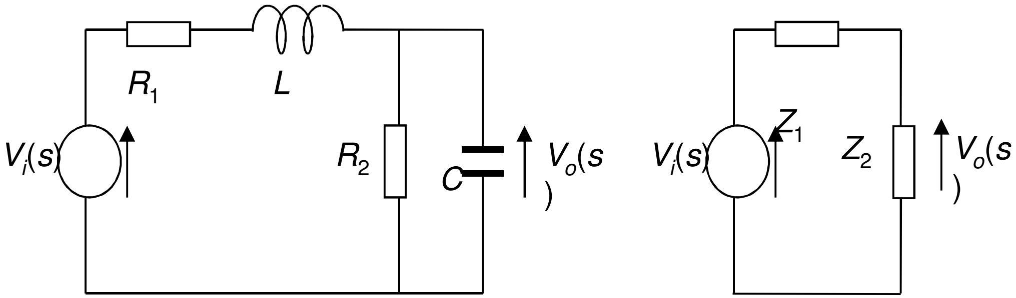
Lecture content

- Laplace Transform 2nd Order Systems
  - Overdamped
  - Critically Damped
  - Underdamped
  - Undamped

2^{nd} order systems



$$\frac{V_o(s)}{V_i(s)} = \frac{1/LC}{s^2 + \left((L + R_1R_2C)/R_2LC\right)s + (R_1 + R_2)/R_2LC}$$

The circuit above is an example of a second order system. The transfer function has a general form

$$H(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

2^{nd} order systems

$$H(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

$$Q = \frac{1}{2\zeta}$$

ω_n is the natural frequency of the system, ζ is the damping factor and $N(s)$ is the numerator polynomial with order less than or equal to that of the denominator polynomial.

2nd order systems

Assuming that $N(s) = k$, $\omega_n > 0$ and $\zeta > 0$

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s-p_1)(s-p_2)}$$

$$as^2 + bs + c = 0$$

$$p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2}$$

$$p_{1,2} = -\zeta\omega_n \pm \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = -\zeta\omega_n \pm \sqrt{\omega_n^2(\zeta-1)^2}$$

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$ are the poles

If $\zeta > 1$, the system will be non-oscillatory and is said to be overdamped. The poles are real but unequal.

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

If $\zeta = 0$, the system has no losses and the oscillation is undamped. The poles are imaginary but unequal and are given by $p_{1,2} = \pm j\omega_n$

If $\zeta = 1$, the system is said to be critically damped with real and equal poles, $p_1 = p_2 = -\omega_n$

If $0 < \zeta < 1$, the system will be oscillatory and is said to be underdamped. The poles cause $H(s) = \infty$, are complex conjugates and are given by $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

Unit step response of 2nd order systems

Both poles are real ($\zeta > 1$)

The system is overdamped and we have,

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n{}^2} = \frac{k}{(s-p_1)(s-p_2)}$$

$$Y(s) = H(s)X(s) = \frac{k}{s(s-p_1)(s-p_2)}$$

Using partial fraction,

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s-p_1} + \frac{k_3}{s-p_2}$$

$$\updownarrow$$

$$y(t) = k_1 + k_2 e^{p_1 t} u(t) + k_3 e^{p_2 t} u(t)$$

Unit step response of 2nd order systems

We can find the real constants k_1, k_2 and k_3 by using partial fraction expansion as follows

$$k_1 = \left.\frac{k}{(s-p_1)(s-p_2)}\right|_{s=0} = \frac{k}{p_1 p_2}$$

$$k_2 = \left.\frac{k}{s(s-p_2)}\right|_{s=p_1} = \frac{k}{p_1(p_1-p_2)}$$

$$k_3 = \left.\frac{k}{s(s-p_1)}\right|_{s=p_2} = \frac{k}{p_2(p_2-p_1)}$$

In general $Y(s) = \dfrac{k_1}{(s-p_1)} + \dfrac{k_2}{(s-p_2)} + \ldots . \dfrac{k_N}{(s-p_N)}$

$$k_i = (s-p_i)X(s)\big|_{s=p_i}$$

$$Y(s) = \frac{k}{s(s-p_1)(s-p_2)} = \frac{k_1(s-p_1)(s-p_2)+k_2 s(s-p_2)+k_3 s(s-p_1)}{s(s-p_1)(s-p_2)}$$

$$k = (k_1+k_2+k_3)s^2 + (-k_1(p_1+p_2)-k_2 p_2 - k_3 p_1)s + k_1 p_1 p_2$$

$$k_1 = \frac{k}{p_1 p_2}$$ Coe. of s^0

$$k_2 + k_3 = -\frac{k}{p_1 p_2}, k_2 = -k_3 - \frac{k}{p_1 p_2}$$ Coe. of s^2

$$-\frac{k}{p_1 p_2}(p_1+p_2) + \left(k_3 + \frac{k}{p_1 p_2}\right)p_2 - k_3 p_1 = 0,$$ Coe. of s^1

$$k_3 = \frac{k}{p_2(p_2-p_1)}$$

$$k_2 = \frac{k}{p_1(p_1-p_2)}$$

Unit step response of 2nd order systems

$$y(t) = k_1 + k_2 e^{p_1 t} u(t) + k_3 e^{p_2 t} u(t)$$

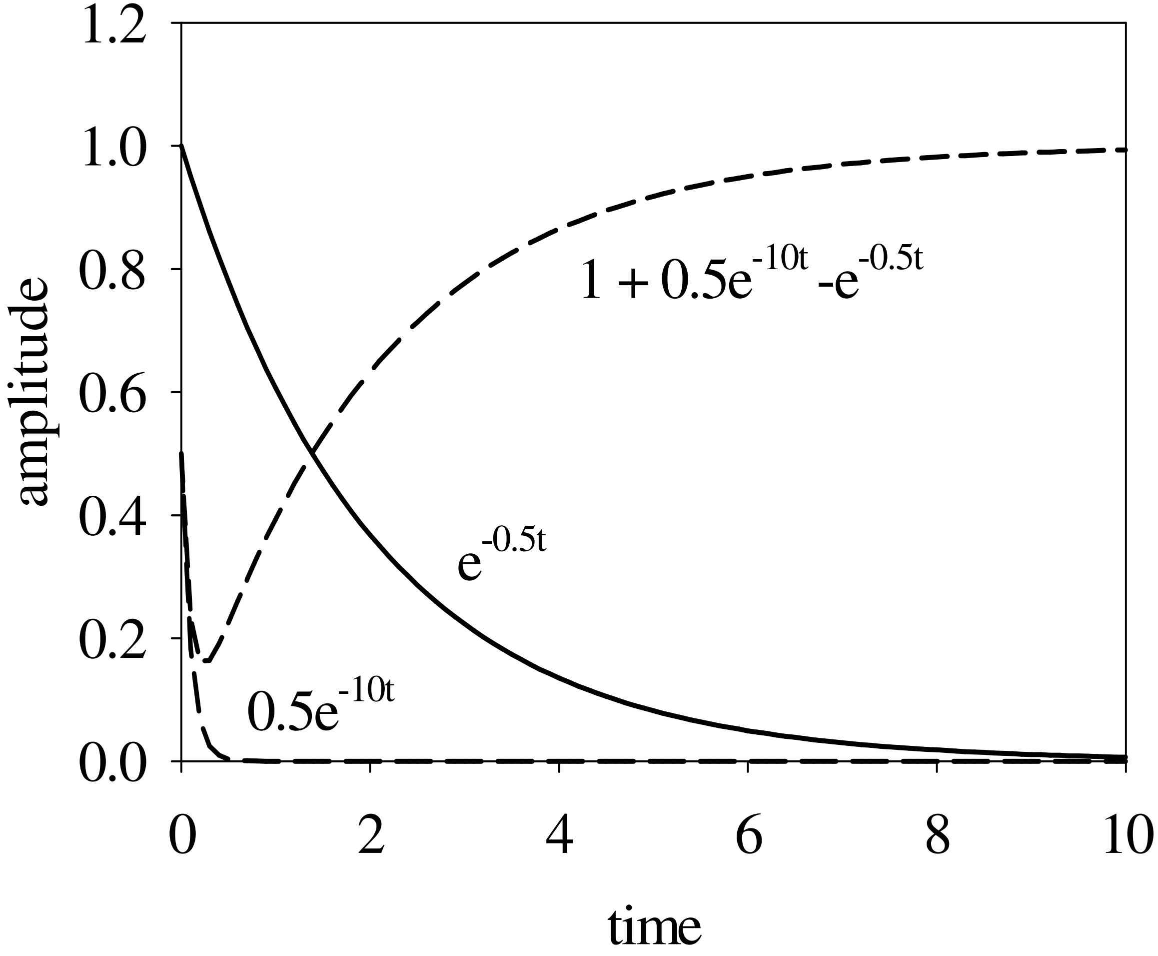
The forced response is $y_{fr}(t) = \dfrac{k}{p_1 p_2}$

The transient response or the natural response is

$$y_{tr}(t) = k_2 e^{p_1 t} u(t) + k_3 e^{p_2 t} u(t)$$

If p_2 is nearer to the $j\omega$-axis it is called the **dominant pole** and the transient response will be dominated by $k_3 e^{p_2 t} u(t)$

Unit step response of 2nd order systems



Unit step response of 2^{nd} order systems

<u>Poles are real and equal ($\zeta = 1$)</u>

The system is critically damped and we have,

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s+\omega_n)^2}$$

The poles are $p_1 = p_2 = -\omega_n$

$$Y(s) = H(s)X(s) = \frac{k}{s(s+\omega_n)^2}$$

$$Y(s) = \frac{k_1}{s} + \frac{k_2}{s+\omega_n} + \frac{k_3}{(s+\omega_n)^2}$$

Unit step response of 2nd order systems

$$\left\{\frac{t^n}{n!}e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^{n+1}}\right\} \qquad k_3 t e^{-at}u(t) \leftrightarrow \frac{k_3}{(s+\omega_n)^2}$$

Therefore

$$y(t) = k_1 + k_2 e^{-\omega_n t}u(t) + k_3 t e^{-\omega_n t}u(t)$$

$$y(t) = k_1 + (k_2 + k_3 t)e^{-\omega_n t}u(t)$$

Unit step response of 2nd order systems

To find the constants,

$$k_1 = \left. \frac{k}{(s+\omega_n)^2} \right|_{s=0} = \frac{k}{\omega_n{}^2}$$

$$k_2 = \frac{1}{(2-1)!} \frac{d}{ds} \left((s+\omega_n)^2 \frac{k}{s(s+\omega_n)^2} \right) \Bigg|_{s=-\omega_n} = -\frac{k}{\omega_n{}^2}$$

$$k_3 = \left. (s+\omega_n)^2 \frac{k}{s(s+\omega_n)^2} \right|_{s=-\omega_n} = -\frac{k}{\omega_n}$$

See p.384 Kamen and Heck

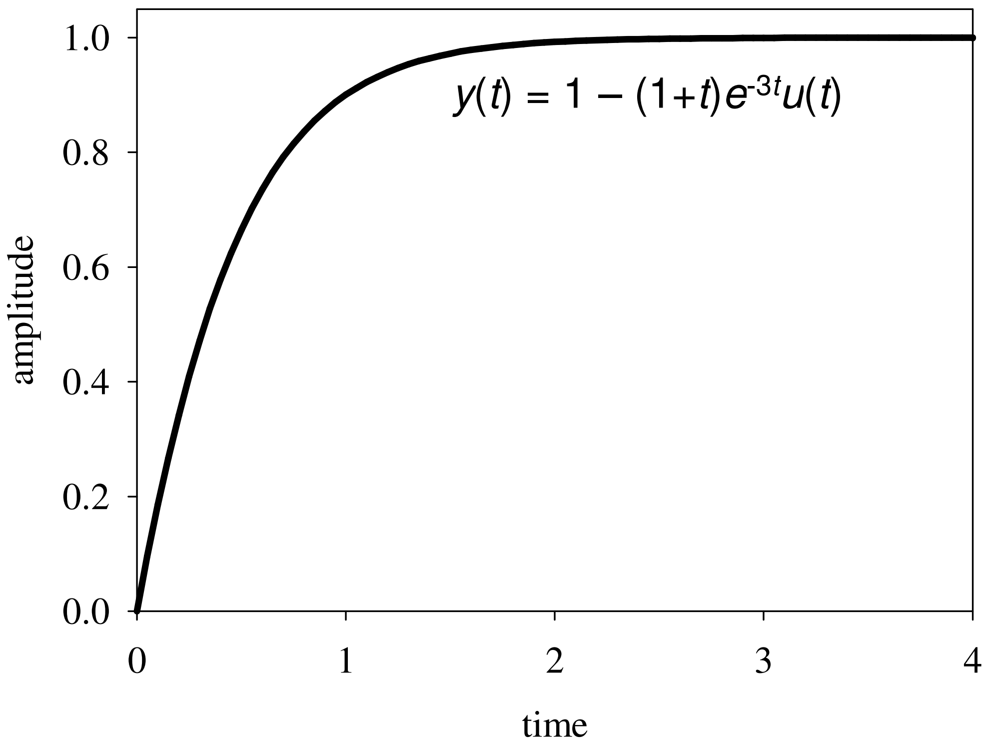
Comparing the coefficients for s

Unit step response of 2nd order systems

$$y(t) = \frac{k}{{\omega_n}^2} - \frac{k}{{\omega_n}^2} e^{-\omega_n t} u(t)[1 + \omega_n t]$$

$$y_{fr}(t) = \frac{k}{{\omega_n}^2}$$

$$y_{tr}(t) = -\frac{k}{{\omega_n}^2} e^{-\omega_n t} u(t)[1 + \omega_n t]$$



Unit step response of 2nd order systems

Poles are complex ($0 < \zeta < 1$)

The system is underdamped and we have,

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{(s+\zeta\omega_n)^2 + \omega_n^2 - (\zeta\omega_n)^2}$$

$$H(s) = \frac{k}{(s+\zeta\omega_n)^2 + \omega_d^2} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

The poles are $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$

$$p_{1,2} = -\zeta\omega_n \pm j\omega_d$$

Unit step response of 2nd order systems

$$Y(s) = \frac{k}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{k_1}{s} + \frac{k_2 s + k_3}{(s^2 + 2\zeta\omega_n s + \omega_n^2)}$$

Comparing the coefficients for *s*:

$k_1 = k/\omega_n^2$

$k_1 + k_2 = 0$, $k_2 = -\, k/\omega_n^2$

$2\zeta\omega_n k_1 + k_3 = 0$, $k_3 = -2\zeta\, k/\omega_n$

$$Y(s) = \frac{(k/\omega_n^2)}{s} - \frac{(k/\omega_n^2)s + 2\zeta k/\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Unit step response of 2nd order systems

$$Y(s)=\frac{(k/\omega_n{}^2)}{s}-\frac{(k/\omega_n{}^2)(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\omega_d{}^2}-\frac{(k\zeta/\omega_n)}{(s+\zeta\omega_n)^2+\omega_d{}^2}$$

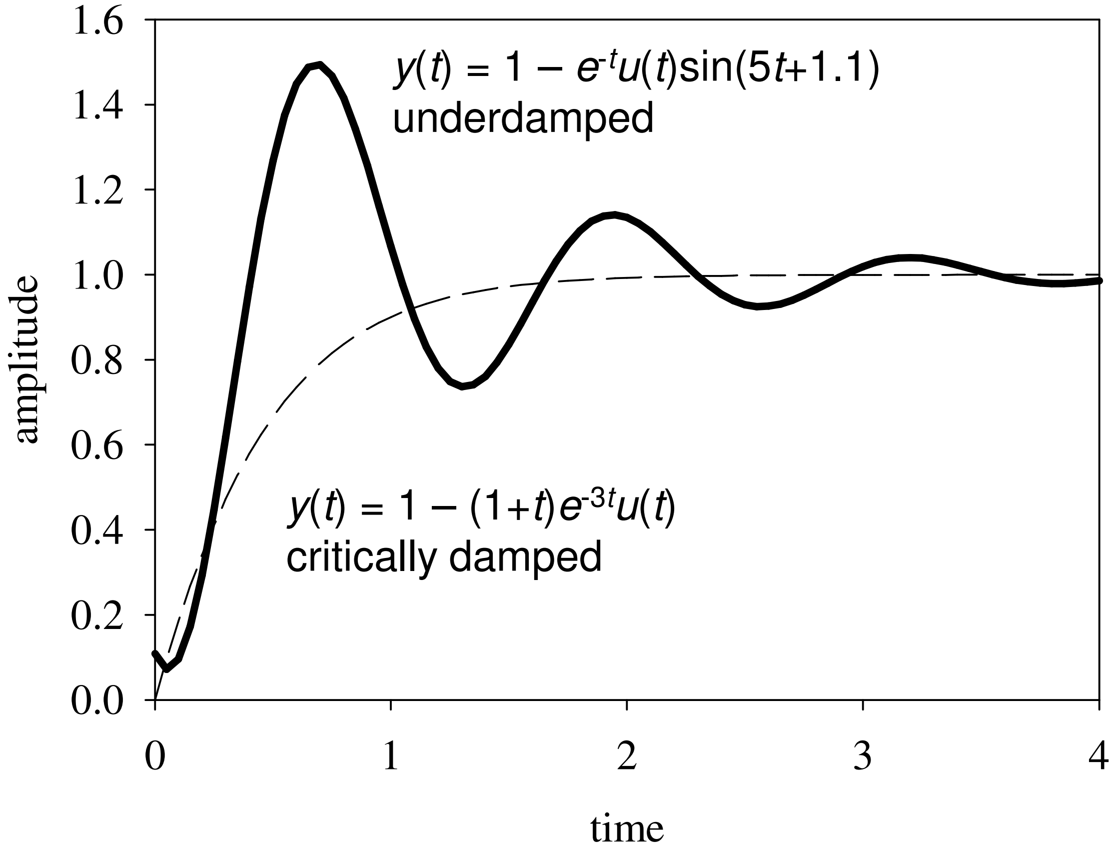
$$y(t)=\frac{k}{\omega_n{}^2}-\frac{k}{\omega_n{}^2}e^{-\zeta\omega_n t}\cos(\omega_d t).u(t)-\frac{k\zeta}{\omega_n\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t).u(t)$$

$$y(t)=\frac{k}{\omega_n{}^2}\left(1-\frac{\omega_n}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)u(t)\right) \qquad \phi=\tan^{-1}(\omega_d/\zeta\omega_n)$$

Unit step response of 2nd order systems

$$y_{fr}(t) = \frac{k}{\omega_n^{\,2}}$$

$$y_{tr}(t) = -\frac{k}{\omega_n \omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) u(t)$$



Unit step response of 2^{nd} order systems

Poles are imaginary ($\zeta = 0$)

The system is lossless and the transfer function is

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k}{s^2 + \omega_n^2}$$

The poles are $p_{1,2} = \pm j\omega_n$

$$Y(s) = \frac{k}{s(s + j\omega_n)(s - j\omega_n)} = \frac{k_1}{s} + \frac{k_2}{s + j\omega_n} + \frac{k_3}{s - j\omega_n}$$

$$y(t) = k_1 + k_2 e^{-j\omega_n t} u(t) + k_3 e^{j\omega_n t} u(t)$$

Unit step response of 2nd order systems

$$k_1 = \left.\frac{k}{(s+j\omega_n)(s-j\omega_n)}\right|_{s=0} = \frac{k}{\omega_n{}^2}$$

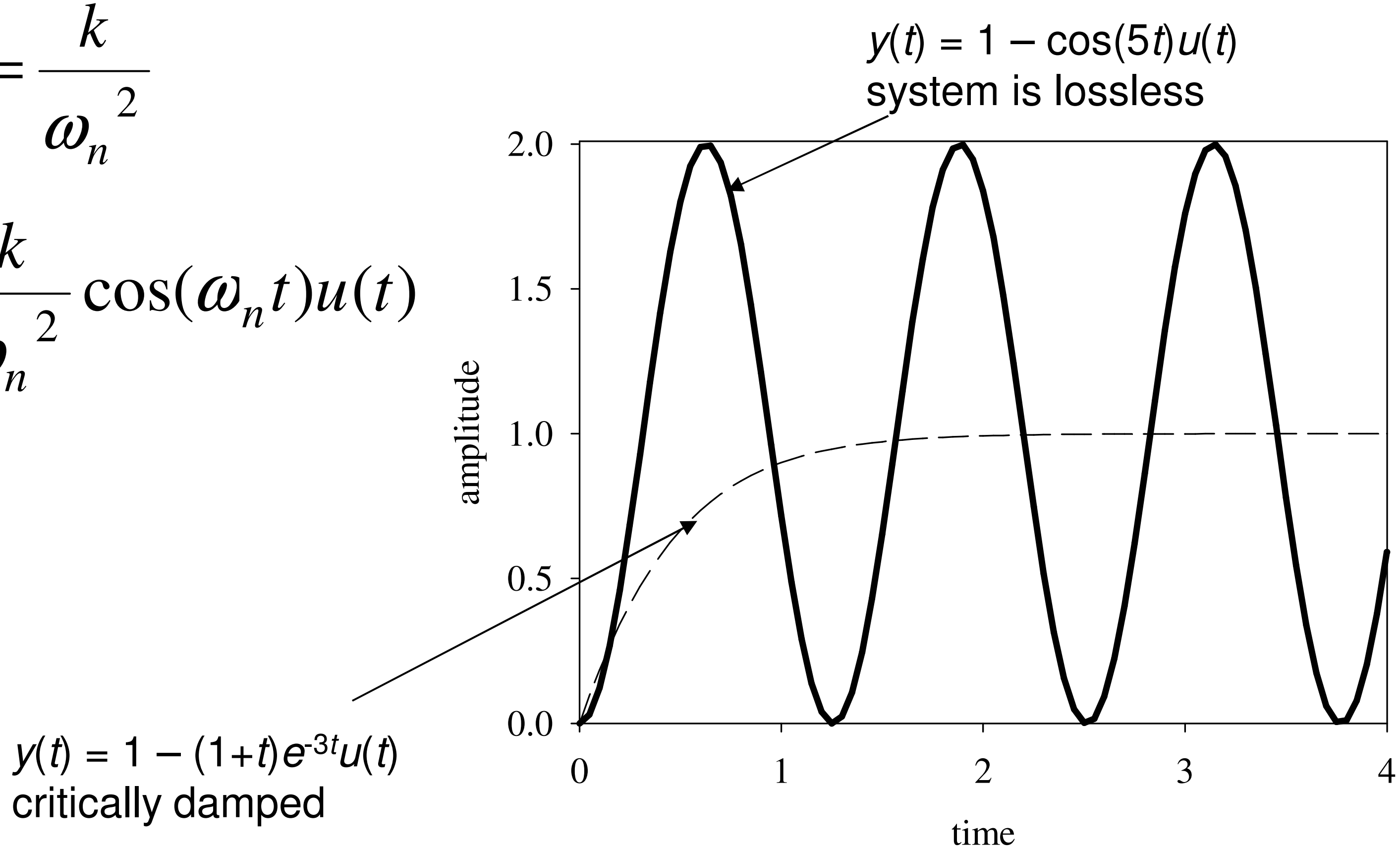
$$k_2 = \left.\frac{k}{s(s-j\omega_n)}\right|_{s=-j\omega_n} = -\frac{k}{2\omega_n{}^2}$$

$$k_3 = \left.\frac{k}{s(s+j\omega_n)}\right|_{s=j\omega_n} = -\frac{k}{2\omega_n{}^2}$$

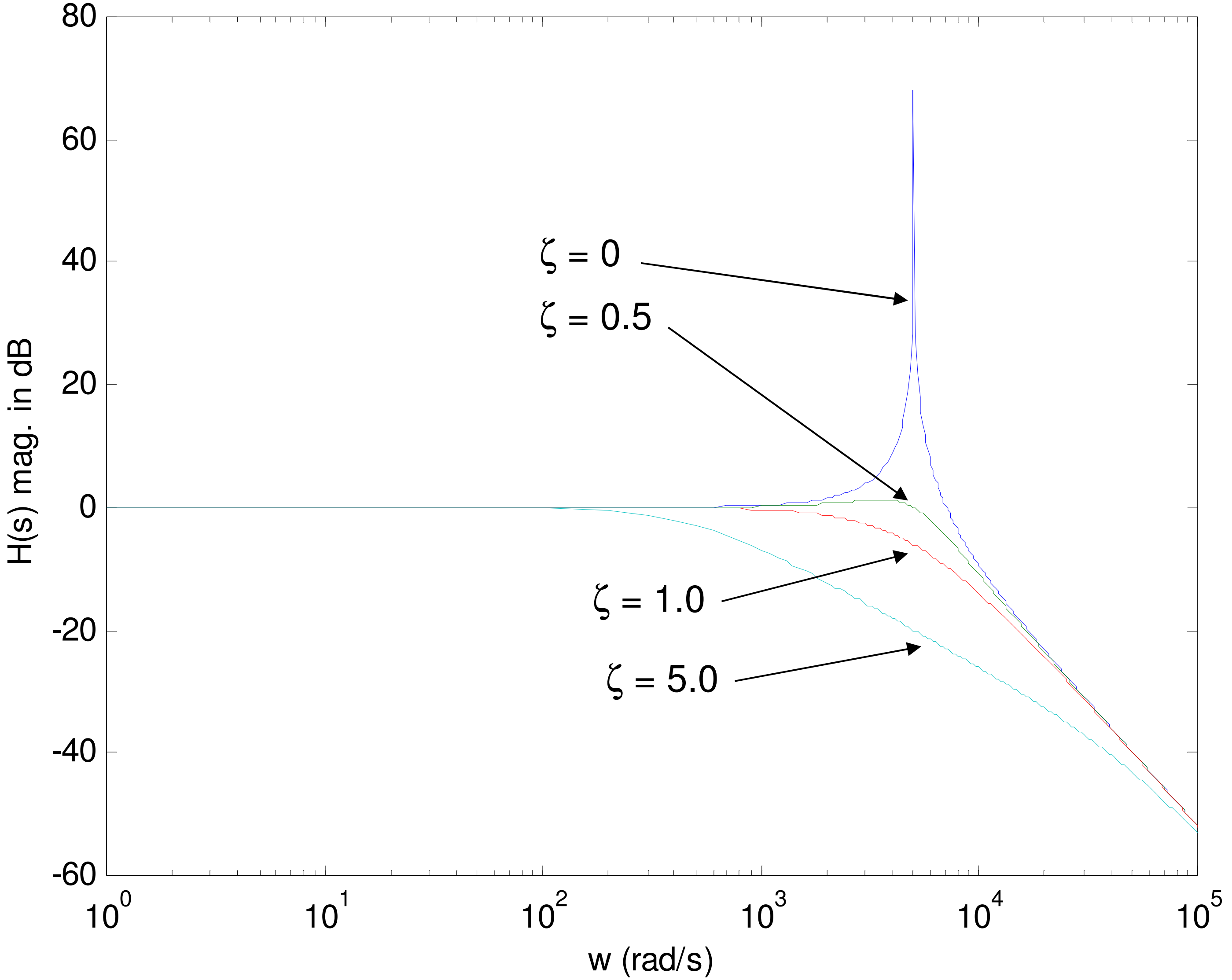
Unit step response of 2^{nd} order systems

$$y_{fr}(t) = \frac{k}{\omega_n^{\ 2}}$$

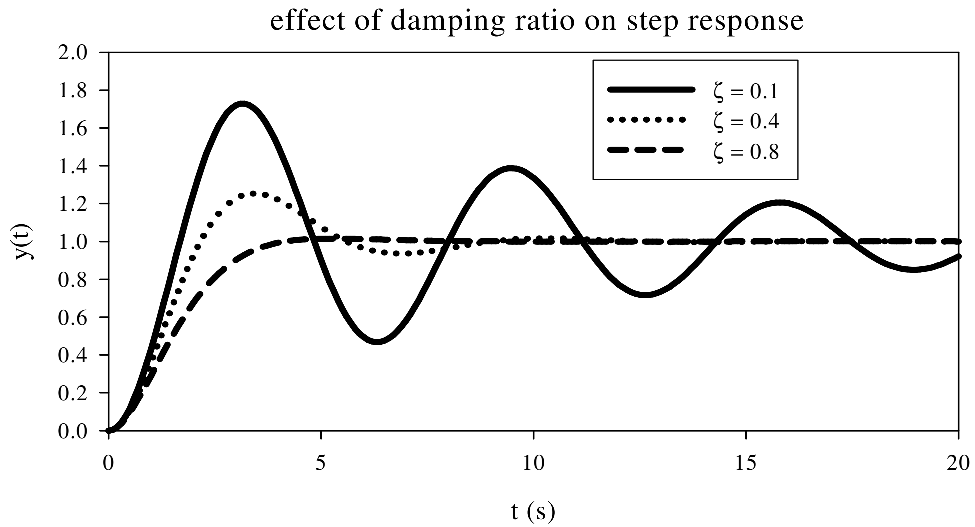
$$y_{tr}(t) = -\frac{k}{\omega_n^{\ 2}}\cos(\omega_n t)u(t)$$

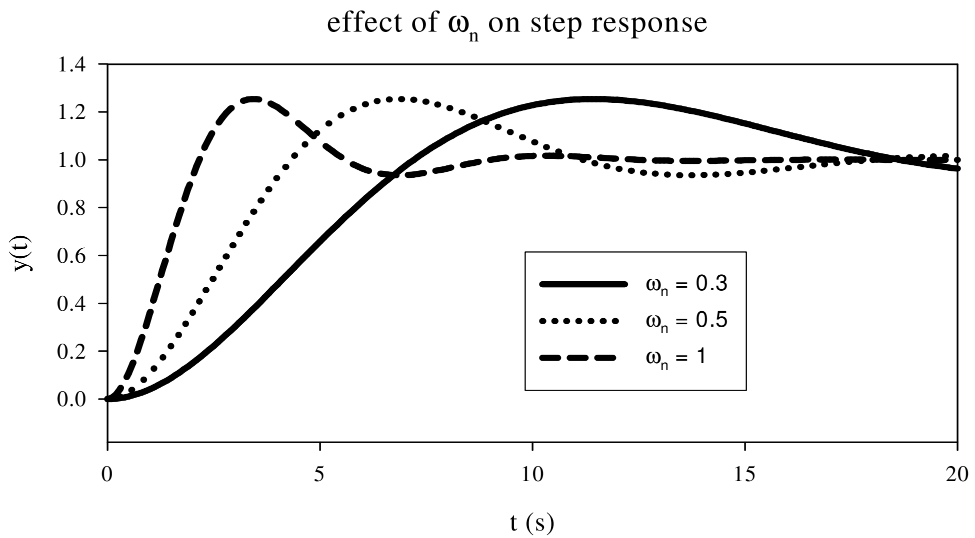


2^{nd} order system transfer function

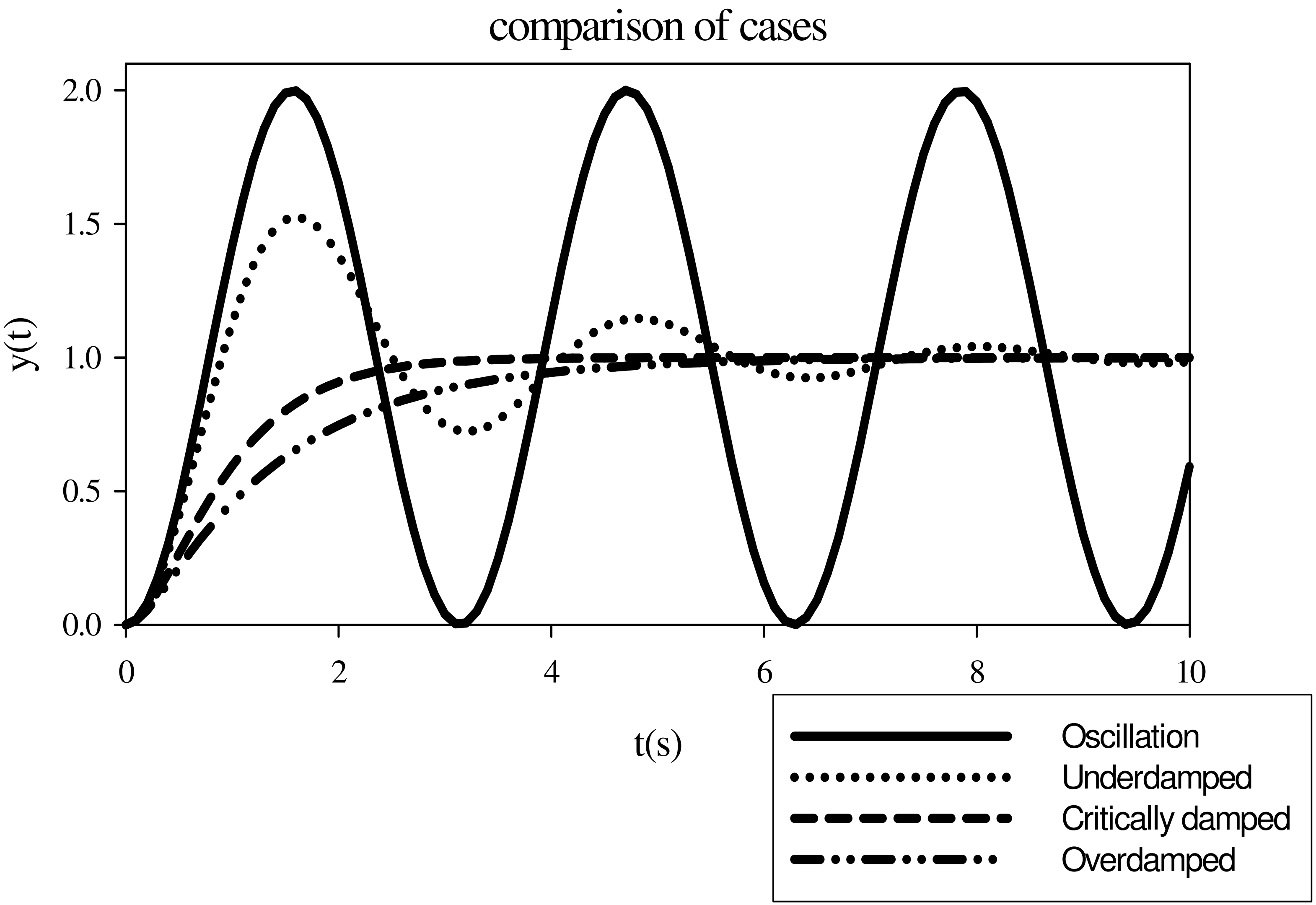


2^{nd} order system unit step response

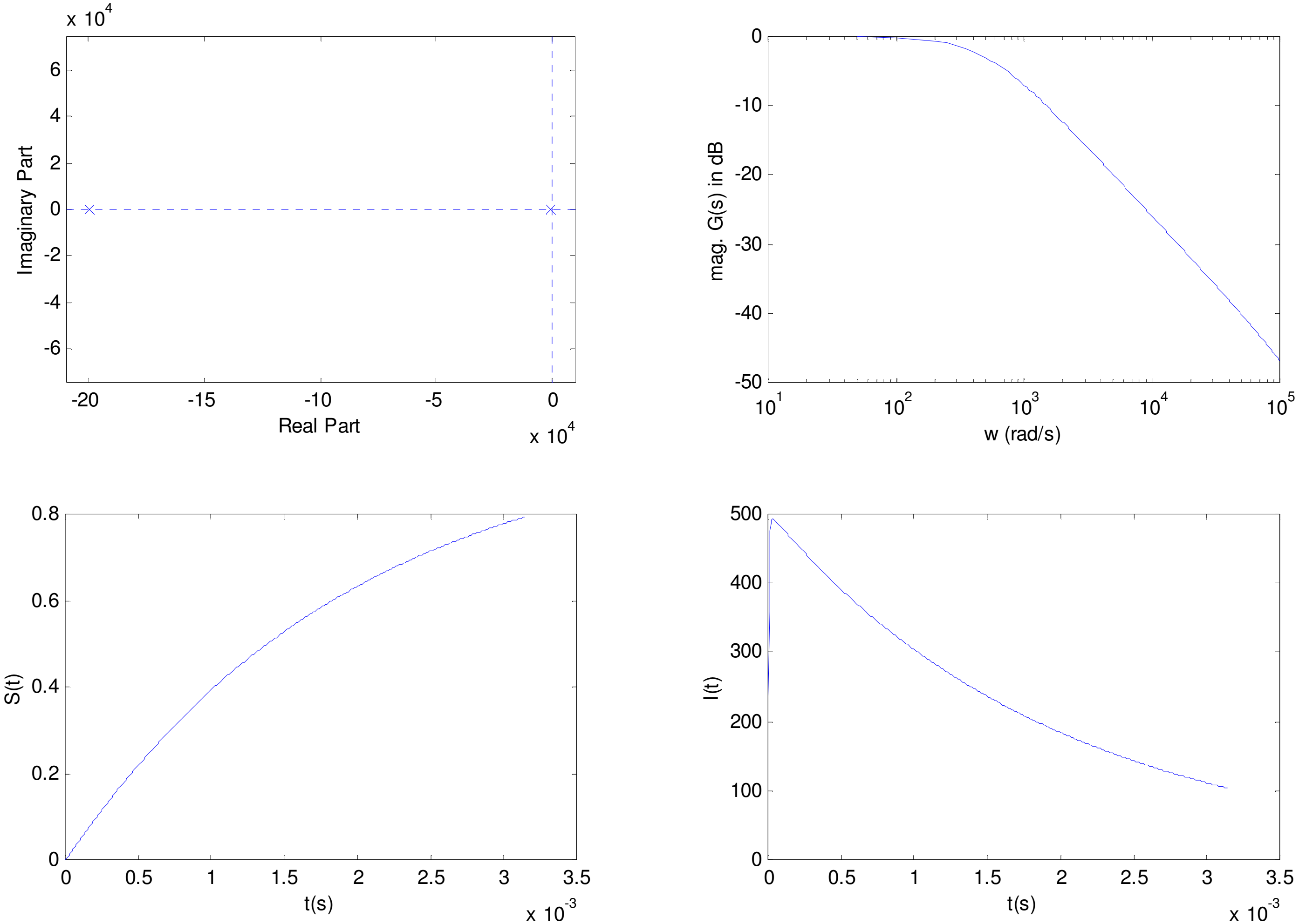




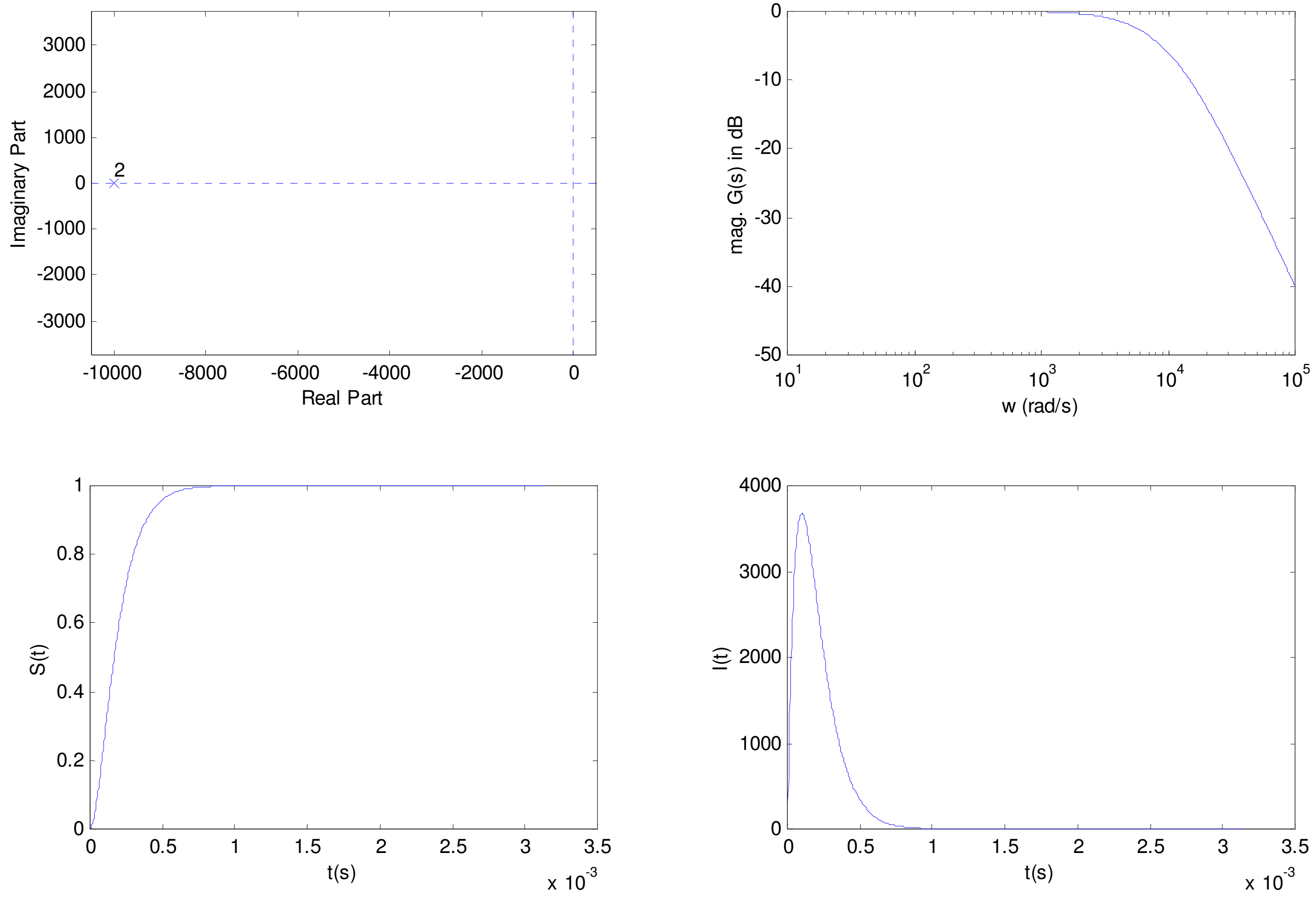
2^{nd} order system unit step response



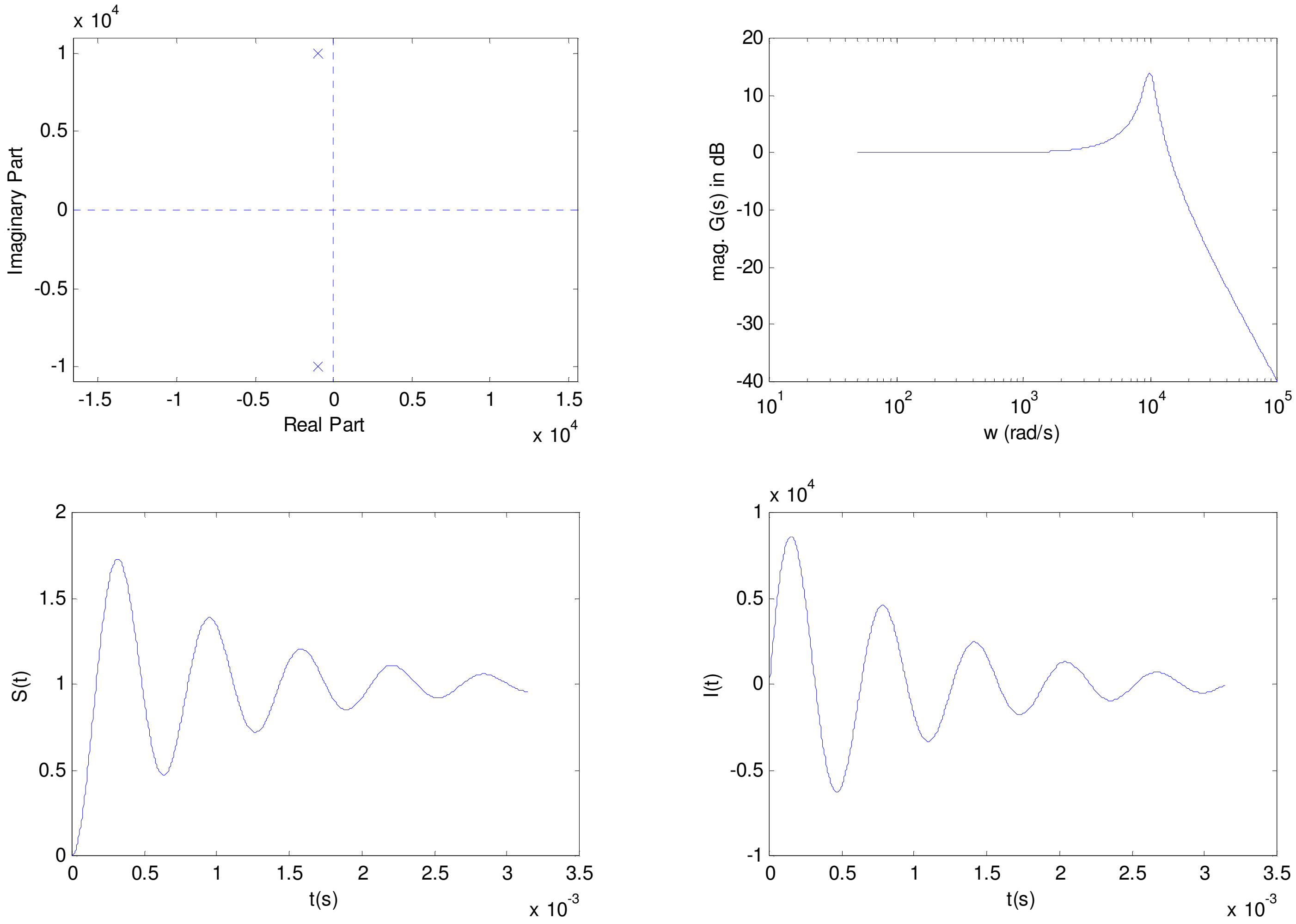
$$\zeta = 10, \omega_n = 10000$$



$$\zeta = 1, \omega_n = 10000$$



$$\zeta = 0.1, \omega_n = 10000$$



$$\zeta = 0.001, \omega_n = 10000$$