Note on coherent state

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This note aims to summarize essential properties of the coherent state. I plan to cover these points:

- Basic propertities for creation and annihilation operators.
- Coherent states under Hamiltonian representation.
- Coherent states under coordinate representation.
- Wigner functions for coherent states and time evolution.
- classical propertities of coherent states.

1 Basic knowledges on oscillators

In this section, I'd like to briefly review some basic properties of some operators. The Hamiltonian is $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$, and the commute relations is $[\hat{x}, \hat{p}] = i\hbar$(Be careful of the sign!). Then the creation and annihilation operator can be defined as

$$\hat{a} = \frac{1}{\sqrt{2m\hbar\omega}}(i\hat{p} + m\omega\hat{x}),\ \hat{a}^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(-i\hat{p} + m\omega\hat{x}) \tag{1}$$

And the commute relations is $[\hat{a}^\dagger, \hat{a}] = -1$(Be careful of the sign!). Inversely, $\hat{x}$, $\hat{p}$ and $\hat{H}$ can be described as

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} &= i\sqrt{\frac{m\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a}) \\ \hat{H} &= (\hat{a}^+\hat{a} + \frac{1}{2})\hbar\omega \end{aligned} \tag{2}$$

Suppose the eigenstate of the Hamiltonian is $|n\rangle$ with eigenvalue $E_n = (n + \frac{1}{2})\hbar\omega$ whose wavefunction is

$$\Psi_n(x) = \big(\frac{m\omega}{\pi\hbar}\big)^{\frac{1}{4}}\frac{1}{\sqrt{2^n n!}}H_n\big(\frac{m\omega}{\hbar}x\big)\exp\big(-\frac{m\omega}{2\hbar}x^2\big) \tag{3}$$

Then the creation and annihilation operators give out

$$\begin{aligned} \hat{a}\,|n\rangle &= \sqrt{n}\,|n-1\rangle \\ \hat{a}^\dagger\,|n\rangle &= \sqrt{n+1}\,|n+1\rangle \end{aligned} \tag{4}$$

2 Important opertor formulas

Then let's review some useful formulas. The first one is Glauber's formula, based on Baker-Hausdorff formula. This formula claims that if operators $\hat{A}$ and $\hat{B}$ satisfies $[\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0$, then

$$\exp(\hat{A}+\hat{B}) = \exp(\hat{A})\exp(\hat{B})\exp\left(-\frac{1}{2}[\hat{A},\hat{B}]\right) \tag{5}$$

The 2nd formula is

$$\exp(\lambda\hat{A})\hat{B}\exp(-\lambda\hat{A}) = \hat{B} + \lambda[\hat{A},\hat{B}] + \frac{\lambda^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + \frac{\lambda^3}{3!}[\hat{A},[\hat{A},[\hat{A},\hat{B}]]] + \dots \tag{6}$$

Specially, if $[\hat{A},\hat{B}] = const =: C$, then

$$\exp(\lambda\hat{A})\hat{B}\exp(-\lambda\hat{A}) = \hat{B} + \lambda C \tag{7}$$

which means a translation of operator $\hat{B}$. More specially, we set $\hat{A} = -\alpha\hat{a}^\dagger + \alpha^*\hat{a}$, $\hat{B} = \hat{a}$ or $\hat{B} = \hat{a}^\dagger$, then we may obtain

$$\exp(-\alpha\hat{a}^+ + \alpha^*\hat{a})\hat{a}\exp(\alpha\hat{a}^+ - \alpha^*\hat{a}) = \hat{a} + \alpha \tag{8}$$

$$\exp(-\alpha\hat{a}^\dagger + \alpha^*\hat{a})\hat{a}^\dagger\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}) = \hat{a}^\dagger + \alpha^* \tag{9}$$

Equation (8) is very important. Denote $\hat{D}(\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})$, then (8) presents

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha \tag{10}$$

Based on (10), we may prove that all the eigenstates of $\hat{a}$ are

$$\hat{a}\big(\hat{D}(\alpha)\left|0\right\rangle\big) = \alpha\big(\hat{D}(\alpha)\left|0\right\rangle\big) \tag{11}$$

Therefore, we may define the coherent states as $\left|\alpha\right\rangle = \hat{D}(\alpha)\left|0\right\rangle$ with ANY complex number eigenvalue $\alpha \in \mathbb{C}$, and $\hat{a}\left|\alpha\right\rangle = \alpha\left|\alpha\right\rangle$.

Since $\hat{a}^\dagger$ and $\hat{a}$ can be described by $\hat{x}$ and $\hat{p}$, we can write the translation operator as another form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}\hat{x} - i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right) \tag{12}$$

It's not difficult to find that $\hat{D}(\alpha)$ is actually a Weyl translation operator $\hat{W}(\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}, \Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}})$. Based on Glauber's formula, we can express $\hat{D}(\alpha)$ as a separated form:

$$\hat{D}(\alpha) = \exp\left(i\Im(\alpha)\sqrt{\frac{2m\alpha}{\hbar}}\hat{x}\right)\exp\left(-i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right)\exp(-i\Im(\alpha)\Re(\alpha)) \tag{13}$$

In the end of this section, I'd like to mention a further appliacation of (6). Set $\hat{A} = \hat{N} = \hat{a}^\dagger\hat{a}$, and $\hat{B} = \hat{a}^\dagger$ or $\hat{a}$. Then we may obtain

$$\exp(\lambda\hat{N})\hat{a}\exp(-\lambda\hat{N}) = \hat{a}e^{-\lambda} \tag{14}$$

$$\exp(\lambda\hat{N})\hat{a}^\dagger\exp(-\lambda\hat{N}) = \hat{a}^\dagger e^{\lambda} \tag{15}$$

Still pay attention to the sign! This formula will give out the squeezed states.

3 Coherent states under Hamiltonian and coordinate representation

Now let's calculate the coherent state under coordinate representation, i.e. wavefunction of the coherent state $\langle x|\alpha\rangle$. Notice the expression (12) of $\hat{D}(\alpha)$ by $\hat{x}$ and $\hat{p}$, we can consider the operation of $\hat{D}(\alpha)$ over coordinate eigenstate $|x\rangle$. Then $\langle x|\alpha\rangle$ can be expressed as $\langle x|\hat{D}(\alpha)|0\rangle$, where the translation operator operates on $\langle x|$. Note that $\hat{D}(\alpha)^\dagger = \hat{D}(-\alpha)$

$$\begin{aligned}\hat{D}(\alpha)|x\rangle &= \exp\left(i\Im(\alpha)\sqrt{\frac{2m\alpha}{\hbar}}\hat{x}\right)\exp\left(-i\Re(\alpha)\sqrt{\frac{2}{m\hbar\omega}}\hat{p}\right)\exp(-i\Im(\alpha)\Re(\alpha))|x\rangle \\ &= \exp\left(i\Im(\alpha)\sqrt{\frac{2m\omega}{\hbar}}(x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}})\right)\exp(-i\Im(\alpha)\Re(\alpha))|x-\Re(\alpha)\sqrt{\frac{2\hbar}{m\omega}}\rangle\end{aligned} \tag{16}$$