Practice Problem Set 4

Probability and Statistics

1 Problems

1.1 Conditional PDF and Expectation, given an event

Problem 1

Let X be the random variable with PDF:

$$f_X(x) = \begin{cases} \frac{x}{4} & 1 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

Let A be an event $\{X \geq 2\}$.

1. Find $f_{X|A}(x)$
2. Find $E[X]$ and $E[X|A]$

Problem 2

Let X be the random variable with PDF:

$$f_X(x) = \begin{cases} cx^{-2} & 1 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

Let A be an event $\{X \geq 1.5\}$.

1. Find $f_{X|A}(x)$
2. Find $E[X^2|A]$

1.2 Exponential Continuous Random Variables

Problem 3

Consider an Exponential Random variable X with a mean of 10. Find the probability that $0.25 \leq X \leq 0.75$.

Problem 4

The average amount of time people spent shopping for anniversary cards is known to be 8 Minutes. The time spent here could be modeled via an exponential random variable. Compute the Probability Distribution.

1.3 Memorylessness of Exponential PDF

Problem 5

Suppose that the number of miles that a car can run before its battery wears out is exponentially distributed with an average value of 10,000 miles. If a person desires to take a 5000-mile trip, what is the probability that he or she will be able to complete the trip without having to replace the car battery? What can be said when the distribution is not exponential?

Problem 6

Consider a post office that is staffed by two clerks. Suppose that when Mr. Smith enters the system, he discovers that Ms. Jones is being served by one of the clerks and Mr. Brown by the other. Suppose also that Mr. Smith is told that his service will begin as soon as either Ms. Jones or Mr. Brown leaves. If the amount of time that a clerk spends with a customer is exponentially distributed with parameter $\lambda = \frac{1}{4}$, what is the probability that, of the three customers, Mr. Smith is the last to leave the post office?

1.4 CDF

Problem 7

Let X be a discrete random variable with range $R_X = \{1, 2, 3, ...\}$. Suppose the PMF of X is given by

$$P_x(k) = \frac{1}{2^k} for\ k = 1, 2, 3...$$

1. Find and plot the CDF of X, $F_X(x)$.
2. Find $P(2 < X \leq 5)$.
3. Find $P(X > 4)$.

1.5 Standard Normal RVs

Problem 8

Let $X \sim N(-5, 4)$.

1. Find $P(X < 0)$.
2. Find $P(-7 < X < 3)$.
3. Find $P(X > -3|X > -5)$.

1.6 Joint PDF

Problem 9

Let X and Y be two jointly continuous random variables with joint PDF

$$f_{XY}(x,y) = \begin{cases} x + cy^2 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

1. Find the constant c.
2. Find $P(0 \le X \le 12, 0 \le Y \le 12)$.

1.7 Joint CRV

Problem 10

Let X and Y be jointly continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx + 1 & x, y \ge 0, x + y < 1 \\ 0 & otherwise \end{cases}$$

1. Show the range of (X, Y), R_{XY}, in the $x - y$ plane.
2. Find the constant c.
3. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$.

1.8 PDF

Problem 11

Let X be a continuous random variable with PDF:

$$f_X(x) = \begin{cases} 4x^3 & 0 < x \le 1 \\ 0 & otherwise \end{cases}$$

Find $P(X \le \frac{2}{3} | X > \frac{1}{3})$.

1.9 Uniform PDF/CRV

Problem 12

Let X $Uniform(-pi2, \pi)$ and $Y = sin(X)$. Find $f_Y(y)$.

1.10 Expectation of CRV

Problem 13

Let X be a continuous random variable with PDF:

$$f_X(x) = \begin{cases} \frac{3}{x^4} & x \geq 1 \\ 0 & otherwise \end{cases}$$

Find the mean of X.

1.11 Variance of CRV

Problem 14

Let X be a continuous random variable with PDF:

$$f_X(x) = \begin{cases} x^2(2x + \frac{3}{2}) & 0 < x \leq 1 \\ 0 & otherwise \end{cases}$$

Find $Var(Y)$, if $Y = 2X + 3$.

2 Solutions

Problem 1 Solution

$$P(A) = \int_2^3 \frac{x}{4}\, dx = \frac{5}{8}$$

Since,

$$f_{X|A}(x) = \begin{cases} \frac{f_X(x)}{P(A)} & \text{if } x \in A \\ 0 & otherwise \end{cases}$$

Therefore,

$$f_{X|A}(x) = \begin{cases} \frac{2x}{5} & 2 \leq x \leq 3 \\ 0 & otherwise \end{cases}$$

$$E[X] = \int_1^3 \left(\frac{x}{2}\right)^2 dx = \frac{13}{6}$$

$$E[X|A] = \int_2^3 x \times \frac{2x}{5}\, dx = \frac{38}{15}$$

Problem 2 Solution

By Total Probability theorem,

$$\int_1^2 cx^{-2}\,dx = 1$$

Solving above, we get,

$$c = 2$$

$$P(A) = \int_{1.5}^2 2x^{-2}\,dx = \frac{1}{3}$$

Therefore,

$$f_{X|A}(x) = \begin{cases} 6x^{-2} & 1.5 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

$$E[X^2|A] = \int_{1.5}^2 x^2 \times 6x^{-2}\,dx = 3$$

Problem 3 Solution

$$E[X] = \frac{1}{\lambda} = 10$$

$$\implies \lambda = \frac{1}{10}$$

$$P(0.25 \leq X \leq 0.75) = P(X \geq 0.25) - P(X \geq 0.75) = e^{-\frac{1}{40}} - e^{-\frac{3}{40}} = 0.0476$$

Problem 4 Solution

Given that

$$E[X] = \frac{1}{\lambda} = 8$$

$$\implies \lambda = \frac{1}{8}$$

We know know that for exponential random variable,

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Therefore,

$$f_X(x) = \begin{cases} \frac{1}{8}e^{-\frac{x}{8}} & x \geq 0 \\ 0 & otherwise \end{cases}$$

Problem 5 Solution

It follows by the memoryless property of the exponential distribution that the remaining lifetime (in thousands of miles) of the battery is exponential with parameter $\lambda = \frac{1}{10}$. Hence the desired probability is,

$$P(\text{remaining lifetime} > 5) = 1 - F(5) = e^{-5\lambda} = e^{\frac{1}{2}} \approx 0.604$$

Problem 6 Solution

The answer is obtained by reasoning as follows: Consider the time at which Mr. Smith first finds a free clerk. At this point, either Ms. Jones or Mr. Brown would have just left, and the other one would still be in service. However, because the exponential is memoryless, it follows that the additional amount of time that this other person (either Ms. Jones or Mr. Brown) would still have to spend in the post office is exponentially distributed with parameter $\lambda = \frac{1}{4}$. That is, it is the same as if service for that person were just starting at this point. Hence, by symmetry, the probability that the remaining person finishes before Smith leaves must equal $\frac{1}{2}$.
We also have,

$$F(x) = P(X \leq x) = 1 - e^{-x/4}$$

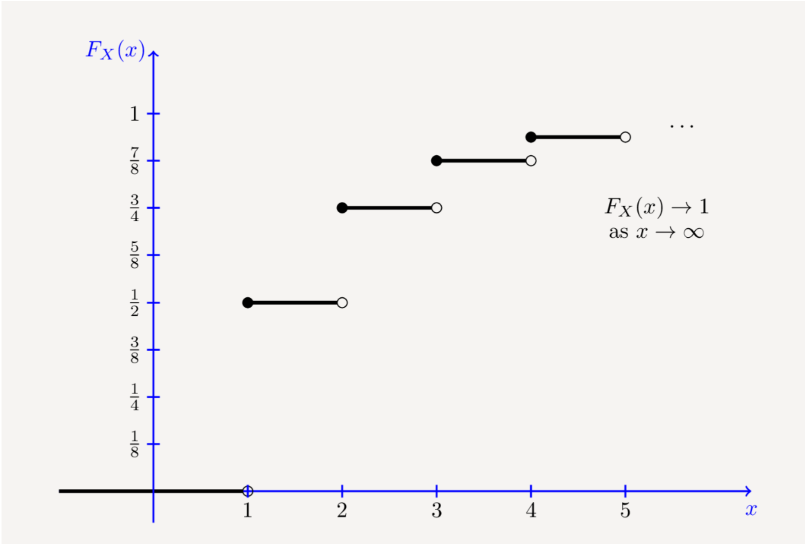
which shows that X is exponentially distributed.

Problem 7 Solution

First Note that this is a valid PMF:

$$\Sigma_{k=1}^{\infty} P_X(k) = \Sigma_{k=1}^{\infty} \frac{1}{2^k} = 1 \text{Geometric Sum}$$

1. To find the CDF, note that
   For $x < 1$, $F_X(x) = 0$
   For $1 \leq x < 2$, $F_X(x) = P_X(1) = \frac{1}{2}$
   For $2 \leq x < 3$, $F_X(x) = P_X(1) + P_X(2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$
   Based on this we can figure out the CDF to be:



2. To find $P(2 < X \leq 5)$, we can write

$$P(2 < X \leq 5) = F_X(5) - F_X(2) = \frac{31}{32} - \frac{3}{4} = \frac{7}{32}$$

We could also do this:

$$P(2 < X \leq 5) = P_X(3) + P_X(4) + P_X(5) = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} = \frac{7}{32}$$

3. To find $P(X > 5)$ we do:

$$P(X > 4) = 1 - P(X \leq 4) = 1 - F_X(4) = 1 - \frac{15}{16} = \frac{1}{16}$$

Problem 8 Solution

X is a normal random variable with $\mu = -5$ and $\sigma = \sqrt{4} = 2$, thus we have

1.
$$\begin{aligned} P(X < 0) &= F_X(0) \\ &= \Phi\left(\frac{0 - (-5)}{2}\right) \\ &= \Phi(2.5) \approx 0.99 \end{aligned}$$

2.
$$\begin{aligned} P(7 < X < 3) &= F_X(3) F_X(7) \\ &= \Phi\left(\frac{(-3) - (-5)}{2}\right) - \Phi\left(\frac{(-7) - (-5)}{2}\right) \end{aligned}$$

$$= \Phi(1) - \Phi(-1)$$
$$= 2\Phi(1) - 1 \text{ since } \Phi(-x) = 1 - \Phi(x)$$
$$\approx 0.68$$

3.
$$P(X > -3) | X > -5) = \frac{P(X > -3), P(X > -5)}{P(X > -5)}$$
$$= \frac{P(X > -3)}{P(X > -5)}$$
$$= 1 - \frac{P(X > -3)}{P(X > -5)}$$
$$= \frac{1 - \Phi\left(\frac{(-3)-(-5)}{2}\right)}{1 - \Phi\left(\frac{(-5)-(-5)}{2}\right)}$$
$$= \frac{1 - \Phi(1)}{1 - \Phi(0)}$$
$$\approx \frac{0.1587}{0.5} \approx 0.32$$

Problem 9 Solution

1. To find c we use:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} = f_{XY}(x, y) dx dy = 1$$

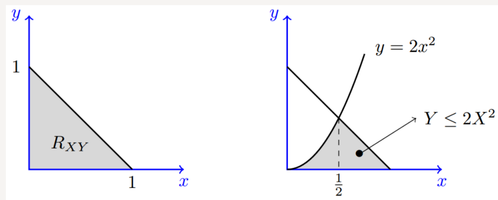
Thus we have,

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} = f_{XY}(x, y) dx dy$$
$$= \int_0^1 \int_0^1 = x + cy^2 dx dy$$
$$= \int_0^1 \left[\frac{1}{2}x^2 + cy^2 x\right]_{x=0}^{x=1} dy$$
$$= \int_0^1 \frac{1}{2} + cy^2 dy$$
$$= \left[\frac{1}{2}y + \frac{1}{3}cy^3\right]_{y=0}^{y=1}$$
$$1 = \frac{1}{2} + \frac{1}{3}c$$

Therefore $c = \frac{3}{2}$

Problem 10 Solution

1. 1. Figure 5.8(a) shows R_{XY} in the $x-y$ plane.

   The figure shows (a) R_{XY} as well as (b) the integration region for finding $P(Y < 2X^2)$ for Solved Problem 1.

2. To find c we use:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} = f_{XY}(x, y)dxdy = 1$$

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} = f_{XY}(x, y)dxdy$$

$$= \int_{0}^{1} \int_{0}^{1-x} = cx + 1dydx$$

$$= \int_{0}^{1} (cx + 1)(1 - x)dx$$

$$1 = \frac{1}{2} + \frac{1}{6}c$$

Therefore we conclude that $c = 3$

3. we note that $R_X = R_Y = [0, 1]$.

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)dy$$

$$= \int_{0}^{1-x} 3x + 1dy$$

$$= (3x + 1)(1 - x), \text{ for } x \in [0, 1]$$

Thus we have,

$$f_X(x) = \begin{cases} (3x + 1)(1 - x) & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$$

Similarly we can obtain marginal $f_Y(y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)dx$$
$$= \int_0^{1-y} 3x + 1dx$$
$$= \frac{1}{2}(5-3y)(1-y), \text{ for } x \in [0,1]$$

Thus we have,

$$f_Y(y) = \begin{cases} \frac{1}{2}(5-3y)(1-y) & 0 \le y \le 1 \\ 0 & otherwise \end{cases}$$

Problem 11 Solution

$$P(X \le \frac{2}{3}|X > \frac{1}{3}) = \frac{P(13 < X \le \frac{2}{3})}{P(X > \frac{1}{3})}$$
$$= \frac{\int_{\frac{1}{3}}^{\frac{2}{3}} 4x^3 dx}{\int_{\frac{1}{3}}^{1} 4x^3 dx}$$
$$= 316$$

Problem 12 Solution

Here $Y = g(X)$, where g is a differentiable function. Although g is not monotone, it can be divided to a finite number of regions in which it is monotone. We note that since $R_X = [-\pi 2, \pi]$, $R_Y = [-1,1]$. By looking at the plot of $g(x) = sin(x)$ over $[-\frac{\pi}{2}, \pi]$, we notice that for $y \in (0,1)$ there are two solutions to $y = g(x)$, while for $y \in (-1,0)$, there is only one solution. In particular, if $y \in (0,1)$, we have two solutions: $x_1 = arcsin(y)$, and $x_2 = \pi - arcsin(y)$. If $y \in (-1,0)$ we have one solution, $x_1 = arcsin(y)$. Thus, for $y \in (-1,0)$, we have:

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|}$$
$$= \frac{f_X(arcsin(y))}{|cos(arcsin(y))|}$$
$$= \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}}$$

For $y \in (0,1)$, we have

$$f_Y(y) = \frac{f_X(x_1)}{|g'(x_1)|} + \frac{f_X(x_2)}{|g'(x_2)|}$$

$$= \frac{f_X(arcsin(y))}{|cos(arcsin(y))|} + \frac{f_X(\pi - arcsin(y))}{|cos(\pi - arcsin(y))|}$$
$$= \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}} + \frac{\frac{2}{3\pi}}{\sqrt{1-y^2}}$$
$$= \frac{4}{3\pi\sqrt{1-y2}}$$

To summarize:

$$f_Y(y) = \begin{cases} \frac{2}{3\pi\sqrt{1-y^2}} & -1 < y < 0 \\ \frac{4}{3\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & otherwise \end{cases}$$

Problem 13 Solution

$$E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$
$$= \int_1^{\infty} \frac{3}{x^3} dx$$
$$= [-\frac{3}{2}x]_1^{\infty}$$
$$= \frac{3}{2}$$

Problem 14 Solution

$$Var(Y) = Var(\frac{2}{X} + 3) = 4Var(\frac{1}{X})$$
$$Var(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2$$
$$E[\frac{1}{X}] = \int 10x(2x+32)dx = 1712$$
$$E[\frac{1}{X^2}] = \int 10(2x+32)dx = 52$$

Thus,

$$Var(\frac{1}{X}) = E[\frac{1}{X^2}] - (E[\frac{1}{X}])^2 = 71144$$

So, we obtain

$$Var(Y) = 4Var(\frac{1}{X}) = 7136$$