Lecture (3)

The homogenous equation

A differential equations is stander form $y^{/} = f(x, y)$ its homogenous if

$f(tx, ty) = f(x, y)$ for any real number (t)

To solution of homogenous equation can be making the substitution

$y = ux$ $\qquad\qquad$ $\dfrac{dy}{dx} = u + x\dfrac{du}{dx}$

The resulting equation in the variables (x) and (u) its solved as Separable equation

Examples

Solving the following Differential equation

1) $y^{/} = \frac{x+y}{x}$

Solution

Let $y = ux$ & $\frac{dy}{dx} = u + x\frac{du}{dx}$

Substitution in the equation

$u + x\frac{du}{dx} = \frac{x+ux}{x}$ $\qquad$ $u + x\frac{du}{dx} = 1 + u$

$x\frac{du}{dx} = 1$ $\qquad$ $\int du = \int \frac{dx}{x}$

$u = \ln x + c$ $\qquad$ $\frac{y}{x} = \ln x + c$

$2) y^{/} = \frac{x^2-2y^2}{xy}$

Solution

Let $y = ux$ & $\frac{dy}{dx} = u + x\frac{du}{dx}$

Substitution in the equation

$u + x\frac{du}{dx} = \frac{x^2-2u^2x^2}{ux^2}$ $u + x\frac{du}{dx} = \frac{1-2u^2}{u}$ $x\frac{du}{dx} = \frac{1-2u^2}{u} - u$

$x\frac{du}{dx} = \frac{1-2u^2-u^2}{u}$ $\int \frac{udu}{1-3u^2} = \int \frac{dx}{x}$

$\frac{-1}{6}\ln(1-3u^2) = \ln x + \ln c$

$\frac{-1}{6}\ln(1-3\left[\frac{y}{x}\right]^2) = \ln xc$

$3) y^{/} = \frac{y}{x+\sqrt{xy}}$

Solution

Let $y = ux$ & $\frac{dy}{dx} = u + x\frac{du}{dx}$

Substitution in the equation

$u + x\frac{du}{dx} = \frac{ux}{x+\sqrt{ux^2}}$ $u + x\frac{du}{dx} = \frac{u}{1+\sqrt{u}}$

$x\frac{du}{dx} = \frac{u}{1+\sqrt{u}} - u$

$x\frac{du}{dx} = \frac{u-(u-u\sqrt{u})}{1+\sqrt{u}}$ $\qquad x\frac{du}{dx} = \frac{u\sqrt{u}}{1+\sqrt{u}}$

$\int \frac{(1+\sqrt{u})du}{u\sqrt{u}} = \int \frac{dx}{x}$

$\int \frac{1du}{u\sqrt{u}} + \int \frac{du}{u} = \int \frac{dx}{x}$ $\qquad \frac{-2}{\sqrt{u}} + \ln u =$

$\ln x + \ln c$ $\qquad \frac{-2}{\sqrt{\frac{y}{x}}} + \ln\frac{y}{x} = \ln xc$

$4) xy^{/} = y + x\sec\left(\frac{y}{x}\right)$

Solution

$y^{/} = \frac{y+x\sec\left(\frac{y}{x}\right)}{x}$

Let $y = ux$ $\quad$ & $\quad \frac{dy}{dx} = u + x\frac{du}{dx}$

Substitution in the equation

$u + x\frac{du}{dx} = \frac{ux+x\sec\left(\frac{ux}{x}\right)}{x}$ $\qquad u + x\frac{du}{dx} = u +$

$\sec u$ $\qquad x\frac{du}{dx} = \sec u$

$\int \frac{du}{\sec u} = \int \frac{dx}{x}$ $\qquad \sin u = \ln x + \ln c$

$\sin\frac{y}{x} = \ln xc$

<u>Non homogenous equation</u>

The equation in the form $\frac{dy}{dx} = \frac{ax+by+c}{a_1x+b_1y+c_1}$ its non homogenous equation

Because $f(tx, ty) \neq f(x, y)$

To solution this equation there are two cases

Case (1) approaches the equation to homogenous equation by substitution

$x = X + h \qquad y = Y + k \qquad \frac{dy}{dx} = \frac{dX}{dY}$ Then

$$\frac{dY}{dX} = \frac{aX + bY + ah + bk + c}{a_1X + b_1Y + a_1h + b_1k + c_1}$$

The condition

$ah + bk + c = 0$ ______________(1)

$a_1h + b_1k + c_1 = 0$ ____________(2)

Solution the equations (1) and (2) to find the values of (h) and (k) after that solution the homogenous equation $\frac{dY}{dX} = \frac{aX+bY}{a_1X+b_1Y}$

Case (2) if the equations (1) and (2) are not solution

$$\begin{vmatrix} a & b \\ a_1 & b_1 \end{vmatrix} = 0 \qquad ab_1 - ba_1 = 0$$

The equation approaches to separated variables by substitution

$z = ax + by$ Then $\frac{dz}{dx} = a + b\frac{dy}{dx}$

Examples

Solving the following Differential equation

1) $y' = \frac{x+y-3}{x-y-1}$

Solution

The determine $\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1 - 1 = -2 \neq 0$

substitution

$x = X + h \qquad y = Y + k \qquad \frac{dy}{dx} = \frac{dX}{dY}$ Then

$\frac{dX}{dY} = \frac{X+h+Y+k-3}{X+h-Y-k-1}$ ....(*)

Solution the equations

$$\begin{array}{l} h + k - 3 = 0 \\ \underline{h - k - 1 = 0} \\ +2h - 4 = 0 \end{array} \qquad h = 2 \;\&\; k = 1$$

Substitution the values of(h & k) in the equation (*)

$\frac{dX}{dY} = \frac{X+Y}{X-Y}$ its humongous equation

Let $Y = uX$ & $\frac{dY}{dX} = u + X\frac{du}{dX}$

Substitution in the equation

$u + X\frac{du}{dX} = \frac{X+uX}{X-uX} \qquad u + X\frac{du}{dX} = \frac{1+u}{1-u} \qquad X\frac{du}{dX} = \frac{1+u}{1-u} - u$

$X\frac{du}{dX} = \frac{1+u-(u-u^2)}{1-u} \qquad X\frac{du}{dX} = \frac{1+u^2}{1-u} \qquad \int \frac{(1-u)du}{1+u^2} = \int \frac{dX}{X}$

$\int \frac{du}{1+u^2} - \int \frac{udu}{1+u^2} = \int \frac{dX}{X}$

$$\tan^{-1} u - \frac{1}{2}\ln(1+u^2) = \ln x + \ln c$$

$$\tan^{-1}(\frac{Y}{X}) - \frac{1}{2}\ln\left(1 + \left[\frac{Y}{X}\right]^2\right) = \ln Xc$$

$$\tan^{-1}\left(\frac{y-1}{x-3}\right) - \frac{1}{2}\ln\left(1 + \left[\frac{y-1}{x-3}\right]^2\right) = \ln(x-3)c$$

2) $y^{/} = \frac{2x+y-1}{4x-y}$

Solution

The determine $\begin{vmatrix} 2 & 1 \\ 4 & -1 \end{vmatrix} = -2-4 = -6 \quad \neq 0$

Substitution

$x = X + h \qquad y = Y + k \qquad \frac{dy}{dx} = \frac{dX}{dY}$ Then

$\frac{dX}{dY} = \frac{2X+2h+Y+k-1}{4X+4h-Y-k}$ ………………..(*)

Solution the equations

$2h + k - 1 = 0$

$4h - k = 0$

$+6h - 1 = 0 \qquad\qquad h = \frac{1}{6} \ \& \ k = \frac{2}{3}$

Substitution the values of(h & k) in the equation (*)

$\frac{dX}{dY} = \frac{2X+Y}{4X-Y}$ its humongous equation

Let $Y = uX$ & $\frac{dY}{dX} = u + X\frac{du}{dX}$

Substitution in the equation

$u + X\frac{du}{dX} = \frac{2X+uX}{4X-uX} \qquad u + X\frac{du}{dX} = \frac{2+u}{4-u} \qquad X\frac{du}{dX} = \frac{2+u}{4-u} - u$

$X\frac{du}{dX} = \frac{2+u-(4u-u^2)}{4-u} \qquad X\frac{du}{dX} = \frac{2-3u+u^2}{4-u} \qquad \int\frac{(4-u)du}{u^2-3u+2} = \int\frac{dX}{X}$

$\int\frac{(4-u)du}{(u-2)(u-1)} = \int\frac{dX}{X} \qquad \int\frac{Adu}{(u-2)} + \int\frac{Bdu}{(u-1)} = \int\frac{dX}{X}$

$Aln(u-2) + Bln(u-1) = lnxc$

$\frac{1}{2}ln\left(\frac{Y}{X} - 2\right) - \frac{3}{2}ln\left(\frac{Y}{X} - 1\right) = lnXc$

$$\frac{1}{2} ln\left(\frac{y-\frac{2}{3}}{x-\frac{1}{6}}-2\right)-\frac{3}{2} ln\left(\frac{y-\frac{2}{3}}{x-\frac{1}{6}}-1\right)=\ln(x-\frac{1}{6})c$$

$3) y^{/}=\frac{x+y-1}{x+y+1}$

Solution

The determine $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}=1-1=0$

The equation approaches to separated variables by substitution

$z=x+y$ Then $\frac{dz}{dx}=1+\frac{dy}{dx}$ $\frac{dz}{dx}-1=\frac{dy}{dx}$

Substitution in the equation

$\frac{dz}{dx}-1=\frac{z-1}{z+1}$ $\frac{dz}{dx}=\frac{z-1}{z+1}+1$ $\frac{dz}{dx}=\frac{z-1+z+1}{z+1}$

$\frac{dz}{dx}=\frac{2z}{z+1}$ $\int\frac{(z+1)dz}{2z}=\int dx$

$$\int\frac{zdz}{2z}+\int\frac{dz}{2z}=\int dx$$

$$\frac{1}{2}z+\frac{1}{2}lnz=x+c$$

$$\frac{1}{2}(x+y)+\frac{1}{2}\ln(x+y)=x+c$$