Option Pricing And The Black-Scholes Model

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Abstract

*This paper is meant to analyze the Black-Scholes Option pricing model. The intent is to go over is assumptions and relate it to computing. I begin by brief history of The B.S.E without delving into its derivation, then by over-viewing its assumptions, show a mock computer calculation using python. From that point, I discuss some further aspects of an options contract and relate it to the B.S.E. If it isn't clear yet, this paper is meant to serve as my final, while poking fun at academic articles and journals(which are okay to read on occasion). I could have chosen to put the title in Latin *obnoxiously* but I refrained from doing so. Excuse the not so perfect grammar. At the very least I hope you find this humours. Finally, I would like to dedicate this to my younger brother who has managed to keep his cool through the Covid-19 pandemic.*

I. Introduction

The Black-Scholes-Merton Model, is a mathematical model that has allowed Economists to describe the price of option contracts, ones with certain assumptions. Two gentlemen by the names of Fischer Black and Myron Scholes created a partial differential equation (named after them) to help investment banks and hedge funds minimize risk when when buying and selling assets. This Black-Scholes equation was later built upon by Robert Merton using stochastic calculus. Their work would award them the 1997 Nobel Prize in Economics[1], with Black labelled as a contributor as he had passed away before then. The model itself does not strictly apply to options contracts, as an internet definition states: "The model simulates the dynamics of a financial market containing derivative financial instruments such as options, futures, forwards and swaps".

For this paper I will focus only on call options. The paper assumes familiarity with financial terms, here's a quick rundown (you can choose to skip over this paragraph). The buyer of the call option has the right, but not the obligation, to buy an agreed quantity of a particular commodity or financial instrument (the underlying) from the seller of the option at a certain time (the expiration date) for a certain price (the strike price). These contracts have characteristics (such as time value) associated with them which can be derived from the Black-Scholes equation, I discuss this more later.

II. Assumptions of BSE

A European style option compared to an American style options contract has key differences that directly affect the model. BSM gives an estimate for the price of a European option contract. An EU contract is such that: the holder has the right to exercise the option only on the expiration date, unlike American options which can be exercised before. In turn, EU options have less associated risk, and loss or profit is more easily calculable.

[1] https://brilliant.org/wiki/black-scholes-merton/

- Log-normal distribution: the take away being that the stock price cannot take on a negative value. A standard normal distribution includes negative values so the prices of the stock can be graphed in a log-normal distribution.
- Expiration date: As stated earlier the option can only be exercised on on the expiration date, this means that the modeling is more relevant* for the European Market.
- No dividends: the stock which the option contract is bought does not pay dividends, or earnings to its shareholder.
- Risk-free interest rate: Interest rates are assumed to be kept constant, making the stock risk-free. This is far from ideal because if the effect of changing interest rates is observable in the market immediately, whereas for the rest of the economy it may take a year or more to see impact. [2]
- Random walk: The link is no way related, but the footnote is highly suggested watch to gain a better understanding of Brownian Motion. [3]
- Frictionless market: There are no transaction costs. Commission is generally paid to brokerages to each time stock or options contract is purchased; this has changed of recent and is seen as something to attract retail investors with: zero in commission!

III. The Equations

The Black-Scholes-Merton model can be described by a second order partial differential equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \quad (1)$$

V is the price of the option as a function of stock price S and time t, r is the risk-free interest rate, and σ is the volatility of the stock. We can re-write the equation to give us a better understanding of how the variables are interacting with each other.

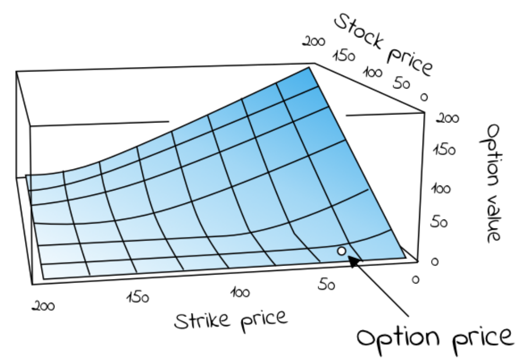

Figure 1: *Visual representation of European call option price/value with respect to strike price and stock price, as calculated using the Black-Scholes equation*[5]

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV - rS\frac{\partial V}{\partial S} \quad (2)$$

The left-hand side represents the change in the value/price of the option V due to time increasing + the convexity[4] of the option's value relative to the stock price. The equation can be written in terms of the Greeks (pointers that offer insight into how sensitive the options price is, more on the Greeks in the Discussion).

$$\Theta + \frac{1}{2}\alpha^2 S^2 \Gamma = rV - rS\Delta \quad (3)$$

Descriptively equation 3 can be read of as $\Theta + \Gamma$ = (risk-free rate) x (price of the option) - (risk-free rate) x (price of stock) x Δ. Solving the pde can allow us to graph the call option as seen in Figure 1. Enough with equation and ancient symbols! I'll attempt to explain the symbols in the first part of the discussion and move to the Greeks just after.

[2] https://www.investopedia.com/investing/how-interest-rates-affect-stock-market/

[3] https://youtu.be/CHsicsvWGQU

[4] Convexity refers to non-linearities in a financial model.The price depends on the second derivative (or higher-order terms) of the modeling function-wikipedia

IV. Discussion of Equations

i. The BSF

I've shown a few variables above, and because I am not deriving them, it may be best to explain them through example. The partial differential equation has an exact solution to it (analytic solution). The BSE was solved by Scholes and Merton, it can also be solved using numerical methods (with the forward finite difference method) or so I read, when it is re-written is a similar form to the heat equation. I provide a link to the derivation in the conclusion.

$$\begin{aligned} C_E(S,t) &= N(d1)S - N(d_2)Xe^{-rT} \\ C_{E,T} &= max(0, S_T - X) \end{aligned} \tag{4}$$

There it is, in all its glory, the solution to the Black-Scholes equation; all that is left is to insert some values, and we will have the ability to model the price of an option. The boundary condition is given in equations 4.

$$\begin{aligned} d_1 &= \frac{ln(\frac{S}{X}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{ln(\frac{S}{X}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} \end{aligned} \tag{5}$$

For a demonstration, I will share a mock calculation of a Tesla (TSLA) call option provided by medium.com. Say we are looking to speculate on the stock and expect it to rise; I the buyer, am interested in purchasing a call option. Looking to make an earining play, we look at the previous years stock and see that from then, the stock has risen 20% over the previous year. If the stock is currently at $245 1.2 times is $294. Looking up the date of the next earnings call, we see that we have 101 days until expiry if we purchase a call today. The risk free interest rate given by ($USGG10YR) is (at the time of mediums's article) paying off 2.12%. From equation 4 and 5 we have the following: $S = 245$, $X = 294$, $T - t = 101$, $r = 0212$. What is missing? The volatility σ. σ for the Black-Scholes equation lies between 0 and 1. By looking at similar stock prices around the same expiry/maturity date we can obtain a value for the volatility, with a values that averaged by 5 different options is 0.38 or 38%. After inserting the values into their appropriate places we obtain a price of about$7. Volatility does rise above 100% on occasion for certain stock, Palantir (PLTR) has on its call options this week in November.

The N from equation 4 represents the normal distribution for random variables. "Very informally, the two terms in the sum given by the Black-Scholes formula may be thought of as 'the current price of the stock weighted by the probability that you will exercise your option to buy the stock' minus 'the discounted price of exercising the option weighted by the probability that you will exercise the option', or simply 'what you are going to get' minus 'what you are going to pay'"[Khan, 2013].

The demonstration was on an American option, but the fact that TSLA does not pay dividends means that the price of the option is the same as if it were a European call option. This is based on what scenario is the most beneficial; if the stock pays dividends it may pay off more to exercise the call before it expires, and choosing to not exercise if it does not.

ii. The Greeks

The Greeks as they are called are option characteristics that can offer insight or general sentiment on a stock. The Greeks describe, in the same way the derivative problems from and Intro to Calculus class give the rate of change and direction on quantities. Δ-delta, Γ-gamma, v-vega, Θ-theta, and ρ-rho. Here is an overview of what each Greek can tell us about the stock provided by the Charles Schwab website.

- Gauge the likelihood that an option you're considering will expire in the money (Delta).
- Estimate how much the Delta will change when the stock price changes (Gamma).
- Get a feel for how much value your option might lose each day as it approaches expiration (Theta).

- Understand how sensitive an option might be to large price swings in the underlying stock (Vega).
- Simulate the effect of interest rate changes on an option (Rho).

I think that this brief over-view is sufficient enough to now demonstrate using python. I will continue discussing the Greeks in the next section. Equation 3 has now been explained using the above.

V. Demonstration

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```
import numpy as np
import scipy.stats as si

def NonDividendCall(S, X, T, r, sigma):

    #S: spot price 245
    #X: strike price 294
    #T: time to maturity 101/365
    #r: interest rate 0.0212
    #sigma: volatility of underlying asset 0.38

    d1 = (np.log(S / X) + (r + 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))
    d2 = (np.log(S / X) + (r - 0.5 * sigma ** 2) * T) / (sigma * np.sqrt(T))

    call = (S * si.norm.cdf(d1, 0.0, 1.0) - X * np.exp(-r * T) * si.norm.cdf(d2, 0.0, 1.0))

    return call

NonDividendCall(245, 294, 101/365, 0.0212, 0.38)

Output: 5.527304771683909
```

VI. Demonstration

For this code the TSLA option shown in the example above was chosen, and the value obtained was roughly 5.5. An online calculator with the same information provided a value of $8.2927. The article that provided the call option example mentions that the true value is around 7, but there could be a number of reasons for this discrepancy: interest rates are quite low right now, and the exact value is not calculated (it can be with python but with square-roots in the final answer). The online calculator used current rates whereas the code used data provided by the article.

VII. Practically Using Greeks

The Delta of an option expresses the change in the price of an option per $1 dollar change in the underlying stock. It has a value between 0 and 1, and can be thought of as the probability that the contract will expire in-the-money. Gamma is the rate of change in an options Delta per $1 change in the price of the underlying stock. Mathematically, Delta is the first order partial derivative of the options price/underlying price, and Gamma is the second partial (in some way analogous to velocity and acceleration). Vega measure the sensitivity of the options price to the volatility; if volatility increases by 1% then vega is the amount that the option price changes by. Theta measures the options price in relation to the time left until maturity. It is often used with words such such time-decay, time-value. Theta is in most scenarios a negative value. Rho measures the sensitivity of the option price relative to interest rates (usually the least discussed Greek).

Here are the Greeks described in mathematical form

$$\Delta = \frac{\partial V}{\partial S} \qquad \Gamma = \frac{\partial^2 V}{\partial^2 S}$$
$$v = \frac{\partial V}{\partial \sigma} \qquad \Theta = -\frac{\partial V}{\partial \tau}$$
$$\rho = \frac{\partial V}{\partial r}$$

The Greeks can tell us about how to approach the option, there are a few charts that give advice on what strategies to employ depending on the signs. Open interest is another aspect to consider when viewing option prices.

VIII. Algorithmic Trading

A study in 2019 showed that around 92% of trading in the Forex market was performed by trading algorithms rather than humans [Kissell and Robert,2020].

There are a few definitions of Algorithmic trading, a Wikipedia definition states "Algorithmic trading is a method of executing or-

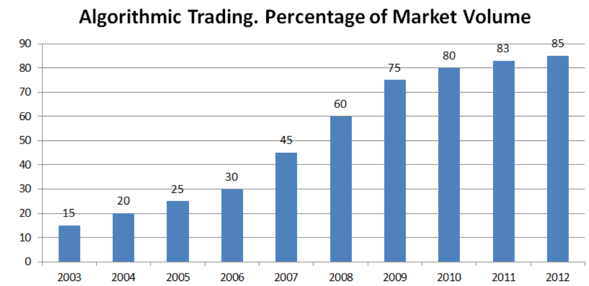

Figure 2: *Algorithmic trading. Percentage of market volume by computers. [Glantz, 2013]*

ders using automated pre-programmed trading instructions accounting for variables such as time, price, and volume". Computerization of the Market began in the early 1970's, with the NYSE. Programs designed with the intention of entering or exiting trades depending on certain characteristics. These only gained popularity from then on, and it wasn't long before they were used in trading between the S&P 500 equity and futures markets (index arbitrage). As in most areas, where novel ways of computing are used (ML/AI), have their criticisms, algorithmic trading or the reliance on it, was blamed for worsening/creating the stock market crash of 1987. Similar sentiment followed for the housing market crash of '07-08.

i. High Frequency Trading

High Frequency Trading, or HFT, is an automated trading system that large institutions such as Banks and Hedge-funds make use of. The idea of these systems is to make use of algorithms to analyze market trends as they occur, and take action in fractions of a second. This has allowed these firms to have high order-to-trade ratios. They gain key insight into market sentiment, and as a result capture gains for each security in their massive volume orders. Some firms using HFT systems are called Market Makers, with the primary goal of profiting on the bid-ask spread, which is the amount by which the ask price exceeds the bid price a market asset. The call options that I discussed have this spread (how much someone is willing to buy/pay for the security).

The Securities and Exchange Commission (SEC) doesn't have a strict definition for HFT but attributes these characteristics to it: High speed/sophisticated programs routing and executing orders, co-location services and data feeds to minimize latency, very short time-frames for establishing and liquidating positions, submission of numerous orders that are cancelled shortly after submission, ending trading day in as flat of a position as possible (minimize un-hedged positions)[6]. HFT has been the subject of controversy.

ii. Impact of HFT on Exchanges

With the advent of more elaborate technology, a gaining an advantage of 4 milliseconds in execution time was enough to spend $300 million USD[7]. This as described in Lewis's book, was the cost of running a 827 mile fibre-optic cable that cuts through mountains from Chicago to New Jersey to reduce transmission time. Front-Running, suppose that Broker Robinhood receives an order to purchase $1 millon dollars of (GME) Game-Stop, an order or such size is going to drive up the price of the stock, enough so that the broker puts the order aside and buys a million worth themselves and then executes the order. Immediately after the purchase of the client's order, the broker sells the bought share for a profit. Front running is highly unethical and considered illegal. A few articles explain the controversy as mischaracterized and that HFT is not the equivalent of front running. Each brokerage has its fiduciary responsibility, to give the best time and price executable and a breach of this would be considered illegal and investigated by the SEC. I read articles from both sides aisle, I'll share what each side offered that I thought to be the pros and cons of such systems.

A key incentive for more HFT systems to be put in place are, as in the NYSE, are rebates for maintaining high liquidity in the market. There is no doubt that computerization has made routing orders and connecting buyers and sellers far easier and quicker. Liquidity allows smaller investors the ability to buy or sell assets quickly, though there are other factor that affect this. Without liquidity, selling contracts which you own can become a real concern when you have un-ordered fills for the lowest bid price. The bid ask spread is directly affected by this, sometimes a large gap means liquidity is low. I didn't read the specifics of how HFT systems earn their rebates for enabling greater liquidity, but the main concern surrounding this is "ghost liquidity". In this Scenario HFT systems provide liquidity in one second only for it to vanish in the next not allowing traders to take part.

IX. Extrapolating from Data

There are a number of libraries built into python for obtaining financial information in real time of assets. Three such libraries are, yahoofinancials 1.6, pandas_datareader, and quandl. I created a demonstration for the Johnson and Johnson Stock price using historical data I obtained from the quandl. I decided to showcase usefulness of a 50 day moving average for a stock, considering big firms make a key point out of using historical data. The moving average is almost like filtering a signal to smooth out noise. JNJ would be perfect for this example because I don't think the price is very smooth and fluctuates quite a bit in my opinion, it looks like a saw blade's teeth. Using the window size and adjusted price, plotting gives us a better idea of the stocks movement (visually). I also provide the superimposed graphs along with the python code. It's worth sharing that some strategies use different time windows to open a short position or long position... For example, If the 30 day mav crosses the 50 day mav in a downward trend its an indication of change (how serious? up for interpretation I would say, I'm not an economist!) but some few articles with similar demonstrations discuss possible scenarios (positions, expecting the stock to fall).

[6] https://www.investopedia.com/ask/answers/09/high-frequency-trading.asp

[7] Lewis,2014

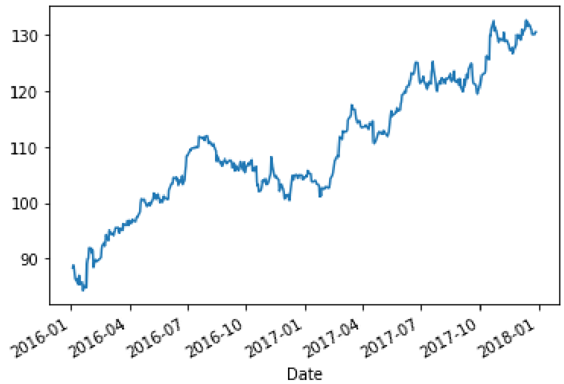

Figure 3: *adjusted price of JNJ in the given time span from 2016-2020*

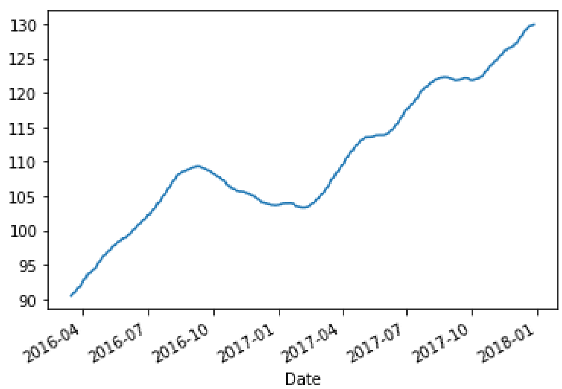

Figure 4: *moving average of JNJ in the given time span from 2016-2020*

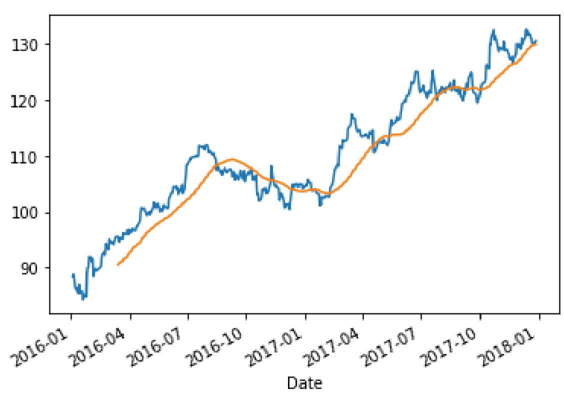

Figure 5: *Superimposed figure 4 on figure 3*

X. Plotting Moving Averages

```
import pandas as pd
import quandl as q
import numpy as np
import matplotlib.pyplot as plt

#set the Api key # my api key(removed this):
q.ApiConfig.api_key = "key"

#send request to get data from jhonson and jhonson
#from 1st Jan 1 2016 to 1st Dec 1 2020
jnj_data = q.get("EOD/JNJ", start_date="2016-01-01", \
end_date="2020-12-01")

#Moving Averages in Trading
adj_price = jnj_data['Adj_Close']

#calculate the moving average
mav = adj_price.rolling(window=50).mean()

#plot the adj_price
adj_price.plot()

#plot the moving average
mav.plot()
```

XI. Proprietary ML

Using moving averages may seem naive but, using historical data is quite useful and considered worthwhile by Big investment firms. From Citadels website: "Citadel's team ... We analyze historical and current information through statistical and quantitative modeling.". I don't discuss ML too much because I haven't taken formal coursework on it but it is the centerpiece at many of theses firms. Stolen proprietary code is recurring complaint in lawsuits these firms make against each other. Often times when employees are recruited at competing ones, there are accusations of sharing software and/or algorithms illegally (pretty funny). I may get extremely luck in finding some proprietary code in a paste bin online, but I doubt legality of putting that in this paper, given the off-chance. I think high level descriptions is enough to share, as its something I can read only about in incomplete detail.

i. Autonomous Market

In an autonomous Market, the advantages are all pointed towards risk-minimization. Back testing, Preserving Discipline, Minimizing Emotion, Improving Entry Speed are all lauded as pros of an Autonomous Market. There are instances where automated systems have sold off assets accidentally plunging stock price (as in 2010), so it has its limitations. Otherwise, It seems that it is quite successful, considering the statistics I shared of how much of market has been automated since 2008. In terms of predicting/anticipating future prices, most resources have offered the same conclusion that using ML in the market offers little to no real advantage. If a firm could for-see prices into the future and where the market was headed, well, I don't know what that would mean for the rest of us ...

XII. Conclusion

I would like to start with a disclaimer: I am in no way an economist or financial advisor and everything contained within this paper I pieced together with valid articles to the best of my knowledge. The breadth of it all could not have all been researched in exact detail in the given time (considering the complexity of certain parts of this paper). I have taken an ODE course but that in no was enough explain through solving the Black Scholes Equation, I viewed the (link->) Solution and was able appreciated it minute. That ends my declaimer, onto the paper.

FUD, Fear, Uncertainty, Doubt. One thing stands out, the Market is far from a closed system from what I've read. Human interaction creates variability that can't be predicted. That may be obvious but, it brings up the limitations of creating algorithms and models to handle financial markets stakes of billion+ of dollars. Human interaction, and human created programs (imperfect ones) create too uncertainty in edge case scenarios. I'm not aware of the proprietary software 2 Sigma uses for example, but how much success do they owe to human decisions over computer generated ones? Everything works until it doesn't.

Citron Research, a one man group handled by Andrew Left tweeted regarding a PLTR rally of $10 calling it a "casino". The stock fell 33% in 2 days. It was likely shorting of the stock, "analysts" often set their prices, and this causes a major shit in price. Market manipulation is something I'm not so sure a model can anticipate/foresee either. There are code examples online, where algorithmic trading can recognize the changes in moving averages (indicating shifts in price) as explained above, but time is backwards-looking. I came across a quote - something about using the past to expect the future in a constantly innovating market. This is especially true when dealing with sudden shifts due to insider information. A Georgia senator (Kelly Lofeller) who is married to the president of NYSE, sold over a million in assets after the private Covid-19 lockdown briefing. There is also negligence on the part of the banks and issuers of big money contracts. A few articles pointed out that lack of responsibility somewhere in the bureaucracy in loan lending, lead to an economic recession in 2007-08.

For these reasons I think this multi-faceted problem has too much input to take account for. Success of the large firms may be due largely to bad practice and unethical dealings, and not an automated algorithm that does what it was designed to do. This is just my opinion, I cannot begin to imagine inner workings of Funds, Banks, and Wall street algo-trading.

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