Practice Problems set - Math 310

March 27, 2022

1 Instructions

These practice problems cover the topics of conditional and independence.

Please try to solve the problems yourself first without looking into the solutions.

The solutions are given at the end of the problems.

***

2 Questions

1. Consider a joint PMF given in the following table. Find the conditional PMF $p_{X|Y}(x|1)$ and the marginal PMF $p_X(x)$.

	Y=1	Y=2	Y=3	Y=4
X=1	1/20	1/20	1/20	0/20
X=2	1/20	2/20	3/20	1/20
X=3	1/20	2/20	3/20	1/20
X=4	0/20	1/20	1/20	1/20

2. Consider two random variables X and Y defined as follows:

$$Y = \begin{cases} 10^2, & \text{with prob } \frac{5}{6} \\ 10^4 & \text{with prob } \frac{1}{6} \end{cases}$$

$$X = \begin{cases} 10^{-4}Y, & \text{with prob } \frac{1}{2} \\ 10^{-3}Y & \text{with prob } \frac{1}{3} \\ 10^{-2}Y & \text{with prob } \frac{1}{6} \end{cases}$$

Find $p_{X|Y}(x|y)$, $p_X(x)$ and $p_{X,Y}(x,y)$.

3. Consider a joint PMF given by the following table. Find $E[X|Y = 10^2]$ and $E[X|Y = 10^4]$.

$Y{=}10^4$	0	0	1/12	1/18	1/36
$Y = 10^2$	5/12	5/18	5/36	0	0
	x=0.01	x=0.1	x=1	x=10	x= 100

4. Let X be a coin and Y be a die. Then the joint PMF is given by the table below

	y=1	y=2	y=3	y=4	y=5	y=6
X=0	1/12	1/12	1/12	1/12	1/12	1/12
X=1	1/12	1/12	1/12	1/12	1/12	1/12

   Are X and Y independent?

5. Consider four independent rolls of a 6-sided die. Let X be the number of 1's and let Y be the number of 2's obtained. What is the joint PMF of X and Y?

6. Consider two independent tosses of a fair coin. Let X be the number of heads and let A be the event that the number of heads is even. Are these events (X and A) independent?

7. Let X be the roll of a fair six-sided die and let A be the event that the roll is an even number. Find the condition PMF $P_{X|A}(x)$.

8. Messages transmitted by a computer in Boston through a data network are destined for New York with probability 0.5, for Chicago with probability 0.3, and for San Francisco with probability 0.2. The transit time X of a message is random. Its mean is 0.05 seconds if it is destined for New York, 0.1 seconds if it is destined for Chicago, and 0.3 seconds if it is destined for San Francisco. Find E(X).

9. Alex and Bob each flips a fair coin twice. Use "1" to denote heads and "0" to denote tails. Let X be the maximum of the two numbers Alex gets, and let Y be the minimum of the two numbers Bob gets.

   (a) Find the joint PMF $p_{X,Y}(x, y)$.

   (b) Find the marginal PMF $p_X(x)$ and $p_Y(y)$.

   (c) Find the conditional PMF $P_{X|Y}(x|y)$. Does $P_{X|Y}(x|y) = P_X(x)$? Why or why not?

10. Roll a die and flip a fair coin. Let X be the result of the die roll. Let Y be 0 if the coin shows a "tail" or 1 if the coin shows a "head.". Are

X and Y independent?

11. Let X indicate whether the first baby born to a certain mother is a girl, i.e., X = 1 if the first baby born is a girl; otherwise, X = 0. Let Y indicate whether the second baby born to a certain mother is a girl, i.e., Y = 1 if the second baby born is a girl; otherwise, Y = 0. (We are not considering the birth of twins, in which one baby's sex might affect the other.). are X and Y independent?

12. Roll a die. Let X be 1 if the outcome is 1, 3, or 5; let X be 0 otherwise. Let Y be 1 if the outcome is 5 or 6; let Y be 0 otherwise. Are X and Y independent?

13. Roll two dice. Let X denote the value of the first die, and let Y denote the value of the sum of the two dice. If we are given Y = 4, then calculate $p_{X|Y}(x|y)$.

14. Roll a die. Let X be 1 if the outcome is 1, 2, or 3; let X be 0 otherwise. Let Y be 1 if the outcome is even (2, 4, or 6); let Y be 0 otherwise. Are X and Y independent?

15. Ten students apply for a job opening, but only 1 of the students will be selected. The employer chooses randomly; all ten outcomes are equally likely. If person 3, 5, 7, or 9 gets the job, let X = 1; otherwise, X = 0. If person 1, 2, 3, 4, or 5 gets the job, let Y = 1; otherwise, Y = 0. Are X and Y independent random variables? Justify your answer.

16. A student flips a fair coin until a head appears. Let X be the number of flips until (and including) this first head. Afterwards, he begins flipping again until he gets another head. Let Y be the number of flips, after the first head, up to (and including) the second head. E.g., if the sequence of flips is TTTTTTHTTH then X = 7 and Y = 3. Are X and Y independent? Justify your answer.

3 Solutions

1. **Solution#1**. To find the marginal PMF, we sum over all the y's for every x:

$$x = 1 : p_X(1) = \sum_{y=1}^{4} P_{X,Y}(1, y) = \frac{1}{20} + \frac{1}{20} + \frac{1}{20} + \frac{0}{20} = \frac{3}{20},$$

$$x = 2 : p_X(2) = \sum_{y=2}^{4} P_{X,Y}(2, y) = \frac{1}{20} + \frac{2}{20} + \frac{2}{20} + \frac{1}{20} = \frac{6}{20},$$

$$x = 3 : p_X(3) = \sum_{y=3}^{4} P_{X,Y}(3, y) = \frac{1}{20} + \frac{3}{20} + \frac{3}{20} + \frac{1}{20} = \frac{8}{20},$$

$$x = 4 : p_X(4) = \sum_{y=4}^{4} P_{X,Y}(4, y) = \frac{0}{20} + \frac{1}{20} + \frac{1}{20} + \frac{1}{20} = \frac{3}{20}.$$

Hence, the marginal PMF is

$$p_X(x) = \left[\frac{3}{20}, \frac{6}{20}, \frac{8}{20}, \frac{3}{20}\right]$$

The conditional PMF $p_{X|Y}(x|1)$ is:

$$p_{X|Y}(x|1) = \frac{p_{X|Y}(x|1)}{p_Y(1)} = \frac{\left[\frac{3}{20}, \frac{6}{20}, \frac{8}{20}, \frac{3}{20}\right]}{\frac{3}{20}} = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right]$$

2. **Solution#2** Since Y takes two different states, we can enumerate Y $= 10^2$ and Y $= 10^4$. This gives us:

$$p_{X|Y}(x|10^2) = \begin{cases} \frac{1}{2}, & \text{if x=0.01,} \\ \frac{1}{3}, & \text{if x=0.1,} \\ \frac{1}{6}, & \text{if x=1.} \end{cases}$$

$$p_{X|Y}(x|10^4) = \begin{cases} \frac{1}{2}, & \text{if x=1,} \\ \frac{1}{3}, & \text{if x=10,} \\ \frac{1}{6}, & \text{if x=100.} \end{cases}$$

The joint PMF $p_{X,Y}(x, y)$ is:

$$p_{X,Y}(x, 10^2) = p_{X|Y}(x|10^2)p_Y(10^2) = \begin{cases} (\frac{1}{2})(\frac{5}{6}), & \text{if x=0.01,} \\ (\frac{1}{3})(\frac{5}{6}), & \text{if x=0.1,} \\ (\frac{1}{6})(\frac{5}{6}), & \text{if x=1.} \end{cases}$$

$$p_{X,Y}(x,10^4) = p_{X|Y}(x|10^4)p_Y(10^4) = \begin{cases} (\frac{1}{2})(\frac{1}{6}), & \text{if x=1,} \\ (\frac{1}{3})(\frac{1}{6}), & \text{if x=10,} \\ (\frac{1}{6})(\frac{1}{6}), & \text{if x=100.} \end{cases}$$

Therefore, the joint PMF is given by the following table.

$Y=10^4$	0	0	1/12	1/18	1/36
$Y=10^2$	5/12	5/18	5/36	0	0
	x=0.01	x=0.1	x=1	x=10	x= 100

The marginal PMF $p_X(x)$ is thus:

$$p_X(x) = \sum_y P_{X,Y}(x,y) = \left[\frac{5}{12}, \frac{5}{18}, \frac{2}{9}, \frac{1}{18}, \frac{1}{36}\right].$$

3. **Solution#3** To find the conditional expectation, we first need to know the conditional PMF.

$$p_{X|Y}(x|10^2) = \left[\frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 0, 0\right],$$

$$p_{X|Y}(x|10^4) = \left[0, 0, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}\right],$$

Therefore, the conditional expectations a

$$E[X|Y=10^2] = (10^{-2})\left(\frac{1}{2}\right) + (10^{-1})\left(\frac{1}{3}\right) + (1)\left(\frac{1}{6}\right) = \frac{123}{600},$$

$$E[X|Y=10^4] = (1)\left(\frac{1}{2}\right) + (10)\left(\frac{1}{3}\right) + (100)\left(\frac{1}{6}\right) = \frac{123}{6}$$

From conditional Expectation we can also fine $E[X]$:

$$E[X] = E[X|Y=10^2]p_Y(10^2) + E[X|Y=10^4]p_Y(10^4)$$
$$= \left(\frac{123}{600}\right)\left(\frac{5}{6}\right) + \left(\frac{123}{6}\right)\left(\frac{1}{6}\right) = 3.5875$$

4. **Solution#4** For any x and y, we have that:

$$p_{X,Y}(x,y) = \frac{1}{12}, And$$

$$p_X(x) * p_Y(y) = \frac{1}{2} * \frac{1}{6}$$

Therefore, the random variables X and Y are independent.

5. **Solution#5** The marginal PMF p_Y is given by the binomial formula

$$p_Y(y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}, y = 0, 1, 2, 3, 4.$$

To compute the conditional PMF $p_{X|Y}$, note that given that Y = y, X is the number of 1's in the remaining 4-y rolls, each of which can take the 5 values 1, 3, 4, 5, 6 with equal probability 1/5. Thus, the conditional PMF $p_{X|Y}$ is binomial with parameters 4-y and p = 1/5:

$$p_{X|Y}(x|y) = \binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x},$$

For all non-negative integers x and y such that $0 <= x + y <= 4$. The joint PMF is now given by:

$$p_{X,Y}(x, y) = p_Y(y)p_{X|Y}(x|y) = \binom{4}{y}\left(\frac{1}{6}\right)^y\left(\frac{5}{6}\right)^{4-y}\binom{4-y}{x}\left(\frac{1}{5}\right)^x\left(\frac{4}{5}\right)^{4-y-x},$$

for all non-negative integers x and y such that $0 <= x + y <= 4$. For other values of x and y, we have $p_{X,Y}(x, y) = 0$.

6. **Solution#6** The (unconditional) PMF of X is:

$$p_X(x) = \begin{cases} \frac{1}{4}, & \text{if x=0,} \\ \frac{1}{2}, & \text{if x=1,} \\ \frac{1}{4}, & \text{if x=2.} \end{cases}$$

And P(A)=1/2. The conditional PMF is obtained from the definition :

$$p_{X|A}(x) = \frac{P(X = x\ and\ A)}{P(A)}$$

$$p_{X|A}(x) = \begin{cases} \frac{1}{2}, & \text{if x=0,} \\ 0, & \text{if x=1,} \\ \frac{1}{2}, & \text{if x=2.} \end{cases}$$

Clearly, X and A are not independent, since the PMFs p_X and $p_{X|A}$ are different. For an example of a random variable that is independent of

A, consider the random variable that takes the value 0 if the first toss is a head, and the value 1 if the first toss is a tail. This is intuitively clear and can also be verified by using the definition of independence.

7. **Solution#7** By applying the formula of conditional PMF we obtain:

$$p_{X|A}(x) = P(X = x|\ roll\ is\ even) = \frac{P(X = x\ and\ X\ is\ even)}{P(roll\ is\ even)}$$

$$= \begin{cases} \frac{1}{3}, & \text{if k=2,4,6,} \\ 0, & \text{otherwise,} \end{cases}$$

8. **Solution#8** E[X] is easily calculated using the total expectation theorem:

$$E(X) = 0.5 \times 0.05 + 0.3 \times 0.1 + 0.2 \times 0.3 = 0.115$$

9. **Solution#9**

   (a) Here our Random Variable are:
   X = (maximum of two numbers) = {0,1}
   Y = (minimum of two numbers) = {0,1}

   $$P_{X,Y}(x, y) = P(\{X = x,\ Y = y\})$$

   As we know that X and Y are independent so:

   $$P_{X,Y}(0, 0) = P(\{X = x,\ Y = y\}) = P(\{x = 0\}) \times P(\{y = 0\}) = \frac{1}{4} \times \frac{3}{4} = \frac{3}{16}$$

   $$P_{X,Y}(0, 1) = P(\{x = 0\}) \times P(\{y = 1\}) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$$

   $$P_{X,Y}(1, 0) = P(\{x = 1\}) \times P(\{y = 0\}) = \frac{3}{4} \times \frac{3}{4} = \frac{9}{16}$$

   $$P_{X,Y}(1, 1) = P(\{x = 1\}) \times P(\{y = 1\}) = \frac{3}{4} \times \frac{1}{4} = \frac{3}{16}$$

   (b) For PMF of x $p_X(x)$:

   $$p_X(x) = \sum_y P_{X,Y}(x, y) = P_{X,Y}(x, 0) + P_{X,Y}(x, 1)$$

   $$p_X(0) = P_{X,Y}(0, 0) + P_{X,Y}(0, 1) = \frac{3}{16} + \frac{1}{16} = \frac{1}{4}$$

$$p_X(1) = P_{X,Y}(1,0) + P_{X,Y}(1,1) = \frac{9}{16} + \frac{3}{16} = \frac{3}{4}$$

So,

$$p_X(k) = \begin{cases} \frac{1}{4}, & \text{if k=0,} \\ \frac{3}{4}, & \text{if k=1,} \end{cases}$$

For PMF of y $= p_Y(y)$:

$$p_Y(y) = \sum_x P_{X,Y}(x,y) = P_{X,Y}(0,y) + P_{X,Y}(1,y)$$

$$p_Y(0) = P_{X,Y}(0,0) + P_{X,Y}(1,0) = \frac{3}{16} + \frac{9}{16} = \frac{3}{4}$$

$$p_Y(1) = P_{X,Y}(0,1) + P_{X,Y}(1,1) = \frac{1}{16} + \frac{3}{16} = \frac{1}{4}$$

So,

$$p_Y(k) = \begin{cases} \frac{3}{4}, & \text{if k=0,} \\ \frac{1}{4}, & \text{if k=1,} \end{cases}$$

(c) The definition of Conditional PMF:

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

So,

$$p_{X|Y}(0|0) = \frac{p_{X,Y}(0,0)}{p_Y(0)} = \frac{\frac{3}{16}}{\frac{3}{4}} = \frac{1}{4}$$

$$p_{X|Y}(1|0) = \frac{p_{X,Y}(1,0)}{p_Y(0)} = \frac{\frac{9}{16}}{\frac{3}{4}} = \frac{3}{4}$$

$$p_{X|Y}(0|1) = \frac{p_{X,Y}(0,1)}{p_Y(1)} = \frac{\frac{1}{16}}{\frac{1}{4}} = \frac{1}{4}$$

$$p_{X|Y}(1|1) = \frac{p_{X,Y}(1,1)}{p_Y(1)} = \frac{\frac{3}{16}}{\frac{1}{4}} = \frac{3}{4}$$

From the above calculations we can clearly see that:

$$p_{X,Y}(x|y) = p_X(x)$$

So therefore X and Y are independent.

10. **Solution#10** We know that

$$p_{X,Y}(x,y) = \frac{1}{12}$$

For all $1 <= x <= 6$ and $0 <= y <= 1$, since all twelve of these outcomes are equally likely,
Also:

$$p_X(x) = 1/6 \;\; for\ 1 <= x <= 6,$$
$$p_Y(y) = 1/2 \;\; for\ 1 <= y <= 2.$$

So $p_{X,Y}(x,Y) = p_X(x) * p_Y(y)$ Therefore X and Y are independent.

11. **Solution#11** Since, $p_{X,Y}(x,y) = \frac{1}{4}$ for $0 <= x, y <= 1$, and since X and Y each take values in the set $\{0,\ 1\}$, we can factor the joint mass $p_{X,Y}(x,y) = \frac{1}{4}$ into $\frac{1}{4} = \frac{1}{2} * \frac{1}{2}$, and we must have $p_X(x) = \frac{1}{2}$ for $x = 0, 1$ and $p_Y(y) = \frac{1}{2}$ for $y = 0, 1$. So X and Y are independent.

12. **Solution#12** Since

$$p_{X,Y}(0,0) = P(\{Outcome\ is\ 2,4\}) = \frac{2}{6}$$
$$p_{X,Y}(0,1) = P(\{Outcome\ is\ 6\}) = \frac{1}{6}$$
$$p_{X,Y}(1,0) = P(\{Outcome\ is\ 1,3\}) = \frac{2}{6}$$
$$p_{X,Y}(1,1) = P(\{Outcome\ is\ 5\}) = \frac{1}{6}$$

Thus $p_{X,Y}(x,y)$ can be factored as $p_{X,Y}(x,y) = p_X(x) * p_Y(y)$, by writing:

$$p_X(x) = \frac{1}{2} \quad for\ x = 0\ or\ x = 1,$$

And

$$p_Y(1) = \frac{1}{3} \quad and\ p_Y(0) = \frac{2}{3}$$

So X, Y are independent random variables.

13. **Solution#13** The set of possible outcomes are:

$$\{\{1,3\}, \{2,2\}, \{3,1\}\}$$

Thus X is either 1, 2, or 3. So the condition mass of X, given Y=4, is:

$$p_{X|Y}(1|4) = \frac{p_{X|Y}(1,4)}{p_Y(4)} = \frac{P(\{(1,3)\})}{P(\{(1,3),(2,2),(3,1)\})} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$$

$$p_{X|Y}(2|4) = \frac{p_{X|Y}(2,4)}{p_Y(4)} = \frac{P(\{(2,2)\})}{P(\{(1,3),(2,2),(3,1)\})} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$$

$$p_{X|Y}(3|4) = \frac{p_{X|Y}(3,4)}{p_Y(4)} = \frac{P(\{(3,1)\})}{P(\{(1,3),(2,2),(3,1)\})} = \frac{\frac{1}{36}}{\frac{3}{36}} = \frac{1}{3}$$

and $p_{X|Y}(x|y) = 0$ otherwise.

14. **Solution#14** To see that X and Y are dependent, first notice that $p_Y(1) = 1/2$ because the outcome is even with probability 1/2. On the other hand, if X = 1, i.e., if the outcome is 1, 2, or 3, then the outcome is even with probability 1/3. so $p_{Y|X}(1|1) = 1/3$. Thus $p_Y(y) = 1/2$. Thus $p_Y(y) = 1/2 \neq 1/3 = p_{Y|X}(y|x)$ when x and y are both equal to 1. So X and Y are not independent. In fact, X and Y are dependent.

15. **Solution#15**
    Yes, since

$$p_{X,Y}(1,1) = \frac{2}{10} = \frac{4}{10} * \frac{5}{10} = p_X(1) * p_Y(1)$$

16. **Solution#16**
    Yes. This makes intuitive sense since the number of flips to the first head has no bearing on the number of flips after the first head to the second head. Indeed, we have

$$p_{X,Y}(x,y) = (\frac{1}{2})^{x+y} = (\frac{1}{2})^x * (\frac{1}{2})^y = p_X(x) * p_Y(y)$$