On the Game Chromatic Number

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Abstract

The *graph coloring game* is a game wherein two players alternate coloring vertices in a graph G. The minimizer aims to use the fewest number of colors possible, and the maximizer aims to prevent the graph from being properly colorable. The *game chromatic number* $\chi_g(G)$ of a graph is the fewest number of colors needed to guarantee that the minimizer can win. There exists no known algorithm to determine $\chi_g(G)$ for an arbitrary graph, but when certain restrictions are placed on G, bounds can be proven. In this paper, we will prove why the game chromatic number of a a forest F is at most 4. We will also show the criteria needed for $\chi_g(F) = 2$, and discuss minimal order trees with $\chi_g(F) = 4$.

1 Background

The *Graph Coloring Game* (GCG) is a two-player game involving a finite graph G and a set of t colors C. The two players play the roles of a minimizer and a maximizer, where the minimizer moves first and selects a color from C and assigns it to a vertex v in G. A vertex v can be assigned a color c from C if and only if none of the neighbors of v have been colored with c. The players alternate until either every vertex in G has been assigned a color or there exists a vertex that cannot be colored from the set C. The goal of the minimizer (player 1) is to use the fewest number of unique colors possible while ensuring every vertex in G is properly colored. The goal of the maximizer (player 2) is to either prevent G from being properly colored or maximize the number of colors used in the process. The *game chromatic number* of a graph G, denoted $\chi_g(G)$, is the smallest t such that the minimizer can be guaranteed to win on G. It is evident that $\chi(G) \leq \chi_g(G) \leq \Delta G + 1$, as the game chromatic number of a graph cannot be lower than its chromatic number, and the number of colors does not need to exceed 1+ the maximum degree in G. [1]

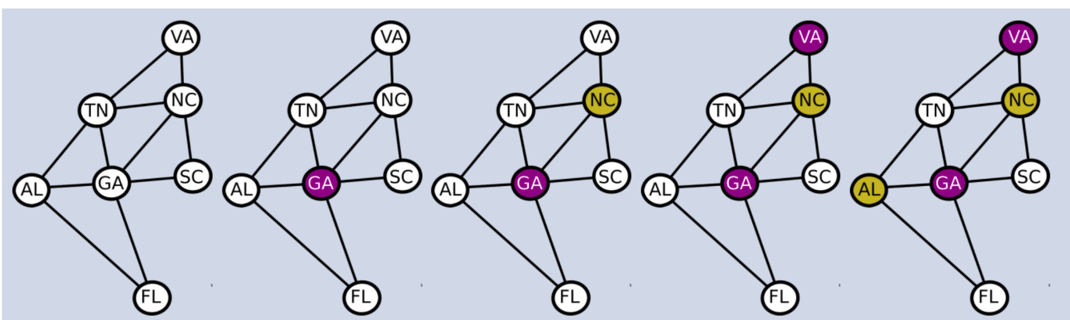

Figure 1: An example of the Graph Coloring Game

The question which remains open is whether or not there exists an algorithm in polynomial time to determine the game chromatic number of a graph. This question has been investigated thoroughly since it was proposed, and much information has been derived from these investigations. Namely, placing restrictions on the graph makes determining its chromatic number much easier, and even allows upper bounds to be set. For example, Faigle et al. [4] proved that for every tree t, $\chi_g(T) \leq 4$. This was an improvement upon Bodlaender's upper bound of 5 [2]. We then see how restricting G can assist in determining $\chi_g(G)$ or the derivation of a polynomial time algorithm for doing so. So we then inquire about this algorithm for graphs with certain restrictions, namely trees and forests.

It is important to note the restrictions being placed on G and the theorems that have already been proven. Specifically, by only considering trees and forests, we know that $\chi_g(G) \leq 4$. We also know that the cases wherein $\chi_g(G) = 0$ or $\chi_g(G) = 1$ are trivial.

Thus we need only consider finding a polynomial time algorithm to determine if the game chromatic number of a graph is 2, 3, or 4.

2 Further Definitions

We say a graph G is *properly colored* if every vertex in G has been assigned a color such that no two vertices who share an edge also share a color. We say a graph is *partially colored* if there exists at least one uncolored vertex in G. The *degree* of a vertex is the number of edges connected to it. A *subgraph* H of a G is a collection of vertices and edges such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$. [5]

Recall that a *tree* is a connected graph that contains no cycles, and a *forest* is a collection of disjoint trees. If a vertex in a tree has a degree of 1, we describe that vertex as being a *leaf*. We can define a *trunk* R of a partially colored forest F if R is a largest possible connected subgraph of F such that every colored vertex in R is also a leaf of R. We can then denote the set of trunks in F by $R(F)$. Note that, unless it consists of two adjacent colored vertices, every trunk will contain at least one uncolored vertex. In addition, every vertex with degree more than 1 is present in at least one trunk.

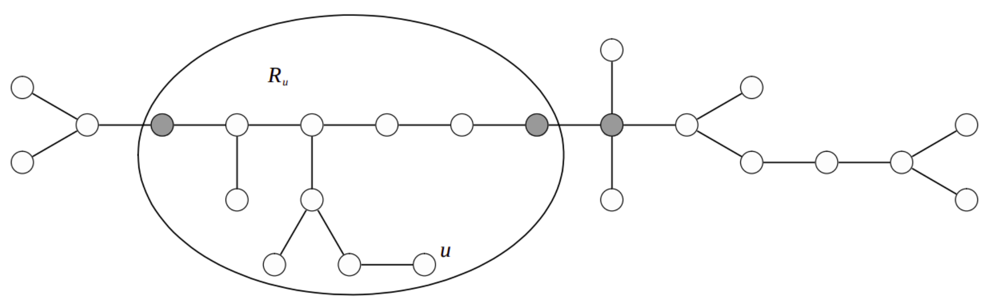

Figure 2: A Forest of partially colored Trees with a Trunk circled

3 Modified Games

One method of proving bounds on the graph coloring game is to explore variations of the game. There exist relationships between these variations and the original game. Namely, the *Modified Coloring Game* and the *Expanded Coloring Game*. In the MCG, the rules are changed so that the maximizer plays first and can choose to pass by not coloring a vertex. In the ECG, the minimizer can choose to not color a vertex and, if so, can add a single colored leaf node to the graph.

Playing the k-coloring game on a partially colored forest F is equivalent to playing the k-MCG on a $R(F)$, as every uncolored vertex in F has the same set of neighbors in $R(F)$ as in F. The minimizer can use the same winning strategy from the k-MCG, as if the maximizer chose to pass initially and never chose to pass again. Given this, we can declare that if the minimizer can with the k-MCG on every trunk in $R(F)$, they can also win the k-coloring game on F.

These modified games allow us to prove the bounds of the k-coloring game on trees and forests, as we will see in the following section.

4 Proving an upper bound

We now have sufficient information to prove the bounds shown by Faigle et al. [4]. We will first prove a useful lemma that will aid in proving the upper of $\chi_g(F) \leq 4$ for any forest F.

Lemma 1. *Let F be a partially colored forest and let the 4-MCG be played on $R(F)$. If every trunk in $R(F)$ has at most 2 colored vertices, the minimizer can win the 4-MCG on $R(F)$.*

Proof. Consider a partially colored forest F such that every trunk in $R(F)$ has at most 2 colored vertices. Let n denote the number of uncolored vertices in $R(F)$ (the minimizer wins when $n = 0$). If $n > 0$, there can be at most one trunk with 3 colored vertices after the maximizer's first move.

If this trunk exists, it is a tree that has 3 colored leaves, and must contain an uncolored vertex of at least 3 whose deletion will disconnect the leaves. Since this vertex has at most 3 colored neighbors, it can be legally colored by the minimizer. Thus, a partially colored graph with trunks containing at most 2 uncolored vertices each is created. The minimizer can now easily win.

If the trunk did not exist, the minimizer may play in any trunk that has an uncolored vertex.

If this trunk has at most 1 colored vertex, the minimizer can color any vertex without issue.

If it has two colored vertices, the minimizer can color any vertex on the unique path between them.

Thus, the minimizer can win the 4-MCG on $R(F)$. □

With the knowledge of these modified games, we can prove the bounds of the k-coloring game for all forests:

Theorem 2. $\chi_g(F) \leq 4$ *for any forest* F.

Proof. Note that every trunk in $R(F)$ has no colored vertices at the start of the game. By lemma 1, the minimizer can win the 4-MCG on $R(F)$. As we have shown earlier, playing the k-MCG on a partially colored forest is equivalent to playing the k-coloring game on $R(F)$. Thus, the minimizer can win the 4-coloring game on F. □

In addition to the k-MCG's usefulness in proving the bounds for the k-coloring game, the k-ECG can be used to simplify the bounds even further using the following lemmas:

Lemma 3. *Let* F *be a partially colored forest and let* F' *be an induced connected subgraph of* F*. If for every vertex* v *in* $V(F')$*.* v *has no colored neighbors in* $V(F)$ $V(F')$ *and bob can win the* k*-ECG on* F'*, then Bob can win the* k*-coloring game on* F*.*

Lemma 4. *The maximizer can win the* 2*-ECG on an uncolored path of length 5* P_5*.*

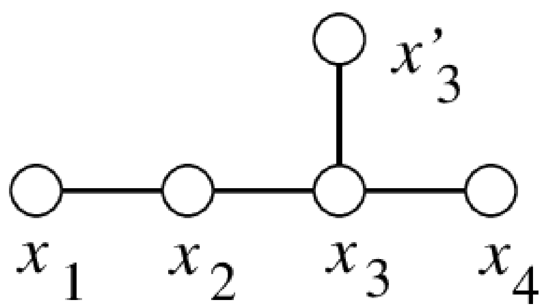

Figure 3: The graph T^+

Lemma 5. *Consider an uncolored tree* T^+ *consisting of vertices* x_1, x_2, x_3, x_4, x'_3 *such that* x_1 *to* x_4 *form a path and* x'_3 *is connected to* x_3*. Then the maximizer can win the* 2*-ECG on* T*. (See Figure 3).*

5 Simplifying the Problem Further

Recall from the beginning of this paper that we need only consider finding a polynomial time algorithm to determine if the game chromatic number of a graph is 2, 3, or 4. In fact, this can be simplified further by classifying all trees with game chromatic number 2, thereby we only need to find an algorithm to determine if the game chromatic number is 3 or 4.

Luckily, it is possible to classify all graphs with game chromatic number 2 by using the modified coloring games and previous lemmas.

Theorem 6. *Let F be a forest and let $l(F)$ be the length of the longest path in F. Then $\chi_g(F) = 2$ if and only if:*

- $1 \leq l(F) \leq 2$ *or*
- $l(F) = 3$, $|V(F)|$ *is odd, and every component with diameter 3 is a path.*

Proof. Consider a forest F that has *not* met the conditions of the theorem. Thus, $l(F) < 1$ (which implies $\chi_g(F) \leq 1$) or $l(F) \geq 3$. If the latter is true, there exist the following possible cases:

Case 1. *If $l(F) > 3$, then a path $P_5 \subseteq F$ and the maximizer can win the 2-ECG on the P_5 by Lemma 4, and thus $\chi_g(F) > 2$ by Lemma 3.*

Case 2. *If $l(F) = 3$ and the aforementioned $T^+ \subseteq F$, the maximizer can win the 2-ECG on the T^+ by Lemma 5, and thus $\chi_g(F) > 2$ by Lemma 3.*

Case 3. *If $l(F) = 3$ and $T+ \not\subseteq F$, then $|V(F)|$ is even and some subgraph of F is a P_4. Therefore, the maximizer can be guaranteed to win or that the minimizer will be the first to play in the P_4. If the latter is true, the maximizer can force the P_4 to be uncolorable.*

If F satisfies the theorem's conditions, then the trunks in $R(F)$ are the components of F. Each trunk R is either a P_4 or has $l(R) \leq 2$. Obviously, the minimizer can win the 2-MCG on any trunk that is not a P_4. If we delete all copies of P_4 from F to obtain a subgraph G, the minimizer can win the 2-MCG on G. If $G = F$, then we have shown that the minimizer can win the 2-coloring game on F. If $G \neq F$, then there is at least one P_4 in F. Because $|V(G)|$ is odd, the maximizer cannot force the minimizer to play first in a P_4. So the minimizer will follow the 2-MCG strategy to start and every time after the maximizer colors a vertex in $V(G)$. If the maximizer colors a vertex in P_4, the minimizer must color a vertex of distance 2 away with the same color.

Therefore, the minimizer can win the 2-coloring game on F[3]. □

Thus we can identify forests with game chromatic number 2. All that is left is differentiating between game chromatic numbers of 3 and 4.

6 Three or Four?

As we have seen, $\chi_g(F) \leq 4$ for any forest F. In addition, determining if $\chi_g(F) = 1$ is a trivial problem. We have also shown the criteria for $\chi_g(F) = 2$. Therefore, all that is left is a method to distinguish between forests with $\chi_g(F) = 3$ and $\chi_g(F) = 4$.

As it turns out, there is a bound for forests with game chromatic number 3:

Theorem 7. *Let F be a forest such that $|V(F)| \leq 13$. Then $\chi_g(F) \leq 3$.*

So any forest F with 13 or fewer vertices is guaranteed to have game chromatic number of at most 3. However, this does not mean that all forests with more than 13 vertices have game chromatic number of 4. We can define a minimal order forest with $\chi_g(F) \leq 4$ with the following theorem:

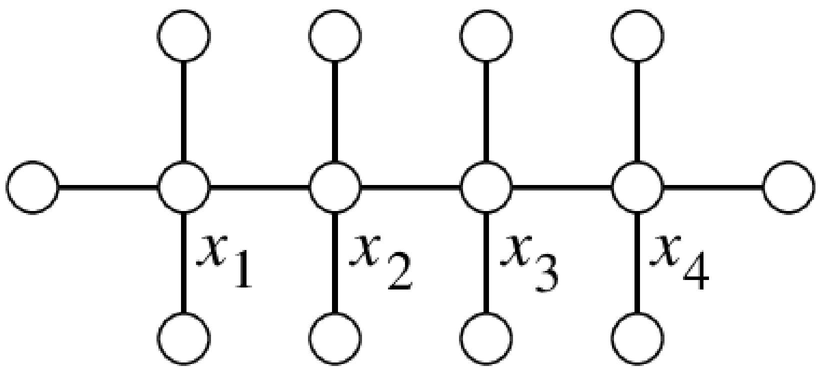

Figure 4: Minimal Order Tree with Game Chromatic Number 4

Theorem 8. *Let T be the caterpillar graph of order 14, as shown in Figure 4. Then Bob can win the 3-ECG on T. Therefore, T is a minimal example of a tree with game chromatic number 4.*

7 Conclusion

In conclusion, we have seen how restricting a graph as a forest F can make proving the game chromatic number bounds possible. We have also seen the process of proving the upper bound of 4 for all forests, identifying the criteria for a game chromatic number of 2, and the maximum order of a graph guaranteed to have game chromatic number of 3. Future inquisitive researchers may wish to investigate other minimal order forests with game chromatic number 4, or explore how more/fewer restrictions on a graph can affect the game chromatic number.

References

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