The University Of Sheffield.

**Data Provided:**
**Laplace and $z$-transforms**
**Compensator design formulae**
**Performance criteria mappings**
**Ziegler-Nichols tuning rules**

**LEAVE THIS EXAM PAPER ON YOUR DESK.**
**DO NOT REMOVE IT FROM THE HALL.**

**DEPARTMENT OF AUTOMATIC CONTROL & SYSTEMS ENGINEERING**
**Autumn Semester 2015–2016**

**ACS342 FEEDBACK SYSTEMS DESIGN** **2 hours**

**Answer THREE questions.**

**No marks will be awarded for solutions to a fourth question.**

**Solutions will be considered in the order that they are presented in the answer book. Trial answers will be ignored if they are clearly crossed out.**

**If more than the required number of questions are attempted, DRAW A LINE THROUGH THE ANSWERS THAT YOU DO NOT WISH TO BE MARKED.**

**All questions are marked out of 20. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.**

**Registration number from U-Card (9 digits) — to be completed by student**

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

1. **a)** Design a PI controller for a plant with the **open-loop** unit step response shown in Figure 1.1. **[6 marks]**

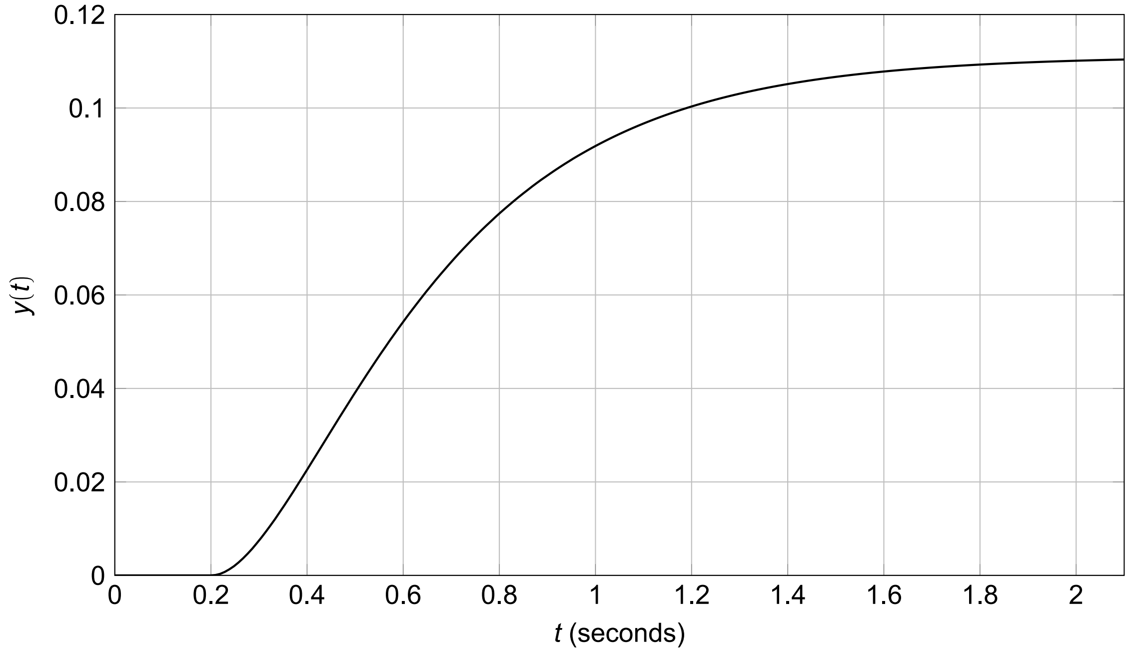


Figure 1.1

**b)** Show that a PI controller is a special case of a phase-lag compensator. Express the locations of the pole and zero of the compensator in terms of the gains $K_P$ and $K_I$. **[4 marks]**

**c)** The plant in part (a) is to be controlled by a digital control system, using a discrete-time (sampled data) version of the PI controller.

(i) Explain why the continuous-time PI controller

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\, d\tau \tag{1.1}$$

must be converted to a discrete-time representation in order to be suitable for use in a digital control system. **[2 marks]**

(ii) Derive the difference equation of the discrete-time PI controller (in terms of $K_P$ and $K_I$) by discretizing the controller in equation (1.1). Assume a sampling period of $T$ seconds. **[5 marks]**

(iii) Using the step response in Figure 1.1, select an appropriate sampling period, $T$, for the digital control system. Justify your answer. **[3 marks]**

**2.** A plant has transfer function

$$G(s) = \frac{20}{s(s+1)(s+10)}$$

**a)** Sketch the Bode diagram of $G(s)$. **[8 marks]**

**b)** Explain why the Bode diagram is usually drawn for the open-loop system rather than the closed-loop one, even though it is the stability and performance of the closed-loop system that is most important. **[2 marks]**

**c)** Calculate or estimate (from your Bode diagram) the gain margin of $G(s)$. **[4 marks]**

**d)** Complete the design of a phase-lead compensator

$$C(s) = K\frac{s\alpha\tau + 1}{s\tau + 1} \text{ where } K = 1$$

in order to increase the system phase margin by 30 degrees. You may use the values in Table 2.1 to guide your design. **[6 marks]**

Table 2.1

| $\omega$ [rad $s^{-1}$] | 0.85 | 1.00 | 1.24 | 1.53 | 1.74 |
|---|---|---|---|---|---|
| $20\log_{10}\lvert G(j\omega)\rvert$ [dB] | $+5.0$ | $+3.0$ | 0.0 | $-3.0$ | $-5.0$ |

**3.** A vehicle's active suspension system is modelled by the *quarter-car* model

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = f$$

where $y$ is the vertical displacement of the car body from its equilibrium position (m), $m$ is the mass of the quarter-car body (kg), $c$ is the damping coefficient ($\mathrm{N\,m^{-1}\,s^{-1}}$), and $k$ is the spring stiffness ($\mathrm{N\,m^{-1}}$). The system includes a hydraulic actuator, which exerts a force $f$ (N) in order to actively control the suspension.

**a)** (i) Derive the open-loop transfer function, $G(s)$, between control input $f$ and displacement $y$. **[3 marks]**

(ii) Obtain expressions for the damping ratio $\zeta$ and natural frequency $\omega_n$ in terms of the model parameters $m$, $c$ and $k$. **[3 marks]**

(iii) Given $m = 250\,\mathrm{kg}$, $c = 1000\,\mathrm{N\,m^{-1}\,s^{-1}}$, $k = 15\,000\,\mathrm{N\,m^{-1}}$, show that the open-loop suspension system is under-damped and estimate the 2% settling time. **[3 marks]**

(iv) The suspension system has an adjustable damper. For what value of damping coefficient, $c$, would the system be critically damped ($\zeta = 1$)? What happens to the settling time in this case? **[3 marks]**

**b)** The closed-loop active suspension system is shown in Figure 3.1. $D(s)$ represents a disturbance force caused by the road surface. The controller provides a hydraulic force proportional to the measured displacement $y(t)$, with the aim of smoothly rejecting disturbances and keeping the displacement small.

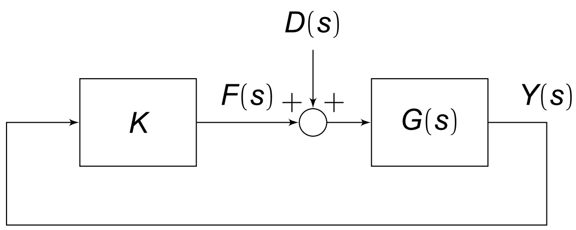


Figure 3.1

(i) Why is a reference input unnecessary for this system? **[1 mark]**

(ii) Derive the closed-loop transfer function between disturbance $D(s)$ and displacement $Y(s)$, expressed in terms of the parameters $m$, $c$ and $k$. **[4 marks]**

(iii) Determine the combination of damping coefficient $c$ and controller gain $K$ that makes the closed-loop system critically damped ($\zeta = 1$) with a settling time of 2 seconds. **[3 marks]**

**4.** A unity feedback system has open-loop transfer function

$$KG(s) = \frac{K}{(s-0.5)(s^2+4s+4)}$$

**a)** Explain why it is open-loop unstable. **[1 mark]**

**b)** Find the range of $K$ for which the closed-loop system is stable. **[5 marks]**

**c)** Determine what is the largest possible value of position error constant $K_p$ – and the corresponding steady-state error $e_{ss}^{step}$ (%) – that maintains closed-loop stability. **[3 marks]**

**d)** Sketch the root locus diagram. (You may use the fact that there is a single break-away point, located at $s \approx -0.35$.) **[8 marks]**

**e)** Explain, with reference to your root locus sketch, why using a phase-lag compensator to improve steady-state performance is problematic here. How else might steady-state performance be improved? **[3 marks]**

## Laplace and $z$-transforms

| Time domain | $s$-domain | $z$-domain |
|---|---|---|
| $f(t)$ | $F(s)$ | $F(z)$ |
| $f(t-T)$ | $e^{-sT}F(s)$ | $z^{-1}F(z)$ |
| $\delta(t)$ | $1$ | – |
| $1$ | $\frac{1}{s}$ | $\frac{z}{z-1}$ |
| $t$ | $\frac{1}{s^2}$ | $\frac{zT}{(z-1)^2}$ |
| $e^{-at}$ | $\frac{1}{s+a}$ | $\frac{z}{z-e^{-aT}}$ |
| $te^{-at}$ | $\frac{1}{(s+a)^2}$ | $\frac{zTe^{-aT}}{(z-e^{-aT})^2}$ |
| $\sin(\omega t)$ | $\frac{\omega}{s^2+\omega^2}$ | $\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$ |
| $\cos(\omega t)$ | $\frac{s}{s^2+\omega^2}$ | $\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$ |
| $e^{-at}\sin(\omega t)$ | $\frac{\omega}{(s+a)^2+\omega^2}$ | $\frac{ze^{-aT}\sin(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$ |
| $e^{-at}\cos(\omega t)$ | $\frac{s+a}{(s+a)^2+\omega^2}$ | $\frac{z^2-ze^{-aT}\cos(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$ |
| $f^{(n)}(t)=\frac{d^n}{dt^n}f(t)$ | $s^nF(s)-s^{n-1}f(0)-\ldots-f^{n-1}(0)$ | Various forms |

## Compensator design formulae

| | |
|---|---|
| Transfer function | $\frac{s\alpha\tau+1}{s\tau+1}$ (lead) $\frac{s\tau+1}{s\alpha\tau+1}$ (lag) |
| Maximum phase lead/lag, $\phi_m$ | $\sin^{-1}\frac{\alpha-1}{\alpha+1}$ |
| Centre frequency, $\omega_m$ | $\frac{1}{\tau\sqrt{\alpha}}$ |

**Performance criteria mappings**

| | |
|---|---|
| 2% settling time, $T_s$ | $\dfrac{4}{\zeta\omega_n}$ |
| 10–90% rise time, $T_r$ | $\dfrac{2.16\zeta + 0.6}{\omega_n}$ for $0.3 \leqslant \zeta \leqslant 0.8$ |
| Percentage overshoot, P.O. | $100\exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$ |
| Peak time, $T_p$ | $\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ for $0 < \zeta < 1$ |
| Peak response, $M_p$ | $1 + \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$ |
| Resonant frequency, $\omega_r$ | $\omega_n\sqrt{1-2\zeta^2}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$ |
| Resonant peak magnitude, $M_{p\omega}$ | $\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$ |
| Phase margin, $\phi_{pm}$ | $100\zeta$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$ |
| Phase margin, $\phi_{pm}$ | $\tan^{-1}\left(\dfrac{2\zeta\omega_n}{\omega_c}\right)$ |
| Phase margin, $\phi_{pm}$ | $\tan^{-1}\left(\dfrac{8}{T_s\omega_c}\right)$ |
| Bandwidth, $\omega_B$ | $(1.85 - 1.19\zeta)\omega_n$ for $0.3 \leqslant \zeta \leqslant 0.8$ |

**Ziegler-Nichols tuning rules**

*First method* (*T* time constant; *L* delay time; *K* process gain)

| | $K_P$ | $T_I$ | $T_D$ |
|---|---|---|---|
| P | $T/KL$ | $\infty$ | 0 |
| PI | $0.9T/KL$ | $L/0.3$ | 0 |
| PID | $1.2T/KL$ | $2L$ | $0.5L$ |

*Second method* (*K* critical gain; *P* critical period of oscillation)

| | $K_P$ | $T_I$ | $T_D$ |
|---|---|---|---|
| P | $0.5K$ | $\infty$ | 0 |
| PI | $0.45K$ | $P/1.2$ | 0 |
| PID | $0.6K$ | $0.5P$ | $0.125P$ |

**END OF QUESTION PAPER**