EEE345/6084 exam 2014: exam questions and model solutions

1. Maxwell's equations in general physics

10 points

a. Using both Maxwell equations for the rotation operators of the electrical and magnetic fields, the materials equations relating corresponding fluxes and fields, and the mathematical identity **rot rot** $\underline{\boldsymbol{E}}$= **grad** div $\underline{\boldsymbol{E}}-\nabla^2 \underline{\boldsymbol{E}}$ show that in vacuum the electric field vector $\underline{\boldsymbol{E}}$ obeys a wave equation.

Solution (similar to lecture where th same was shown for $\underline{\boldsymbol{B}}$):
The relevant Maxwell equations are:

$\mathbf{rot}\ \underline{\boldsymbol{E}} = -\partial \underline{\boldsymbol{B}}/\partial t$ (i) and
$\mathbf{rot}\ \underline{\boldsymbol{H}} = \underline{\boldsymbol{j}} + \partial \underline{\boldsymbol{D}}/\partial t$ (ii) where
$\underline{\boldsymbol{B}} = \mu_0 \mu_r \underline{\boldsymbol{H}}$ (iii) and
$\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\mathbf{E}}$ (iv).

Applying the rot operator to (i) inserting (iii), then (ii) and finally (iv) yields:

$\mathbf{rot\ rot}\ \underline{\boldsymbol{E}} = -\ \mathbf{rot}\ \partial \underline{\boldsymbol{B}}/\partial t = -\ \mu_0 \mu_r\ \mathbf{rot}\ \partial \underline{\boldsymbol{H}}/\partial t = -\ \mu_0 \mu_r\ \partial/\partial t\ (\mathbf{rot}\ \underline{\boldsymbol{H}})$
$= -\ \mu_0 \mu_r\ [\partial/\partial t\ \underline{\boldsymbol{j}} + \partial^2 \underline{\boldsymbol{D}}/\partial t^2]$
$= -\ \mu_0 \mu_r\ [\partial/\partial t\ \underline{\boldsymbol{j}} + \varepsilon_0 \varepsilon_r\ \partial^2 \underline{\boldsymbol{E}}/\partial t^2]$ (v)

In vacuum, $\mu_r = 1 = \varepsilon_r$ (vi)
and without any currents $\underline{\boldsymbol{j}} = 0$ (vii),
hence the right side $= -\ \mu_0 \varepsilon_0\ \partial^2 \underline{\boldsymbol{E}}/\partial t^2$.
The left side is

$\mathbf{rot\ rot}\ \underline{\boldsymbol{E}} = \mathbf{grad}\ \mathrm{div}\ \underline{\boldsymbol{E}} - \nabla^2 \underline{\boldsymbol{E}} =$ (provided above)
$= \mathbf{grad}\ \rho/\varepsilon_0\ - \nabla^2 \underline{\boldsymbol{E}}$

where div $\underline{\boldsymbol{E}} = \rho/\varepsilon_0$ (viii) has been used, with the charge density vanishing in empty space, i.e. $\rho=0$, (ix) we get for the left side:

$\mathbf{rot\ rot}\ \underline{\boldsymbol{E}} = \ -\nabla^2 \underline{\boldsymbol{E}}$

Hence, $\nabla^2 \underline{\boldsymbol{E}} = \mu_0 \varepsilon_0\ \partial^2 \underline{\boldsymbol{E}}/\partial t^2$, which is the desired wave equation (x).

6 points

b. Use Maxwell's equation for the rotation of the magnetic field, together with Ohm's Law and complex expressions for both the dielectric constant $\boldsymbol{\varepsilon}_r$ and a planar wave of form $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \exp(\mathrm{j}\omega t)$ to derive an expression for $\boldsymbol{\varepsilon}_r$. Interpret the imaginary part of $\boldsymbol{\varepsilon}_r$ physically: what does it mean?

Solution:
Starting from

$\varepsilon_0 \boldsymbol{\varepsilon}_r\ \partial \underline{\mathbf{E}}/\partial t\ = \mathrm{rot}\ \underline{\boldsymbol{H}} = \underline{\boldsymbol{j}} + \partial \underline{\boldsymbol{D}}/\partial t$ (i)
$= \sigma \underline{\boldsymbol{E}} + \varepsilon_0 \varepsilon_r\ \partial \underline{\boldsymbol{E}}/\partial t$ (ii)

For $\underline{\boldsymbol{E}} = \underline{\boldsymbol{E}}_0 \exp(\mathrm{j}\omega t)$ we get $\partial \underline{\boldsymbol{E}}/\partial t = \mathrm{j}\omega \underline{\boldsymbol{E}}$ (iii).
Inserting this yields, for complex $\boldsymbol{\varepsilon}_r = \varepsilon_r' + \mathrm{j}\ \varepsilon_r''$ (iv)
$\mathrm{j}\varepsilon_0 (\varepsilon_r' + \mathrm{j}\ \varepsilon_r'')\ \omega \underline{\boldsymbol{E}}\ = (\sigma + \mathrm{j}\omega\varepsilon_0\varepsilon_r) \underline{\boldsymbol{E}}$
Comparing coefficients yields for the real part: $\varepsilon_r' = \varepsilon_r$
and for the purely imaginary part: $\varepsilon_r'' = -\sigma/(\omega\varepsilon_0)$ (v)
The imaginary part of the dielectric constant means an exponential dampening of the planar wave if the material has finite conductivity. (vi)

4 points

c. Using Maxwell's modification of Ampere's Law calculate the divergence of the current density and interpret the result in terms of changes of the electrical charge.

Solution:
Maxwell's modification of Ampere's Law states: $\textbf{rot}\ \underline{\boldsymbol{H}} = \underline{\boldsymbol{j}} + \partial \underline{\boldsymbol{D}}/\partial t$
Applying the div operator to both sides yields
$\text{div}\ \textbf{rot}\ \underline{\boldsymbol{H}} = \text{div}\ \underline{\boldsymbol{j}} + \text{div}\ \partial \underline{\boldsymbol{D}}/\partial t$
As div **rot (any vector)**=0 and div $\boldsymbol{D} = q_{\text{free}}$ this yields $\text{div}\ \boldsymbol{j} = -\partial \underline{q}_{\text{free}}/\partial t$.
Any source of current density means a change in the free charge density, which guarantees charge conservation.

2. Transmission Lines

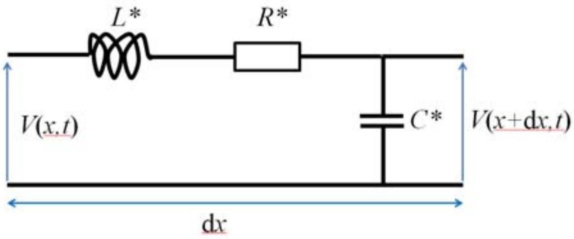
9 points

a. Sketch and annotate a short elementary length dx of a lossy transmission line where the only resistive component to be considered is the Ohmic resistance R^* per unit length along the line.
Show that the propagation constant k' for a fixed frequency source ω is approximately given by the expression

$$k' = \omega\,(L^*C^*)^{1/2}\,[1-\mathrm{j}\,R^*/(2\omega L^*)]$$

where ω is the angular frequency, L^* the inductance per unit length and C^* the capacitance per unit length.

Solution (similar to 2011 exam)



(i)

voltage drop along the line: $-\mathrm{d}V/\mathrm{d}x = L^*\,\mathrm{d}I/\mathrm{d}t + R^*\,I$ (ii)
current drop between the lines: $-\mathrm{d}I/\mathrm{d}x = C^*\,\mathrm{d}V/\mathrm{d}t$ (iii)
differentiation of (ii) w.r.t. x yields: $\mathrm{d}^2V/\mathrm{d}x^2 = -L^*\,\mathrm{d}^2I/(\mathrm{d}x\,\mathrm{d}t) - R^*\,\mathrm{d}I/\mathrm{d}x$
differentiation of (iii) w.r.t. t yields: $\mathrm{d}^2I/(\mathrm{d}t\,\mathrm{d}x) = -C^*\,\mathrm{d}^2V/\mathrm{d}t^2$ (iv)
insert (v) and (iii) into (iv):
$\mathrm{d}^2V/\mathrm{d}x^2 = L^*C^*\,\mathrm{d}^2V/\mathrm{d}t^2 + R^*C^*\,\mathrm{d}V/\mathrm{d}t$ (v)
Ansatz: $V=V_0\exp[\mathrm{j}(\omega t-k'x)]$ (vi)
double differentiation yields: $\mathrm{d}^2V/\mathrm{d}x^2 = -k'^2\,V$ and $\mathrm{d}^2V/\mathrm{d}t^2 = -\omega^2 V$ (vii)
Inserting into (vii) gives: $-k'^2 = -\omega^2L^*C^* + \mathrm{j}\omega R^*C^*$
Hence, $k'^2 = \omega^2L^*C^* - \mathrm{j}\omega R^*C^* = \omega^2L^*C^*\,[1-\mathrm{j}R^*/(\omega L^*)]$ (viii)
Taking the square root of both sides: $k' = \omega\,(L^*C^*)^{1/2}\,\sqrt{[1-\mathrm{j}R^*/(\omega L^*)]}$
Using the approximation $\sqrt{(1-x)} \approx 1-x/2$ for small x (ix)
gives the desired result: $k' \approx \omega\,(L^*C^*)^{1/2}\,[1-\mathrm{j}R^*/(2\omega L^*)]$

4 points

b. A 50Hz signal is fed into the lossy transmission line with the characteristic given in Question 2a above with C*=1nF/m, L*=1mH/m, R*=1Ω/m. Over what length can the signal be transferred so that at the end of the cable at least 95% of the voltage of the input signal arrives?

Solution:
Use $\omega=2\pi f$ where f=50Hz and insert numbers in above equation and get $k' = \omega\,(L^*C^*)^{1/2}\,[1-\mathrm{j}\,R^*/(2\omega L^*)]=3.142\times10^{-4}\,\mathrm{m}^{-1}(1-1.592\mathrm{j})$. If $k' = k_1-\mathrm{j}k_2$ with real components k_1 and $k_2=5.002\times10^{-4}\,\mathrm{m}^{-1}$, then $0.95=|V/V_0|=|\exp[\mathrm{j}(\omega t-k'x)]| = |\exp[\mathrm{j}(\omega t-k_1x)]|\,|\exp(-k_2x)|=\exp(-k_2x)$. This yields $x=-(\ln 0.95)/k_2=102.54$m

7 points

c. A 30cm short coaxial cable with inner and outer cable diameters of 0.5mm and 3mm, respectively, and a non-magnetic dielectric with a relative permittivity (dielectric constant) of ε_r=2 is to be used for high frequency measurements. Write down equations for and calculate:

i) its capacity,
ii) its inductance,
iii) its approximate real-valued impedance in the lossless case
and
iv) the voltage reflection coefficient for Ohmic loads of Z_L=50 or Z_L=75Ω. Which of the two loads would be the better termination choice and why?

Solution:
i) $C=2\pi\varepsilon_0\varepsilon_r\, l\, /\, \ln(R/r)$=18.63 pF (equation derived and discussed in lecture 3)
ii) $L=\mu_0\mu_r\, l\, \ln(R/r)\, /(2\pi)$=0.108 μH (with μ_r=1 for non-magnetic material)
iii) $Z_0 \approx (L^*/C^*)^{½} = (L/C)^{½}$ =75.96 Ω ≈ 76 Ω (cf. lecture 4)
iv) $\Gamma=(Z_L-Z_0)/(Z_L+Z_0)$
$\Gamma(Z_L=50\Omega) = -0.206$ and $\Gamma(Z_L=75\Omega) = -0.006$. The 75Ω termination will be much better, as less than 1% of the signal will be reflected, whereas for the smaller resistor about 20% of the voltage amplitude would be reflected.

3. Electric potential and electronic devices

6 points

a. The electric potential in a region of free space may be given as
$V(x,y,z) = (x^3+2y^3+2z^2) \times 100\text{V}$.

(i) determine whether it satisfies the Laplace equation.

(ii) Calculate the electric field strength E and the charge density ρ at the point $(x,y,z)=(1,2,3)$m for a permittivity of $\varepsilon_0=8.8542\times10^{-12}$ As/(Vm).

Solution (similar to 2010 exam, but with proper units):
(i) Calculate second derivates:
$\partial^2 V/\partial x^2 = 6x \times 100\text{V/m}^2$, $\partial^2 V/\partial y^2 = 12y \times 100\text{V/m}^2$, $\partial^2 V/\partial z^2 = 400\text{V/m}^2$
The Laplace equation would demand $\nabla^2 V=0$.
We get $\nabla^2 V = (\partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2) = (6x, 12y, 4)\times 100\text{V} \neq 0$, so this potential does <u>not</u> satisfy the Laplace equation $\nabla^2 V=0$.
(ii) $\underline{\boldsymbol{E}} = -\text{grad } V$
Differentiation gives for the individual components:
$E_x = -\partial V/\partial x = -3x^2 \times 100\text{V/m}$
$E_y = -\partial V/\partial y = -6y^2 \times 100\text{V/m}$
$E_z = -\partial V/\partial z = -4z \times 100\text{V/m}$
At point $(x=1, y=2, z=3)$m this yields $\underline{\boldsymbol{E}} = -(300, 2400, 1200)$ V/m
div $\underline{\boldsymbol{E}} = \rho/\varepsilon_0$ is Coloumb's Law, hence
$\rho = \varepsilon_0 (\partial E_x/\partial x + \partial E_y/\partial y + \partial E_z/\partial z) = -(6x+12y+4)\times 100\text{V/m}^2\ \varepsilon_0$
At point $(x=1, y=2, z=3)$m this yields $\rho = -3\times10^{-8}$ As/(m^3) $= -30$ nC/m^3

7 points

b. Show that the function $V(x) = (2ax - x^2)\ \rho_{\text{free}}/(2\varepsilon_0\,\varepsilon_r)$ solves the 1-dimensional Poisson equation for a semiconducting pn-junction of total depletion layer width $2a$ along the x-direction.
Calculate

(i) the voltage drop across the whole junction and

(ii) the junction capacitance

for a depletion layer width of 100nm, a free charge density of 8000C/m^3, a dielectric constant of 9 and a cross-sectional area of 10^{-8} m^2. Assume $\varepsilon_0=8.8542\times10^{-12}$ F/m.

(iii) Compare the junction capacitance quantitatively to that of a standard plate capacitor.

Solution (similar to 2012 exam):
Poisson's equation $\nabla^2 V = -\rho/\varepsilon_0$ in x-direction means $\mathrm{d}^2V/\mathrm{d}x^2 = -\rho/\varepsilon_0$. (i)
The obvious solution by double integration would be a 2nd order polynomial of form $V(x) = \mathrm{A}x^2 + \mathrm{B}x + \mathrm{C}$ where the constants are given by the boundary conditions ($V(0)=0$ and $E(\pm a) = -\mathrm{d}V/\mathrm{d}x$ *(at $\pm a$)*$=0$). Alternatively, directly differentiating the given V(x) twice solves the Poisson equation,
as $\mathrm{d}V/\mathrm{d}x = (2a - 2x)\ \rho_{\text{free}}/(2\varepsilon_0\,\varepsilon_r)$
and
$\mathrm{d}^2V/\mathrm{d}x^2 = -\rho_{\text{free}}/(\varepsilon_0\,\varepsilon_r) = -\rho/\varepsilon_0$.
Finally, use $\rho_{\text{free}} = \text{div } \boldsymbol{D} = \varepsilon_0\varepsilon_r \text{ div } \boldsymbol{E} = \varepsilon_0\varepsilon_r\,\rho/\varepsilon_0 = \varepsilon_r\,\rho$ for the free charge. (ii)

(i) the voltage drop across the whole pn-junction is
$\Delta V=V(a)-V(-a)[a^2-(-3a^2)]\,\rho_{free}/(2\varepsilon_0\,\varepsilon_r)=2a^2\,\rho_{free}/(\varepsilon_0\,\varepsilon_r)$. (iii)
Inserting numbers (a=50nm) yields 0.5V. (iv)

(ii) The charge contained at either side of the pn-junction is $Q=\rho aA=\rho_{free}\,aA/\varepsilon_r$ (v)
The capacitance is $C=Q/\Delta V=[\rho_{free}\,aA/\varepsilon_r]/[2a^2\,\rho_{free}/(\varepsilon_0\,\varepsilon_r)]=\varepsilon_0 A/(2a)$ =0.89pF. (vi)
This is the capacitance of a plate capacitor with ε_r=1 and effective (average) plate distance 2a. (vii)

7 points

c. The potential of a static electric dipole consisting of a pair of two charges $-q$ and $+q$ is given by the equation

$$V(\underline{\boldsymbol{r}})=\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}\,/(4\pi\varepsilon_0 r^3)$$

where $r=|\underline{\boldsymbol{r}}|$ is the distance from charge $+q$ and $\underline{\boldsymbol{p}}=q\underline{\boldsymbol{ds}}$ is defined as the dipole moment where the vector $\underline{\boldsymbol{ds}}$ points from $-q$ to $+q$. Provide a sketch of the dipole geometry and calculate its electric field vector, using the identity grad($\underline{\boldsymbol{r}}^n$)= $nr^{n-1}\underline{\boldsymbol{e}}_r$ where $\underline{\boldsymbol{e}}_r=\underline{\boldsymbol{r}}/r$ is the radial unity vector pointing outwards. Compare the electric field along and perpendicular to the dipole axis.

Solution:

$\underline{\boldsymbol{E}} = -\mathrm{grad}\,V(\underline{\boldsymbol{r}}) = -1/(4\pi\varepsilon_0)\ \mathrm{grad}\,(\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}\,/r^3)$

$= -1/(4\pi\varepsilon_0)\,[1/r^3\ \mathrm{grad}\,(\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}) + \underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}\ \ \mathrm{grad}\,(1/r^3)]$

$= -1/(4\pi\varepsilon_0)\ \ [1/r^3\,\underline{\boldsymbol{p}}\,\underline{\boldsymbol{e}}_r\,\mathrm{grad}\,\underline{\boldsymbol{r}} + \underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}\ \ \mathrm{grad}\,(1/r^3)]$

Now use grad $\underline{\boldsymbol{r}}=\underline{\boldsymbol{e}}_r$, $\underline{\boldsymbol{e}}_r\underline{\boldsymbol{e}}_r$=1 and grad $(1/r^3)$ = grad $r^{-3}=-3r^{-4}\,\underline{\boldsymbol{e}}_r$

$\underline{\boldsymbol{E}} = -1/(4\pi\varepsilon_0)\ \ [1/r^3\,\underline{\boldsymbol{p}}\ -3\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}\,\underline{\boldsymbol{e}}_r/r^4]$

$= \ 1/(4\pi\varepsilon_0)\,[3(\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}})\,\underline{\boldsymbol{r}}/r^5-\underline{\boldsymbol{p}}/r^3]$

The first term in the bracket points along $\underline{\boldsymbol{e}}_r$, i.e. outwards, the second along $\underline{\boldsymbol{p}}=q\underline{\boldsymbol{ds}}$, i.e. along the dipole axis.
Along the dipole axis $\boldsymbol{r}$ is parallel to $\boldsymbol{p}$, so $\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}$=p r and the bracket yields $(3p/r^3-p/r^3)=+2p/r^3$
Perpendicular to the dipole axis when $\underline{\boldsymbol{r}}\perp\underline{\boldsymbol{p}}$ we get $\underline{\boldsymbol{p}}\,\underline{\boldsymbol{r}}$=0 and the bracket is simply $-p/r^3$. This is half as small as along the dipole axis, and the direction is reversed.

4. Waves

5 points

a. Which of the following $f(x,t)$ functions (where x= spatial coordinate , t=time, a,b,c=constants, h=any function) represent travelling or standing waves? Explain your answers.

(i) $f(x,t) = \sin(4xt+a)$
(ii) $f(x,t) = b\cos(2x+t^2)$
(iii) $f(x,t) = \exp j(3at-bx)$
(iv) $f(x,t) = \sin(4x)\exp(-3x)$
(v) $f(x,t) = [g(bt-x)]^2$
(vi) $f(x,t) = g(at+x^2)$

Solution:
A wave travelling in $+x$-direction must be of form $f(x,t)=g(vt-x)$ where v is the velocity. A standing wave has no time dependence anymore and is only periodic in x. Hence, (iii) and (v) are travelling waves and (iv) is a damped standing wave.

7 points

b. Show explicitly by double differentiation that the function $f(r,t)=[\exp j(\omega t-kr)]/r$ fulfils the wave equation, using the mathematical operator identity $\nabla^2_{\mathrm{r}}= 1/r^2\,[\partial/\partial r\,(r^2\,\partial/\partial r)]$ for the radial component of the second derivative ∇^2 in spherical coordinates. What is the physical meaning of $f(r,t)$ if $\underline{\boldsymbol{r}}$ is the usual radial vector with $r=|\underline{\boldsymbol{r}}|$?

Solution:
$\partial^2 f/\partial t^2 = -\omega^2 f(r,t)$
and
$\partial f/\partial r = \exp j(\omega t-kr)\,(-1/r^2 - jk/r) = -f(r,t)\,(1/r+jk)$
Multiplication with r^2 yields
$r^2\,\partial f/\partial r = -\exp j(\omega t-kr)\,(1+jkr)$
Another differentiation gives:

$$
\begin{aligned}
\partial/\partial r\,(r^2\,\partial/\partial r)\,f &= -\exp j(\omega t-kr)\,(-jk)\,(1+jkr) - \exp j(\omega t-kr)\,jk \\
&= \exp j(\omega t-kr)\,[jk\,(1+jkr)-jk] \\
&= \exp j(\omega t-kr)\,(-k^2 r)
\end{aligned}
$$

Division by r^2 then finally yields for the radial component:

$$
\begin{aligned}
\nabla^2_{\mathrm{r}} f(r,t) &= 1/r^2\,[\partial/\partial r\,(r^2\,\partial/\partial r) f] \\
&= \exp j(\omega t-kr)\,(-k^2)/r \\
&= f(r,t)\,(-k^2)
\end{aligned}
$$

All other second derivatives are functions of angles θ and φ and therefore vanish.
Hence, $\partial^2 f/\partial t^2 = (-\omega^2)/(-k^2)\;\nabla^2 f(r,t) = (\omega/k)^2\,\nabla^2 f(r,t)$.
That's a wave equation with $\omega/k= (2\pi f)/(2\pi/\lambda)=\lambda f=v$ where v is the wave velocity.
The function f describes a spherical wave emanating from the point of origin, as for given time t the phase is constant on a spherical shell around the origin and only depends on the distance r.

8 points

c. For an oscillating electric dipole $\underline{\boldsymbol{p}}$ the magnetic flux in the far field at position $\underline{\boldsymbol{r}}$ is given by the equation

$$\underline{\boldsymbol{B}}_{\mathrm{f}} \approx \mu_0\,(\ddot{\underline{\boldsymbol{p}}} \times \underline{\boldsymbol{e}}_{\mathrm{r}})\,/\,(4\pi c r)$$

where $\ddot{\underline{\boldsymbol{p}}} = \partial^2\underline{\boldsymbol{p}}/\partial t^2$ and $\underline{\boldsymbol{e}}_r=\underline{\boldsymbol{r}}/r$ is the radial unity vector. Using the additional relationships

$$\underline{\boldsymbol{E}}_f=c\underline{\boldsymbol{B}}_f\times\underline{\boldsymbol{e}}_r \qquad \text{and} \qquad \underline{\boldsymbol{B}}_f=\underline{\boldsymbol{e}}_r\times\underline{\boldsymbol{E}}_f/c$$

between the electrical field and the magnetic flux in the far field, calculate the Poynting vector. Express the result in terms of the angle θ between $\underline{\boldsymbol{p}}$ and $\underline{\boldsymbol{r}}$ and interpret the result physically.

Solution:
the Poynting vector is

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{E}} \times \underline{\boldsymbol{H}} = 1/\mu_0\, \underline{\boldsymbol{E}} \times \underline{\boldsymbol{B}}$$

For the far-field components we get here

$$\underline{\boldsymbol{S}} = 1/\mu_0\, \underline{\boldsymbol{E}}_f \times \underline{\boldsymbol{B}}_f = c/\mu_0\, (\underline{\boldsymbol{B}}_f\times\underline{\boldsymbol{e}}_r)\times\underline{\boldsymbol{B}}_f = -c/\mu_0\, \underline{\boldsymbol{B}}_f\times(\underline{\boldsymbol{B}}_f\times\underline{\boldsymbol{e}}_r)$$

Use $\underline{\boldsymbol{B}}_f\times(\underline{\boldsymbol{B}}_f\times\underline{\boldsymbol{e}}_r)= \underline{\boldsymbol{B}}_f\,(\underline{\boldsymbol{B}}_f\underline{\boldsymbol{e}}_r) - \underline{\boldsymbol{e}}_r\,(\underline{\boldsymbol{B}}_f\,\underline{\boldsymbol{B}}_f) = -\underline{\boldsymbol{e}}_r\,\underline{\boldsymbol{B}}_f^{\,2}$ and get

$$\underline{\boldsymbol{S}} = c/\mu_0\, \underline{\boldsymbol{e}}_r\, \underline{\boldsymbol{B}}_f^{\,2}$$

Now insert $\underline{\boldsymbol{B}}_f^{\,2} \approx \mu_0^2\, (\ddot{\underline{\boldsymbol{p}}} \times \underline{\boldsymbol{e}}_r)^2 / (16\pi^2 c^2 r^2)$ to get

$$\underline{\boldsymbol{S}} = \mu_0 / (16\pi^2 c r^2)\, (\ddot{\underline{\boldsymbol{p}}} \times \underline{\boldsymbol{e}}_r)^2\, \underline{\boldsymbol{e}}_r$$

Now $(\underline{\boldsymbol{p}}\times\underline{\boldsymbol{e}}_r)^2=|\underline{\boldsymbol{p}}\times\underline{\boldsymbol{e}}_r|^2= (|\underline{\boldsymbol{p}}|\,|\underline{\boldsymbol{e}}_r|\, \sin\theta\,)^2= p^2 \sin^2\theta$, hence

$$\underline{\boldsymbol{S}} = \mu_0 / (16\pi^2 c r^2)\, \ddot{\underline{\boldsymbol{p}}}^{\,2} \sin^2\theta\, \underline{\boldsymbol{e}}_r$$

This means the energy is radiated non-isotropically: virtually none along the dipole axis (where $\sin\theta = \sin 0° = 0$) and most perpendicular to the dipole ($\sin \pm 90° = \pm 1$). The radiation reduces with distance from the source as $1/r^2$. The physical reason for the radiation is the acceleration of the charge (second time derivative!).

Alternative part solution:
Consider that $\underline{\boldsymbol{B}}_f\times\underline{\boldsymbol{e}}_r$ is perpendicular to both vectors, i.e. $\underline{\boldsymbol{S}} = -c/\mu_0\, \underline{\boldsymbol{B}}_f\times(\underline{\boldsymbol{B}}_f\times\underline{\boldsymbol{e}}_r)$.
With angle θ between them,
$S=|\underline{\boldsymbol{S}}| = c/\mu_0\, |\underline{\boldsymbol{B}}_f^2|\, \sin^2\theta = =\mu_0 / (16\pi^2 c r^2)\, (\partial p^2/\partial^2 t)\, \sin^2\theta.$