On the Game Chromatic Number

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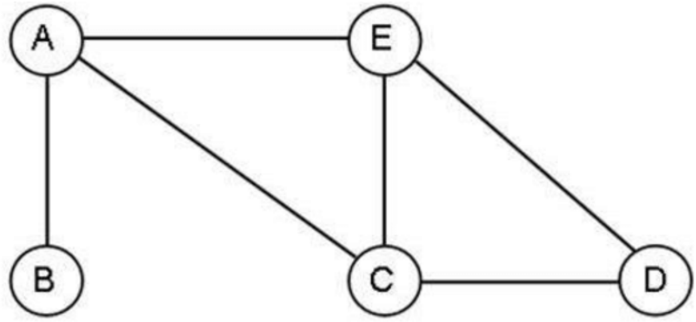

Figure: A simple graph

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- Game ends when either the graph is fully colored or no proper coloring exists.[1]
- The Minimizer (P1) aims to use the fewest colors possible and fully color the graph.
- The Maximizer (P2) aims to use the most colors possible or make the graph uncolorable.

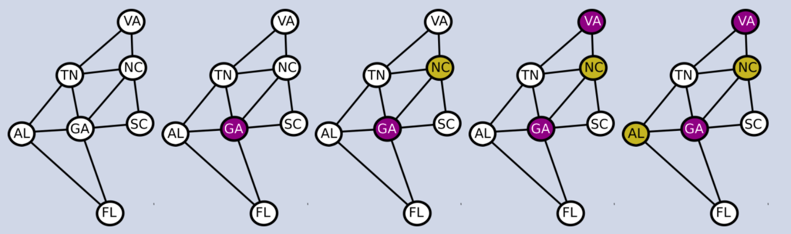

Figure: An example of the Graph Coloring Game with 2 colors

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- Minimizer decides number of available colors k.
- No player can color a vertex if any adjacent vertices share the same color.
- Minimizer moves first.*
- Each player must color a vertex on their turn.*

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*Variations of the GCG exist which modify this rule.

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- The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimum number of colors needed to properly color G.
- The *game chromatic number* of a graph G, denoted $\chi_g(G)$, is the minimum number of colors needed to guarantee victory for the Minimizer.

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- A *subgraph* H of a graph G is a collection of vertices and edges such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$. [5]
- The *degree* of a vertex $deg(v)$ is the number of neighbors it has.

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Note: Every trunk will contain at least one uncolored vertex, and every vertex over degree 1 is in at least 1 trunk.

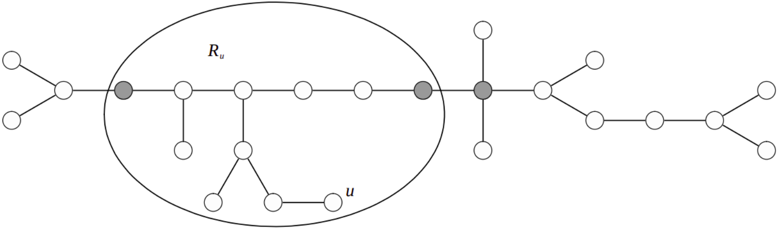

Figure: A forest of partially colored trees with a trunk circled.

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- In the *Modified Coloring Game*, the Maximizer plays first and can choose to pass a turn.
- In the *Expanded Coloring Game*, the Minimizer can choose to pass and, if so, can add a single colored leaf to the graph.

- The k-coloring game on a forest F is equivalent to the k-MCG on a $R(F)$, as every uncolored vertex in F has the same set of neighbors in $R(F)$ as in F.

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  - Minimizer pretends that the Maximizer chose to pass initially and uses the same strategy from the k-MCG.
- Thus, if the Minimizer can win the k-MCG on every trunk in $R(F)$, they can also win the k-coloring game on F.

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- Clearly, $\chi(G) \leq \chi_g(G) \leq \Delta G + 1$
- With restrictions on G, we can prove upper bounds.

Theorem (Faige et al. [4])

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- This is an improvement upon an earlier upper bound of 5.[2]
- We will first prove a useful lemma, and then prove this upper bound for all forests.

Lemma

Let F be a partially colored forest and let the 4-MCG be played on $R(F)$. If every trunk in $R(F)$ has at most 2 colored vertices, the minimizer can win the 4-MCG on $R(F)$.

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- If $n > 0$, there can be at most one trunk with 3 colored vertices after the Maximizer's first move.

- If this trunk exists, it is a tree containing 3 colored leaves and an uncolored vertex v of degree $deg(v) \geq 3$ whose deletion will disconnect the leaves.

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- Since v has at most 3 colored neighbors, it can be legally colored by the Minimizer.
- This creates a partially colored graph with trunks containing at most 2 uncolored vertices each.
- The Minimizer can now easily win.

- If this trunk did not exist, the Minimizer may play on any trunk that has an uncolored vertex.

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- Thus, the Minimizer can win the 4-MCG on $R(F)$

$\square$

Now that we've proved our lemma and shown some equivalence between the k-coloring game and the k-MCG, we can prove our upper bounds of 4.

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- Thus, the Minimizer can win the 4-coloring game on F.
- Therefore, $\chi_g(F) \leq 4$ for any forest F.

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- So we need a way to distinguish game chromatic numbers of 2, 3, and 4.

Theorem (Dunn et al [3])

Let F be a forest and let $l(F)$ be the length of the longest path in F. Then $\chi_g(F) = 2$ if and only if:

- $1 \leq l(F) \leq 2$ *or*
- $l(F) = 3$, $|V(F)|$ *is odd, and every component with diameter 3 is a path.*

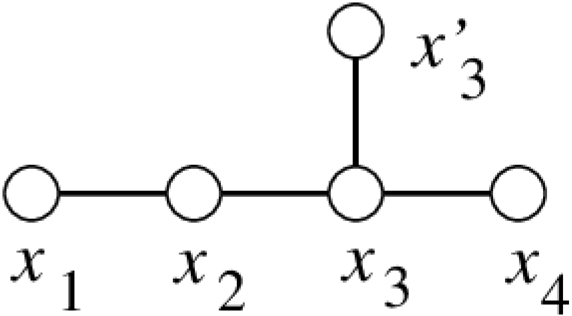

Figure: The graph T^+, AKA the Minimizer's Kryptonite

Note that this violates the last condition because T^+ is not a path.

Theorem

Let F be a forest such that $|V(F)| \leq 13$. Then $\chi_g(F) \leq 3$.

This means that *any* collection of disconnected acyclic graphs with 13 or fewer vertices has a game chromatic number of at most 3.

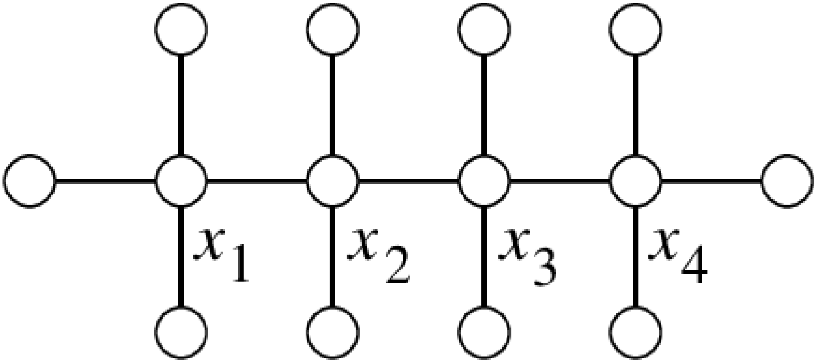

Figure: Minimal Order Tree with Game Chromatic Number 4

Theorem

Let T be the caterpillar graph of order 14, as shown in Figure 5. Then Bob can win the 3-ECG on T. Therefore, T is a minimal example of a tree with game chromatic number 4.

Note this is not the only minimal order forest with $\chi_g(F) = 4$, it is just one such graph.

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- More definitive methods for distinguishing between 3 and 4 are areas for future work.
- Introducing cycles breaks everything!

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