CONVOLUTION

Convolution of Discrete-time (DT) signals

Any DT signal can be constructed using a sequence of DT unit impulses.

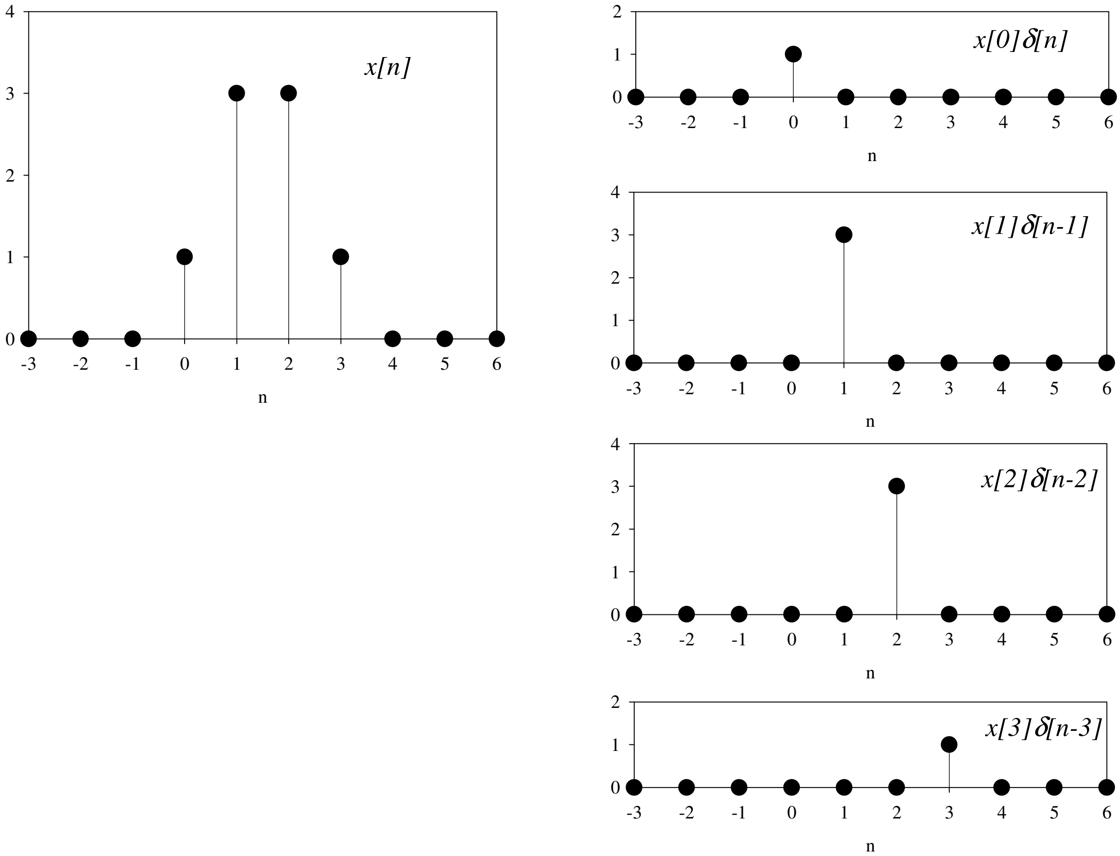


Figure 1: Decomposition of a DT signal x[n] into a weighted sum of shifted impulses.

Figure 1 shows that a DT signal $x[n]$ can be decomposed into;

$$x[n] = \ldots + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k].$$

This corresponds to a representation of an arbitrary sequence as a linear combination of shifted impulses $\delta[n-k]$ with amplitudes or weights of $x[n]$. Since $\delta[n-k]$ is non-zero only when $n = k$, $\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ ***sifts*** through the sequence $x[n]$ and preserves only the value when $n = k$. This property is therefore known as the ***sifting property*** of DT unit impulse.

We have shown that a DT signal $x[n]$ can be represented as a superposition of scaled versions of shifted impulse $\delta[n-k]$. We shall now show that it is possible to compute the LTI system response to any input if the impulse response is known. Let the impulse response of a LTI system be $h[n]$.

We have

input		**response**	
$\delta[n]$	$\rightarrow$	$h[n]$	(definition),
$\delta[n-k]$	$\rightarrow$	$h[n-k]$	(time shifting),
$x[k]\delta[n-k]$	$\rightarrow$	$x[k]h[n-k]$	(homogeneity),
$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$	$\rightarrow$	$\sum_{k=-\infty}^{\infty}x[k]h[n-k]$	(additivity).

Thus, the response of the LTI system to an input $x[k]$ is

$$y[n] = \sum_{k=-\infty}^{\infty}x[k]h[n-k].$$

This result is referred to as the ***convolution sum*** and the operation on the right hand side is called the ***discrete convolution*** of the sequences $x[n]$ and $h[n]$, which is usually represented symbolically as

$$y[n] = x[n] * h[n].$$

Example:
Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated in figure 2.

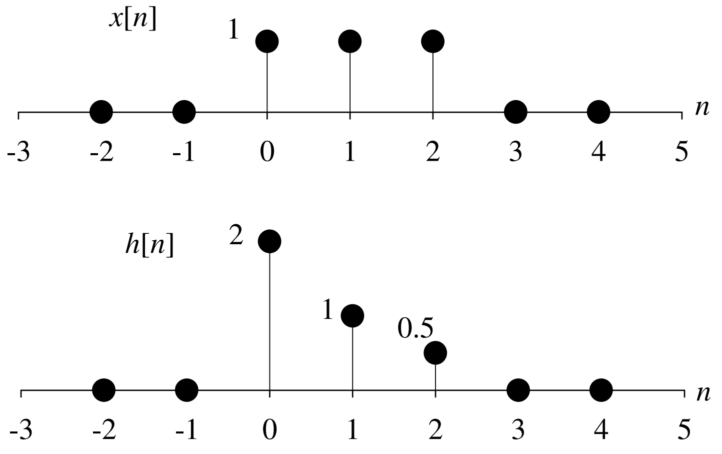


Figure 2: Input x[n] and impulse response of an LTI system h[n].

The procedures to compute $y[n]$ are:

1. Replace the variable n with k.
2. Flipping $h[k]$ with respect to $k = 0$ to obtain $h[-k]$.
3. Shifting $h[-k]$ to n to give $h[n-k]$.
4. Multiply $h[n-k]$ and $x[k]$ for all k.
5. Summing all non-zero product of $h[n-k]x[k]$ to yield $y[n]$.

The procedures can be outlined in a table form as:

	k	-2	-1	0	1	2	3	4	$\Sigma h[n-k]x[k]$
	$x[k]$	0	0	1	1	1	0	0	
	$h[k]$	0	0	2	1	0.5	0	0	
$n = 0$	$h[-k]$								
$n = 1$	$h[1-k]$								
$n = 2$	$h[2-k]$								
$n = 3$	$h[3-k]$								
$n = 4$	$h[4-k]$								

$y[n] = x[n] * h[n]$ is sketched in figure 3

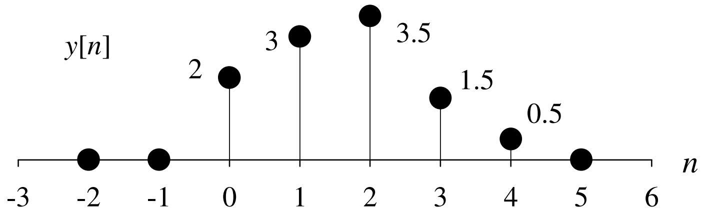


Figure 3: Computed output y[n].

We can also perform this discreet time convolution graphically.

Step 1

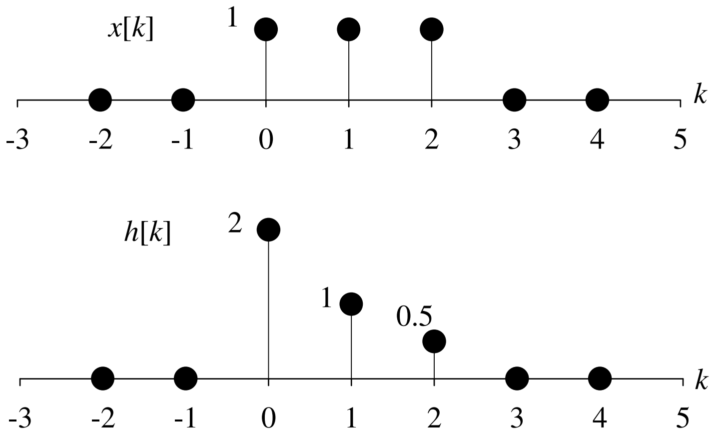


Figure 4: Replacing variable n with k.

Step 2

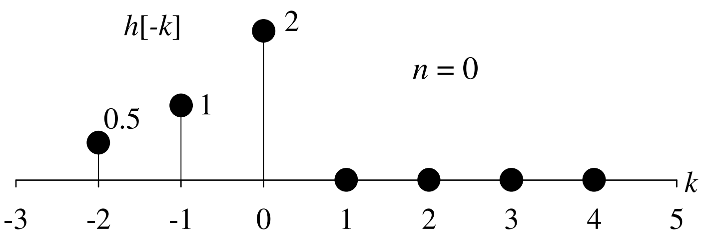


Figure 5: Flipping h[k] with respect to k = 0 to obtain h[-k].

Step 3

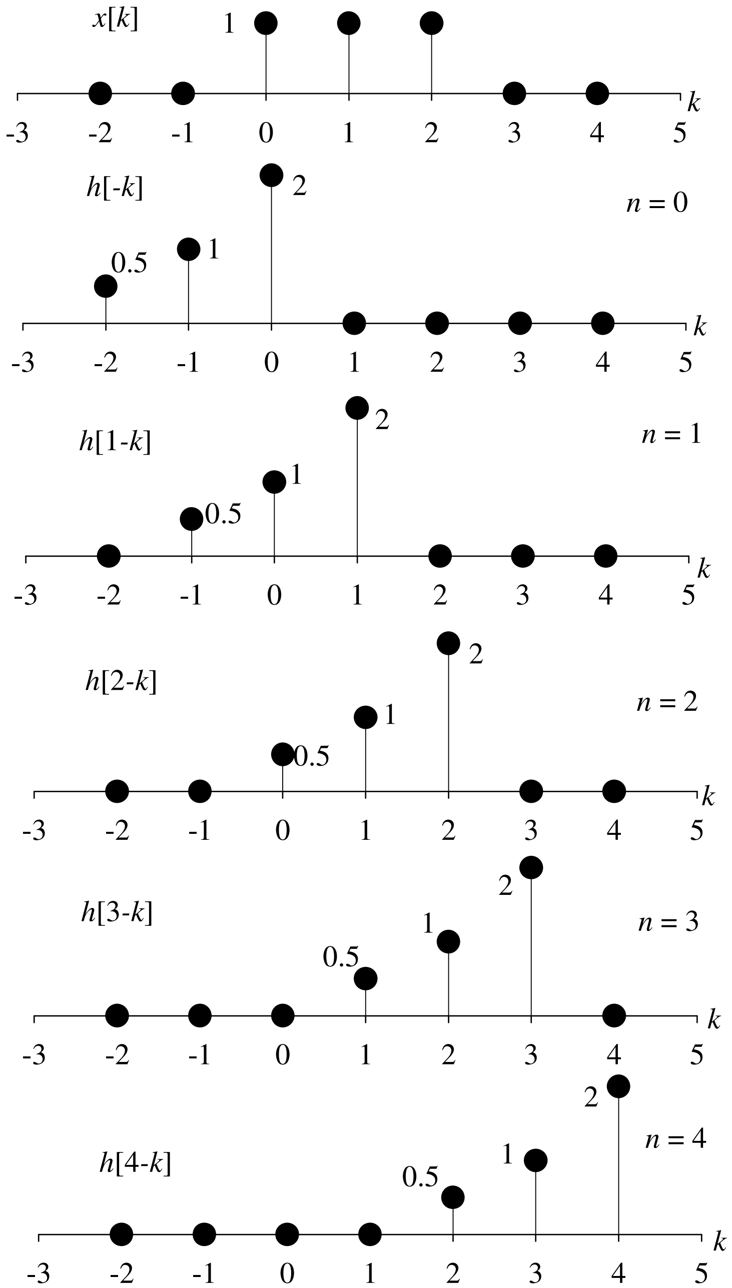


Figure 6: Shifting h[-k] to n to give h[n-k].

Step 4 and Step 5

Find the products $h[n\text{-}k]x[k]$ for all k and summing all the non-zero products.

$n = 0$: $y[0] = h[0]x[0] = 2\times1 = 2$

$n = 1$: $y[1] = h[1]x[0] + h[0]x[1] = (1\times1) + (2\times1) = 3$

$n = 2$: $y[2] = h[2]x[0] + h[1]x[1] + h[0]x[2] = (0.5\times1) + (1\times1) + (2\times1) = 3.5$

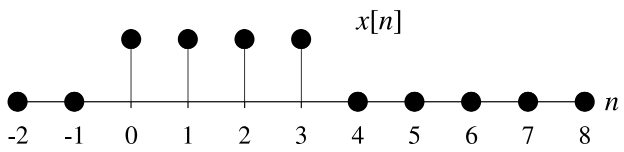
$n = 3$: $y[3] = h[2]x[1] + h[1]x[2] = (0.5\times1) + (1\times1) = 1.5$

$n = 4$: $y[4] = h[3]x[2] = (0.5\times1) = 0.5$

More examples:

1. Consider the two sequences:

$$x[n] = \begin{cases} 1, & 0 \le n \le 3 \\ 0, & otherwise \end{cases} \quad \text{and} \quad h[n] = \begin{cases} n, & 0 \le n \le 4 \\ 0, & otherwise \end{cases}.$$



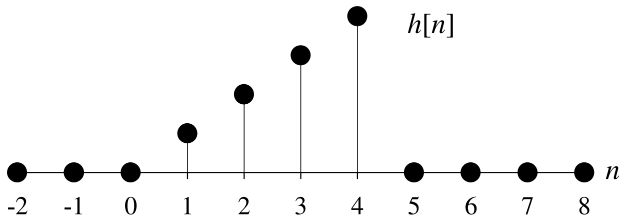


Figure 7: The signals x[n] and h[n] to be convolved.

Using a table to perform the discreet time convolution gives

	k	-2	-1	0	1	2	3	4	5	6	7
	$x[k]$	0	0	1	1	1	1	0	0	0	0
	$h[k]$	0	0	0	1	2	3	4	0	0	0
n=0	$h[-k]$										
n=1	$h[1-k]$										
n=2	$h[2-k]$										
n=3	$h[3-k]$										
n=4	$h[4-k]$										
n=5	$h[5-k]$										
n=6	$h[6-k]$										
n=7	$h[7-k]$										

	$\Sigma h[n-k]x[k]$
n=0	
n=1	
n=2	
n=3	
n=4	
n=5	
n=6	
n=7	

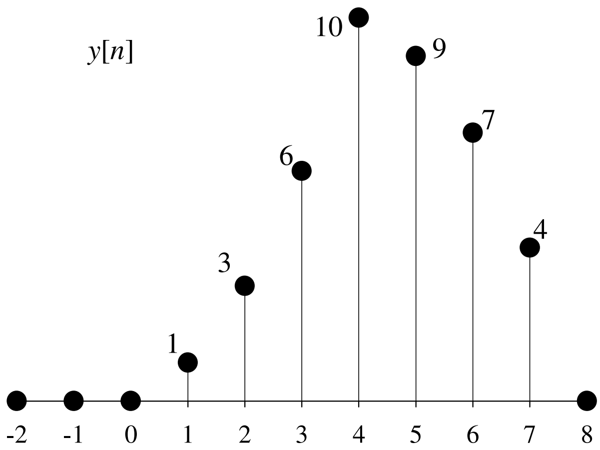
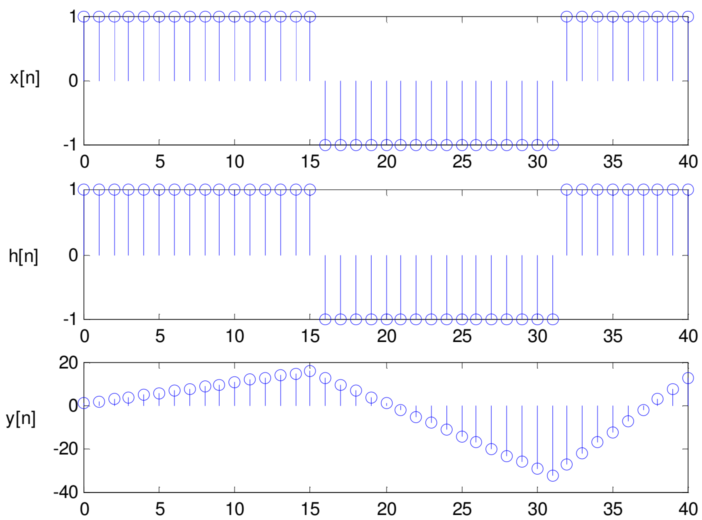
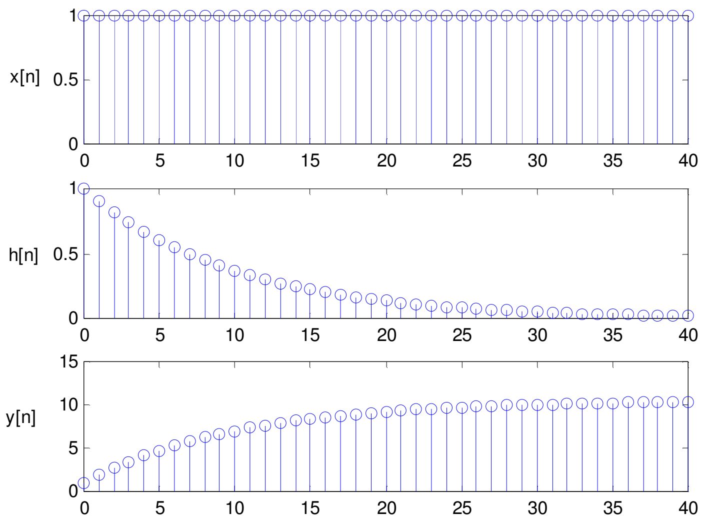


Figure 8: Computed response y[n].

2. Convolution of two square waveforms.



3. Convolution of an exponential impulse response $h[n] = \alpha^n u[n]$ with $\alpha = 0.95$ and a unit step input $x[n]$ $u[n]$.



Convolution of Continuous-time (CT) signals

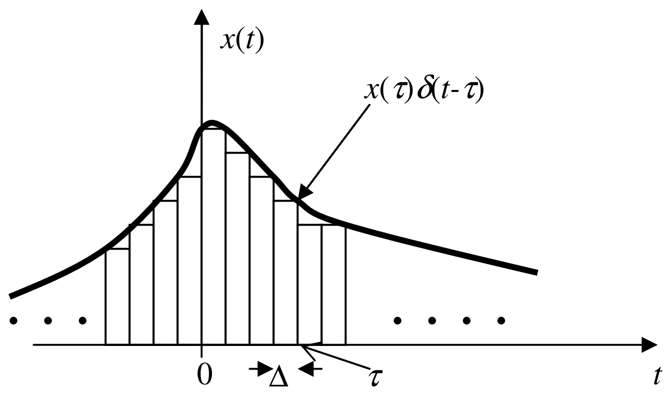


Figure 9: Staircase approximation to a CT signal x(t).

Any CT signal can be approximated by a combination of delayed impulses as illustrated in figure 9 if the impulse is defined as

$$\delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases},$$

where $\Delta \to 0$. Using the sifting property of impulse the signal $x(t)$ can be represented as $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$. If the impulse response of an LTI system is $h(t)$ we have

input		**response**	
$\delta(\tau)$	$\rightarrow$	$h(\tau)$	(definition),
$\delta(t-\tau)$	$\rightarrow$	$h(t-\tau)$	(time shifting),
$x(\tau)\delta(t-\tau)$	$\rightarrow$	$x(\tau)h(t-\tau)$	(homogeneity),

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \rightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \quad \text{(additivity).}$$

Thus, the response of the LTI system to an input $x(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau .$$

This equation is known as the ***convolution integral*** and the convolution of two signals will be represented symbolically as

$y(t) = x(t) * h(t)$.

The procedures for evaluating convolution in CT are very similar to those for in DT.

exercise: Let $h(t) = u(t)$ and $x(t) = e^{-at}u(t)$, $a > 0$. Evaluate $y(t) = h(t)*x(t)$.

1. Replacing the variable t with τ to yield $h(\tau)$ and $x(\tau) = e^{-a\tau}u(\tau)$.

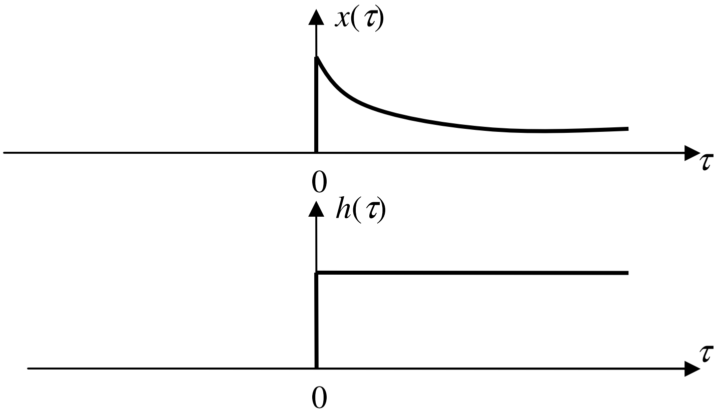


Figure 10: Changing the variable t to τ.

2. Flipping $h(\tau)$ with respect to $\tau = 0$ to obtain $h(-\tau)$.

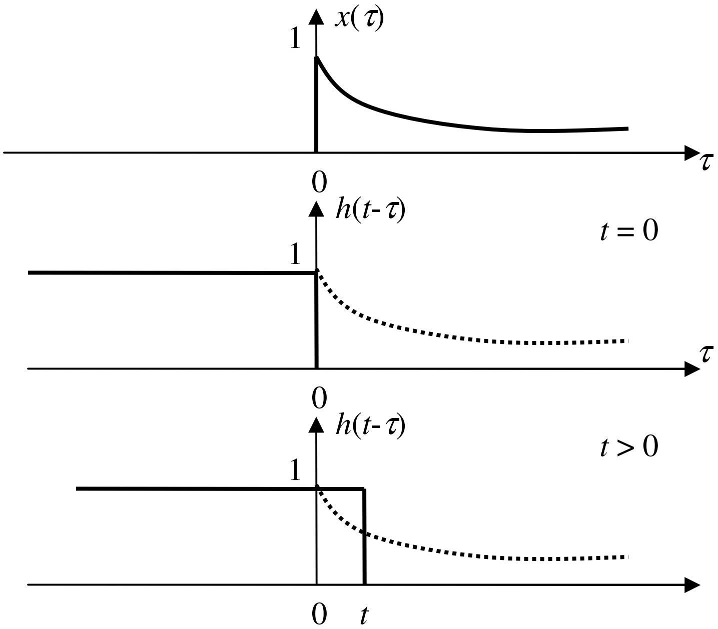


Figure 11: Flipping h(τ) with respect to τ = 0 and shifting h(-τ) by t to obtain h(t-τ).

3. Shift $h(\tau)$ along the τ-axis by t to give $h(t-\tau)$.
4. Multiply $x(\tau)$ and $h(t-\tau)$ for all τ. For t > 0,

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & otherwise \end{cases}.$$

5. Integrate $x(\tau)h(t-\tau)$ to yield

$$y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t e^{-a\tau}d\tau = -\frac{1}{a}(e^{-at} - e^{-0}) = \frac{1}{a}(1-e^{-at}).$$

For all t, the response is

$$y(t) = \frac{1}{a}(1-e^{-at})u(t).$$

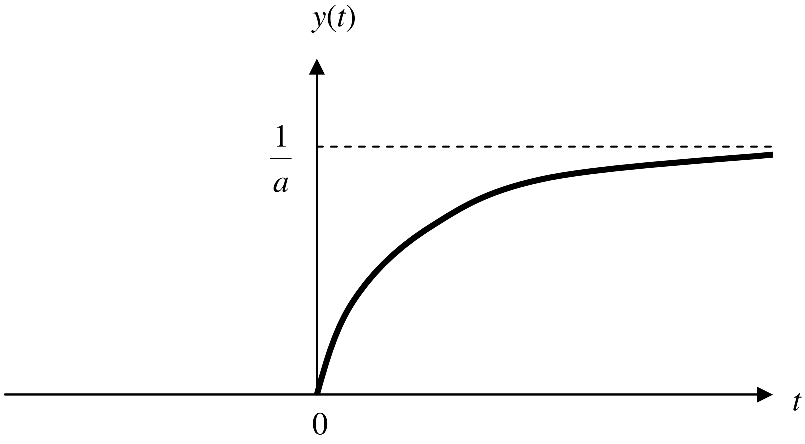


Figure 12: Computed response y(t).

More examples:
1. Consider the input signal $x(t)$ and impulse $h(t)$ illustrated in figure 13.

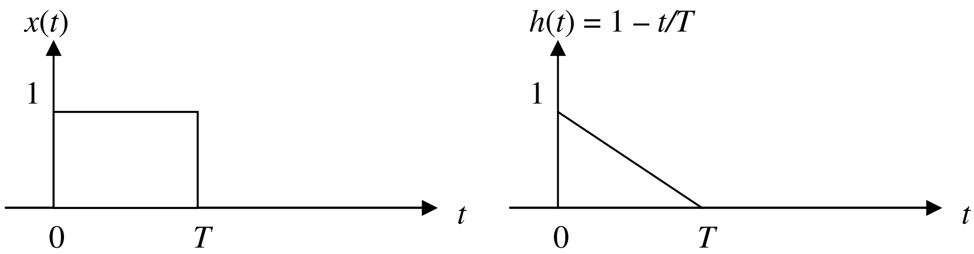


Figure 13: Input signal x(t) and impulse h(t).
Consider the following intervals:

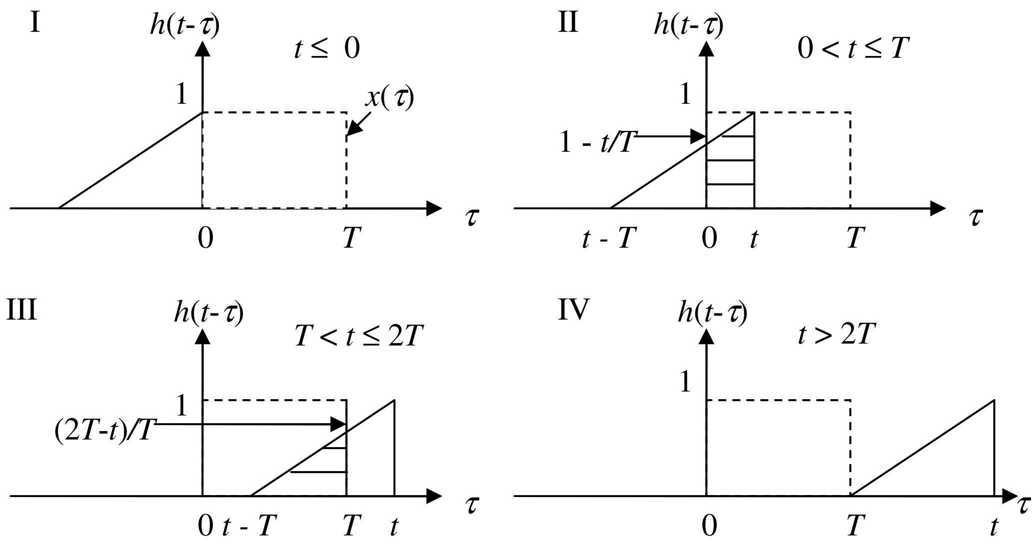


Figure 14: Signals $x(\tau)$ and $h(t\text{-}\tau)$ for different intervals.

The signals are $x(t)=\begin{cases}1, & 0<t<T\\ 0, & otherwise\end{cases}$ and $h(t)=\begin{cases}1-\dfrac{t}{T}, & 0<t<T\\ 0, & otherwise\end{cases}$. In order to evaluate the convolution it is convenient to consider four separate intervals for t.

Interval I: For $t \le 0$, $x(\tau)h(t\text{-}\tau) = 0$, hence $y(t) = 0$.

Interval II: For $0 < t \le T$, $x(\tau)h(t-\tau)=\begin{cases}1-\dfrac{(t-\tau)}{T}, & 0<\tau\le t\\ 0, & otherwise\end{cases}$.

Hence $y(t)=\int_0^t\left(1-\dfrac{(t-\tau)}{T}\right)d\tau$ = overlapping area in figure 14 II.

$$y(t)=\frac{1}{2}t\left(1+1-\frac{t}{T}\right)=t-\frac{t^2}{2T}.$$

Interval III: For $T < t \le 2T$, $x(\tau)h(t-\tau)=\begin{cases}1-\dfrac{(t-\tau)}{T}, & t-T<\tau\le T\\ 0, & otherwise\end{cases}$.

Hence $y(t)=\int_{t-T}^{T}\left(1-\dfrac{(t-\tau)}{T}\right)d\tau$ = overlapping area in figure 14 III.

$$y(t)=\frac{1}{2}\left(T-(t-T)\right)\left((2T-t)/T\right)=\frac{1}{2}(2T-t)(2T-t)/T=\frac{1}{2T}(2T-t)^2$$

Interval IV: For $t > 2T$, $x(\tau)h(t\text{-}\tau) = 0$, hence $y(t) = 0$.

In summary we have $y(t)=\begin{cases}0, & t\le 0\\ t-\dfrac{t^2}{2T} & 0<t\le T\\ \dfrac{1}{2T}(2T-t)^2 & T<t\le 2T\\ 0, & t>2T\end{cases}$.

Here we have evaluated the convolution $y(t) = x(t)*h(t)$. We will now show that convolution is a commutative operation, i.e $x(t)*h(t) = h(t)*x(t)$.
As before consider the following intervals:

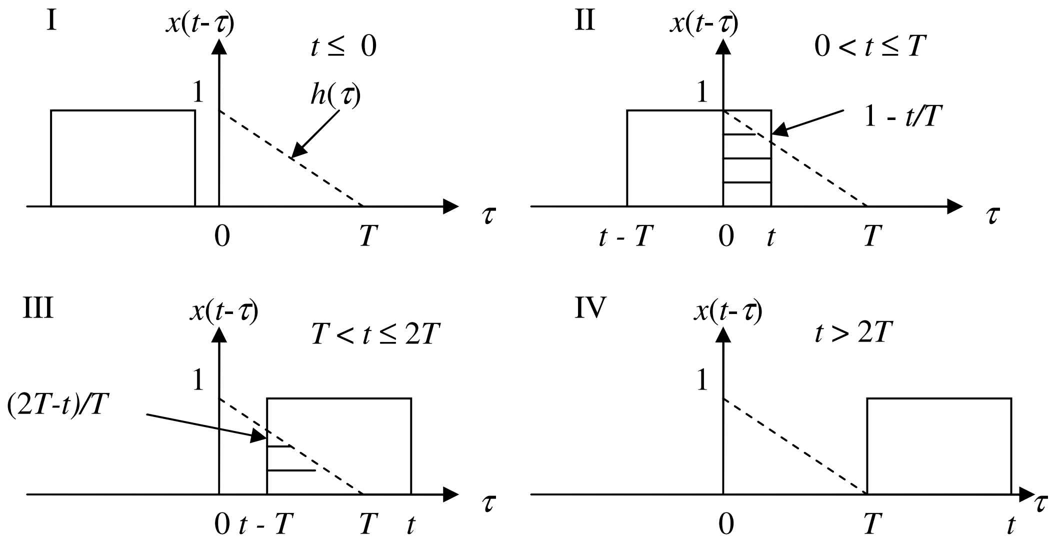


Figure 15: Signals h(τ) and x(t-τ) for different intervals.

Interval I: For $t \le 0$, $h(\tau)x(t-\tau) = 0$, hence $y(t) = 0$.

Interval II: For $0 < t \le T$, $h(\tau)x(t-\tau) = \begin{cases} 1-\frac{\tau}{T}, & 0 < \tau \le t \\ 0, & otherwise \end{cases}$.

Hence $y(t) = \int_0^t \left(1-\frac{\tau}{T}\right)d\tau$ = overlapping area in figure 15 II.

$$y(t) = \frac{1}{2}t\left(1+1-\frac{t}{T}\right) = t-\frac{t^2}{2T}.$$

Interval III: For $T < t \le 2T$, $h(\tau)x(t-\tau) = \begin{cases} 1-\frac{\tau}{T}, & t-T < \tau \le T \\ 0, & otherwise \end{cases}$.

Hence $y(t) = \int_{t-T}^{T} \left(1-\frac{\tau}{T}\right)d\tau$ = overlapping area in figure 15 III.

$$y(t) = \frac{1}{2}(T-(t-T))((2T-t)/T) = \frac{1}{2}(2T-t)(2T-t)/T = \frac{1}{2T}(2T-t)^2$$

Interval IV: For $t > 2T$, $h(\tau)x(t-\tau) = 0$, hence $y(t) = 0$.

In summary we have $y(t) = \begin{cases} 0, & t \le 0 \\ t-\frac{t^2}{2T} & 0 < t \le T \\ \frac{1}{2T}(2T-t)^2 & T < t \le 2T \\ 0, & t > 2T \end{cases}$ as before.

t	$T/4$	$T/2$	$3T/4$	T	$5T/4$	$3T/2$	$7T/4$	$2T$
$y(t)$	$7T/32$	$3T/8$	$15T/32$	$T/2$	$9T/32$	$T/8$	$T/32$	0

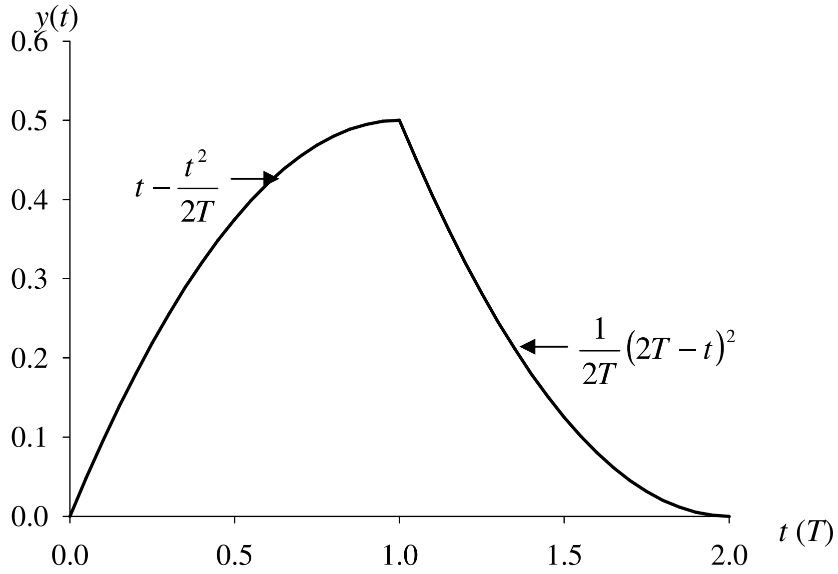
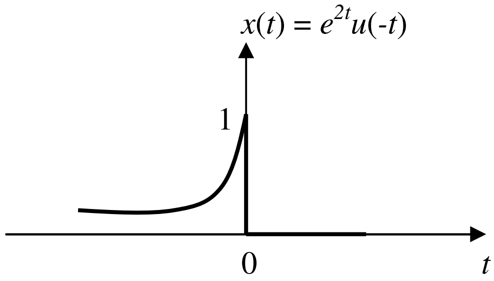
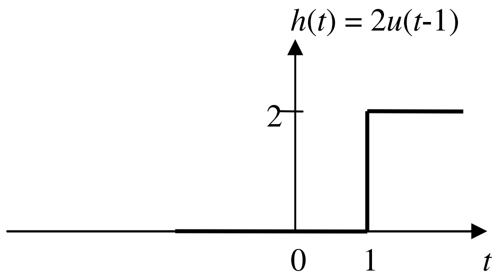
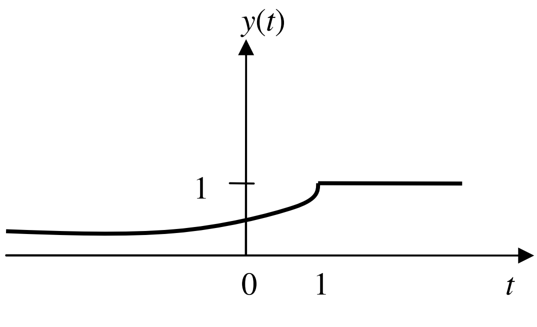


*Figure 16: Response y(t) = h(t)*x(t).*

2. $x(t) = e^{2t}u(-t)$ and $h(t) = 2u(t-1)$.







1. A basic property of convolution in both CT and DT is that it is a ***commutative*** operation. That is, in DT

$$x[n]*h[n] = h[n]*x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

and in CT

$$x(t)*h(t) = h(t)*x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau .$$

2. Another basic property of convolution is the ***distributive*** property. In DT

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n] + x[n]*h_2[n]$$

and in CT

$$x(t)*(h_1(t) + h_2(t)) = x(t)*h_1(t) + x(t)*h_2(t).$$

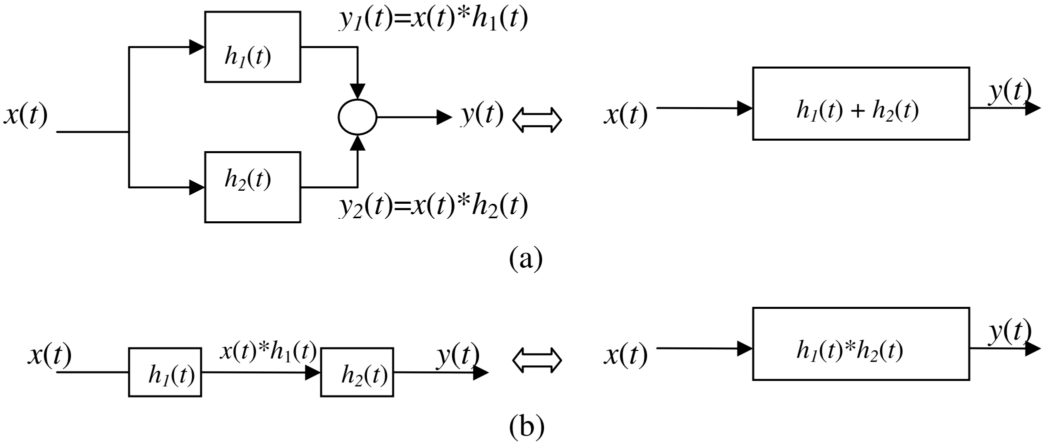


Figure 17: (a) Distributive property and (b) associative property of convolution process.

3. Another useful property of convolution is that it is ***associative***. In DT

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$$

and in CT

$$x(t)*(h_1(t)*h_2(t)) = (x(t)*h_1(t))*h_2(t).$$

Notes: