THE FIVE COLOR THEOREM

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Abstract. In Mathematics, a graph is defined as a set of vertices and edges wherein the vertices represent entities in a collection and edges represent connections between them. A planar graph is a graph that can be depicted on a two-dimensional surface such that no edges overlap. The chromatic number $\chi(G)$ of a graph G is the fewest number of colors needed to color every vertex in a G such that no two vertices connected by an edge share the same color. The problem of finding $\chi(G)$ for generalized graphs remains unsolved, but work has been done on computing it for graphs with imposed restrictions. The Five Color Theorem provides a value for $\chi(G)$ for all planar graphs G. It is proved using a Euler's Formula as a basis and is the strictest coloring bounds for planar graphs provable by traditionally mathematical techniques.

1. Introduction

Imagine that you have made reservations for a Michelin 3-star restaurant. The evening arrives and, after being seated, you are handed the menu. To your surprise, on the back side of the menu is a blank map of the nations of the world. At the end of the table you see a pack of crayons. Upon opening the pack, three uniquely colored crayons fall to the table. Filled with disbelief, you immediately excuse yourself from the restaurant and vow never to return, as you know it is impossible to properly color a map of the world with only three unique colors.

But why is this? Why is it impossible to properly color a map of the world with only three colors? How many colors would you need to successfully color the entire map? Surely if there were three dozen crayons you would be able to achieve success. But it would be impractical for the aforementioned restaurant to give out three dozen crayons for every menu. So how many crayons would they need to give out to allow you to color your map and enjoy your meal alongside your newly created artwork? To answer this, we must delve into the Mathematical field of Graph Theory.

2. Background

Graph Theory, as the name suggests, is an area of Mathematics that involves the study of structures known as graphs. A graph $G = (V, E)$ is defined as a set of vertices V and edges E, where a vertex (or node) is simply an entity within the graph and an edge represents some form of connection between vertices. Two vertices are said to be adjacent if there exists an edge between them, and an edge stemming from a vertex is said to be incident with that edge. A walk is defined as a sequence of connected vertices in a graph. Figure 1 illustrates a simple graph with four vertices and four edges. The number of vertices in a graph G is known as its order and is denoted by $|V|$ while the number of edges is known as its size and is denoted $|E|$.

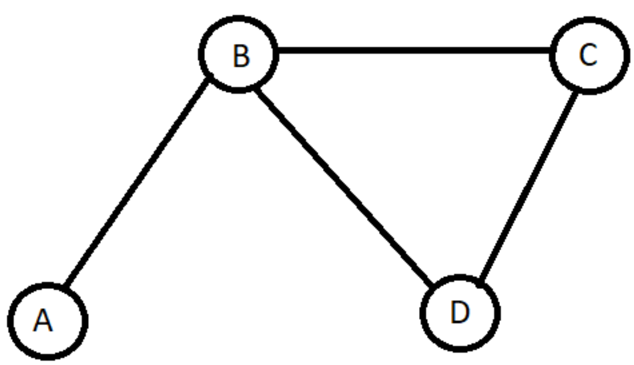

Figure 1. A simple graph

If a graph can be illustrated on a two dimensional plane such that no two edges are overlapping, it is known as a planar graph[3]. In a planar depiction, the area enclosed by a set of edges is known as a face. The outside region of a planar graph is also considered a face. Depicting a planar graph G in

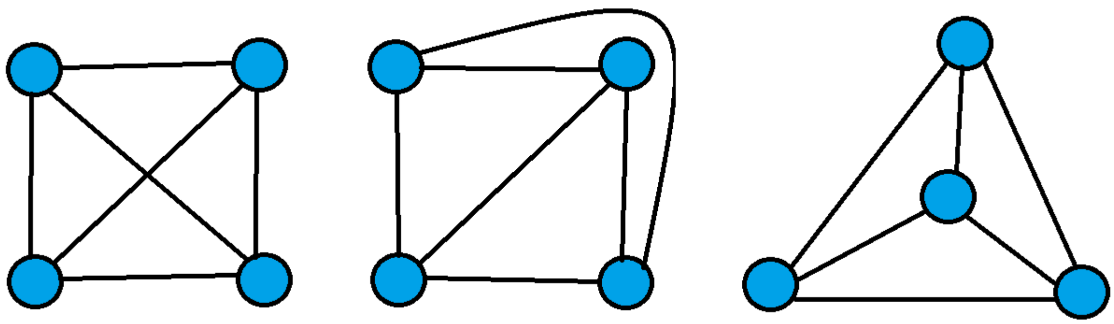

FIGURE 2. A graph depicted as planar (middle, right) and non-planar (left)

such a way allows certain properties to hold true. Figure 2 depicts a planar graph in three ways; two of which are planar. Geographical maps can be translated into planar graphs by considering nations as vertices and edges as the borders between them[1]. With the knowledge of a graph, vertices, edges, and faces, we can state a key theorem in the study of planar graphs.

Theorem 1 (Euler's Formula). *Let G be a connected planar graph with size e and order v. Any planar depiction of G has $e - v + 2$ faces.*

Euler's formula serves as the basis for many other theorems and corollaries in Graph Theory due to its simplicity and significance. Consider Figure 2. Recall that the three depictions are the same graph. The depictions on the center and right both have four faces (including the unbounded region outside of the graph). So $e - v + 2 = 6 - 4 + 2 = 4$. The depiction on the left has five faces because it is not a planar depiction. This distinction is important as Euler's Formula only applies to planar depictions of graphs. From Euler's formula we can derive two important corollaries.

Corollary 1.1 (1). *If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.*

Corollary 1.2 (2). *Every planar graph G has a vertex of degree ≤ 5.*

These corollaries play an important role in the coloring of graphs. A graph coloring is the assignment of a unique color to every vertex in a graph such that no two adjacent vertices share the same color. The chromatic number $\chi(G)$ of a graph G is the fewest number of colored needed to color a graph and meet this criteria. There is no generalized formula for computing $\chi(G)$ for any graph G, but much work has been done to compute this value for graphs with specified restrictions. For example, there exists a known $\chi(G)$ for all planar graphs G.

Theorem 2 (The Five Color Theorem). *Every planar graph G has $\chi(G) \leq 5$.*

The Five Color Theorem provides an upper bound for the chromatic number of planar graphs. The order and size of the graph do not matter so long as it remains planar.

3. Proof

The proof of this theorem is rather long and utilizes mathematical induction. It relies heavily on Corollary 1.2 from Euler's Formula.

Proof. We want to show that $\chi(G) \leq 5$ for every planar graph G. To do this, we will use induction.

Base Case: Let $v = 1$, so G consists of a single vertex.
This is trivial as a single vertex with no neighbors can be colored with a single color, thus $\chi(G) = 1 \leq 5$. We may also note that the same is true for $v = 2, 3, 4$, or 5, as a graph's chromatic number will never be greater than its order.

Inductive Hypothesis: Assume $\chi(G) \leq 5$ for every planar graph G of order v. Consider a planar graph H with order $v + 1$.
From 1.1 we know that H contains a vertex with order ≤ 5. Denote this vertex by x. Now consider the graph generated by the removal of this vertex and all edges incident with it. Denote this graph by J. We know that J is a planar graph of order v, which, by our inductive hypothesis, has a chromatic number of ≤ 5. Suppose J has been colored with at most 5 colors. Now we must show that it is possible to rejoin x into J without introducing a new color.

Case 1: J is colored using < 5 colors.
This case is simple as J is colored using at most 4 colors, so we can reintroduce x and its incident edges using the remaining fifth color.

Case 2: J is colored using 5 colors and x has degree < 5.
This case is also simple, x is adjacent to at most 4 nodes, and therefore can be reintroduced and colored using the fifth remaining color at no consequence.

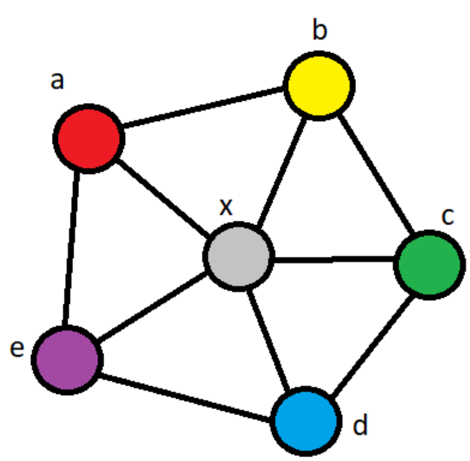

Figure 3. Planar depiction of subgraph of H with vertices colored

Case 3: J is colored using 5 colors and x has degree 5.
This case is more difficult, as we cannot simply reintroduce x with the fifth color. Instead, let us denote the five vertices adjacent to x in a clockwise rotation by a, b, c, d and e. Figure 3 depicts this orientation of vertices. Note that the edges between the neighbors of x, such as (a, b), do not necessarily exist and are only shown to demonstrate a worst-case scenario. If any two of these vertices are the same color, then there exists a legal color for x and we are done. So let us then assume that they have been colored with colors 1, 2, 3, 4, and 5, respectively. We must recolor one of these vertices without disrupting the legal coloring of the existing graph. To do this, let us consider, without loss of generality, two nonadjacent vertices a, and c (colored 1 and 3, respectively). If there is no walk from a to c consisting solely of alternating colors 1 and 3, then we can simply color a with color 3 and alternate the coloring of nodes on any walks adjacent to a so that nodes colored 1 become 3 and vice versa. Thus, color 1 has become available for x, so x can be reintroduced with color 1 and we are done. Figure 5 illustrates what the subgraph would look like after recoloring.

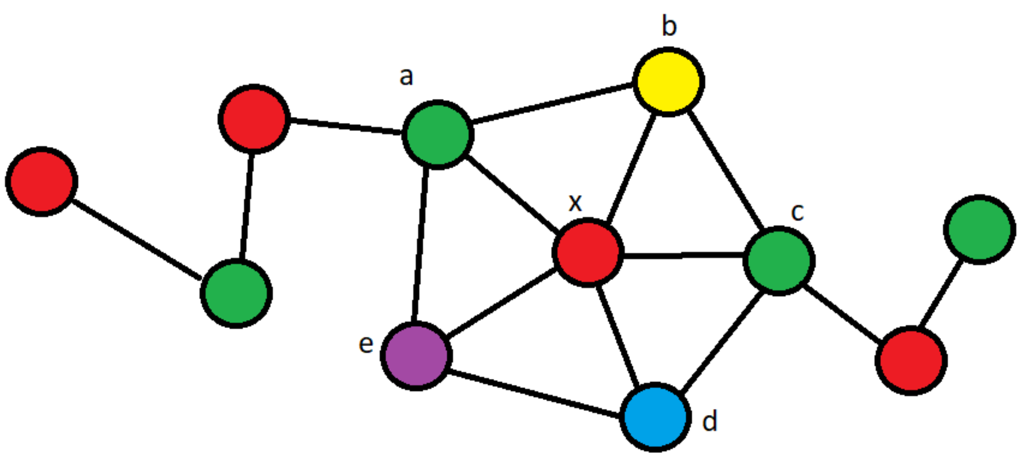

FIGURE 4. If there is no walk between a and c, a can be recolored

Suppose, however, that there is a walk joining a to c consisting solely of colors 1 and 3. We then look at two more non-adjacent vertices b and d, colored 2 and 4, respectively. Since G is planar, we can assume (without loss of generality) that b is surrounded by a cycle containing a and c. We have already assumed that this cycle consists of only colors 1 and 3, therefore there can exist no walk between b and d consisting solely of colors 2 and 4. Thus, we can recolor b with 4 and repeat the earlier process of alternating the coloring of the nodes adjacent to it until we have a legal coloring. Thus, color 2 is now legal for x. Figure 4 illustrates what the subgraph would look like after recoloring.

Therefore if G is a planar graph of order v such that $\chi(G) \leq 5$, then $\chi(H) \leq 5$ holds true as well for any planar graph H of order $v + 1$. Thus we see that any planar graph can be colored with at most five colors[2]. $\square$

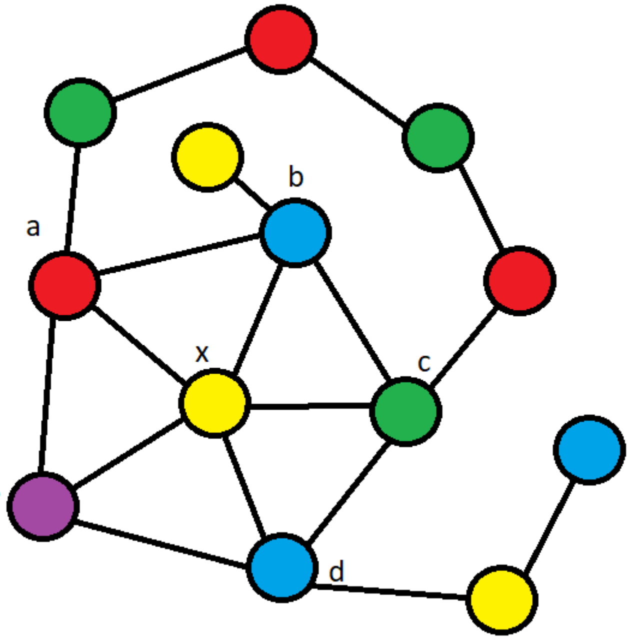

FIGURE 5. If there is a walk between a and c, b can be recolored

4. Conclusion

So now it is clear that you would need at most five crayons to color the world map on the back of the menu during your disappointing and expensive restaurant visit. In fact, you can properly color any geographical map of any location with only four colors. The Five Color Theorem is the strictest traditionally provable coloring bounds for planar graphs, but there exists a Four Color Theorem, which states that $\chi(G) \leq 4$ for any planar graph G. However, it has not been proven in a traditional mathematical sense. It was determined to be true by computational brute force on a few thousand cases and has faced much scrutiny in the world of Mathematics research as a result[2]. Recall that the basis of the Five Color Theorem was Corollary 1.2 which stated that every planar graph G has a vertex of order ≤ 5. As there exists no such claim that there exists a vertex of order ≤ 4, the Four Color Theorem cannot be proven in the same fashion. Thus, the Five Color Theorem can remain your excuse for abandoning your evening dinner plans.

References

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