Data Provided: None

Autumn Semester 2011-12 (2.0 hours)

EEE6440 Advanced Signal Processing

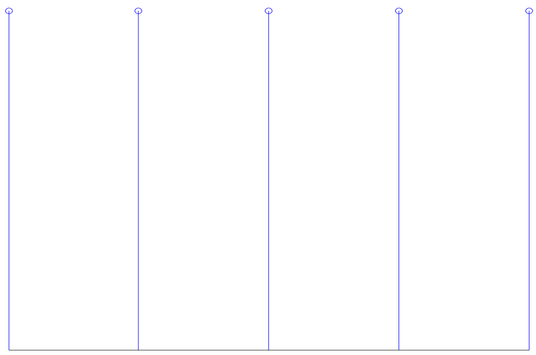
Solutions:

1.

a. Impulse response:

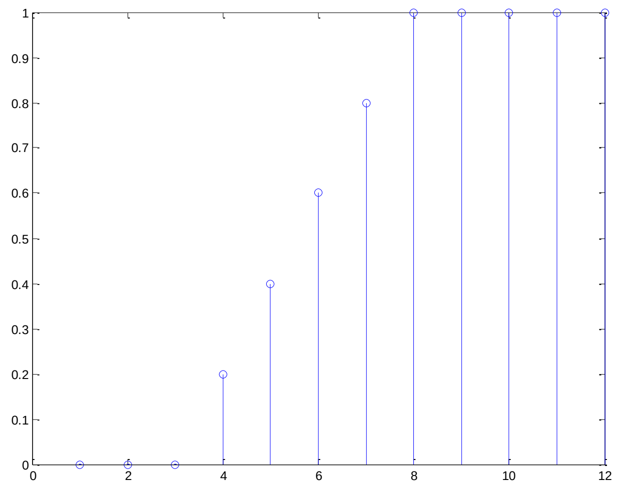
y(n) = 1/5(x[n-2]+x[n-1]+x[n]+x[n+1]+x[n+2])

h(n) = { 1/5, 1/5, 1/5, 1/5, 1/5} the third element is at n=0.



Step response:

Convolve the h(n) with step function u(n). In other words, taking the discrete integral of h(n). Results in { ... 0, 1/5, 2/5, 3/5, 4/5, 1, ...}



(5)

Frequency response:

$y(n) = 1/5(x[n-2]+x[n-1]+x[n]+x[n+1]+x[n+2])$

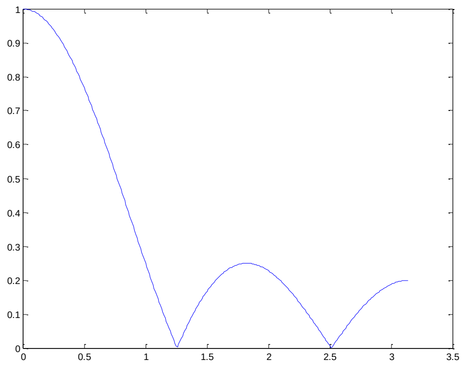
$h(n) = 1/5, 1/5, 1/5, 1/5, 1/5$

$h(z) = 1/5\ (z^{-2} + z^{-1} + 1 + z^{1} + z^{2})$

$z= e^{-j\omega}$,

$H(j\omega) = 1/5\ (e^{2j\omega} + e^{j\omega} + 1 + e^{j\omega} + e^{-2j\omega}) = (1+2\cos\omega+2\cos 2\omega)/5$

$|H(j\omega)| = |\ (1+2\cos\omega+2\cos 2\omega)/5|$



b.

$y(n) = (x[n-2]+x[n-1]+x[n]+x[n+1]+x[n+2])/5$

$y[i] = (x[i-2]+x[i-1]+x[i]+x[i+1]+x[i+2])$

$y[i-1] = (x[i-3]+x[i-2]+x[i-1]+x[i]+x[i+1])$

$y[i] = y[i-1]+(x[i+2]-x[i-3])$

all y(n) values are multiplied by 1/5

Number of multiplications: L (L is the length of y(n)) -----------------(A)

Number of additions:

4 for y[0] and 2x(L-1) for the rest ---------------(B)

(5)

For non-recursive implementation

Number of multiplications: 5L ----------------(C)

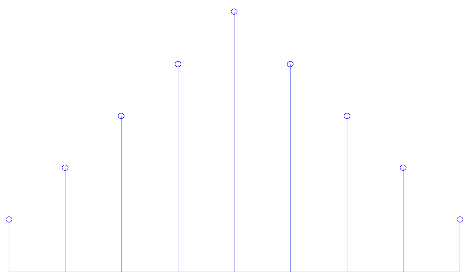
Number of additions: 4L ---------------------------(D)

(A) and (B) are much smaller than (C) and (D).

e. Convolve h(n{ 1/5, 1/5, 1/5, 1/5, 1/5} with itself.

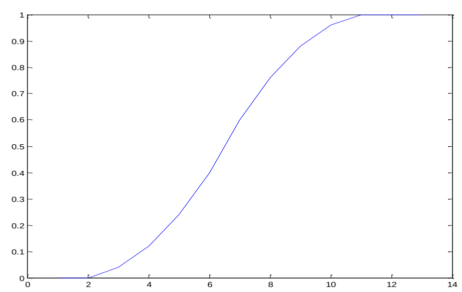
{ 1/5, 1/5, 1/5, 1/5, 1/5} *{ 1/5, 1/5, 1/5, 1/5, 1/5}

= { 1/25, 2/25, 3/25, 4/25, 5/25, 4/25, 3/25, 2/25, 1/25}



Time domain properties:

The step response is as follows:



Smooth rise. Since the kernel is larger, more emphasis is on centre data points in the filter kernel. Therefore sharp changes are preserved, while smoothing out noise.

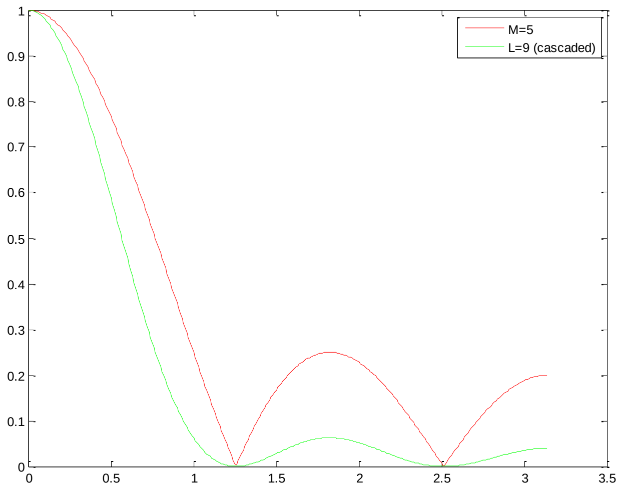
L=5

Now compare M=3, M=5 and L=5

For M=5, $|H(j\omega)| = |(1+2\cos(\omega) + 2\cos(2\omega))/5|$

For L=9, $|H(j\omega)| = |(5+8\cos(\omega) + 6\cos(2\omega)+ 4\cos(3\omega)+ 2\cos(4\omega))/25|$

(5)



(Only an estimated sketch is required to explain the performance difference.)

L=9 provides a faster time-domain response compared to M=5 with better stop-band attenuation.

2.

a. In T each row corresponds to a basis vector f_v.

$$f_0 = [h \quad h \quad h \quad h] \qquad f_1 = [h \quad h \quad -h \quad -h]$$
$$f_2 = [h \quad -h \quad 0 \quad 0] \quad f3 = [0 \quad 0 \quad h \quad -h]$$

(2)

b. For the orthogonality condition

If the inner product $\langle f_n, f_m \rangle$ = 1 when n = m and
= 0 when n $\neq$ m.

In other words $\sum_{i=0}^{3} f_{ni} f_{nm} = \delta_{nm}$

<f0,f0> =<f1,f1> =<f2,2> =<f3,f3> = ((hxh)+(hxh)+(hxh)+(hxh)) = $4h^2$=1
h = +/- 1/2

<f1,f2> = 0
<f1,f3> = ((hxh)+(-hxh)+(0)+(0)) = 0
<f1,f4> = 0
Similarly, <f2,f2>=<f3,f3>=<f4,f4>=1 and
<f2,f3>=<f2,f4>=<f4,f3>=0

(3)

c. F is orthogonal. Therefore, the inverse matrix is the transpose.

$$T^{-1} = \begin{bmatrix} h & 0 & h & 0 \\ h & 0 & -h & 0 \\ 0 & h & 0 & h \\ 0 & h & 0 & -h \end{bmatrix}$$

Compute the $T^{-1}T$ and show it is the Identity matrix (I)

$$TT^{-1} = \begin{bmatrix} h & h & 0 & 0 \\ 0 & 0 & h & h \\ h & -h & 0 & 0 \\ 0 & 0 & h & -h \end{bmatrix} \begin{bmatrix} h & 0 & h & 0 \\ h & 0 & -h & 0 \\ 0 & h & 0 & h \\ 0 & h & 0 & -h \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(3)

d. $y_{0=}$(x0+x1+x2+x3)*h

mean(x0+x1+x2+x3)= (x0+x1+x2+x3)/4

= y0/(4h)

=1/2 x y0

(2)

e. Do the forward transform Y=TX

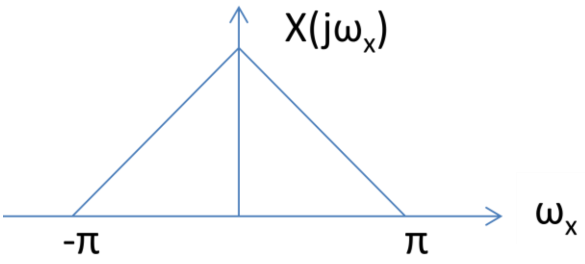
Keep y0

For y1, y2 and y3 keep the value only if they are greater than a threshold. Otherwise set to 0.

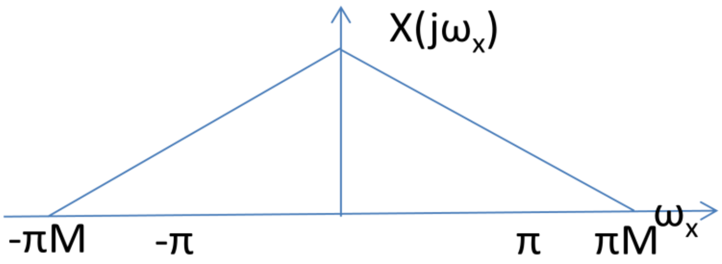
Take the inverse transform of the new transform coefficients

Denoised X= KY , K is the inverse transform matrix **(5)**

3. a. They are used as anti-aliasing filters.



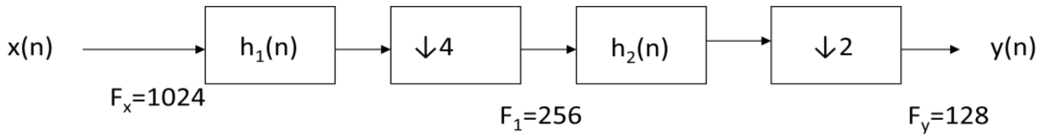
When downsampled by M , the spectrum will be spread to cause aliasing.



By choosing a low pass filter to restrict the signal frequency content to less than pi/M bandwidth, can avoid aliasing when the sample rate is decimated by a factor of M.

(3)

b.



Passband deviation: 0.01dB → 0.00115

Stopband atteneuation: 80dB → 0.0001

For both filters we choose

δ_p=0.00115/2 =0.00058

δ_s=0.0001

Filter length given by $N \approx \frac{-10\log(\delta_p\, \delta_s)-13}{14.6(\Delta f)} + 1$

$$N \approx \frac{-10\log(0.0005 \text{ x } 0.0001) - 13}{14.6(\Delta f)} + 1$$

(7)

$$N \approx \frac{4.066}{(\Delta f)} + 1$$

For h_2:

Passband 0 - 60 Hz

Stopband 64 - 128 Hz

Transition band 60Hz – 64Hz

Normalised transition bandwidth (64-60)/128 = 4/128

Therefore $N_2 \approx \frac{4.066}{\left(\frac{4}{128}\right)} + 1 = 131$

For h_1:

Passband 0 - 60 Hz

Stopband (256-64) -256 Hz = 192-256

Transition band 60Hz – 192Hz

Normalised transition bandwidth (192-60)/1024 = 132/1024

Therefore $N_1 \approx \frac{4.066}{\left(\frac{132}{1024}\right)} + 1 = 32$

c MPS = $\sum_{i=1}^{2} F_i N_i$ = 128x131 + 256x32 = 32 960

N is inversely proportion to Δf. If a single-stage was used Δf would have been (64-60)/1024. To make this value larger, we need to make the numerator bigger and the denominator smaller. This can be achieved by factoring F into a product of several smaller sampling rates. Each of the early stage filetrs the transition bandwidth is large because the corresponding sampling rates are closer to F. **(5)**

4. **a.** Question 4 (or the last question) or part thereof should appear on the last page. This table is preceded by a section break and this page is in section 2 – this has a different footer identifying that this is the END OF PAPER rather than TURN OVER or CONTINUE).

delete the contents of this box to begin working. **()**

Solutions for Part B

Q4 a.

Mean: (1.39+1.63+1.87+2.75+0.68)/5=1.6640
(1 mark)

Variance: ((1.39-1.6640)^2+(1.63-1.6640)^2+(1.87-1.6640)^2+(2.75-1.6640)^2+(0.68-1.6640)^2)/5=0.4533
(1 mark)

Q4 b.

Two random processes are uncorrelated if
E[x(n)y(k)]=E[x(n)]E[y(k)],
where E is the expectation operation.
(1 mark)

Two random processes are independent if
p(x(n),y(k))=p(x(n))p(y(k)),
where p(x(n),y(k)) is the joint probability density function.
(1 mark)

Since the two random processes are independent, then from p(x(n),y(k))=p(x(n))p(y(k)), we have

$$E[x(n)y(k)] = \iint x(n)y(k)\,p(x(n),y(k))dx(n)dy(k)$$
$$= \iint x(n)y(k)\,p(x(n))\,p(y(k))dx(n)dy(k)$$
$$= \int x(n)\,p(x(n))dx(n)\int y(k)\,p(y(k))dy(k) = E[x(n)]E[y(k)]$$

(2 marks)

Q4 c.

i) $H_1(z)=2-3z^{-1}$
z-transform of the autocorrelation at the output
$S_{y_1 y_1}(z) = H_1(z)\ H_1(z^{-1})\ \sigma_x^2$
$=(2-3z^{-1})(2-3z)=4-6z^{-1}-6z+9=-6z+13-6z^{-1}$
(2 marks)

Inverse z-transform by inspection to give autocorrelation sequence:

$$\phi_{y_1 y_1}(m) = Z^{-1}[\, S_{y_1 y_1}(z) \,]$$

Autocorrelation sequence: -6 for m=-1, 13 for m=0, -6 for m=1 and zero for other values of m
(1 mark)

ii)
$H_1(z)=2-3z^{-1}$
$H_2(z)=3-2z^{-2}$
Cross-correlation sequence $\phi_{y_1 y_2}(m) = E[\, y_1(n)\, y_2(n+m) \,]$.
z-transform of the cross-correlation at the outputs
$S_{y_1 y_2}(z) = H_1(z^{-1})\, H_2(z)\, \sigma_x^2$
$=(2-3z)(3-2z^{-2})=6-4z^{-2}-9z+6z^{-1}$
$=-9z+6+6z^{-1}-4z^{-2}$
(2 marks)

Inverse z-transform yields: $\phi_{y_1 y_2}$
-9 for m=-1, 6 for m=0, 6 for m=1, -4 for m=2 and zero for other values of m
(1 mark)

The second cross-correlation is most easily obtained by using the property that $\phi_{xy}(m) = \phi_{yx}(-m)$ i.e.

$$\phi_{y_2 y_1}(m) = \phi_{y_1 y_2}(-m)$$

(2 marks)

-4 for m=-2, 6 for m=-1, 6 for m=0, -9 for m=1, zeros for other values of m
(1 mark)

Q5 a.

The power spectral density function:

$$S_{xx}(\omega) = \sum_{m=-\infty}^{\infty} \phi_{xx}(m) \exp(-j\omega m \Delta t)$$

(1 mark)

Its inverse transform is given by:

$$\phi_{xx}(m) = \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} S_{xx}(\omega) \exp(\, j\omega m \Delta t)\, d\omega$$

(1 mark)

Then, for a zero mean stationary random process, it is variance (the average power) is given by

$$\sigma_x^2 = \phi_{xx}(0)$$
$$= \frac{\Delta t}{2\pi} \int_0^{2\pi/\Delta t} S_{xx}(\omega)\, d\omega$$

(1 mark)

The average power is the integral of $S_{xx}(\omega)$ over the whole frequency range.

$S_{xx}(\omega)$ is the distribution of average power with respect to frequency - the POWER SPECTRAL DENSITY.

(1 mark)

Q5 b.

For cosine wave input, the dynamic range R_D of the quantiser can be calculated from the equation in Section 7.5.2 since sine wave and cosine wave have the same power given the same amplitude.

Then, for a 8-bit A/D converter (M=8):

R_D=1.76+6M dB=1.76+8*6=49.76dB,

(3 marks)

Q5 c.

i)

A Time Recursion

$$\mathbf{h}(n) = \mathbf{h}(n-1) - \mu\, \hat{\underline{\nabla}}(n-1)\,.$$

The Exact Gradient

$$\begin{aligned}\underline{\nabla}(n) &= -\,2\ \mathrm{E}[\ \mathbf{y}(k)\ (\ x(k) - \mathbf{h}^T(n)\ \mathbf{y}(k)\ )\ ] \\ &= -\,2\ \mathrm{E}[\ \mathbf{y}(k)\ e(k)\ ]\end{aligned}$$

A Simple Estimate of the Gradient

$$\hat{\underline{\nabla}}(n) = -\,2\ \mathbf{y}(n+1)\ e(n+1)$$

The Error

$$e(n+1) = x(n+1) - \mathbf{h}^T(n)\ \mathbf{y}(n+1)$$

(3 marks)

Then the updated equation of the LMS algorithm is given by

$\mathbf{h}(n)=\mathbf{h}(n-1)+2\mu\mathbf{y}(n)e(n)$

(1 mark)

ii)

$e(15)=x(14)-\mathbf{h}^T(14)\mathbf{y}(15)=-0.2-[1\ 6][0.3\ 0.25]^T$

$=-2$

(2 marks)

The impulse response is then updated by

$\mathbf{h}(15)=\mathbf{h}(14)+2\mu\mathbf{y}(15)e(15)$

$=[1\ 6]^T+0.2*(-2)*[0.3\ 0.25]^T$

$=[0.88\quad 5.9]^T$

(2 marks)

Q6 a.

i)
Suppose the z-transform of the filter is given by H(z), then the relationship is given by

$$S_{xy}(z) = H(z)\ S_{xx}(z)$$

(2 marks)

ii)
For a white input, we have

$$S_{xy}(z) = H(z)\ \sigma_x^2$$

where σ_x^2 is variance of the input.
Taking inverse transforms gives:

$$\phi_{xy}(m) = h_m\ \sigma_x^2$$

(1 mark)
where h_m is the impulse response of the filter. It can be measured by estimating the cross-correlation directly from the data with the following three steps:

$$\hat{\phi}_{xy}(m) = \frac{1}{M} \sum_{n=0}^{M-1} x(n)\ y(n+m)$$

$$\hat{\sigma}_x^2 = \frac{1}{M} \sum_{n=0}^{M-1} x^2(n)$$

$$\hat{h}_m = \frac{\hat{\phi}_{xy}(m)}{\hat{\sigma}_x^2}$$

(One mark for each step)

Q6 b.

$$e(n) = x(n) - \hat{x}(n)$$

The mean-square error (MSE) cost function

$$\xi(n) = E[\ e^2(n)\ ]$$

(1 mark)

$$\begin{aligned}
\hat{x}(n) &= \sum_{i=0}^{N-1} h_i\ y(n-i) \\
&= [\ h_0\ h_1 \cdots h_{N-1}\ ] \begin{bmatrix} y(n) \\ y(n-1) \\ \cdot \\ \cdot \\ \cdot \\ y(n-N+1) \end{bmatrix} \\
&= \mathbf{h}^T\ \mathbf{y}(n) = \mathbf{y}^T(n)\ \mathbf{h}
\end{aligned}$$

Differentiate

$$\frac{\partial \xi}{\partial h_j} = \frac{\partial}{\partial h_j} E[\ \{\ e^2(n)\ \}\ ]$$

$$= E[\ \frac{\partial}{\partial h_j}\ \{\ e^2(n)\ \}\ ]$$

$$= E[\ 2\ e(n)\ \frac{\partial e(n)}{\partial h_j}\ ]$$

$$= E[\ 2\ e(n)\ \frac{\partial}{\partial h_j}\ \{\ x(n) - \mathbf{h}^T\ \mathbf{y}(n)\ \}\ ]$$

$$= E[\ 2\ e(n)\ \frac{\partial}{\partial h_j}\ \{\ -h_j\ y(n-j)\ \}\ ]$$

$$= E[\ 2\ e(n)\ y(n-j)\ \}\ ]$$

$$= 0$$

for j=0, 1, …, N-1.

In vector form, the gradient is given by

$$\underline{\nabla} = -2\ E[\ \mathbf{y}(n)\ e(n)\ ]$$

$$= -2\ E[\ \mathbf{y}(n)\ (\ x(n) - \mathbf{y}^T(n)\ \mathbf{h}\ )\ ]$$

$$= -2\ E[\ \mathbf{y}(n)\ x(n)\ ] + 2\ E[\ \mathbf{y}(n)\mathbf{y}^T(n)\ ]\ \mathbf{h}$$

$$= -2\ \mathbf{\Phi}_{yx} + 2\ \mathbf{\Phi}_{yy}\ \mathbf{h}$$

$$= \underline{0}$$

where

Autocorrelation matrix

$$\mathbf{\Phi}_{yy} = E[\ \mathbf{y}(n)\ \mathbf{y}^T(n)\ ]$$

Cross-correlation vector

$$\mathbf{\Phi}_{yx} = E[\ \mathbf{y}(n)\ x(n)\ ]$$

Optimal Solution

$$\Phi_{yy}\ \mathbf{h}_{opt} = \Phi_{yx}$$

Alternative formulation

$$\mathbf{h}_{opt} = \Phi_{yy}^{-1}\ \Phi_{yx}$$

GCKA / WL