Probability & Statistics: Practice Questions
Spring 2022

Simple Probability Mass Function

Question 1
Let S be a set of integers and $g(x)$ defined as :

$$g(x) = \begin{cases} k(7x+3) & x = 1, 2, 3...7 \\ 0 & otherwise \end{cases}$$

For what value of k is $g(x)$ a valid PMF?

Question 2
Determine the constant c for which the random variable Y is a valid PMF.

$$f(y) = \begin{cases} c(\frac{1}{4})^y & y = 1, 2, 3, ... \\ 0 & otherwise \end{cases}$$

Bernoulli RV and Binomial RV

Question 3

(a) If a Bernoulli distribution has a parameter 0.45 then find its mean.

(b) A basketball player can shoot a ball into the basket with a probability of 0.6. What is the probability that he misses the shot?

(c) If a Bernoulli distribution has a parameter 0.72 then find its variance.

Question 4
Let X = number of heads after a coin is flipped three times. $X \sim Bin(3, 0.5)$. What is the probability of each of the different values of X?

Question 5
When sending messages over a network, there is a chance that the bits will be corrupted. A Hamming code allows for a 4 bit code to be encoded as 7 bits, with the advantage that if 0 or 1 bit(s) are corrupted, then the message can be perfectly reconstructed. You are working on the Voyager space mission and the probability of any bit being lost in space is 0.1. How does reliability change when using a Hamming code?

Geometric RV

Question 6
You draw cards, one at a time, with replacement (i.e., placing them randomly back into the deck after they are drawn), from a shuffled, standard deck of playing cards. Let be the number of cards that are drawn to get the first heart that appears.

(a) How many cards do you expect to draw, to see the first heart? Also find the variance of the number of cards.

(b) Now suppose that you draw five cards (again, with replacement), and none of them are hearts. How many additional cards (not including the first five) do you expect to draw to see the first heart?

Question 7
During a bad economy, a graduating ECE student goes to career fair booths in the technology sector (e.g., Google, Apple, Qualcomm, Texas Instruments, Motorola, etc) - and his/her likelihood of receiving an off-campus interview invitation after a career fair booth visit depends on how well he/she did in ECE 313. Specifically, an A in 313 results in a probability p = 0.95 of obtaining an invitation, whereas a C in 313 results in a probability of p = 0.15 of an invitation.

(a) Give the pmf for the random variable Y that denotes the number of career fair booth visits a student must make before his/her first invitation including the visit that results in the invitation. Express your answer in terms of p.

(b) On average, how many booth visits must an A student make before getting an off-campus interview invitation? How about a C student?

__Poisson RV__

Question 8
Suppose the number of hits a web site receives in any time interval is a Poisson random variable. A particular site gets on average 5 hits per second.

(a) What is the probability that there will be no hits in an interval of two seconds?

(b) What is the probability that there is at least one hit in an interval of one second?

Question 9
Approximately 6.85 left-handed people are killed each day by using an object or machinery designed for right-handed people. Let X be the number of left-handed people killed this way in one day.

(a) What is the probability that exactly 7 left-handed people will be killed using a right-handed object tomorrow? What is the standard deviation for the number of left-handed people killed using a right-handed object tomorrow?

(b) What is the expected number of left-handed people killed using a right-handed object over the next week? What is the standard deviation of the number of left-handed people killed using a right-handed object over the next week?

__Uniform RV__

Question 10
Let $Z = 3W + 11$ where is uniformly distributed on the interval [0,5]. Find the Expected value of Z.

Question 11
X is a random variable with a uniform PMF in the interval [-2, 10]. Find $P(Y \leq 7|X \geq 5)$ given $Y = 2X - 6$.

__Functions of RV__

Question 12
Let the random variable X be the outcome of the roll of the first six sided die and random variable Y be the outcome of the roll of the second die which is four sided. Consider a new random variable $Z = X + Y$. Evaluate

1. Evaluate the PMF of Z.
2. Expected value of Z.
3. $P(Z \geq 6)$

Question 13
Let Y_n be a random variable such that:

$$Y_n = min(X_1, X_2, X_3, \dots)$$

Where X_i represent the outcome of a six-sided die in the i^{th} throw. Find:

1. $P(Y_6 > 2)$
2. $P(Y_2 < 4)$

3. $P(Y_4 = 3)$

Variance

Question 14
The Expected value of a random variable X is given by $E[X] = X^2 + 2X$. Find the variance of the random variable i.e., $var(X)$.
Question 15
Find the variance of:

- Z from Question 12
- Y_3 from Question 13

Joint PMF

Question 16
Two six-sided dice are rolled .Consider a random variable X which is the outcome of a throw of the first six-sided die and Z being the outcome of the throw of the second die. Consider another random variable $Y = (X + Z)\%3$ [(X+Z) mod 3]. Draw a Table for the Joint PMF of X and Y and determine $p_{x,y}(2, 6)$
Question 17
Consider the following given Joint PMF of two random variables X and Y.

Y \ X	1	2	3	4
1	$3/50$	$5/50$	$2/50$	$4/50$
2	$5/50$	$4/50$	$3/50$	$6/50$
3	$4/50$	$1/50$	$1/50$	$3/50$
4	$1/50$	$2/50$	$5/50$	$1/50$

1. Determine whether the given Joint PMF is valid or not.
2. If it is a valid PMF, then consider a function $f(X, Y) = X - Y$. Compute the Expected value of $f(X, Y)$.

Solutions

Solution 1

For a valid PMF first we know that $f(x) \geq 0$, so k is non-negative. Secondly, we know that $\sum_{x \in S} f(x) = 1$. Therefore:

$$\sum_{x=1}^{7} f(x) = 1$$

$$f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + f(7) = 1$$

$$k(7+3) + k(14+3) + k(21+3) + k(28+3) + k(35+3) + k(42+3) + k(49+3) = 1$$

$$10k + 17k + 24k + 31k + 38k + 45k + 52k = 1$$

$$217k = 1$$

$$k = \frac{1}{217}$$

Solution 2

For a valid PMF first we know that $f(y) \geq 0$, so c is non-negative. Secondly, we know that $\sum_{y=1}^{\infty} f(y) = 1$. Therefore:

$$\sum_{y=1}^{\infty} f(y) = 1$$

$$\sum_{y=1}^{\infty} c(\frac{1}{4})^y = 1$$

$$c \times \sum_{y=1}^{\infty} (\frac{1}{4})^y = 1$$

$$c \times \left(\frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^4} + ... \right)$$

It can be seen that the integers follow a geometric series so we can use the formula for sum to infinity :

$$S = \frac{a}{1-r} \text{ where } a = \frac{1}{4}, r = \frac{1}{4}$$

$$c \times \left(\frac{a}{1-r} \right) = 1$$

$$c \times \left(\frac{\frac{1}{4}}{1 - \frac{1}{4}} \right) = 1$$

$$c \times \left(\frac{\frac{1}{4}}{\frac{3}{4}} \right) = 1$$

$$c \times \frac{1}{3} = 1$$

$$c = 3$$

Solution 3

(a) $X \sim Bernoulli(p)$ or $X \sim Bernoulli(0.45)$.

$$E[X] = p$$

Therefore:

$$E[X] = 0.45$$

(b) We know that success probability $P(X = 1) = p = 0.6$.
Thus, probability of failure is $P(X = 0) = 1 - p = 1 - 0.6 = 0.4$. The probability of failure of the Bernoulli distribution is 0.4.

(c) $X \sim Bernoulli(p)$ or $X \sim Bernoulli(0.72)$. Variance $Var[X] = p(1 - p) = 0.72(0.28) = 0.2016$
Therefore, Variance = 0.2016

Solution 4

A coin is flipped three times so we have 3! possibilities, where we can get heads either 0, 1, 2, or 3 times. Therefore:

$$P(X = 0) = \binom{3}{0} p^0 (1 - p)^3 = \frac{1}{8}$$

$$P(X = 0) = \binom{3}{1} p^1 (1 - p)^2 = \frac{3}{8}$$

$$P(X = 0) = \binom{3}{2} p^2 (1 - p)^1 = \frac{3}{8}$$

$$P(X = 0) = \binom{3}{3} p^3 (1 - p)^0 = \frac{1}{8}$$

Solution 5

Image we use error correcting codes. Let $X \sim Bin(7, 0.1)$.

$$P(X = 0) = \binom{7}{0} (0.1)^0 (0.9)^7 \approx 0.468$$

$$P(X = 0) = \binom{7}{1} (0.1)^1 (0.9)^6 = 0.372$$

$$P(X = 0) + P(X = 1) = 0.850$$

What if we didn't use error correcting codes? Let $X \sim Bin(4, 0.1)$.

$$P(X = 0) = \binom{4}{0} (0.1)^0 (0.9)^4 \approx 0.656$$

Using Hamming Codes improves reliability by about 30%!

Solution 6

(a) We keep on playing till we win so this is a Geometric Random Variable. Since there are 13 hearts, $p = 0.25$

$$X \sim Geo(0.25)$$

$$E[X] = \frac{1}{p} = 4$$

$$Var(X) = \frac{1 - p}{p^2} = \frac{0.75}{0.25^2}$$

(b) If we get failure on first 5, doesn't matter. Due to independence (memorylessness property). The additional number of cards we are expected to draw will still be 4.

Solution 7

(a)

$$pY(k) = p(1 - p)^{k-1}$$

for $k \geq 1$

(b)

$$E[Y] = \frac{1}{p} = \begin{cases} 1.0526 & \text{A student} \\ 6.6667 & \text{C student} \end{cases}$$

Solution 8

(a)

$$X \sim Poisson(2 \times 5)$$

$$P(X = k) = e^{-10}\frac{10k}{k!}$$

$$P(X = 0) = e^{-10}\frac{100}{0!} = e^{-10}$$

(b)

$$\lambda = 5$$

$$X \sim Poisson(5)$$

$$P(X \geq 1) = 1 - P(X = 0) = 1 - e^{-5}\frac{5^0}{0!} = 1 - e^{-5}$$

Solution 9

(a) This is Poisson Random Variable since things have a rate at which they happen.

$$X \sim Poisson(6.85)$$

$$P(X = 7) = \frac{6.85^7 e^{-6.85}}{7!}$$

$$st.dev(X) = \sqrt{Var(X)} = \sqrt{6.85}$$

(b) We can define a new random variable for one week.

$$Y \sim Poisson(6.85 \times 7)$$

$$E[Y] = 6.85 \times 7$$

$$st.dev(Y) = \sqrt{Var(Y)} = \sqrt{7 \times 6.85}$$

Solution 10

$$E[W] = \frac{a+b}{2} = \frac{0+5}{2} = 2.5$$

$$E[Z] = 2 \times E[W] + 11 = 2 \times 2.5 + 11 = 16$$

Solution 11

The area under the curve must be 1. We know that the curve makes a rectangle in case of uniform rv, so using the area of a rectangle formula:

$$(10 + 2) \times h = 1$$

$$h = \frac{1}{12}$$

$$f_x(x) = \begin{cases} \frac{1}{12} & -2 \le x \le 10 \\ 0 & \text{otherwise} \end{cases}$$

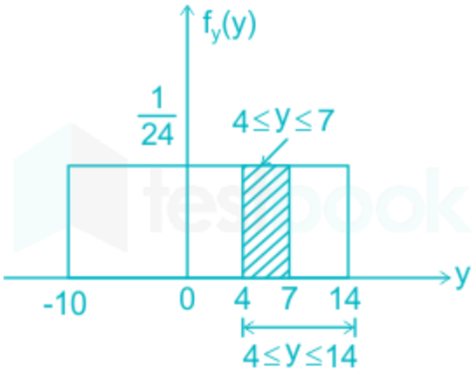
Now, y = 2x-6

$$f_y(Y) = \begin{cases} \frac{1}{24} & -10 \le x \le 14 \\ 0 & \text{otherwise} \end{cases}$$

If $x \ge 5$, then $y \ge 4$

So,

$$P(Y \le 7 | X \ge 5) = P(Y \le 7 | Y \ge 4)$$



$$P(Y \le 7 | X \ge 5) = \frac{P(Y \le 7 \cap Y \ge 4)}{P(4 \le Y \le 14)} = \frac{P(4 \le Y \le 7)}{P(4 \le Y \le 14)}$$

$$= \frac{3}{10} = 0.3$$

Solution 12
Part 1

$$p_Z(2) = p_X(1)p_Y(1) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

$$p_Z(3) = p_X(1)p_Y(2) + p_X(2)p_Y(1) = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{18}$$

$$p_Z(4) = p_X(1)p_Y(3) + p_X(3)p_Y(1) + p_X(2)p_Y(2) = \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{12}$$

$$p_Z(5) = p_X(1)p_Y(4) + p_X(4)p_Y(1) + p_X(2)p_Y(3) + p_X(3)p_Y(2) = \frac{4}{36} = \frac{1}{9}$$

$$p_Z(6) = p_X(1)p_Y(5) + p_X(5)p_Y(1) + p_X(2)p_Y(4) + p_X(4)p_Y(2) + p_X(3)p_Y(3) = \frac{5}{36}$$

$$p_Z(7) = p_X(1)p_Y(6) + p_X(6)p_Y(1) + p_X(2)p_Y(5) + p_X(5)p_Y(2) + p_X(3)p_Y(4) + p_X(4)p_Y(3) = \frac{6}{36} = \frac{1}{6}$$

$$p_Z(8) = p_X(2)p_Y(6) + p_X(6)p_Y(2) + p_X(3)p_Y(5) + p_X(5)p_Y(3) + p_X(4)p_Y(4) = \frac{5}{36}$$

$$p_Z(9) = p_X(3)p_Y(6) + p_X(6)p_Y(3) + p_X(4)p_Y(5) + p_X(5)p_Y(4) = \frac{4}{36} = \frac{1}{9}$$

$$p_Z(10) = p_X(4)p_Y(6) + p_X(6)p_Y(4) + p_X(5)p_Y(5) = \frac{3}{36} = \frac{1}{12}$$

$$p_Z(11) = p_X(5)p_Y(6) + p_X(6)p_Y(5) = \frac{2}{36} = \frac{1}{18}$$

$$p_Z(12) = p_X(6)p_Y(6) = \frac{1}{36}$$

Part 2

$$\begin{aligned} E[Z] = &\left(2 \times \frac{1}{36}\right) + \left(3 \times \frac{1}{18}\right) + \left(4 \times \frac{1}{12}\right) + \left(5 \times \frac{1}{9}\right) + \left(6 \times \frac{5}{36}\right) + \left(7 \times \frac{1}{6}\right) \\ &+ \left(8 \times \frac{5}{36}\right) + \left(9 \times \frac{1}{9}\right) + \left(10 \times \frac{1}{12}\right) + \left(11 \times \frac{1}{18}\right) + \left(12 \times \frac{1}{36}\right) \end{aligned}$$

$$E[Z] = 7$$

Part 3

$$P(Z \geq 6) = p_Z(2) + p_Z(3) + p_Z(4) + p_Z(5) + p_Z(6) = \frac{5}{12}$$

Solution 13

Part 1

$P(Y_6 > 2)$ refers to the probability that we get a minimum greater than 2 in 6 rolls of the die. This means that in all the 6 throws we can have a number greater than 2, i.e. 4 possibilities (3,4,5,6).

$$\implies P(Y_6 > 2) = \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6} \times \frac{4}{6}. = \left(\frac{4}{6}\right)^6$$

Part 2

$$P(Y_2 < 4) = 1 - P(Y_2 \geq 4) = 1 - P(Y_2 > 3)$$

$$P(Y_2 < 4) = 1 - \left(\frac{3}{6}\right)^2$$

Part 3

We can break $P(Y_4 = 3)$ into $P(Y_4 > 2) - P(Y_4 > 3)$. Hence, we have:

$$P(Y_4 = 3) = \left(\frac{4}{6}\right)^4 - \left(\frac{3}{6}\right)^4$$

Solution 14

Given that,

$$E[X] = X^2 + 2X$$

$$E[X^2] = (X^2)^2 + 2(X^2)$$
$$E[X^2] = X^4 + 2X^2$$
$$\text{Let } \mu = E[X] = X^2 + 2X$$

$$\because var(X) = E[(X - \mu)^2]$$

$$\therefore var(X) = E[(X - (X^2 + 2X))^2]$$
$$var(X) = E[(X - X^2 - 2X)^2]$$
$$var(X) = E[(X^2 - X)^2]$$
$$var(X) = E[X^4 - 2X^3 + X^2]$$
$$var(X) = (X^4 - 2X^3 + X^2)^2 + 2(X^4 - 2X^3 + X^2)$$

Solution 15

Part 1

$$\because var(X) = E[X^2] - (E[X])^2$$

$$E[Z^2] = \sum (x+y)^2 p_Z(x,y) = (1+1)^2 p_Z(1,1) + (1+2)^2 p_Z(1,2) + \cdots + (6+6)^2 p_Z(6,6)$$
$$E[Z^2] = 63.722$$
$$\implies var(X) = 63.722 - 7^2 = 14.722$$

Part 2

$$P(Y_3 = 1) = \left(\frac{6}{6}\right)^3 - \left(\frac{5}{6}\right)^3 = \frac{91}{216}$$

$$P(Y_3 = 2) = \left(\frac{5}{6}\right)^3 - \left(\frac{4}{6}\right)^3 = \frac{61}{216}$$

$$P(Y_3 = 3) = \left(\frac{4}{6}\right)^3 - \left(\frac{3}{6}\right)^3 = \frac{37}{216}$$

$$P(Y_3 = 4) = \left(\frac{3}{6}\right)^3 - \left(\frac{2}{6}\right)^3 = \frac{19}{216}$$

$$P(Y_3 = 5) = \left(\frac{2}{6}\right)^3 - \left(\frac{1}{6}\right)^3 = \frac{7}{216}$$

$$P(Y_3 = 6) = \left(\frac{1}{6}\right)^3 - \left(\frac{0}{6}\right)^3 = \frac{1}{216}$$

Therefore,

$$E[Y_3] = \left(1 \times \frac{91}{216}\right) + \left(2 \times \frac{61}{216}\right) + \left(3 \times \frac{37}{216}\right) + \left(4 \times \frac{19}{216}\right) + \left(5 \times \frac{7}{216}\right) + \left(6 \times \frac{1}{216}\right)$$

$$E[Y_3] = \frac{49}{24}$$

$$E[Y_3^2] = \left(1^2 \times \frac{91}{216}\right) + \left(2^2 \times \frac{61}{216}\right) + \left(3^2 \times \frac{37}{216}\right) + \left(4^2 \times \frac{19}{216}\right) + \left(5^2 \times \frac{7}{216}\right) + \left(6^2 \times \frac{1}{216}\right)$$

$$E[Y_3^2] = \frac{1183}{216}$$

$$\implies var(Y_3) = \frac{1183}{216} - \left(\frac{49}{24}\right)^2$$

$$var(Y_3) = \frac{2261}{1728} = 1.30845$$

Solution 16

Y \ X	1	2	3	4	5	6
0	1/18	1/18	1/12	1/18	1/18	1/12
1	1/12	1/18	1/18	1/12	1/18	1/18
2	1/18	1/12	1/18	1/18	1/12	1/18

From the table, we have,

$$P_{X,Y}(2,6) = \frac{1}{18}$$

Solution 17

Part 1

The Given Joint PMF is a valid PMF because summing all the values above yields 1 which is in accordance with the Total Probability Theorem.

Part 2

Given That,

$$f(X,Y) = X - Y$$

$$E[f(X,Y)] = f(1,1)p_{X,Y}(1,1) + f(1,2)p_{X,Y}(1,1) + f(1,3)p_{X,Y}(1,1) + \cdots + f(5,6)p_{X,Y}(1,1) + f(6,6)p_{X,Y}(1,1)$$

After computing the above sum, we get

$$E[f(X,Y)] = \frac{13}{50}$$