<u>Practice Problems (Set 5)</u>

Intro to Probability and Statistics
EE 354 / CE 361 / MATH 310

Question 1

X and Y are random variables with a joint PDF given by:

$$f_{X,Y}(x,y) = \frac{2}{c}$$

for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. The joint PDF is zero for all other values of x and y.

i. Find the value of c.
ii. Calculate the marginal PDFs of X and Y.

Question 2

W and X are independent continuous random variables that are uniformly distributed with W$\sim U[0,1]$ and X$\sim U[0,2]$. Find the Cumulative Distribution Function, CDF, and Probability Density Function, PDF, for the following random variables:

a) $Y = \max(W, X)$
b) $Z = \max(W, X) + 1$

Question 3

Let X and Y be two independent continuous random variables with uniform PDFs:

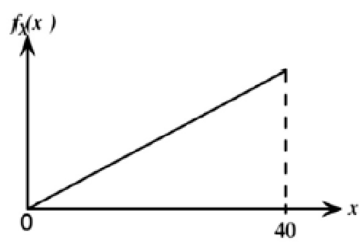
$$X \sim U[0,1]$$
$$Y \sim U[0,2]$$

a)
Find $P(max(X,Y) \leq 0.8)$

b)
Find $f_{X,Y}(x,y)$

Question 4

A random variable X has the following PDF:



Conditional on random variable $X = x$, the continuous random variable Y is uniformly distributed between 0 and $3x$. Determine the joint PDF $f_{X,Y}(x,y)$ for $0 < x < 40$ and $0 < y < 3x$

Question 5

Let X be a standard Normal random variable i.e. $X \sim N(0,1)$. Let Y be an independent Bernoulli random variable taking on values 0 or 1 with equal probability, p=1/2. Also, consider the random variable Z = X + Y.

a) Find the conditional PDF of X given Y=1.

b) Find the conditional PDF of Z given Y=1.

Question 6

On a sunny day, it takes a student between 30 to 45 minutes to get to Habib University campus with all times being equally likely. On a rainy day, it takes the same student between 40 to 60 minutes to get to campus with all times being equally likely. Assume a day is sunny with probability 0.75 and rainy with probability 0.25.

a) Find the PDF of the time T that it takes the student to get to campus.
b) On a given day, it took the student 42 minutes to get to campus. What is the probability that this particular day was rainy?

Question 7

A binary signal S is transmitted and we are given that P(S = 1) = 0.5 and P(S = -1) = 0.5. The received signal is Y = N + S, where N is normal noise with zero mean and unit variance, independent of S. What is the probability that S = 1 if the observed value of received signal Y is 0.5.

Question 8

Smartphones produced by a particular manufacturer are known to have an exponentially distributed lifetime Y. However, the manufacturing plant has had some quality control problems lately. As a result, on any given day, the parameter λ of PDF of Y is actually a random variable uniformly distributed in the interval [1,1.5]. We decide to test a smartphone and record its lifetime. If the lifetime turns out to be 3 years, what is the updated PDF of the underlying parameter λ.

SOLUTIONS:

Question 1

i)

$$f_{X,Y}(x,y) = \begin{cases} \frac{2}{c} & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\because \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$$

$$\Rightarrow \quad f_{X,Y}(x,y) = \begin{cases} 1 & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\Rightarrow \quad c = 2$$

iii)

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) \cdot dy$$

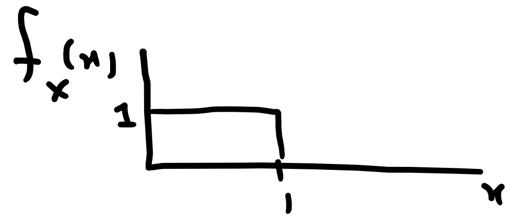
$$= \int_0^1 1 \cdot dy = y \Big|_0^1$$

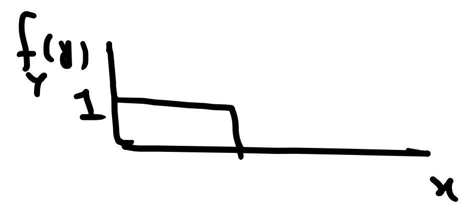
$$f_X(x) = 1 \qquad 0 \le x \le 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx$$

$$= \int_0^1 1 \cdot dx = x \Big|_0^1$$

$$f_Y(y) = 1 \qquad 0 \le y \le 1$$





Question 2

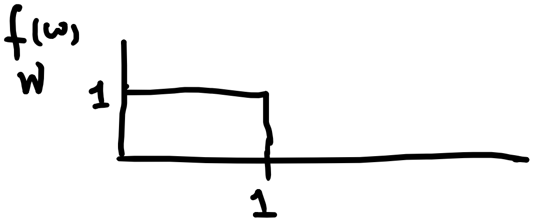
(a)

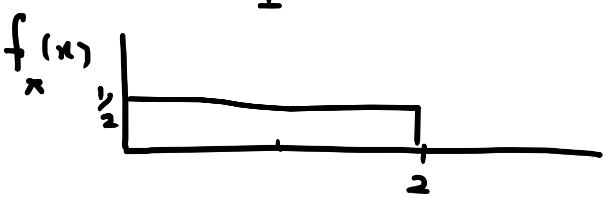
$$F_Y(y) = P(Y \le y) = P(\max(W, X) \le y)$$

$$= P(W \le y, X \le y)$$

$\because$ W & X are independent

$$F_Y(y) = P(W \le y) \cdot P(X \le y) \quad \text{---(1)}$$





Now

$$P(W \le y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y < 1 \\ 1 & 1 \le y \end{cases}$$

$$P(X \le y) = \begin{cases} 0 & y < 0 \\ y/2 & 0 \le y < 2 \\ 1 & 2 \le y \end{cases}$$

From (1)

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^2/2 & 0 \le y < 1 \\ y/2 & 1 \le y < 2 \\ 1 & 2 \le y \end{cases}$$

$$\because \quad f_Y(y) = \frac{d}{dy}\left[F_Y(y)\right]$$

$$f_Y(y) = \begin{cases} 0 & y < 0 \\ y & 0 \le y < 1 \\ 1/2 & 1 \le y < 2 \\ 0 & 2 \le y \end{cases}$$

(b)

Note that

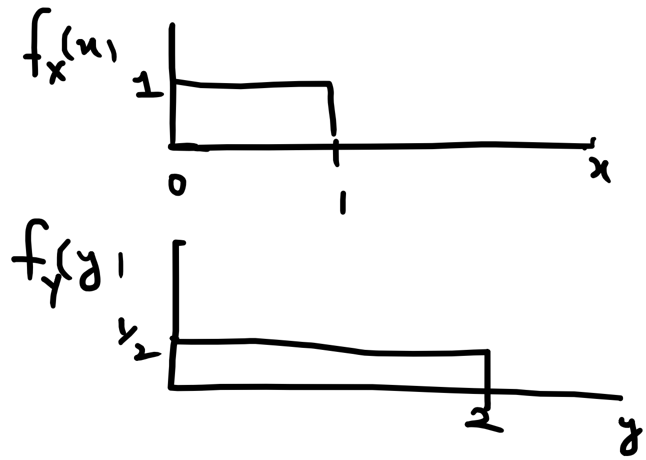
$$Z = Y + 1$$

$\Rightarrow$ $F_Z(z)$ and $f_Z(z)$ are just shifted (by 1) versions of

$F_Y(y)$ and $f_Y(y)$

Question 3

(a)



$P(\max(X,Y) \leq 0.8)$

$= P(X \leq 0.8, Y \leq 0.8)$

By Independence:

$= P(X \leq 0.8) . P(Y \leq 0.8)$

$= 1(0.8) . \frac{1}{2}(0.8)$

$= 0.32$ — Ans

(b)

By Independence:

$$f_X(x,y) = f_X(x)\, f_Y(y)$$

$$f_X(x,y) = \begin{cases} 1 . \frac{1}{2} & 0 < x \leq 1,\ 0 \leq y \leq 2 \\ 0 . \frac{1}{2} & 1 < x \leq 2,\ 0 \leq y \leq 2 \\ 0 . 0 & \text{otherwise} \end{cases}$$

$$f_X(x,y) = \begin{cases} \frac{1}{2} & 0 \leq x \leq 1,\ 0 \leq y \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Question 4

$$f_{X,Y}(x,y) = f_X(x) \cdot f_{Y|X}(y|x) \qquad \text{—(1)}$$

Now,

$$f_X(x) = mx \qquad 0 \le x \le 40$$

when

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1/20 - 0}{40 - 0} = \frac{1}{800}$$

$$\Rightarrow \quad f_X(x) = \frac{1}{800}x \qquad \text{for } 0 \le x \le 40 \qquad \text{—(2)}$$

Also,

$$f_{Y|X}(y|x) = U(0, 3x)$$

$$= \frac{1}{3x} \qquad 0 \le y \le 3x \qquad \text{—(3)}$$

Using (1)

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{800}x \cdot \frac{1}{3x} = \frac{1}{2400} & 0 \le x \le 40,\ 0 \le y \le 3x \\ 0 & \text{otherwise} \end{cases}$$

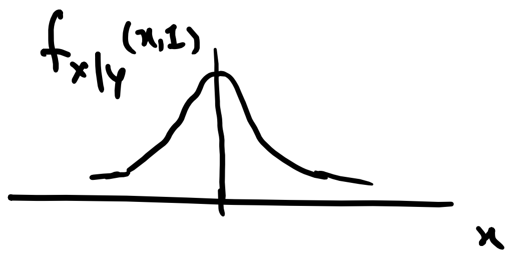
— Ans

Question 5

(a)

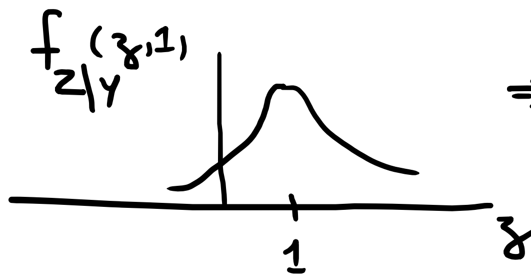
By Independence.

$$f_{X|Y}(x,y) = f_X(x)$$



$$\Rightarrow f_{X|Y}(x,1) = N(0,1)$$

(b)



$$\Rightarrow f_{Z|Y}(z,1) = N(1,1)$$

Question 6

(a)

Let X be the random variable that takes the value 0 when its a sunny day and value 1 when its a rainy day.

$$p_X(x) = \begin{cases} 0.75 & \text{when } x = 0 \\ 0.25 & \text{when } x = 1 \end{cases}$$

Now,

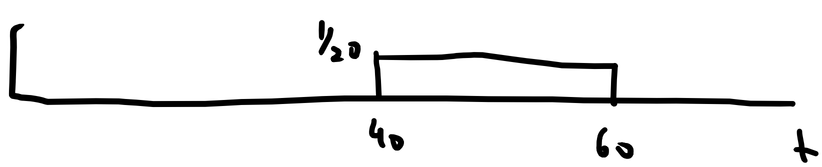
$$f_T(t) = \sum_x p_X(x)\, f_{T|X}(t|x)$$

$$= p_X(0)\, f_{T|X}(t|0) + p_X(1)\, f_{T|X}(t|1)$$

$$f_T(t) = 0.75\left[U(30,45)\right] + 0.25\left[U(40,60)\right] \quad \text{—(1)}$$

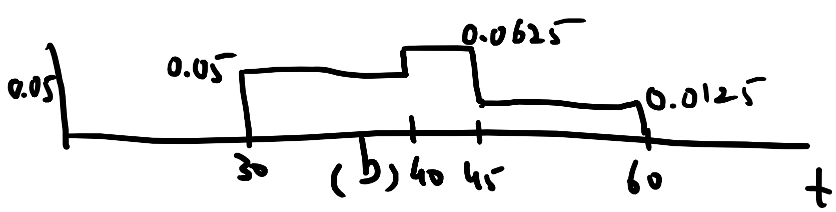
$f_{T|X}(t|0)$

$f_{T|X}(t|1)$



From 1,

$f_T(t)$



(b)

$$p_{X|T}(1|42) = ?$$

Using Bayes' Rule :- (Discrete X, Continuous Y)

$$p_{X|Y}(x|y) = \frac{p_X(x)\, f_{Y|X}(y|x)}{f_Y(y)}$$

$$\Rightarrow \quad p_{X|T}(x|t) = \frac{p_X(x)\, f_{T|X}(t|x)}{f_T(t)}$$

$$p_{X|T}(1|42) = \frac{p_X(1)\, f_{T|X}(42|1)}{f_T(42)} \quad \text{— (2)}$$

$$p_X(1) = 0.25$$

$$f_{T|X}(42|1) = \frac{1}{20} = 0.05$$

$$f_T(42) = 0.0625$$

From (2)

$$p_{X|T}(1|42) = \frac{0.25 \times 0.05}{0.0625}$$

$$p_{X|T}(1|42) = 0.2 \quad \text{—— Ans}$$

Question 7

Let S be the random variable representing the transmitted signal and Y be the random variable representing the received signal.

$$p_S(s) = \begin{cases} 0.5 & \text{when } s = 1 \\ 0.5 & \text{when } s = -1 \end{cases}$$

$$f_{Y|S}(y|1) = \frac{1}{\sqrt{2\pi}} e^{-(y-1)^2/2}$$

$$f_{Y|S}(y|-1) = \frac{1}{\sqrt{2\pi}} e^{-(y+1)^2/2}$$

By Total Probability Theorem :-

$$f_Y(y) = \sum_s p_S(s) f_{Y|S}(y|s)$$

$$= p_S(1) f_{Y|S}(y|1) + p_S(-1) f_{Y|S}(y|-1)$$

$$f_Y(y) = \frac{0.5}{\sqrt{2\pi}} e^{-(y-1)^2/2} + \frac{0.5}{\sqrt{2\pi}} e^{-(y+1)^2/2}$$

$$p_{S|Y}(1|0.5) = ?$$

Using Bayes' Rule :-

$$p_{S|Y}(s|y) = \frac{p_S(s) f_{Y|S}(y|s)}{f_Y(y)}$$

$$p_{S|Y}(1|0.5) = \frac{p_S(1)\, f_{Y|S}(0.5|1)}{f_Y(0.5)}$$

$$= \frac{0.5\left(\frac{1}{\sqrt{2\pi}}\, e^{-(0.5-1)^2/2}\right)}{\frac{0.5}{\sqrt{2\pi}}\, e^{-(0.5-1)^2/2} + \frac{0.5}{\sqrt{2\pi}}\, e^{-(0.5+1)^2/2}}$$

$$= \frac{e^{-0.125}}{e^{-0.125} + e^{-1.125}}$$

$$= \frac{0.8825}{0.8825 + 0.3247}$$

$$p_{S|Y}(1|0.5) = 0.731$$ — Ans

Question 8

$$f_\lambda(\lambda) = \begin{cases} 2 & 1 \le \lambda \le 1.5 \\ 0 & \text{otherwise} \end{cases} \qquad \text{— (1)}$$

$$f_{Y|\lambda}(y|\lambda) = \lambda e^{-\lambda y} \qquad y \geq 0 \qquad \text{— (2)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_\lambda(t)\, f_{Y|\lambda}(y|t).dt$$

$$f_Y(y) = \int_{1}^{1.5} 2 t e^{-ty} .dt \qquad \text{— (3)}$$

$$f_{\lambda|Y}(\lambda|3) = ?$$

Using Bayes' Rule (Continuous X, Continuous Y): -

$$f_{\lambda|Y}(\lambda|y) = \frac{f_\lambda(x)\, f_{Y|\lambda}(y|\lambda)}{f_Y(y)}$$

$$f_{\lambda|Y}(\lambda|3) = \frac{f_\lambda(\lambda)\, f_{Y|\lambda}(3|\lambda)}{f_Y(3)}$$

$$f_{\lambda|y}(\lambda|3) = \frac{2\lambda e^{-3\lambda}}{\int_1^{1.5} 2t e^{-3t} \cdot dt} \qquad 1 \leq \lambda \leq 1.5$$

— Ans