BUFFON'S NEEDLE

DANNY HAMMER, JULIA JANUCHOWSKI

ABSTRACT. If a number of needles of equal length were dropped onto a ruled surface, what percentage of the needles would land such that they are crossing a line? The answer involves the constant π. In fact, this situation is the premise for the Buffon's Needle theorem. The probability that a needle will cross a line is related to the length of the needle, distance between lines, and π. Thus, by reordering this formula, one can generate an estimation of π by dropping needles onto a ruled surface and counting the number of needles that cross a line. This experiment illustrates the relationship between π and probability. A famous recreation of this experiment is widely believed to be a hoax, which relates this theorem with importance of peer review in Mathematics.

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1. Background

Suppose you drop a needle on a sheet of paper with evenly spaced lines across it. What is the probability that the needle will land in a such a way that it crosses one of the lines? As it turns out, this probability is related to the number π. The probability itself depends on the length of the needle l and distance between the lines d. The needle's length must not be larger than the distance between the lines $l \leq d$. In the end, the probability can be used to estimate the value of π. [1]

Theorem 1 (Buffon's Needle)**.** *If a needle of length l is dropped on a paper with equally spaced lines of distance $d \geq l$, then the probability p that the needle lands in such a way that it rests on a line is*

$$p = \frac{2l}{\pi d}$$

Consider dropping n needles and, of the n needles dropped, h of them land on one of the lines. Then the ratio between successes and total drops $\frac{h}{n} \approx \frac{2l}{\pi d}$. Given this, we can rearrange to find that $\pi \approx \frac{2ln}{dh}$. Buffon's needle problem is an example of one of the first Monte Carlo Simulations which utilizes large sample sizes in order to gain information about a specific problem. Of course, in order to obtain an accurate approximation for π, a very large number of needles must be dropped. Consider the approximation $\pi \approx \frac{355}{113} = 3.1415929\ldots$ and note that in order to obtain a more accurate estimation, larger numbers will be present on the numerator and denominator, which is easiest to obtain by increasing the number of needles dropped n.

Previous tests have been done to prove this approximation. In 1901, Italian mathematician Mario Lazzarini experimented by dropping 3,408 needles with 1,808 successes. He chose a needle length of 5 and a distance of 6, which yielded $\frac{2\times5\times3408}{6\times1808} = 3.14159\ldots$. There has been some speculation on the validity of Lazzarini's experiment, as 3,408 is a very peculiar number of needles to drop. Given that the needle length $l = 5$ and distance $d = 6$, if Lazzarini was hoping to obtain the well-known approximation of $\frac{355}{113}$, he could simply solve Buffon's fomula for n as follows: $\frac{355}{113} = \frac{5n}{6h} \to h = \frac{113n}{213}$. This means that, so long as n is a multiple of 213, the formula will provide an estimation of π to 6 decimal places. Given this information, if one drops 213 needles and obtains 113 successes, then an accurate estimation of π is achieved. If not, one can repeat the "trial" with 213 more needles until a multiple of 113 successes is obtained. As $3408 = 16 \times 213$, it is hypothesized that Lazzarini used a strategy like this to obtain his estimation. This hypothesis is further supported by the fact that a single decrease or increase in the number of successes h would lead to significantly different results: $\frac{5\times3408}{6\times1807} = 3.1433\ldots$, $\frac{5\times3408}{6\times1809} = 3.1398\ldots$. [2]

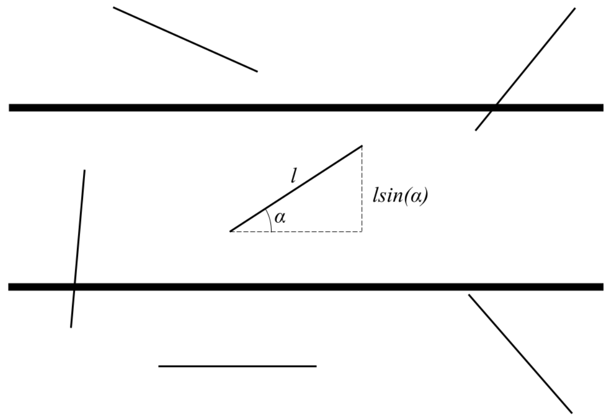

FIGURE 1. reference image

2. PROOF

Proof. Consider an arbitrary needle dropped onto a plane with evenly ruled horizontal lines of distance d apart, as illustrated in figure 1. In order to know if a specific needle crosses one of the lines we must find the slope of the needle and it's distance from the closest horizontal line. In order to find the slope of the needle we must first find the angle α that the needle makes with the horizontal. We can assume that α is within the range $0 \leq \alpha \leq \pi$ by the fact that every needle can be shown using a negative slope as well as a positive slope. We can then see that the slope of any given needle is equal to $l\sin(\alpha)$. We will then use probability to say the probability that a needle crosses a line is $p = \frac{l\sin(\alpha)}{d}$. Finally we will find the average possibility that a needle crosses a line by taking the integral of p over all the possible angles of α.

$$p = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{l\sin\alpha}{d}d\alpha = \frac{2}{\pi}\frac{l}{d}[\cos\alpha]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\frac{l}{d}$$

Finally we get that the probability of any given needle crossing a line is equal to $\frac{2l}{\pi d}$, thus proving our original statement. □

3. BARBIER'S SOLUTION

Another notable solution is one of the French mathematician Joseph Émile Barbier. His focus was on how many times a given needle (of any shape or length) passed over a line rather than whether a needle crossed a line or not. His most popular solution involved circles with a diameter the same length as the distance between the lines (d). He then inscribed and circumscribed polygons with n sides as shown by figure 2. These polygons were then used as the needles in the experiment.

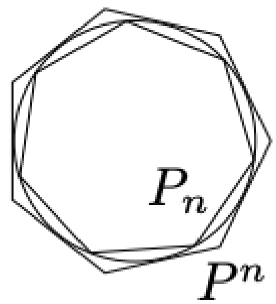

FIGURE 2. Example of inscribed and circumscribed polygons

Barbier then solved for how many times this given needle would cross one of the lines using limits for n as it approaches ∞. He was then able to conclude that the number of times a needle would cross a line was $\frac{2l}{\pi d}$.

4. Conclusion

Buffon's Needle demonstrates a relationship between π and probability. It illustrates how seemingly unrelated mathematical concepts can be related to one another. Lazzarini's alleged hoax is noteworthy in showing how simple it was to falsify evidence and reiterates the importance of peer review in mathematical research. Buffon's Needle is also one of the first Monte Carlo simulations. A Monte Carlo simulation uses the law of large numbers in order to make approximations based on random events[4]. Through this experiment we are able to use Monte Carlo simulations in order to estimate π. In practice, Buffon's Needle can be used in filtration problems in order to determine the particle size or mesh fineness needed to ensure a desired probability of filtration. This concept has also been translated into three dimensions[3].

References

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