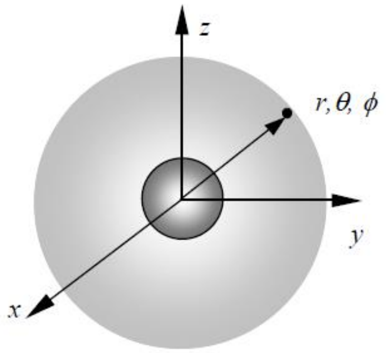
EEE349 – Power Engineering Electromagnetics

2013/14 Examination – Worked Solutions

1. The region surrounding the point charge is shown below



If we surround the charge by an imaginary sphere with radius r_o and centre at the position of the charge, the electric field at any point on its surface is:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r_o^2}\vec{r}$$

The electric flux density is given by $\vec{D} = \varepsilon_0 \vec{E}$, or

$$\vec{D} = \frac{\varepsilon_0 Q}{4\pi\varepsilon_0 r_o^2}\vec{r} = \frac{Q}{4\pi r_o^2}\vec{r}$$

Therefore, the surface integration of the normal component of $\vec{D}$ (in this case the radial component) over the surface of this sphere is:

$$\oiint_S \vec{D} \bullet d\vec{S} = \oiint_S \frac{Q}{4\pi r_o^2}(\vec{r} \bullet \vec{r})dS = \frac{Q}{4\pi r_0^2}4\pi r_0^2 = Q$$

which is Gauss's Law, i.e. the surface integral of the electric flux density over any closed surface equals the total charge enclosed.

(4 marks for full derivation starting from E field)

b) Given V = 3x2+xy3+5yz (V) then:

i) The electric field strength is given by:

$$\vec{E} = \overrightarrow{u_x}\frac{\partial V}{\partial x} + \overrightarrow{u_y}\frac{\partial V}{\partial y}\overrightarrow{u_z}\frac{\partial V}{\partial z} = \overrightarrow{u_x}(6x + y^3) + \overrightarrow{u_y}(3xy^2 + 5z) + \overrightarrow{u_z}(5y)\ V/m$$

Substituting in for the point defined gives:

$$\vec{E} = \overrightarrow{u_x}9.125\ + \overrightarrow{u_y}2.125 + \overrightarrow{u_z}2.5\ V/m$$

(3 marks – 2 for establishing correct expression and indicating that it is a vector and 1 for substitution. Half mark deduction for not including units)

ii) The charge density is given by:

$$\rho = \nabla \cdot \vec{D} = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \varepsilon_0 (6 + 6xy) \;\; C/m^3$$

Substituting in for the point defined and ε_0 gives:

$$\rho = 9.30 \times 10^{-11} \; C/m^3$$

(2 marks for correct expression and a further 1 mark for correct substitution –OK to leave in terms of ε_0-full marks awarded. ½ mark deduction for no or incorrect units)

c. Starting from Gauss's Law in cylindrical coordinates and assuming that the cable is infinitely long:

$$\vec{D} = \frac{Q}{2\pi r L_c} \vec{r}$$

[Important to start from Gauss's Law and indicate that $\vec{D}$ is a vector with direction $\vec{r}$]

i) The electric field is hence given by:

$$\vec{E} = \frac{\vec{D}}{\varepsilon_r \varepsilon_0} = \frac{Q}{2\varepsilon_r \varepsilon_0 \pi r L_c} \vec{r}$$

ii) The voltage across the insulation layer is given by:

$$V = \int_{Rc}^{Ri} \frac{Q}{2\varepsilon_r \varepsilon_0 \pi r L_c} dr = \left[\frac{Q}{2\varepsilon_r \varepsilon_0 \pi L_c} log_e(r) \right]_{R_c}^{R_i} = \frac{Q}{2\varepsilon_r \varepsilon_0 \pi L_c} log_e \left(\frac{R_i}{R_c} \right)$$

iii) The capacitance is given by:

$$C = \frac{Q}{V} = \frac{2\varepsilon_r \varepsilon_0 \pi L_c}{log_e \left(\frac{R_i}{R_c} \right)}$$

(6 marks in total - 2 marks for each section)

d. The breakdown voltage is most likely to occur on the surface of the insulation, i.e. at r = R_i. At this radius the electrical field needs to be 5 times less than that required for breakdown of the air, i.e. 600×10^3 V/m

Starting from the expression derived in part c for the electric field and taking its magnitude and rearranging leads to:

$$\varepsilon_r = \frac{Q}{2|\vec{E}| \varepsilon_0 \pi R_i L_c}$$

The total charge on the conductor Q is given by:

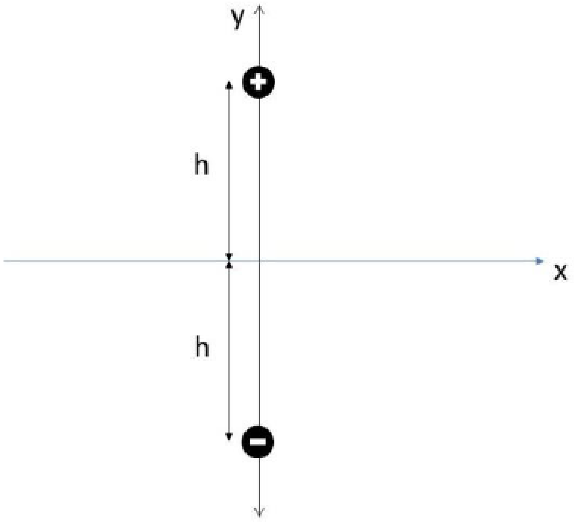
$$Q = \pi {R_c}^2 L q = \pi \times 0.01^2 \times 10 \times 0.1275 = \; 4.01 \times 10^{-4} C$$

Substituting into the expression for the relative permittivity and noting that R_i=50mm, yields:

$$\varepsilon_r = \frac{Q}{2|\vec{E}|\varepsilon_0 \pi R_i L_c} = \frac{4.01 \times 10^{-4}}{2 \times 600 \times 10^3 \times 8.85 \times 10^{-12} \times \pi \times 0.05 \times 10} = 24.0$$

(4 marks in total – 2 for recognising the need to calculate E field and remainder for substation and final calculation. ½ mark deduction for incorrect units -noting ε_r is dimensionless)

2. The method of images allows the geometry of the ground plane to be represented by an image charge as shown below.



Assume that Rc <<h.

Consider a general point outside the conductor. Applying Gauss's Law to an imaginary circular surface at a radius r yields:

$$\overrightarrow{D_r} 2\pi r L_c = \pi {R_c}^2 L_c q$$

Hence, the electric field strength is given by:

$$\overrightarrow{E_r} = \frac{\overrightarrow{D_r}}{\varepsilon_0} = \frac{{R_c}^2 q}{2\varepsilon_0 r}$$

By specifying a vertical integration path, the voltage between the conductors due to the charge on the upper transmission line (u) is given by:

$$V_{u-l,u} = \int_{R_c}^{2h} \frac{{R_c}^2 q}{2\varepsilon_0 r} dr = \left[\frac{{R_c}^2 q}{2\varepsilon_0} log_e r\right]_{R_c}^{2h} = \frac{{R_c}^2 q}{2\varepsilon_0} log_e\left(\frac{2h}{R_c}\right)$$

A similar procedure can be applied to the lower image conductor, but this will simply produce an equal and opposite voltage *[Strictly not necessary to perform the integral].*

$$V_{l-u,l} = \int_{R_c}^{2h} \frac{-{R_c}^2 q}{2\varepsilon_0 r} dr = \left[\frac{-{R_c}^2 q}{2\varepsilon_0} log_e r\right]_{R_c}^{2h} = \frac{-{R_c}^2 q}{2\varepsilon_0} log_e\left(\frac{2h}{R_c}\right)$$

Hence the voltage difference is given by:

$$V_{u-l,u} - V_{l-u,l} = \frac{{R_c}^2 q}{\varepsilon_0} log_e\left(\frac{2h}{R_c}\right)$$

However, from symmetry the potential between the upper conductor and ground is only half the voltage between the upper and lower charges.

$$V_{u-gnd} \quad = \frac{{R_c}^2 q}{2\varepsilon_0} log_e\left(\frac{2h}{R_c}\right)$$

Hence, capacitance of transmission line to ground is:

$$C_{u-gnd} = \frac{Q}{V_{u-gnd}} = \frac{\pi {R_c}^2 L_c q}{\frac{{R_c}^2 q}{2\varepsilon_0} log_e\left(\frac{2h}{R_c}\right)} = \frac{2\pi\varepsilon_0 L_c}{log_e\left(\frac{2h}{R_c}\right)}$$

(8 marks for total derivation – 2 for recognising method of images is required; 4 for formulation of voltage expressions; 2 for establishing expressions for capacitance)

b.

i) Substituting the parameters for the transmission line gives:

$$C = \frac{2\pi\varepsilon_0 L_c}{log_e\left(\frac{2h}{R_c}\right)} = \frac{2\pi \times 8.85 \times 10^{-12} \times 1000}{log_e\left(\frac{18}{25 \times 10^{-3}}\right)} = 8.45nF$$

At a voltage of 500kV, the corresponding total charge on the 1km transmission line is:

$$Q = CV = 8.45 \times 10^{-9} \times 500 \times 10^3 = 4.23mC$$

ii) In order to use the standard equations for the electric field, it is easier (though stricitly not necessary) to convert the total charge to a charge density.

$$q = \frac{Q}{\pi {R_c}^2 L_c} = \frac{4.23 \times 10^{-3}}{\pi \times 0.025^2 \times 1000} = 2.15 \times 10^{-3}\ C/m^3$$

The electric field 1m from the ground and directly below the transmission line due to the upper conductor is:

$$\overrightarrow{E_{ur}} = \frac{\overrightarrow{D_{ur}}}{\varepsilon_0} = \frac{{R_c}^2 q}{2\varepsilon_0 r} = \frac{0.025^2 \times 2.15 \times 10^{-3}}{2 \times 8.85 \times 10^{-12} \times 8} = 9500\ V/m$$

The electric field strength due to the image charge is:

$$\overrightarrow{E_{lr}} = \frac{\overrightarrow{D_{ur}}}{\varepsilon_0} = \frac{{R_c}^2 q}{2\varepsilon_0 r} = -\frac{0.025^2 \times 2.15 \times 10^{-3}}{2 \times 8.85 \times 10^{-12} \times 10} = 7600\ V/m$$

The net electric field is field is the hence 17,100 V/m

(3 marks for part (i) and 5 marks for part (ii))

c. To solve this analytically, the basic process used above based on Gauss'a Law must be repeated to calculate the capacitance between each and every pair of conductors in order. This series of interactions are used to construct a capacitance matrix that captures the interactions between the individual conductors as a basis for establishing the static charge density on each conductor. Once these charge densities are established, it become relatively straightforward to calculate the electric field by superposition. The drawback of this approach is that the capacitance matrix increases with the square of the number of conductor. Even for three conductors, it is likely to be more practical to resort to the solution of Laplace's equation in two-dimensions using finite element analysis.

(4 – 3 for describing elements on the process and 1 for identifying FEA (or indeed other numerical approaches) as an alternative although an analytical solution of Laplace's equation in two-dimensions is also possible).

3.

$$\vec{A} = \overrightarrow{u_x}3xyz + \overrightarrow{u_y}3x^2z + \overrightarrow{u_z}z^2$$

i) The magnetic flux density is given by:

$$\vec{B} = \nabla \times \vec{A} = \overrightarrow{u_x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \overrightarrow{u_y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \overrightarrow{u_z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$\vec{B} = \overrightarrow{u_x}(0 - 3x^2) + \overrightarrow{u_y}(3xy - 0) + \overrightarrow{u_z}(6xz - 3xz)$$

Substituting in for the defined point yields:

$$\vec{B} = \overrightarrow{u_x}(-0.75) + \overrightarrow{u_y}(3.0) + \overrightarrow{u_z}(4.5)\ T$$

(3)

ii) The magnetic field strength at the point is given by:

$$\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0}\left(\overrightarrow{u_x}(-0.75) + \overrightarrow{u_y}(3.0) + \overrightarrow{u_z}(4.5)\right)\ A/m$$

(1)

iii) The current density is given by:

$$\vec{J} = \nabla \times \vec{H} = \frac{1}{\mu_0}\left(\overrightarrow{u_x}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \overrightarrow{u_y}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \overrightarrow{u_z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right)$$

$$\vec{B} = \overrightarrow{u_x}(0 - 3x^2) + \overrightarrow{u_y}(3xy - 0) + \overrightarrow{u_z}(3xz)$$

$$\vec{J} = \frac{1}{\mu_0}\left(\overrightarrow{u_x}(0 - 0) + \overrightarrow{u_y}(0 - 3z) + \overrightarrow{u_z}(3y - 0)\right)$$

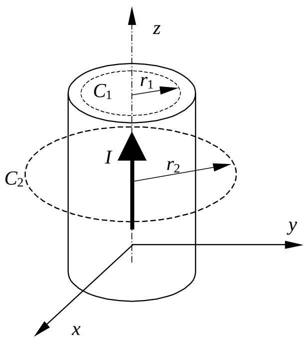
Substituting in for the coordinates of the point yields:

$$\vec{J} = \frac{1}{\mu_0}\left(\overrightarrow{u_x}(0) + \overrightarrow{u_y}(-9) + \overrightarrow{u_z}(6)\right)\ A/m^2$$

[OK if μ0 is multiplied out – full marks awarded]

(4)

b. Applying Ampere's Law to the section of conductor shown below.



For ($r > a$), we choose path C_2, which encloses all the current I. Hence,

$$\oint_{C2} \vec{H}_2 \cdot d\vec{l}_2 = \int_0^{2\pi} H_2(e_\theta \cdot e_\theta) r_2 d\theta = 2\pi r_2 H_2 = I$$

and

$$\vec{H}_2 = H_2 e_\theta = \frac{I}{2\pi r_2} e_\theta \qquad (\text{ for } r_2 > a)$$

(3)

For the operator exposure, the expression above can be modified to calculate flux density and rearranged in terms of the distance

$$r = \frac{\mu_0 I}{2\pi B_{max}} = \frac{4\pi \times 10^{-7} \times 300 \times 10^3}{2\pi \times 50 \times 10^{-3}} = 1.2m$$

(2)

If this distance cannot be accommodated then some shielding of the field could be employed, e.g. introduce a steel sheet between the operator and the cable.

(1)

c. Starting from the expression

$$A_z(x, y) = \frac{\mu_0 J_m}{30} sin(30x)e^{-30y}$$

There is only 1 component of A, and hence the expressions for curl simplify such that the the x and y components of flux density are given by:

$$B_x = \frac{\partial A_z}{\partial y} = -\mu_0 J_m sin(30x)e^{-30y}$$

$$B_y = -\frac{\partial A_z}{\partial x} = -\mu_0 J_m cos(30x)e^{-30y}$$

Hence, for the specified B_x:

$$J_m = \frac{B_x}{-\mu_0 sin(30x)e^{-30y}}$$

Substituting for B_x, x and y yields:

$$J_m = \frac{-0.5}{-4\pi \times 10^{-7} \times sin(4.5)e^{-0.3}} = 5.49 \times 10^5\ A/m$$

[Note units]

(4)

Subsituting in for J_m at the point (0.3,0.05) yields:

$$B_x = -\mu_0 \times 5.49 \times 10^5 sin(9)e^{-1.5} = -63.4 \times 10^{-3} T$$

(1)

$$B_y = -\mu_0 \times 5.49 \times 10^5 cos(9)e^{-1.5} = 0.140T$$

(1)

4.

a) The significance of the phrase 'thick' is that the dimension of plate in the negative z-direction is at least several multiples of the skin-depth at the particular frequency of excitation.

(2)

b) The classical skin-depth is given by:

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{1}{\pi f \sigma\mu}}$$

Rearranging this equation yields:

$$f = \frac{1}{\pi\delta^2\sigma\mu} = \frac{1}{\pi(2 \times 10^{-3})^2 \times 1 \times 10^7 \times 4\pi \times 10^{-7} \times 1000} = 633Hz$$

(3)

c)

i) The general expression for the curl of the magnetic field strength is:

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) e_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) e_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) e_z = \vec{J}$$

Since the only field component $\boldsymbol{H}_z$ is independent of *x* and *z*, then:

$$\therefore \nabla \times \vec{H} = a_x \frac{\partial H_z}{\partial y} = J_x$$

ie. only J_x exists

(3)

d) The diffusion equation is given by:

$$\nabla^2 H = \sigma\mu \frac{\partial H}{\partial t}$$

But since only J_z exists

$$\frac{\partial^2 H_z}{\partial y^2} = \sigma\mu \frac{\partial H_z}{\partial t}$$

But $H_z = H_z e^{j\omega t}$ (ie. applied field is sinusoidally time-varying)

$$\therefore \frac{\partial^2 H_z}{\partial y^2} = j\omega\sigma\mu H = \alpha^2 H_z$$

where $\alpha^2 = j\omega\sigma\mu$

(4)

d. The DC resistance is given by:

$$R_{dc} = \frac{\rho l}{A} = \frac{4\rho l}{\pi d^2} = 1.89 m\Omega$$

The skin depth for these conditions is given by:

$$\delta = \sqrt{\frac{2}{\omega\sigma\mu}} = \sqrt{\frac{\rho}{\pi f \mu}} = \sqrt{\frac{1.78 \times 10^{-8}}{\pi \times 15 \times 10^3 \times 4\pi \times 10^{-7}}} = 0.55mm$$

This is << conductor radius of 3mm and hence it is reasonable to apply the simplified form of the expression for AC resistance which is based on a 1D planar simplification. The ratio of the AC to DC resistance is given by:

$$\frac{R_{ac}}{R_{dc}} \approx \frac{a}{2\delta}$$

There is more exact solution for a circular conductor can be derived in terms of Bessel Functions, which in turn approximates to :

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} - \frac{1}{4} + \frac{3\delta}{32a}$$

Either of these is fine, but once it is shown that skin depth << radius, then the simplified version is adequate:

$$\frac{R_{ac}}{R_{dc}} \approx \frac{3 \times 10^{-3}}{2 \times 0.55 \times 10^{-3}} = 2.74$$

Hence, $R_{ac} = 2.74 \times R_{dc} = 5.17 m\Omega$

(5)

e. [*The key to this section is realising that the skin depth needs to be greater than, or at least equal to, the conductor radius (at which point strictly the simplified formula is in error) – hence the reference to estimate. Hence a conductor diameter of ~1mm would meet this condition. Should a candidate choose a slightly smaller diameter by assuming some fraction of the skin depth then this is also fine providing there is some reasonable rationale behind the decision]*

Taking 1mm as a conductor diameter (i.e twice the skin depth), then 36 conductors would be required to give the same overall cross-sectional area. This is arguably a minimum number of conductors of maximum diameter.

(3)