EEE 345 : Summary of important equations and relationships

Maxwell equations

i) Coulomb's Law : $\text{div}\,\underline{D} = \rho_{free}$

ii) no magnetic monopoles : $\text{div}\,\underline{B} = 0$

iii) Faraday's Law : $\text{rot}\,\underline{E} = -\dot{\underline{B}}$

iv) Ampere-Maxwell Law: $\text{rot}\,\underline{H} = \underline{j} + \dot{\underline{D}}$

flux equations

i) $\underline{D} = \varepsilon_0 \varepsilon_r \underline{E}$

ii) $\underline{B} = \mu_0 \mu_r \underline{H}$

iii) $\underline{j} = \sigma \underline{E}$

mathematical theorems:

a) Gauß's theorem: $\iiint_V \text{div}\,\underline{A}\, d^3v = \oiint \underline{A}\, d\underline{s}$

b) Stoke's theorem: $\iint_S \text{rot}\,\underline{A}\, d\underline{s} = \oint \underline{A}\, d\underline{r}$

operator algebra: $\nabla^2 = \text{div grad}$, $\text{rot rot} = \text{grad div} - \text{div grad}$

⇒ **device electrostatics:**

(i) Poisson's equation: $\nabla^2 V = -\frac{\rho}{\varepsilon_0}$ (if $\rho = 0$, then Laplace: $\nabla^2 V = 0$)

(ii) $\underline{E} = -\text{grad}\, V$

⇒ **general field equations:**

a) $\underline{E} = -\text{grad}\, V - \dot{\underline{A}}$

b) $\underline{B} = \text{rot}\,\underline{A}$

wave equation: $\ddot{\underline{f}} = v^2 \nabla^2 \underline{f}$ with velocity v, $\underline{f} = \underline{A}$ or $\underline{B}$ or $\underline{E}$

propagation speed: $v = \frac{\omega}{k}$ (in vacuum: $= \frac{2\pi f}{2\pi/\lambda} = \lambda f = c$), $c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}$

⇒ **transmission lines**

(i) voltage drop along lines: $-\frac{\partial V}{\partial x} = R^* I + L^* \dot{I}$

(ii) leakage current between lines: $-\frac{\partial I}{\partial x} = G^* V + C^* \dot{V}$

lossless line: $k_0 = \omega \sqrt{L^* C^*}$

lossy line: $\tilde{k} = \sqrt{Z^* \cdot Y^*} = \sqrt{R^* + j\omega L^*}\sqrt{G^* + j\omega C^*} \approx \omega\sqrt{L^* C^*}\left[1 - \frac{j}{2\omega}\left(\frac{G^*}{C^*} + \frac{R^*}{L^*}\right)\right]$

characteristic impedance: $Z_0 = \frac{Z^*}{j\tilde{k}} = \sqrt{\frac{L^*}{C^*}}$

voltage reflection coefficient: $\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$

i) matched impedance: $Z_L = Z_0 \rightarrow \Gamma = 0$

ii) open line: $Z_L = \infty \rightarrow \Gamma = +1$

iii) short-circuited line: $Z_L = 0 \Rightarrow \Gamma = -1$

voltage standing wave ratio: $\text{VSRW} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$

where $V(x,t)$ is given by $|V(x)| = V_0^+ \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2kx + \varphi)\right]^{1/2}$ where φ = angle in $\Gamma = \gamma \exp(j\varphi)$ at position $x = d$

apparent impedance at source:

$$Z_A = Z_0 \frac{Z_L + j Z_0 \tan kd}{Z_0 + j Z_L \tan kd}$$

coax cables: $C^* = \frac{C}{\ell} = \frac{2\pi\varepsilon_0\varepsilon_r}{\ln\frac{R}{r}}$, $L^* = \frac{L}{\ell} = \frac{\mu_0\mu_r}{2\pi}\ln\frac{R}{r}$, $v = \frac{c}{\sqrt{\varepsilon_r \mu_r}}$

parallel plates: $C^* = \varepsilon_0\varepsilon_r \frac{w}{d}$, $L^* = \mu_0\mu_r \frac{d}{w}$, v as above, $Z_0 = \sqrt{\frac{L^*}{C^*}}$

⇒ **absorption in materials**

$\frac{E}{H} = \frac{E_0}{H_0} = \frac{w}{d} Z_0 = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$ ($\approx 377\,\Omega$ in vacuo) energy density of $\underline{E}$ field: $\tilde{w}_{elec} = \frac{1}{2}\varepsilon_0\varepsilon_r E^2 = \frac{1}{2} ED$

" " " $\underline{B}$ " $\tilde{w}_{mag} = \frac{1}{2}\mu_0\mu_r H^2 = \frac{1}{2} HB$

" " " electromagnetic field: $w = \frac{1}{2}(ED + HB)$

Poynting vector: $\underline{S} = \underline{E} \times \underline{H}$ where $|\underline{S}| = |\underline{E}|\cdot|\underline{H}|$ = power density

specific conductance: $G^* = \frac{G}{l} = \frac{\sigma w}{d}$

absorption length: $\beta = \frac{\sigma}{2}\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$ so that $|\underline{S}| \propto \exp(-2\beta x)$

$\rightarrow$ optics

weak absorption: $\sigma \ll \omega\varepsilon_0\varepsilon_r \Rightarrow k = k_0(1-j\beta)$

strong " : $\sigma \gg \omega\varepsilon_0\varepsilon_r \Rightarrow k = (1-j)\sqrt{\frac{\omega\sigma\mu_0\mu_r}{2}}$

skin depth : $\delta = \sqrt{\frac{2}{\omega\sigma\mu_0\mu_r}}$

complex permittivity: $\varepsilon_r = \varepsilon_r' + j\varepsilon_r''$ with $\varepsilon_r'' = -\frac{\sigma}{\omega\varepsilon_0}$

refractive index : $n = n' + j\varkappa$ with $n' = \sqrt{\varepsilon_r}$

$\Rightarrow \varepsilon_r' = n'^2 - \varkappa^2$

$\varepsilon_r'' = 2n'\varkappa$

reflection: $R = \left(\frac{|\underline{E}_r|}{|\underline{E}_i|}\right)^2$, transmission: $T = \left(\frac{|\underline{E}_t|}{|\underline{E}_i|}\right)^2$

absorption: $A = 1-R$, $\underline{E} = c\,\underline{B}\times\underline{e}_r$, $\underline{B} = \frac{1}{c}\,\underline{e}_r\times\underline{E}$ orthonormal

all depend on polarisation

$$R_\perp = \frac{(n_1\cos\theta_1 - n_2\cos\theta_2)^2 + (\varkappa_1\cos\theta_1 - \varkappa_2\cos\theta_2)^2}{(n_1\cos\theta_1 + n_2\cos\theta_2)^2 + (\varkappa_2\cos\theta_1 + \varkappa_2\cos\theta_2)^2}$$ monotonic in θ_1

$$R_\parallel = \frac{(n_2\cos\theta_1 - n_1\cos\theta_2)^2 + (\varkappa_2\cos\theta_1 - \varkappa_1\cos\theta_2)^2}{(n_2\cos\theta_1 + n_1\cos\theta_2)^2 + (\varkappa_2\cos\theta_1 + \varkappa_1\cos\theta_2)^2}$$ has minimum at

Brewster angle $\theta_1 = \arctan\frac{n_2}{n_1}$

for unpolarised light: $R = \langle R\rangle = \frac{1}{2}(R_\perp + R_\parallel)$

for vertical incidence $(\theta_1 = 0)$:

$$R = R_\perp = R_\parallel = \frac{(n_1-n_2)^2 + (\varkappa_1-\varkappa_2)^2}{(n_1+n_2)^2 + (\varkappa_1+\varkappa_2)^2}$$

$$T = T_\perp = T_\parallel = \frac{4(n_1^2+\varkappa_1^2)}{(n_1+n_2)^2 + (\varkappa_1+\varkappa_2)^2}$$

$\varkappa = \frac{1}{k\delta}$

$\rightarrow$ dielectrics:

$\underline{D} = \varepsilon_0\varepsilon_r\underline{E} = \varepsilon_0\underline{E} + \underline{P}$ where $\underline{P}$ is Polarisation

macroscopic: $\underline{P} = \varepsilon_0 \underbrace{(\varepsilon_r - 1)}_{\chi} \underline{E}$

microscopic : $\underline{P} = n\underline{p}$ where $\underline{p} = q\,d\underline{s}$ is dipole $\underline{p}\updownarrow^{+q}_{-q}$ and n density

electrostatic: $V = \frac{1}{4\pi\varepsilon_0}\frac{\underline{p}\cdot\underline{r}}{r^3} \Rightarrow \underline{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\underline{p}\cdot\underline{r})\underline{r}}{r^5} - \frac{\underline{p}}{r^3}\right)$

electrodynamic: $V = \frac{1}{4\pi\varepsilon_0}\left(\frac{\dot{\underline{p}}\cdot\underline{r}}{cr^2} + \frac{\underline{p}\,\underline{r}}{r^3}\right)$

dipole far-field: $\underline{S} = \frac{\mu_0}{16\pi^2 c}\frac{\ddot{p}^2\sin^2\theta}{r^4}\underline{e}_r$ $\propto \sin^2\theta$ is no longer radial