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Summer Reading:
Group Theory - Application to the Physics of Condensed Matter

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0 Introduction

This is the summary of the most important points we took away from the reading of "Group Theory - Application to the Phyzsics of Condensed Matter" by M. S. Dresselhaus, G. Dresselhaus and A. Jorio published by Springer in 2008. It is more of a "cheat sheet" we can refere back to than a complete summary. Hence, many examples from the book will be missing and only the main defintions, theorems and lemmas will be listed without much explanation. For the respective derivations and proofs we refer to the actual book.

Each section corresponds to a chapter in the book with the same title. However, not all chapters from the book are summarized here. We start with the first for chapters on the mathematical foundations of group theory and will cherry pick the chapters along the way.

0.1 Quantifiers

For a concise notation of defintions and theorems it is helpful to use the quantifier notation. This enables a transformation from a more "prosaic" way of expressing logical statements to a very precise notation. There are two important quantifiers

Definition 0.1 (All quantifier)**.** The all quantifier $\forall \ldots$ states "for all ..."

For example, $\forall\ z \in \mathbb{C}$ means "for all complex numbers z". To specify further properties or implications of the objects addressed by the quantifier either ":" or "," is used, e.g. $\forall\ x \in \mathbb{R}^+ : \sqrt{x} \in \mathbb{R}$ means "for all positive real numbers it holds true that the square root of that number is a real number".

Definition 0.2 (Existence quantifier)**.** The **existence quantifier** $\exists \ldots$ states "there exists (at least one) ...". A further limitation of the existence quantifier is the **uniquness quantifier** $\exists! \ldots$ stating there "exists only one ..."

For example $\exists!\ q \in \mathbb{R} : q^2 = 0$ says that there is only one real number whose square equals zero, which is of course 0.

The combination of the quantifiers enable a very concise statement of mathematical relations. Some examples:

Example 0.1 (Quantifier Examples)**.** Two lines through the origin have only one intersection:

$$\forall\ m_1, m_2 \in \mathbb{R}\ \exists! x_0 \in \mathbb{R} : m_1 x = m_2 x$$

The condition for a continuous function can be given by the "epsilon-delta-criterium":

A function $f : D_f \to \mathbb{R}$ is called **continuous** at $x_o \in D_f$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D_f :\ |x - x_0| < \delta \ \Rightarrow\ |f(x) - f(x_0)| < \epsilon$$

This reads as "*For all ϵ greater than zero there exists a δ greater than zero so that for all elements from the domain of the function f it holds true that if the distance between any value from the domain of the function and x_0 is less than δ the distance between the two image values is less than ϵ.*"

1 Basic Mathematical Background: Introduction

1.1 Basic Definitions

Definition 1.1 (Group)**.** Let $G \neq \emptyset$ be a nonempty set and $\circ : \; G \times G \to G$ a binary operation. $(G, \circ)$ is a *group* if the following conditions are full filled:

1. neutral element: $\exists\, e \in G \;\; \forall\, g \in G : \; e \circ g = g \circ e = g$
2. inverse element: $\forall\, g \in G \;\; \exists\, g^{-1} \in G : \; g^{-1} \circ g = g \circ g^{-1} = e$
3. associative law: $\forall a, b, c \in G : \; (a \circ b) \circ c = a \circ (b \circ c)$

G is *Abelian* if elements commute: $\forall a, b \in G : \; a \circ b = v \circ a$

Definition 1.2 (Subgroup)**.** Let $(G, \circ)$ be a group and U a set with $U \subset G$. $(U, \circ)$ is a *subgroup* of $(G, \circ)$ if

1. $(U, \circ)$ is a group by definition 1.1
2. $(U, \circ)$ is closed: $\forall\, u, v \in U : u \circ v \in U$

Often the notation for a group $(G, \circ)$ is shortened to G if the operation is clear from the context or does not need to be specified.

Definition 1.3 (Order and period)**.** Let $(G, \circ)$ be a group.

- *order of the group G*: $\#G = |G|$
- *order n of the element* $g \in G$: $g^n = e$
  in finite groups the order of each element is finite
- *period of the element* $g \in G$: $\langle g \rangle = \{e, g, g^2, \dots, g^{n-1}\}$
  $\langle g \rangle$ is an abelian subgroup of G.

Theorem 1.1 (Rearrangement Theorem)**.** *Let $G = \{e, g_1, g_2, \dots g_n\}$ be a group.*
For any $g_i \in G : \; g_i \circ G = \{g_i \circ e, g_i \circ g_1, \dots g_i \circ g_n\}$ contains each element og G once and only once.

A consequence of the rearrangement theorem is that each coloumn of a multiplication table contains each element only once.

1.2 Cosets and Classes

Definition 1.4 (Coset)**.** Let G be a group and $U \subseteq G$ a subgroup of G.
For $x \in G : \; Ux = \{e \circ x, u_1 \circ x, \dots u_r \circ x\}$ is the *right coset* of U

The same rational leads to the definition of the *left coset* as xU. A coset is not necessarily a subgroup but if $x \in U$ then $Ux = U$ by the rearrangement theorem.

Theorem 1.2 (Distinct cosets)**.** *Let Ux, Uy be right cosets of a subgroup $U \subseteq G$. Then either $Ux \cap Uy = \emptyset$ or $Ux = Uy$.*

Theorem 1.3 (Group order divisor)**.** *The order of a subgroup is a divisor of the group order.*

1.3 (Self-)Conjugacy

Definition 1.5 (Conjugate)**.** Let G be a group and $a, b \in G$.
b conjugate a $\quad\Leftrightarrow\quad \exists\, x \in G: \; b = xax^{-1}$

Theorem 1.4. *Conjugacy is an equivalence relation:*

1. *Reflexivity: a conj. a*
2. *Symmetric: a conj. b* $\quad\Rightarrow\quad$ *b conj. a*
3. *Transitivity: a conj. b* $\wedge$ *b conj. c* $\quad\Rightarrow\quad$ *a conj. c*

Definition 1.6 (Class)**.** A *class* is the totality of elements which can be obtained from a group element by conjugation.

Theorem 1.5 (Class order)**.** *All elements of the same class have the same order.*

Definition 1.7 (self-conjugate subgroup)**.** Let $N \subseteq G$ be a subgroup of group G.
N is self-conjugate $\quad\Leftrightarrow\quad \forall\, x \in G : xNx^{-1} = N$

This means a subgroup is self-conjugate if it is invariant under conjugation with all elements of the group. A self-conjugate group is sometimes called *invariant, normal or normal divisor*

Definition 1.8 (Simple group)**.** A group with no self-conjugate subgroup is *simple*.

Theorem 1.6. *The right and left cosets of the self-conjugate subgroup $N \subseteq G$ are the same: $xN = Nx, \; x \in G$.*

Theorem 1.7. *If $N \subseteq G$ is a self-conjugate subgroup of G,the multiplication of two right cosets of N (i.e. the multiplication of all elements of the cosets) yields another right coset of N.*

1.4 Factor groups

Definition 1.9 (Factor group)**.** Let $N \subseteq (G, \bullet)$ be a self-conjugated subgroup of G. The set $F : \{Nx | x \in G\}$ contains all right cosets of N. We define a binary operation
$\circ : \; F \times F \to F$ with

$$\begin{aligned} Nx \circ Ny &= \{n_1 \bullet x, n_2 \bullet x, \ldots, n_r \bullet x\} \circ \{n_1 \bullet y, n_2 \bullet y, \ldots, n_r \bullet y\} \\ &= \{n_1 \bullet x \bullet n_1 \bullet y, n_1 \bullet x \bullet n_2 \bullet y, \ldots, n_1 \bullet x \bullet n_r \bullet y, dots, n_r \bullet x \bullet n_r \bullet y\} \quad = N(x \bullet y). \end{aligned}$$

$(F, \circ) = G/N$ is a group called *Factor group*.

The neutral element of the factor group G/N is the coset which contains the neutral element of G. The inverse element of $Nx \in G/N$ is Nx^{-1}.

Definition 1.10 (Index)**.** The index of a subgroup $U \subseteq G$ is the total number of cosets which is the order of the group G devided by the order of the subgroup U.
$idxU = |\{Ux | x \in G\}| = \frac{|G|}{|U|}$

The order of a factor group G/N is the index of the normal divisor N.

2 Representation Theory and Basic Theorems

2.1 Representations

Definition 2.1 (homomorphic/isomorphic groups)**.** Two groups $(G, \circ), (H, \times)$ are *homomorphic*
$\Leftrightarrow \exists\, \varphi : G \to H \;\forall\, a, b \in G : \;\varphi(a \circ b) = \varphi(a) \times \varphi(b)$
$(G, \circ), (H, \times)$ are *isomorphic* $G \cong H$, if G and H are homomorphic and

1. $\varphi(a) = \varphi(b) \Rightarrow a = b$ (injective)
2. $\varphi(G) = H$ (surjective)

i.e. the homomorphism φ is bijective, hence creating a one-to-one correspondence.

Definition 2.2 (Representation of a group)**.** A *representation of a group* G is a substitution group (group of square matrices) which is homomorphic/isomorphic to G.
$\forall\, g \in G : \; g \mapsto D(g)$ where $D(g)$ is the matrix representation of element g.

Attention: A one-to-one correspondence is not *not necessary*! For example, the unity matrix is a trivial representation of any group.

Definition 2.3 (Dimensionality of a representation)**.** The *dimensionality of a representation* is equal to the dimensionality of its matrices.

Definition 2.4 ((Ir)reducible representation)**.** *reducible representation*:
all matrices of the representation can be transformed to the same block form by one and the same equivalence representation.
irreducible representation: not reducible, i.e. and irreducible representation cannot be expressed in terms of lower dimensionality.

Theorem 2.1 (Unitary representations)**.** *Every representation with matrices having non-vanishing determinents can be brought into unitary form by an equivalence transformation.*

2.2 Wonderful Orthogonality Theorem

Theorem 2.2 (Wonderful orthogonality theorem)**.**

$$\sum_R D^{\Gamma_j}_{\mu\nu}(R) D^{\Gamma_{j'}}_{\mu'\nu'}(R^{-1}) = \frac{h}{l_j} \delta_{\Gamma_j \Gamma_{j'}} \delta_{\mu\mu'} \delta_{\nu\nu'}$$

is obeyed for all inequivalent, irreducible representations of a group, summing over all h elements of the group. l_j *and* $l_{j'}$ *are the dimensionalities of the representations* Γ_j *and* $\Gamma_{j'}$*, respectively.*

3 Template examples

Here you can see how to use the theorem environments. Find more of them in the style.sty file

3.1 Examples

Usually a chapter in mathematics does not start with a theorem but with a defintiion. For example

Definition 3.1 (Continous functions). A function $f : D_f \to \mathbb{R}$ is called **continuous** at $x_o \in D_f$ if

$$\forall \epsilon > 0 \quad \exists \delta > 0 \quad \forall x \in D_f : \ |x - x_0| < \delta \ \Rightarrow \ |f(x) - f(x_0)| < \epsilon$$

With the label tag you can referenence the defintion of a continous function 3.1 later on in the manuscript which is a nice feature. The following parts are from the original author of the template showing some more features.

There are many famous theorems in Mathematics. One of the most famous theorems is Fermat's Last Theorem.

Theorem 3.1 (Fermat's Last Theorem). *If $n > 2$, there are no integers a, b, c with $abc \neq 0$ such that $a^n + b^n = c^n$.*

3.2 More examples

Theorem 3.1 is one of the most famous theorems in Mathematics. But most undergraduate students do not learn Fermat's Last Theorem. Instead, many students learn formulas such as:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot dS$$

But Mathematics starts far more simple than that. The first topic in Mathematics that one typically sees is Arithmetic. For instance, students typically will learn "FOIL."

$$(x + y)^2 = x^2 + 2xy + y^2 \tag{3.1}$$

However, (3.1) tends to be a stumbling block for students. Many students will instead claim, incorrectly, that $(x + y)^2 = x^2 + y^2$. The purpose of this course will be to prove Dirichlet's Unit Theorem, which states:

Theorem 3.2 (Dirichlet's Unit Theorem). *Let K be a number field of degree n with r real embeddings and s conjugate pairs of complex embeddings. Then the abelian group $\mathcal{O}_K^\times$ is a finitely generated abelian group with rank $r + s - 1$ and $\mathcal{O}_K^\times \cong \mu_K \times \mathbb{Z}^{r+s-1}$, where μ_K are the roots of unity in $\mathcal{O}_K$.*

However, it will take some time to prove Theorem 3.2.

3.3 Even more examples

Recall that the goal of this course was to prove Dirichlet's Unit Theorem:

Theorem 3.2 (Dirichlet's Unit Theorem). *Let K be a number field of degree n with r real embeddings and s conjugate pairs of complex embeddings. Then the abelian group $\mathcal{O}_K^\times$ is a finitely generated abelian group with rank $r + s - 1$ and $\mathcal{O}_K^\times \cong \mu_K \times \mathbb{Z}^{r+s-1}$, where μ_K are the roots of unity in $\mathcal{O}_K$.*

Proof. L.T.R. $\square$

Example 3.1. If $K = \mathbb{Q}$, then $r = 1$ and $s = 0$ so that $r + s - 1 = 0$. Therefore, $\mathcal{O}_\mathbb{Q}^\times = \mathbb{Z}^\times = \{\pm 1\}$. Of course, this is the most trivial possible example. $\triangleleft$