Definition:-

A vector is a quantity having both magnitude and direction, such as displacement, velocity, force and acceleration.

Graphically a vector is represented by an arrow OP (Fig. 1) defining the direction, the magnitude of the vector being indicated by the length of the arrow. The tail end O of the arrow is called the origin or initial point of the vector, and the head P is called the terminal point or terminus.

Analytically a vector is represented by a letter with an arrow over it, as $\vec{A}$ in (Fig. 1), and its magnitude is denoted by $|\vec{A}|$. The vector OP is also indicated as $\vec{OP}$, in such case we shall denote its magnitude by $|\vec{OP}|$.

A Scalar: is a quantity having magnitude but no direction, e.g. mass, length, time, temperature, and any real number. Scalars are indicated by letters in ordinary type as in elementary algebra. Operation with scalars follow the same rules as in elementary algebra.

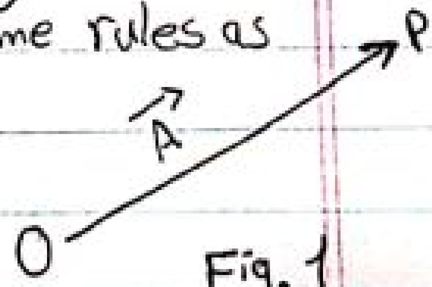


Fig. 1

Vector Algebra:

The operations of addition, Subtraction and multiplication familiar in the algebra of numbers or Scalar are, with suitable definition, capable of extension to an algebra of vectors.

The following definitions are fundamental.

1- Tow vectors $\vec{A}$ and $\vec{B}$ are equal if they have the Same magnitude and direction regardless of the position of their initial points. Thus $\vec{A}=\vec{B}$ (Fig.2)

2- A vector having direction opposite to that of vector $\vec{A}$ but having the Same magnitude is denoted by $-\vec{A}$ (Fig.3)

3- The Sum or resultant of vectors $\vec{A}$ and $\vec{B}$ is a vector $\vec{C}$ formed by placing the initial Point of $\vec{B}$ on the terminal point of $\vec{A}$ and then joining the initial point of $\vec{A}$ to the terminal point of $\vec{B}$ (Fig.4)
This sum is written $\vec{A}+\vec{B}$ i.e. $\vec{C}=\vec{A}+\vec{B}$.

4. The difference of vectors $\vec{A}$ and $\vec{B}$, represented by $\vec{A}-\vec{B}$, is the vector $\vec{C}$ which added to $\vec{B}$ yields vector $\vec{A}$. Equivalently, $\vec{A}-\vec{B}$ Can be defined as the Sum $\vec{A}+(-\vec{B})$.
If $\vec{A}=\vec{B}$, then $\vec{A}-\vec{B}$ is defined as the null or Zero vector and is represented by the Symble $\vec{O}$. It has magnitude and no specific direction.

5- The product of a vector $\vec{A}$ by a scalar m is a vector $m\vec{A}$ with magnitude $|m|$ times the magnitude of $\vec{A}$ and with direction the same as or opposite to that of $\vec{A}$, according as m is positive or negative. If $m=0$, $m\vec{A}$ is the zero vector.

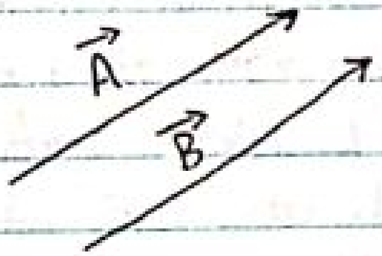


Fig-2

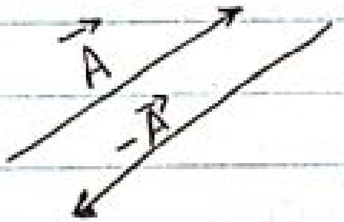


Fig.3

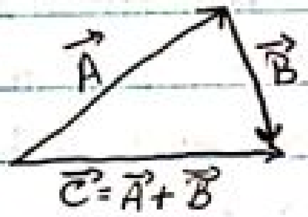


Fig.4

Laws of vectors Algebra :-

If $\vec{A}$, $\vec{B}$ and $\vec{C}$ are vectors and m and n are scalars, then

$\vec{A}+\vec{B}=\vec{B}+\vec{A}$	Commutative law for addition
$\vec{A}+(\vec{B}+\vec{C})=(\vec{A}+\vec{B})+\vec{C}$	Associative law for addition
$m\vec{A}=\vec{A}m$	Commutative law for multiplication
$m(n\vec{A})=(mn)\vec{A}$	Associative law for multiplication
$(m+n)\vec{A}=m\vec{A}+n\vec{A}$	Distributive Law
$m(\vec{A}+\vec{B})=m\vec{A}+m\vec{B}$	Distributive Law

Note that in these laws only multiplication of a vector by one or more scalars is used.

These laws enable use to treat vector equations in the same way as ordinary algebraic equations.

A unit vector

is a vector having unit magnitude, if $\vec{A}$ is a vector with magnitude $|\vec{A}| \neq 0$, then $\vec{A}/|\vec{A}|$ is a unit vector having the same direction as $\vec{A}$.

Any vector $\vec{A}$ can be represented by a unit vector $\vec{a}$ in the direction of $\vec{A}$ multiplied by the magnitude of $\vec{A}$. In symbols, $\vec{A} = |\vec{A}|\vec{a}$.

The Rectangular Unit vectors i, j, k :-

An important set of unit vectors are those having the directions of the positive x, y and z axes of a three dimensional rectangular coordinate system, and are denoted respectively by i, j and k. (Fig. 5)

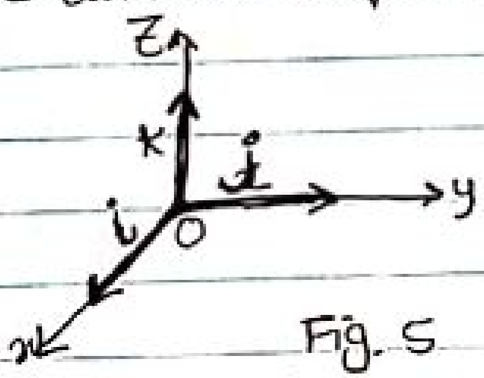


Fig. 5

We shall use right handed rectangular coordinate systems unless otherwise stated. Such a system derives its name from the fact that a right-threaded screw rotated through 90° from Ox to Oy will advance in the positive z direction

Components of a Vectors:-

Any vector $\vec{A}$ in 3 dimensions can be represented with initial point at the origin O of a rectangular coordinate system (Fig.6). Let (A_1, A_2, A_3) be the rectangular coordinates of the terminal point of vector $\vec{A}$ with initial point at O. The vectors $A_1 i$, $A_2 j$, and $A_3 k$ are called the rectangular component vectors or simply component vectors of $\vec{A}$ in the x, y, and z directions respectively. A_1, A_2 and A_3 are called the rectangular components or simply components of $\vec{A}$ in the x, y and z directions respectively.

The sum or resultant of $A_1 i$, $A_2 j$ and $A_3 k$ is the vector $\vec{A}$ so that we can write

$$\vec{A} = A_1 i + A_2 j + A_3 k$$

The magnitude of $\vec{A}$ is

$$|\vec{A}| = \sqrt{A_1^2 + A_2^2 + A_3^2}$$

In particular, the position vector or radius vector $\vec{r} = xi + yj + zk$ and has magnitude

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

Example

State which of the following are scalars and which are vectors.

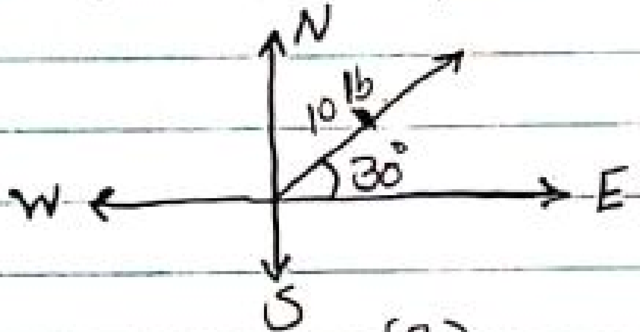
(1) momentum (2) specific heat (3) density (4) volume (5) speed.

Example

represent graphically

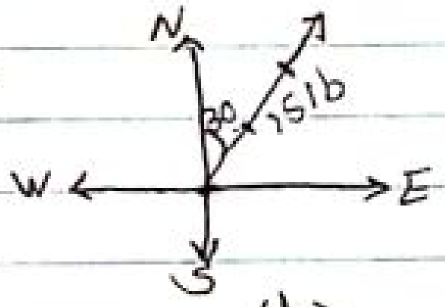
(a) a force of 10 lb in a direction 30° north of east.

(b) a force of 15 lb in a direction 30° east of north.



(a)

unit = 5 lb

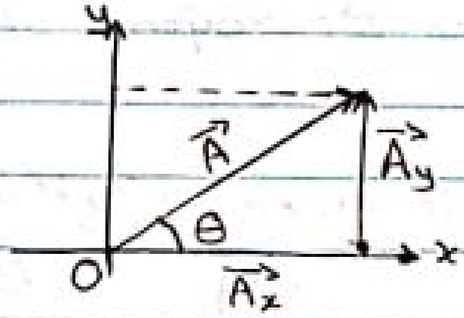


(b)

Components of a vector:-

Assume you are given a vector $\vec{A}$. It can be expressed in terms of two other vectors $\vec{A_x}$ and $\vec{A_y}$. These three vectors from a right triangle (Fig(1))

$$\vec{A} = \vec{A_x} + \vec{A_y}$$



Fig(1)

- The Y-component is moved to the end of the x-component. This is due to the fact that any vector can be moved parallel to itself without being affected.
- The x-component of the vector is the projection along the x-axis

$$A_x = A \cos \theta$$

- The y-component of a vector is the projection along the y-axis

$$A_y = A \sin \theta$$

This assumes the angle θ is measured with respect to the x-axis.

The components are the legs of the right triangle whose hypotenuse is the length of $\vec{A}$ (Fig(1))

$$A = \sqrt{A_x^2 + A_y^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{A_y}{A_x}$$

- The components can be positive or negative and (Fig(2)) will have the same units as the original vector.
- The sign of the components will depend on the angle

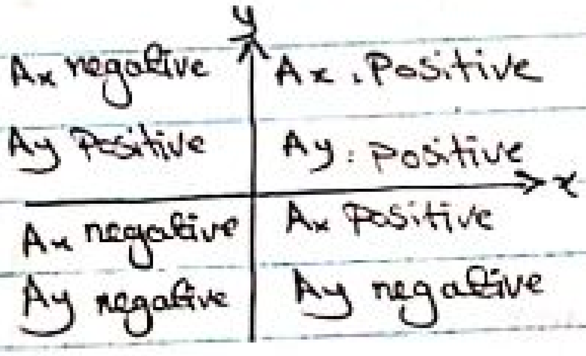


Fig (2)

Adding Vectors Using Unit vectors:-

using $\vec{R} = \vec{A} + \vec{B}$

then $\vec{R} = (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j})$

$$\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j}$$

$$\vec{R} = R_x\hat{i} + R_y\hat{j}$$

where $R_x = A_x + B_x$ and $R_y = A_y + B_y$

$$R = \sqrt{R_x^2 + R_y^2}, \quad \theta = \tan^{-1} \frac{R_y}{R_x}$$

Three-Dimensional Extension :-

using $\vec{R} = \vec{A} + \vec{B}$

then $\vec{R} = (A_x\hat{i} + A_y\hat{j} + A_z\hat{k}) + (B_x\hat{i} + B_y\hat{j} + B_z\hat{k})$

$$= (A_x+B_x)\hat{i} + (A_y+B_y)\hat{j} + (A_z+B_z)\hat{k}$$

$$\vec{R} = R_x\hat{i} + R_y\hat{j} + R_z\hat{k}$$

where $R_x = A_x + B_x$, $R_y = A_y + B_y$ and $R_z = A_z + B_z$

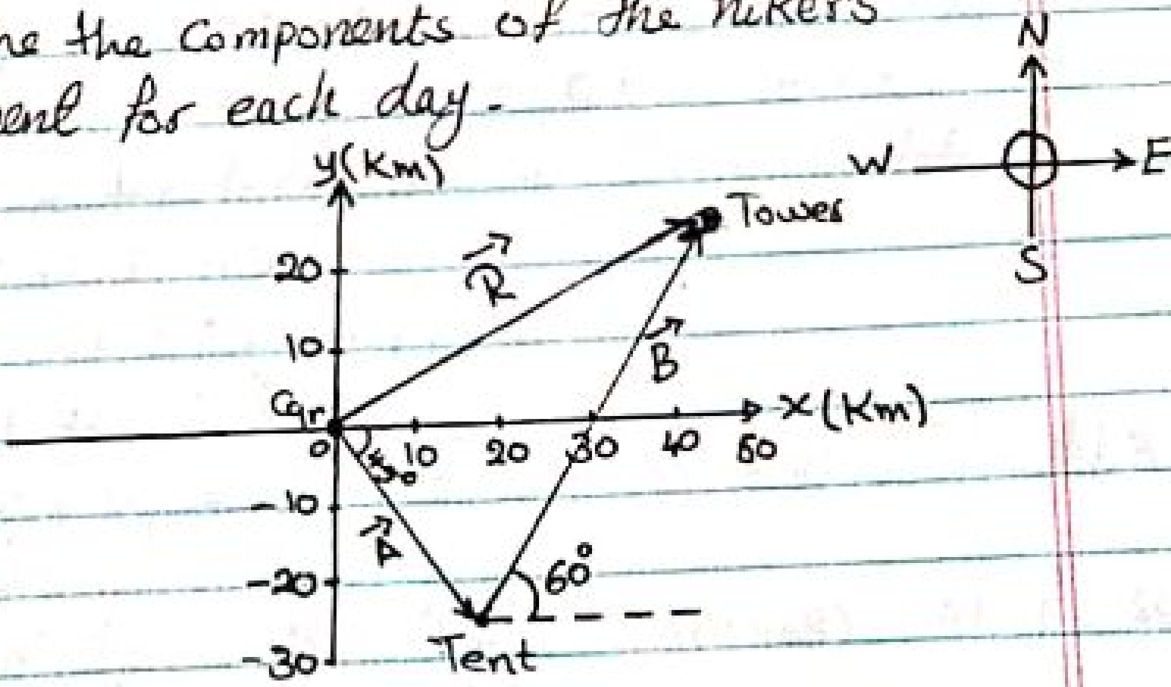
$$R = \sqrt{R_x^2 + R_y^2 + R_z^2} \quad , \quad \theta = \cos^{-1}\frac{R_x}{R}$$

Example :-

A hiker begins a trip by first walking 25.0 km southeast from her car. She stops and sets up her tent for the night. On the second day, she walks 40.0 km in a direction 60° north of east, at which point she discovers a forest ranger's tower.

Sol

(A) Determine the components of the hiker's displacement for each day.



Its components are:

$A_x = A\cos(-45^\circ) = (25\,km)(0.707) = 17.7\ km$

$A_y = A\sin(-45^\circ) = (25\,km)(-0.707) = -17.7\ km$

The negative value of A_y indicates that the hiker walks in the negative y direction on the first day.

The second displacement $\vec{B}$ has a magnitude of 40 km and is 60° north of east

Its components are

$B_x = B\cos 60^\circ = (40\,km)(0.5) = 20\ km$

$B_y = B\sin 60^\circ = (40\,km)(0.866) = 34.6\ km$

(B) Determine the components of the hiker's resultant displacement $\vec{R}$ for the trip. Find the expression for $\vec{R}$ in terms of unit vectors.

The resultant displacement for the trip $\vec{R} = \vec{A} + \vec{B}$ has components given by

$R_x = A_x + B_x = 17.7\,km + 20\,km = 37.7\ km$

$R_y = A_y + B_y = -17.7\,km + 34.6\,km = 16.9\ km$

In unit vector form, we can write the total displacement as

$$\vec{R} = (37.7\,\hat{i} + 16.9\,\hat{j})\ km$$

• The magnitude of $\vec{R}$ is

$$|\vec{R}| = \sqrt{R_x^2 + R_y^2} = \sqrt{(37.7)^2 + (16.9)^2} = \sqrt{1706.9} = 41.3\ km$$

• The direction

$$\theta = \tan^{-1}\frac{R_y}{R_x} = \tan^{-1}\frac{16.9}{37.7} = \tan^{-1}(0.448) = 24.1^\circ \text{ north of east}$$

The Dot and Cross product

Definition

The dot or scalar product of two vectors $\vec{A}$ and $\vec{B}$, denoted by $\vec{A}.\vec{B}$ (read $\vec{A}$ dot $\vec{B}$), is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the cosine of the angle θ between them. In Symbols,

$$\vec{A}.\vec{B} = |A||B|\cos\theta \quad . \quad 0 \leq \theta \leq \pi$$

Note that $\vec{A}.\vec{B}$ is a scalar and not a vector.

The following laws are valid:

1- $\vec{A}.\vec{B} = \vec{B}.\vec{A}$ "Commutative law for dot products"
2- $\vec{A}.(\vec{B}+\vec{C}) = \vec{A}.\vec{B} + \vec{A}.\vec{C}$ "Distributive law"
3- $m(\vec{A}.\vec{B}) = (m\vec{A}).\vec{B} = \vec{A}.(m\vec{B}) = (\vec{A}.\vec{B})m$ "where m is a scalar"
4- $\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1 \ , \ \vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{k}.\vec{i} = 0$
5- If $\vec{A} = A_1\vec{i} + A_2\vec{j} + A_3\vec{k}$ and $\vec{B} = B_1\vec{i} + B_2\vec{j} + B_3\vec{k}$, then

$$\vec{A}.\vec{B} = A_1B_1 + A_2B_2 + A_3B_3$$
$$\vec{A}.\vec{A} = \vec{A}^2 = A_1^2 + A_2^2 + A_3^2$$
$$\vec{B}.\vec{B} = \vec{B}^2 = B_1^2 + B_2^2 + B_3^2$$

6- If $\vec{A}.\vec{B} = 0$ and $\vec{A}$ and $\vec{B}$ are not null vectors, then $\vec{A}$ and $\vec{B}$ are Perpendicular.

Definition

The cross or vector product of $\vec{A}$ and $\vec{B}$ is a vector $\vec{C} = \vec{A}\times\vec{B}$ (read $\vec{A}$ cross $\vec{B}$). The magnitude of $\vec{A}\times\vec{B}$ is defined as the product of the magnitudes of $\vec{A}$ and $\vec{B}$ and the sine of the angle θ between them. The direction of the vector $\vec{C} = \vec{A}\times\vec{B}$ is perpendicular to the plane of $\vec{A}$ and $\vec{B}$ and such that $\vec{A}, \vec{B}$ and $\vec{C}$ form a right-hand system. In Symbols,

$$\vec{A}\times\vec{B} = AB\sin\theta\, \mathbf{u} \qquad 0 \leq \theta \leq \pi$$

where u is a unit vector indicating the direction of $\vec{A}\times\vec{B}$.

If $\vec{A}=\vec{B}$, or if $\vec{A}$ is parallel to $\vec{B}$, then $\sin\theta=0$ and we define $\vec{A}\times\vec{B}=0$.

The following laws are valid:

1- $\vec{A}\times\vec{B} = -\vec{B}\times\vec{A}$

2- $\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B}+\vec{A}\times\vec{C}$ "Distributive law"

3- $m(\vec{A}\times\vec{B}) = (m\vec{A})\times\vec{B} = \vec{A}\times(m\vec{B}) = (\vec{A}\times\vec{B})m$ "where m is scalar"

4. $\vec{i}\times\vec{i} = \vec{j}\times\vec{j} = \vec{k}\times\vec{k} = 0$, $\vec{i}\times\vec{j}=\vec{k}$, $\vec{j}\times\vec{k}=\vec{i}$, $\vec{k}\times\vec{i}=\vec{j}$

5. If $\vec{A}= A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ and $B = B_1\vec{i}+B_2\vec{j}+B_3\vec{k}$. then

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}$$

6- The magnitude of $\vec{A}\times\vec{B}$ is the Same as the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$.

7. if $\vec{A}\times\vec{B}=0$, and $\vec{A}$ and $\vec{B}$ are not null vector, then $\vec{A}$ and $\vec{B}$ are parallel.

<u>Example</u>

1) If $A= A_1\vec{i}+A_2\vec{j}+A_3\vec{k}$ and $\vec{B}= B_1\vec{i}+B_2\vec{j}+B_3\vec{k}$, prove that
$\vec{A}\cdot\vec{B} = \vec{A}_1\vec{B}_1+\vec{A}_2\vec{B}_2+\vec{A}_3\vec{B}_3$

2) Find the angle between $\vec{A}=2\vec{i}+2\vec{j}-\vec{k}$ and $\vec{B}=6\vec{i}-3\vec{j}+2\vec{k}$

3) If $\vec{A}=2\vec{i}-3\vec{j}-\vec{k}$ and $\vec{B}=\vec{i}+4\vec{j}-2\vec{k}$, find
(a) $\vec{A}\times\vec{B}$ (b) $\vec{B}\times\vec{A}$, (c) $(\vec{A}+\vec{B})\times(\vec{A}-\vec{B})$

Triple products:-

Dot and cross multiplication of three vectors $\bar{A}, \vec{B}$ and $\vec{C}$ may produce meaningful products of the form $(\bar{A}.\bar{B})\bar{C}$, $\bar{A}.(\bar{B}\times\bar{C})$ and $\bar{A}\times(\bar{B}\times\bar{C})$. The following laws are valid:

1. $(\bar{A}.\bar{B})\bar{C} \neq \bar{A}(\bar{B}.\bar{C})$
2. $\bar{A}.(\bar{B}\times\bar{C}) = \bar{B}.(\bar{C}\times\bar{A}) = \bar{C}.(\bar{A}\times\bar{B})$ = volume of a parallelepiped having $\bar{A}, \bar{B}$ and C as edges, or the negative of this volume, according as A, B and C do or do not form a right handed system.

   If $\bar{A} = A_1\bar{i} + A_2\bar{j} + A_3\bar{k}$, $\bar{B} = B_1\bar{i} + B_2\bar{j} + B_3\bar{k}$ and $\bar{C} = C_1\bar{i} + C_2\bar{j} + C_3\bar{k}$ then

$$\bar{A}.(\bar{B}\times\bar{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

3. $\bar{A}\times(\bar{B}\times\bar{C}) \neq (\bar{A}\times\bar{B})\times\bar{C}$
4. $\bar{A}\times(\bar{B}\times\bar{C}) = (\bar{A}.\bar{C})\bar{B} - (\bar{A}.\bar{B})\bar{C}$

   $(\bar{A}\times\bar{B})\times\bar{C} = (\bar{A}.\bar{C})\bar{B} - (\bar{B}.\bar{C})\bar{A}$

The product $\bar{A}.(\bar{B}\times\bar{C})$ is sometimes called the scalar triple product or box product and may be denoted $[\bar{A}\bar{B}\bar{C}]$. The product $\bar{A}\times(\bar{B}\times\bar{C})$ is called the vector triple product.

Example

Determine a unit vector perpendicular to the plane of $\bar{A} = 2i - 6j - 3k$ and $\bar{B} = 4i + 3j - k$.

Sol

$A\times B$ is a vector perpendicular to the plane of $\bar{A} = 2i - 6j - 3k$ and $\bar{B} = 4i + 3j - k$

$$\bar{A}\times\bar{B} = \begin{vmatrix} \bar{i} & \bar{j} & k \\ 2 & -6 & -3 \\ 4 & 3 & -1 \end{vmatrix} = 15i - 10j + 30K$$

A unit vector Parallel to $\bar{A}\times\bar{B}$ is $\frac{\bar{A}\times B}{|\bar{A}\times\bar{B}|}$

$$\frac{\bar{A}\times\bar{B}}{|\bar{A}\times\bar{B}|} = \frac{15i - 10j + 30K}{\sqrt{15^2+(-10)^2+30^2}} = \frac{3}{7}i - \frac{2}{7}j + \frac{6}{7}K$$

Example

Evaluate $(2i-3j)\cdot[(i+j-k)\times(3i-k)]$

$$= \begin{vmatrix} 2 & -3 & 0 \\ 1 & 1 & -1 \\ 3 & 0 & -1 \end{vmatrix} = 4$$

Vector Differentiation

Ordinary Derivative of vectors:

let $R(u)$ be a vector depending on a single scalar variable u.
Then

$$\frac{\Delta R}{\Delta u} = \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

where Δu denotes an increment in u.
The ordinary derivative of the vector $R(u)$ with respect to the scalar u is given by

$$\frac{dR}{du} = \lim_{\Delta u \to 0} \frac{\Delta R}{\Delta u} = \lim_{\Delta u \to 0} \frac{R(u+\Delta u) - R(u)}{\Delta u}$$

if the limit exists.
Since $\frac{dR}{du}$ is itself a vector depending on u, we can consider its derivative with respect to u. If this derivative exists it is denoted by $\frac{d^2R}{du^2}$.

Space Curves:

If in particular $R(u)$ is the position vector $r(u)$ joining the origin O of a coordinate system and any point (x, y, z), then

$$r(u) = x(u)\bar{i} + y(u)\bar{j} + z(u)\bar{k}$$

and specification of the vector function $r(u)$ defines x, y and z as functions of u.
As u changes, the terminal point of $\bar{r}$ describes a space curve having parametric equations

$$x = x(u), \quad y = y(u), \quad z = z(u)$$

Then $\frac{\Delta r}{\Delta u} = \frac{r(u+\Delta u) - r(u)}{\Delta u}$ is a vector in the direction of Δr.

If $\lim_{\Delta u \to 0} \frac{\Delta r}{\Delta u} = \frac{dr}{du}$ exists, the limit will be a vector in the direction of the tangent to the space curve at (x, y, z) and is given by

$$\frac{dr}{du} = \frac{dx}{du}\bar{i} + \frac{dy}{du}\bar{j} + \frac{dz}{du}\bar{k}$$

If u is the time t, $\frac{dr}{dt}$ represents the velocity v with which the terminal point of r describes the curve. Similarly $\frac{dv}{dt} = \frac{d^2r}{dt^2}$ represents its acceleration a along the curve.

Differentiation Formulas:

If $\bar{A}$, $\bar{B}$ and $\bar{C}$ are differentiable vector functions of a Scalar u, and ϕ is a differentiable scalar function of u, then

1. $\frac{d}{du}(\bar{A}+\bar{B}) = \frac{d\bar{A}}{du} + \frac{d\bar{B}}{du}$

2- $\frac{d}{du}(\bar{A}.\bar{B}) = \bar{A}.\frac{d\bar{B}}{du} + \frac{d\bar{A}}{du}.\bar{B}$

3- $\frac{d}{du}(A \times B) = \bar{A} \times \frac{d\bar{B}}{du} + \frac{d\bar{A}}{du} \times \bar{B}$

4- $\frac{d}{du}(\phi \bar{A}) = \phi \frac{d\bar{A}}{du} + \frac{d\phi}{du}\bar{A}$

5- $\frac{d}{du}(\bar{A}.\bar{B} \times \bar{C}) = \bar{A}.\bar{B} \times \frac{d\bar{C}}{du} + \bar{A}.\frac{d\bar{B}}{du} \times \bar{C} + \frac{d\bar{A}}{du}.\bar{B} \times \bar{C}$

6- $\frac{d}{du}\{\bar{A} \times (\bar{B} \times \bar{C})\} = \bar{A} \times (\bar{B} \times \frac{d\bar{C}}{du}) + \bar{A} \times (\frac{d\bar{B}}{du} \times \bar{C}) + \frac{d\bar{A}}{du} \times (\bar{B} \times \bar{C})$

The order in these products may be important.

Example:

Given $R = \sin t\,\bar{i} + \cos t\,\bar{j} + t\,\bar{k}$, find

(1) $\frac{dR}{dt}$, (2) $\frac{d^2R}{dt^2}$ (3) $\left|\frac{dR}{dt}\right|$ (4) $\left|\frac{d^2R}{dt^2}\right|$

Sol

(1) $\frac{dR}{dt} = \cos t\,\bar{i} - \sin t\,\bar{j} + \bar{k}$

(2) $\frac{d^2R}{dt^2} = \frac{d}{dt}\left(\frac{dR}{dt}\right) = -\sin t\,\bar{i} - \cos t\,\bar{j}$

(3) $\left|\frac{dR}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + 1^2} = \sqrt{2}$

(4) $\left|\frac{d^2R}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2} = 1$

Example:

A particle moves along a curve whose parametric equations are $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$, where t is the time.

(a) Determine its velocity and acceleration at any time.

(b) Find the magnitudes of the velocity and acceleration at $t=0$.

Sol

(a) The position vector r of the particle is $r = x\bar{i} + y\bar{j} + z\bar{k}$

$$r = e^{-t} i + 2\cos 3t\, j + 2\sin 3t\, k$$

The velocity is $v = \dfrac{dr}{dt} = -e^{-t} i - 6\sin 3t\, j + 6\cos 3t\, k$

The acceleration is $a = \dfrac{d^2 r}{dt^2} = e^{-t} i - 18\cos 3t\, j - 18\sin 3t\, k$

(b) At $t=0$, $\dfrac{dr}{dt} = -i + 6k$ and $\dfrac{d^2 r}{dt^2} = i - 18j$. Then

magnitude of velocity at $t=0$ is $\sqrt{(-1)^2 + (6)^2} = \sqrt{37}$

magnitude of acceleration at $t=0$ is $\sqrt{1^2 + (-18)^2} = \sqrt{325}$

Example:

If $\bar{A} = 5t^2 i + t j - t^3 k$ and $\bar{B} = \sin t\, i - \cos t\, j$. Find

(a) $\dfrac{d}{dt}(\bar{A} \cdot \bar{B})$ (b) $\dfrac{d}{dt}(\bar{A} \times \bar{B})$ (c) $\dfrac{d}{dt}(A \cdot A)$

Partial Derivatives of Vectors

If $\bar{A}$ is a vector depending on more than one scalar variable, say x, y, z for example, then we write $\bar{A} = \bar{A}(x, y, z)$. The partial derivative of $\bar{A}$ with respect to x is defined as

$$\frac{\partial \bar{A}}{\partial x} = \lim_{\Delta x \to 0} \frac{\bar{A}(x+\Delta x, y, z) - \bar{A}(x, y, z)}{\Delta x}$$

if this limit exist. Similarly,

$$\frac{\partial \bar{A}}{\partial y} = \lim_{\Delta y \to 0} \frac{\bar{A}(x, y+\Delta y, z) - \bar{A}(x, y, z)}{\Delta y}$$

$$\frac{\partial \bar{A}}{\partial z} = \lim_{\Delta z \to 0} \frac{\bar{A}(x, y, z+\Delta z) - \bar{A}(x, y, z)}{\Delta z}$$

are the partial derivatives of $\bar{A}$ with respect to y and z respectively if these limits exist.

Higher derivatives can be defined as in the calculus. Thus, for example,

$$\frac{\partial^2 \bar{A}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{A}}{\partial x}\right), \quad \frac{\partial^2 \bar{A}}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial \bar{A}}{\partial y}\right), \quad \frac{\partial^2 \bar{A}}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{\partial \bar{A}}{\partial z}\right)$$

$$\frac{\partial^2 \bar{A}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \bar{A}}{\partial y}\right), \quad \frac{\partial^2 \bar{A}}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial \bar{A}}{\partial x}\right), \quad \frac{\partial^3 \bar{A}}{\partial x \partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial^2 \bar{A}}{\partial z^2}\right)$$

If A has continuous partial derivatives of the second order at least, then $\frac{\partial^2 \bar{A}}{\partial x \partial y} = \frac{\partial^2 \bar{A}}{\partial y \partial x}$, i.e. the order of differentiation does not matter.

Rules for partial differentiation of vectors are similar to those used in elementary calculus for scalar functions. Thus if $\bar{A}$ and $\bar{B}$ are functions of x, y, z then, for example,

(1) $\frac{\partial}{\partial x}(\bar{A} \cdot \bar{B}) = \bar{A} \cdot \frac{\partial \bar{B}}{\partial x} + \frac{\partial \bar{A}}{\partial x} \cdot \bar{B}$

(2) $\frac{\partial}{\partial x}(\bar{A} \times \bar{B}) = \bar{A} \times \frac{\partial \bar{B}}{\partial x} + \frac{\partial \bar{A}}{\partial x} \times \bar{B}$

(3) $\frac{\partial^2}{\partial y \partial x}(\bar{A}\cdot\bar{B}) = \frac{\partial}{\partial y}\left[\frac{\partial}{\partial x}(\bar{A}\cdot\bar{B})\right] = \frac{\partial}{\partial y}\left[\bar{A}\cdot\frac{\partial \bar{B}}{\partial x} + \frac{\partial \bar{A}}{\partial x}\cdot\bar{B}\right]$

$$= \bar{A}\cdot\frac{\partial^2 \bar{B}}{\partial y \partial x} + \frac{\partial \bar{A}}{\partial y}\cdot\frac{\partial \bar{B}}{\partial x} + \frac{\partial \bar{A}}{\partial x}\cdot\frac{\partial \bar{B}}{\partial y} + \frac{\partial^2 \bar{A}}{\partial y \partial x}\cdot\bar{B}$$

Differentials of vectors:

Follow rules similar to those of elementary Calculus. For example,

(1) If $\bar{A} = A_1 i + A_2 j + A_3 k$, then $d\bar{A} = dA_1 i + dA_2 j + dA_3 k$

(2) $d(\bar{A}\cdot\bar{B}) = \bar{A}\cdot d\bar{B} + d\bar{A}\cdot B$

(3) $d(\bar{A}\times\bar{B}) = \bar{A}\times d\bar{B} + d\bar{A}\times\bar{B}$

(4) If $\bar{A} = \bar{A}(x, y, z)$, then $d\bar{A} = \frac{\partial \bar{A}}{\partial x}dx + \frac{\partial \bar{A}}{\partial y}dy + \frac{\partial \bar{A}}{\partial z}dz$

<u>Example</u>

If $\bar{A} = (2x^2y - x^4)i + (e^{xy} - y\sin x)j + (x^2\cos y)k$, Find:

$\frac{\partial \bar{A}}{\partial x}$, $\frac{\partial \bar{A}}{\partial y}$, $\frac{\partial^2 \bar{A}^2}{\partial x^2}$, $\frac{\partial^2 A}{\partial y^2}$, $\frac{\partial^2 \bar{A}}{\partial x \partial y}$, $\frac{\partial^2 \bar{A}}{\partial y \partial x}$