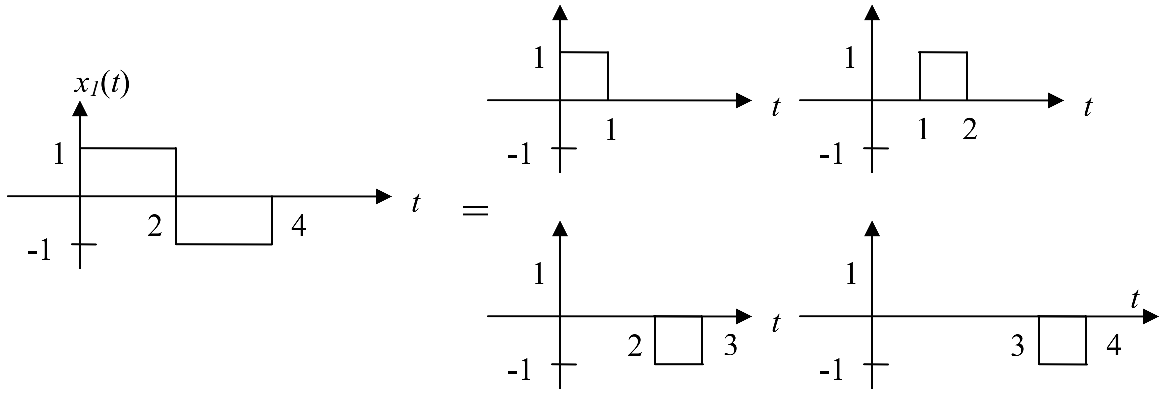
Q1.a.

i) Let x(t) be the input to a system and y(t) be the corresponding output of the system. The system is said to be linear if it satisfies the additivity and the homogeneity properties as follows:

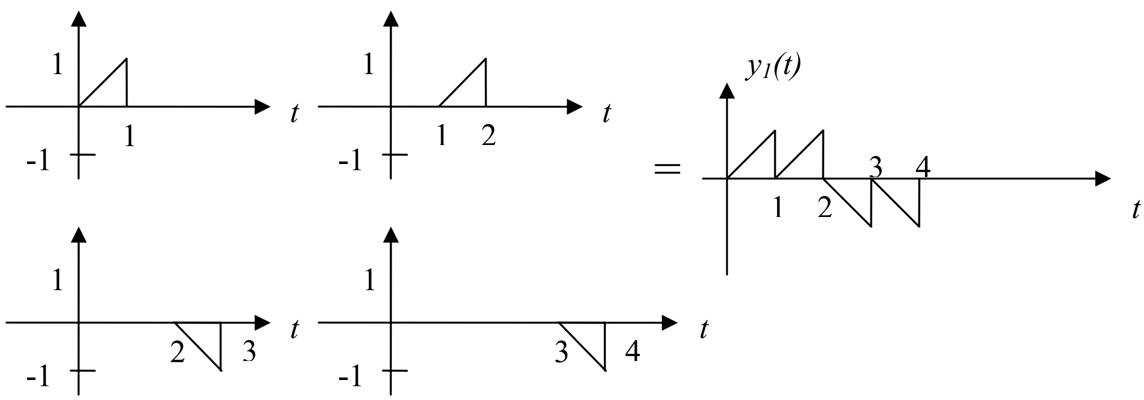
1) if input = $x_1(t) + x_2(t)$ then output = $y_1(t) + y_2(t)$, ADDITIVITY PROPERTY.
2) if input = $ax_1(t)$ then output = $ay_1(t)$, HOMOGENEITY PROPERTY.

A system is said to be time invariant if the characteristics of the system are independent of time.

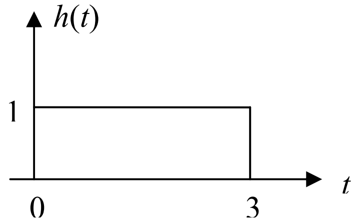
ii)

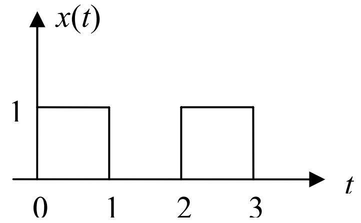


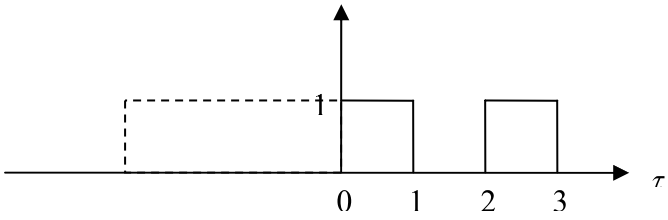
Therefore the output is



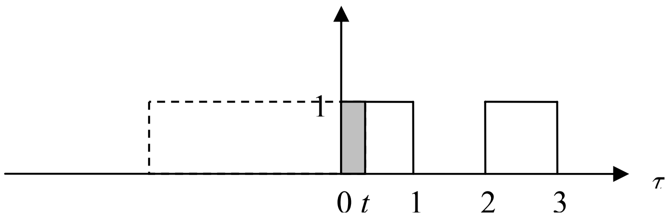
Q1.b.



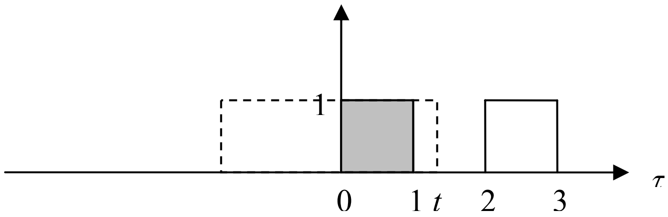




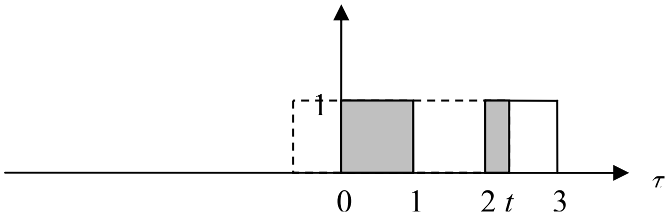
For $t \le 0$, area = 0, $y(t) = 0$



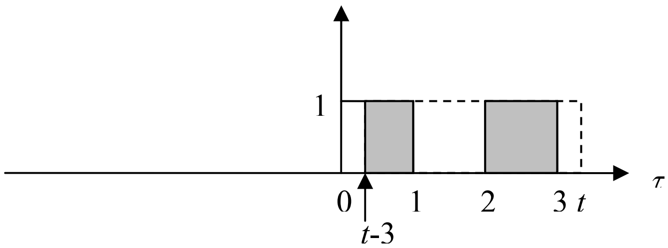
For $0 < t \le 1$, area = t, $y(t) = t$



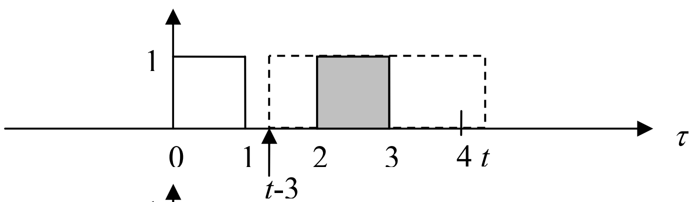
For $1 < t \le 2$, area = 1, $y(t) = 1$



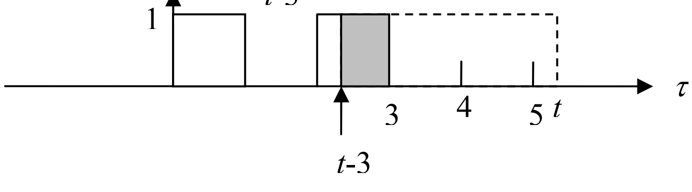
For $2 < t \le 3$, area = $1 + (t - 2) = t - 1$, $y(t) = t - 1$



For $3 < t \le 4$, area = $1 + (1 - (t - 3)) = 5 - t$, $y(t) = 5 - t$



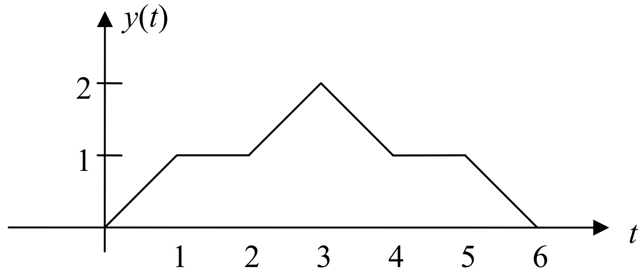
For $4 < t \le 5$, area = 1, $y(t) = 1$



For $5 < t \le 6$, area = $3 - (t-3) = 6-t$, $y(t) = 6-t$

Q1.b.

In summary $y(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t \le 1 \\ 1 & 1 < t \le 2 \\ t-1 & 2 < t \le 3 \\ 5-t & 3 < t \le 4 \\ 1 & 4 < t \le 5 \\ 6-t & 5 < t \le 6 \\ 0 & t > 6 \end{cases}$



Q2.

a. The d.c. term is $a_o = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \int_{-\tau}^{\tau} 1 dt = 2\tau$, since T = 1.

This is an even signal, $b_n = 0$.

$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t dt$, where $\omega_0 = 2\pi/T = 2\pi$.

$$= \frac{2}{T}\int_{-\tau}^{\tau} x(t)\cos n\omega_0 t dt = \frac{2}{T}\int_{-\tau}^{\tau} \cos n\omega_0 t dt$$

$$= \frac{2}{n\omega_0}\sin n\omega_0 t\Big|_{-\tau}^{\tau} = \frac{2}{2n\pi}\left[\sin(2n\pi\tau) - \sin(-2n\pi\tau)\right] = \frac{2}{n\pi}\sin(2n\pi\tau).$$

Therefore the Trigonometric Fourier Series is

$$x(t) = 2\tau + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(2n\pi\tau)\cos(2n\pi t).$$

b. If $\tau = 1/4$,

$a_n = \sin(2n\pi/4) = \sin(n\pi/2) = 0$ when n = 0,2,4… and $a_n = \pm 1$ when n = 1,3,5…..

Therefore we have

$$x(t) = \frac{1}{2} + \frac{2}{\pi}\left[\cos(2\pi t) - \frac{1}{3}\cos(6\pi t) + \frac{1}{5}\cos(10\pi t)\right].$$

c. At $\omega = \omega_0$, $H(\omega_0) = \frac{1}{1 + j\omega_0/\omega_c} = \frac{1}{1 + j2\pi/\pi} = \frac{1}{\sqrt{5}}\angle -1.107 \text{ rad}.$

The amplitude of the fundamental after low pass filtering = $\frac{2}{\pi} \times \frac{1}{\sqrt{5}} = \frac{2}{\pi\sqrt{5}}$.

At $\omega = 3\omega_0$, $H(3\omega_0) = \dfrac{1}{1 + j3\omega_0/\omega_c} = \dfrac{1}{1 + j6\pi/\pi} = \dfrac{1}{\sqrt{37}} \angle -1.406 \text{ rad}.$

The amplitude of the 3rd harmonic after low pass filtering = $\dfrac{2}{3\pi} \times \dfrac{1}{\sqrt{37}} = \dfrac{2}{3\pi\sqrt{37}}$.

At $\omega = 5\omega_0$, $H(5\omega_0) = \dfrac{1}{1 + j5\omega_0/\omega_c} = \dfrac{1}{1 + j10\pi/\pi} = \dfrac{1}{\sqrt{101}} \angle -1.471 \text{ rad}.$

The amplitude of the 5th harmonic after low pass filtering = $\dfrac{2}{5\pi} \times \dfrac{1}{\sqrt{101}} = \dfrac{2}{5\pi\sqrt{101}}$.

Therefore after filtering $v(t)$ becomes

$$v(t) = \frac{1}{2} + \frac{2}{\pi}\left[\frac{1}{\sqrt{5}}\cos(2\pi t - 1.107) - \frac{1}{3\sqrt{37}}\cos(6\pi t - 1.406) + \frac{1}{5\sqrt{101}}\cos(10\pi t - 1.471)\right].$$

Q.3

a. i) $W(\omega) = \int_{-\infty}^{\infty} w(t) e^{-j\omega t} dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t} dt$

$$W(\omega) = \frac{1}{j\omega}\left[-e^{-j\omega t}\right]_{-\tau/2}^{\tau/2} = \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} = \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j2\omega/2} = \frac{\tau}{\omega\tau/2}\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j2}$$

$$W(\omega) = \tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}.$$

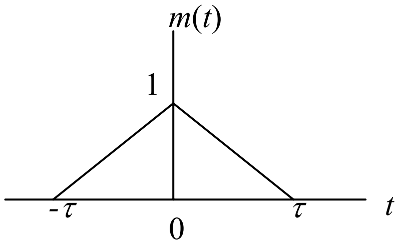
ii) Peak amplitude of $W(\omega)$ occur when $\omega = 0$. Using l'Hopital's rule,

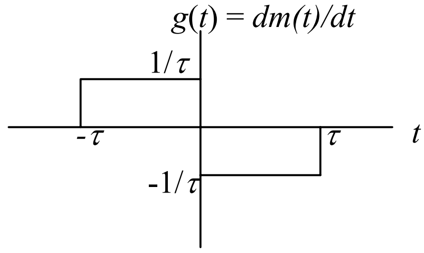
$\left.\dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right|_{\omega=0} = 1$. Therefore the peak amplitude $W(0) = \tau$.

The nulls occur when $\sin(\omega\tau/2) = 0$, i.e when $\omega\tau/2 = k\pi$, where k is an integer.

Therefore nulls occur at frequencies $\omega = \pm 2\pi/\tau, \pm 4\pi/\tau, \pm 6\pi/\tau, \ldots$.

b.





We know that the Fourier Transform of a rectangular pulse with duration of τ and amplitude of 1 is $\tau\dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)}$. Using the time shift property,

$$G(\omega) = \tau\left(\frac{1}{\tau}\right)\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}e^{j\omega\tau/2} - \tau\left(\frac{1}{\tau}\right)\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}e^{-j\omega\tau/2}$$

$$= \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}(e^{j\omega\tau/2} - e^{-j\omega\tau/2})$$

$$= \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}(2j\sin(\omega\tau/2)) = j\omega\tau\left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right]^2.$$

$$m(t) = \int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega).$$

$M(\omega) = G(\omega)/j\omega$ since $G(0) = 0$.

Finally we have, $M(\omega) = \tau\left[\dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right]^2$.

Q4.

a. $\dfrac{V_o(s)}{V_i(s)} = \dfrac{1/sC}{R + sL + 1/sC} = \dfrac{1}{sRC + s^2LC + 1} = \dfrac{1/LC}{s^2 + (R/L)s + 1/LC}$.

b. Comparing $\dfrac{1/LC}{s^2 + (R/L)s + 1/LC}$ with $\dfrac{\omega_n^{\ 2}}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$ we have

$\omega_n = 1/\sqrt{LC}$ and $2\zeta\omega_n = R/L$. Therefore the natural frequency is

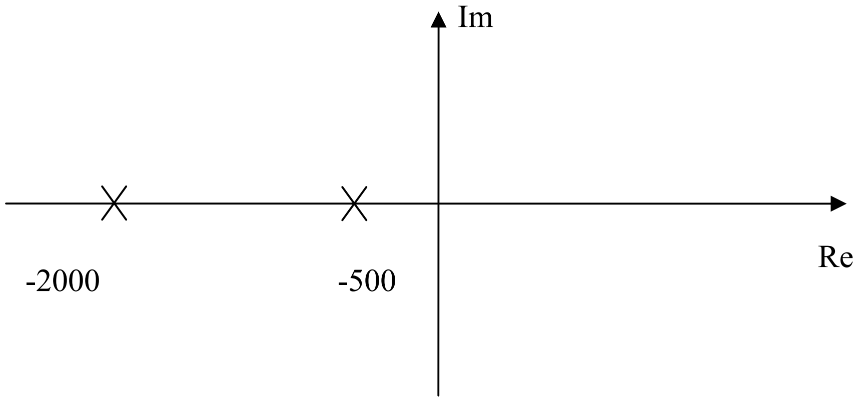
$\omega_n = 1/\sqrt{LC} = 1/\sqrt{10\times10^{-3}\times100\times10^{-6}}$ = 1000 rad/s or f_n = 159 Hz.

and the damping factor is $\zeta = \dfrac{R}{L\left(2/\sqrt{LC}\right)} = \dfrac{25}{10\times10^{-3}\left(2\times1000\right)} = 1.25$.

c. The system is overdamped, the poles are given by $p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$.

$\zeta = 1.25$, $\omega_n = 1000$ rad/s.

Therefore the poles $= -1.25\times1000 \pm \left(1000\times\sqrt{(1.25)^2 - 1}\right)$= -500 and -2000.



d. The unit step response is given by $y(t) = \dfrac{k}{p_1 p_2} + k_2 e^{p_1 t}.u(t) + k_3 e^{p_2 t}.u(t)$ where p_1 = -500 and p_2 = -2000, $k = 1/LC$, k_2 and k_3 are constants. Since p_1 is the dominant pole the response is

$$y(t) \approx \frac{k}{p_1 p_2} + k_2 e^{p_1 t}.u(t)$$