Quantum Mechanics Assignment 3

Muhammad Meesum Ali Qazalbash - mq06861

1. $\boxed{\text{15 points}}$ Consider the wave function:

$$\Psi(x,t) = Ae^{-\lambda|x|}e^{-i\omega t}$$

where A, λ and ω are positive real numbers.

1. Normalize Ψ.
2. Determine the expression values of x and x^2.
3. Find the standard deviation of x. Sketch the graph of Ψ as a function of x, and mark the points $\langle\langle x\rangle + \sigma\rangle$ and $\langle\langle x\rangle - \sigma\rangle$ to to illustrate the sense in which σ represent the s̈preadöf x. What is the probability that the particle would be found outside this range?

Solution: We would first find the coefficient A for normalization,

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \\
\iff &\int_{-\infty}^{\infty} \Psi(x,t)\Psi^*(x,t) dx = 1 \\
\iff &\int_{-\infty}^{\infty} \left(Ae^{-\lambda|x|}e^{-i\omega t}\right)\left(Ae^{-\lambda|x|}e^{i\omega t}\right) dx = 1 \\
\iff &\int_{-\infty}^{\infty} \left(Ae^{-\lambda|x|}\right)\left(Ae^{-\lambda|x|}\right) dx = 1 \\
\iff &A^2 \int_{-\infty}^{\infty} e^{-2\lambda|x|} dx = 1 \\
\iff &A^2 \int_{-\infty}^{0} e^{2\lambda x} dx + A^2 \int_{0}^{\infty} e^{-2\lambda x} dx = 1 \\
\iff &\frac{A^2}{2\lambda} e^{2\lambda x}\Big|_{-\infty}^{0} - \frac{A^2}{2\lambda} e^{-2\lambda x}\Big|_{0}^{\infty} = 1 \\
\implies &\frac{A^2}{\lambda} = 1 \\
\implies &A = \sqrt{\lambda}
\end{aligned}$$

Therefore the normalized wavefunction is,

$$\Psi(x,t) = \sqrt{\lambda} e^{-\lambda|x| - i\omega t}$$

The probability density function for a given wave function is defined as,

$$P[X = x] = |\Psi(x,t)|^2$$

From the analysis in the previous part,

$$P[X = x] = \lambda e^{-2\lambda|x|}$$

The expectation of x is therefore given by,

$$\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} x P[X = x] dx \\
&= \int_{-\infty}^{\infty} x \lambda e^{-2\lambda|x|} dx \\
&= \int_{-\infty}^{0} x \lambda e^{2\lambda x} dx + \int_{0}^{\infty} x \lambda e^{-2\lambda x} dx \\
&= -\frac{1}{4\lambda} + \frac{1}{4\lambda} \\
&= 0
\end{aligned}$$

And the expectation of x^2 is given by;

$$\begin{aligned}
\langle x^2 \rangle &= \int_{-\infty}^{\infty} x^2 P[X = x] dx \\
&= \int_{-\infty}^{\infty} x^2 \lambda e^{-2\lambda|x|} dx \\
&= \int_{-\infty}^{0} x^2 \lambda e^{2\lambda x} dx + \int_{0}^{\infty} x^2 \lambda e^{-2\lambda x} dx \\
&= \frac{1}{4\lambda^2} + \frac{1}{4\lambda^2} \\
&= \frac{1}{2\lambda^2}
\end{aligned}$$

The plot for the probability density function is as follows:

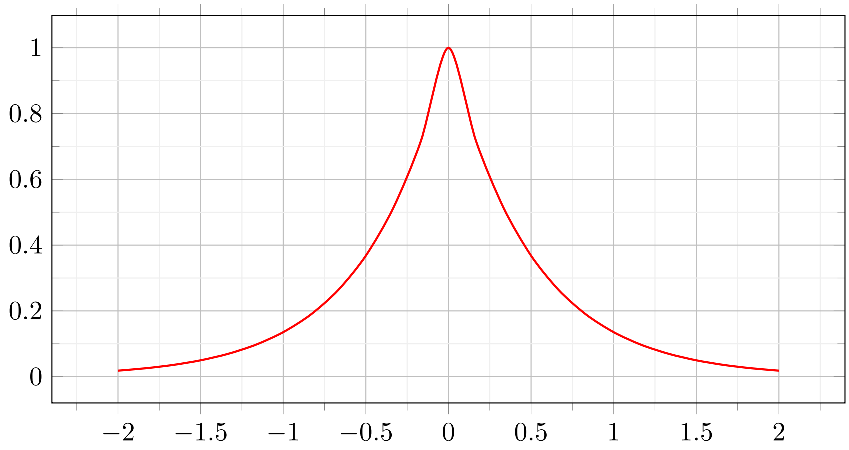

Figure 1: Probability Distribution function

The standard deviation for $P[X = x]$ is given as,

$$\begin{aligned}\sigma &= \sqrt{\langle x^2\rangle - \langle x\rangle^2} \\ &= \sqrt{\frac{1}{2\lambda^2} - 0} \\ &= \frac{1}{\sqrt{2}\lambda}\end{aligned}$$

According to the plot attached above, the probability that the particle would be found outside this range can be calculated as;

$$\begin{aligned}P[|X - \langle x\rangle| > \sigma] &= \left(\int_{-\infty}^{\langle x\rangle-\sigma} + \int_{\langle x\rangle+\sigma}^{\infty}\right) |\Psi(x,t)|^2 dx \\ &= \lambda\int_{-\infty}^{-\frac{1}{\sqrt{2}\lambda}} e^{2\lambda x}dx + \lambda\int_{\frac{1}{\sqrt{2}\lambda}}^{\infty} e^{-2\lambda x}dx \\ &= \frac{e^{-\sqrt{2}}}{2} + \frac{e^{-\sqrt{2}}}{2} \\ &= e^{-\sqrt{2}} \approx 0.243\end{aligned}$$

2. $\boxed{\text{15 points}}$ At the time $t = 0$, the particle waver function is represented by :

$$\Psi(x,0) = \begin{cases} Ax/a & \text{if } 0 \leq x \leq a \\ A(b-x)/(b-a) & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

where A, a, and b are constants.

1. Normalise Ψ, that is A in terms of a and b.
2. Sketch $\Psi(x,0)$ as a function of x.
3. Where is the particle most likely to be found at $t = 0$?
4. what is the probability of finding the particle to the left of a ? Check your results in the limiting cases when $b = a$ and $b = 2a$.
5. What is the expectation value of x ?

Solution:

We would first find the coefficient A for normalization,

$$\begin{aligned}
&\int_{-\infty}^{\infty} |\Psi(x,t)|^2 dx = 1 \\
\iff &\left(\int_{-\infty}^{0} + \int_{0}^{a} + \int_{a}^{b} + \int_{b}^{\infty}\right) |\Psi(x,t)|^2 dx = 1 \\
\iff &\int_{0}^{a} |\Psi(x,t)|^2 dx + \int_{a}^{b} |\Psi(x,t)|^2 dx = 1 \\
\iff &\int_{0}^{a} \left|A\frac{x}{a}\right|^2 dx + \int_{a}^{b} \left|A\frac{b-x}{b-a}\right|^2 dx = 1 \\
\iff &\frac{A^2}{a^2}\int_{0}^{a} x^2 dx + \frac{A^2}{(b-a)^2}\int_{a}^{b} (b-x)^2 = 1 \\
\iff &A^2\left(\frac{a^3}{3a^2} + \frac{(b-a)^3}{3(b-a)^2}\right) = 1 \\
\iff &A^2\left(\frac{a+b-a}{3}\right) = 1 \\
\iff &A = \sqrt{\frac{3}{b}}
\end{aligned}$$

The Schrodinger equation at t = 0 is given by;

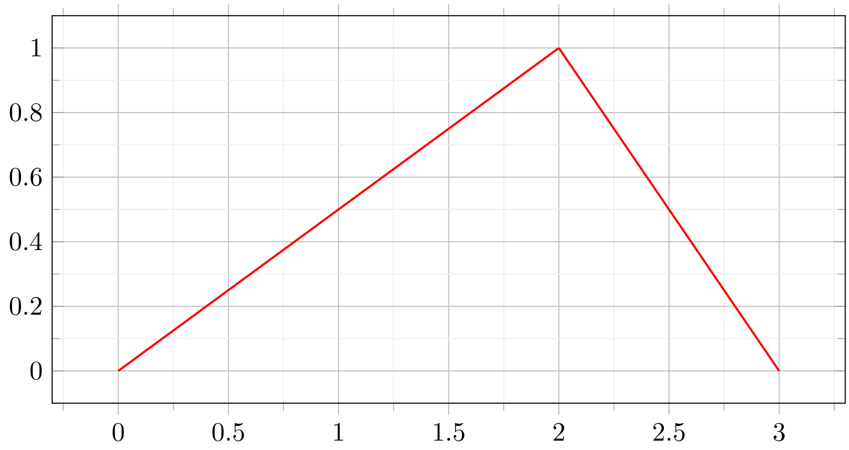

Figure 2: Probability Distribution function

The plot for the probability density function is as follows:

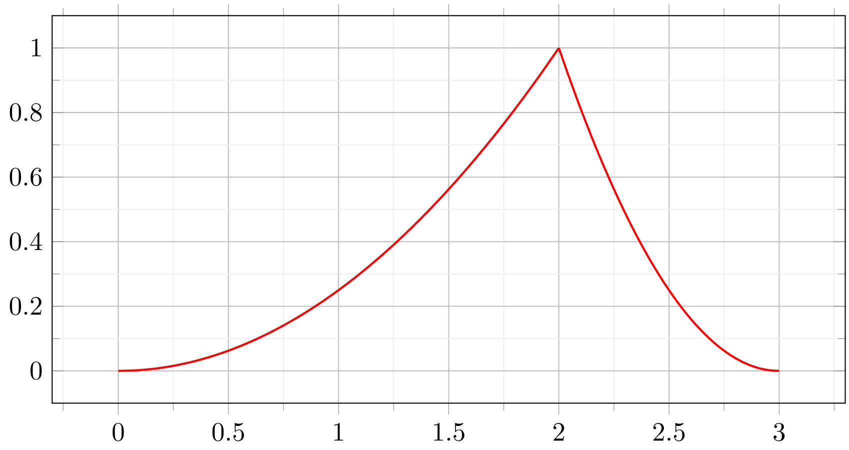

Figure 3: Probability Distribution function

It can be seen in the graph attached above that at $t = 0$, the particle is most likely to be found at $x = a$. The probability of finding the particle to the left of a can be given as;

$$\begin{aligned} P[X < a] &= \int_{-\infty}^{a} |\Psi(x,t)|^2 dx \\ &= \int_0^a \left(\sqrt{\frac{3}{b}}\frac{x}{a}\right)^2 dx \\ &= \frac{a}{b} \end{aligned}$$

When $b = a$, $P[X < a] = 1$ and when $b = 2a$, $P[X < a] = 0.5$. The expectation of x can be found as;

$$
\begin{aligned}
\langle x \rangle &= \int_{-\infty}^{\infty} xP[X = x]dx \\
&= \int_0^a x\left(\sqrt{\frac{3}{b}}\frac{x}{a}\right)^2 dx + \int_a^b x\left(\sqrt{\frac{3}{b}}\frac{b-x}{b-a}\right)^2 dx \\
&= \frac{3}{a^2b}\frac{a^4}{4} - \frac{3}{b(b-a)^2}\left(\frac{b^4}{12} - \frac{6a^2b^2 - 8a^3b + 3a^4}{12}\right) \\
&= \frac{3a^2}{4b} + \frac{3}{b(b-a)^2}\frac{(b-a)^3(b+3a)}{12} \\
&= \frac{3a^2}{4b} + \frac{3(b-a)(b+3a)}{12b} \\
&= \frac{9a^2 + 3b^2 + 6ab - 9a^2}{12b} \\
&= \frac{6a + 3b}{12} \\
&= \frac{2a + b}{4}
\end{aligned}
$$