The Probability Distribution for a Continuous Random Variable

The distribution function (or cumulative distribution function):
Definition:
Let Y denote any random variable. The distribution function of Y, denoted by $F(y)$, is such that $F(y) = P(Y \leq y)$ for $-\infty < y < \infty$.
Example (1):
Suppose that Y has a binomial distribution with $n = 2$ and $p = 1/2$. Find $F(y)$.
Solution: The probability function for Y is given by

$$P(y) = \binom{2}{y}(1/2)^y(1/2)^{2-y}, \quad y = 0,1,2$$

Which yields $P(0) = \frac{1}{4}, P(1) = \frac{1}{2}, P(2) = \frac{1}{4}$

$$F(y) = P(Y = y)\begin{cases} \frac{1}{2}, & for\ y < 0, \\ \frac{1}{4}, & for\ 0 \leq y < 1, \\ \frac{3}{4}, & for\ 1 \leq y < 2, \\ 1, & for\ y \geq 2 \end{cases}$$

In Example(1) the points between 0 and 1 or between 1 and 2 all had probability 0 and contributed nothing to the cumulative probability depicted by the distribution function. As a result, the cumulative distribution function stayed flat between the possible values of Y and increased in jumps or steps at each of the possible values of Y. Functions that behave in such a manner are called step functions. Distribution functions for discrete random variables are always step functions because the cumulative distribution function increases only at the finite or countable number of points with positive probabilities.
Because the distribution function associated with any random variable is such that $F(y) = P(Y \leq y)$, from a practical point of view it is clear that $F(-\infty) = lim_{y\to-\infty}P(Y \leq y)$ must equal zero. If we consider any two values $y1 < y2$, then $P(Y \leq y1) \leq P(Y \leq y2)$—that is, $F(y1) \leq F(y2)$. So, a distribution function, $F(y)$, is always a monotonic, nondecreasing function. Further, it is clear that $(\infty) = lim_{y\to\infty}P(Y \leq y) = 1$. These three characteristics define the properties of any distribution function and are summarized in the following theorem.

Theorem:

Properties of a Distribution Function1 If $F(y)$ is a distribution function, then
1. $F(-\infty) \equiv lim_{y\to\infty}F(y) = 0.$
2. $F(\infty) \equiv lim_{y\to\infty}F(y) = 1.$
3. $F(y)\ is\ a\ nondecreasing\ function\ of\ y.\ [If\ y_1\ and\ y_2\ are\ any\ values\ such\ that\ y_1 < y_2, then\ F(y_1) \leq F(y_2).]$

Definition:
A random variable Y with distribution function $F(y)$ is said to be continuous if $F(y)$ is continuous, for $-\infty < y < \infty$.
Definition:
Let $F(y)$ be the distribution function for a continuous random variable Y. Then $f(y)$, given by

$$f\ (y) = \frac{dF(y)}{dy}$$

wherever the derivative exists, is called the probability density function for the random variable Y .

Theorem:

Properties of a Density Function If $f\ (y)$ is a density function for a continuous random variable, then

1. $f\ (y)\ \geq\ 0\ for\ all\ y, -\infty\ <\ y\ <\ \infty$.
2. $\int_{-\infty}^{\infty} f(y)dy = 1$

The next example gives the distribution function and density function for a Continuous random variable.

Example(1):

Suppose that

$$F(y) = \begin{cases} 0, & for\ \ y < 0, \\ Y, & for\ \ 0 \leq y < 1, \\ 1, & for\ \ y > 1 \end{cases}$$

Find the probability density function for Y.

Solution Because the density function $f\ (y)$ is the derivative of the distribution function $F(y)$, when the derivative exists,

$$F(y) = \frac{dF(y)}{dy} \begin{cases} \dfrac{d(0)}{dy} = 0, & for\ \ y < 0, \\ \dfrac{d(y)}{dy} = 1, & for\ \ 0 \leq y < 1, \\ \dfrac{d(1)}{dy} = 0, & for\ \ y > 1 \end{cases}$$

Example(2):

Let Y be a continuous random variable with probability density function given by

$$f(y) = \begin{cases} 3y^2 & ,0 \leq y \leq 1, \\ 0, & elsewhere\ , \end{cases}$$

Find $F(y)$.

Solution:

$$F(y) = \begin{cases} \int_{-\infty}^{y} 0dt = 0\ , & for\ y < 0 \\ \int_{-\infty}^{y} 0dt + \int_{0}^{t} 3t^2dt = 0 + t^3]_0^y = y^3, & for\ 0 \leq y \leq 1, \\ \int_{-\infty}^{y} 0dt + \int_{0}^{1} 3t^2dt + \int_{0}^{y} o\ dt = 0 + t^3]_0^1 + 0 = 1, & for\ y > 1 \end{cases}$$

Example(3):

Given $f\ (y)\ =\ cy^2, 0\ \leq\ y\ \leq\ 2$, and $f\ (y)\ =\ 0$ elsewhere, find the value of c for which $f\ (y)$ is a valid density function.

Solution We require a value for c such that

$$F(\infty) = \int_{-\infty}^{\infty} f(y)dy = 1 = \int_{0}^{2} y^2dy = \frac{cy^3}{3}\bigg]_0^2 = \left(\frac{8}{3}\right)c$$

Thus, $(8/3)c\ =\ 1$, and we find that $c\ =\ 3/8$.

Example(4):

Find $P(1 \leq Y \leq 2)$ for Example (3). Also find $P(1 < Y < 2)$.

Solution

$$P(1 \leq Y \leq 2) = \int_{\mathbf{1}}^{\mathbf{2}} f(y)dy = 3/8 \int_{1}^{2} y^2 dy = 7/8$$

Because Y has a continuous distribution, it follows that $P(Y = 1) = P(Y = 2) = 0$ and, therefore, that

$$P(1 < Y < 2) = P(1 \leq Y \leq 2) = 3/8 \int_{1}^{2} y^2 dy = 7/8$$

Expected Values for Continuous Random Variables

The next step in the study of continuous random variables is to find their means, variances, and standard deviations.

Definition:

The expected value of a continuous random variable Y is

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy,$$

provided that the integral exists.

Theorem:

Let $g(Y)$ be a function of ; then the expected value of $g(Y)$ is given by

$$Eg(Y) = \int_{-\infty}^{\infty} g(y) f(y) dy,$$

provided that the integral exists.

Theorem:

Let c be a constant and let $g(Y), g1(Y), g2(Y), \ldots, gk(Y)$ be functions of a continuous random variable Y. Then the following results hold:

1. $E(c) = c$.
2. $E[cg(Y)] = cE[g(Y)]$.
3. $E[g_1(Y) + g_2(Y) + \cdot \cdot \cdot + g_k(Y)] = E[g_1(Y)] + E[g_2(Y)] + \cdot \cdot \cdot + E[g_k(Y)]$.

Example(1):

In Example(3) we determined that $f(y) = (3/8)y^2 for\ 0 \leq y \leq 2, f(y) = 0$ elsewhere, is a valid density function. If the random variable Y has this density function, find $\mu = E(Y)\ and\ \sigma^2 = V(Y)$.

$$E(Y) = \int_{-\infty}^{\infty} y f(y) dy,$$

$$= \int_{0}^{2} y(\frac{3}{8})y^2\, dy = \frac{3}{8} * \frac{1}{4} y^4 \Big]_0^2 = 1.5$$

The variance of Y can be found once we determine$E(y^2)$. In this case,

$$E(y^2) = \int_{-\infty}^{\infty} y^2 f(y)dy = \int_0^2 y^2 (\frac{3}{8}) y^2\, dy = \frac{3}{8} * \frac{1}{5} y^5 \Big]_0^2 = 2.4$$

Thus $\sigma^2 = V(Y) = E(y^2) - (E(y))^2 = 2.4 - 1.5^2 = .15$