LECTURE NO. 2

CALCULUS & ANALYTICAL GEOMETRY

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Limit

By limit of function we mean how a function behaves as indepemndent variables as we move toward a certain point.

WHY WE STUDY LIMIT

When we are to find a value at certain point like

$$f(x) = \frac{\sin x}{x} \ (\frac{0}{0}), \quad x = 0$$

We cant determine value at $x = 0$, so we study bhavior of such functions near $x = 0$.

Thers are two method to study this phenomena

Method 1

$$\lim_{x \to 0} \frac{\sin x}{x} = (\frac{0}{0})$$

x	$f(x) = \frac{\sin x}{x}$
± 2.0	0.454648713
± 1.0	0.841470985
± 0.5	0.95885107
± 0.3	0.973545856
± 0.2 ± 0.1	0.993346654 0.998334166
± 0.01	0.999983333
± 0.001	0.9999999833
± 0.0001	0.9999999998

When $x \to 0, then\ f(x) \to 1$

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Method 2

$$\lim_{x\to 0}\frac{\sin x}{x} = (\frac{0}{0})$$

Applying L-Hopital's rule

$$\lim_{x\to 0}\frac{\frac{d}{dx}\sin x}{\frac{dx}{dx}} = \lim_{x\to 0}\frac{\cos x}{1}$$

$$\lim_{x\to 0}\cos x = \cos 0 = 1$$

Medthod 3

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\lim_{x\to 0}\frac{1}{x}(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots)$$

$$\lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = 1 - 0 + 0 - \cdots = 1$$

Exercise # 1.1

Ex. 11-24 Use an algebraic simplification to help find the limit, if it exists.

15. $\lim\limits_{r \to 1} \frac{r^2 - r}{2r^2 + 5r - 7}$

$$= \lim_{r \to 1} \frac{r(r-1)}{2r^2 + 7r - 2r - 7}$$

$$= \lim_{r \to 1} \frac{r(r-1)}{r(2r+7) - 1(2r+7)}$$

$$= \lim_{r \to 1} \frac{r(r-1)}{(2r+7)(r-1)}$$

$$= \lim_{r \to 1} \frac{r}{(2r+7)} = \frac{1}{9}$$

20. $\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$, $(\frac{0}{0})$

$$= \lim_{h\to 0}\frac{x^3+h^3+3xh(x+h)-x^3}{h}$$

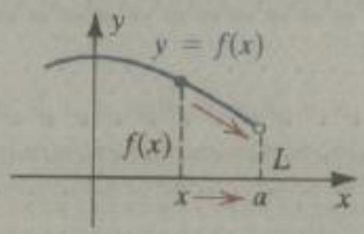
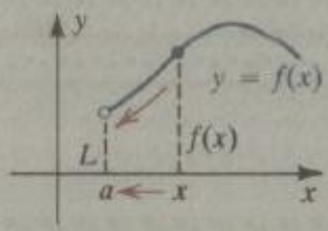
$$= \lim_{h\to 0}\frac{h^3+3xh(x+h)}{h}$$

$$= \lim_{h\to 0} h(\frac{h^2+3x(x+h)}{h})$$

$$= \lim_{h\to 0}(\frac{h^2+3x(x+h)}{1})$$

$$= 3x^2$$

One-Sided Limits 1.2

Notation	Intuitive meaning	Graphical interpretation
$\lim_{x \to a^-} f(x) = L$ (left-hand limit)	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a, and $x < a$.	
$\lim_{x \to a^+} f(x) = L$ (right-hand limit)	We can make $f(x)$ as close to L as desired by choosing x sufficiently close to a, and $x > a$.	

$$L.H.L = \lim_{x \to a^-} f(x) \ = L \quad , \quad x < a$$

$$R.H.L = \lim_{x \to a^+} f(x) \ = L \quad , \quad x > a$$

$$|x - a| = \begin{cases} -(x - a), & x < a \\ (x - a), & x \geq a \end{cases}$$

$$|x - 4| = \begin{cases} -(x - 4), & x < 4 \\ (x - 4), & x \geq 4 \end{cases}$$

Ex 25-30 Find the limit, if it exists.

(a) $\lim_{x\to a^-} f(x)$ (b) $\lim_{x\to a^+} f(x)$ (c) $\lim_{x\to a} f(x)$

25. $f(x) = \frac{|x-4|}{x-4}$

We express above function like

$$f(x) = \frac{|x-4|}{x-4} = \begin{cases} \frac{-(x-4)}{x-4} = -1, & x < 4 \\ \frac{(x-4)}{x-4} = 1, & x \geq 4 \end{cases}$$

(a) $\lim_{x\to a^-} f(x) = \lim_{x\to 4^-} f(x) = \lim_{x\to 4} -1 = -1$

(b) $\lim_{x\to a^+} f(x) = \lim_{x\to 4^+} f(x) = \lim_{x\to 4} 1 = 1$

(c) As $\lim_{x\to 4^-} f(x) \neq \lim_{x\to 4^+} f(x)$

So $\lim_{x\to a} f(x)$ does not exists

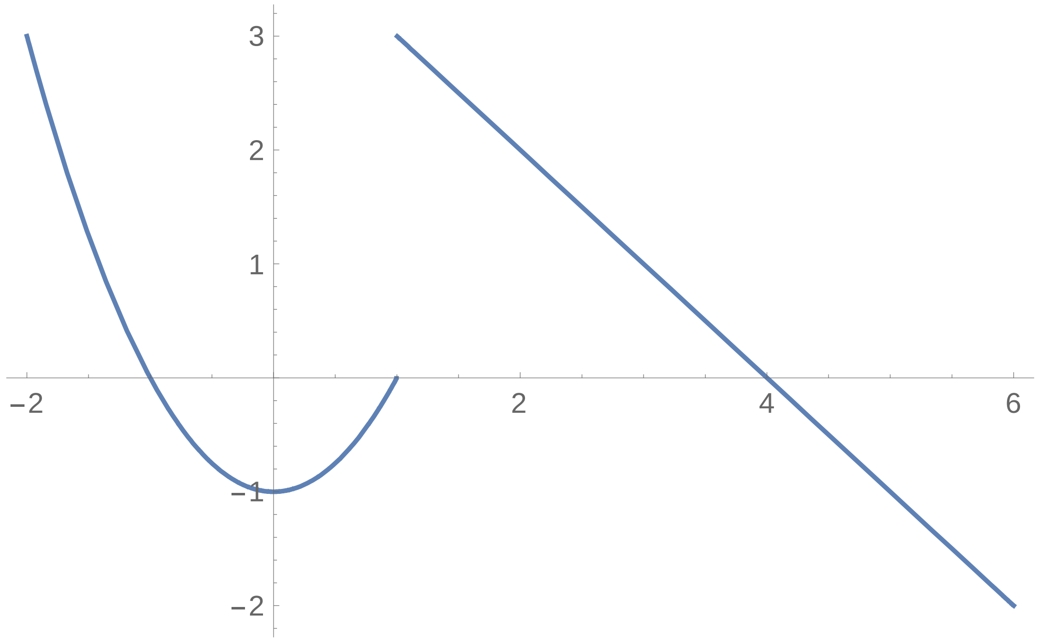
26.

$$f(x) = \frac{x+5}{|x+5|}, \qquad a = -5$$

Solve it in 10min

Ex 41-46: Sketch the graph of f and find each limit, if it exists;

41. $f(x) = \begin{cases} x^2 - 1, & if\ x < 1 \\ 4 - x, & if\ x \geq 1 \end{cases}$



(a) $\lim_{x \to a^-} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} x^2 - 1 = 0$

(b) $\lim_{x \to a^+} f(x) = \lim_{x \to 1^+} f(x) = \lim_{x \to 1} (4 - x) = 3$

(c) As $\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$

So $\lim_{x \to 1} f(x)$ does not exists