Discrete Random Variables and Their Probability Distributions

Basic Definition

A random variable is a real-valued function defined over a sample space. Consequently, a random variable can be used to identify numerical events that are of interest in an experiment. For example, the event of interest in an opinion poll regarding voter preferences is not usually the particular people sampled or the order in which preferences were obtained but Y = the *number* of voters favoring a certain candidate or issue. The observed value of this random variable must be zero or an integer between 1 and the sample size. Thus, this random variable can take on only a finite number of values with nonzero probability. A random variable of this type is said to be discrete.

Definition:

A random variable Y is said to be discrete if it can assume only a finite or countably infinite number of distinct values.

- The Probability Distribution for a Discrete Random Variable

Notationally, we will use an uppercase letter, such as Y , to denote a random variable and a lowercase letter, such as y, to denote a particular value that a random variable may assume. For example, let Y denote any one of the six possible values that could be observed on the upper face when a die is tossed. After the die is tossed, the number actually observed will be denoted by the symbol y. Note that Y is a random variable, but the specific observed value, y, is not random.

The expression $(Y = y)$ can be read; the set of all points in S assigned the value y by the random variable Y .

It is now meaningful to talk about the probability that Y takes on the value y, denoted by $P(Y = y)$. This probability is defined as the sum of the probabilities of appropriate sample points in S.

Definition:

The probability that Y takes on the value y, $P(Y = y)$, is defined as the sum of the probabilities of all sample points in S that are assigned the value y. We will sometimes denote $P(Y = y)$by $P(y)$.

Because $P(y)$.is a function that assigns probabilities to each value y of the random variable, it is sometimes called the probability function for Y.

Definition:

The probability distribution for a discrete variable Y can be represented by a formula, a table, or a graph that provides $P(y) = P(Y = y)$ for all y.

Notice that $P(y) \geq 0$ for all y, but the probability distribution for a discrete random variable assigns nonzero probabilities to only a countable number of distinct y values. Any value y not explicitly assigned a positive probability is understood to be such that $P(y) = 0$. We illustrate these ideas with an example.

Example:

A supervisor in a manufacturing plant has three men and three women working for him. He wants to choose two workers for a special job. Not wishing to show any biases in his selection, he decides to select the two workers at random. Let Y denote the number of women in his selection. Find the probability distribution for Y

Solution

The supervisor can select two workers from six in $\binom{6}{2} = 15$ ways. Hence, S contains 15 sample points, which we assume to be equally likely because random sampling was employed. Thus, $\mathrm{P(E_i)} = \frac{1}{15}$, for i = 1,2, … … . ,15. The values for Y that have nonzero probability are 0, 1, and 2. The number of ways of selecting $Y = 0$ women is $\binom{3}{0}\binom{3}{2}$ because the supervisor must select zero workers from the three women and two from the three men. Thus, there are $\binom{3}{0}\binom{3}{2} = 1 * 3 = 3$ sample points in the event $Y = 0$, and

$$\mathrm{P(0) = P(Y = 0)} = \frac{\binom{3}{0}\binom{3}{2}}{15} = \frac{3}{15} = \frac{1}{5}$$

Similarly,

$$\mathrm{P(1) = P(Y = 1)} = \frac{\binom{3}{1}\binom{3}{1}}{15} = \frac{9}{15} = \frac{3}{5}$$

$$\mathrm{P(2) = P(Y = 2)} = \frac{\binom{3}{2}\binom{3}{0}}{15} = \frac{3}{15} = \frac{1}{5}$$

Notice that $(Y = 1)$ is by far the most likely outcome. This should seem reasonable since the number of women equals the number of men in the original group.

Theorem:

For any discrete probability distribution, the following must be true:

1. $0 \le P(y) \le 1$ for all y.
2. $\sum_y P(y) = 1$, where the summation is over all values of y with nonzero probability.

- ## The Expected Value of a Random Variable or a Function of a Random Variable

Definition:

Let Y be a discrete random variable with the probability function $P(y)$. Then the expected value of Y , $E(Y)$, is defined to be

$$\mathrm{E(Y)} = \sum_{\mathrm{y}} \mathrm{yP(y)}.$$

If $P(y)$ is an accurate characterization of the population frequency distribution, then $E(Y) = \mu$, the population mean.

Example:

if Y is the random variable with probability function $P(y)$ compute the expected value.

y	$P(y)$
0	1/4
1	1/2
2	1/4

$$\mathrm{E(Y)} = \sum_{\mathrm{y}} \mathrm{yP(y)}$$

$$E(Y) = 0 * \frac{1}{4} + 1 * \frac{1}{2} + 2 * \frac{1}{4} = 1$$

Theorem:

Let Y be a discrete random variable with probability function $P(y)$ and $g(Y)$ be a real-valued function of Y. Then the expected value of $g(Y)$ is given by

$$E[g(Y)] = \sum_{\text{all } y} g(y)P(y).$$

Definition:

If Y is a random variable with mean $E(Y) = \mu$, the variance of a random variable Y is defined to be the expected value of. That is,

$$V(Y) = E[(Y - \mu)^2] = E(Y^2) - E(Y)^2].$$

The standard deviation of Y is the positive square root of $V(Y)$.

If $P(y)$ is an accurate characterization of the population frequency distribution (and to simplify notation, we will assume this to be true), then $E(Y) = \mu, V(Y) = \sigma^2$, the population variance, and σ is the population standard deviation.

Example:

The probability distribution for a random variable Y is given in the belowTable Find the mean, variance, and standard deviation of Y .

y	$P(y)$
0	1/8
1	1/4
2	3/8
3	1/4

$$\mu = E(y) = \sum_{y=0}^{3} yP(y) = (0)\left(\frac{1}{8}\right) + (1)\left(\frac{1}{4}\right) + (2)\left(\frac{3}{8}\right) + (3)\left(\frac{1}{4}\right) = 1.75$$

$$\sigma^2 = E(y^2) - E(y)^2$$

$$E(y)^2 = (1.75)^2 = 3.0625$$

$$E(y^2) = \sum_{y=0}^{3} y^2P(y) = (0^2)\left(\frac{1}{8}\right) + (1^2)\left(\frac{1}{4}\right) + (2^2)\left(\frac{3}{8}\right) + (3^2)\left(\frac{1}{4}\right) = 4$$

$$\sigma^2 = 4 - 3.0625 = 0.9375$$

Standard deviation of Y=$\sigma = +\sqrt{0.9375} = 0.97$

Theorem:

Let Y be a discrete random variable with probability function $p(y)$ and c be a Constant. Then $E(c) = c$.

Proof

Consider the function $g(Y) \equiv c$

$$E(c) = \sum_{y} cP(y) = c\sum_{y} P(y)$$

But $\sum_y P(y) = 1$ and, hence, $E(c) = c(1) = c$.

Theorem:

Let Y be a discrete random variable with probability function $P(y)$, $g(y)$ be a function of Y and c be a Constant. Then

$$E\{cg(y)\} = cE\{g(y)\}.$$

$$E\{cg(y)\} = \sum_y cg(y)P(y) = c\sum_y g(y)P(y) = cE\{g(y)\}$$

Theorem:

Let Y be a discrete random variable with probability function $P(y)$, $g_1(y), g_2(y), \ldots\ldots g_k(y)$ be k function of Y. Then $E[g_1(y) + g_2(y) + \cdots + g_k(y)] = E[g_1(y)] + E[g_2(y)] + \cdots + E[g_K(y)]$]

Example:

Let Y be a discrete random variable with probability function $P(y)$, and X is the function of Y

$$X = 3y$$

Compute $E(X)$.

Y	0	1	2
$P(y)$	0.3	0.3	0.4

$$E(X) = E(3y) = 3E(y) = 3\sum_{y=0}^{2} yP(y) = 3[(0)(0.3) + (1)(0.3) + (2)(0.4)] = 3.3$$

The Uniform probability Distribution:

Sample space $S = \{1, 2, 3, .., k\}$.

Probability measure: equal assignment $(1/k)$ to all outcomes, ie all outcomes are equally likely.

Definition:

A random variable Y is said to have a Uniform probability distribution function if and only if

$$P(Y = y) = \frac{1}{k}, \quad y = 1,2,3, \ldots.., k$$

Theorem:

Let Y be a uniform. Then

$$\mu = E(Y) = \frac{k+1}{2} and\ \sigma^2 = V(Y) = \frac{k^2+1}{12}.$$

For example, if Y is the score on a fair die, $P(Y = y) = \frac{1}{6}$, $for\ y = 1,2, \ldots.., 6$

The Bernoulli probability Distribution:

A Bernoulli trial is an experiment which has (or can be regarded as having) only two possible outcomes – s (‘success’) and f (‘failure’).

Sample space $\{s, f\}$. The words ‘success’ and ‘failure’ are merely labels – they do not necessarily carry with them the ordinary meanings of the words.

For example in life insurance, a success could mean a death.

Probability measure: $P(\{s\}) = p$, $P(\{f\}) = 1 - p$

Random variable Y defined by $(s) = 1, Y(f) = 0$. X is the number of successes that occur (0 or 1).

Definition:

A random variable Y is said to have a Bernoulli probability distribution function if and only if

$$P(Y = y) = p^y 1 - p^{1-y} \quad y = 0,1, \quad 0 < p < 1$$

Theorem:

Let Y be a Bernoulli random variable Then

$$\mu = E(Y) = p \text{ and } \sigma^2 = V(Y) = p(1-p).$$

The Binomial Probability Distribution:

Some experiments consist of the observation of a sequence of identical and independent trials, each of which can result in one of two outcomes. Each item leaving a manufacturing production line is either defective or nondirective. Each shot in a sequence of firings at a target can result in a hit or a miss, and each of n persons Questioned prior to a local election either favors candidate Jones or does not. In this Section we are concerned with experiments, known as *binomial experiments*, that Exhibit the following characteristics.

Definition:

A binomial experiment possesses the following properties:
1. The experiment consists of a fixed number, n, of identical trials.
2. Each trial results in one of two outcomes: success, S, or failure, F.
3. The probability of success on a single trial is equal to some value$P$ and Remains the same from trial to trial. The probability of a failure is equal to

$$q = (1 - p).$$

4. The trials are independent.
5. The random variable of interest is Y , the number of successes observed during the n trials.

the binomial expansion

$$(q+p)^n = \binom{n}{0} q^n + \binom{n}{1} p^1 q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \cdots + \binom{n}{n} p^n$$

You will observe that $\binom{n}{0} q^n = p(0)$, $\binom{n}{1} p^1 q^{n-1} = p(1)$, and in general $P(y) = \binom{n}{y} p^y q^{n-y}$ It also follows that $p(y)$ satisfies the necessary properties for a probability function because $p(y)$ is positive for $y = 0, 1, \dots, n$ and [because $(q + p) = 1]$

$$\sum_y P(y) = \sum_{y=0}^{n} \binom{n}{y} p^y q^{n-y} = (q+p)^n = 1^n = 1$$

Theorem:

Let Y be a binomial random variable based on n trials and success probability p. Then

$$\mu = E(Y) = np \text{ and } \sigma^2 = V(Y) = npq.$$

Example(1):

A fair coin is tossed 6 times find the probability of obtaining

a) Exactly 4 heads;
b) At least 5 heads;
c) At most 2 heads;

$$P(X=r) = \binom{n}{r} p^r q^{n-r}$$

$$p = \frac{1}{2} \qquad q = \frac{1}{2} \qquad n = 6$$

a) $P(X=4)=\binom{6}{4}\frac{1}{2}^{4}*\frac{1}{2}^{2}=15*\frac{1}{16}*\frac{1}{4}=\frac{15}{64}$

b) $P(X\geq 5)=P(X=5)orP(X=6)$

$$=\binom{6}{5}\frac{1}{2}^{5}*\frac{1}{2}^{1}+\binom{6}{6}\frac{1}{2}^{6}*\frac{1}{2}^{0}=6*\frac{1}{32}*\frac{1}{2}+1*\frac{1}{64}*1=\frac{7}{64}$$

a) $P(X\leq 2)=P(X=0)orP(X=1)orP(X=2)$

$$=\binom{6}{0}\frac{1}{2}^{0}*\frac{1}{2}^{6}+\binom{6}{1}\frac{1}{2}^{1}*\frac{1}{2}^{5}+\binom{6}{2}\frac{1}{2}^{2}*\frac{1}{2}^{4}=\frac{22}{64}$$

Example (2):

An unbiased die with 6 faces is thrown 5 times. Find the probability that a:

a) Factor of 6 appears Exactly 3 times;

b) Perfect square appears at most 4 times;

$$P(X=r)=\binom{n}{r}p^{r}q^{n-r}$$

a) $S=\{1,2,3,4,5,6\}\ and\ n(s)=6$

$F=[1,2,3,6] and\ n(F)=4$

$p=\frac{4}{6}=\frac{2}{3}\qquad q=\frac{2}{6}=\frac{1}{3}\quad ,n=5\,,r=3$

$$P(X=3)=\binom{5}{3}\frac{2}{3}^{3}\frac{1}{3}^{2}=10*\frac{8}{27}*\frac{1}{9}=\frac{80}{243}$$

b) $S=\{1,2,3,4,5,6\}\ and\ n(s)=6$

$E=[1,4] and\ n(E)=2$

$p=\frac{2}{6}=\frac{1}{3}\qquad q=\frac{4}{6}=\frac{2}{3}\quad ,n=5\,,r=0,1,2,3\ and\ 4$

$$P(X\leq 4)=1-P(X=5)$$

$$1-\binom{5}{5}\frac{1}{3}^{5}\frac{2}{3}^{0}=1-1*\frac{1}{243}*1=\frac{242}{243}$$

Example (3):

A test contains 10 multiple choice questions comprising of 4 options in which only one options is correct. Find the probability that a candidate can guess 7 out of the 10 questions correctly.

$p=\frac{1}{4}\qquad q=\frac{3}{4}\quad ,n=10\,,r=7$

$$P(X=7)=\binom{10}{7}\frac{1}{4}^{7}\frac{3}{4}^{3}=120*\frac{1}{16384}*\frac{27}{64}=0.003089$$

Example (4):

The probability that a patient will be cured of corona virus when injected with the new vaccine is 0.8. find the probability that exactly 3 out of 8 corona virus patient will be cured being injected with the vaccine.

$$p = 0.8 \qquad q = 0.2 \quad , n = 8 \, , r = 3$$

$$P(X = 3) = \binom{8}{3} 0.8^3 0.2^5 = 56 * 0.512 * 0.00032 = 0.00917$$

The Geometric Probability Distribution

The random variable with the geometric probability distribution is associated with an experiment that shares some of the characteristics of a binomial experiment. This experiment also involves identical and independent trials, each of which can result in one of two outcomes: success or failure. The probability of success is equal to p and is constant from trial to trial. However, instead of the number of successes that occur in n trials, the geometric random variable Y is the number of the trial on which the first success occurs. Thus, the experiment consists of a series of trials that concludes with the first success. Consequently, the experiment could end with the first trial if a success is observed on the very first trial, or the experiment could go on indefinitely.

Definition:

A random variable Y is said to have a geometric probability distribution if and only if

$$P(y) = q^{y-1}p, y = 1, 2, 3, \ldots, 0 \leq p \leq 1.$$

Theorem:

Let Y be a random variable with a geometric distribution,

$$\mu = E(Y) = \frac{1}{p}$$

$$and\ \sigma^2 = V(Y) = \frac{1-p}{p^2}$$

Example (1):

Suppose that the probability of engine malfunction during any one-hour period is $p = .02$. Find the probability that a given engine will survive two hours. And the mean & standard deviation

Solution Letting Y denote the number of one-hour intervals until the first malfunction, we have

$$P(survive\ two\ hours) = P(Y \geq 3) = \sum_{y=3}^{\infty} P(y)$$

Because $\sum_{y=1}^{\infty} P(y) = 1$

$$P(survive\ two\ hours) = 1 - \sum_{y=1}^{2} P(y) = 1 - p - qp = 1 - (0.02) - (0.98)(0.02)$$

$$= 0.9604$$

$$E(Y) = \frac{1}{P} = \frac{1}{0.02} = 50$$
$$\sigma = \sqrt{\frac{1-P}{P^2}} = \sqrt{\frac{0.98}{0.0004}} = 49.49$$

The Poisson Probability Distribution:

Definition:

A random variable Y is said to have a Poisson probability distribution if and only if

$$P(y) = \frac{\lambda^y}{y!}e^{-\lambda} \quad , y = 0,1,2, \dots , \qquad \lambda > 0$$

Example (1):

Suppose that a random system of police patrol is devised so that a patrol officer may visit a given beat location $Y = 0, 1, 2, 3, \dots$ times per half-hour period, with each location being visited an average of once per time period. Assume that Y possesses, approximately, a Poisson probability distribution. Calculate the probability that the patrol officer will miss a given location during a half-hour period. What is the probability that it will be visited once? Twice? At least once?

Solution: For this example the time period is a half-hour, and the mean number of visits per half-hour interval is $\lambda = 1$. Then

$$P(y) = \frac{1^y}{y!}e^{-1} = \frac{e^{-1}}{y!} , \qquad y = 0,1,2, \dots.$$

The event that a given location is missed in a half-hour period corresponds to $(Y = 0)$, and

$$P(y = 0) = \frac{e^{-1}}{0!} = e^{-1} = 0.368$$

Similarly,

$$P(1) = e^{-1} = 0.368$$
$$P(2) = \frac{e^{-1}}{2!} = 0.184$$

The probability that the location is visited at least once is the event$(Y \geq 1)$. Then

$$P(Y \geq 1) = \sum_{y=1}^{\infty} P(y) = 1 - P(0) = 1 - e^{-1} = 0.632$$

Theorem:

Let Y be a random variable with a Poisson distribution,

$$\mu = E(Y) = \lambda$$
$$and\ \sigma^2 = V(Y) = \lambda$$

Exercise:

Let Y denote a random variable that has a Poisson distribution with mean $\lambda = 2$. Find

a P(Y $=$ 4).

b P(Y $\geq$ 4).

c P(Y $<$ 4).

d $P(Y \geq 4 | Y \geq 2)$.