EEE117: Electrical Circuits and Networks

6 Introduction to a.c. Circuits

The term "a.c. circuit analysis" usually means the analysis of circuits that are driven by sinusoidal sources. All periodic signals can be represented as the sum of harmonics of the fundamental frequency of the signal (ie, the reciprocal of the periodic time) so if the response of a circuit to an input signal consisting of a sinusoid of arbitrary frequency can be worked out, the response to a general periodic signal can be worked out.

In circuits containing only resistors the analytical process used for solving a.c. problems is exactly the same as that for d.c problems. In a.c. circuit analysis problems, however, there are almost always one or both of **inductance** and **capacitance** to deal with and it is the behaviour of these elements and how to deal with it that is the subject of the next section.

7 Reactive Components

7.1 Capacitors

A capacitor is a circuit element that exhibits the physical property of capacitance. It stores energy in the form of charge and electric field. Its basic form is two parallel conducting plates separated by a non-coducting medium called the "dielectric" as shown in figure 2.1. Capacitors made of flexible materials are usually rolled up into a cylindrical shape in order to achieve a high surface area in as small as possible a volume. The capacitance of the structure of figure 2.1 is

$$C = \frac{A\,\varepsilon}{d} \text{ where } \varepsilon = \text{permittivity} = \varepsilon_r \varepsilon_0 \qquad \mathbf{(2.1)}$$

Permittivity is a property of a dielectric that describes its ability to concentrate electric flux. It is often written as the product of ε_r, the "**relative permitivity**" or "**dielectric constant**", and ε_0, the permittivity of free space. The dielectric can be air, inorganic materials (such as mica, ceramic or, as is common in digital and analogue integrated circuits, silicon dioxide) or organic polymers (such as polyester, polycarbonate or polystyrene); all these dielectrics give rise to non-polarised capacitors that can be put in the circuit either way round. The symbol for a non-polarised capacitor is shown in figure 2.2 (i).

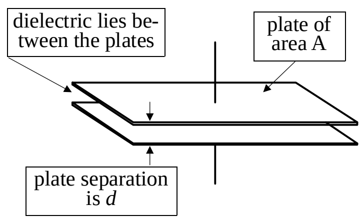


Figure 2.1

A class of capacitors known as "electrolytic" capacitors are polarised and must be inserted into the circuit in the correct polarity - ie the capacitor terminal designed to be the positive one must be positively biased with respect to the other (negative) terminal. Failure to observe this rule usually leads to a short capacitor life and a violent failure. Electrolytic capacitors, usually made of aluminium or tantalum, achieve very large values of capacitance by using as a dielectric an electrolytically grown layer of oxide which is extremely thin. Incorrect biasing reverses the layer forming process until the layer becomes too thin to withstand the voltage drop across it and hence fails. The symbol for an electrolytic capacitor is shown in figure 2.2 (ii) - the hollow electrode is the one that must be positive.

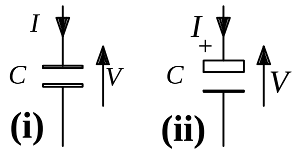


Figure 2.2

In fact all capacitors have a maximum working voltage set by the ability of the dielectric to withstand electric fields and specified by the manufacturer.

The relationship between current through and voltage across a capacitor is given by,

$$I = C\frac{dV}{dt} \text{ or } V = \frac{1}{C}\int Idt + \text{ constant} \qquad \textbf{(2.2)}$$

and the relationship between charge stored and capacitor voltage is

$$Q = CV \qquad \textbf{(2.3)}$$

The unit of capacitance is the **Farad**, **F**. The Farad is a large unit and until recently most capacitors that could be bought in shops lay in the range 1pF (10^{-12} F) to tens of mF ($\approx 10^{-2}$ F). Recently a new breed of capacitor, known as a super-capacitor, has emerged for special applications and these devices have capacitances of the order of kF (10^{3} F).

Stored energy in a capacitor

Consider a capacitor such as the one in figure 2.2 **(i)**. Imagine that at some instant of time, the voltage across the capacitor is v and the current through it is i. The incremental energy stored in a time δt is then $\delta E = vi\,\delta t$ and since $i = C\frac{dv}{dt}$, $\delta E = v\,C\frac{dv}{dt}\delta t$. In the limit $\delta t \to 0$, $\delta t \to dt$ and $\delta E \to dE$ so the energy increment becomes $dE = v\,C\frac{dv}{dt}dt = Cv\,dv$.

The stored energy in the capacitor at a voltage V V is the sum of all the increments of energy between 0 V and V V, or

$$E = C\int_0^V v\,dv = \frac{C\,V^2}{2} \text{ J} \qquad \textbf{(2.4)}$$

where J is the energy unit, Joules. The total energy stored in the capacitor at any time is related only to the voltage across the capacitor at that time and not to the history of how the voltage reached that value.

7.2 Inductors

Inductors, like capacitors, are energy storage elements. In an inductor energy is stored in the magnetic field associated with a wire or a coil of wire. The magnetic flux might be contained within a magnetic circuit as shown in figure 2.3 or alternatively the coil might be air cored in which case the magnetic flux paths will not be so tightly confined. Magnetic circuits are usually made from special iron alloys or ceramic materials based on magnetic iron oxides and other trace elements; these materials are usually called "ferrite materials". Iron tends to be used at power frequences (50 or 60 Hz) whereas ferrite tends to be used from a few kHz to a few MHz.

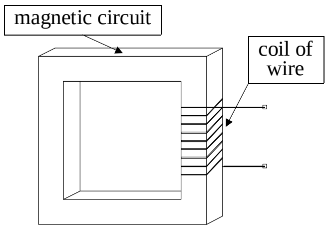


Figure 2.3

In power and power management circuits inductors are common because

- Electromechanical energy converters are based on electro-magnets, magnetic circuits and the interaction between current carrying conductors and magnetic fields.
- Magnetically coupled inductors - i.e., transformers - are used extensively in power systems.
- High values of current mean that modest values of inductance can store relatively large amounts of energy.

In electronic circuitry designed to process or condition signals inductors are less commonly used. (One notable exception to this is the area of radio circuitry at frequencies of MHz and above where inductors are often used as part of resonant circuits.) There are many reasons why designers of non radio circuitry avoid the use of inductors in signal conditioning circuits:

- Inductors are relatively bulky in comparison to resistors and capacitors .
- The presence of a magnetic circuit introduces non-linear behaviour into a circuit but without a magnetic circuit, excessive amounts of wire are needed for quite modest inductance values.
- The magnetic field from a air cored inductor can easily interact with nearby circuitry.

The circuit symbols used for some of the commonest types of inductor are shown in figure 2.4. Figure 2.4 **(i)** shows an air cored inductor, figure 2.4 **(ii)** a ferrite cored inductor and figure 2.4 **(iii)** an iron cored inductor. In each case, L is the algebraic variable used to represent inductance and this usage is universal. The convention for V and I is as shown in figure 2.4.

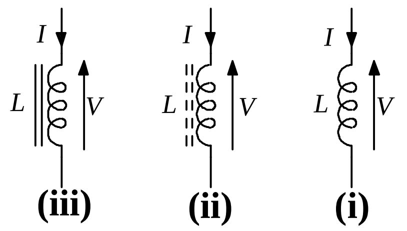


Figure 2.4

The relationship between current through and voltage across an inductor is given by,

$$V = L\frac{dI}{dt} \text{ or } I = \frac{1}{L}\int V dt + \text{ constant} \qquad \textbf{(2.5)}$$

The unit of inductance is the **Henry**, **H**. Inductance values from μH (10^{-6} H) to around 10^{2} H are common in various applications. Inductances of nH (10^{-9} H) tend to be parasitic and troublesome - especially in high frequency or high current circuitry.

Stored energy in an inductor

Consider an inductor such as the one of those in figure 2.4. Imagine that at some instant of time, the voltage across the inductor is v and the current through it is i. The incremental energy stored in a time δt is then $\delta E = vi\,\delta t$ and since $v = L\frac{di}{dt}$, $\delta E = i\,L\frac{di}{dt}\,\delta t$. In the limit $\delta t \to 0$, $\delta t \to dt$ and $\delta E \to dE$ so the energy increment becomes $dE = i\,L\frac{di}{dt}\,dt = Li\,di$.

The stored energy in the inductor at a current I A is the sum of all the increments of energy between 0 A and I A, or

$$E = L\int_0^I i\,di = \frac{L\,I^2}{2} \text{ J} \qquad \textbf{(2.6)}$$

where, as for the capacitor case, J is the energy unit, Joules. The total energy stored in the inductor at any time is related only to the current through the inductor at that time and not to the history of how the current reached that value.

Note the similarity between the capacitor relationships of equations **(2.2)** and **(2.4)** and the inductor relationships of equations **(2.5)** and **(2.6)** respectively. The shapes of the realtionships are the same but where there is L in equations **(2.5)** and **(2.6)** there is C in equations **(2.2)** and **(2.4)**, where there is V in equations **(2.5)** and **(2.6)** there is I in equations **(2.2)** and (**2.4)** and where there is I in equations **(2.5)** and **(2.6)** there is V in equations **(2.2)** and **(2.4).**

7.3 Inductors and capacitors at d.c.

Dealing with reactive components under d.c. conditions is easy;

- Inductors behave like short circuits - i.e., their dc resistance = 0 Ω

This is because for the inductor, V must be zero since dI/dt must be zero - nothing changes with time for d.c. conditions.

- Capacitors behave like open circuits - i.e., their resistance is infinite

This is because for the capacitor, I must be zero since dV/dt must be zero - nothing changes with time for d.c. conditions.

For example, consider the circuit of figure 2.5. Under d.c. conditions the circuit can be drawn as figure 2.6 where all Ls have been replaced by short circuits and all Cs by open circuits.

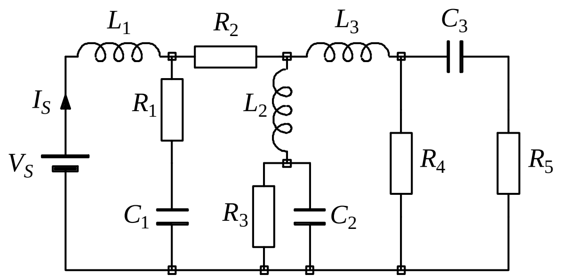


Figure 2.5

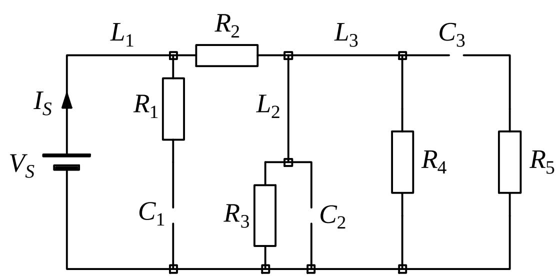


Figure 2.6

Inspection of the circuit then reveals that

$$\frac{V_S}{I_S} = R_2 + R_3 \,//\, R_4$$

Note that although the capacitors and inductors are open circuits and short circuits respectively, they will all store energy because in this circuit, all the Ls will be carrying a current and there will be a voltage drop across all the Cs.

8 Representing a.c. Quantities

The sinusoid is the basic a.c. signal. Sinusoidal sources can be of the voltage source or the current source variety but voltage sources are more common. This is probably because power systems use sinusoidal waveforms and most people have encountered power systems. From a signal point of view, sinusoids have properties that make them particularly attractive as test signals and as the signals of interest - but those special properties are beyond the scope of this module.

The basic sinusoid comes in two forms, a sine wave and a cosine wave. These two are really the same thing - the only difference between them is the position of the time origin, as can be seen in figure 2.7.

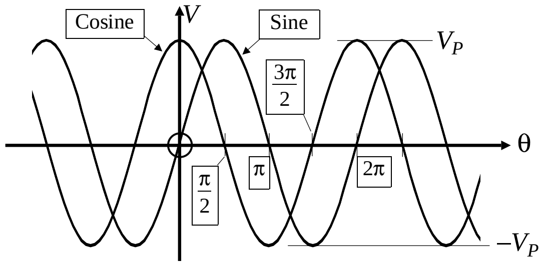


Figure 2.7

The sine wave is given by $V(\theta) = V_P \sin\theta$ and it is easy to see that the only difference between the sine and cosine waves is an angular shift of $\pi/2$ radians. The cosine wave can be expressed in terms of the sine wave by adding $\pi/2$ to θ. For the cosine

$$V(\theta) = V_P \cos\theta = V_P \sin\left(\theta + \frac{\pi}{2}\right) \qquad \textbf{(2.7)}$$

In the electronic engineering area, the angle on which the sinusoid operates is usually linearly increasing with time at a rate ω radians per second so equation **(2.7)** becomes

$$V(t) = V_P \cos\omega t = V_P \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \textbf{(2.8)}$$

8.1 Sinusoids and reactive components.

(i) Capacitors

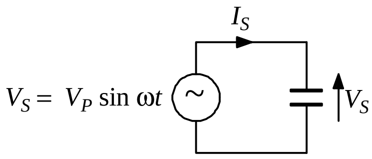


Figure 2.8

The current through a capacitor that has a sinusoidal voltage imposed across it, as shown in figure 2.8, can be derived with the help of equation **(2.2).**

$$I_S = C\frac{dV_S}{dt} = C\frac{d\,(V_P \sin\omega t)}{dt}$$

$$= C\,\omega V_P \cos\omega t = C\,\omega V_P \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \textbf{(2.9)}$$

The modulus of the capacitor's impedance is $\left|Z_C\right| = \left|\frac{V_S}{I_S}\right| = \frac{V_P}{V_P\,\omega C} = \frac{1}{\omega C}$ **(2.10)**

and V_S lags I_S by $\frac{\pi}{2}$ radians or 90^o. **(2.11)**

(ii) Inductors

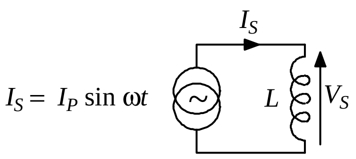


Figure 2.9

The voltage across an inductor that has a sinusoidal current driven through it, as shown in figure 2.9, can be derived with the help of equation **(2.5).**

$$V_S = L\frac{dI_S}{dt} = L\frac{d\,(I_P \sin\omega t)}{dt}$$

$$= \omega L\, I_P\, Cos\,\omega t = \omega L\, I_P \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \textbf{(2.12)}$$

The modulus of the inductor's impedance is $\left|Z_L\right| = \left|\frac{V_S}{I_S}\right| = \frac{I_P\,\omega L}{I_P} = \omega L$ **(2.13)**

and V_S leads I_S by $\frac{\pi}{2}$ radians or 90^o. **(2.14)**

Note the similarity between the definitions of **resistance** given by equation **(1.4)** in section **2.6** of handout 1 and the **impedance** definitions above. **Impedance** is the general form of a *V/I* relationship as a function of frequency - it has both an amplitude and a phase shift associated with it. **Resistance** and **reactance** are two special cases of **impedance** - for **resistive** circuits *V* and *I* are always in phase and there is no frequency dependent behaviour; for **reactive** circuits or circuit elements *V* and *I* are out of phase by $+90^o$ or -90^o and behaviour is frequency dependent. Impedance and reactance are frequency domain quantities - they assume a sinusoidal drive.

8.2 Adding voltages of different phase - the notion of "phasors"

One of the main problems in dealing with a.c. circuit analysis lies in combining quantities that have relative phases other than zero. The problem could be solved graphically by adding the two quantities together at each instant of time. This approach would be very tedious. An alternative approach could be to use trigonometry.

Consider two sinusoids, $a \sin \omega t$ and $a \sin(\omega t + \phi)$. Using the standard trigonometrical identity $\sin C + \sin D = 2 \sin \frac{C+D}{2} \cos \frac{C-D}{2}$, their sum can be written as

$$a \sin\omega t + a \sin(\omega t + \phi) = 2a \sin\left(\frac{\omega t + \omega t + \phi}{2}\right)\cos\left(\frac{\omega t - \omega t - \phi}{2}\right)$$

$$= 2a \sin\left(\frac{2\omega t + \phi}{2}\right)\cos\left(\frac{-\phi}{2}\right) = 2a \sin\left(\omega t + \frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right) \qquad \textbf{(2.15)}$$

Note that the frequency of the sum is unchanged. This is a property of linearity; if a linear circuit is excited by one or more sources at the same frequency, all currents and voltage differences in that circuit will have the same frequency as the exciting sources. In general the sources may be phase shifted with respect to each other and the currents and voltage differences in the circuit will have phase relationships that are a function of driving source phases and the nature of the network.

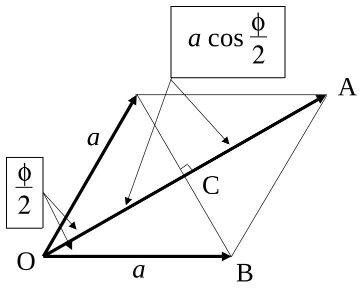


Figure 2.10

Equation **(2.15)** could be regarded as the result of a vector addition. Figure 2.10 represents the two sinusoids which when summed vectorially have the same resultant magnitude and phase as in equation 2.15. Their sum, OA, has a magnitude of $2\,a \cos \frac{\phi}{2}$ and OA is phase shifted by $\frac{\phi}{2}$ with respect to OB, the reference direction.

Consider the sum of two sinusoids $a \sin \omega t + b \cos \omega t$. Here the amplitudes of the two sinusoids is different and the task of working out the sum proceeds as follows

$$a \sin \omega t + b \cos \omega t = \sqrt{a^2 + b^2}\left(\frac{a}{\sqrt{a^2+b^2}} \sin \omega t + \frac{b}{\sqrt{a^2+b^2}} \cos \omega t\right) \tag{2.16}$$

The amplitudes $\frac{a}{\sqrt{a^2+b^2}}$ and $\frac{b}{\sqrt{a^2+b^2}}$ can be interpreted in terms of a right angled triangle as shown in figure 2.11. Equation (2.16) can then be written as

$$a \sin \omega t + b \cos \omega t = \sqrt{a^2 + b^2}\,(\cos \theta \sin \omega t + \sin \theta \cos \omega t)$$
$$= \sqrt{a^2 + b^2}\,(\sin \omega t + \theta), \text{ where } \theta = \tan^{-1} \frac{b}{a}. \tag{2.17}$$

Thus sinusoids of the same frequency but of different phase and different amplitude can be summed vectorially. Note that the ωt part of the phase does not explicitly appear in the vector diagrams; the diagrams illustrate the relative phases between various terms of interest. In fact, the whole vector diagram is rotating in an anticlockwise direction at a rate of ω radians per second. A rotating vector diagram of this type is usually called a "**phasor**" diagram. Figure 2.12 illustrates the idea of a rotating vector diagram.

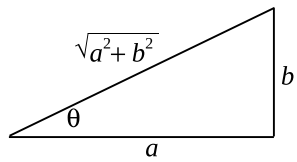
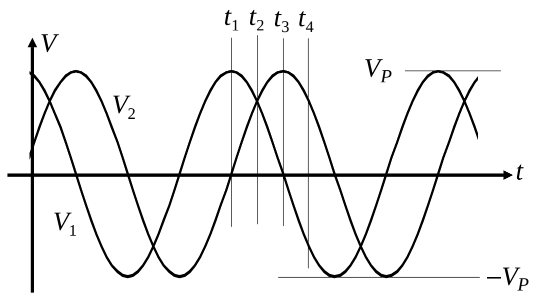


Figure 2.11

The length of the phasors can be their peak value (as in figure 2.12) or their rms value - the latter measure is commonly used in power systems because in that subject area the primary objective is to transmit power and rms has meaning only in terms of power. It is important to know what voltage measure is being used for phasor diagrams and indeed for all ac circuit calculations.



(a)

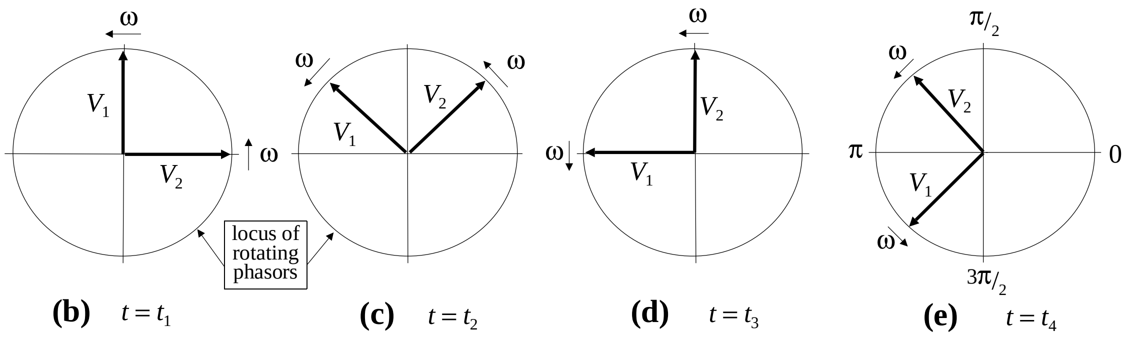


Figure 2.12

Two sinusoids, V_1 and V_2 in figure 34a, are plotted as vector diagrams at t_1, t_2, t_3 and t_4 in figures 34b to 34e. Notice that the difference in phase between V_1 and V_2 is unaffected by the rotation.

8.3 Complex number representation of a.c. circuits

Sometimes called the "*j*" notation, the complex number representation of reactive elements is a powerful tool that makes a.c. analysis much easier than it would otherwise be. "*j*" is the square root of (– 1). Mathematicians call it "*i* " but electronic engineers call it *j* to avoid the confusion that would arise if *i* were used both for current and for the imaginary number operator.

It is tempting to ask what $\sqrt{-1}$ has to do with electronics; the answer is that it offers a way of coding the phase behaviour of a network so that it can be dealt with using the algebra of complex numbers.

The use of complex numbers is really an extension of the phasor idea. Figure 2.13 shows an Argand diagram - a two dimensional representation of a complex number. A point $(a+jb)$ can be plotted on the Argand diagram as shown. The same point can also be expressed as a complex exponential $re^{j\theta}$ where r and θ are as shown. Euler's theorem, $re^{j\theta} = r\cos\theta + jr\sin\theta$, gives a relationship between the polar (r,θ or magnitude and phase) and Cartesian ($a+ jb$) forms of the same number and thus $a = r\cos\theta$ and $b = r\sin\theta$. So as θ increases, a cosine function is projected on to the real axis and a sine function onto the imaginary axis - behaviour very similar to that associated with the phasor.

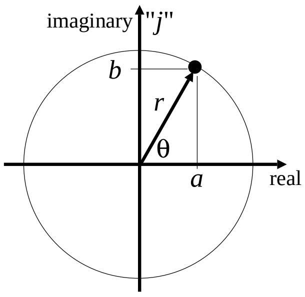
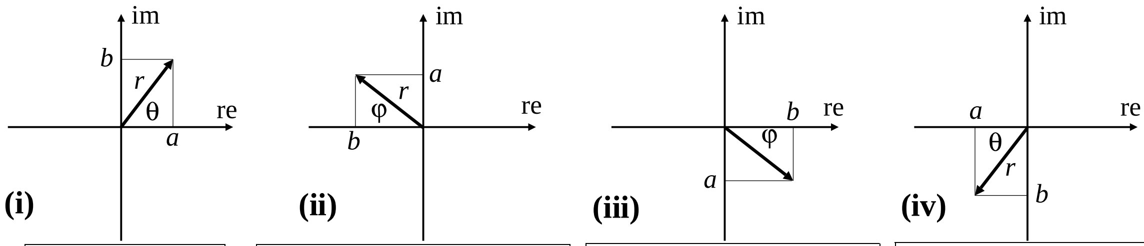


Figure 2.13

If θ increases linearly with time at a rate ω radians per second, $r\,e^{j\theta}$ becomes $r\,e^{j\omega t}$. This source can be advanced or retarded in time simply by adding a phase term $r\,e^{j(\omega t+\phi)}$ which in turn can be expressed as $r\,e^{j\omega t}e^{j\phi}$.

From a practical point of view, multiplying a complex number by j adds a phase shift of 90^o to that quantity whilst dividing by j adds a phase shift of -90^o. In other words j acts as a 90^o phase shifting operator. This behaviour is illustrated in figure 2.14.



(i)	(ii)	(iii)	(iv)
$(a+jb)$ or in polar form $r\,e^{j\theta}$ $\theta = \tan^{-1}\frac{b}{a}$	$j(a+jb) = (-b+ja)$ or in polar form $r\,e^{j(\theta+\frac{\pi}{2})}$ $\varphi = \tan^{-1}\frac{a}{-b} = -\tan^{-1}\frac{a}{b}$	$(-j)(a+jb) = (b-ja)$ or in polar form $r\,e^{j(\theta-\frac{\pi}{2})}$ $\varphi = \tan^{-1}\frac{-a}{b} = -\tan^{-1}\frac{a}{b}$	$(j^2)(a+jb) = (-a-jb)$ or in polar form $r\,e^{j(\theta\pm\pi)}$ $\theta = \tan^{-1}\frac{-b}{-a} = \tan^{-1}\frac{b}{a}$

Notes: The angle θ is positive in case **(i)** and **(iv)** because it is measured in an anticlockwise direction with respect to its reference in both cases.

The angle φ is negative in cases **(ii)** and **(iii)** because it is measured in a clockwise direction with respect to its reference in both cases.

Figure 2.14

The behaviour of resistors, capacitors and inductors in terms of the "j" notation approach is as follows.

(i) Resistance

Resistance is straightforward to deal with because there are no reactive effects. V is proportional to I whatever the form of V or I. For example, if a voltage $V_S = V_P\, e^{j(\omega t + \theta)}$ is imposed across a resistor, the resulting current will be

$$I = \frac{V_P\, e^{j(\omega t + \theta)}}{R} \quad \text{so } Z = \frac{V_S}{I} = \frac{V_P\, e^{j(\omega t + \theta)}}{\dfrac{V_P\, e^{j(\omega t + \theta)}}{R}} = R$$

(ii) Capacitance

If a voltage V_S is imposed across a capacitor as shown in figure 2.15, the capacitor current is

$$I_S = C\,\frac{dV_S}{dt} = C\, j\, \omega V_P\, e^{j(\omega t + \theta)}$$

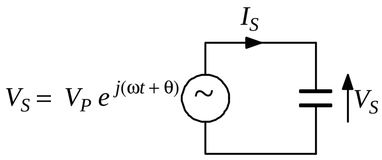


Figure 2.15

and so the impedance of the capacitor, Z_C , is given by

$$Z_C = \frac{V_S}{I_S} = \frac{V_P\, e^{j(\omega t + \theta)}}{C\, j\, \omega V_P\, e^{j(\omega t + \theta)}} = \frac{1}{j\, \omega C} = X_C \text{ , a pure capacitive reactance.} \qquad \textbf{(2.15)}$$

Notice that the driving function, $e^{j\omega t}$, and its arbitrary phase shift, θ, have cancelled out leaving a modulus of $Z_C = 1/\omega C$ and a phase relationship described by $1/j$ or $-j$. Since j is a 90^o phase shift operator this indicates a phase of V_S with respect to I_S of -90^o which is consistent with equation **(2.11).** Because Z_C is purely imaginary it is also called the capacitive reactance, X_C.

(iii) Inductance

If a current I_S is driven through an inductor, as shown in figure 2.16, the inductor voltage is

$$V_S = L\,\frac{dI_S}{dt} = L\, j\, \omega I_P\, e^{j(\omega t + \theta)}$$

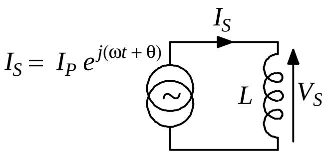


Figure 2.16

and so the impedance of the inductor, Z_L , is given by

$$Z_L = \frac{V_S}{I_S} = \frac{L\, j\, \omega I_P\, e^{j(\omega t + \theta)}}{I_P\, e^{j(\omega t + \theta)}} = j\, \omega L = X_L \text{ , a pure inductive reactance} \qquad \textbf{(2.16)}$$

As in the case of the capacitor, the driving function, $e^{j\omega t}$, and its arbitrary phase shift, θ, have cancelled out leaving a modulus of $Z_L = \omega L$ and a phase relationship described by j . Since j is a 90^o phase shift operator this indicates a phase of V_S with respect to I_S of 90^o which is consistent with equation **(2.14).**

8.4 Some a.c. circuit examples

(i) An *L-R* series circuit

Phasor approach. A phasor diagram can be used to investigate the relationship between the various quantities in figure 2.17.

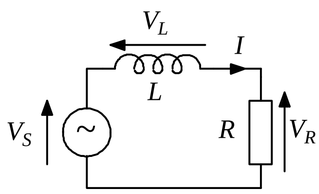


Figure 2.17

- The first thing to do is choose a reference direction. For a series circuit this would normally be the current because it is common to all the circuit elements.
- Next draw the phasors representing the circuit quantities - in this case V_R and V_L.
- Summing votages around the circuit gives $V_L + V_R = V_S$ so a construction that vectorially adds V_L and V_R will have V_S as the resultant.
- Next sketch the resultant.

The result is shown in figure 2.18. Since V_R and V_L are 90^o out of phase, with V_L leading V_R, the triangle is a right angled triangle and so

$$V_S^2 = (IR)^2 + (I\,\omega L)^2.$$

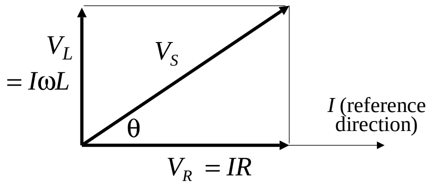


Figure 2.18

The phase of V_S with respect to I is given by the angle θ, i.e.,

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

Complex number approach. Using the "j" notation, $V_L = IX_L = I\,j\omega L$ so summing voltages shown in figure 2.17 gives $V_S = V_L + V_R$ or,

$$V_S = I\,j\,\omega L + IR = I\,(j\,\omega L + R) \text{ and thus,}$$

$$Z = \frac{V_S}{I} = j\,\omega L + R$$

the modulus or magnitude of this function is

$$|Z| = (R^2 + \omega^2 L^2)^{0.5} \text{ and the phase of } V_S \text{ with respect to } I \text{ is } \theta = \tan^{-1}\frac{\omega L}{R}$$

(ii) A *C-R* parallel circuit

Phasor approach. In the case of a parallel circuit such as that of figure 2.19, the voltage across each element is the same so it makes sense to choose the voltage direction as the reference and sum the currents.

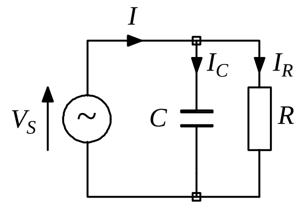


Figure 2.19

- Draw the reference phasor
- Draw I_R and I_C
- Sum these to find I as the resultant

The completed diagram is shown in figure 2.20. Again the result is a right angled triangle so

$$I^2 = \frac{V_S^2}{R^2} + V_S^2\,\omega^2 C^2$$

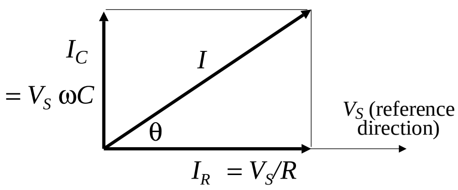


Figure 2.20

and the phase of I with respect to V_S is

$$\theta = \tan^{-1}\omega CR$$

Complex number approach. Using the "j" notation the capacitive reactance combines with the resistance in the same way as a parallel resistor pair to form an impedance given by,

$$Z = R \,//\, X_C = \frac{RX_C}{R + X_C} = \frac{\dfrac{R}{j\,\omega C}}{R + \dfrac{1}{j\,\omega C}} = \frac{R}{1 + j\,\omega CR}$$

The magnitude and phase of Z would normally be worked out directly from this expression to give

$$|Z| = \frac{R}{(1+\omega^2 C^2 R^2)^{0.5}} \text{ and phase is } \theta = -\tan^{-1}\omega CR$$

(iii) An *L-C-R* circuit with a series and parallel mix

Phasor approach. Phasor diagrams can get quite complicated. Combinations of components that are purely in series or in parallel always reduce to a relatively simple vector summation with an obvious reference direction choice. In circuits that are mixtures of series and parallel elements, such as that of figure 2.21, the choice of reference is not so obvious . . .

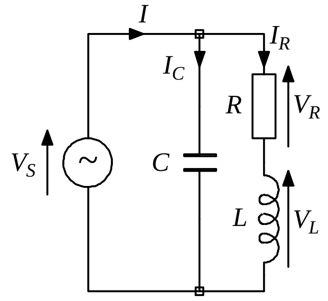


Figure 2.21

The diagram for this circuit can be constructed as follows,

- Draw a partial phasor diagram to perform the series arm summation, $V_L + V_R = V_S$ - it makes sense to use the direction of IR as the reference direction here. This produces a diagram similar to figure 2.18 for the series *L-R* case which shows the phase relationship between V_S and I_R.
- Armed with knowledge of the phase relationship between V_S and I_R, it is possible to draw a diagram to do the summation $I = I_C + I_R$. The most convenient reference direction here is that of V_S. A diagram similar to figure 2.20 is the result, the main difference being that I_R is not in the same direction as V_S, as it is in figure 2.20.

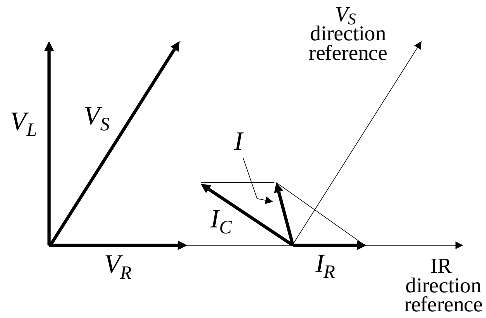


Figure 2.22

The complete phasor diagram is shown in figure 2.22. Notice how the left hand diagram defines the direction needed for the right hand one. Notice also that if C is reduced, the resultant, I, will rotate in a clockwise direction, getting smaller as it does so. At some particular value of C, I and V_S are in phase with each other - a condition known as "resonance". The phenomenon of resonance will be discussed in more detail in a later section.

The ability to get V_S and I in phase is also inportant for power factor correction, a technique for ensuring that the load presented to the power source, V_S, is purely resistive.

Complex number approach. Using the j approach to find Z,

$$Z = X_C \,//(R + X_L) = \frac{\dfrac{R + j\,\omega L}{j\,\omega C}}{R + j\,\omega L + \dfrac{1}{j\,\omega C}}$$

By multiplying top and bottom by $j\omega C$ this simplifies to

$$Z = \frac{R + j\,\omega L}{1 - \omega^2 LC + j\,\omega CR}$$

$$\text{so } |Z| = \left(\frac{R^2 + \omega^2 L^2}{(1 - \omega^2 LC\,)^2 + \omega^2 C^2 R^2}\right)^{0.5} \text{ and } \theta = \tan^{-1}\frac{\omega L}{R} - \tan^{-1}\frac{\omega CR}{1 - \omega^2 LC}$$

Notice that the phase term is the phase of the numerator minus the phase of the denominator

In the phasor diagram approach to this circuit it was demonstrated that it was possible to choose a value of C for a given frequency that would make I in phase with V_S. This is equivalent to identifying the C that makes the j terms disappear from Z for a given frequency. Alternatively one could look for the value of frequency that, with a given C, will make the j terms vanish. When the j terms vanish, the circuit is said to be "resonant". The phenomenon of resonance will be discussed in more detail later in the module.

In this example, to find the condition that will make the j terms vanish, Z must be expressed in the form $(a + jb)$ and then b must be equated to zero,

$$Z = \frac{R + j\,\omega L}{1 - \omega^2 LC + j\,\omega CR} = \frac{(R + j\,\omega L)\,((1 - \omega^2 LC) - j\,\omega CR)}{(1 - \omega^2 LC)^2 + (\omega CR)^2}$$

$$Z = \frac{R\,(1 - \omega^2 LC) + \omega^2 LCR \;+\; j\,\omega(L\,(1 - \omega^2 LC) - CR^2)}{(1 - \omega^2 LC)^2 + (\omega CR)^2}$$

equating j terms to zero

$$L\,(1 - \omega^2 LC) - CR^2 = 0$$

which for the resonant frequency at a given C gives

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)^{0.5}$$

and for the C necessary for resonance at a given ω gives

$$C = \frac{L}{\omega^2 L^2 + R^2}$$

8.5 Polar and Cartesian representations

So far the complex quantities used to describe voltage current and impedance have been in $(a + jb)$ form but in some cases - especially multiplication and division - the polar form is more useful. The relationship between $(a + jb)$ and $r\,e^{j\theta}$ is

polar to Cartesian: $a = r\cos\theta$ and $b = r\sin\theta$ **(2.17)**

Cartesian to polar: $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\dfrac{b}{a}$ **(2.18)**

Most calculators have a dedicated Cartesian to polar (and vice - versa) transformation facility - you can use this to perform conversions if you wish but practice the conversion process thoroughly so that you know exactly what keys you need to press and in what order they should be pressed. If you would rather not use your calculator to perform these transformations, you can use equations **(2.17)** and **(2.18)** to achieve the same function.

(i) Addition and subtraction

In general, the linear operations of addition and subtraction are most effectively performed using the Cartesian form because it is easier to add complex numbers in this form. For example,

$$(a + jb) + (c + jd) - (f + jg) = (a + c - f) + j\,(b + d - g) \qquad \textbf{(2.19)}$$

ie, the real parts are added together and the imaginary parts are added together.

(ii) Multiplication and division

These are both multiplicative operations which, although possible in Cartesian form, are most conveniently done in polar form. For example, multiply $r_1\angle\theta_1$ by $r_2\angle\theta_2$. Remembering that $r\angle\theta$ is shorthand for $r\,e^{j\theta}$,

$$r_1\angle\theta_1 \times r_2\angle\theta_2 = r_1\,e^{j\theta_1} \times r_2\,e^{j\theta_2} = r_1\,r_2\,e^{j(\theta_1+\theta_2)} = (r_1\,r_2)\angle(\theta_1+\theta_2) \qquad \textbf{(2.20)}$$

so to multiply in polar co-ordinates, **multiply magnitudes and add phases**.

[To do the same thing in Cartesian form,

$(a + jb)(c + jd) = (ac - bd) + j(bc + ad)$ - more complicated.]

Division is very similar to multiplication. As an example, divide $r_1\angle\theta_1$ by $r_2\angle\theta_2$.

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1\,e^{j\theta_1}}{r_2\,e^{j\theta_2}} = \frac{r_1}{r_2}\,e^{j(\theta_1-\theta_2)} = \frac{r_1}{r_2}\angle(\theta_1-\theta_2) \qquad \textbf{(2.21)}$$

ie, for division, **divide the magnitudes and subtract the phases.**

[To do the same thing in Cartesian form,

$\dfrac{a + jb}{c + jd} = \dfrac{a + jb}{c + jd} \times \dfrac{c - jd}{c - jd} = \dfrac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$ - much more complicated!]

9 Power in ac circuits

9.1 Introduction

Working out power dissipation in dc circuits is relatively straightforward because the voltage and current variables in the circuit have values that do not vary with time. When ac sources are involved there are two issues that have to be faced:

- How is the time varying nature of V (or I) accommodated?
- How are the effects of phase accommodated?

9.2 Time variation

For ac circuits the power is governed by the power integral first introduced in secion **2.3**.

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt \text{ , where } T \text{ is the periodic time of the signal,} \qquad \textbf{(2.22)}$$

If a resistor has a voltage V_P sin ωt across it, the current that flows through it is $I(t) = I_P \sin \omega t = \frac{V_P}{R} \sin \omega t$. Thus the power integral becomes,

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt = \frac{1}{R}\frac{V_P^2}{T}\int_0^T \sin^2\omega t\,dt = \frac{\text{mean squared voltage}}{\text{R}} \qquad \textbf{(2.23)}$$

[Note that one could have supposed that the resistor had a current $I(t) = I_P \sin \omega t$ flowing through it and in that case the voltage across the resistor would be $V(t) = V_P \sin \omega t = R\, I_P \sin \omega t$. Putting these expressions for $V(t)$ and $I(t)$ into equation **(2.22)** would give

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt = R\,\frac{I_P^2}{T}\int_0^T \sin^2\omega t\,dt = \text{mean squared current} \times R \qquad \textbf{(2.23a)}$$]

Equation **(2.23)** is important because it makes it clear that the power dissipated in a resistive circuit element is proportional to the mean squared voltage across the resistor (or, alternatively, is proportional to the mean squared current through the resistor). The relationship between the peak voltage across the resistor and the mean squared voltage is a function of the voltage waveshape and is found by proceeding with the integral in equation **(2.23)** . . .

$$P = \frac{1}{R}\frac{V_P^2}{T}\int_0^T \sin^2\omega t\,dt = \frac{1}{R}\frac{V_P^2}{T}\int_0^T \frac{(1-\cos 2\omega t)}{2}\,dt = \frac{1}{R}\frac{V_P^2}{T}\left[\frac{t}{2} - \frac{1}{4\omega}\sin 2\omega t\right]_0^T$$

where T is the periodic time and $T = \frac{1}{f} = \frac{2\pi}{\omega}$. Putting in the limits leads to

$$P = \frac{1}{R}\frac{V_P^2}{2} = \frac{V_{rms}^2}{R} \ . \qquad \textbf{(2.24)}$$

The mean squared value of the sinusoidal voltage used here is thus $\frac{V_P^2}{2}$. **(2.25)**

The power could also be expressed as $P = \frac{I_P^2 R}{2} = I_{rms}^2 R$. **(2.26)**

It is very common for the mean squared voltage to be expressed in terms of V rather that V^2 and when expressed in terms of volts it is referred to as the "root mean square" voltage, a name often abbreviated to "rms". It is important to remember that although rms is commonly used as a voltage measure, it means nothing except in the context of the power delivering capability of the waveshape concerned. The rms voltage associated with equation **(2.25)** is

$$V_{rms} = \frac{V_P}{\sqrt{2}} \ . \qquad \textbf{(2.27)}$$

Many text books deal with rms as a separate entity rather than as part of the power calculation. This is unfortunate because rms has no meaning other than in the context of average power. The rms value of a quantity $V(t)$ is found by evaluating

$$V_{rms} = \left[\frac{1}{T}\int_0^T [V(t)]^2\,dt\right]^{\frac{1}{2}} \qquad \textbf{(2.28)}$$

so at least the name - root-mean-square - describes what this measure is.

9.3 Phase effects

The effects of phase are also worked out by using the power integral of **(2.22)**. Suppose a voltage source V_S is driving a current I through an impedance Z such that

$V_S = V_P \sin \omega t$ and

$I = I_P \sin(\omega t + \varphi)$

Notice that there is a phase difference between V_S and I and the purpose of this section is to work out the effect of this phase difference on power dissipation.

The power dissipation in Z (which equals the power delivered by V_S) is

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt = \frac{V_P I_P}{T}\int_0^T \sin \omega t \ \sin(\omega t + \varphi)\,dt$$

$$= \frac{V_P I_P}{2T}\int_0^T (\cos(-\varphi) - \cos(2\omega t + \varphi))\,dt = \frac{V_P I_P}{2T}\left[\ (t\cos\varphi - \frac{\sin(2\omega t + \varphi)}{2\omega})\ \right]_0^T$$

where,
- $T = \dfrac{2\pi}{\omega}$,
- the first term in line 2 is the sine product in line 1 expressed as a cosine sum, and
- the second term in line 2 recognises that $\cos(-\varphi) = \cos\varphi$.

Putting in the limits gives

$$P = \frac{V_P I_P}{2}\cos\varphi = V_{rms} I_{rms}\cos\varphi \qquad \textbf{(2.29)}$$

and since $V_P = I_P|Z|$, power can also be written as

$$P = \frac{V_P^2}{2|Z|}\cos\varphi = \frac{V_{rms}^2}{|Z|}\cos\varphi = \frac{I_P^2}{2}|Z|\cos\varphi = I_{rms}^2|Z|\cos\varphi \qquad \textbf{(2.30)}$$

This means that power is the product of the **in phase** components of V and I. **cos** φ is called the "**power factor**" of the circuit - a very important parameter of power systems.

Remember that if the current through or the voltage across a purely resistive circuit element is known, the power dissipated within that element can be worked out simply from equations **(2.24)** or **(2.26)** respectively.

Note that these results are for a sinusoid. For non sinusoidal waveforms the concept of phase doesn't really mean much and an apparent phase shift would normally be expressed in terms of time delay. The power delivered by such a waveform can be calculated using equation **(2.22) .**

9.4 Power dissipation due to more than one source

The way in which multiple sources combine to give rise to power dissipation within a resistive circuit element depends upon the nature of the sources. Four example combinations of different sources are described below to illustrate the key points. Assume that in each case we are dealing with the voltage across a resistor and remember that $P \propto V^2$.

(i) $\boldsymbol{V = V_{DC1} + V_{DC2}}$

$$\begin{aligned} P &\propto \overline{V}^2 \propto \overline{(V_{DC1} + V_{DC2})^2} \\ &\propto \overline{V_{DC1}^2} + \overline{2V_{DC1}V_{DC2}} + \overline{V_{DC2}^2} \end{aligned}$$

There are three terms in this mean square expansion. The first and the last are the mean squared values of the components due to each source - these are always positive numbers for non-zero source magnitudes. The middle term is a cross product term. Cross product terms always appear when the sum of two or more terms is squared. If V_{DC1} and V_{DC2} are non zero then the mean value of their product will also be non zero and will therefore contribute to power dissipation.

For two (or more) dc sources contributing to a voltage across a resistor,

the contributions must be added (as voltages) before mean squaring.

(ii) $\boldsymbol{V = V_{DC} + V_P \sin \omega t}$

$$\begin{aligned} P &\propto \overline{V}^2 \propto \overline{(V_{DC} + V_P \sin \omega t)^2} \\ &\propto \overline{V_{DC}^2} + \overline{2V_{DC}V_P \sin \omega t} + \overline{V_P^2 \sin^2 \omega t} \\ &\propto \overline{V_{DC}^2} + \overline{V_P^2 \sin^2 \omega t} \quad \text{since} \quad \overline{2V_{DC}V_P \sin \omega t} = 0 \end{aligned}$$

In this case the first and last terms are non zero, as for case **(i)**. The cross product term multiplies V_{DC} and V_P with the mean value of $\sin \omega t$. The mean, or average, value of a sinusoid is zero so the cross product is zero and power is proportional to the sum of the first and third terms.

For a dc and an ac source contributing to a voltage across a resistor

the contributions due to each source must be mean squared before adding.

(iii) $\boldsymbol{V = V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_2 t}$

$$\begin{aligned} P &\propto \overline{V}^2 \propto \overline{(V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_2 t)^2} \\ &\propto \overline{V_{P1}^2 \sin^2 \omega_1 t} + \overline{2V_{P1}V_{P2} \sin \omega_1 t \sin \omega_2 t} + \overline{V_{P2}^2 \sin^2 \omega_2 t} \\ &\propto \overline{V_{P1}^2 \sin^2 \omega_1 t} + \overline{V_{P2}^2 \sin^2 \omega_2 t} \quad \text{since} \quad \overline{2V_{P1}V_{P2} \sin \omega_1 t \sin \omega_2 t} = 0 \end{aligned}$$

This case is very similar to case **(ii)**. The cross product here is equal to zero because the mean value of the product of two sinusoids of different frequency is zero. The easiest way to see why this is so is to use the identity, $2 \sin\omega_1 t \sin \omega_2 t = \cos(\omega_1 - \omega_2)t - \cos(\omega_1 + \omega_2)t$. The mean value of the two cosine functions is zero.

If two or more ac sources of different frequencies contribute to the voltage across a resistor

the contributions due to each source must be mean squared before adding.

(iv) $\boldsymbol{V = V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_1 t}$

$$P \propto \overline{V}^2 \propto \overline{(V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_1 t)^2}$$
$$\propto \overline{V_{P1}^2 \sin^2\omega_1 t} + \overline{2V_{P1}V_{P2} \sin \omega_1 t \sin \omega_1 t} + \overline{V_{P2}^2 \sin^2\omega_1 t}$$
$$\propto (\overline{V_{P1}^2} + \overline{2V_{P1}V_{P2}} + \overline{V_{P2}^2})\ \overline{\sin^2\omega_1 t}$$

This case is similar to case **(i)**. When two contributing ac sources are at the same frequency, a $\sin^2$ term appears in the cross product and this gives a non zero cross product.

So for two or more sources at the same frequency and with the same phase,

the magnitudes must be added before mean squaring.

If the sources are not in the same phase, the magnitudes must be added vectorially using phasor diagrams or complex numbers to find the resultant magnitude.

9.5 Maximum power transfer

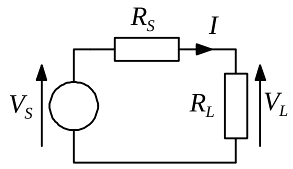


Figure 2.23

Consider the circuit of figure 2.23 where a source with a series resistance R_S supplies a load R_L.

- If $R_L \Rightarrow \infty$ and R_S is finite, $V_L \Rightarrow V_S$ and $I \Rightarrow 0$. Thus P_{RL} (which $= V_S I$) $\Rightarrow 0$ as $R_L \Rightarrow \infty$.
- If $R_L \Rightarrow 0$, $I \Rightarrow V_S/R_S$ and $V_{RL} \Rightarrow 0$. Once again, $P_{RL} \Rightarrow 0$ but this time it happens as $R_L \Rightarrow 0$.

Between these two extremes in value of R_L the product of V_L and I will not be zero - in fact there must be a particular value of R_L that maximises the power transferred to R_L.

To identify the value of R_L that will cause maximum power transfer to R_L, we need to derive an expression that describes the power dissipated in R_L in terms of the other circuit elements and then differentiate to find the maximum value. First write down the current I . . .

$I = \dfrac{V_S}{R_S + R_L}$ and then use this expression to write down the power dissipated in R_L,

$$P = I^2 R_L = \frac{V_S^2 R_L}{(R_S + R_L)^2} \qquad \textbf{(2.31)}$$

There are several ways of finding the maximum value of P. One way would be to differentiate directly using the quotient rule. Another is to recognise that a maximum in P is the same as a minimum in $1/P$ and this has the advantage that it is easier to differentiate $1/P$ than it is to differentiate P directly. There are several situations in electronics where this trick is useful so it will be pursued here. From equation **(2.31)** . . .

$$\frac{1}{P} = \frac{(R_S + R_L)^2}{V_S^2 R_L} = \frac{1}{V_S^2} \frac{R_S^2 + 2R_S R_L + R_L^2}{R_L} = \frac{1}{V_S^2}\left(\frac{R_S^2}{R_L} + 2R_S + R_L\right)$$

Differentiating, $\dfrac{d\,(1/P)}{d\,R_L} = \dfrac{1}{V_S^2}\left(-\dfrac{R_S^2}{R_L^2} + 0 + 1\right)$ and equating this to zero gives $1/P$ is a minimum

(i.e., P is a maximum) when $\boldsymbol{R_S = R_L}$.

Notes

(i) This rule is especially important at high frequencies. In general the elements R_S and R_L will be impedances Z_S and Z_L both with real and imaginary parts. In this case it can be shown that maximum power is transferred when Z_L is the complex conjugate of Z_S, i.e., if $Z_S = a + jb$, $Z_L = Z_S^* = a - jb$ for maximum power transfer. You don't need to know the $Z_S^* = Z_L$ rule for this module but it is easy to remember and you'll recognise it when you come across it in later modules.

(ii) For power supply applications the maximum power transfer theorem is irrelevant. For example, figure 2.23 could be interpreted as a battery supplying a load through its own internal resistance or a generator supplying the power grid through its own internal resistance. In these instances if $R_L = R_S$ half the available power will be lost in the internal resistance of the source - the last thing one wants to happen in a power drill, mobile phone or generator. Power sources are usually designed to approximate to ideal voltage sources - their internal resistances are designed to be very low and the current that would flow if they were loaded by their own internal resistance would be sufficient to cause severe damage to them.