EEE117: Electrical Circuits and Networks

10 Resonance

10.1 Introduction

Most physical systems exhibit resonance. In order to be capable of resonance, a system must be able to store energy in potential and kinetic forms and to allow transfer of energy from one to the other. Resonant systems abound in the field of music; guitars, flutes, violins, pianos (real ones!) and horns are all examples that either use the mass and elasticity of stretched string or the mass and elasticity of columns of air. Pendulums were once commonly used in clocks but are now largely confined to grandfather clocks. In electric circuits it is inductor - capacitor (*L-C*) circuits that form the basis of resonant systems. They were originally used in radio sets to selectively "magnify" one particular frequency but are now used extensively over a whole range of specialisms from communication systems at one end to power management systems at the other.

The purpose of this section is to describe electrical resonance and identify the parameters used to quantify resonant behaviour. There are two basic forms of electrical resonant behaviour, series resonance and parallel resonance.

10.2 Series resonance

The circuit of figure 3.1 is a classic series resonant circuit. The impedance of the circuit is

$$Z = \frac{V_S}{I} = j\,\omega L + \frac{1}{j\,\omega C} + R = j\left(\omega L - \frac{1}{\omega C}\right) + R \qquad \textbf{(3.1)}$$

The condition for resonance is

- **V_S and I must be in phase with each other**

or in different words

- **the collected "j" terms in Z must equal zero at the resonant frequency**

so at the resonant frequency, Z is purely real and the circuit behaves like a resistance.

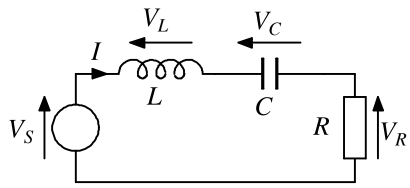


Figure 3.1

The resonant frequency, ω_r, of a circuit is found by equating the "j" term of its impedance to zero. In the case of equation **(3.1)** this means putting $\left(\omega L - \frac{1}{\omega C}\right) = 0$, to give

$$\omega_r^2 = \frac{1}{LC} \text{ or } \omega_r = \frac{1}{\sqrt{LC}} \text{ or } f_r = \frac{1}{2\pi\sqrt{LC}} \qquad \textbf{(3.2)}$$

Phasor diagram view of series resonance

It is sometimes conceptually useful to consider resonance from a phasor diagram point of view. For the circuit of figure 3.1 the current is common to all circuit elements so it makes sense to choose current as the reference direction as shown in figure 3.2. In figure 3.2, the voltage across the inductance, which leads the current by 90°, is larger than the voltage across the capacitor, which lags the current by 90°. This leads to a resultant V_S as shown. As frequency reduces, the voltage across the inductance will decrease and that across the capacitor will increase. Resonance occurs when $|I\,\omega L| = |I/(\omega C)|$ - i.e., the 90° terms sum to zero and V_S lies in the same direction as I.

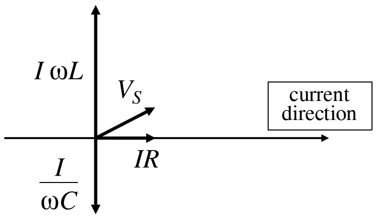


Figure 3.2

Frequency dependence of impedance and current

The way in which the circuit impedance changes with frequency is shown in figure 3.3. Also shown on figure 3.3 is the magnitude and phase of the voltage across the resistor which is proportional to the circuit current.

For frequencies below f_r the circuit behaves capacitively because $|V_C| > |V_L|$ so the sum $V_C + V_L$ is in a V_C direction. For frequencies above f_r the circuit behaves inductively - this is the situation in figure 3.2. On a logarithmic frequency scale these graphs are symmetrical about f_r.

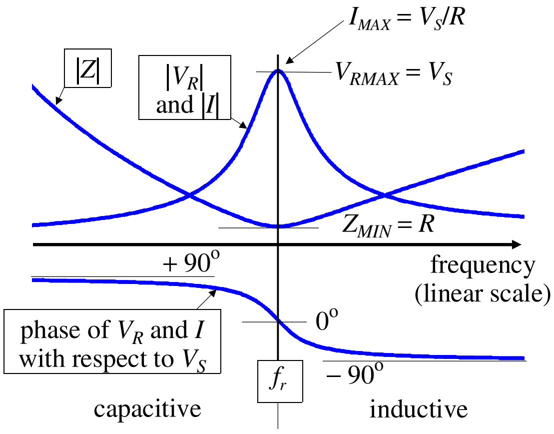


Figure 3.3

"Q" or "magnification" factor

The sharpness of the resonant peak in V_R or I is dependent upon the particular values of L, C and R used. In the early radio applications of resonant circuits, circuits that produced a very narrow peak, and as a consequence a large magnification, were regarded as high quality resonators. It wasn't long before someone defined a technically sensible figure of merit for resonant circuits . . . and called it Q factor. Q is an abbreviation for quality; high Q circuits had narrow peaks - a sign of quality in early radio applications.

There are several ways of defining Q factor; they all measure the same thing. The simplest of these is often used experimentally and relates to the shape of the resonant peak as shown in figure 3.4. Firstly the resonant frequency is identified by finding the frequency at which the maximum value of V_R occurs. Secondly the frequencies f_1 and f_2 at which the value of V_R has fallen by a factor of 0.707 (or -3dB) from its maximum value are determined. The Q factor is then given by:

$$Q = \frac{f_r}{f_2 - f_1} = \frac{f_r}{\Delta f} \qquad \textbf{(3.3)}$$

f_r is not half way between f_1 and f_2; it is the geometric mean of f_1 and f_2,

$$f_r = \sqrt{f_1 f_2} \qquad \textbf{(3.4)}$$

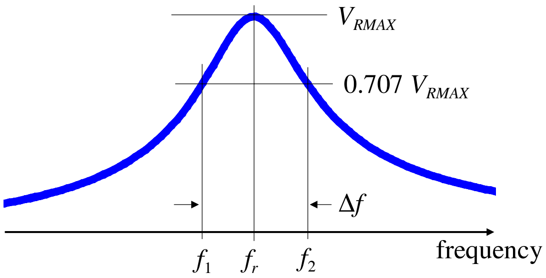


Figure 3.4

The Q factor can be expressed in terms of the circuit components by using the magnification definition of Q factor. For the ideal circuit of figure 3.1

$$Q = \frac{|V_L|}{|V_S|} = \frac{|V_L|}{|V_R|} = \frac{I\,\omega_r\,L}{IR} = \frac{\omega_r\,L}{R} \qquad \textbf{(3.5)}$$

The voltages are measured at the resonant frequency where, for the circuit of figure 3.1, $|V_L| = |V_C|$ so V_C could equally well have been used in equation **(3.5)**. Using $\omega_r = 1/(LC)^{0.5}$,

$$Q = \frac{\omega_r\,L}{R} = \frac{1}{\omega_r\,CR} = \frac{1}{R}\left(\frac{L}{C}\right)^{0.5} \qquad \textbf{(3.6)}$$

The $1/(\omega_r\,CR)$ term in equation **(3.6)** is the result of using V_C rather than V_L to calculate the magnification in equation **(3.5)**. The justification of equation **(3.4)** and of the fact that equation **(3.3)** is consistent with equations **(3.5)** and **(3.6)** is given in Appendix 1. Two other definitions of Q are given in Appendix 2.

Non-ideal components

In reality all three components in the circuit of figure 3.1 will deviate from their ideal behaviour. Resistors have a series inductance and a parallel capacitance, capacitors have series inductance and series and parallel resistance and inductors have a series resistance and a parallel capacitance. It is usually possible to find a technology of resistor or capacitor that makes their defects insignificant for the application of interest but the deviations from ideality associated with inductors are less easy to reduce to insignificant proportions. In this module we will look at the effects of inductor series resistance which is important in communication systems because it limits Q factor (and alters f_r for parallel circuits) and is important in power management systems because of the energy that is dissipated within it.

Figure 3.5 is the circuit of figure 3.1 with the inductor series resistance included. The circuit behaves in the same basic way as that of figure 3.1. At resonance the impedance of the series combination of C and the ideal part, L, of the inductor is zero so at resonance the circuit consists of two resistors in series. V_R at resonance is thus

$$V_R = V_S\,\frac{R}{R + R_L}$$

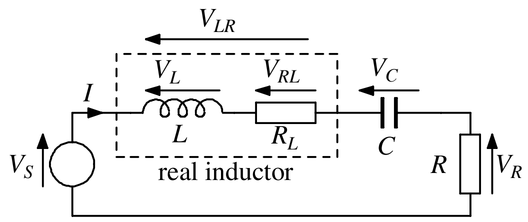


Figure 3.5

and if V_S and V_R are measured and R is known, this provides an easy way of measuring R_L at the resonant frequency.

It is important to know the resistance **at the resonant frequency** because the inductor resistance will tend to increase as frequency increases. This means that the resistance of the inductor at dc (0 Hz) will not be the same as its resistance at the resonant frequency.

> The increase happens in part because of something called the "skin effect" - as frequency increases, current flow is confined to a "skin" on the outside surface of the conductor. The thickness of this skin, called the "skin depth", is inversely proportional to the square root of frequency and is of the order of 10^{-2} m at 100 Hz - power frequencies - and 10^{-6} m at 10 GHz - satellite television frequencies. To find out more, look in any good book on electromagnetism.

The Q of the circuit is affected by R_L because the total series resistance is now $(R + R_L)$. Consequently one would expect the measured Q (measured perhaps by $f_r/\Delta f$) to be

$$Q_E = \frac{1}{R + R_L}\left(\frac{L}{C}\right)^{0.5} \qquad \textbf{(3.7)}$$

Dividing equation **(3.7)** by the expression for the Q factor with ideal components, equation **(3.6)**, leaves $\frac{Q_E}{Q} = \frac{R}{R + R_L}$. If Q_E is measured and Q is calculated from equation **(3.6)** assuming ideal components, this offers another route to finding R_L experimentally. If the magnification approach is used to measure Q, it must be $\frac{|V_C|}{|V_S|}$ that is used since V_R is no longer interchangeable with V_S at resonance (because some voltage is lost across R_L at resonance) and the terminals of the ideal part of L are not accessible (so the voltage measured across L will contain a component due to R_L).

10.3 Parallel resonance

Parallel resonant circuits are used much more commonly than series resonant circuits in communications systems but they are not as common as series resonant circuits in power management systems. The basic parallel resonant circuit is shown in figure 3.6. Sometimes the circuit is drawn as in figure 3.7 but a moment's thought will reveal that Thevenin's theorem identifies the two circuits as functionally the same if $V_S = I_S R$. The approach to identifying the resonance condition is the same as for the series circuit, i.e., write down the impedance and force j terms to zero. For the figure 3.6 form of the circuit,

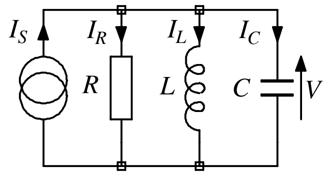


Figure 3.6

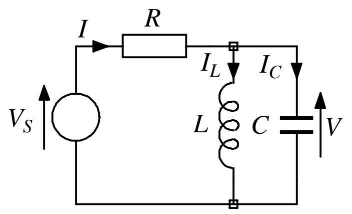


Figure 3.7

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{j\,\omega L} + j\,\omega C = \frac{1}{R} + j\,(\omega C - \frac{1}{\omega L})$$

The objective is to make the j terms equal zero and if they can be made to equal zero in $1/Z$, they will also equal zero in Z. Thus, as for the series case

$$\omega_r^2 = \frac{1}{LC} \text{ or } \omega_r = \frac{1}{\sqrt{LC}} \text{ or } f_r = \frac{1}{2\pi\sqrt{LC}} \qquad \textbf{(3.8)}$$

Phasor diagram view of parallel resonance

For the parallel circuit the voltage is the same across all the elements so it makes sense to use voltage as the reference direction as in figure 3.8. The currents through L and C are in antiphase, I_C leading the voltage by 90° and I_L lagging the voltage by 90°. The current through R is in phase with V. Since in the diagram of figure 3.8 I_L has a slightly bigger magnitude than I_C, the resultant I has a slightly inductive behaviour. At resonance, the magnitudes of I_C and I_L are equal so they sum to zero.

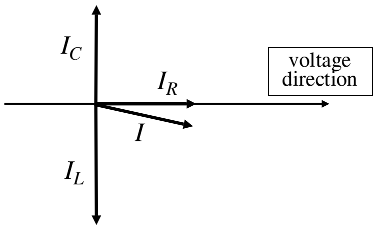


Figure 3.8

For the parallel circuit the total current flowing through the L-C combination at resonance is zero - i.e., **the impedance of the parallel *L-C* circuit at resonance is infinite.**

Frequency dependence of impedance and voltage

Figure 3.9 shows, for the circuit of figure 3.6, the relationship between V, $|Z|$ and the phase of V with respect to I_S for an I_S of varying frequency but constant amplitude. Since $V = I_S Z$, the shape of $|V|$ and $|Z|$ as a function of frequency is the same.

At resonance, since the L-C combination has infinite impedance, $I_S = I_R$. At frequencies below resonance $I_L > I_C$ so the circuit behaves inductively, at frequencies above resonance, $I_C > I_L$ so the circuit behaves capacitively. Since the maximum V is proportional to R, the Q factor of the parallel circuit will increase with increasing R.

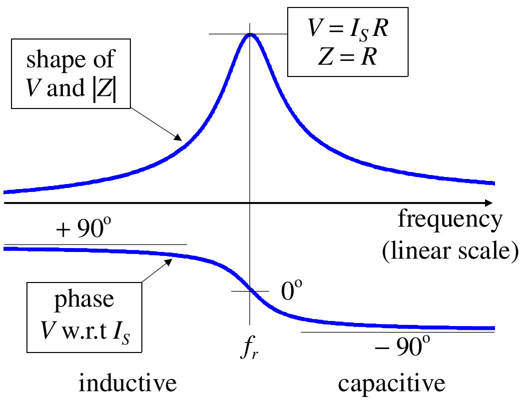


Figure 3.9

"Q" or "magnification" factor

For a parallel circuit the Q factor is either measured using a similar approach to the one used for a series circuit or calculated (or measured) using magnification behaviour. Figure 3.9 indicates that the voltage across the parallel network has a similar frequency dependent shape to that of the current through a series resonant circuit such as that shown in figure 3.4. The $f_r/\Delta f$ measurement approach follows the same process as was described in the series case. The magnification approach must be based on current magnification since all elements have the same voltage across them. So

$$Q = \frac{|I_L|}{|I_S|} = \frac{|I_C|}{|I_S|} \quad \text{at resonance}$$

$$= \frac{V\omega_r C}{\frac{V}{R}} = \omega_r CR = R\left(\frac{C}{L}\right)^{0.5} \qquad \textbf{(3.9)}$$

Again, the same result would be obtained from $|I_L|/|I_S|$.

Notice that the expression for the Q of a parallel circuit is the inverse of that for a series circuit. In terms of remembering which is which, it might help to remember that in a series circuit, a large Q arises when only a small fraction of the supply voltage is lost across R. For a parallel circuit, a large Q occurs when the parallel R draws only a small current compared with the magnitudes of I_L and I_C at resonance.

Inductor with series resistance

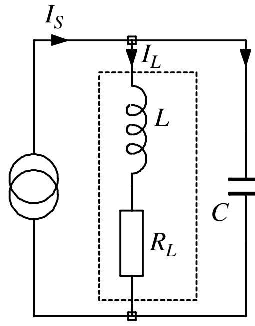


Figure 3.10

A parallel resonant circuit including the effects of inductor series resistance is shown in figure 3.10. The R in parallel with the real L and C has been omitted - it does not affect the resonant frequency*(see below) but it does affect the Q factor. The circuit impedance is the impedance of the real L in parallel with the purely reactive impedance of C,

$$Z = \frac{V}{I_S} = \frac{\dfrac{R_L + j\,\omega L}{j\,\omega C}}{R_L + j\,\omega L + \dfrac{1}{j\,\omega C}} = \frac{R_L + j\,\omega L}{j\,\omega C R_L - \omega^2 LC + 1}$$

Since this is complex in both the numerator and denominator it must be rationalised to get the complex part in the numerator alone,

$$Z = \frac{(R_L + j\,\omega L)\,((1 - \omega^2 LC) - j\,\omega C R_L)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_L^2} = \frac{R_L + j\,\omega(L - \omega^2 L^2 C - C R_L^2)}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_L^2} \qquad \textbf{(3.10)}$$

which is of the form $R + jX$. To find the resonant frequency, the j terms must be forced to zero

so $\quad (L - \omega^2 L^2 C - C R_L^2) = 0$ which gives

$$\omega_r^2 = \frac{L - C R_L^2}{L^2 C} \text{ or } \omega_r^2 = \frac{1}{LC} - \frac{R_L^2}{L^2} \qquad \textbf{(3.11)}$$

Equation **(3.11)** shows the modification to the resonant frequency referred to in the "non-ideal components" part of section **10.2**. For high Q circuits - i.e., ones containing carefully designed low loss inductors, the modification is small because R_L is small.

The real part of Z is $Z_{re} = \dfrac{R_L}{(1 - \omega^2 LC)^2 + \omega^2 C^2 R_L^2}$ which reduces to $Z_{re} = \dfrac{L}{C\,R_L}$ at resonance.

Z_{re} is the R term in the $R + jX$ mentioned below equation **(3.10)** so all the resistive aspects of circuit behaviour have been transformed into a single equivalent series resistance. It is also possible to represent a circuit like this as a parallel circuit by calculating the circuit admittance. Admittance is introduced in the next section.

When components are not ideal, defining resonant parameters can become very awkward. You will be pleased to hear that you don't need to worry about that for EEE117 - except of course the material covered in this handout.

* This is so because if a parallel R existed, as in figure 3.6, the current through it would always be in phase with V. Since the resonant condition is that I_S be in phase with V, the presence of an extra current that is always in phase with V will not alter the condition for resonance. It will, however alter the rate of energy loss in the circuit and hence will alter Q.

11 Admittance, Conductance and Susceptance

11.1 Definitions

Admittance, conductance and susceptance are the inverse of impedance, resistance and reactance. In other words

admittance, $Y = 1/Z = 1/(\text{impedance})$ S

conductance, $G = 1/R = 1/(\text{resistance})$ S

susceptance, $B = 1/X = 1/(\text{reactance})$ S

The unit of impedance, resistance and reactance is Ohm, Ω. The unit of admittance, conductance and susceptance is 1/(Ohm), Ω^{-1}. For many years Ω^{-1} was called "mho" and was sometimes represented by an upside down Ω symbol. The official SI unit name for admittance, conductance and susceptance is the Siemen, S, but as this official unit was introduced at a relatively late stage one needs to be aware of the variety of units that might be encountered. The important point is that all the unit names mean the same thing; (current in amps)/(voltage difference in volts).

Admittance is the general term for an I/V ratio. In general it will be a complex quantity - ie it will have real and imaginary parts.

For the circuit of figure 3.11(a), $V_S = V_R + V_X = IR + IjX$ which leads easily to $Z = \frac{V_S}{I} = R + jX$. Z will always have this form and will thus always represent the circuit as a series combination of R and X even if the circuit has some parallel elements as in the case of the circuit of figure 3.10.

Consider now figure 3.11(b). For this circuit $I_S = I_G + I_B = VG + VjB$ and hence $Y = \frac{I_S}{V} = G + jB$. Y will always have this form and hence will always represent the circuit as a parallel combination of G and B.

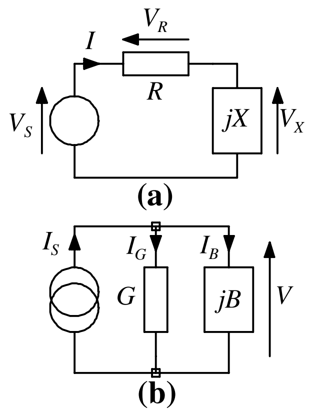


Figure 3.11

As an example of applying an admittance approach we will look at figure 3.10 from an admittance point of view.

$Y = \frac{1}{Z_{RL}} + \frac{1}{Z_C} = \frac{1}{R_L + j\omega L} + j\omega C = \frac{1 + j\omega CR_L - \omega^2 LC}{R_L + j\omega L}$ which is the inverse of the expression for Z derived prior to equation (**3.10**). This is expected because $Y = 1/Z$. Rationalising and putting in real and imaginary parts gives

$$Y = \frac{R_L}{R_L^2 + \omega^2 L^2} + j\,\frac{\omega(CR_L^2 - L\,(1 - \omega^2 LC))}{R_L^2 + \omega^2 L^2} \equiv G + jB\,.$$

Although G is clearly not the same in general terms as $1/Z_{re}$ it is interesting to note that at resonance $G = 1/Z_{re} = RC/L$. This makes sense because at resonance neither the reactance nor the susceptance play any part in the circuit so the ration of V to I at resonance must be the same however it is calculated.

The sign of the susceptive part of an admittance defines the type of susceptance. A $+j$ multiplier indicates a capacitive suceptance and a $-j$ multiplier indicates an inductive susceptance. For the circuit of figure 3.10, the sign of the reactive component of admittance (and impedance) is a function of frequency. For frequencies above ω_r the susceptive part of the admittance is positive (capacitive) whereas below ω_r the susceptive part of admittance is negative (inductive). The significance of "j" is the same as it is for impedance except that the phase of Y, $\tan^{-1}\left(\frac{\text{imag}Y}{\text{real}Y}\right)$, is the phase of I with respect to V.

11.2 Series to Parallel Transformation

In the previous section it was shown that a network with two terminals can be represented as a series combination of resistance and reactance or as a parallel combination of conductance and susceptance. It is sometimes useful to transform from a series representation to a parallel one and this section describes the transformation process.

Consider figure 3.12. The series to parallel process aims to find L_p and R_p that will make the admittance of figure 3.12b the same as that for figure 3.12a, in both cases looking into terminals **A** and **B** .

Thus we want to achieve $Y_{3.12\mathrm{a}} = Y_{3.12\mathrm{b}}$

$$Y_{3.12a} = \frac{1}{Z_{3.12a}} = \frac{1}{R_S + j\omega L_S} = \frac{R_S - j\omega L_S}{R_S^2 + \omega^2 L_S^2} \qquad \textbf{(3.12)}$$

and $$Y_{3.12b} = \frac{1}{R_P} + \frac{1}{j\omega L_P} = \frac{1}{R_P} - \frac{j}{\omega L_P} \qquad \textbf{(3.13)}$$

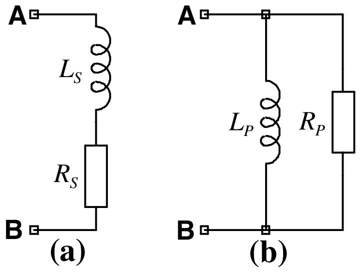


Figure 3.12

Both $Y_{3.12a}$ and $Y_{3.12b}$ are of the form $Y = G - jB$. The transformation proceeds by recognising that the G and B parts of each expression must be equal. In other words we want to make

$$G_{3.12a} = \frac{R_S}{R_S^2 + \omega^2 L_S^2} = G_{3.12b} = \frac{1}{R_P} \qquad \textbf{(3.14)}$$

and $$B_{3.12a} = \frac{\omega L_S}{R_S^2 + \omega^2 L_S^2} = B_{3.12b} = \frac{1}{\omega L_P} \qquad \textbf{(3.15)}$$

Thus $$R_P = \frac{R_S^2 + \omega^2 L_S^2}{R_S} \text{ and } L_P = \frac{R_S^2 + \omega^2 L_S^2}{\omega^2 L_S} \qquad \textbf{(3.16)}$$

will make figure 3.12b electrically the same as figure 3.12a when looking into terminals **A** and **B**.

A similar process, this time equating impedances, will find the R_S and L_S required to make figure 3.12a the same as figure 3.12b when looking into terminals **A** and **B**. The result of this parallel to series conversion is

$$R_S = \frac{R_P\,\omega^2 L_P^2}{R_P^2 + \omega^2 L_P^2} \text{ and } L_S = \frac{L_P R_P^2}{R_P^2 + \omega^2 L_P^2}$$

As an application example, the series to parallel transformation can be used to transform the circuit of figure 3.10 into an ideal parallel resonant circuit of the form of figure 3.6. The transformed circuit is shown in figure 3.12c. The inductance and resistance have been labelled with the transformed equivalents of equation **(3.16)** but the subscripts have been altered to suit the labelling of figure 3.10. The resonant frequency of the ideal parallel circuit is given by equation **(3.8)** as

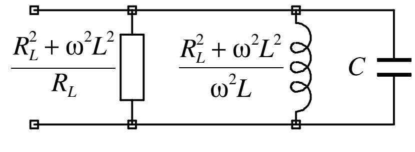


Figure 3.12c

$\omega_r^2 = \dfrac{1}{LC}$ which in this case becomes $\omega_r^2 = \dfrac{1}{C\,\dfrac{R_L^2 + \omega_r^2 L^2}{\omega_r^2 L}} = \dfrac{\omega_r^2 L}{C\,(R_L^2 + \omega_r^2 L^2)}$. This leads to the result of equation **(3.11)**. It is also possible to calculate the circuit Q factor using equation **(3.9)**, **(3.11)** and the transformed R and L of equation **(3.16)**.

Since these transformation procedures aim to transform R + jX into G + jB or vice versa they can be used to realise series to parallel and parallel to series transformations on any two terminal circuit. Note that even in the relatively straightforward case of an *LR* combination, the transformed equivalent components, such as those given in equation **(3.16)**, are themselves functions of frequency and therefore do not conform to normal resistive or inductive frequency dependent behaviour. These expressions rapidly expand if more components are involved.

12 Transient Analysis

12.1 Introduction

The analysis performed using the "$j\omega$" or phasor diagram approach is known as a frequency domain analysis. The driving source is assumed to be an ideal sinusoid and the independent variable is frequency; the analysis evaluates gain and phase as a function of frequency and time is not available as an explicit variable.

(You will discover in later years that there are methods that allow the time response to be deduced from knowledge of the frequency domain behaviour - time and frequency are in fact intimately linked - but we will not deal with those issues here.)

To work out circuit performance as a function of time one must perform a transient analysis. For transient problems, concepts such as impedance and admittance have no meaning so inductors and capacitors must be described by their differential (or integral) *V-I* relationships. Resistors, of course, obey Ohm's law irrespective of whether the variable of interest is time or frequency. Transient analysis usually investigates the response of a circuit to a driving source input in the form of an ideal step. An ideal step input is an instantaneous change in input voltage from one value that has existed from $t = -\infty$ to $t = 0$ to another that exists fron $t = 0$ to $t = +\infty$. Although it might appear that this is a fairly unrealistic sort of signal it is very useful tool for helping to work out the response due to pulses, a much more commonly encountered and useful signal form.

12.2 Working out initial conditions

One of the things one must define in order to complete the solution of a step response problem is the set of initial conditions of voltages across and sometimes current through the various elements in the circuit. There are three sets of conditions that are relatively easy to work out:

- the conditions at $t = 0^-$ (immediately before the instant of the step)
- the conditions at $t = 0^+$ (immediately after the instant of the step)
- the conditions at $t = \infty$ (a very long time after the step)

Assume that the step input voltage changes from V_1 for all $t < 0$ to V_2 for all $t > 0$. The conditions at $t = 0^-$ are easy to work out because the step voltage has been at V_1 for all time before $t = 0$ so is effectively a dc input. **At dc, the voltage across any (ideal) inductors must be zero and the current through any (ideal) capacitors must be zero**. By a similar argument, the conditions at $t = \infty$ can be found by working out the dc conditions resulting from an input of V_2. The conditions at $t = 0^+$ are not hard to evaluate but one must bear in mind that both the Ls and Cs will have some stored energy. **At the instant of the step, each L will want to keep constant the current through it and each C will want to keep constant the voltage across it**. Thus the voltage across the Cs will not change during the transient (ie, V_C @ $t = 0^+ = V_C$ @ $t = 0^-$ will be true) and the current through the Ls will not change during the transient (ie, I_L @ $t = 0^+ = I_L$ @ $t = 0^-$). So at $t = 0^+$ (and only at $t = 0^+$) the Ls can be represented as current sources and the Cs as voltage sources, both keeping the currents and voltages that they had at $t = 0^-$. As t evolves from $t = 0^+$ towards $t = \infty$, the stored energy in the Ls and Cs changes slowly from its $t = 0$ value to its $t = \infty$ value. An example on how to apply these ideas is given below.

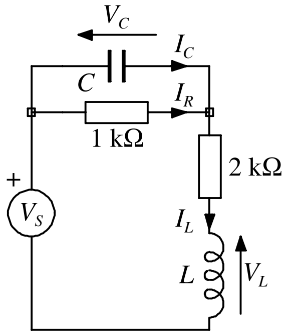


Figure 3.13

Example: For the circuit of figure 3.13, work out I_R, I_C, I_L, V_C and V_L at $t = 0^-$, $t = 0^+$ and $t = \infty$ for an input step waveform that changes from −6 V to +12 V at $t = 0$.

At $t = 0^-$, $V_S = -6$ V

$I_C = 0$ (since capacitors are open circuit to dc)

$V_L = 0$ (since inductors are short circuit to dc)

Thus $I_R = I_L = V_S/(1\text{k}\Omega + 2\text{k}\Omega) = -6/3\text{k}\Omega = -2$ mA

and $$V_C = -6\frac{1\text{k}\Omega}{1\text{k}\Omega + 2\text{k}\Omega} = -2\text{ V}$$

At $t = 0^+$, $V_S = 12$ V

V_C and I_L remain unchanged during the transition from $t = 0^-$ to $t = 0^+$ and at the $t = 0^+$ instant can be regarded as sources having their $t = 0^-$ values as shown in figure 3.14. The voltage across the 1kΩ resistor is V_C, which is unchanged at $t = 0^+$, so the current, I_R, through that resistor is also unchanged. Since $I_R + I_C = I_L$ and I_R and I_L are unchanged, I_C in this circuit must be unchanged.

Thus $I_R = I_L = -2$ mA

$I_C = 0$

$V_C = -2$ V

$V_L + I_L\,2\text{k}\Omega + V_C = V_S$ or $V_L = V_S - V_C - I_L\,2\text{k}\Omega$

or $V_L = 12\text{ V} - (-2\text{V}) - (-2\text{mA}\,2\text{k}\Omega) = 18$ V

At $t = \infty$, $V_S = 12$ V and the problem is a dc problem

$I_C = 0$ (since capacitors are open circuit to dc)

$V_L = 0$ (since inductors are short circuit to dc)

Thus $I_R = I_L = V_S/(1\text{k}\Omega + 2\text{k}\Omega) = 12/3000 = 4$ mA

and $$V_C = 12\frac{1\text{k}\Omega}{1\text{k}\Omega + 2\text{k}\Omega} = 4\text{ V}$$

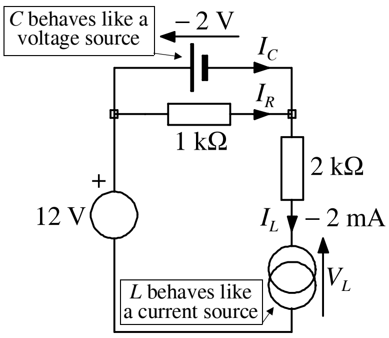


Figure 3.14

The equivalent circuit of figure 3.13 at the instant $t = 0^+$

12.3 Transient response examples

A simple *CR* circuit. Consider the circuit of figure 3.15. Here the step is produced by switching the switch from position **A** to position **B** (or position **B** to position **A**) at $t = 0$. We will assume here that the switch has been in position **B** for a long time before being switched to position **A** at $t = 0$.

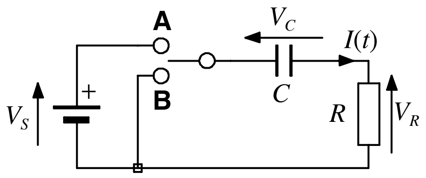


Figure 3.15

Summing voltages round the loop immediately after the switch, ie, at $t = 0^+$,

$$V_S = V_C + V_R = \frac{1}{C}\int I(t)dt + I(t)R \qquad \textbf{(3.17)}$$

The first thing to do is to change this integral equation into a differential equation by differentiating both sides . . . .

$$\frac{dV_S}{dt} = \frac{I(t)}{C} + R\,\frac{dI(t)}{dt} = 0 \text{ (since } V_S = \text{constant)} \qquad \textbf{(3.18)}$$

then separate the variables to give $\dfrac{dI(t)}{I(t)} = -\dfrac{dt}{CR}$ **(3.19)**

and integrate both sides of equation **(3.19)** to give

$$\int \frac{dI(t)}{I(t)}dt = -\int \frac{dt}{CR} + \text{ constant}$$

$$\text{or} \quad \ln I(t) = -\frac{t}{CR} + \text{constant}$$

$$\text{or} \quad I(t) = e^{\left(-\frac{t}{CR} + \text{const.}\right)} = Ae^{\left(-\frac{t}{CR}\right)} \qquad \textbf{(3.20)}$$

We now need to identify initial conditions that can be used to find A. Essentially this involves finding $I(t)$ at $t = 0^+$. In this case, at $t = 0^+$, $V_C = 0$ (since the voltage across C at $t = 0^-$ was 0 V and there has been no time for it to change) and so $I(0^+) = V_S/R$.

Thus for these conditions, $I(0^+) = \dfrac{V_S}{R} = A\,e^{-0} = A$

$$\text{so} \quad I(t) = \frac{V_S}{R}\,e^{-\frac{t}{RC}} \quad \text{for } t > 0 \qquad \textbf{(3.21)}$$

A simple *LR* circuit. The process of finding the transient behaviour of figure 3.16 is similar to the *CR* case of figure 3.15. Assume that the switch has been in position **B** for a long time before being suddenly switched to position **A**. Summing voltage around the loop at $t = 0^+$,

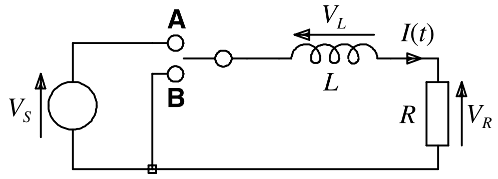


Figure 3.16

$$V_S = V_L + V_R = L\,\frac{dI(t)}{dt} + I(t)R$$

$$\text{or} \quad \frac{dI(t)}{dt} = -\frac{R}{L}\left(I(t) - \frac{V_S}{R}\right) \quad \text{or} \quad \frac{dI(t)}{\left(I(t) - \frac{V_S}{R}\right)} = -\frac{R}{L}\,dt \qquad \textbf{(3.22)}$$

Integrating both sides gives $\ln\left(I(t) - \dfrac{V_S}{R}\right) = -\dfrac{R}{L}t + \text{const}$ which can be written as

$$I(t) - \frac{V_S}{R} = e^{-\frac{R}{L}t + \text{const}} = A\,e^{-\left(\frac{R}{L}\right)t} \qquad \textbf{(3.23)}$$

Again, initial conditions must be identified in order to find A. $I(0^-) = 0$ because prior to $t = 0$ the driving source = 0 V. The inductor will ensure that $I(0^+) = I(0^-)$ so $I(0^+) = 0$.

Thus $A = -\dfrac{V_S}{R}$ and $I(t) = \dfrac{V_S}{R}\left(1 - e^{-\left(\frac{R}{L}\right)t}\right)$ **(3.24)**

A slightly more complicated *LR* circuit. In figure 3.17 the switch is in position **A** for all $t < 0$ and position **B** for all $t > 0$. Find V_{R2} as a function of time. At $t = 0^+$

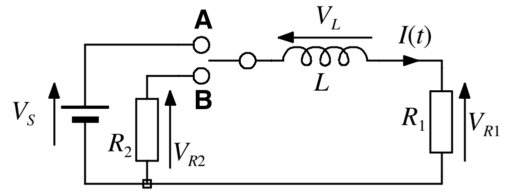


Figure 3.17

$$V_{R2} = V_L + V_{R1}$$

or $$-I(t)R_2 = L\frac{dI(t)}{dt} + I(t)R_1$$

or $$L\frac{dI(t)}{I(t)} = -\frac{R_1 + R_2}{L}\,dt \qquad \textbf{(3.22)}$$

Integrating both sides of equation **(3.22)** and expressing t as an exponent,

$$I(t) = A\,e^{-\left(\frac{R_1+R_2}{L}\right)t} \qquad \textbf{(3.23)}$$

The initial condition is $I(0^+)$, which because of L is equal to $I(0^-)$. $I(0^-)$ is given by

$$I(0^-) = \frac{V_S}{R_1} \text{ and so } A = \frac{V_S}{R_1} \text{ giving } I(t) = \frac{V_S}{R_1}\,e^{-\left(\frac{R_1+R_2}{L}\right)t} \qquad \textbf{(3.24)}$$

This solution for $I(t)$ can then be used to find V_{R2} . . . .

$$V_{R2} = -\,I(t)R_2 = -\,\frac{V_S\,R_2}{R_1}\,e^{-\left(\frac{R_1+R_2}{L}\right)t} \qquad \textbf{(3.25)}$$

Note that the peak value of V_{R2} can be bigger in magnitude than the supply voltage, V_S. This capability to produce voltage transients that are of larger magnitude than the driving source is exploited in some types of power supply.

The transient performance of many other first order circuits can be found in this way. To summarise the process,

- Sum voltages around the loop at $t = 0^+$,
- Write those voltages down in terms of current,
- Differentiate (if necessary) to get rid of integrals
- Work out the $t = 0^+$ initial conditions to find the constants in the solution.

The key integral is

$$\int \frac{dx}{a + bx} = \frac{1}{b}\ln(a + bx) + \text{constant}$$

12.4 Exponentials and pulses

The solutions to all the transient analyses of section 12.3 were "simple exponentials". The step responses of first order systems are always of this form. A first order system is one described by a differential equation involving only first derivatives and having coefficients that are constant. This section explores the geometry of exponential shapes and identifies their key properties.

Figure 3.18 shows a rising exponential. The main features of the exponential are labelled on the diagram. The particular point to notice is that a tangent to the exponential at $t = 0$ meets the aiming level at point **C** after a time τ seconds. In fact, a tangent drawn at any point along the exponential will meet the aiming voltage τ seconds later. As an example, the tangent at $t = t_1$ meets the aiming level at **D**. Another way of looking at this behaviour is to recognise that for any starting point on the exponential, after τ seconds the exponenential will have moved by $(1 - 1/e)$, (approximately 63%), of the difference between that starting point and the aiming level. The time τ is evidently an important parameter of the exponential and is called the "**time constant**". A general exponential has the form

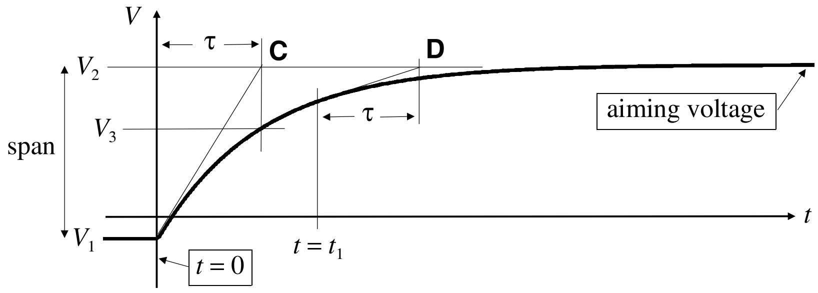


Figure 3.18

span = $(V_2 - V_1)$ $V_3 = (1 - 1/e)(V_2 - V_1) + V_1$

$$V(t) = (V_{start} - V_{aiming})\, e^{-\left(\frac{t}{\tau}\right)} + V_{aiming}$$

so in the three examples of section 12.3 the time constants are CR, L/R and $L/(R_1 + R_2)$ respectively.

The exponential gets closer and closer to its aiming level as time progresses but in principle it never exactly reaches it. From an engineering point of view it soon gets sufficiently close to its aiming value that the difference is negligible. For an exponential with a span of 1 V, the difference between the exponential and the aiming level after five time constants is 6.7 mV and after ten time constants is 45 μV. For most purposes the exponential is taken as having reached its aiming level after five time constants but for high precision work it may be necessary to calculate the number of time constants necessary to achieve a given accuracy.

Time constant is not always an easy parameter to measure. As an alternative, the rise time (or, for a falling exponential, the fall time) is often measured and most digital oscilloscopes will measure risetime automatically if asked to do so. The rise time of an upwards going response to a step input is defined as the time taken for the rising output to travel between 10% and 90% of its span. It is possible to measure a rise time for any rising edge but **rise time is only related to time constant if the rising shape is a simple exponential (ie, caused by a first order system) and the span includes the aiming voltage**. In other words the exponential must be visible over a time of at least five time constants from the start of the rise. The relationship between rise time and time constant is derived below;

The exponential rise of figure 3.19 is defined by $V(t) = (V_1 - V_2)\,(1 - e^{-\left(\frac{t}{\tau}\right)}) + V_1$

At the 90% point $V(t_2) = 0.9(V_1 - V_2) + V_1 = (V_1 - V_2)\,(1 - e^{-\left(\frac{t_2}{\tau}\right)}) + V_1$

or $0.9 = (1 - e^{-\left(\frac{t_2}{\tau}\right)})$ or $e^{-\left(\frac{t_2}{\tau}\right)} = 0.1$ or $-t_2 = \tau \ln 0.1$

and at the 10% point $V(t_1) = 0.1(V_1 - V_2) + V_1 = (V_1 - V_2)\,(1 - e^{-\left(\frac{t_1}{\tau}\right)}) + V_1$

which, by following the process for the 90% case, gives $-t_1 = \tau \ln 0.9$.

Risetime $t_r = t_2 - t_1 = \tau(\ln 0.9 - \ln 0.1) = \tau \ln 9 = 2.2\,\tau$

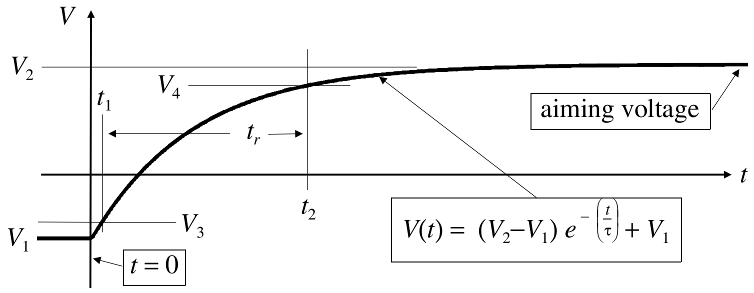


Figure 3.19

$V_4 = 0.9(V_2 - V_1) + V_1$ $\qquad\qquad$ $V_3 = 0.1(V_2 - V_1) + V_1$

13 Decibels (dB), Standard Forms and Filter Responses

13.1 The dB scale

The dB (decibel) is a logarithmic unit used to express ratios of quantities such as current, voltage and power. It was originally devised as a measure of sound level in the context of loss of signal power in early telephone systems. The dB, being a logarithmic measure, is capable of representing a huge range of values in a managable number set. For example, operational amplifiers have very large voltage gains, typically between 10^5 and 10^7 V/V. A gain of unity in dB terms would be 0 dB and a gain of 10^7 V/V is 140 dB; the logarithmic process has condensed the range 1 to 10^7 into a range 0 to 140.

definitions (a) as a power ratio $\qquad$ $\text{dB} = 10\log\left(\dfrac{P_1}{P_2}\right)$

(b) as a voltage ratio $\qquad$ $\text{dB} = 10\log\left(\dfrac{V_1^2}{V_2^2}\right) = 20\log\left(\dfrac{V_1}{V_2}\right)$

(c) as a current ratio $\qquad$ $\text{dB} = 10\log\left(\dfrac{I_1^2}{I_2^2}\right) = 20\log\left(\dfrac{I_1}{I_2}\right)$

note that the relationship between power and voltage or current is given here as power being proportional to V^2 or I^2. This is of course true but unless both V_1 and V_2 are measured across the same value of resistance there will not be a straightforward relationship between a dB voltage ratio and a dB power ratio.

The lower part of the ratio is often a fixed reference level in which case the dB is augmented by an extra letter to identify the reference. Some common examples are

dBV	reference level is 1Vrms
dBu	reference level is 1 mW into 600 Ω (0.775 Vrms)
dBm	reference level is 1 mW into 50 Ω (0.223 Vrms)
dB(spl)	reference level is 20 μPa (threshold of human hearing)
dB(swl)	reference level is 10^{-12} W (threshold of human hearing)

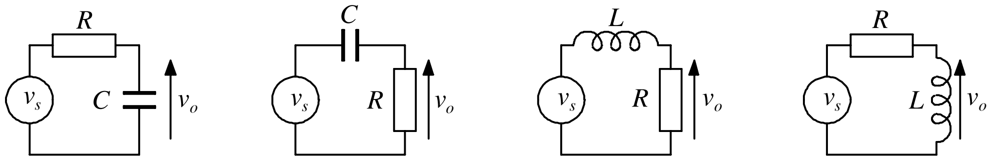
The dBm is used as an absolute power measure in 50 Ω impedance matched systems such as satellite, radar and other microwave systems.

The dBu is used as an absolute power measure in 600 Ω impedance matched audio systems.

dB(spl) and dB(swl) are used to measure sound levels - (swl) is a direct sound power measure whereas (spl) is a sound pressure level measure. (spl) has a similar relationship to sound power as voltage has to electrical power - ie, sound power (swl) is proportional to (spl)^2

13.2 Standard forms and filter responses

A filter is a circuit that is designed to have a particular frequency dependent magnitude (or sometimes phase) response. The three main types of filter are low-pass, band-pass and high-pass and in each case the name describes the range of frequencies that can pass easily through the filter. The simplest frequency dependent circuits are first order circuits and these are the circuits of interest in this module. First order circuits cannot exhibit a band-pass (resonant) response. Consider the circuits of figure 3.20.



(a) **(b)** **(c)** **(d)**

Figure 3.20

$$\frac{v_o}{v_s} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\omega\tau} = \frac{1}{1 + \frac{j\omega}{\omega_c}} = \frac{1}{1 + \frac{jf}{f_c}}$$

where $\omega_c = 2\pi f_c = 1/\tau$ and $\tau = RC$

$$\frac{v_o}{v_s} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega CR}{1 + j\omega CR} = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}} = \frac{\frac{jf}{f_c}}{1 + \frac{jf}{f_c}}$$

where $\omega_c = 2\pi f_c = 1/\tau$ and $\tau = RC$

$$\frac{v_o}{v_s} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\frac{L}{R}} = \frac{1}{1 + j\omega\tau} = \frac{1}{1 + \frac{j\omega}{\omega_c}} = \frac{1}{1 + \frac{jf}{f_c}}$$

where $\omega_c = 2\pi f_c = 1/\tau$ and $\tau = L/R$

$$\frac{v_o}{v_s} = \frac{j\omega L}{R + j\omega L} = \frac{j\omega\frac{L}{R}}{1 + j\omega\frac{L}{R}} = \frac{j\omega\tau}{1 + j\omega\tau} = \frac{\frac{j\omega}{\omega_c}}{1 + \frac{j\omega}{\omega_c}} = \frac{\frac{jf}{f_c}}{1 + \frac{jf}{f_c}}$$

where $\omega_c = 2\pi f_c = 1/\tau$ and $\tau = L/R$

The transfer function, v_o/v_s, of each circuit is worked out below each diagram. Since the transfer function describes circuit gain as a function of frequency, the time constant, τ, has been expressed in the form of a frequency domain constant, ω_c.

The circuits of figures 3.20 (a) and (c) have the same form of transfer function which is known as a "low pass" function. The circuits of figures 3.20 (b) and (d) also have the same form of transfer function, this time known as a "high pass" function. All four transfer functions are in the "standard form" for first order transfer functions. All first order circuit transfer functions will reduce to one of the standard forms,

low-pass $$\frac{v_o}{v_i} = k \cdot \frac{1}{1 + j\frac{\omega}{\omega_c}} = k \cdot \frac{1}{1 + j\frac{f}{f_c}} \qquad \textbf{(3.26)}$$

high-pass $$\frac{v_o}{v_i} = k \,.\, \frac{j\dfrac{\omega}{\omega_c}}{1+j\dfrac{\omega}{\omega_c}} = k \,.\, \frac{j\dfrac{f}{f_c}}{1+j\dfrac{f}{f_c}} \qquad \textbf{(3.27)}$$

A third form, which is a linear sum of **(3.26)** and **(3.27)**, is often called a "pole-zero" or "lead lag" function.

$$\frac{v_o}{v_i} = k \,.\, \frac{1+j\dfrac{\omega}{\omega_1}}{1+j\dfrac{\omega}{\omega_c}} = k \,.\, \frac{1+j\dfrac{f}{f_1}}{1+j\dfrac{f}{f_c}} = k \,.\, \frac{1}{1+j\dfrac{f}{f_c}} + k \,.\, \frac{f_c}{f_1} \,.\, \frac{j\dfrac{f}{f_c}}{1+j\dfrac{f}{f_c}} \qquad \textbf{(3.28)}$$

If the values of the frequency independent gain, k, and the frequency f_c can be identified, the frequency response is completely specified. In equation **(3.28)** complete specification is achieved either with f_c and two gains (high pass and low pass) or f_1, f_c and a single gain.

There are two parts to the frequency response of these circuits; the ***magnitude*** or ***amplitude*** response and the ***phase*** response.

Low pass magnitude response. The magnitude response is a plot of the modulus of v_o/v_i as a function of frequency. $|v_o/v_i|$ is usually plotted as a logarithmic scale in dB and the frequency scale is usually plotted logarithmically. Consider the low pass response of equation **(3.26)**

$$\left|\frac{v_o}{v_i}\right| = k \,.\, \left|\frac{1}{1+j\dfrac{f}{f_c}}\right| = k \,.\! \left[\frac{1}{1+\left(\dfrac{f}{f_c}\right)^2}\right]^{\frac{1}{2}} \qquad \textbf{(3.29)}$$

Ignoring for the moment the constant multiplier, k, the frequency dependent part of equation **(3.29)** can be explored under three different conditions,

1) when $f \ll f_c$, $(f/f_c)^2 \ll 1$ and $|v_o/v_i| \approx 1$

Thus when frequency becomes small compared to f_c, $|v_o/v_i|$ asymptotically approaches unity from below.

2) when $f = f_c$, $(f/f_c)^2 = 1$ and $|v_o/v_i| = \dfrac{1}{\sqrt{2}}$

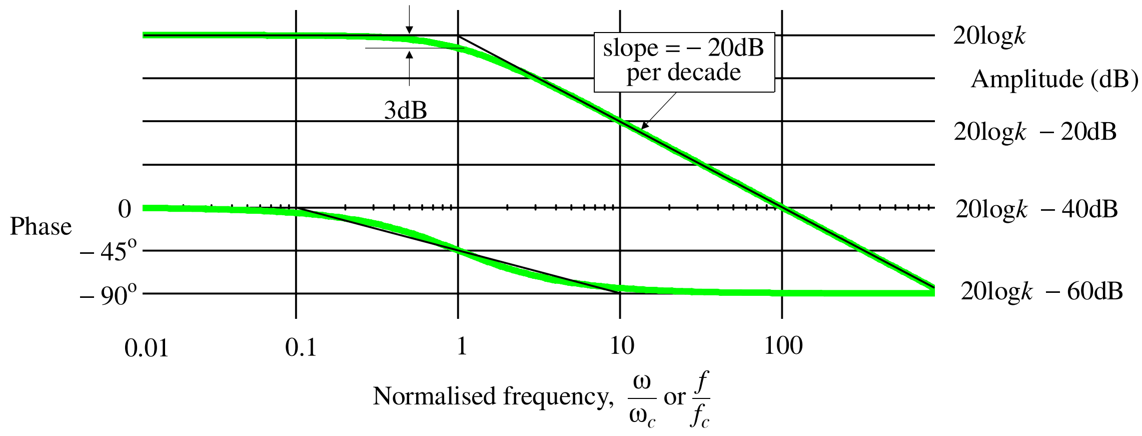
Thus when frequency is equal to f_c, $|v_o/v_i| = \dfrac{1}{\sqrt{2}}$.

3) when $f \gg f_c$, $(f/f_c)^2 \gg 1$ and $|v_o/v_i| \approx 1/(f/f_c) = f_c/f$.

Thus when frequency becomes large compared to f_c, $|v_o/v_i|$ asymptotically approaches an inversely proportional relationship with frequency. In other words, if f increases by a factor of 10, $|v_o/v_i|$ reduces by the same factor. A factor of 10 increase in frequency is called a "decade" and in dB terms a factor of 10 reduction in gain is – 20 dB. In region 3), the gain is said to "roll off" at – 20 dB per decade. Note that $f_c/f = 1$ (0 dB) when $f = f_c$ so the – 20 dB per decade roll off asymptote crosses the 0 dB gain line at $f = f_c$. A factor of two change in frequency is an octave. A factor of two reduction in gain corresponds to – 6 dB. Roll off is sometimes expressed as – 6 dB per octave; this means exactly the same thing as – 20 dB per decade. Since k is a constant multiplier it simply adds 20 log k to the whole amplitude response. Figure 3.21 illustrates this behaviour.

Figure 3.21

Low-pass amplitude and phase response



Low pass phase response. Since the circuits of figure 3.20 lead to transfer functions that have real and imaginary parts the phase of the signal will be modified as it travels through the circuit. The difference in phase between input and output signals is known as the "phase response" of the circuit. Its general behaviour can be deduced by the same process as was used for the amplitude response. Using the low pass response of equation **(3.26)**,

1) when $f \ll f_c$, phase shift $= -\tan^{-1}\dfrac{imaginary}{real} = -\tan^{-1}\dfrac{f}{f_c} \Rightarrow 0^{\circ}$

2) when $f = f_c$, phase shift $= -\tan^{-1}\dfrac{imaginary}{real} = -\tan^{-1}\dfrac{f}{f_c} = -45^{\circ}$

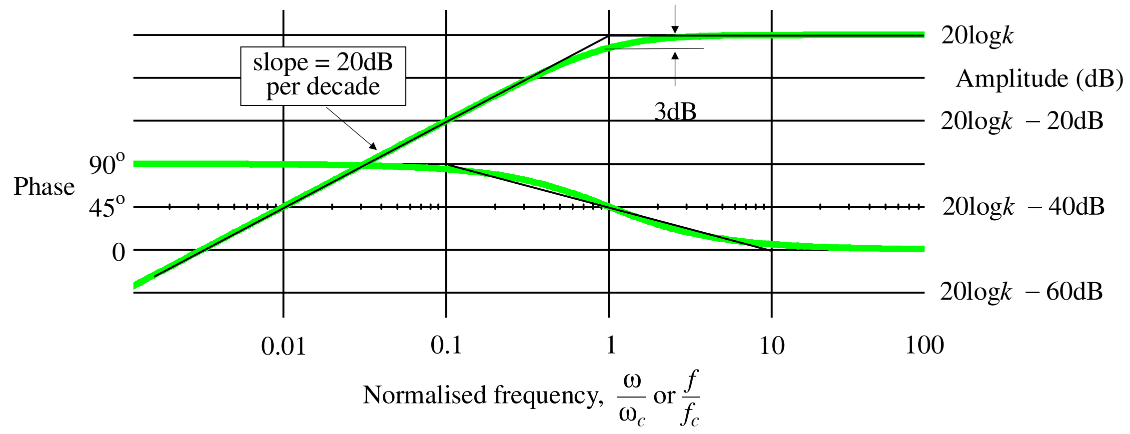
3) when $f \gg f_c$, phase shift $= -\tan^{-1}\dfrac{imaginary}{real} = -\tan^{-1}\dfrac{f}{f_c} \Rightarrow -90^{\circ}$

In each case the angle calculated is the phase of the output signal with respect to that of the input. A low pass phase response is shown underneath the amplitude response in figure 3.21.

High pass responses. High pass responses are deduced using the same process as for low-pass but the process is applied to equation **(3.27)**. The result is shown in figure 3.22.

Figure 3.22

High- pass amplitude and phase response



In figures 3.21 and 3.22 the grey line is the actual response. The thinner black line in the amplitude response uses the asymptotes as an approximation to the response. This gives a plot that changes slope abruptly at $f = f_c$. f_c is sometimes called the **"corner frequency"** because of this abrupt corner and sometimes is is called the **"– 3 dB frequency"** because the real gain has fallen by 3 dB at this frequency. The thin black line in the phase response graphs is a -45^o per decade approximation to the real phase relationships in grey. The error involved in this approximation has a maximum value of around 6^o. Notice that the phase shift at the corner frequency is -45^o for the low-pass and 45^o for the high-pass circuits. The thin black line approximations are called **"Bode approximations"** and the plots of figures 3.21 and 3.22 are sometimes called **"Bode plots"**. The Bode approximation is a very useful tool for sketching approximate response shapes to give an idea of circuit magnitude and phase behaviour.

Appendix 1 - Proof that $Q = f_r/\Delta f$

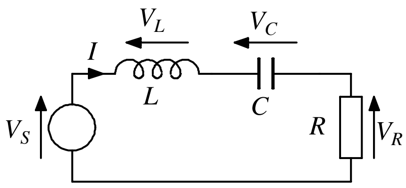


Figure A1.1

For the circuit of figure A1.1, the response V_R/V_S is shown in figure A1.2 . For frequencies higher than f_r, the circuit behaves inductively; for frequencies below f_r it behaves capacitively. Since $V_R = IR$ and $V_S = IZ$,

$$\frac{V_R}{V_S} = \frac{IR}{IZ} = \frac{R}{Z}$$

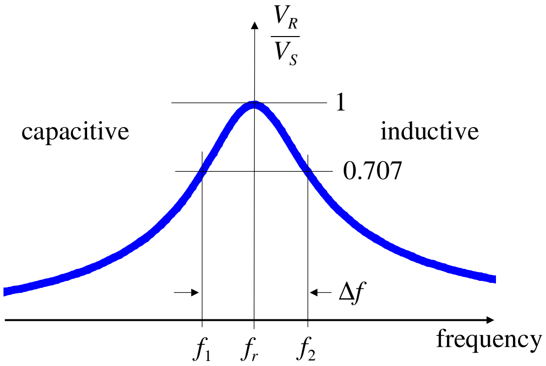


Figure A1.2

and the maximum value of V_R/V_S, which occurs when $f = f_r$, is unity. We want to find the frequencies at which V_R/V_S falls by $1/\sqrt{2}$ - labelled as f_2 and f_1 in figure A1.2. Considering f_2 first, the circuit behaves inductively at f_2 so ωL . $1/(\omega C)$. Thus we need to solve

$$\left|\frac{R}{Z}\right| = \frac{R}{\left[R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2\right]^{\frac{1}{2}}} = \frac{1}{\sqrt{2}}$$

$$\text{or} \quad R\sqrt{2} = \left[R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2\right]^{\frac{1}{2}}$$

$$\text{or} \quad 2R^2 = R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2$$

$$\text{or} \quad R^2 = \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2 \quad \text{or} \quad R = \omega_2 L - \frac{1}{\omega_2 C} \quad \text{or} \quad 0 = \omega_2^2 LC - \omega_2 CR - 1$$

$$\text{which gives} \quad \omega_2 = \frac{CR \pm \sqrt{C^2R^2 + 4LC}}{2LC}.$$

This result gives one positive and one negative value of frequency. Keeping the positive value and discarding the negative one leaves,

$$\omega_2 = \frac{CR + \sqrt{C^2R^2 + 4LC}}{2LC}\,.$$

For ω_1 the circuit behaves capacitively so $1/(\omega C) > \omega L$ and the equation to solve is

$$R\sqrt{2} = \left[R^2 + \left(\frac{1}{\omega_1 C} - \omega_1 L\right)^2\right]^{\frac{1}{2}} \quad \text{or, following the last procedure,} \quad R = \frac{1}{\omega_1 C} - \omega_1 L$$

which gives $\quad \omega_1 = \dfrac{-CR + \sqrt{C^2R^2 + 4LC}}{2LC}\,.$

Hence $\omega_2 - \omega_1 = \dfrac{2CR}{2LC} = \dfrac{R}{L}$ and so $\dfrac{\omega_r}{\omega_2 - \omega_1} = \dfrac{\omega_r L}{R} = Q$ as defined in equation (**3.6**).

The results of the preceding analysis can also be used to justify the relationship of equation (3.4), $f_r = \sqrt{f_1 f_2}$, that defines the resonant frequency f_r as the geometric mean of the two $-$ 3dB or half power frequencies, f_1 and f_2. Using the results derived for ω_1 and ω_2,

$$\sqrt{\omega_1\omega_2} = \left[\frac{-CR + \sqrt{C^2R^2 + 4LC}}{2LC} \cdot \frac{CR + \sqrt{C^2R^2 + 4LC}}{2LC}\right]^{\frac{1}{2}}$$

$$= \left[\frac{-C^2R^2 + CR\sqrt{C^2R^2 + 4LC} - CR\sqrt{C^2R^2 + 4LC} + C^2R^2 + 4LC}{4L^2C^2}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{LC}\right]^{\frac{1}{2}} = \omega_r$$

Appendix 2 - other definitions of *Q*

The Q factor of a resonant circuit is a fundamental parameter of the system that relates to its ability to store energy. There are two other definitions of Q that are sometimes useful,

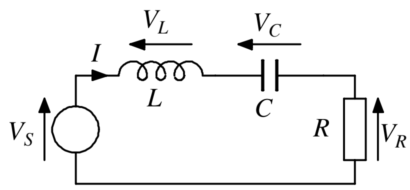


Figure A2.1

1) $Q = 2\pi\left[\dfrac{\text{energy stored}}{\text{energy dissipated per cycle}}\right]$

2) $Q = \dfrac{\omega}{2G}\dfrac{dB}{d\omega}$ for a parallel circuit or

$Q = \dfrac{\omega}{2R}\dfrac{dX}{d\omega}$ for a series circuit.

In case **1)**, the instantaneous energy stored at resonance in C and L is constant throughout a cycle. If $I(t) = I_P \sin \omega t$ then $V_C(t) = V_{CP} \cos \omega t$ where $V_{CP} = I_P X_C = I_P/(\omega C)$. Thus at an instant t_1 the total energy stored is

$$E_{TOT} = \frac{L\,(I_P \sin \omega t_1)^2}{2} + \frac{C\,(V_{CP} \cos \omega t_1)^2}{2}$$

$$= \frac{I_P^2}{2}\left[L \sin^2 \omega t_1 + \frac{C\,(\cos^2 \omega t_1)}{\omega_r^2 C^2}\right] = \frac{I_P^2}{2}\left[L \sin^2 \omega t_1 + L \cos^2 \omega t_1\right] = \frac{L I_P^2}{2}$$

Note that the energy stored in the circuit at any instant is the same as the peak energy stored in L or in C. Thus

$$Q = 2\pi \frac{\frac{L I_P^2}{2}}{I_{RMS}^2 R T} \quad \text{where } T = \frac{2\pi}{\omega_r} \text{ is the cycle time of } \omega_r \text{ and } I_{RMS} = \frac{I_P}{\sqrt{2}}.$$

Thus Q simplifies to $Q = \dfrac{L\,\omega_r}{R}$ which agrees with equation **(3.6)**. This definition of Q can readily be applied to any resonant system, mechanical or electrical, because it is based on considerations of energy storage and energy loss rather than explicit components.

Case **2)** is useful when a circuit is not simply an LCR circuit because it deals with resistance and reactance - ie, L and C do not have to be explicit. It is particularly useful in its conductance and susceptance form when dealing with some types of microwave circuits. (Microwaves are signals at frequencies between about 1GHz and 100GHz).

Consider again the circuit of figure A2.1. The combination of L and C could be represented by a reactance X given by

$$X = \omega L - \frac{1}{\omega C} \ .$$

According to case **2)**,

$$Q = \frac{\omega}{2R} \frac{dX}{d\,\omega}$$

$$\text{so} \qquad Q = \frac{\omega}{2R} \frac{d\left(\omega L - \frac{1}{\omega C}\right)}{d\,\omega} = \frac{\omega_r}{2R}\left(L + \frac{1}{\omega_r^2 C}\right) \text{ at resonance.}$$

Thus $Q = \dfrac{\omega_r}{2R}(L + L) = \dfrac{\omega_r L}{R}$ which once again agrees with equation **(3.6)**.