EEE331 Analogue Electronics

9th lecture:

- passive analogue filters
  - general filter specification
  - passive *LC* filters: only inductors & capacitors (difficult at low *f*)
  - leapfrog design principle
  - active RC filters: R, C & op-amps (thick or hybrid thin-film technol.)
  - switched capacitor filters: fully IC-compliant

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Filter transfer function: definition

definition of filter **transfer function**: $T(s)$=output signal /input signal $=V_o(s)/V_i(s)$
-> get transmission for physical frequencies f by setting $s=j\omega=2\pi jf$:

$$T(j\omega)=|T(j\omega)|\exp[j\phi(\omega)] \text{ where}$$

$G(\omega)=20\log|T(j\omega)|$ is the **gain** [dB],
$A(\omega)=-G(\omega)$ the **attenuation** [dB] and
$\phi(\omega)$ is the **phase** of transmission

-> $T(s)=(a_m s^m+a_{m-1}s^{m-1}+\ldots+a_o)/(s^n+b_{n-1}s^{n-1}+\ldots+b_o)$ is a ratio of 2 polynomials.
The degree of the denominator, n, is called the **filter order**.
-> condition for filter circuit to be stable: **m≤n is stability criterion**
The polynomials of nominator and denominator can be factorised:

$$\boxed{T(s)=[a_m(s-z_1)(s-z_2)\ldots(s-z_m)]\ /\ [(s-p_1)(s-p_2)\ldots(s-p_n)]}$$

zeros poles (or natural modes)

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Filter transfer function: symmetry

statement 1: zeros & poles can be real or complex, but the latter must always occur as **conjugate pairs** to make the expression in the bracket vanish

proof: assume p is a pole, i.e. $(s-p)=0$. **Write $p=\sigma+j\omega$** with real σ and real ω.

then:

$p^*=\sigma-j\omega$ and

$$0=(s-p)(s-p^*)$$
$$=s^2-sp-sp^*+pp^*$$
$$=s^2-s(\sigma+j\omega)-s(\sigma-j\omega)+(\sigma+j\omega)(\sigma-j\omega)$$
$$=s^2-s\sigma\cancel{-js\omega}-s\sigma\cancel{+js\omega}+\sigma^2\cancel{-j\sigma\omega}\cancel{+j\sigma\omega}+\omega^2$$
$$=(s-\sigma)^2+\omega^2$$

implies that

$\omega^2=-(s-\sigma)^2$,

hence $\omega=\pm j(s-\sigma)$,

i.e. the absolute sign of ω is not fixed,

i.e. ***p** must also be a pole**.

NB: The same reasoning ca be applied to the zeros.

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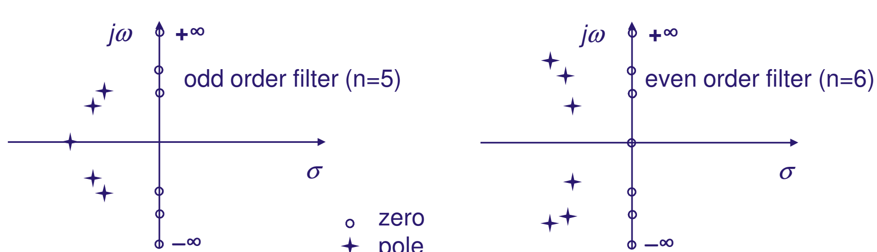
Filter transfer function: symmetry

statement 2: **all poles must lie in the left half of the s-plane** for a stable filter

proof: We have just seen that with $p=\sigma+j\omega$ also $p=\sigma-j\omega$ is a pole solution. This means complex poles always occur in conjugate pairs and we can factorise the denominator.

The quadratic equation $s^2+as+b=0$ has the known solution $s=-a/2\pm(a^2/4-b)^{1/2}$.

Hence $s=-\sigma\pm j\omega$ is the solution to $(s+p)(s+p^*)=s^2+2\sigma s+(\sigma^2+\omega^2)=0$, so the real part of the solution always has the opposite sign of the terms in the brackets.



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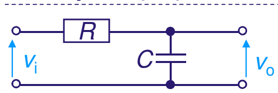
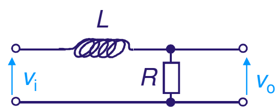
1^{st} order filters: low and high-pass filters

Low pass (LP)



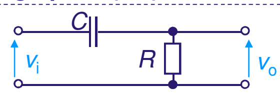
$T(s)=v_o/v_i=(1/sC)/[R+1/(sC)]=1/(1+sCR)$



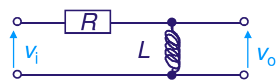
$T(s)=v_o/v_i=R/[R+sL)=1/(1+sL/R)$

general form of LP: $T(s)=1/(1+s/\omega_p)$

High pass (HP)



$T(s)=v_o/v_i=R/[R+1/(sC)]=sCR/(1+sCR)$



$T(s)=v_o/v_i=sL/[R+sL)]=(sL/R)/(1+sL/R)$

general form of HP: $T(s)=(s/\omega_p)/(1+s/\omega_p)$

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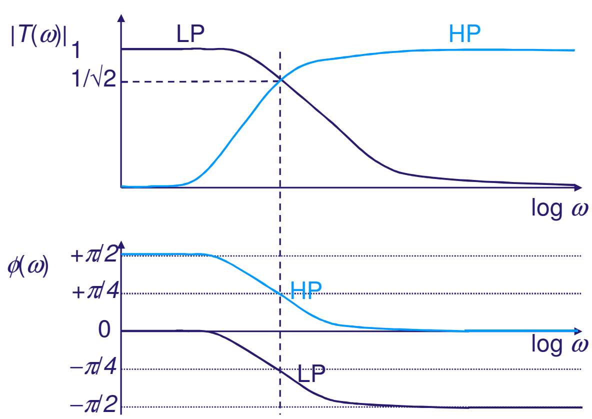
1^{st} order filters: low and high-pass filters



Note: A transfer function can be made up of *RC* or *LR* combinations. While the frequency behaviour may be identical, the impedances to the source and output impedances are different.

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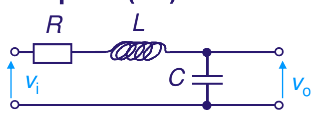
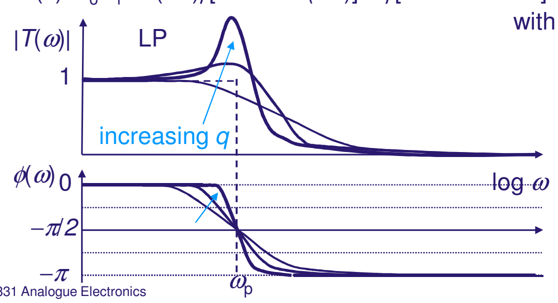
2nd order filters: low, high, band-pass and band stop (notch) filters

Low pass (LP)



$T(s)=v_o/v_i=1/(sC)/[R+sL+1/(sC)]=1/[1+sCR+s^2LC]=\boxed{1/[1+s/(\omega_p q)+s^2/\omega_p^2]}$

with $\boxed{\omega_p=(LC)^{-1/2} \text{ and } q=1/(\omega_p CR)}$



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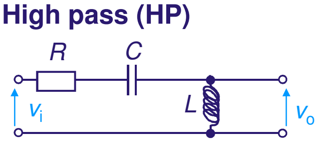
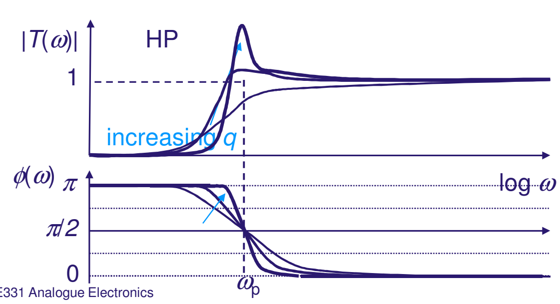
2nd order filters: low, high, band-pass and band stop (notch) filters

High pass (HP)



$T(s)=v_o/v_i=sL/[R+sL+1/(sC)]=(s^2LC)/[1+sCR+s^2LC]=\boxed{(s^2/\omega_p^2)/[1+s/(\omega_p q)+s^2/\omega_p^2]}$



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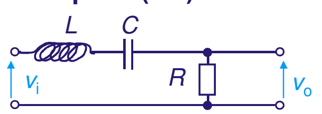
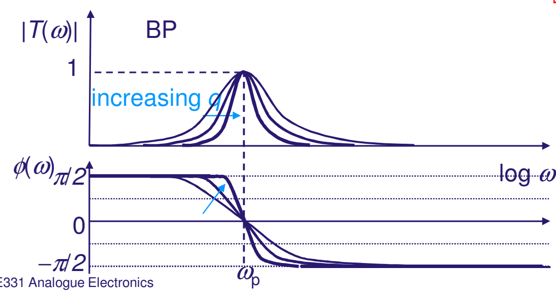
2^{nd} order filters: low, high, band-pass and band stop (notch) filters

Band pass (BP)



$T(s)=v_o/v_i=R/[R+sL+1/(sC)]=sCR/[1+sCR+s^2LC]=\boxed{s/(\omega_p q)/[1+s/(\omega_p q)+s^2/\omega_p^2]}$



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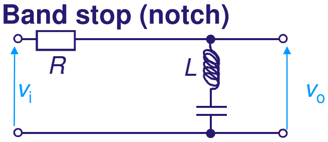
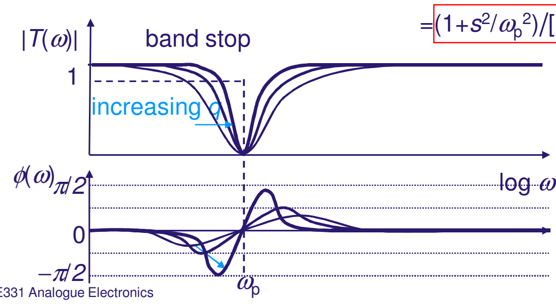
2^{nd} order filters: low, high, band-pass and band stop (notch) filters

Band stop (notch)



$T(s)=v_o/v_i=[sL+1/(sC)]/[R+sL+1/(sC)]=(1+s^2LC)/[1+sCR+s^2LC]$

$=\boxed{(1+s^2/\omega_p^2)/[1+s/(\omega_p q)+s^2/\omega_p^2]}$



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2^{nd} order filters: the standard from

All 2^{nd} order filters are biquadratic transfer function of the standard form

$T(s)$= numerator/ $(1+s\tau/q+s^2\tau^2)$

where the **numerator** decides what type of filter we have:

1	= low pass
$s\tau$	= band pass
$s^2\tau^2$	= high pass
$1+s^2\tau^2$	= band stop or notch

and the **denominator** decides the

time constant: $\tau=(LC)^{1/2}$,

pole frequency: $\omega_0=1/\tau=(LC)^{-1/2}$ and

quality factor: $q=1/(\omega_0 RC)=\tau/(RC)=(LC)^{1/2}/(RC)=(L/C)^{1/2}/R=\omega_0 L/R$

It is important to get the second order transfer function into a standard form, as this allows us to define the important parameters and decide which type of filter we have. Other standard notations (for the example of the low pass filter version) are:

$T(s)=\omega_0^2/(\omega_0^2+s\omega_0/q+s^2)=\omega_0^2/(\omega_0^2+2\xi\omega_0 s+s^2)$

with a damping factor $\xi=1/(2q)$

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2^{nd} order filters: the meaning of the quality factor q

The **quality factor** $q=1/(\omega_0 RC)$ of a filter has several meanings:

a) It is a measure of the magnification of the signal amplitude at resonance:
Consider series connection of L, C and R. Then $X_C=1/(j\omega C)$ and $X_L=j\omega L$.
At resonance: $\omega=\omega_0$. Then: $q=1/(\omega_0 RC)=jX_C/R=(i|X_C|)/(iR)=|v_C|/|v_R|$ $(=|v_L|/|v_R|)$

b) It is the product of resonance frequency, stored energy and rate of energy dissipation:
Consider a current $i=i_p\sin\omega_0 t$ *at resonance frequency* $\omega_0=(LC)^{-1/2}$
The instantaneous energy stored in L then is: $W_L=\frac{1}{2}Li^2=i_p^2\sin^2\omega_0 t/(2\omega_0^2 C)$
The voltage across C is given by: $V=Q/C=1/C\int i\,dt=-1/(\omega_0 C)\, i_p\cos\omega_0 t$
The instantaneous energy stored in C is: $W_C=\frac{1}{2}CV^2=i_p^2\cos^2\omega_0 t/(2\omega_0^2 C)$
The total stored energy thus is the sum: $W=i_p^2/(2\omega_0^2 C)\ \cancel{(\sin^2\omega_0 t+\cos^2\omega_0 t)}$
The rate of energy dissipation is: $P=VI=Ri^2=\frac{1}{2}Ri_p^2$
Thus $\omega_0 W/P=q$, i.e. q increases with the stored energy.

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2nd order filters: the meaning of the quality factor *q*

The **quality factor** $q=1/(\omega_0 RC)$ of a filter has several meanings:

c) It measures the rate change of reactance:
Consider magnitude of reactance: $|X|=\omega L-1/(\omega C)$
Differenttaion yields: $dX/d\omega=L+1/(\omega^2 C)$
At resonance $\omega=\omega_0=(LC)^{-1/2}$: $dX/d\omega|_{at\ \omega=\omega_0}=1/(\omega_0^2 C)+1/(\omega_0^2 C)=2/(\omega_0^2 C)$
Thus $q=1/(\omega_0 CR)=\omega_0/(2R)\times 2/(\omega_0^2 C)=\omega_0/(2R)\ dX/d\omega|_{at\ \omega=\omega_0}$
This means that q increases with the rate of change of reactance.

d) It measures the stability of the system:
$\lim_{q\to\infty} T(s)=$ numerator$/(1+s\tau/q+s^2\tau^2)$ = numerator $/(1+s^2\tau^2)$ describes a simple harmonic oscillator without damping (equivalent to unity gain at $\phi=180º$)

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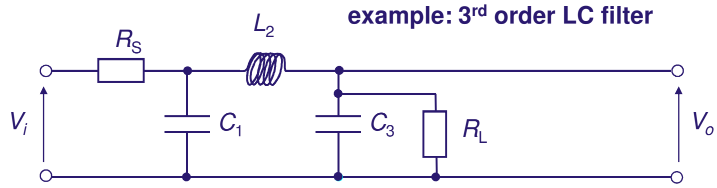
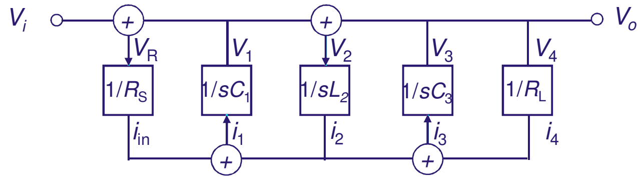
Leap-frog ladder design

example: 3rd order LC filter





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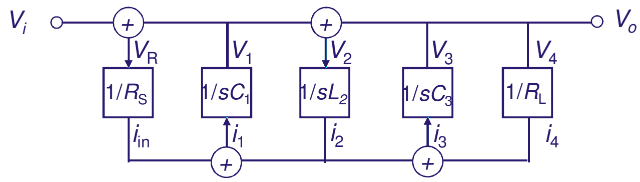
Leap-frog ladder design

With input and output resistors there are **5 components**. For each of the 5 components we can write down two equations, one from Ohm's Law and one from Kirchhoff's Law. Hence, there are **10 equations** altogether to describe the interdependence of 6 voltages V_i, V_R, $V_{1\text{-}3}$ and $V_4=V_o$ and 5 currents i_{in}, $i_{1\text{-}3}$ and $i_4= i_{out}$. **So, if input voltage V_i and a frequency ω are given, it is possible to solve the system of 10 equations with 10 unknowns and calculate the output voltage V_o.**

(i) $i_{in}=V_R/R_S$; $V_R=V_i-V_1$
(ii) $V_1=i_1/(sC_1)$; $i_1=i_{in}-i_2$
(iii) $i_2=V_2/(sL_2)$; $V_2=V_1-V_3$
(iv) $V_3=i_3/(sC_3)$; $i_3=i_2-i_4$
(v) $i_4\equiv i_{out}=V_4/R_L$; $V_4\equiv V_o=V_3$

Due to the alternating addition (or subtraction) of voltages and currents this is called a **leap-frog topology**. A computer program can then simulate $T(j\omega)$ over the whole frequency range.



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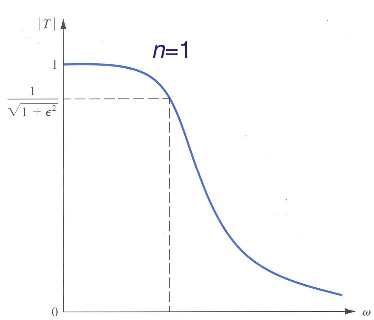
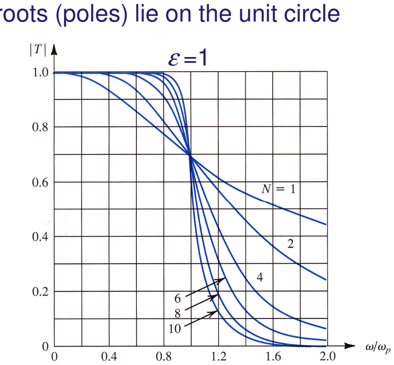
Butterworth filters

Butterworth filter: $\boxed{T(s)=1/[1+\varepsilon^2(\omega/\omega_0)^{2n}]^{½}}$

characteristics: all pole filter with low-pass characteristic (**monotonically decreasing with maximally-flat magnitude response**); $T(j\omega_0)$ is down –3dB; all roots (poles) lie on the unit circle





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normalised Butterworth polynomials

order n	denominator $B_n(s)$ (normalised to ω_0=1 rad/s)
1	$(s+1)$
2	$(s^2+1.414s+1)$
3	$(s+1)\ (s^2+s+1)$
4	$(s^2+0.765s+1)\ (s^2+1.848s+1)$
5	$(s+1)\ (s^2+0.618s+1)\ (s^2+1.618s+1)$
6	$(s^2+0.518s+1)\ (s^2+1.414s+1)\ (s^2+1.932s+1)$
7	$(s+1)\ (s^2+0.445s+1)\ (s^2+1.247s+1)\ (s^2+1.802s+1)$
8	$(s^2+0.390s+1)\ (s^2+1.111s+1)\ (s^2+1.663s+1)\ (s^2+1.962s+1)$

example: want a low-pass Butterworth filter with –40dB attenuation of $T(s)=1/B(s)$ at ω/ω_0=2:

$20 \log \{[1/(1+2^{2n})]^{½}\}=-40$ gives $2^{2n}=10^4-1$, i.e. n=6.64≈7

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Chebyshev filters

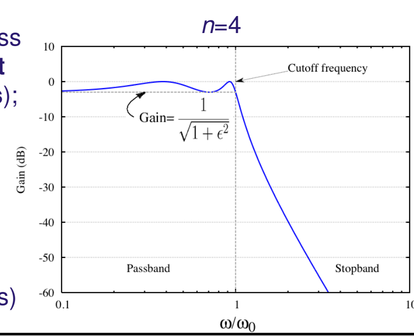
Chebyshev filter: $\boxed{T(s)=1/[1+\varepsilon^2 C_n^{\,2}(\omega/\omega_0)]^{½}}$

where the C_n are Chebyshev polynomials defined by

$$C_n(\omega/\omega_0)=\begin{cases}\cos(n\cos^{-1}\omega/\omega_0) & \text{for } 0\le\omega/\omega_0\le1\\ \cosh(n\cosh^{-1}\omega/\omega_0) & \text{for } \omega/\omega_0>1,\end{cases}$$

roots (poles) lie on an ellipse

characteristics: all pole filter with low-pass characteristic (**steep but with significant ripples**); $T(j\omega_0)$ is down –3dB; parameter ε is related to passband ripple γ in dB by $\varepsilon^2=10^{0.1\gamma}-1$ (ε=0.3493 for 0.5dB, ε=0.5089 for 1dB and ε=0.7648 for 2dB ripples)



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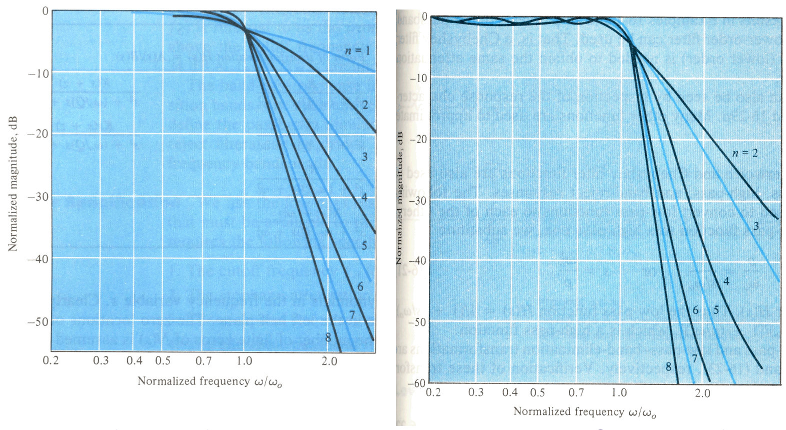
comparison of Butterworth and Chebyshev filter (on same scale)



Butterworth filters (left) are smoother and without ripples; Chebyshev filters (right) provide steeper edges and improved high-*f* suppression, for the same order *n*.