Siwick Research Group

Departments of Physics and Chemistry
www.physics.mcgill.ca/siwicklab

Summer Reading:
Group Theory - Application to the Physics of Condensed Matter

by M. S. Dresselhaus, G. Dresselhaus and A. Jorio

Summer 2022

Template adapted from "course-notes-template" by Caleb McWhorter
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0 Introduction

This is the summary of the most important points we took away from the reading of "Group Theory - Application to the Phyzsics of Condensed Matter" by M. S. Dresselhaus, G. Dresselhaus and A. Jorio published by Springer in 2008. It is more of a "cheat sheet" we can refere back to than a complete summary. Hence, many examples from the book will be missing and only the main defintions, theorems and lemmas will be listed without much explanation. For the respective derivations and proofs we refer to the actual book.

Each section corresponds to a chapter in the book with the same title. However, not all chapters from the book are summarized here. We start with the first for chapters on the mathematical foundations of group theory and will cherry pick the chapters along the way.

0.1 Quantifiers

For a concise notation of defintions and theorems it is helpful to use the quantifier notation. This enables a transformation from a more "prosaic" way of expressing logical statements to a very precise notation. There are two important quantifiers

Definition 0.1 (All quantifier)**.** The all quantifier $\forall \dots$ states "for all ..."

For example, $\forall\ z \in \mathbb{C}$ means "for all complex numbers z". To specify further properties or implications of the objects addressed by the quantifier either ":" or "," is used, e.g. $\forall\ x \in \mathbb{R}^+ : \sqrt{x} \in \mathbb{R}$ means "for all positive real numbers it holds true that the square root of that number is a real number".

Definition 0.2 (Existence quantifier)**.** The **existence quantifier** $\exists \dots$ states "there exists (at least one) ...". A further limitation of the existence quantifier is the **uniquness quantifier** $\exists! \dots$ stating there "exists only one ..."

For example $\exists!\ q \in \mathbb{R} : q^2 = 0$ says that there is only one real number whose square equals zero, which is of course 0.

The combination of the quantifiers enable a very concise statement of mathematical relations. Some examples:

Example 0.1 (Quantifier Examples)**.** Two lines through the origin have only one intersection:

$$\forall\ m_1, m_2 \in \mathbb{R}\ \exists! x_0 \in \mathbb{R} : m_1 x = m_2 x$$

The condition for a continuous function can be given by the "epsilon-delta-criterium":

A function $f : D_f \to \mathbb{R}$ is called **continuous** at $x_o \in D_f$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D_f : \ |x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \epsilon$$

This reads as "*For all ϵ greater than zero there exists a δ greater than zero so that for all elements from the domain of the function f it holds true that if the distance between any value from the domain of the function and x_0 is less than δ the distance between the two image values is less than ϵ.*"

1 Basic Mathematical Background: Introduction

2 Template examples

Here you can see how to use the theorem environments. Find more of them in the style.sty file

2.1 Examples

Usually a chapter in mathematics does not start with a theorem but with a defintiion. For example

Definition 2.1 (Continous functions)**.** A function $f : D_f \to \mathbb{R}$ is called **continuous** at $x_o \in D_f$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D_f : \ |x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \epsilon$$

With the label tag you can reference the defintion of a continous function 2.1 later on in the manuscript which is a nice feature. The following parts are from the original author of the template showing some more features.

There are many famous theorems in Mathematics. One of the most famous theorems is Fermat's Last Theorem.

Theorem 2.1 (Fermat's Last Theorem)**.** *If $n > 2$, there are no integers a, b, c with $abc \neq 0$ such that $a^n + b^n = c^n$.*

2.2 More examples

Theorem 2.1 is one of the most famous theorems in Mathematics. But most undergraduate students do not learn Fermat's Last Theorem. Instead, many students learn formulas such as:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot dS$$

But Mathematics starts far more simple than that. The first topic in Mathematics that one typically sees is Arithmetic. For instance, students typically will learn "FOIL."

$$(x + y)^2 = x^2 + 2xy + y^2 \tag{2.1}$$

However, (2.1) tends to be a stumbling block for students. Many students will instead claim, incorrectly, that $(x + y)^2 = x^2 + y^2$. The purpose of this course will be to prove Dirichlet's Unit Theorem, which states:

Theorem 2.2 (Dirichlet's Unit Theorem)**.** *Let K be a number field of degree n with r real embeddings and s conjugate pairs of complex embeddings. Then the abelian group $\mathcal{O}_K^\times$ is a finitely generated abelian group with rank $r + s - 1$ and $\mathcal{O}_K^\times \cong \mu_K \times \mathbb{Z}^{r+s-1}$, where μ_K are the roots of unity in $\mathcal{O}_K$.*

However, it will take some time to prove Theorem 2.2.

2.3 Even more examples

Recall that the goal of this course was to prove Dirichlet's Unit Theorem:

Theorem 2.2 (Dirichlet's Unit Theorem)**.** *Let K be a number field of degree n with r real embeddings and s conjugate pairs of complex embeddings. Then the abelian group $\mathcal{O}_K^\times$ is a finitely generated abelian group with rank $r + s - 1$ and $\mathcal{O}_K^\times \cong \mu_K \times \mathbb{Z}^{r+s-1}$, where μ_K are the roots of unity in $\mathcal{O}_K$.*

Proof. L.T.R. □

Example 2.1. If $K = \mathbb{Q}$, then $r = 1$ and $s = 0$ so that $r + s - 1 = 0$. Therefore, $\mathcal{O}_\mathbb{Q}^\times = \mathbb{Z}^\times = \{\pm 1\}$. Of course, this is the most trivial possible example. ◁