Buffon's Needle

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- Hint: This probability is related to π!

Theorem (Buffon's Needle)

If a needle of length l is dropped on a paper with equally spaced lines of distance $d \geq l$, then the probability P that the needle lands in such a way that it rests on a line is

$$P = \frac{2l}{\pi d}$$

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- Since l and d are constants, we can estimate π by dropping needles! [1]

- Consider an arbitrary needle dropped onto a plane with evenly ruled horizontal lines of distance d apart.

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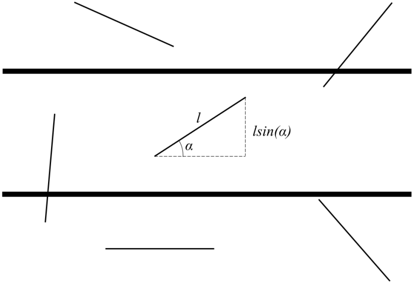
- We can then find the angle the needle makes with the horizontal (α) and the height of the needle using trigonometry



We will then use probability to say the probability that a needle crosses a line is:

$$P = \frac{l \sin(\alpha)}{d}$$

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- $f(x) = P = \frac{l\sin(\alpha)}{d}$
- $a = 0$
- $b = \frac{\pi}{2}$

Therefore, the average probability that a given needle of length l crosses a line at any α is:

$$P = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{l\sin(\alpha)}{d}\, d\alpha$$

$$P = \frac{2}{\pi}\frac{l}{d}[-\cos\alpha]_0^{\frac{\pi}{2}}$$

$$P = \frac{2}{\pi}\frac{l}{d}$$

$\square$

Additional solution: E. Barbier (1860) “Buffon’s noodle”

- Drop any size or shaped needle
- Focused on the probability of the number of times a needle crossed a line
  - utilizes limits
- Circular needle where the diameter = d

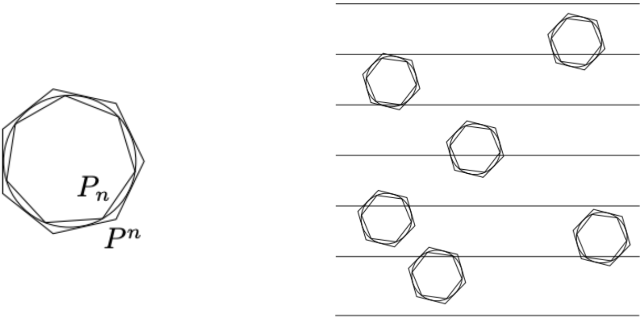

Figure: Example of “noodles” dropped on a sheet of paper

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- But why $3,408$?

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- So long as n is a multiple of 213, h will be an integer and the formula will estimate π to six decimal places.
- All one needs to do is drop 213 needles at a time until the number of successes h is a multiple of 113.
- Note $3,408 = 16 \cdot 213$ [2]

Significance

- Estimate π by simply dropping sticks
- Monte Carlo Simulation

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