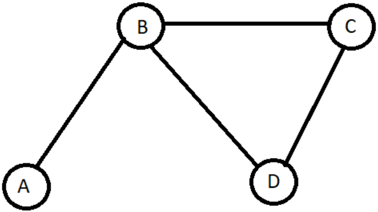
The Five Color Theorem

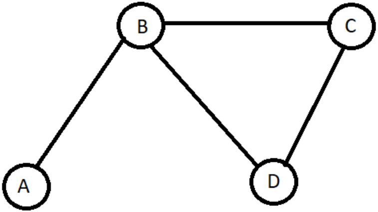
Danny Hammer

August 26, 2021

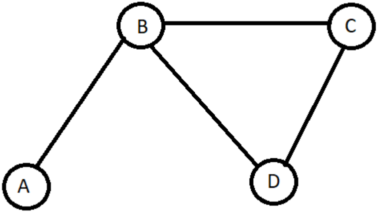
- Recall that a **Graph** is a mathematical structure consisting of a set of vertices V and edges E.



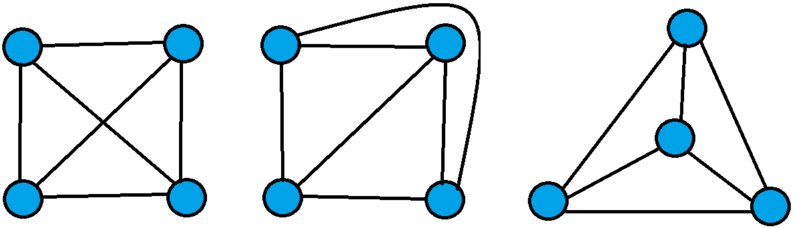
- Recall that a **Graph** is a mathematical structure consisting of a set of vertices V and edges E.
  - A **Vertex** (or **Node**) represents an entity in the graph.



- Recall that a **Graph** is a mathematical structure consisting of a set of vertices V and edges E.
  - A **Vertex** (or **Node**) represents an entity in the graph.
  - An **Edge** represents connection between vertices.



- A **Planar Graph** is a graph that can be drawn on a two dimensional surface such that no edges overlap. [3]



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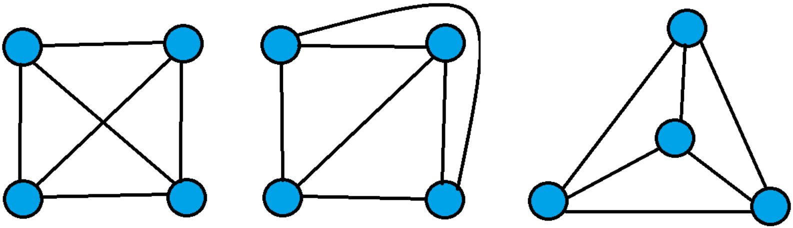
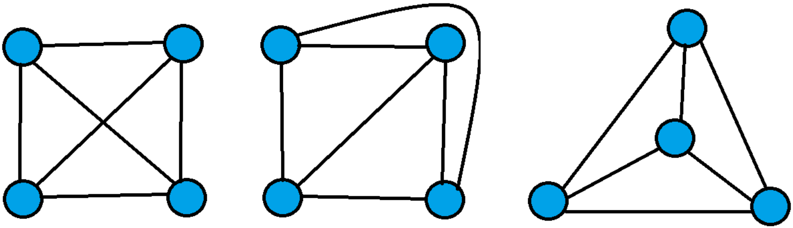
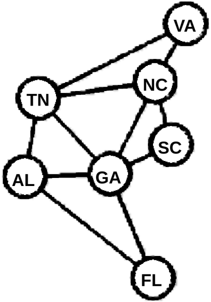


- A **Planar Graph** is a graph that can be drawn on a two dimensional surface such that no edges overlap. [3]
  - The area enclosed by a set of edges is called a **Face**.
  - If a graph cannot be drawn this way, it is not planar.



- Geographical maps can be translated into planar graphs.
- Nations are nodes and their borders are edges.



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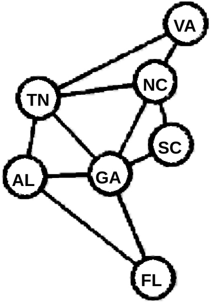
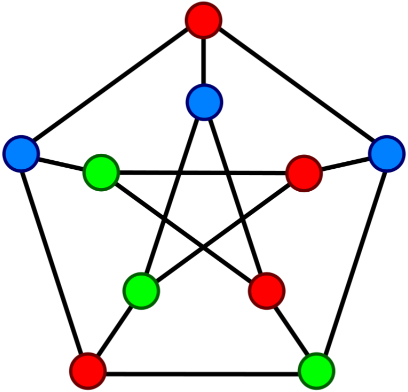
- Nations are nodes and their borders are edges.
    - This translation allows us to use graph theory when constructing maps. [1]

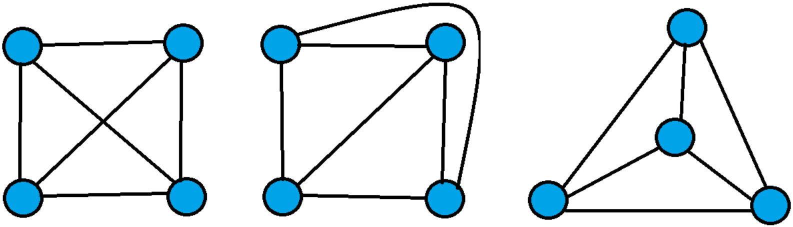


- The **chromatic number** $\chi(G)$ of a graph G is the minimum number of colors needed to assign a color to every vertex in G such that no two vertices sharing an edge are the same color.



Theorem (Euler's Formula)

Let G be a connected planar graph with size e and order v. Any planar depiction of G has $e - v + 2$ faces.



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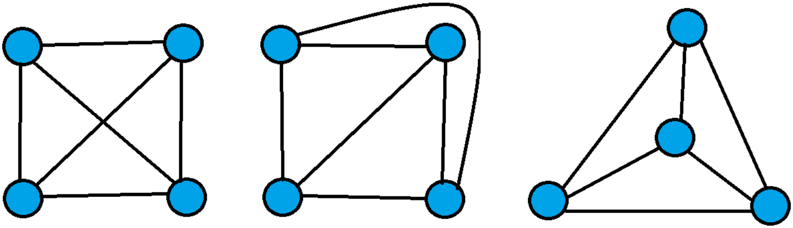
- e is the number of edges (size)
- v is the number of vertices (order)



Corollary (1)

If G is a connected planar graph with $e > 1$, then $e \leq 3v - 6$.

Corollary (2)

Every planar graph G has a vertex of degree ≤ 5.

- Derived from Euler's formula
- Basis for the Five Color Theorem

Theorem (The Five Color Theorem)

Every planar graph G has $\chi(G) \leq 5$.

- Trivially true for any G with order ≤ 5.
- Will use induction on the order of the graph v for the non-trivial cases. [2]

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- From Corollary 3 we know H contains a vertex x with degree ≤ 5.
- Now look at J, which is $H - x$ and all edges incident with it.
- We know J is planar and of order v, so $\chi(J) \leq 5$.
- Suppose J has been colored with at most 5 colors, and now we must attempt to rejoin x into J.

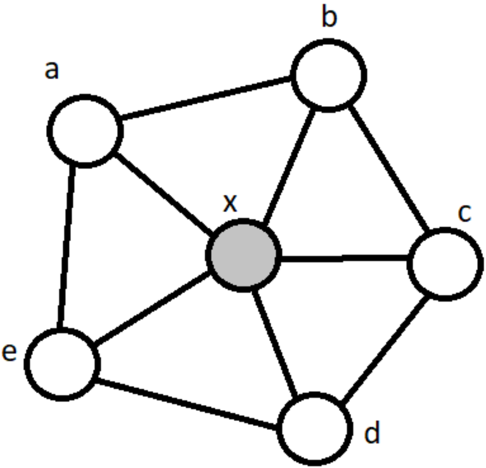

Figure: Planar Depiction of a subgraph of H

Case 1: J is colored using < 5 colors.

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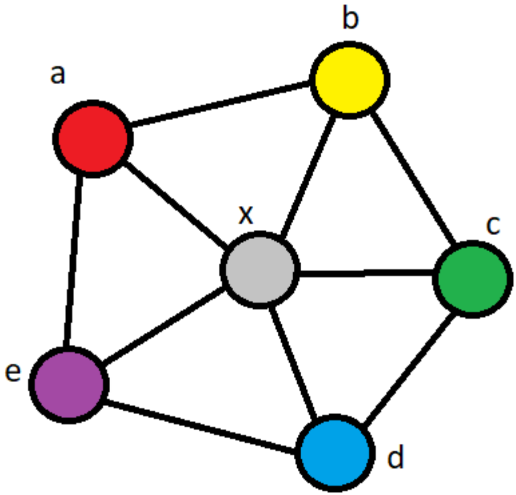
Case 3: J is colored using 5 colors and x has degree 5.

- Identify the five nodes (clockwise) adjacent to x as a, b, c, d, and e and the five available colors as 1, 2, 3, 4, and 5.
- If any two of these vertices are the same color, x can use the remaining fifth color.


a
b
x
c
e
d

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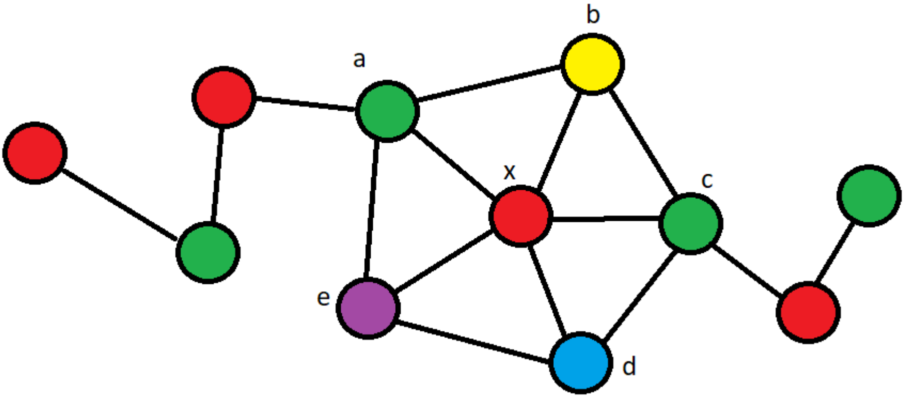
- Thus, color 1 is now available for x.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- Since J is planar, there cannot exist a walk connecting b and d consisting only of colors 2 and 4.

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- So alternate the coloring of b and its adjacent vertices.

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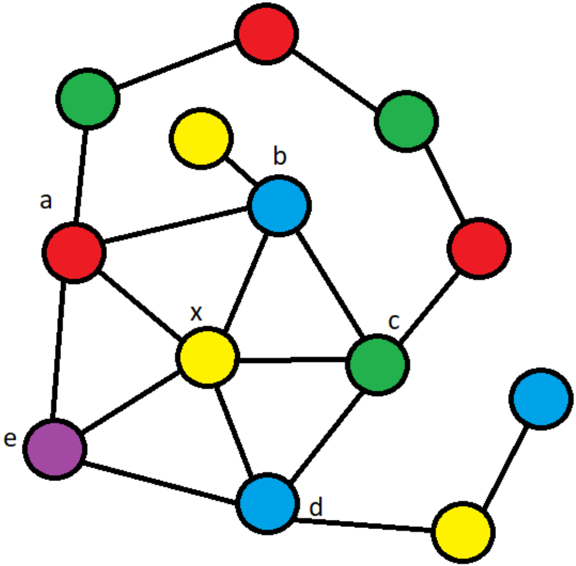
- Consider two other non-adjacent vertices b and d (colored 2 and 4).
- Since J is planar, there cannot exist a walk connecting b and d consisting only of colors 2 and 4.
- So alternate the coloring of b and its adjacent vertices.
- Thus, color 2 is now available for x.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- Therefore if G is a planar graph of order v such that $\chi(G) \leq 5$, then $\chi(H) \leq 5$ for a planar graph H of order $v+1$.

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- Thus we can see that any planar graph can be colored with at most five colors.

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- There is an improved Four Color Conjecture, but can only be proven by computational brute-force.
- All map projections can be colored with ≤ 5 colors.

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