Approximation Algorithms and Local Search for CVRP

CSCI2951O: Foundations of Prescriptive Analytics

Hammad Izhar (carlisle)

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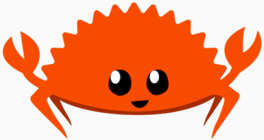
Brown University

Implementation Overview

- ∼700 lines of Rust code
- TSP Approximation Algorithm to determine an initial feasible solution
- Naive Local Search Heuristics



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- Limited by the runtime of determining a perfect matching

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```
fn mst() {
  let mut included_clients = HashSet::new([0]);
  let mut pq = PriorityQueue::new();
  let mut tree = Vec::new();
  // Push into pq all of the incident edges to the depot
  ...
  while included_clients.len() < self.clients.len() {
      let (edge, _) = pq.pop().unwrap();
      // The fringe vertex we include by adding this edge
      let new_client = ...
      ...
      tree.push(edge);
      for client in 0..self.clients.len() {
        // Add edges to vertices not yet visited in the tree
      }
  }
  tree
}
```

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    - At a high-level, find augmenting paths while contracting odd cycles (a.k.a blossoms) along the way
- Can be viewed as a generalization of the Ford-Fulkerson Max-Flow algorithm to find maximum matchings in bipartite graphs

Eulerian Tour (Hierholzer's) Algorithm

```
pub fn find_eulerian_tour(mst: &[Edge], matching: &[Edge]) -> Vec<ClientId> {
    let mut vertex_to_edges = HashMap::new();
    // Associate each edge to its endpoints
    ...
    let mut eulerian_tour = Vec::new();
    let mut stack = vec![mst[0].first];
    while !stack.is_empty() {
        let edges = vertex_to_edges.get_mut(stack.last()).unwrap();
        if edges.is_empty() {
            eulerian_tour.push(stack.pop().unwrap());
        } else {
            let edge = *edges.iter().next().unwrap();
            // The next vertex in the path
            let next_vertex = ...
            stack.push(next_vertex);
        }
    }
    eulerian_tour
}
```

Shortcutting and Partitioning

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- We pack the clients into k "bins" where k is the number of vehicles we have access to using a first-fit strategy
- Works for all but a couple of instances! For these remaining couple, a random solution is generated

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- The vertices are placed in the position that minimizes the cost of the new route
- Only neighboring solutions with strictly better costs are ever accepted
- Runs for all remaining available time