Standard and Canonical Forms

- Sum of Products (SOP)
- Product of Sums (POS)
- Gate-Level Minimisation

Sum of Products (SOP)

An expression such as $F = A.\overline{B}.C + B.C$ which consists of AND terms OR'd together, is known as a ***sum of products***.

A **minterm** is defined as a Boolean AND function containing exactly one instance of each input variable or its complement.

e.g. for a function of three variables A,B,C

$\overline{A}.B.C$, $\overline{A}.B.\overline{C}$, $A.B.C$ are all minterms, $A.B$ is not.

A logic function expressed by minterms is known as a fundamental sum-of-products. This is a **standard canonical form**.

Conversion to standard SOP form

Some product terms in an expression may not contain all of the variables present in the function.

A non standard SOP is converted to a standard form by using the rules

$$\boxed{X + \overline{X} = 1 \quad \text{and} \quad X.1 = X}$$

Each non standard term is multiplied by a term containing the sum of the missing term and its complement.

For example,

$$F(A,B,C) = A.B + A.\overline{B}.C$$

$$F(A,B,C) = A.B.(C + \overline{C}) + A.\overline{B}.C$$

$$F(A,B,C) = A.B.C + A.B.\overline{C} + A.\overline{B}.C$$

This is repeated until each term contains all variables in either their true or complemented forms.

As each row of a truth table corresponds to a product term, if a row evaluates to a ‘1’, then there is a minterm at that location.

A B C	F	minterm	notation
0 0 0	0		
0 0 1	1	$\overline{A}.\overline{B}.C$	m_1
0 1 0	1	$\overline{A}.B.\overline{C}$	m_2
0 1 1	0		
1 0 0	1	$A.\overline{B}.\overline{C}$	m_4
1 0 1	0		
1 1 0	0		
1 1 1	1	$A.B.C$	m_7

$$F = \overline{A}.\overline{B}.C + \overline{A}.B.\overline{C} + A.\overline{B}.\overline{C} + A.B.C$$

or we can write

$$F(A,B,C) = m_1 + m_2 + m_4 + m_7$$

$$F(A,B,C) = \Sigma\, m(1,2,4,7)$$

or simply

$$F(A,B,C) = \Sigma\, (1,2,4,7)$$

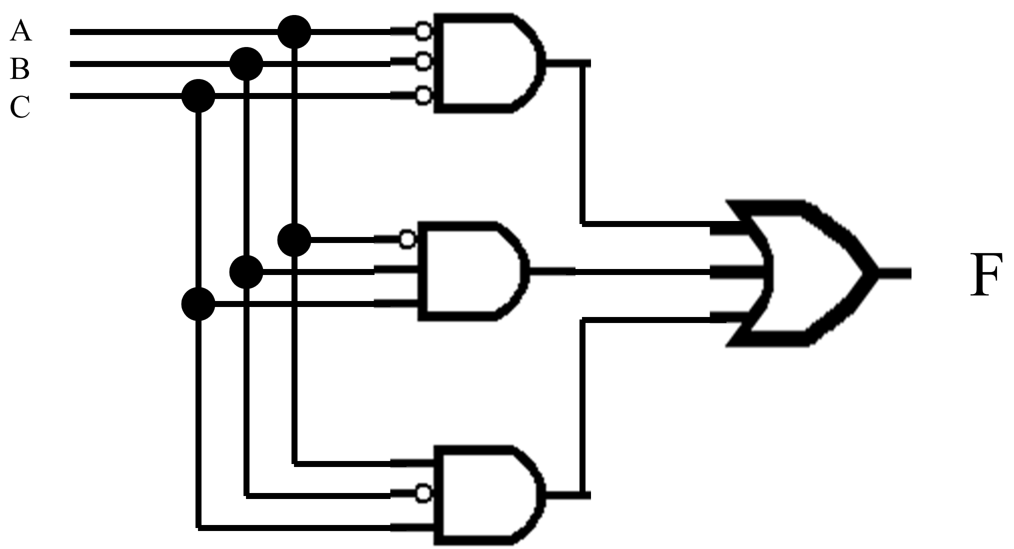
For a fundamental sum of products expression, if any minterm evaluates to ‘1’ then the expression will evaluate to a ‘1’.

Minterm examples

1. Draw the circuit for the function F(A,B,C) = Σ (0,3,5)

   This represents the expression:

$$F = \overline{A}.\overline{B}.\overline{C} + \overline{A}.B.C + A.\overline{B}.C$$



2. Express the Boolean function $Y = \overline{A}.\overline{B} + C$ as a minterm list.

The first product term is missing C so multiply it by $(C + \overline{C})$

$$Y = \overline{A}.\overline{B}.(C + \overline{C}) + C = \overline{A}.\overline{B}.C + \overline{A}.\overline{B}.\overline{C} + C$$

Similarly multiply C by $(B + \overline{B})$

$$Y = \overline{A}.\overline{B}.C + \overline{A}.\overline{B}.\overline{C} + C.(B + \overline{B})$$

$$= \overline{A}.\overline{B}.C + \overline{A}.\overline{B}.\overline{C} + B.C + \overline{B}.C$$

Then multiply the last two terms by $(A + \overline{A})$

$$Y = \overline{A}.\overline{B}.C + \overline{A}.\overline{B}.\overline{C} + B.C.(A + \overline{A}) + \overline{B}.C.(A + \overline{A})$$

$$= \overline{A}.\overline{B}.C + \overline{A}.\overline{B}.\overline{C} + A.B.C + \overline{A}.B.C + A.\overline{B}.C + \overline{A}.\overline{B}.C$$

$$= \overline{A}.\overline{B}.C + \overline{A}.\overline{B}.\overline{C} + A.B.C + \overline{A}.B.C + A.\overline{B}.C \qquad (\text{as } X + X = X)$$

$$= \Sigma\,(0,1,3,5,7)$$

Product of Sums (POS)

An expression such as $F = (A + B).(\overline{A} + B + C)$ is known as a ***product of sums*** expression.

A **maxterm** is defined as a Boolean OR function containing exactly one instance of each input variable or its complement.

e.g. for a function of three variables A,B,C

$A + \overline{B} + C$, $\overline{A} + \overline{B} + \overline{C}$, $A + B + C$ are all maxterms, $A + B$ is not.

A logic function expressed by maxterms is known as a fundamental product-of-sums.

$F = (A + \overline{B} + C).(\overline{A} + \overline{B} + \overline{C}).(A + B + C)$ is a fundamental product-of-sums. This is a **standard canonical form**.

Conversion to standard POS form

Some sum terms in an expression may not contain all of the variables present in the function.

A non standard POS is converted to a standard form by using the rules

$$\boxed{X.\overline{X} = 0 \quad \text{and} \quad X + 0 = X}$$

Each non standard term is summed with a term containing the product of the missing term and its complement.

For example,

$$F(A,B,C) = (A + B) \,.\, (A + \overline{B} + C)$$

$$F(A,B,C) = (A + B + C.\overline{C}).(A + \overline{B} + C)$$

$$F(A,B,C) = (A + B + C).(A + B + \overline{C}).(A + \overline{B} + C)$$

This is repeated until each term contains all variables in either their true or complemented forms.

Generating maxterms

A B C	F
0 0 0	0
0 0 1	1
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1

Consider the truth table shown. The expression for $\overline{F}$ is obtained by selecting the minterms for which F = 0.

$$\overline{F} = \overline{A}.\overline{B}.\overline{C} + \overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C}$$

Invert both sides and apply De Morgan’s theorems to obtain an expression for F.

$$\overline{\overline{F}} = \overline{\overline{A}.\overline{B}.\overline{C} + \overline{A}.B.C + A.\overline{B}.C + A.B.\overline{C}}$$

$$F = \overline{\overline{A}.\overline{B}.\overline{C}} \,.\, \overline{\overline{A}.B.C} \,.\, \overline{A.\overline{B}.C} \,.\, \overline{A.B.\overline{C}}$$

$$F = (A + B + C).(A + \overline{B} + \overline{C}).(\overline{A} + B + \overline{C}).\ (\overline{A} + \overline{B} + C)$$

A B C	F
0 0 0	0
0 0 1	1
0 1 0	1
0 1 1	0
1 0 0	1
1 0 1	0
1 1 0	0
1 1 1	1

$$F = (A + B + C).(A + \overline{B} + \overline{C}).\ (\overline{A} + B + \overline{C}).(\overline{A} + \overline{B} + C)$$

$$F(A,B,C) = M_0M_3M_5M_6 \quad \text{or} \quad F(A,B,C) = \Pi\ (0,3,5,6)$$

$$= \Sigma\ (1,2,4,7)$$

For a fundamental product of sums expression, if any maxterm evaluates to ‘0’ then the expression will evaluate to a ‘0’.

N.B. minterms are denoted by a lower case m and maxterms are denoted by an upper case M.

The Greek letter sigma (Σ) is used to denote a sum and the Greek letter pi (Π) is used to denote a product.

maxterm example

Write out the Boolean expression for $Y = A.(B + C)$ in fundamental product of sums form.

$$Y = A.(B + C)$$

$$= (A + B.\overline{B})(B+C)$$

$$= (A + B)(A + \overline{B})(B + C)$$

$$= (A + B + C.\overline{C})(A + \overline{B} + C.\overline{C})(A.\overline{A} + B + C)$$

$$= (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(A + \overline{B} + \overline{C})(A + B + C)(\overline{A} + B + C)$$

$$= (A + B + C)(A + B + \overline{C})(A + \overline{B} + C)(A + \overline{B} + \overline{C})(\overline{A} + B + C) \qquad (\text{as } X.X = X)$$

$$= \Pi(0,1,2,3,4)$$

You can check the answer by expressing the function as a minterm list and checking that the minterms and maxterms are mutually exclusive.

	Minterms		Maxterms	
X Y Z	Term	Designation	Term	Designation
0 0 0	$\bar{X}.\bar{Y}.\bar{Z}$	m_0	$X + Y + Z$	M_0
0 0 1	$\bar{X}.\bar{Y}.Z$	m_1	$X + Y + \bar{Z}$	M_1
0 1 0	$\bar{X}.Y.\bar{Z}$	m_2	$X + \bar{Y} + Z$	M_2
0 1 1	$\bar{X}.Y.Z$	m_3	$X + \bar{Y} + \bar{Z}$	M_3
1 0 0	$X.\bar{Y}.\bar{Z}$	m_4	$\bar{X} + Y + Z$	M_4
1 0 1	$X.\bar{Y}.Z$	m_5	$\bar{X} + Y + \bar{Z}$	M_5
1 1 0	$X.Y.\bar{Z}$	m_6	$\bar{X} + \bar{Y} + Z$	M_6
1 1 1	$X.Y.Z$	m_7	$\bar{X} + \bar{Y} + \bar{Z}$	M_7

The minterms and maxterms of a function are easily obtained from the truth table. The minterm indices and maxterm indices are mutually exclusive. A minterm is the complement of a maxterm.

Example

A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

Express the function F as a fundamental sum-of-products and as a fundamental product-of-sums.

Solving the truth table, there is a minterm at A=B=1

$F = A.B = m_3$ giving $F(A,B) = \sum(3)$

If the minterm evaluates to '1' then the function evaluates to '1'

Because we know that minterms and maxterms are mutually exclusive, we can deduce $F(A,B) = \prod(0,1,2)$

A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

Alternatively, finding the minterms that are at locations where there is a ‘0’ in the truth table will give an expression for NOT F.

$$\overline{F} = \overline{A}.\overline{B} + \overline{A}.B + A.\overline{B}$$

To obtain F invert both sides as $\overline{\overline{F}} = F$

$$F = \overline{\overline{A}.\overline{B} + \overline{A}.B + A.\overline{B}}$$ apply De Morgan’s theorem

$$F = \overline{\overline{A}.\overline{B}}\ .\ \overline{\overline{A}.B}\ .\ \overline{A.\overline{B}}$$ apply De Morgan’s theorem again

$$F = (A + B).(A + \overline{B}).(\overline{A} + B) = M_0M_1M_2 = \prod(0,1,2)$$

Or, read the maxterms directly from the truth table. Find the ‘0’s in the output column and take the variable if there is a ‘0’ for the input and the complement of the variable if there is a ‘1’ for the input.

A	B	A.B
0	0	0
0	1	0
1	0	0
1	1	1

$$F = (A + B).(A + \overline{B}).(\overline{A} + B) = M_0M_1M_2 = \prod(0,1,2)$$

Remember, if a maxterm is ‘0’ then the function is ‘0’ because

$$\mathbf{X.0 = 0}$$

To make an individual sum term ‘0’ there must not be a ‘1’ in it as

$$\mathbf{X + 1 = 1}$$

Design Problem

Consider the following function which has been derived to solve a particular problem.

$$F = A.B.\overline{C} + A.B.C + A.\overline{B}.\overline{C} + A.\overline{B}.C$$

If the cost of a 3-input gate is 9p, and a 4-input gate is 14p, how much does it cost to implement this function?

What is the annual cost if the company ships 250,000 parts per year?

Can this cost be reduced?

Design solution

Can the original design equation be simplified?

$$Y = A.B.\overline{C} + A.B.C + A.\overline{B}.\overline{C} + A.\overline{B}.C$$

apply distributive rule

$$= A.B.(\overline{C} + C) + A.\overline{B}.(\overline{C} + C)$$

$$= A.B + A.\overline{B}$$

complement property

$$= A.(B + \overline{B})$$

$$= A$$

In this case, the solution does not actually require any logic gates !

The input A is simply routed to the output Y.

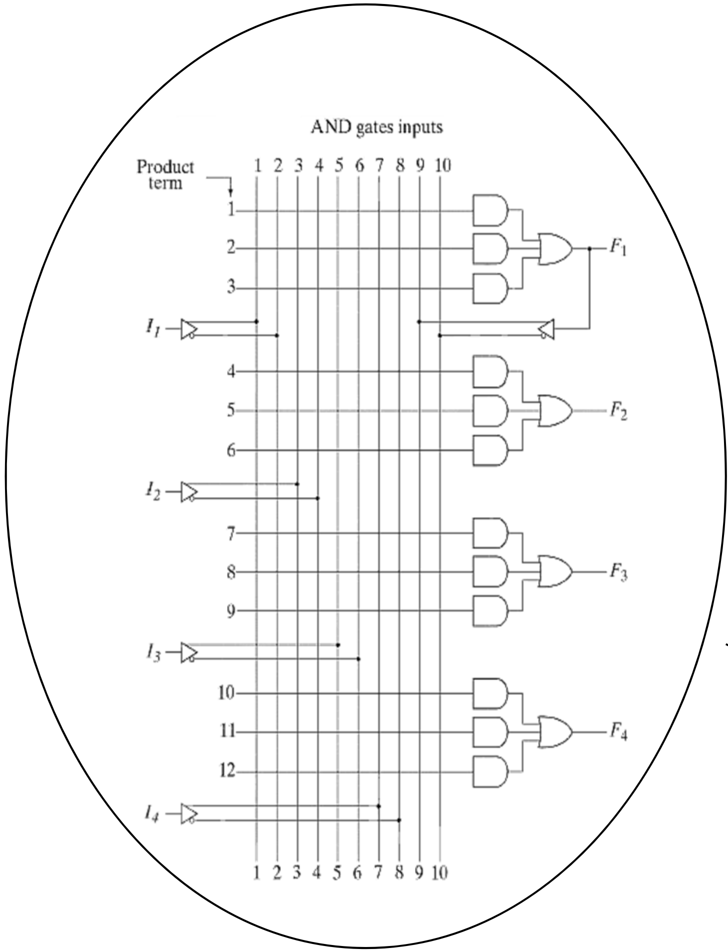
The complexity of the logic circuit required to implement a Boolean function is related to the complexity of the logic expression that describes the function. This directly affects the cost of a product.

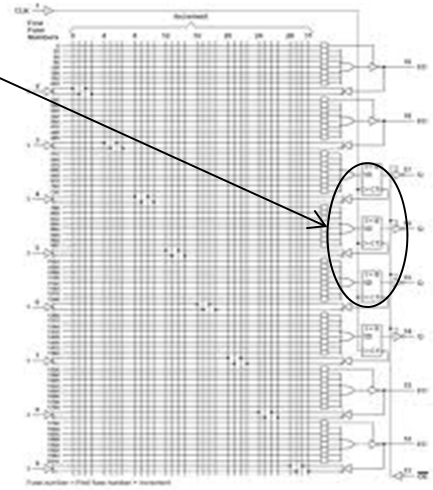
Minimisation can sometimes provide a cheaper, more reliable solution.

The number of potential faults is reduced because there are less connections in the circuit.

Minimisation can be done manually for a small number of variables by a graphical method called a Karnaugh Map. This technique becomes impractical for more than five variables. EDA tools currently available provide the best solution for circuit optimisation.

Then why expand to a SOP form?





Summary

- Logic expressions can be presented in a Standard Canonical Form as a list of minterms or maxterms.
- It may be possible to minimise logic expressions which can result in fewer gates to implement the function.