Gradient, Divergence and Curl

The Vector Differential Operator Del:

Written ∇, is defined by

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

This vector operator possesses properties analogous to those of ordinary vectors. The operator ∇ is also known as nabla.

The Gradient:

Let $\phi(x,y,z)$ be defined and differentiable at each point (x,y,z) in certain region of space. Then the gradient of ϕ, written $\nabla\phi$ or grad ϕ, is defined by

$$\nabla\phi = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right)\phi = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

Note that $\nabla\phi$ defines a vector field.

The Divergence:

Let $V(x,y,z) = V_1 i + V_2 j + V_3 k$ be defined and differentiable at each point (x,y,z) in a certain region of space. The divergence of V, written $\nabla . V$ or div V, is defined by

$$\nabla . V = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) . (V_1 i + V_2 j + V_3 k) = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$$

Note that $\nabla . V \neq V . \nabla$

The Curl:

If $V(x,y,z)$ is a differentiable vector field then Curl or rotation of V, written $\nabla \times V$, curl V or rot V, is defined by

$$\nabla \times V = \left(\frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k\right) \times (V_1 i + V_2 j + V_3 k)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v_1 & v_2 \end{vmatrix} k$$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right) i + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right) j + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) k$$

Example

If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\nabla\phi$ (or grad ϕ) at the point $(1, -2, -1)$.

Sol

$$\nabla\phi = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(3x^2y - y^3z^2)$$

$$= i\frac{\partial}{\partial x}(3x^2y - y^3z^2) + j\frac{\partial}{\partial y}(3x^2y - y^3z^2) + k\frac{\partial}{\partial z}(3x^2y - y^3z^2)$$

$$= 6xy\,i + (3x^2 - 3y^2z^2)j - 2y^3z\,k$$

$$= 6(1)(-2)i + [3(1)^2 - 3(-2)^2(-1)^2]j - 2(-2)^3(-1)k$$

$$= -12i - 9j - 16k$$

Example

If $A = x^2z\,i - 2y^3z^2 j + xy^2z\,k$, find $\nabla \cdot A$ (or div A) at the point $(1, -1, 1)$

Sol

$$\nabla \cdot A = \left(\frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k\right)(x^2z\,i - 2y^3z^2 j + xy^2z\,k)$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$= 2xz - 6y^2z^2 + xy^2$$

$$= 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2 = \underline{-3}$$

Example

If $A = xz^3 i - 2x^2yz\,j + 2yz^4 k$, find $\nabla \times A$ (or Curl A) at the point $(1, -1, 1)$

Sol

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz^3 & -2x^2yz & 2yz^4 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(2yz^4) - \frac{\partial}{\partial z}(-2x^2yz)\right]i - \left[\frac{\partial}{\partial x}(2yz^4) - \frac{\partial}{\partial z}(xz^3)\right]j + \left[\frac{\partial}{\partial x}(-2x^2yz) - \frac{\partial}{\partial y}(xz^3)\right]k$$

$$= (2z^4 + 2x^2y)i + 3xz^2 j - 4xyz\,k$$

$$= 3j + 4k \quad \text{"at } (1, -1, 1)\text{"}.$$

Vector Integration

Ordinary Integrals of vectors :-

Let $R(u) = R_1(u)i + R_2(u)j + R_3(u)k$ be a vector depending on a single scalar variable u, where $R_1(u), R_2(u), R_3(u)$ are supposed continuous in a specified interval. Then

$$\int R(u)\,du = i\int R_1(u)\,du + j\int R_2(u)\,du + k\int R_3(u)\,du$$

is called an indefinite integral of $R(u)$.

If there exists a vector $S(u)$ such that $R(u) = \frac{d}{du}(S(u))$, then

$$\int R(u)\,du = \int \frac{d}{du}(S(u))\,du = S(u) + C$$

where C is an arbitrary constant vector independent of u. Then finite integral between limits $u = a$ and $u = b$ can in such case be written

$$\int_a^b R(u)\,du = \int_a^b \frac{d}{du}(S(u))\,du = S(u) + C\Big|_a^b = S(b) - S(a)$$

Line Integrals :

Let $r(u) = x(u)i + y(u)j + z(u)k$, where $r(u)$ is the position vector of (x, y, z), define a curve C joining points P_1 and P_2, where $u = u_1$ and $u = u_2$ repectively.

We assume that C is composed of a finite number of curves for each of which $r(u)$ has a continuous derivative. let $A(x, y, z) = A_1 i + A_2 j + A_3 k$ be vector function of position defined and continuous along C. Then the integral of the tangential component of A along C from P_1 to P_2, written as

$$\int_{P_1}^{P_2} A \cdot dr = \int_C A \cdot dr = \int_C A_1\,dx + A_2\,dy + A_3\,dz$$

is an example of a line integral.

Example

If $R(u) = (u-u^2)i + 2u^3 j - 3k$, find

(a) $\int R(u)\,du$

(b) $\int_1^2 R(u)\,du$

Sol

(a) $\int R(u)\,du = \int [(u-u^2)i + 2u^3 j - 3k]\,du$

$= i\int (u-u^2)\,du + j\int 2u^3\,du - k\int 3\,du$

$= \left(\frac{u^2}{2} - \frac{u^3}{3}\right)i + \frac{u^4}{2}j - 3uk + C$

where C is the constant vector.

(b) $\int_1^2 R(u)\,du = \left(\frac{u^2}{2} - \frac{u^3}{3}\right)i + \frac{u^4}{2}j - 3uk + C \Big|_1^2$

$= \left[\left(\frac{2^2}{2} - \frac{2^3}{3}\right)i + \frac{2^4}{2}j - 3(2)k + C\right] - \left[\left(\frac{1^2}{2} - \frac{1^3}{3}\right)i + \frac{1^4}{2}j - 3(1)k + C\right]$

$= -\frac{5}{6}i + \frac{15}{2}j - 3k$

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