frac{1}{|\eta|}, \tag{72}$$

where we have used the summation formula

$$\sum_{r=0}^{\infty}\frac{(2r)!}{2^{2r}(r!)^2}x^r = (1-x)^{-\frac{1}{2}}.$$

Therefore,

$$|T(0\to 0)| = |\langle 0;f\,|\,0;\infty\rangle| = |\eta|^{-\frac{1}{2}} = (4\Omega_2)^{\frac{1}{4}}\left[\left(\frac{1}{\rho} + \Omega_2\rho\right)^2 + M^2\dot\rho^2\right]^{-\frac{1}{4}} = \left(\frac{2}{1+\cosh\delta}\right)^{\frac{1}{4}}, \tag{73}$$

where we have used Eqs. (23′), (47), and (67b). This corresponds to a transition probability

$$P_{00} = \left(\frac{2}{1+\cosh\delta}\right)^{\frac{1}{2}}. \tag{74}$$

Similarly, Eqs. (71) lead to the more general transition probability

$$P_{0m} = \frac{m!}{2^m[(m/2)!]^2}\left(\frac{\cosh\delta - 1}{\cosh\delta + 1}\right)^{m/2}\left(\frac{2}{\cosh\delta + 1}\right)^{\frac{1}{2}}, \quad m \text{ even},$$
$$= 0, \quad m \text{ odd}. \tag{75}$$

By repeatedly applying the raising operator $a^\dagger(\infty)$ to Eq. (68), we may express any eigenstate $|s;\infty\rangle$ of I in terms of the states $|m;f\rangle$, and thus we may compute the general amplitude $T(s \to m)$. The resulting transition probability is, according to Eq. (24), given by

$$\begin{aligned}P_{sm} &= |\langle m;f\,|\,s;\infty\rangle|^2 \\ &= \frac{1}{s!}\left|\sum_{n=0}^{\infty}\langle m;f|\,(\zeta^* b + \eta^* b^\dagger)^s\,|n;f\rangle\langle n;f\,|\,0;\infty\rangle\right|^2 \\ &= \frac{1}{s!}\left(\frac{2}{\cosh\delta + 1}\right)^{\frac{1}{2}} \\ &\quad\times\left|\sum_{r=0}^{\infty}\left(-\frac{\zeta}{2\eta}\right)^r\frac{[(2r)!]^{\frac{1}{2}}}{r!}\right. \\ &\quad\left.\times\,\langle m;f|\,(\zeta^* b + \eta^* b^\dagger)^s\,|2r;f\rangle\right|^2.\end{aligned} \tag{76}$$

These probabilities, of course, obey the sum rule $\sum_m P_{sm} = 1$. This is easily verified by direct summation of Eq. (76):

$$\begin{aligned}\sum_{m=0}^{\infty} P_{sm} &= \frac{1}{s!}\left(\frac{2}{\cosh\delta + 1}\right)^{\frac{1}{2}}\sum_{r=0}^{\infty}\sum_{r'=0}^{\infty}\left(-\frac{\zeta^*}{2\eta^*}\right)^{r'}\left(-\frac{\zeta}{2\eta}\right)^r \\ &\quad\times\frac{[(2r)!\,(2r')!]^{\frac{1}{2}}}{r!\,r'!} \\ &\quad\times\langle 2r';f|\,(\eta b + \zeta b^\dagger)^s(\zeta^* b + \eta^* b^\dagger)^s\,|2r;f\rangle.\end{aligned}$$

On the other hand, to within a phase factor Eq. (68) leads to

$$\begin{aligned}|s;\infty\rangle &= (s!)^{-\frac{1}{2}}\left(\frac{2}{\cosh\delta + 1}\right)^{\frac{1}{4}} \\ &\quad\times\sum_{r=0}^{\infty}\left(-\frac{\zeta}{2\eta}\right)^r\frac{[(2r)!]^{\frac{1}{2}}}{r!}(\zeta^* b + \eta^* b^\dagger)^s\,|2r;f\rangle,\end{aligned}$$

so that we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} P_{sm} &= \langle s;\infty\,|\,s;\infty\rangle \\ &= 1.\end{aligned}$$

Of course, P_{sm} vanishes unless the initial and final states have the same parity.

The case $\delta = 0$ corresponds to a situation in which $\lim_{t\to\infty}\rho(t)$ is equal to the constant $\Omega_2^{-\frac{1}{2}}$, so that $\eta = 1$, and $\zeta = 0$. Eq. (76) then yields

$$P_{sm}\,(\delta = 0) = \delta_{sm}, \tag{77}$$

which also is the result given by the adiabatic approximation. We conclude that the rigorous transition probability coincides with the adiabatic transition probability whenever the continuous time evolution of the auxiliary function $\rho(t)$ leads to a final form $\rho(\infty) = \Omega_2^{-\frac{1}{2}}$, starting from the initial form $\rho(-\infty) = \Omega_1^{-\frac{1}{2}}$. It is clear that only a restricted class of $\Omega(t)$ functions will produce such a result, but the members of this class need by no means satisfy any adiabaticity requirement. The time evolution of such systems, while leading to Eq. (77), will in general be nonadiabatic.

D. The Adiabatic and Sudden Approximations

In the adiabatic limit,

$$\frac{M}{\Omega^2(t)}\frac{d\Omega}{dt} \equiv \theta(t), \quad |\theta(t)| \ll 1, \tag{78}$$

it has been shown for the classical theory[1] that the leading term in the expansion in powers of θ of the invariant of Eq. (44) is the usual adiabatic invariant, energy divided by frequency. In the quantum theory this statement becomes the assertion that the quantum number remains constant, implying Eq. (77). This equation, of course, holds independently of any particular representation for $\rho(t)$; however, the choice $\rho(-\infty) = \Omega_1^{-\frac{1}{2}}$ is especially convenient, and with this assumption the adiabatic condition implies $\delta = 0$. For the sake of completeness we furnish an outline of a simple proof of the adiabatic theorem.

We let the frequency parameter $\Omega(t)$ evolve continuously from an initial value Ω_1 in the remote past to a final value Ω_2 in the distant future, such that Eq. (78) remains valid for all times $-\infty < t < \infty$. Since, according to Eq. (78), the frequency cannot change sign, we take Ω_1 and Ω_2 both to be positive constants. Eq. (45) may be formally integrated to yield

$$\begin{aligned}M^2\dot\rho^2 + \Omega^2(t)\rho^2 + \frac{1}{\rho^2} &= 2\Omega_1 + 2\int_{-\infty}^{t} dt'\rho^2(t')\Omega(t')\frac{d\Omega}{dt'} \\ &= 2\Omega_1 + \frac{2}{M}\int_{-\infty}^{t} dt'\rho^2(t')\Omega^3(t')\theta(t'),\end{aligned} \tag{79}$$

provided that $\rho(-\infty) = \Omega_1^{-\frac{1}{2}}$. We make the ansatz

$$\begin{aligned}\rho(t) &= \Omega^{-\frac{1}{2}}(t)[1 + \nu(t)], \\ \nu(-\infty) &= 0.\end{aligned} \tag{80}$$

In the limit that θ vanishes so that $\Omega(t)$ becomes constant, the function $\nu(t)$ also must vanish. Hence, if θ is an infinitesimal quantity, so is ν. The essence of the adiabatic theorem is that ν is a higher-order infinitesimal than θ. By differentiating Eq. (80) with respect to time, we obtain

$$\dot{\rho} = -(1/2M)\Omega^{\frac{1}{2}}\theta(1+\nu) + \Omega^{-\frac{1}{2}}\dot{\nu}, \tag{81}$$

which implies that $\dot{\nu}$ is an infinitesimal of the same order as θ. To see that ν is of higher order than θ, we substitute the ansatz of Eq. (80) into Eq. (79), retaining terms only up to first order in θ:

$$\Omega\left[(1+\nu)^2 + \frac{1}{(1+\nu)^2}\right] = 2\Omega_1 + 2\int_{-\infty}^{t} dt' \frac{d\Omega}{dt'}[1+\nu(t')]^2. \tag{82}$$

Since the left-hand side of this integral equation has no first-order contributions in ν, it follows that the solution to first order in θ consistent with the condition $\Omega(-\infty) = \Omega_1$ is

$$\nu(t) = 0, \tag{83}$$

establishing the theorem. Equation (80) then asserts $\rho(\infty) = \Omega_2^{-\frac{1}{2}}$, leading to $\delta = 0$ and the result of Eq. (77).

The sudden approximation is used to describe the time evolution of systems in which the Hamiltonian operator experiences a rapid change during a time interval which is short compared to the characteristic periods of the system. Such a change might be represented by a jump discontinuity in the function $\Omega(t)$ at a specified instant of time. The sudden theory asserts that the state vector remains constant across such a discontinuity; the transition amplitudes bridging the discontinuity are therefore given by simple overlap integrals of eigenstates of the Hamiltonian just before and after the discontinuity. This result, which is rigorous for instantaneous discontinuities, is the basis of the sudden approximation for "fast," but not instantaneous, changes in the Hamiltonian operator.

The rigorous transition amplitude of Eq. (19) contains all the features of the sudden theory for $\Omega(t)$ whose time histories involve jump discontinuities. We may easily derive the result of the sudden theory for the harmonic oscillator on the basis of a simple example. The continuity of $I(t)$ as an explicit function of time is guaranteed by requiring ρ and $\dot{\rho}$ to be continuous. The possibility of jump discontinuities in the piecewise-continuous Hamiltonian operator $H(t)$ is retained by making $\ddot{\rho}$ discontinuous.

Our example is conveniently based on a representation of ρ in the form

$$\begin{aligned}\rho(t) &= G_1(t)\Omega_1^{-\frac{1}{2}}, \quad t<0,\\ &= G_2(t)\gamma_1\Omega_2^{-\frac{1}{2}}\\ &\quad\times[\cosh\delta + \gamma_2\sinh\delta\sin((2\Omega_2/M)t+\varphi)]^{\frac{1}{2}},\\ &\qquad t>0,\end{aligned} \tag{84}$$

where $G_1(t)$ and $G_2(t)$ are continuous functions of time with continuous derivatives, possessing the limits

$$\lim_{t\to-\infty} G_1(t) = 1, \qquad \lim_{t\to\infty} G_2(t) = 1, \tag{85}$$

and the remaining symbols are defined as in Eq. (48). Thus $\rho(t)$ corresponds to a time history with constant, positive frequency parameters Ω_1 and Ω_2 in the remote past and future, respectively, and arbitrary behavior at intermediate times. We demand that ρ and $\dot{\rho}$ be continuous at $t=0$ and retain the possibility that $\ddot{\rho}$, and thus $\Omega(t)$, experience a jump discontinuity at this point that depends on the behavior of the G functions. For compactness we introduce the notation

$$g_{1,2} \equiv G_{1,2}(0), \qquad \Delta \equiv \begin{vmatrix} G_1(0) & G_2(0) \\ \dfrac{dG_1}{dt}(0) & \dfrac{dG_2}{dt}(0)\end{vmatrix}. \tag{86}$$

The continuity conditions at $t=0$ determine the parameters φ and δ, the latter of which, after some algebra, may be written in the form

$$\cosh\delta = \frac{1}{2}\left(\frac{g_1^2\Omega_2}{g_2^2\Omega_1} + \frac{g_2^2\Omega_1}{g_1^2\Omega_2} + \frac{M^2\Delta^2}{g_2^4\Omega_1\Omega_2}\right). \tag{87}$$

It is easily verified that the right-hand side of Eq. (87) is greater than unity for positive frequencies. Substitution into the transition probability of Eq. (75) yields

$$P_{0,2n} = \frac{(2n)!}{2^{2n}(n!)^2}\left[\frac{(g_1^2\Omega_2 - g_2^2\Omega_1)^2 + M^2\Delta^2\left(\dfrac{g_1}{g_2}\right)^2}{(g_1^2\Omega_2 + g_2^2\Omega_1)^2 + M^2\Delta^2\left(\dfrac{g_1}{g_2}\right)^2}\right]^n \times\left[\frac{4g_1^2g_2^2\Omega_1\Omega_2}{(g_1^2\Omega_2 + g_2^2\Omega_1)^2 + M^2\Delta^2\left(\dfrac{g_1}{g_2}\right)^2}\right]^{\frac{1}{2}}. \tag{88}$$

The special case $G_1(t)=1$, $G_2(t)=1$ corresponds to a step-function discontinuity in the frequency Ω at $t=0$, from a constant value Ω_1 to a constant value Ω_2. In this case we have $\Delta = 0$, and the exact formula of Eq. (88) reduces to the usual result of the sudden theory:

$$P_{0,2n}(\text{sudden}) = \frac{(2n)!}{2^{2n}(n!)^2}\left(\frac{\Omega_2 - \Omega_1}{\Omega_2 + \Omega_1}\right)^{2n} \frac{2(\Omega_1\Omega_2)^{\frac{1}{2}}}{\Omega_2 + \Omega_1}. \tag{89}$$

The right-hand side of Eq. (89) is, of course, simply the modulus squared of the overlap between the ground state of the initial oscillator Hamiltonian and the $2n$th state of the final oscillator Hamiltonian, in accordance with the fundamental assertion of the sudden theory. The more general transition probability of Eq. (76) can be calculated similarly.

IV. APPLICATION TO CHARGED PARTICLE MOTION IN A TIME-DEPENDENT ELECTROMAGNETIC FIELD

A. The Physical System

We consider a particle of mass M and charge e moving in a classical, axially symmetric electromagnetic field defined by the vector potential

$$\mathbf{A} = \tfrac{1}{2}B(t)\mathbf{k} \times \mathbf{r} \tag{90a}$$

and the scalar potential

$$\varphi = \frac{1}{2}\frac{e}{Mc^2}\eta(t)r^2 = \frac{1}{2}\frac{e}{Mc^2}\eta(t)(x^2 + y^2), \tag{90b}$$

where $\mathbf{r}$ is the position vector, $\mathbf{k}$ is a unit vector along the symmetry axis, r is perpendicular distance from the symmetry axis, x and y are Cartesian coordinates perpendicular to the symmetry axis, $B(t)$ and $\eta(t)$ are arbitrary piecewise-continuous functions of time, and c is the speed of light. The potential φ corresponds to an axially symmetric, time-dependent uniform charge density equal to $-(1/2\pi)(e/Mc^2)\eta(t)$. The electric and magnetic fields are

$$\mathbf{E} = -\nabla\varphi - \frac{1}{c}\dot{\mathbf{A}} = -\frac{e}{Mc^2}\eta(t)(x\mathbf{i} + y\mathbf{j}) - \frac{1}{2c}\dot{B}(t)\mathbf{k} \times \mathbf{r} \tag{91}$$

and

$$\mathbf{B} = \nabla \times \mathbf{A} = B(t)\mathbf{k},$$

where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the positive x and y directions, respectively, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$. Since the axial motion of a particle in these fields is trivial, we shall ignore it and treat only the motion perpendicular to the symmetry axis. The Hamiltonian for this system is

$$H = \frac{1}{2M}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\varphi = \frac{1}{2M}(p_x^2 + p_y^2) + \frac{e^2}{2Mc^2}\left(\frac{B^2}{4} + \eta\right) \times (x^2 + y^2) + \frac{eB}{2Mc}(yp_x - xp_y), \tag{92}$$

where the operator $\mathbf{p} = \mathbf{i}p_x + \mathbf{j}p_y$ is the canonical momentum of the particle. The only nonvanishing commutators between the coordinates and momenta are, as usual,

$$[x, p_x] = [y, p_y] = i\hbar. \tag{93}$$

We introduce cylindrical coordinates r and θ and their conjugate momenta p_r and p_θ by the definitions

$$\begin{aligned} r &= (x^2 + y^2)^{\frac{1}{2}}, \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right), \\ p_r &= \frac{1}{2}\left(\frac{x}{r}p_x + p_x\frac{x}{r} + \frac{y}{r}p_y + p_y\frac{y}{r}\right) \\ &= \frac{1}{r}(xp_x + yp_y) - \frac{i\hbar}{2}\frac{1}{r}, \\ p_\theta &= xp_y - yp_x. \end{aligned} \tag{94}$$

These operators are Hermitian, and the only nonvanishing commutators between them are

$$[r, p_r] = [\theta, p_\theta] = i\hbar. \tag{95}$$

Expressed in terms of these variables, the Hamiltonian given by Eq. (92) is

$$H = \frac{1}{2M}\left[p_r^2 + \frac{\left(p_\theta - \frac{\hbar}{2}\right)\left(p_\theta + \frac{\hbar}{2}\right)}{r^2}\right] + \frac{e^2}{2Mc^2}\left(\frac{B^2}{4} + \eta\right)r^2 - \frac{eB}{2Mc}p_\theta. \tag{96}$$

Because of the axial symmetry, p_θ is a constant of the motion, as is clearly evident from this form of the Hamiltonian. The usual wave equation[5] can be obtained from Eq. (96) by substituting

$$p_r \to -i\hbar\left(\frac{\partial}{\partial r} + \frac{1}{2r}\right),$$

which is the standard coordinate representation of the *Hermitian* operator p_r.

B. Connection with the Time-Dependent Harmonic Oscillator

The Cartesian operator variables for the particle can be related to variables that satisfy the same equations of motion as time-dependent harmonic oscillator variables by means of the *noncanonical* transformation

$$Q = (x + iy)\exp\left[i\frac{e}{2Mc}\int^t B(t')\,dt'\right],$$

$$P = \frac{c}{e}(p_x + ip_y)\exp\left[i\frac{e}{2Mc}\int^t B(t')\,dt'\right]. \tag{97}$$

It is easily verified that the variables Q and P satisfy

$$\dot{Q} = \frac{e}{Mc} P,$$
$$\dot{P} = -\frac{e}{Mc} \Omega^2(t) Q, \tag{98}$$

where $\Omega^2(t)$ is defined by

$$\Omega^2(t) = \tfrac{1}{4}B^2(t) + \eta(t). \tag{99}$$

Equations (98) are identical in form to Eqs. (31) for the time-dependent harmonic oscillator. However, we emphasize that the transformation given by Eqs. (97) is *not canonical*, because Q and P satisfy the commutation relation $[Q, P] = 0$. Nevertheless, the transformation can be used to obtain an invariant for the charged particle because the I defined by Eq. (44) for the oscillator is an invariant as long as q and p satisfy Eqs. (31). It is not necessary that the canonical commutation relation, Eq. (30), be satisfied, nor that q and p be Hermitian. The invariant that we obtain from Eq. (44) (and also denote by I) is

$$I(t) = \frac{1}{2}\left\{\rho^{-2}(x+iy)^2 + \left(\frac{Mc}{e}\right)^2 \times \left[\frac{1}{M}\rho(p_x + ip_y) - \dot{\rho}(x+iy)\right]^2\right\} \times \exp\left[i\frac{e}{Mc}\int^t B(t')\,dt'\right], \tag{100}$$

where ρ is *any* particular solution of

$$\left(\frac{Mc}{e}\right)^2 \ddot{\rho} + \Omega^2(t)\rho - \rho^{-3} = 0, \tag{101}$$

with $\Omega^2(t)$ given by Eq. (99). It is easily verified by direct computation that the I defined by Eq. (100) satisfies

$$\dot{I} \equiv \frac{\partial I}{\partial t} + \frac{1}{i\hbar}[I, H] = 0$$

with H given by Eq. (92). The invariant I is neither Hermitian nor anti-Hermitian, but we shall derive from it a Hermitian invariant to which the theory discussed in Sec. II is applicable.

C. Derivation of a Hermitian Invariant[9]

We introduce time-dependent Cartesian coordinates and momenta such that the explicit time dependence of $I(t)$ is contained solely in a phase factor. These coordinates and momenta are defined by

$$X = \frac{1}{\rho}x, \qquad Y = \frac{1}{\rho}y,$$
$$P_X = \rho p_x - M\dot{\rho}x, \quad P_Y = \rho p_y - M\dot{\rho}y. \tag{102}$$

In order that the new variables be Hermitian, we choose ρ to be a *real* solution of Eq. (101). The transformation is canonical because the only nonvanishing commutators between the variables are

$$[X, P_X] = [Y, P_Y] = i\hbar. \tag{103}$$

Expressed in terms of the new variables, $I(t)$ can be written as

$$I(t) = \frac{1}{2}\left\{(X+iY)^2 + \left(\frac{c}{e}\right)^2 (P_X + iP_Y)^2\right\} \times \exp\left[i\frac{e}{Mc}\int^t B(t')\,dt'\right]. \tag{104}$$

Because of the axial symmetry, it is also convenient to introduce cylindrical operators associated with X, Y, P_X, and P_Y:

$$R = (X^2+Y^2)^{\frac{1}{2}} = \frac{1}{\rho} r,$$
$$\theta = \tan^{-1}\left(\frac{Y}{X}\right) = \tan^{-1}\left(\frac{y}{x}\right),$$
$$P_R = \frac{1}{2}\left(\frac{X}{R}P_X + P_X\frac{X}{R} + \frac{Y}{R}P_Y + P_Y\frac{Y}{R}\right) = \frac{1}{R}(XP_X + YP_Y) - \frac{i\hbar}{2}\frac{1}{R} = \rho p_r - M\dot{\rho}r, \tag{105}$$
$$p_\theta = XP_Y - YP_X = xp_y - yp_x.$$

This transformation is also canonical because, as before, the only nonvanishing commutators are

$$[R, P_R] = [\theta, p_\theta] = i\hbar. \tag{106}$$

In terms of these operators, $I(t)$ can be written as

$$I(t) = \frac{1}{2}\left\{\exp\left[2i\left(\theta + \frac{e}{2Mc}\int^t B(t')\,dt'\right)\right]\right\}(C + iD), \tag{107}$$

where C and D are Hermitian operators given by

$$C = R^2 + \left(\frac{c}{e}\right)^2\left(P_R^2 - \frac{(p_\theta+\hbar)^2 - \frac{3}{4}\hbar^2}{R^2}\right),$$
$$D = \left(\frac{c}{e}\right)^2 (p_\theta + \hbar)\left(P_R\frac{1}{R} + \frac{1}{R}P_R\right). \tag{108}$$

Equation (107) can also be rewritten in the similar form

$$I(t) = \tfrac{1}{2}(C_1 + iD_1)\exp\left[2i\left(\theta + \frac{e}{2Mc}\int^t B(t')\,dt'\right)\right], \tag{109}$$

where C_1 and D_1 are obtained from C and D, respectively, by replacing $(p_\theta + \hbar)$ by $(p_\theta - \hbar)$.

From Eqs. (107) and (109) it follows that the Hermitian operators $I^\dagger I$ and $II^\dagger$ are invariants that are independent of θ and differ from each other only by a constant operator depending on p_θ. Motivated by results for the corresponding classical system,[9] we seek to construct a θ-independent invariant of biquadratic form in R, P_R, and R^{-1} which can be written as a linear function of $I^\dagger I$:

$$\left[R^2 + \left(\frac{c}{e}\right)^2 P_R^2 + \frac{\beta}{R^2}\right]^2 = 4I^\dagger I + \gamma, \tag{110}$$

where β and γ are Hermitian constant operators that may depend on p_θ but not on θ, P_R, or R. The numerical factor multiplying $I^\dagger I$ in Eq. (110) is immediately obtained from the normalization of I given by Eqs. (107) and (108). Use of the operator $II^\dagger$ in Eq. (110) would merely have changed the value of γ. The solution of Eq. (110) is

$$\beta = \left(\frac{c}{e}\right)^2 (p_\theta - \tfrac{1}{2}\hbar)(p_\theta + \tfrac{1}{2}\hbar),$$
$$\gamma = 4\left(\frac{c}{e}\right)^2 (p_\theta + \hbar)^2. \tag{111}$$

Therefore the operator

$$I_1^2 = \left\{R^2 + \left(\frac{c}{e}\right)^2 \left[P_R^2 + \frac{(p_\theta - \frac{1}{2}\hbar)(p_\theta + \frac{1}{2}\hbar)}{R^2}\right]\right\}^2 \tag{112}$$

is the desired biquadratic invariant. Moreover, direct calculation shows that the operator I_1 [defined as the inside of the curly brackets of Eq. (112)] is itself a Hermitian invariant, i.e., it satisfies Eqs. (1) and (2) for the Hamiltonian of Eq. (96).

Finally, instead of working with I_1 directly, we define another Hermitian invariant K by

$$\begin{aligned} K &= \tfrac{1}{4}(|e|/c)I_1 - \tfrac{1}{2}sp_\theta \\ &= \frac{c}{4\,|e|}\left[P_R^2 + \frac{(p_\theta - \frac{1}{2}\hbar)(p_\theta + \frac{1}{2}\hbar)}{R^2}\right] \\ &\quad + \frac{|e|}{4c}R^2 - \tfrac{1}{2}sp_\theta, \end{aligned} \tag{113}$$

where

$$s = e/|e|. \tag{114}$$

The form of K is identical to the form of the Hamiltonian for a particle moving in a *time-independent* magnetic field [see Eq. (96)]. The general theory developed in Sec. II is applicable to K, and the eigenvalues and eigenvectors of K can be found elegantly by operator methods.

[9] This derivation is closely related to a derivation of an analogous invariant for the corresponding classical system. The treatment of the classical system is given in H. R. Lewis, Jr., Phys. Rev. **172**, 1313 (1968).

D. Eigenvalues and Eigenstates of K

Since the invariant K has the form of the Hamiltonian for a particle moving in a time-independent magnetic field, its eigenvalues and eigenvectors are known. The usual derivation[4,5] is in terms of confluent hypergeometric functions. However, it is possible to derive the eigenvalues and eigenvectors by purely operator techniques. This derivation, which we present here, was motivated by the work of Infeld,[10] although his method differs somewhat in detail from ours.

We define operators a, $a^\dagger$, b, and $b^\dagger$ by

$$\begin{aligned} b &= \frac{1}{2}\left(\frac{sc}{|e|\,\hbar}\right)^{\frac{1}{2}}\left\{P_R - i\left[\frac{e}{c}R + \frac{(p_\theta + \frac{1}{2}\hbar)}{R}\right]\right\}, \\ b^\dagger &= \frac{1}{2}\left[\left(\frac{sc}{|e|\,\hbar}\right)^{\frac{1}{2}}\right]^*\left\{P_R + i\left[\frac{e}{c}R + \frac{(p_\theta + \frac{1}{2}\hbar)}{R}\right]\right\}, \\ a &= be^{-i\theta}, \quad a^\dagger = e^{i\theta}b^\dagger. \end{aligned} \tag{115}$$

The commutator of a with $a^\dagger$ is

$$[a, a^\dagger] = s. \tag{116}$$

In terms of a and $a^\dagger$, the expression for K can be written as

$$\begin{aligned} K &= \hbar aa^\dagger - s(p_\theta + \tfrac{1}{2}\hbar) \\ &= \hbar a^\dagger a - s(p_\theta - \tfrac{1}{2}\hbar). \end{aligned} \tag{117}$$

The commutators of K with a and $a^\dagger$ are

$$[K, a] = [K, a^\dagger] = 0, \tag{118}$$

which implies that operation with a or $a^\dagger$ on an eigenstate of K produces another eigenstate of K with the same eigenvalue. The commutators of p_θ with a and $a^\dagger$ are

$$[p_\theta, a] = -\hbar a$$

and

$$[p_\theta, a^\dagger] = \hbar a^\dagger. \tag{119}$$

Therefore, a and $a^\dagger$ are, respectively, lowering and raising operators for the eigenvalues of p_θ.

Since K and p_θ commute, we can define simultaneous eigenstates of these two operators. Let $|j, n\rangle$ denote a normalized eigenstate for which the eigenvalue of K is $(j + \frac{1}{2})\hbar$ by definition, and the eigenvalue of p_θ is $n\hbar$, where n is an integer:

$$\begin{aligned} \langle j, n \,|\, j, n\rangle &= 1, \\ K\,|j, n\rangle &= (j + \tfrac{1}{2})\hbar\,|j, n\rangle, \\ p_\theta\,|j, n\rangle &= n\hbar\,|j, n\rangle. \end{aligned} \tag{120}$$

[10] L. Infeld, Phys. Rev. **59**, 737 (1941). Later developments of this interesting method can be found in the following references: (a) T. Inui, Progr. Theoret. Phys. (Kyoto) **3**, 168, 244 (1948); (b) L. Infeld and T. E. Hull, Rev. Mod. Phys. **23**, 21 (1951); (c) A. Joseph, *ibid.* **39**, 829 (1967); (d) C. A. Coulson and A. Joseph, *ibid.* **39**, 838 (1967).

Because a and $a^\dagger$ are lowering and raising operators for the eigenvalues of p_θ, the states $|j, n-1\rangle$ and $|j, n+1\rangle$ are proportional to $a\,|j, n\rangle$ and $a^\dagger\,|j, n\rangle$, respectively. Therefore, the requirement that all admissible states be normalizable means that the matrix elements $\langle j, n|\, a^\dagger a\,|j, n\rangle$ and $\langle j, n|\, aa^\dagger\,|j, n\rangle$ must be nonnegative:

$$\begin{aligned}\langle j, n|\, a^\dagger a\,|j, n\rangle &= \frac{1}{\hbar}\langle j, n|\, K + s(p_\theta - \tfrac{1}{2}\hbar)\,|j, n\rangle \\ &= (j + \tfrac{1}{2}) + (n - \tfrac{1}{2})s \geq 0, \\ \langle j, n|\, aa^\dagger\,|j, n\rangle &= \frac{1}{\hbar}\langle j, n|\, K + s(p_\theta + \tfrac{1}{2}\hbar)\,|j, n\rangle \\ &= (j + \tfrac{1}{2}) + (n + \tfrac{1}{2})s \geq 0. \qquad (121)\end{aligned}$$

From this we immediately conclude that j must be an integer; otherwise, by repeated application of a or $a^\dagger$ we could obtain an unnormalizable state from an admissible state. The first of the inequalities (121) is the more restrictive for $s = 1$ and the second is the more restrictive for $s = -1$. Therefore we can replace these inequalities by the single inequality

$$j + ns \geq 0. \qquad (122)$$

We can restrict the values of j further by expressing K in terms of X, Y, P_X, and P_Y as

$$K = \frac{c}{4\,|e|}\left[\left(P_X + \frac{e}{c}Y\right)^2 + \left(P_Y - \frac{e}{c}X\right)^2\right]. \qquad (123)$$

Thus we see that

$$(j + \tfrac{1}{2})\hbar = \langle j, n|\, K\,|j, n\rangle$$

is the expectation value of the sum of the squares of two Hermitian operators and therefore cannot be negative.

With these results we can write the allowable solutions of the inequalities (121) as

$$j = l + \tfrac{1}{2}(|n| - sn), \qquad (124)$$

where l is an integer that can assume any nonnegative value $(0, 1, 2, \cdots)$. For fixed j and $s = 1$ the minimum value of n that is allowed is $-j$, whereas the maximum value of n allowed for fixed j and $s = -1$ is j. Therefore the state $|j, -j\rangle$ for $s = 1$ and the state $|j, j\rangle$ for $s = -1$ are determined by

$$\left.\begin{aligned} p_\theta\,|j, -j\rangle &= -j\hbar\,|j, -j\rangle, \\ a\,|j, -j\rangle &= 0, \end{aligned}\right\} \quad \text{for } s = 1, \qquad (125a)$$

and

$$\left.\begin{aligned} p_\theta\,|j, j\rangle &= j\hbar\,|j, j\rangle, \\ a^\dagger\,|j, j\rangle &= 0, \end{aligned}\right\} \quad \text{for } s = -1. \qquad (125b)$$

All other admissible states are obtained by repeated operation with $a^\dagger$ on $|j, -j\rangle$ (for $s = 1$) or with a on $|j, j\rangle$ (for $s = -1$). Finally, by using Eqs. (121) and making a suitable choice of relative phases for the states, we obtain the following recursion formulas for the admissible normalized eigenstates:

$$\left.\begin{aligned} |j, n+1\rangle &= (j + n + 1)^{-\frac{1}{2}}a^\dagger\,|j, n\rangle \quad \text{for } s = 1, \\ |j, n-1\rangle &= (j - n + 1)^{-\frac{1}{2}}a\,|j, n\rangle \quad \text{for } s = -1.\end{aligned}\right\} \qquad (126)$$

E. Calculation of the Phases

The matrix element $\langle j, n'|\, i\hbar(\partial/\partial t) - H\,|j, n\rangle$ vanishes for $n' \neq n$ because p_θ commutes with $\partial/\partial t$ and with H. Therefore the state $e^{i\alpha_{jn}}\,|j, n\rangle$ will be a normalized solution of the Schrödinger equation if we choose $\alpha_{jn}(t)$ as a solution of Eq. (15):

$$\hbar\frac{d\alpha_{jn}}{dt} = \langle j, n|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n\rangle. \qquad (15')$$

We begin by finding a recursion formula for the right-hand side of Eq. (15′) for $s = 1$ and for $s = -1$. For $s = 1$ we have

$$\begin{aligned}&\langle j, n|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n\rangle \\ &= \frac{1}{j+n}\langle j, n-1|\, a\left(i\hbar\frac{\partial}{\partial t} - H\right)a^\dagger\,|j, n-1\rangle \\ &= \frac{1}{j+n}\langle j, n-1|\left\{\left(i\hbar\frac{\partial}{\partial t} - H\right)a + \left[a, i\hbar\frac{\partial}{\partial t} - H\right]\right\}a^\dagger\,|j, n-1\rangle \\ &= \langle j, n-1|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n-1\rangle \\ &\quad + \frac{1}{j+n}\langle j, n-1|\left[a, i\hbar\frac{\partial}{\partial t} - H\right]a^\dagger\,|j, n-1\rangle. \qquad (127)\end{aligned}$$

Similarly, for $s = -1$ we obtain

$$\begin{aligned}&\langle j, n|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n\rangle \\ &= \langle j, n+1|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n+1\rangle \\ &\quad + \frac{1}{j-n}\langle j, n+1|\left[a^\dagger, i\hbar\frac{\partial}{\partial t} - H\right]a\,|j, n+1\rangle. \qquad (128)\end{aligned}$$

We only need calculate the commutator $[a, i\hbar(\partial/\partial t) - H]$ that appears in Eq. (127) because the commutator in Eq. (128) is related to it by

$$\left[a^\dagger, i\hbar\frac{\partial}{\partial t} - H\right] = -\left[a, i\hbar\frac{\partial}{\partial t} - H\right]^\dagger.$$

To evaluate $[a, i\hbar(\partial/\partial t) - H]$ we use the expressions for H and a in terms of r, p_r, and p_θ. The expression for H is given by Eq. (96), and the expression for a is

$$a = \frac{1}{2}\left(\frac{sc}{|e|\,\hbar}\right)^{\frac{1}{2}} \times \left\{\rho p_r - M\dot{\rho} r - i\left[\frac{e}{c}\frac{r}{\rho} + \rho\frac{(p_\theta + \frac{1}{2}\hbar)}{r}\right]\right\}e^{-i\theta}. \tag{129}$$

The commutator may be expressed in the form

$$\left[a, i\hbar\frac{\partial}{\partial t} - H\right] = -i\hbar\dot{a}, \tag{130}$$

where the right-hand side refers to the total time derivative operator. The evaluation is straightforward and particularly simple if one uses the Heisenberg equations of motion in evaluating $\dot{a}$. The result is

$$\left[a, i\hbar\frac{\partial}{\partial t} - H\right] = \frac{e\hbar}{Mc}\left(\frac{B}{2} - \frac{1}{\rho^2}\right)a$$

and

$$\left[a^\dagger, i\hbar\frac{\partial}{\partial t} - H\right] = -\frac{e\hbar}{Mc}\left(\frac{B}{2} - \frac{1}{\rho^2}\right)a^\dagger. \tag{131}$$

We now substitute the commutators given by Eqs. (131) into Eqs. (127) and (128) and use Eqs. (121) to obtain

$$\langle j, n|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n\rangle = \langle j, n-1|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n-1\rangle + \frac{e\hbar}{Mc}\left(\frac{B}{2} - \frac{1}{\rho^2}\right) \quad \text{for} \quad s = 1 \tag{132}$$

and

$$\langle j, n|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n\rangle = \langle j, n+1|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n+1\rangle - \frac{e\hbar}{Mc}\left(\frac{B}{2} - \frac{1}{\rho^2}\right) \quad \text{for} \quad s = -1. \tag{133}$$

We are still free to choose the phase of $|j, -j\rangle$ for $s = 1$ and the phase of $|j, j\rangle$ for $s = -1$ arbitrarily. We choose these phases in such a way that the solution of Eqs. (132) and (133) is

$$\langle j, n|\, i\hbar\frac{\partial}{\partial t} - H\,|j, n\rangle = [n + (j + \tfrac{1}{2})s]\frac{e\hbar}{Mc}\left(\frac{B}{2} - \frac{1}{\rho^2}\right). \tag{134}$$

The expression for the phase $\alpha_{jn}(t)$ that we obtain by substituting the matrix element given by Eq. (134) into Eq. (15′) is

$$\alpha_{jn}(t) = [n + (j + \tfrac{1}{2})s]\frac{e}{Mc}\int^t dt'[\tfrac{1}{2}B(t') - \rho^{-2}(t')]. \tag{135}$$

Using Eq. (135), we may construct the time-dependent Schrödinger state vector according to the prescription of Eq. (16), and hence we may compute transition probabilities for processes analogous to those treated in Sec. III. It should be pointed out that the definition of Eq. (99) may lead to both positive and negative values of $\Omega^2(t)$. The latter situation arises when the sign of the particle charge is the same as the sign of the background charge density, provided that the instantaneous Larmor frequency $|eB|/2Mc$ is less than the "electrostatic oscillation frequency" $(2\pi e\sigma/M)^{\frac{1}{2}}$, where σ is the background charge density. Under these circumstances the asymptotic form for real $\rho(t)$ given by Eq. (48) has to be modified appropriately. The transition probability formalism of Sec. III is directly transcribable to the present case of charged particle motion only if $\Omega^2(\pm\infty)$ is positive.