LAPLACE TRANSFORM

A generalised or extended Fourier Transform which is known as the bilateral or two-sided Laplace Transform of a signal $x(t)$ is defined as

$$X_B(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

where $s = \sigma + j\omega$. If $\sigma = 0$, $X(s)$ becomes $X(j\omega)$, sometimes written as $X(\omega)$, the Fourier Transform. In practice most systems are causal, that is $x(t) = 0$ for $t < 0$, resulting in the single-sided (uni-lateral) form of the Laplace Transform

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt\ .$$

The inverse Laplace Transform is defined as

$$x(t) = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}dt$$

where c is a constant chosen to be within the ***region of convergence*** (ROC).

Examples:

1. Consider a signal $x(t) = e^{2t}$, defined for $t \geq 0$. Its Laplace Transform is

$$X(s) = \int_{0}^{\infty} e^{2t}e^{-st}dt = \int_{0}^{\infty} e^{-(s-2)t}dt\ .$$

Substituting $s = \sigma + j\omega$ into $e^{-(s-2)t}$ we have

$$X(s) = \int_{0}^{\infty} e^{2t}e^{-st}dt = \int_{0}^{\infty} e^{2t}e^{-\sigma t}e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(\sigma-2)t}e^{-j\omega t}dt\ .$$

We see that the Laplace Transform can be interpreted as the Fourier Transform of the signal $e^{-(\sigma-2)t}$. If $\sigma < 2$, $e^{-(\sigma-2)t}$ is a growing exponential and $X(s)$ does not converge. However for $\sigma > 2$,

$$X(s) = \int_{0}^{\infty} e^{-(s-2)t}dt = \frac{-1}{s-2}e^{-(s-2)t}\Big|_{t=0}^{\infty} = \frac{1}{s-2}\left[1 - e^{-(s-2)t}\Big|_{t=\infty}\right] = \frac{1}{s-2}\ .$$

Hence, $X(s)$ is not defined if $\sigma = \mathrm{Re}\{s\} < 2$. If $\mathrm{Re}\{s\} > 2$, the Laplace Transform of $x(t)$ becomes

$$X(s) = \frac{1}{s-2},\ \mathrm{Re}\{s\} > 2.$$

The region $\mathrm{Re}\{s\} > 2$ is called ***region of convergence*** (ROC) and is displayed as

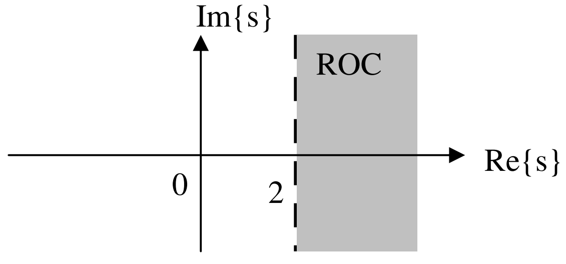


Figure 1: Region of convergence.

2. Let $x(t) = e^{-at}u(t)$. The Laplace Transform is

$$X(s) = \int_0^\infty e^{-at}e^{-st}dt = \int_0^\infty e^{-(s+a)t}dt = \frac{-1}{s+a}e^{-(s+a)t}\Big|_{t=0}^{\infty} = \frac{1}{s+a}, \ \mathrm{Re}\{s\} > -a.$$

3. Let $x(t) = -e^{-at}u(-t)$. The Laplace Transform is

$$X(s) = -\int_{-\infty}^0 e^{-at}e^{-st}dt = -\int_{-\infty}^0 e^{-(s+a)t}dt = \frac{1}{s+a}e^{-(s+a)t}\Big|_{t=-\infty}^{0} = \frac{1}{s+a}, \ \mathrm{Re}\{s\} < -a.$$

The Laplace Transforms are identical but the ROCs are different in examples 2 and 3. This demonstrates that the ROC is needed to compute the inverse Laplace Transform. Without specifying the ROC the inverse Laplace transform may not produce the original signal $x(t)$. However, we will not be computing the inverse Laplace Transform but instead we will use a lookup table containing Laplace Transform pairs.

4. Consider a signal that is the sum of two real exponentials:

$$x(t) = 2e^{-t}u(t) + 5e^{-3t}u(t).$$

The Laplace Transform is

$$X(s) = \int_0^\infty (2e^{-t} + 5e^{-3t})e^{-st}dt = 2\int_0^\infty e^{-(s+1)t}dt + 5\int_0^\infty e^{-(s+3)t}dt$$

$$= \frac{2}{s+1} + \frac{5}{s+3} = \frac{7s+11}{(s+1)(s+3)}, \ \mathrm{Re}\{s\} > -1.$$

Note that

$$2e^{-t}u(t) \leftrightarrow \frac{2}{s+1}, \ \mathrm{Re}\{s\} > -1,$$

$$5e^{-3t}u(t) \leftrightarrow \frac{5}{s+3}, \ \mathrm{Re}\{s\} > -3.$$

Therefore the ROC of $X(s)$ is defined by $\mathrm{Re}\{s\} > -1$.

The Laplace Transform in each of the examples 1 to 4 is rational, i.e it can be written as a ratio of polynomial

$$X(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are the numerator polynomial and denominator polynomial, respectively. The roots of $N(s)$ are called zeros and usually indicated with "O" while the roots of $D(s)$ are called poles and are usually indicated with "X" as illustrated in figure 2. The s-plane representation of $X(s)$ via the poles and zeros is also known as the pole-zero plot of $X(s)$.

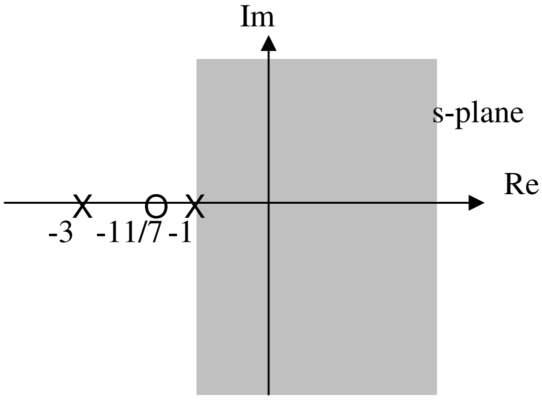


Figure 2: s-plane representation of Laplace Transform for example 4 with zero at s = -11/7 and poles at s = -1 and s =-3.

Unilateral Laplace Transform pairs

Signal	Transform
Unit step: $u(t)$	$\frac{1}{s}$
Unit impulse: $\delta(t)$	1
Unit ramp: $tu(t)$	$\frac{1}{s^2}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$t^n e^{-at}u(t)$	$\frac{n!}{(s+a)^{n+1}}$
$(\cos\omega_o t)u(t)$	$\frac{s}{(s^2+\omega_o^2)}$
$(\sin\omega_o t)u(t)$	$\frac{\omega_o}{(s^2+\omega_o^2)}$
$(e^{-at}\cos\omega_o t)u(t)$	$\frac{s+a}{((s+a)^2+\omega_o^2)}$
$(e^{-at}\sin\omega_o t)u(t)$	$\frac{\omega_o}{((s+a)^2+\omega_o^2)}$
$(t\cos\omega_o t)u(t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$
$(t\sin\omega_o t)u(t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$

Properties of Laplace Transform

Property	Transform Property
Linearity	$ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$.
Time shift	$x(t - t_o)\, u(t - t_o) \leftrightarrow X(s)e^{-st_o}$, $t_o > 0$
Multiplication by a complex exponential	$x(t)e^{s_o t} \leftrightarrow X(s - s_o)$
Time scaling	$x(at) \leftrightarrow X(s/a)/\lvert a\rvert$
Differentiation in time domain	$\dfrac{dx(t)}{dt} \leftrightarrow sX(s) - x(0)$
	$\dfrac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0) - \left.\dfrac{dx(t)}{dt}\right\rvert_{t=0}$
Differentiation in s domain	$t^n x(t) \leftrightarrow \dfrac{d^n X(s)}{ds^n}(-1)^n$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \dfrac{1}{s}X(s)$
Convolution in time domain	$x(t)*h(t) \leftrightarrow X(s).H(s)$
Initial value theorem	$x(0) = \lim_{s\to\infty} sX(s)$
Final value theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$
(if $x(t)$ has a finite value as $t \to \infty$)	

Transfer Function

Recall that the convolution of a signal $x(t)$ with another signal $h(t)$ is given by

$$y(t) = \int_0^\infty x(\tau)h(t-\tau)d\tau ,$$

where $x(t)$ and $h(t)$ are assumed to be zero for $t < 0$. Applying the Laplace Transform to $y(t)$ gives

$$\mathrm{Y(s)} = \mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}dt = \int_0^\infty \left(\int_0^\infty x(\tau)h(t-\tau)d\tau \right) e^{-st}dt$$

$$= \int_0^\infty \left(\int_0^\infty x(\tau)h(t-\tau)d\tau \right) e^{-s(t+\tau-\tau)}dt = \int_0^\infty x(\tau)e^{-s\tau}d\tau \int_0^\infty h(t-\tau)e^{-s(t-\tau)}dt .$$

Let $\lambda = t - \tau$.

$$Y(s) = \int_0^\infty x(\tau)e^{-s\tau}d\tau \int_0^\infty h(\lambda)e^{-s\lambda}d\lambda = X(s)H(s) .$$

This shows that convolution in time domain is equivalent to multiplication in s domain. The function $H(s) = Y(s)/X(s)$ is known as the transfer function

Example:

1. Consider a system with a step response given by

$$y(t) = 1- e^{-t}u(t).$$

In theory the impulse response can be obtained as $h(t) = dy(t)/dt = e^{-t}u(t)$. However the differentiation process is not desirable in practice because of high frequency noise as illustrated in figure 3.

Alternatively, we can obtain the impulse response by taking the ratio of the Laplace Transform of (output step response/input step function), i.e the transfer function.

The Laplace Transform of $y(t)$ is $Y(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{1}{s(s+1)}$ and the Laplace Transform of the step input is $\mathcal{L}[u(t)] = X(s) = \frac{1}{s}$. Hence, the transfer function is

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s(s+1)}}{\frac{1}{s}} = \frac{1}{s+1} .$$

Taking the inverse Laplace Transform gives, $h(t) = e^{-t}u(t)$, which is the impulse response obtained before.

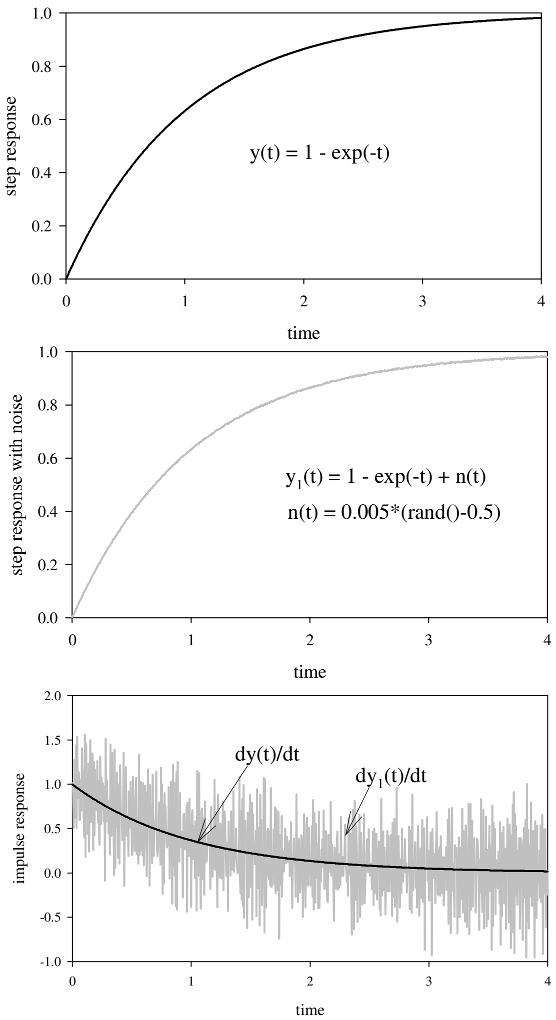


Figure 3: Impulse response obtained with ($dy_1(t)/dt$) and ($dy(t)/dt$) without the presence of high frequency noise.

2. Consider an RC circuit shown in figure 4 with zero initial condition.

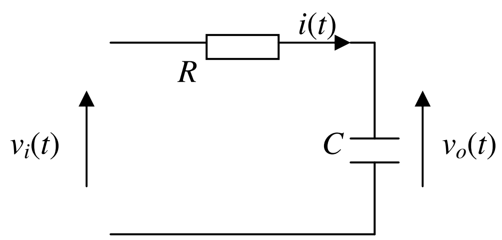


Figure 4: RC circuit.

$$v_i(t) = i(t)R + v_o(t)$$

$$v_i(t) = RC\frac{dv_o(t)}{dt} + v_o(t) = \tau\frac{dv_o(t)}{dt} + v_o(t)\text{, where } \tau = RC.$$

We can solve the differential equation to find $v_o(t)$ when $v_i(t) = u(t)$. However we will not do that. Instead we will apply the Laplace Transfrom to the differential equation.

$$V_i(s) = \tau s V_o(s) + V_o(s) = (1 + s\tau)V_o(s)$$

$$V_o(s) = \frac{1}{1 + s\tau}V_i(s) = H(s)V_i(s).$$

If $v_i(t) = u(t)$, $V_i(s) = 1/s$ and we have,

$$V_o(s) = \frac{1}{s(1 + s\tau)} = \frac{a}{s(s + a)}\text{, where } a = 1/\tau.$$

$$V_o(s) = \frac{1}{s} - \frac{1}{(s + a)}.$$

Taking the inverse Laplace Transform we have,

$$v_o(t) = 1 - e^{-at}u(t) = 1 - e^{-t/\tau}u(t).$$

forced response

natural response of the system.

We see that the first order differential equation can be solved using the Laplace Transform. The algebraic operations (addition and multiplication) used in the Laplace Transform are much simpler than the calculus operations (differentiation and integration) required in solving the differential equation.

Note that the computed step response $v_o(t)$ consists of two clearly identifiable parts, one is the forced response resulting from the input and the second is the natural output response of the system.

Transform impedance

We have shown that the transfer function of an RC circuit can be obtained by first deriving the differential equation and then take its Laplace Transform. However, it is generally simpler if we compute the transfer function using the transform impedances. Consider the following relationships

$$v(t) = L\frac{di(t)}{dt}, \qquad i(t) = C\frac{dv(t)}{dt} \quad \text{and} \quad v(t) = i(t)R.$$

Taking their Laplace Transforms and assuming zero initial conditions yield

$$V(s) = LsI(s) \qquad I(s) = CsV(s) \qquad V(s) = I(s)R$$

$$\frac{V(s)}{I(s)} = sL \quad , \quad \frac{V(s)}{I(s)} = \frac{1}{sC} \quad \text{and} \quad \frac{V(s)}{I(s)} = R \quad .$$

If we consider the current as the input and the voltage as the output the transfer functions of L, C and R are $Z(s) = sL$, $Z(s) = 1/sC$ and $Z(s) = R$, respectively. These are called transform impedances, Laplace impedances or simply impedances.

Example:
Compute the transfer function of the circuit shown in figure 5.

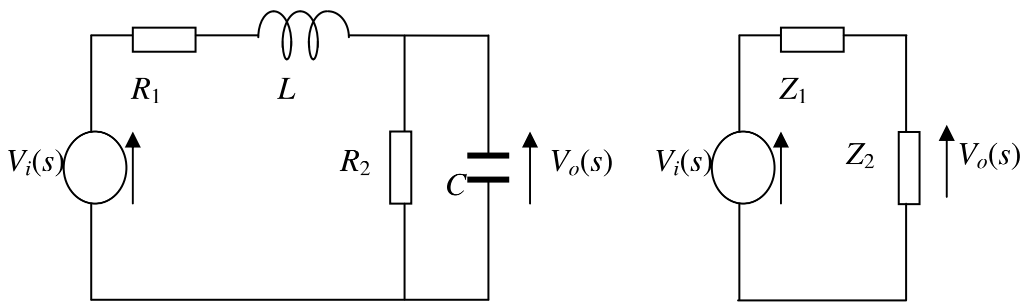


Figure 5: Circuit and its equivalent circuit on the left.

$Z_1 = R_1 + sL$ and $Z_2 = R_2 \| 1/sC = \dfrac{R_2/sC}{R_2 + 1/sC} = \dfrac{R_2}{1 + sR_2C}$.

Clearly the transfer function can be obtained as

$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{\dfrac{R_2}{1 + sR_2C}}{R_1 + sL + \dfrac{R_2}{1 + sR_2C}} = \frac{R_2}{R_2LCs^2 + (L + R_1R_2C)s + (R_1 + R_2)} .$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1/LC}{s^2 + \left((L + R_1R_2C)/R_2LC\right)s + (R_1 + R_2)/R_2LC}$$

First Order Systems

The RC circuit in figure 4 is an example of a first order system. The transfer function is

$$H(s) = \frac{1/\tau}{s + 1/\tau}, \ \mathrm{Re}\{s\} > -1/\tau$$

and the impulse response is

$$h(t) = \frac{dy(t)}{dt} = \frac{1}{\tau} e^{-t/\tau} u(t).$$

The pole-zero plot is

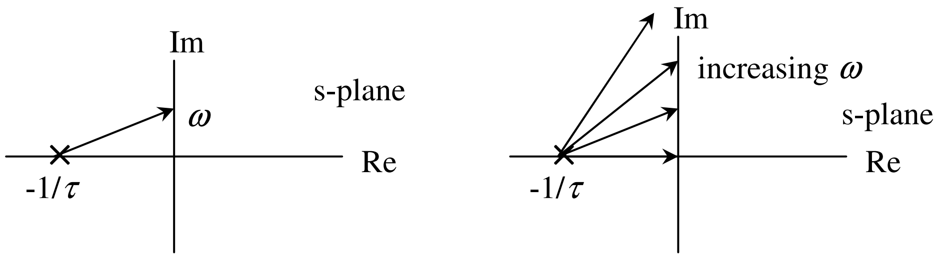


Figure 6: Pole-zero plot for a first order system.

The length of the pole vector corresponds to the magnitude of the denominator polynomial of $H(s)$ which is minimal for $\omega = 0$ and increases with ω. The angle of the pole increases from 0 to $\pi/2$ as ω increases from 0 to ∞. Assuming that $s = j\omega$ we have,

$$H(j\omega) = H(\omega) = \frac{1/\tau}{j\omega + 1/\tau} = \frac{1}{1 + j\omega/\omega_c}, \text{ where } \omega_c = 1/\tau.$$

For $\omega << \omega_c$, $|H(\omega)| \approx 1$ and $\angle H(\omega) \approx -\tan^{-1}(0) = 0$.

For $\omega = \omega_c$, $|H(\omega)| = \dfrac{1}{\sqrt{2}}$ and $\angle H(\omega) = -\tan^{-1}(1) = -\pi/4$.

For $\omega >> \omega_c$, $|H(\omega)| \approx \dfrac{1}{\omega}$ and $\angle H(\omega) \approx -\tan^{-1}(\infty) = -\pi/2$.

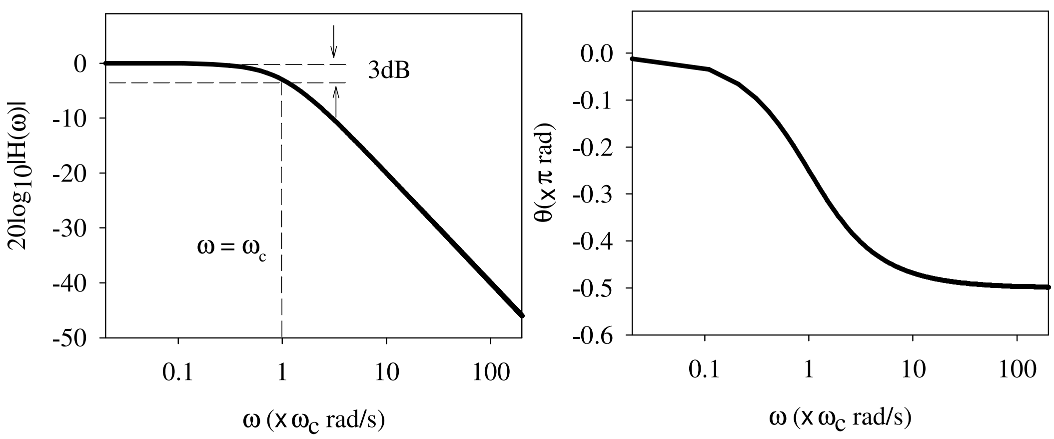


Figure 7: Frequency response for a first order system.

From figure 7 and the transfer function we see that changing the time constant τ or equivalently changing the position of the pole $s = -1/\tau$ changes the characteristics of

$H(s)$. When τ is reduced the pole moves farther to the left hand plane corresponding to a larger cut-off frequency ω_c and a faster decay in the impulse response $h(t)$. In general, if the poles are farther away from the $j\omega$-axis, the cut-off frequency is higher and the impulse response decays faster.

The numerator of $H(s)$ may also be a polynomial with degree/order of one such as in a high pass filter, $H(s) = \dfrac{s+C}{s+B}$ where B and C are constants.

Second order systems

The circuit in figure 5 is an example of a second order system. The transfer function has a general form

$$H(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where ω_n is the natural frequency of the system, ζ is the damping factor and $N(s)$ is the numerator polynomial with order less than or equal to that of the denominator polynomial.

Assuming that $N(s) = k$, $\omega_n > 0$ and $\zeta > 0$

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s-p_1)(s-p_2)},$$

where p_1 and p_2 are the poles and are given by

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}.$$

If $\zeta > 1$, the system will be non-oscillatory and is said to be overdamped. The poles are real but unequal.

If $\zeta = 0$, the system has no losses and the oscillation is undamped. The poles are imaginary but unequal and are given by $p_{1,2} = \pm j\omega_n$.

If $\zeta = 1$, the system is said to be critically damped with real and equal poles, $p_1 = p_2 = -\omega_n$.

If $0 < \zeta < 1$, the system will be oscillatory and is said to be underdamped. The poles cause $H(s) = \infty$, are complex conjugates and are given by $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$.

Step response of second order systems

Both poles are real ($\zeta > 1$)

The system is overdamped and we have,

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s-p_1)(s-p_2)}$$

and the transform of the step response is

$$Y(s) = H(s)X(s) = \frac{k}{s(s-p_1)(s-p_2)}$$

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s-p_1} + \frac{k_3}{s-p_2}.$$

Taking the inverse Laplace Transform yields,

$$y(t) = k_1 + k_2 e^{p_1 t}u(t) + k_3 e^{p_2 t}u(t)$$

.

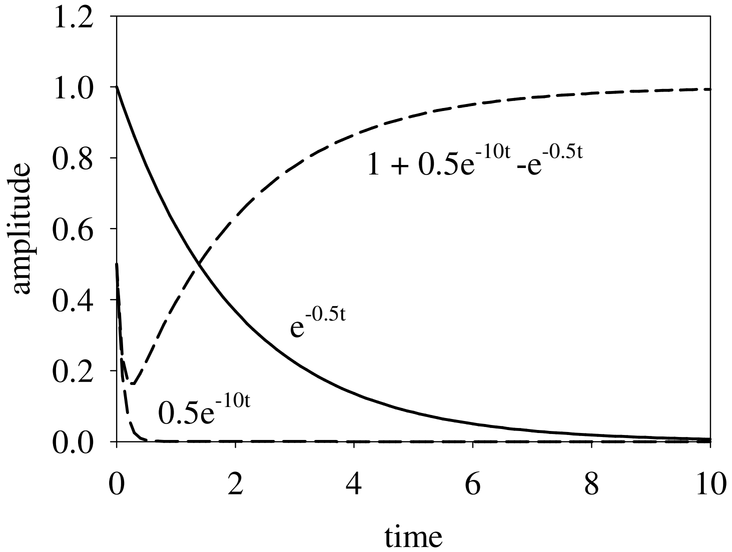
We can find the real constants k_1, k_2 and k_3 by using partial fraction expansion as follows:

$$k_1 = \left.\frac{k}{(s-p_1)(s-p_2)}\right|_{s=0} = \frac{k}{p_1 p_2},$$

$$k_2 = \left.\frac{k}{s(s-p_2)}\right|_{s=p_1} = \frac{k}{p_1(p_1-p_2)},$$

$$k_3 = \left.\frac{k}{s(s-p_1)}\right|_{s=p_2} = \frac{k}{p_2(p_2-p_1)}.$$

The forced response is $y_{fr}(t) = \dfrac{k}{p_1 p_2}$ and the transient response or the natural response is $y_{tr}(t) = k_2 e^{p_1 t} u(t) + k_3 e^{p_2 t} u(t)$, which is a sum of two exponentials. If p_2 is nearer to the $j\omega$-axis it is called the dominant pole and the transient response will be dominated by $k_3 e^{p_2 t} u(t)$.



Poles are real and equal ($\zeta = 1$)

The system is critically damped and we have,

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s+\omega_n)^2}$$, poles are $p_1 = p_2 = -\omega_n$,

and the transform of the step response is

$$Y(s) = H(s)X(s) = \frac{k}{s(s+\omega_n)^2}$$

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+\omega_n} + \frac{k_3}{(s+\omega_n)^2}.$$

Using the Laplace Transform pairs $\left\{ \frac{t^n}{n!} e^{-at} u(t) \leftrightarrow \frac{1}{(s+a)^{n+1}} \right\}$ we have,

$$k_3 t e^{-at} u(t) \leftrightarrow \frac{k_3}{(s+\omega_n)^2}.$$

Taking the inverse Laplace Transform of $Y(s)$ yields,

$$y(t) = k_1 + k_2 e^{-\omega_n t} u(t) + k_3 t e^{-\omega_n t} u(t)$$

$$y(t) = k_1 + (k_2 + k_3 t) e^{-\omega_n t} u(t).$$

We can find the real constants k_1, k_2 and k_3 by using partial fraction method. We have[1]

$$k_1 = \left. \frac{k}{(s+\omega_n)^2} \right|_{s=0} = \frac{k}{\omega_n^2},$$

$$k_2 = \frac{1}{(2-1)!} \frac{d}{ds} \left. \left((s+\omega_n)^2 \frac{k}{s(s+\omega_n)^2} \right) \right|_{s=-\omega_n} = -\frac{k}{\omega_n^2},$$

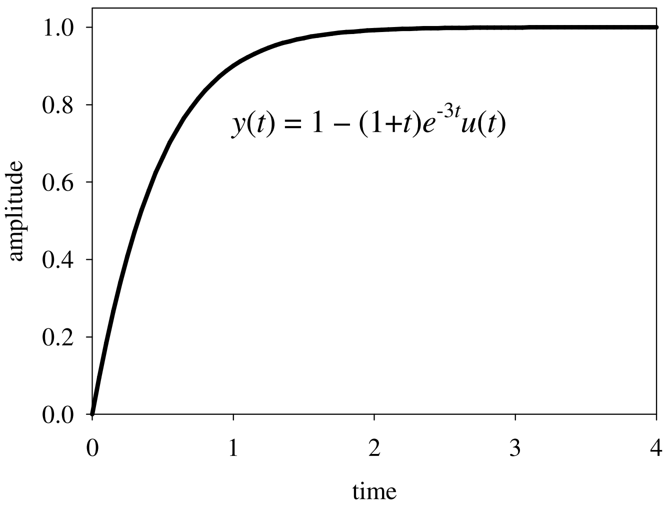
$$k_3 = (s+\omega_n)^2 \left. \frac{k}{s(s+\omega_n)^2} \right|_{s=-\omega_n} = -\frac{k}{\omega_n}.$$

Therefore the step response becomes

$$y(t) = \frac{k}{\omega_n^2} - \frac{k}{\omega_n^2} e^{-\omega_n t} u(t) [1 + \omega_n t].$$

The forced response is $y_{fr}(t) = \frac{k}{\omega_n^2}$ and the transient response or the natural response is $y_{tr}(t) = -\frac{k}{\omega_n^2} e^{-\omega_n t} u(t) [1 + \omega_n t]$.

[1] See p.384 Kamen and Heck



<u>Poles are complex ($0 < \zeta < 1$)</u>

The system is underdamped and we have,

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s+\zeta\omega_n)^2 + \omega_n^2 - (\zeta\omega_n)^2}$$

$$H(s) = \frac{k}{(s+\zeta\omega_n)^2 + \omega_d^2}\text{, where } \omega_d = \omega_n\sqrt{1-\zeta^2}\,.$$

The poles are $p_{1,2} = -\zeta\omega_n \pm j\omega_d$.

The transform of the step response is

$$Y(s) = \frac{k}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}\,.$$

Comparing the coefficients for s:

$k_1 = k/\omega_n^2$

$k_1 + k_2 = 0$, $k_2 = -\,k/\omega_n^2$

$2\zeta\omega_n k_1 + k_3 = 0$, $k_3 = -2\zeta\, k/\omega_n$

So we have,

$$Y(s) = \frac{(k/\omega_n^2)}{s} - \frac{(k/\omega_n^2)s + 2\zeta k/\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$Y(s) = \frac{(k/\omega_n^2)}{s} - \frac{(k/\omega_n^2)(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{(k\zeta/\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2}\,.$$

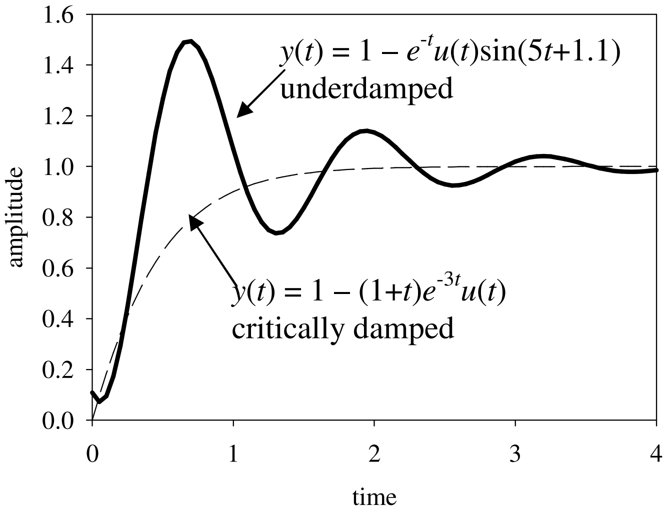
The inverse Laplace Transform is

$$y(t) = \frac{k}{\omega_n^2} - \frac{k}{\omega_n^2}e^{-\zeta\omega_n t}\cos(\omega_d t).u(t) - \frac{k\zeta}{\omega_n\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t).u(t)\,.$$

Using the trigonometric identity $\{\, C\cos\beta + D\sin\beta = \sqrt{C^2 + D^2}\,\sin(\beta + \theta)$ where $\theta = \tan^{-1}(C/D)\,\}$, the step response becomes

$$y(t) = \frac{k}{\omega_n^2}\left(1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)u(t)\right)\text{, where } \phi = \tan^{-1}(\omega_d/\zeta\omega_n)\,.$$

The forced response is $y_{fr}(t) = \dfrac{k}{\omega_n^{\,2}}$ and the transient response is an exponentially decaying sinusoid $y_{tr}(t) = -\dfrac{k}{\omega_n \omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)u(t)$.



<u>Poles are imaginary ($\zeta = 0$)</u>
The system is lossless and the transfer function is

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^{\,2}} = \frac{k}{s^2 + \omega_n^{\,2}}.$$

The poles are $p_{1,2} = \pm j\omega_n$. The transform of the step response is

$$Y(s) = \frac{k}{s(s + j\omega_n)(s - j\omega_n)} = \frac{k_1}{s} + \frac{k_2}{s + j\omega_n} + \frac{k_3}{s - j\omega_n}.$$

Taking the inverse Laplace Transform gives,

$$y(t) = k_1 + k_2 e^{-j\omega_n t}u(t) + k_3 e^{j\omega_n t}u(t).$$

Using partial fraction expansion,

$$k_1 = \left.\frac{k}{(s + j\omega_n)(s - j\omega_n)}\right|_{s=0} = \frac{k}{\omega_n^{\,2}},$$

$$k_2 = \left.\frac{k}{s(s - j\omega_n)}\right|_{s=-j\omega_n} = -\frac{k}{2\omega_n^{\,2}},$$

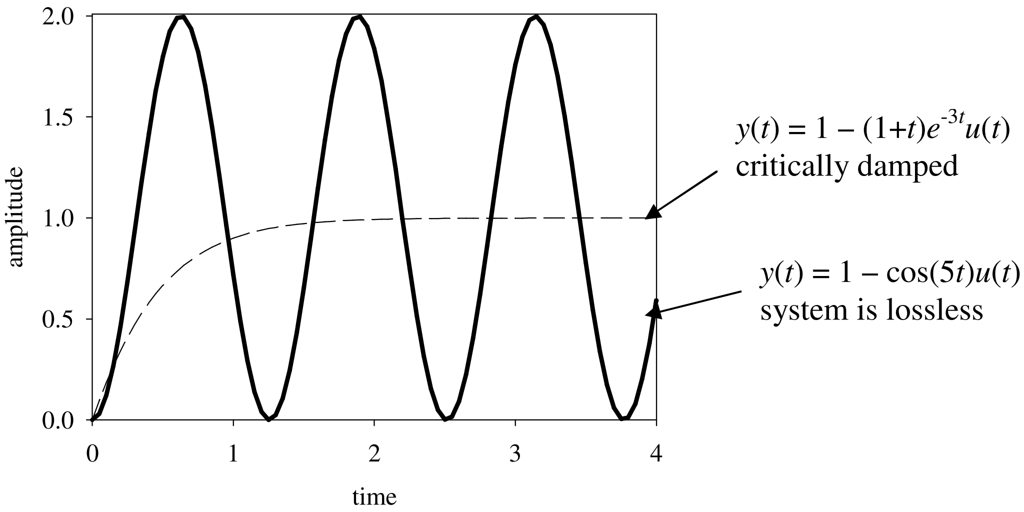
$$k_3 = \left.\frac{k}{s(s + j\omega_n)}\right|_{s=j\omega_n} = -\frac{k}{2\omega_n^{\,2}}.$$

So we have,

$$y(t) = \frac{k}{\omega_n^{\,2}} - \frac{k}{\omega_n^{\,2}}(e^{-j\omega_n t} + e^{j\omega_n t})u(t) = \frac{k}{\omega_n^{\,2}} - \frac{k}{\omega_n^{\,2}}\cos(\omega_n t)u(t).$$

The forced response is $y_{fr}(t) = \dfrac{k}{\omega_n^{\,2}}$ and the transient response is

$$y_{tr}(t) = -\frac{k}{\omega_n^{\ 2}}\cos(\omega_n t)u(t)\ .$$



Transfer function	Damping factor	Poles	Step response	Comments
$\dfrac{k}{(s-p_1)(s-p_2)}$	$\zeta > 1$	$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$	$y(t) = \dfrac{k}{p_1 p_2} + k_2 e^{p_1 t}.u(t) + k_3 e^{p_2 t}.u(t)$ k_2 and k_3 are real constants whose values depend on p_1 and p_2.	The system is overdamped. The transient response is a sum of two exponentials The pole nearer to the Im. axis is the dominant pole.
$\dfrac{k}{(s+\omega_n)^2}$	$\zeta = 1$	$p_1 = p_2 = -\omega_n$	$y(t) = \dfrac{k}{\omega_n^2}\left(1 - (1+\omega_n t)e^{-\omega_n t}.u(t)\right)$	The system is critically damped.
$\dfrac{k}{(s+\zeta\omega_n)^2 + \omega_d^2}$	$0 < \zeta < 1$	$p_{1,2} = -\zeta\omega_n \pm j\omega_d$	$y(t) = \dfrac{k}{\omega_n^2}\left(1 - \dfrac{\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin(\omega_d t + \theta).u(t)\right)$ where $\theta = \tan^{-1}(\omega_d / \zeta\omega_n)$.	The system is underdamped. The transient is a sinusoid with a frequency of $\omega_d = \omega_n\sqrt{1-\zeta^2}$ and decays exponentially with a time constant $1/\zeta\omega_n$.
$\dfrac{k}{(s^2 + \omega_n^2)}$	$\zeta = 0$	$p_{1,2} = \pm j\omega_n$	$y(t) = \dfrac{k}{\omega_n^2}\left(1 - \cos(\omega_n t).u(t)\right)$	The system is lossless and the oscillation at ω_n will be undamped.

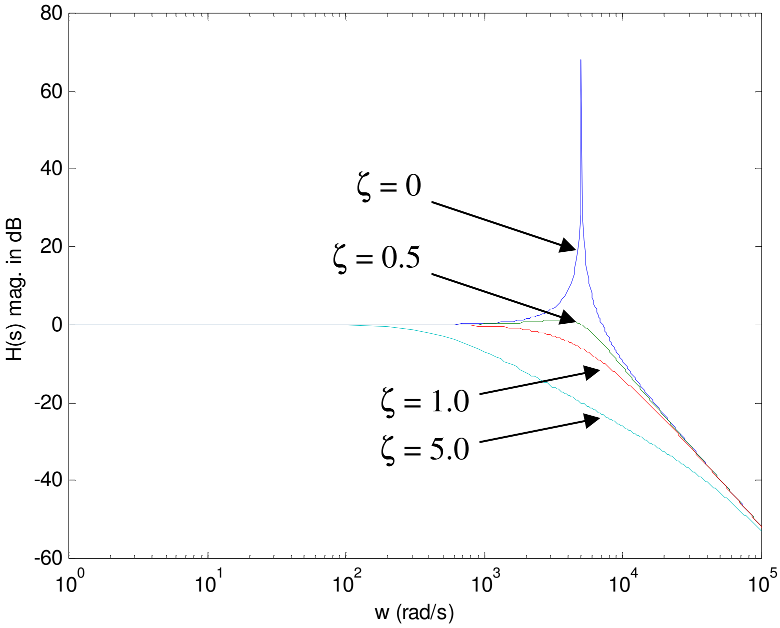
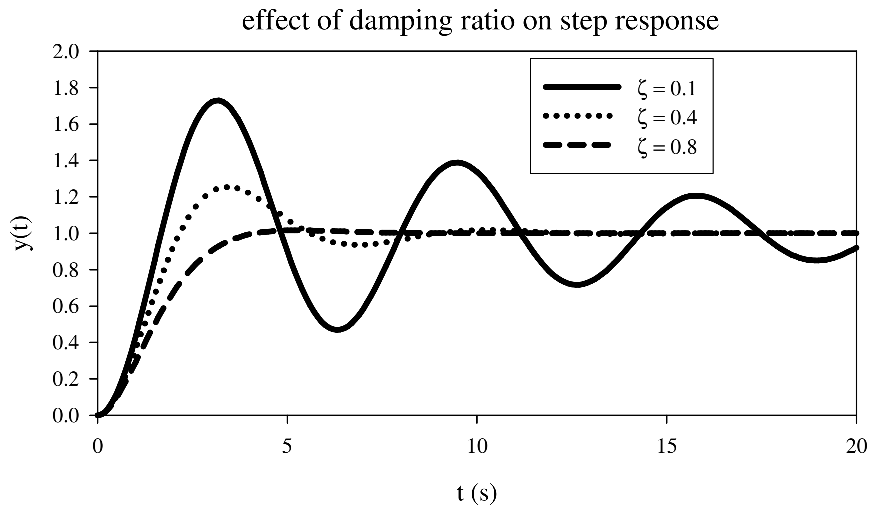
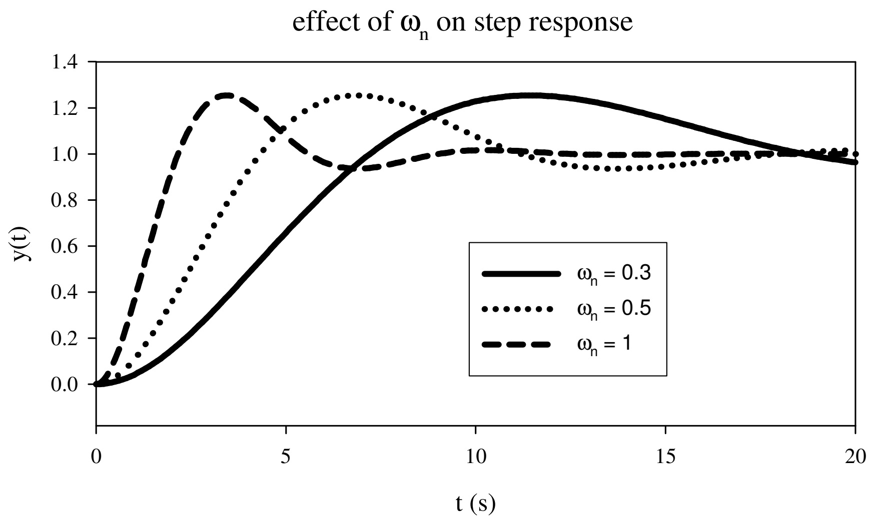
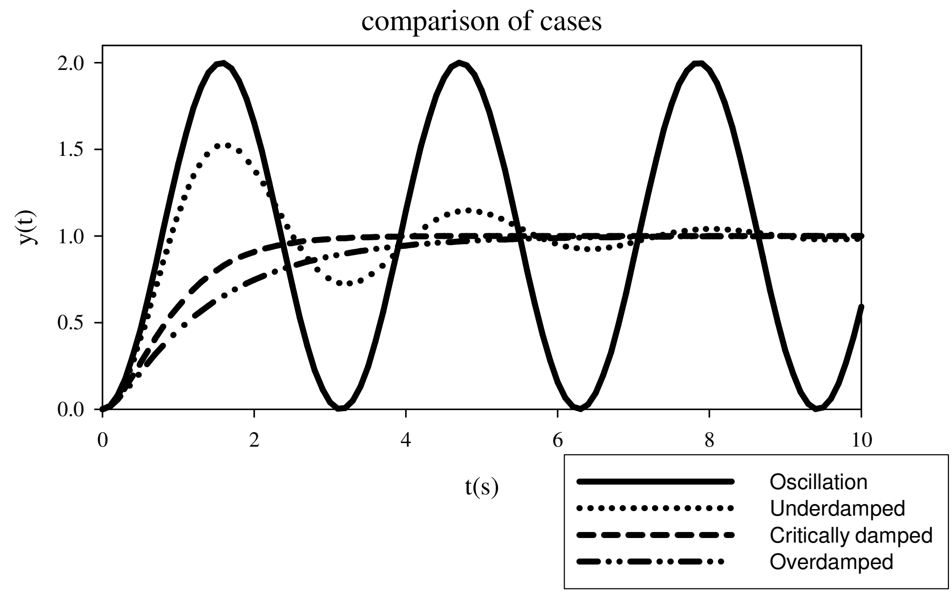


Figure 8: Transfer function with different values of ζ.


effect of damping ratio on step response
y(t)
t (s)
ζ = 0.1
ζ = 0.4
ζ = 0.8


effect of ωn on step response
y(t)
t (s)
ωn = 0.3
ωn = 0.5
ωn = 1


comparison of cases
y(t)
t(s)
Oscillation
Underdamped
Critically damped
Overdamped

Examples:

1. Suppose an LTI system is described by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t)$$

with zero initial conditions. Taking the Laplace Transform gives,

$$s^2Y(s) + 3sY(s) + 2Y(s) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s+2)(s+1)} .$$

The poles are -2 and -1. Consider an input $x(t) = Au(t)$. The response is

$$Y(s) = H(s)X(s)$$

$$Y(s) = \frac{A}{s(s+2)(s+1)} = \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+1}$$

where

$$A_1 = \left.\frac{A}{(s+2)(s+1)}\right|_{s=0} = \frac{A}{2} ,$$

$$A_2 = \left.\frac{A}{s(s+1)}\right|_{s=-2} = \frac{A}{-2(-2+1)} = \frac{A}{2} ,$$

$$A_3 = \left.\frac{A}{s(s+2)}\right|_{s=-1} = \frac{A}{-1(-1+2)} = -A .$$

Therefore we have,

$$Y(s) = \frac{A}{2s} + \frac{A}{2(s+2)} - \frac{A}{(s+1)} .$$

The response in time domain is

$$y(t) = \left(\frac{A}{2} + \frac{A}{2}e^{-2t} - Ae^{-t} \right) u(t) .$$

2. Consider the second order system described by

$$H(s) = \frac{17}{s^2 + 2s + 17}.$$

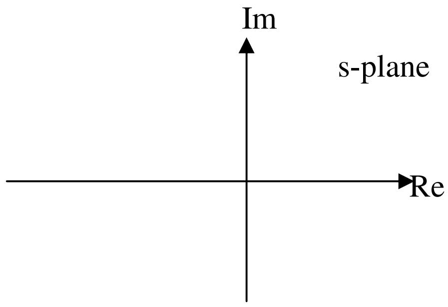
By inspection k = 17, $\omega_n = \sqrt{17}$, ζ = 1/ω_n = 0.242. This shows that the poles are complex and are given by

$$p_{1,2} = -\zeta\omega_n \pm j\omega_d$$

where $\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{17}\sqrt{1-\frac{1}{17}} = \sqrt{17}\sqrt{\frac{16}{17}} = 4$. Therefore the poles are

$$p_{1,2} = -1 \pm j4.$$

The s-plane representation is



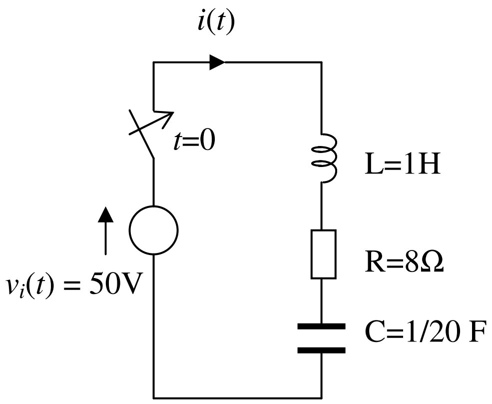
From the second order systems table,

$$y(t) = \frac{k}{\omega_n^2}\left(1 - \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta) u(t)\right)$$

where $\theta = \tan^{-1}\left(\frac{\omega_d}{\zeta\omega_n}\right) = \tan^{-1}(4) = 1.326$. Hence,

$$y(t) = \frac{17}{17}\left(1 - \frac{\sqrt{17}}{4} e^{-t} \sin(4t + 1.326) u(t)\right) = 1 - \frac{\sqrt{17}}{4} e^{-t} \sin(4t + 1.326) u(t).$$

3. Determine the current and capacitor voltage for the RLC circuit shown below.



$$V_i(s) = I(s)\left[R + sL + 1/sC\right]$$ (using the transform impedance).

$$I(s) = \frac{V_i(s)}{sL + R + 1/sC} = \frac{V_i(s)Cs}{LCs^2 + RCs + 1} = \frac{\frac{V_i(s)s}{L}}{s^2 + \frac{R}{L}s + \frac{1}{LC}}.$$

Substituting $V_i(s) = 50/s$, $R/L = 8$ and $1/LC = 20$ into $I(s)$ gives

$$I(s) = \frac{50}{s^2 + 8s + 20} = \frac{50}{(s+4)^2 + 2^2} = 25\left[\frac{2}{(s+4)^2 + 2^2}\right].$$

Using the Laplace Transform table the current is given by

$$i(t) = 25e^{-4t}\sin(2t).u(t).$$

The capacitor voltage is given by

$$V_c(s) = \frac{I(s)}{sC} = \frac{50}{(s^2 + 8s + 20)(s/20)} = \frac{20}{(s^2 + 8s + 20)}\frac{50}{s} = H(s)\frac{50}{s}.$$

$\omega_n = \sqrt{20}$, $\zeta = \dfrac{4}{\sqrt{20}}$, k =20 and $\omega_d = \sqrt{20}.\sqrt{1 - \dfrac{16}{20}} = \sqrt{4} = 2$.

The poles for $H(s)$ are $p_{1,2} = -4 \pm j2$. From the second order systems table, the step response is

$$v_{cs}(t) = \frac{k}{\omega_n^{\ 2}}\left(1 - \frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \theta)u(t)\right)$$

where $\theta = \tan^{-1}\left(\dfrac{\omega_d}{\zeta\omega_n}\right) = \tan^{-1}\left(\dfrac{2}{4}\right) = 0.464$. Hence,

$$v_{cs}(t) = \frac{20}{20}\left(1 - \frac{\sqrt{20}}{2}e^{-4t}\sin(2t + 0.464)u(t)\right) = 1 - \frac{\sqrt{20}}{2}e^{-4t}\sin(2t + 0.464)u(t),$$

and the response to a 50V input is

$$v_c(t) = 50\left[1 - \frac{\sqrt{20}}{2}e^{-4t}\sin(2t + 0.464)u(t)\right].$$

Notes: