Stat215

Exercises 6

Q1) The joint probability function of two discrete random variables X and Y is given by

f(x,y) = cxy for x = 1, 2, 3 and y = 1, 2, 3 and equals zero otherwise. Find:

a) **The constant c.**

$\sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i, y_j) = 1$

c+2c+3c+2c+4c+6c+3c+6c+4c = 1 ➔ c = 1/36

6c+12c+18c = 1 ➔ c = 1/36

x \ y	1	2	3	$f_Y(y)$
1	1c	2c	3c	6c
2	2c	4c	6c	12c
3	3c	6c	9c	18c
$f_X(x)$	6c	12c	18c	1 = total

b) **P(X = 2 ,Y = 3)**

= f(2,3) = 6c = 6/36 = 1/6

c) **$P(1 \le X \le 2, Y \le 2)$**

x = 1,2 ; y = 1,2

$P(1 \le X \le 2, Y \le 2)$ = P(X = 1,Y = 1)+P(X = 1,Y = 2)+P(X = 2,Y = 1)+P(X = 2,Y = 2)

= f(1,1) + f(1,2) + f(2,1) + f(2,2) = 1c+2c+2c+4c = 9c = 9/36

d) **$(X \ge 2)$**

x = 2,3

$(X \ge 2)$ = P(X = 2) + P(X = 3) = $f_X(2)$+ $f_X(3)$ =12c+18c = 30c = 30/36

e) **P(Y<2)**

y = 1

P(Y<2) = P(Y = 1) = $f_Y(1)$ = 6c = 6/36 = 1/6

f) **P(X = 1)**

= $f_X(1)$ = 6c = 6/36 = 1/6

g) **P(Y = 3)**

= $f_Y(3)$ = 18c = 18/36

Q2)For the random variables of Problem 1, find the marginal probability function of X and Y.

Determine ether X and Y are independent.

Marginal dis.of X:

X	X_1=1	X_2 =2	X_3= 3	Total
F_x(x) =P(X=x)	6/36	12/36	18/36	1

Marginal dis.of Y:

Y	Y_1=1	Y_2=2	Y_3= 3	Total
$F_y(y)$ =P(Y=x)	6/36	12/36	18/36	1

Remark:

Are X and Y independent?

If $f(x,y) = f_x(x)\, f_y(y)$

For all x=1,2,3 y=1,2,3 ,Then x and y are independent.

But if there some values of x and y which make that If $f(x,y) \neq f_x(x)\, f_y(y)$

then x and y are not independent.

In this example we have:

$f(1,1) = f_x(1)f_y(1)$

$f(1,2) = f_x(1)f_y(2)$

.

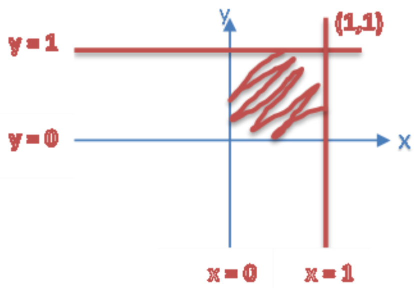
.

.

$f(3,3) = f_x(3)f_y(3)$

So as If $f(x,y) = f_x(x)\, f_y(y)$ for all x and y , then x and y are independent.

Q3) $f(x,y) = c(x^2, y^2)$, $0 \leq x \leq 1$, $0 \leq y \leq 1$

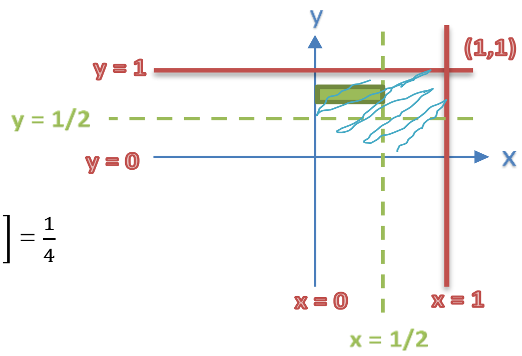


a) $\int_0^1 \int_0^1 c(x^2, y^2)dx\,dy = 1 \Rightarrow c\int_0^1 [\int_0^1 (x^2, y^2)dx]\,dy = 1$

$\Rightarrow c\int_0^1 [\frac{x^3}{3} + xy^2]_0^1 dy = 1 \Rightarrow c\int_0^1 \left(\frac{1}{3} + y^2\right) dy = 1$

$\Rightarrow c\,[\frac{1}{3}y + \frac{y^3}{3}]_0^1 = 1 \Rightarrow c\,[\frac{1}{3} + \frac{1}{3}] = 1 \Rightarrow c = \frac{3}{2}$

b) $P(X < \frac{1}{2}, Y > \frac{1}{2})$



$= \frac{3}{2}\int_{\frac{1}{2}}^1 [\int_0^{\frac{1}{2}}(x^2, y^2)dx]\,dy = \frac{3}{2}\int_{\frac{1}{2}}^1 [\frac{x^3}{3} + xy^2]_0^{\frac{1}{2}} dy$

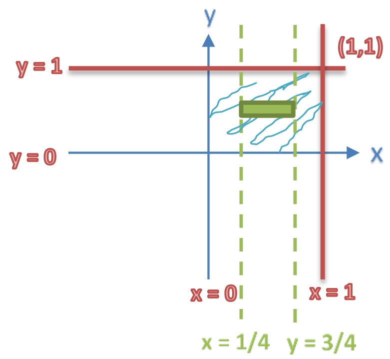
$= \frac{3}{2}\int_{\frac{1}{2}}^1 \left(\frac{1}{24} + \frac{1}{2}y^2\right) dy = \frac{3}{2}\,[\frac{y}{24} + \frac{y^3}{6}]_{\frac{1}{2}}^1 = \frac{3}{2}\left[\left(\frac{1}{24} + \frac{1}{6}\right) - \left(\frac{1}{48} + \frac{1}{48}\right)\right] = \frac{1}{4}$

c) P $(\frac{1}{4} < X < \frac{3}{4})$

$= \frac{3}{2}\int_0^1 [\int_{\frac{1}{4}}^{\frac{3}{4}} (x^2, y^2)dx]\, dy = \frac{39}{128}$

?

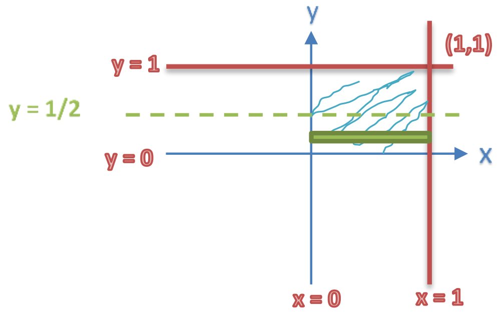
29/64



d) P $(Y < \frac{1}{2})$

$= \frac{3}{2}\int_0^{\frac{1}{2}} [\int_0^1 (x^2, y^2)dx]\, dy = \frac{3}{2}\int_0^{\frac{1}{2}} [\frac{x^3}{3} + xy^2]_0^1 dy$

$= \frac{3}{2}\int_0^{\frac{1}{2}} (\frac{1}{3} + y^2)dy = \frac{3}{2}\ [\frac{1}{3}y + \frac{y^3}{3}]_0^{\frac{1}{2}} = \frac{1}{2}\left[\frac{1}{2} + \frac{1}{2^3}\right] = \frac{5}{16}$



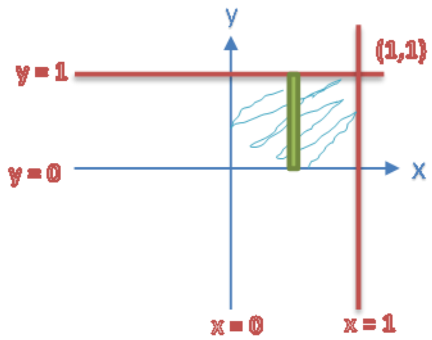
e) Whether X and Y are independent

f(x,y) ≠ g(x) h(y)

X and Y are not independent

Q4)For the random variables of Problem 3, find the marginal probability function of X and Y

Marginal dis. of X:



$f_x(x) = \int_0^1 f(x,y)\, dy$

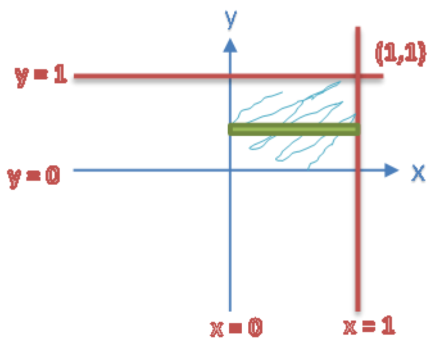
$= 3/2\int_0^1 (x^2, y^2)\, dy\ = 3/2\ [x^2y + \frac{y^3}{3}]_0^1$

$= 3/2\ [x^2 + 1/3] = 3/2x^2 + 1/2$

then

$f_x(x) = 3/2\ x^2 + 1/2\ , \quad 0 < x < 1$

Marginal dis. of Y:



$f_y(y) = \int_0^1 f(x,y)\, dx$

$= 3/2\ \int_0^1 (x^2, y^2)\, dx = 3/2\ [y^2x + \frac{x^3}{3}]_0^1$

$= 3/2\ [y^2 + 1/3]\ = 3/2\ y^2 + 1/2$

then

$f_y(y) = 3/2\ y^2 + 1/2\ , \quad 0 < y < 1$

Q5) For the distribution of problem 1, find the conditional probability function of X given Y, Y given X .

Dis. of X|Y : $f_{X|Y=y}(x) = \frac{f(x,y)}{f_Y(y)}$, $x = 1,2,3$

1) if Y=1: $f_{X|Y=1}(x) = \frac{f(x,1)}{f_Y(1)} = \frac{f(x,1)}{6/36} \longrightarrow$

x	1	2	3	total
$f_{X\|Y=1}(x)$	$\frac{f(1,1)}{6/36} = 1/6$	$\frac{f(2,1)}{6/36} = 1/3$	$\frac{f(3,1)}{6/36} = 1/2$	1

2) if Y=2: $f_{X|Y=2}(x) = \frac{f(x,2)}{f_Y(2)} = \frac{f(x,2)}{12/36} \longrightarrow$

x	1	2	3	total
$f_{X\|Y=2}(x)$	$\frac{f(1,2)}{12/36} = 2/12$	$\frac{f(2,2)}{12/36} = 4/12$	$\frac{f(3,2)}{12/36} = 6/12$	1

3)if Y=3: $f_{X|Y=3}(x) = \frac{f(x,3)}{f_Y(3)} = \frac{f(x,3)}{18/36} \longrightarrow$

x	1	2	3	total
$f_{X\|Y=3}(x)$	$\frac{f(1,3)}{18/36} = 3/18$	$\frac{f(2,3)}{18/36} = 6/18$	$\frac{f(3,3)}{18/36} = 9/18$	1

Dis. of Y|X : $f_{Y|X=x}(y) = \frac{f(x,y)}{f_X(x)}$, $y = 1,2,3$

1) if X=1: $f_{Y|X=1}(y) = \frac{f(1,y)}{f_X(1)} = \frac{f(1,y)}{6/36} \longrightarrow$

y	1	2	3	total
$f_{Y\|X=1}(y)$	$\frac{f(1,1)}{6/36} = 1/6$	$\frac{f(1,2)}{6/36} = 2/6$	$\frac{f(1,3)}{6/36} = 3/6$	1

2) if X=2: $f_{Y|X=2}(y) = \frac{f(2,y)}{f_X(2)} = \frac{f(2,y)}{12/36} \longrightarrow$

y	1	2	3	total
$f_{Y\|X=2}(y)$	$\frac{f(2,1)}{12/36} = 2/12$	$\frac{f(2,2)}{12/36} = 4/12$	$\frac{f(2,3)}{12/36} = 6/12$	1

3) if X=3: $f_{Y|X=3}(y) = \frac{f(3,y)}{f_X(3)} = \frac{f(3,y)}{18/36} \longrightarrow$

y	1	2	3	total
$f_{Y\|X=3}(y)$	$\frac{f(3,1)}{18/36} = 3/18$	$\frac{f(3,2)}{18/36} = 6/18$	$\frac{f(3,3)}{18/36} = 9/18$	1

Q6) Let $f(x, y) = \{ x + y \; 0 \le x \le 1, 0 \le y \le 1$ Find the conditional probability function of X given Y, Y given X.

$$f(x,y) = x + y$$

Marginal of x,y

$$f(x) = \int_0^1 (x+y)dy = xy + \frac{y^2}{2} = x + \frac{1}{2}$$

$$f(y) = y + \frac{1}{2}$$

Conditional

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{x+y}{y+\frac{1}{2}} = \frac{2(x+y)}{2y+1}$$

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{x+y}{x+\frac{1}{2}} = \frac{2(x+y)}{2x+1}$$

Q7) For the distribution of Problem 3, find the conditional probability function of X given Y, Y given X.

conditional dis of x|y:

$$f_{x|y=y}(x) = \frac{f(x,y)}{f_y(y)} = \frac{x+y}{y+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}y^2+\frac{1}{2}} = \frac{x^2+y^2}{y^2+\frac{1}{3}}$$

For 0<x<1 where 0<y<1 fixed value

conditional dis of y|x:

$$f_{y|x=x}(y) = \frac{f(x,y)}{f_x(x)} = \frac{x+y}{y+\frac{1}{2}} = \frac{\frac{3}{2}(x^2+y^2)}{\frac{3}{2}x^2+\frac{1}{2}} = \frac{x^2+y^2}{x^2+\frac{1}{3}}$$

For o<y<1 where 0<x<1 fixed value

8) Let $f(x, y) = \{ e^{-(x+y)} \; x \ge 0, y \ge 0$ be the joint density function of X and Y. Find the conditional probability function of X given Y, Y given X.

$$f(x,y) = e^{-(x,y)}$$

$$f(x) = \int_0^\infty e^{-x} e^{-y} dy = e^{-y}(-e^{-y}) = e^{-x}[1-0] = e^{-x}$$

$$f(y) = \int_0^\infty e^{-x} e^{-y} dx = e^{-y}$$

Conditional x|y

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{e^{-(x+y)}}{e^{-y}} = e^{-x}$$

Conditional y|x

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{e^{(x+y)}}{e^{-x}} = e^{-y}$$

Q9) Let X and Y be random variables having joint density function

f$(x, y) = c(2x + y)$ $0 < x < 1, 0 < y < 2$ Find:

a. The constant c.

b. $P(X > 1/2 , Y < 3/2)$.

c .The (marginal) density function of X.

d. The (marginal) density function of Y.

a. $\boldsymbol{c \int_0^2 \int_0^1 (2x+y)\, dx\, dy = 1}$

$c \int_0^2 [\frac{2x^3}{3} + xy] |_0^1 \; dy = 1$ ➔ $c \int_0^2 [\frac{2}{3} + y] \; dy = 1$

➔ $c \; [\frac{2}{3}y + \frac{y^2}{2}]|_0^2 = 1$ ➔$c \; \left[\frac{4}{3} + 2\right] = 1$ ➔ $c \; \frac{10}{6} = 1$ ➔ $c = \frac{6}{10}$

b. $\boldsymbol{f_{X\,(x)= \frac{6}{10} \int_0^2 (2x+y)\, dy}}$

$\boldsymbol{\frac{6}{10} \; (2xy + \frac{y^2}{2})|_0^2 = \frac{6}{10} \; [4x+2] = \frac{6}{10} \; [4x+2]} = \frac{12}{5}x + \frac{6}{5} \qquad 0 < x < 1$

c. $\boldsymbol{p(x > \frac{1}{2} \text{ ، } y < \frac{3}{2})}$

$\frac{6}{10} \int_0^{\frac{3}{2}} \int_{\frac{1}{2}}^1 (2x+y)\, dx\, dy = \frac{6}{10} \int_0^{3/2} \left[\frac{2x^3}{3} + xy\right] |_{\frac{1}{2}}^1 \text{dy} = \frac{6}{10} \int_0^{3/2} \left[\left(\frac{2}{3} + \text{y}\right) - \left(\frac{1}{12} + \frac{1}{2}\text{y}\right)\right] \text{dy}$

$= \frac{6}{10} \int_0^{3/2} \left[\frac{2}{3} + \text{y} - \frac{1}{12} - \frac{1}{2}\text{y}\right] dy \quad = \frac{6}{10} \int_0^{3/2} \left[\frac{1}{2}\text{y} - \frac{7}{12}\right] dy$

$= \frac{6}{10} \left[\frac{1}{4}y^2 - \frac{7}{12}\text{y}\right] |_0^{3/2} = \frac{6}{10} \left[\frac{13}{48}\right] = \frac{13}{80}$

d. $\boldsymbol{f_{y(y)} = \frac{6}{10} \int_0^1 (2x+y)\, dx}$

$= \frac{6}{10} \left[\frac{2}{3} + \text{y}\right] \quad = \frac{2}{5} + \frac{6}{10}\text{y} \qquad 0 < \text{y} < 2$

Q10)The joint probability function for the random variables X and Y is given in following table, then find:

A)The marginal probability functions of X and Y.

f(x)=

y y / x	0	1	2	f(x)
0	1/18	1/9	1/6	1/3
1	1/9	1/18	1/9	5/18
2	1/6	1/6	1/18	7/18
f(y)	1/3	1/3	1/3	1

X	0	1	2
f(x)	1/3	5/18	7/18

f(y)=

Y	0	1	2
f(y)	1/3	1/3	1/3

B)p(1≤ $x < 3, y \geq 1$)

f(1,1)+f(1,2)+f(2.2)+f(2,1)=1/18+1/6+1/18+1/9=0.389

c)Determine whether X and Y are independent.

f(1,1) → فاصلة وليست نقطة

$f(\cancel{1.1})=f_x(1)f_y(1)$

$$\frac{1}{18}=\frac{1}{3}\times\frac{5}{18}$$

$$\frac{1}{18}\neq\frac{5}{54}$$

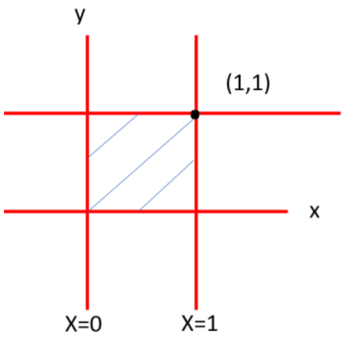
X and y not independent

Q11) Let X and Y be random variables having joint density function

$$f(x,y) = \begin{cases} x + y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Find: a. Var(X). b. Var(Y). c. σX. d. σY. e. σXY. f. p

solution (11):



f(x,y)= x+y

f(x)= $\int_0^1 x + y \, dy$ = xy+($\frac{y^2}{2}$)|)|$_0^1$= [x+$\frac{1}{2}$], 0<x<1

and the same

f(y))= $\int_0^1 (x + y) \, dx$ = [y+$\frac{1}{2}$], 0<y<1

a,b

E(x)=$\int_0^1 x$. f(x)dx=$\int_0^1 x$.(x+$\frac{1}{2}$)dx=$\int_0^1 x^2$ +($\frac{x}{2}$)dx

=($\frac{x^3}{3}$)+($\frac{x^4}{4}$)|$_0^1$=$\frac{1}{3}$+ $\frac{1}{4}$ = $\frac{7}{12}$

And the same

E(y) = $\int_0^1 y$. f(y) dy = $\frac{7}{12}$

E(x^2)= $\int_0^1 x^2$. f(x)dx=$\int_0^1 x^2$ (x+$\frac{1}{2}$)dx=$\int_0^1 x^3$ +($\frac{x^2}{2}$)dx

=($\frac{x^4}{4}$)+($\frac{x^3}{6}$)|$_0^1$=$\frac{1}{4}$ + $\frac{1}{16}$ = $\frac{10}{24}$ = $\frac{5}{12}$

And the same

E(y^2) = $\int_0^1 y^2$. f(y) dy = $\frac{5}{12}$

V(x) = E(x^2) – [E(x)]^2=$\frac{5}{12}$– ($\frac{7}{12}$)^2 = $\frac{11}{144}$

V(y) = E(y^2) – [E(y)]^2=$\frac{5}{12}$– ($\frac{7}{12}$)^2 = $\frac{11}{144}$

C.σ_x = $\sqrt{var(x)}$ = $\sqrt{\frac{11}{144}}$ = 0.2764

d.σ_y = $\sqrt{var(y)}$ = $\sqrt{\frac{11}{144}}$ = 0.2764

e.cov(x,y) = E(XY) – E(X)E(Y)

E(XY)= $\int_0^1\int_0^1 XY(x+y) \, dx \, dy = \int_0^1\int_0^1 x^2y + xy^2 \, dx \, dy$

$= \int_0^1 (\frac{x^3}{3})y + (\frac{x^2}{2})y^2 |_0^1 \, dy$

$= \int_0^1 \frac{1}{3}y + \frac{1}{2} y \, dy = \frac{y^2}{6} + y^3/6|_0^1 = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$

Cov(x,y) = E(X,Y) – E(X).E(Y) = $\frac{1}{3} - \frac{7}{12} \cdot \frac{7}{12} = \frac{1}{3} - \frac{49}{144} = \frac{-1}{144}$

f.= cov(x,y) / $\sqrt{var(x).var(y)} = \frac{\frac{-1}{144}}{\sqrt{\frac{11}{144} \cdot \frac{11}{144}}}$

$= \frac{\frac{-1}{144}}{\frac{11}{144}} = \frac{-1}{11} = -0.091$

Relation weak negative

Q12)Work Problem 11 if the joint density function is f(x,y) = $e^{-(x+y)}$ $x \geq 0$, $y \geq 0$ Find:
a) Var(X).
b) Var(Y).
c) σX.
d) σY.
e) σXY.
f) ρ.

$f_x(x) = \int_0^\infty e^{-(x+y)} \, dy = -e^{-(x+y)} |_0^\infty = e^{-x}$

$f_y(y) = \int_0^\infty e^{-(x+y)} \, dx = -e^{-(x+y)} |_0^\infty = e^{-y}$

a) $E(X) = \int_0^\infty x e^{-x} \, dx$

u=x dv=e^{-x}

du=dx v= -e^{-x}

$I = -xe^{-x} |_0^\infty + \int_0^\infty e^{-x} \, dx = -e^{-(x)} |_0^\infty = 1$

E(x^2)=$\int_0^\infty x^2 \, e^{-x}$ dx

u=x^2 dv=e^{-x}

du= 2x dx v= -e^{-x}

$I = -x^2 e^{-x} |_0^\infty + 2 \int_0^\infty x e^{-x} \, dx = 2(1) = 2$

V(x) = E(x^2) – (E(x))^2 = 2-(1)^2 = 1

b) E(y) = $\int_0^\infty y\, e^{-y}$ dy = 1 (was solved)

E(y^2) =$\int_0^\infty y^2 \;\; e^{-y}$ = 2 (was solved)

V(y) = E(y^2) – (E(y))^2 = 2-(1)^2 = 1

c) σX = $\sqrt{V(x)}$ = 1

d) σY = $\sqrt{V(y)}$ = 1

e) Con (x,y) = E (xy) –E (x) E (y)

E (xy) = $\int_0^\infty \int_0^\infty xy\, e^{-(x+y)}\, dx\, dy = \int_0^\infty\; y[\int_0^\infty x\, e^{-(x+y)}\, dx]\, dy$

u=x dv=$e^{-(x+y)}$

du=dx v= -$e^{-(x+y)}$

$$I = \int_0^\infty y[\, -xe^{-(x+y)}|_0^\infty + \int_0^\infty e^{-(x+y)}\, dx]dy \; = \int_0^\infty y\, e^{-(y)}\, dy$$

u=y dv=e^{-y}

du= dx v= -e^{-y}

$= -ye^{-(y)}|_0^\infty + \int_0^\infty e^{-(y)}\, dy$ = 1

Con (x,y) = E (xy) –E (x) E (y) = 1 – (1)(1) = 0

$\rho = \frac{\text{Con (x,y)}}{\sqrt{V(x)V(y)}} = \frac{0}{\sqrt{(1)(1)}} = 0$

Q13) Find a. The covariance. b. The correlation coefficient of two random variables X and Y. If

E(x)=2 , E(y)=3 ,E(xy)=10 ,E (x^2) $=10$, E (y^2) $=16$

solution (13):

a.Var(x)= E (x^2) $-$ [E (x)]^2

= 10-2^2 $=10$

Var(x)= E (y^2) $-$ [E (y)]^2

= 16-3^2 $=7$

Cov(x,y) =E(xy) -E(x)E(y)

4= = 10-[2$\times$ 3]

b. f=$\frac{\text{Cov(x,y)}}{\sqrt{var(x)var(y)}} = \frac{4}{\sqrt{6\times 7}} = \frac{4}{\sqrt{42}} = 0.6172$

Q14) The correlation coefficient of two random variables X and Y is -1/4 while their variances are 3 and 5. Find the covariance .

Answer:

Var(x)=3 Var(y)=5 $\rho X,Y = -\frac{1}{4}$ $Cov(X,Y)$=??

$$\boxed{\rho X,Y = \frac{Cov(X,Y)}{\sqrt{V(X)V(Y)}}}$$

$-\frac{1}{4} = \frac{\text{Cov(X,Y)}}{\sqrt{3*5}} \Longrightarrow -\frac{1}{4}\sqrt{15} = Co(X,Y) \Longrightarrow Cov(X,Y)$=-0.9682

Q15) be the joint density function of X and Y. Find the conditional probability function of X given Y, Y given X.

$$f(x,y)=\begin{cases} \frac{xy}{36} & x = 1,2,3 \quad y = 1,2,3 \\ 0 & otherwise \end{cases}$$

f(x,y)		y		
		1	**2**	**3**
	1	**36/1**	**1/18**	**1/12**
x	**2**	**18/1**	**9/1**	**6/1**
	3	**12/1**	**6/1**	**4/1**

y	1	2	3
$f_y(y)$	**6/1**	**3/1**	**2/1**

x	1	2	3
$f_x(x)$	**6/1**	**3/1**	**2/1**

x given y:

$f_{x|y}(x|y) = f_{1|1}(1|1) = \frac{f(x,y)}{f(y)} = \frac{f(1,1)}{f(1)} = \frac{1/36}{1/6} = \frac{1}{6}$

$f_{x|y}(x|y) = f_{2|1}(2|1) = \frac{f(x,y)}{f(y)} = \frac{f(2,1)}{f(1)} = \frac{1/18}{1/6} = \frac{1}{3}$

$f_{x|y}(x|y) = f_{3|1}(3|1) = \frac{f(x,y)}{f(y)} = \frac{f(3,1)}{f(1)} = \frac{1/12}{1/6} = \frac{1}{2}$

$f_{x|y}(x|y)= f_{1|2}(1|2)=\frac{f(x,y)}{f(y)}=\frac{f(1,2)}{f(2)}=\frac{1/18}{1/3}=\frac{1}{6}$

$f_{x|y}(x|y)= f_{2|2}(2|2)=\frac{f(x,y)}{f(y)}=\frac{f(2,2)}{f(2)}=\frac{1/9}{1/3}=\frac{1}{3}$

$f_{x|y}(x|y)= f_{3|2}(3|2)=\frac{f(x,y)}{f(y)}=\frac{f(3,2)}{f(2)}=\frac{1/6}{1/3}=\frac{1}{2}$

$f_{x|y}(x|y)= f_{1|3}(1|3)=\frac{f(x,y)}{f(y)}=\frac{f(1,3)}{f(3)}=\frac{1/12}{1/2}=\frac{1}{6}$

$f_{x|y}(x|y)= f_{2|3}(2|3)=\frac{f(x,y)}{f(y)}=\frac{f(2,3)}{f(3)}=\frac{1/6}{1/2}=\frac{1}{3}$

$f_{x|y}(x|y)= f_{3|3}(3|3)=\frac{f(x,y)}{f(y)}=\frac{f(3,3)}{f(3)}=\frac{1/4}{1/2}=\frac{1}{2}$

x	1	2	3
$f_{x\mid 2}(x\mid 2)$	**6/1**	**3/1**	**2/1**

$f_{x\mid y}(x\mid y)$		y		
		1	**2**	**3**
	1	**6/1**	**1/6**	**1/6**
x	**2**	**3/1**	**3/1**	**3/1**
	3	**2/1**	**2/1**	**2/1**

x	1	2	3
$f_{x\mid 1}(x\mid 1)$	**6/1**	**3/1**	**2/1**

X	1	2	3
$f_{x\mid 3}(x\mid 3)$	**6/1**	**3/1**	**2/1**

y given x:

- $f_{y|x}(y|x)= f_{1|1}(1|1)=\frac{f(x,y)}{f(x)}=\frac{f(1,1)}{f(1)}=\frac{1/36}{1/6}=\frac{1}{6}$
- $f_{y|x}(y|x)= f_{2|1}(2|1)=\frac{f(x,y)}{f(x)}=\frac{f(2,1)}{f(1)}=\frac{1/18}{1/6}=\frac{1}{3}$

- $f_{y|x}(y|x)= f_{3|1}(3|1)=\frac{f(x,y)}{f(x)}=\frac{f(3,1)}{f(1)}=\frac{1/12}{1/6}=\frac{1}{2}$
- $f_{y|x}(y|x)= f_{1|2}(1|2)=\frac{f(x,y)}{f(x)}=\frac{f(1,2)}{f(2)}=\frac{1/18}{1/3}=\frac{1}{6}$
- $f_{y|x}(y|x)= f_{2|2}(2|2)=\frac{f(x,y)}{f(x)}=\frac{f(2,2)}{f(2)}=\frac{1/9}{1/3}=\frac{1}{3}$
- $f_{y|x}(y|x)= f_{3|2}(3|2)=\frac{f(x,y)}{f(x)}=\frac{f(3,2)}{f(2)}=\frac{1/6}{1/3}=\frac{1}{2}$
- $f_{y|x}(y|x)= f_{1|3}(1|3)=\frac{f(x,y)}{f(x)}=\frac{f(1,3)}{f(3)}=\frac{1/12}{1/2}=\frac{1}{6}$
- $f_{y|x}(y|x)= f_{2|3}(2|3)=\frac{f(x,y)}{f(x)}=\frac{f(2,3)}{f(3)}=\frac{1/6}{1/2}=\frac{1}{3}$
- $f_{y|x}(y|x)= f_{3|3}(3|3)=\frac{f(x,y)}{f(x)}=\frac{f(3,3)}{f(3)}=\frac{1/4}{1/2}=\frac{1}{2}$

$f_{y\|x}(x\|y)$		**y**		
		1	**2**	**3**
	1	**6/1**	**1/3**	**2/1**
x	**2**	**6/1**	**3/1**	**2/1**
	3	**6/1**	**3/1**	**2/1**

Y	1	2	3
$f_{y\|1}(y\|1)$	**6/1**	**3/1**	**2/1**

Y	1	2	3
$f_{y\|2}(y\|2)$	**6/1**	**3/1**	**2/1**

Y	1	2	3
$f_{y\|3}(y\|3)$	**6/1**	**3/1**	**2/1**

Q16) The joint probability function of two discrete random variables X and Y is given by $(x, y) = c(2x + y)$, where x and y can assume all integers such

that $0 \le x \le 2$, $0 \le y \le 3$ and $f(x, y) = 0$ otherwise. Find:

a. The value of the constant c.

b. $(X = 2, Y = 1)$.

c. $(X \ge 1, Y \le 2)$

solution (16):

f(x)=c(2x+y) $0 \le X \le 2$ $0 \le y \le 3$

a. the value of the constant c

$$\sum_{y=0}^{3}\sum_{x=0}^{2} c(2X+Y)=1$$

f(0,0)+f(0,1)+f(0,2)+f(0,3)+ f(1,0)+f(1,1)+f(1,2)+f(1,3)+

f(2,0)+f(2,1)+f(2,2)+f(2,3)=1

C+2C+3C+2C+3C+4C+5C+4C+5C+6C+7C=1

42C=1 C=$\frac{1}{42}$

b. p(X=2,Y=1)=f(2,1)=$\frac{1}{42}(2(2)+1) = \frac{5}{42}$

c. p($X \geq 1, Y \leq 2$)

= f(1,0)+f(1,1)+f(1,2)+ f(2,0)+f(2,1)+f(2,2)

= $\frac{4}{7}$

Q17) For the Problem 16, find :

a. E(X) . b E(Y).

c. E(XY). d. E(X2).

e. E(Y2). f. Var(X).

g. Var(Y). h. Cov(X,Y). i. *ρ*.

Y \ x	0	1	2	
0	0	1/21	2/21	1/7
1	1/12	1/14	5/42	3/14
2	1/21	2/21	1/7	2/7
3	1/14	5/42	1/6	5/14
	1/7	1/3	11/21	1

x	0	1	2	
f(x)	0.14285	0.33333	0.5238	
Xf(x)	0	0.3333	1.04761	E(x)=1.38
X^2f(x)	0	0.3333	2.09523	E(x^2)= 2.428

Var(x)= E(x^2)-[E(x)]^2 = 2.428 –(1.38)^2 =0.5236

Y	0	1	2	3
F(y)	0.14285	0.21428	0.285714	0.35714
y f(y)	0	0.21428	0.571428	1.0714
y^2 f(y)	0	0.21428	1.142857	3.21428

E(y)= 1.857

E(y^2)= 4.571412

Var(y) = 1.122968

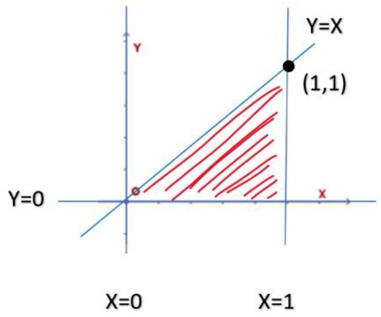
$(1)(0)\frac{1}{21} + (2)(0)\frac{2}{21} + (1)(0)\frac{1}{42} + (1)(1)\frac{1}{14} + (1)(2)\frac{5}{42} + (2)(0)\frac{1}{21} + (2)(1)\frac{2}{21} + (2)(2)\frac{1}{7} + (3)(0)\frac{1}{14} + (3)(1)\frac{5}{42} + (2)(3)\frac{1}{6} = E(x,y) = \frac{17}{7} = 2.428$

Cov(x,y) = E(x,y) – E(x).E(y) = 2.428 – (1.38 X 1.857) = - 0.13466

$$P_{xi\ y} = \frac{Cov(x,y)}{\sqrt{v(x).v(y)}} = \frac{-0.13466}{\sqrt{1.122968\ X\ 0.5236}} = -0.1756$$

Q18) The joint density function of X and Y is given by $(x, y) = \begin{cases} 8xy & 0 \le x \le 1\ ,\quad 0 \le y \le x \\ 0 & \text{otherwise} \end{cases}$

Find:



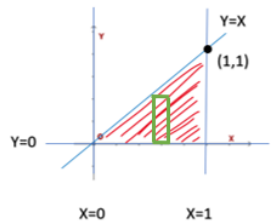
a. The marginal density of X.

b. The marginal density of Y.

c. The conditional density of X.

d. The conditional density of Y.

a. Answer:

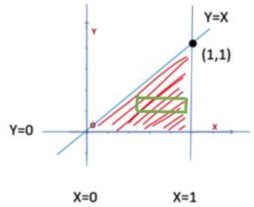


$f_x(x) = \int_0^x f(x,y)dy = 8x\int_0^x y\,dy$

$= 8x\,\frac{y^2}{2}\Big|_0^x = 4x[x^2]$

$= 4x^3 \quad for \quad 0 < x < 1$

b. Answer:

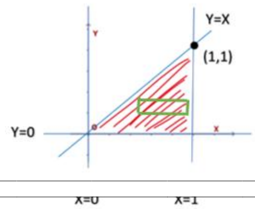


$f_y(y) = \int_y^1 f(x,y)dx = 8y\int_y^1 x\,dx$

$= 8y\,\frac{x^2}{2}\Big|_y^1 = 4y[1-y^2] = 4y-4y^3$

$= 4y(y-y^3) \quad for \;\; 0 < y < 1$

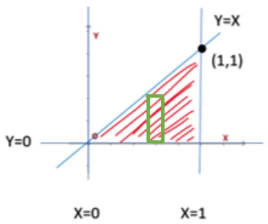
c. Answer:



$f_{x|y=y}(x) = \frac{f(x,y)}{fy(y)} = \frac{8xy}{4(y-y^3)} = \frac{2x\cancel{y}}{\cancel{y}(1-y^2)} = \frac{2x}{1-y^2}$

for $y < x < 1$ *where* $0 < y < 1$ *fined value*

d. Answer:



$f_{y|x=x}(y) = \frac{f(x,y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}$

for $0 < y < x$ *where* $0 < x < 1$ *fined value*

Q19) Find the conditional expectation of X given Y and Y given X in Problem 18

$$E(X|Y=y) = \int_y^1 x\ f_{x|y=y}(x)\,dx = \frac{2}{1-y^2}\int_y^1 x^2\,dx = \frac{2}{1-y^2}\left.\frac{x^3}{3}\right|_y^1 = \frac{2}{3}\frac{1}{1-y^2}(1-y^3) = \frac{2}{3}\frac{(1-y^2)}{(1-y^2)}$$

$$E(Y|X=x) = \int_0^y y\,f_{Y|X=x}(y)\,dy = \frac{2}{x^2}\int_0^x y^2\,dy = \frac{2}{x^2}\left.\frac{y^3}{3}\right|_0^x = \frac{2}{3}\frac{1}{x^2}(x^3) = \frac{2}{3}x$$

Q20) Find the conditional variance of Y given X for Problem 18.

$$Var(Y|X) = E(Y^2|X) = E(Y|X)^2$$

$$E(Y^2|X) = \int_0^x y^2 f_{Y|X=x}(y)dy = \frac{2}{x^2}\int_0^x y^3\,dy = \frac{2}{x^2}\left.\frac{y^4}{4}\right|_0^x = \frac{2}{4}\frac{1}{x^2}[x^4] = \frac{1}{2}x^2$$

$$Var(Y|X) = \frac{1}{2}x^2 - \left[\frac{2}{3}x\right]^2 = \left[\frac{1}{2} - \frac{2^2}{3^2}\right]x^2 = \frac{1}{18}x^2$$