Network Analysis

There are many methods available to use for solving electrical networks. On this course we will limit our study to the following methods:

- Superposition Theorem
- Kirchhoff's Laws
- Thévenin and Norton equivalent circuits

At this stage we will only consider networks comprising of resistors and d.c. sources. However the theorems and methods may be applied to more complex circuits containing inductance, capacitance and fed from a.c. or d.c. sources.

A network comprises a number of branches or circuit elements connected together and considered as a unit.

If the network contains no source of e.m.f. it is termed a PASSIVE network
If the network does contain an e.m.f. source it is termed an ACTIVE network

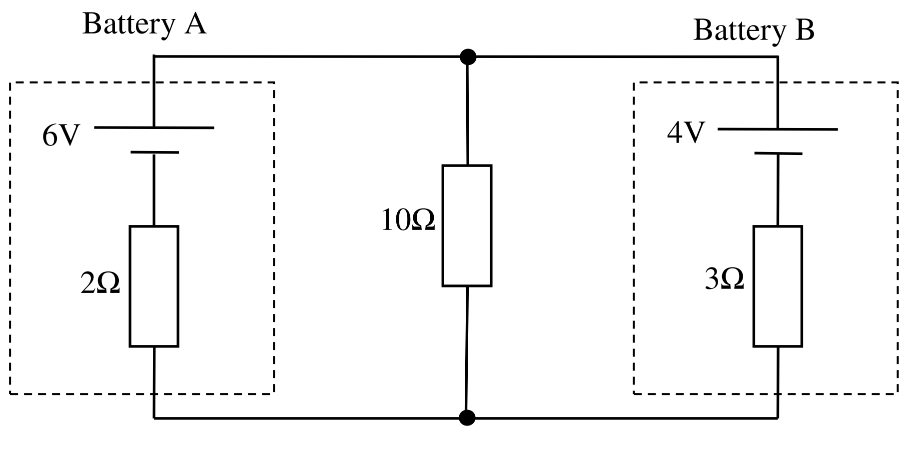
Superposition Theorem

"An emf. acting on any linear network produces the same effect whether it acts alone or in conjunction with other emfs." (only linear networks will be considered on this course).

What this means is that a network containing several emf sources may be analysed by considering the effect of each source in turn, with all other sources being represented by their internal resistances, then the resultant current is obtained from the algebraic sum of the currents from each source. This is best illustrated by means of an example.

Example

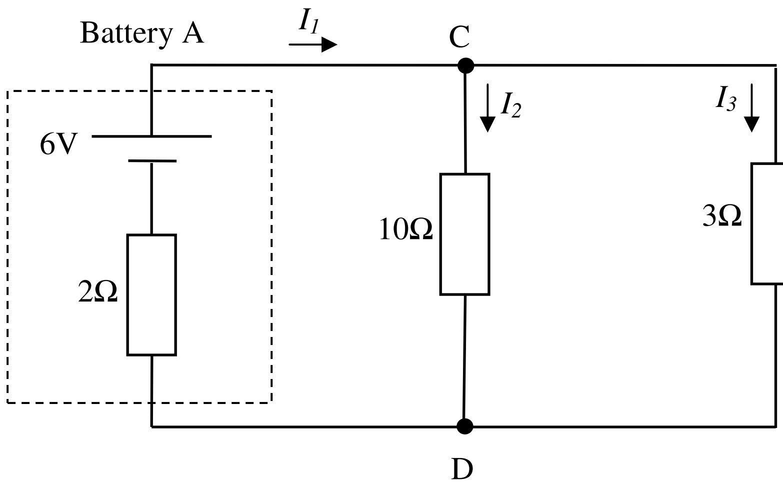
A circuit consists of two batteries, each having an internal resistance, and a single 10Ω load resistor as shown in the figure. What is the current in the 10Ω resistor?



Using the superposition theorem, first find the currents in each branch of the circuit when supplied by each source in turn, then add the results to find the total currents.

First consider the effect of battery A alone.

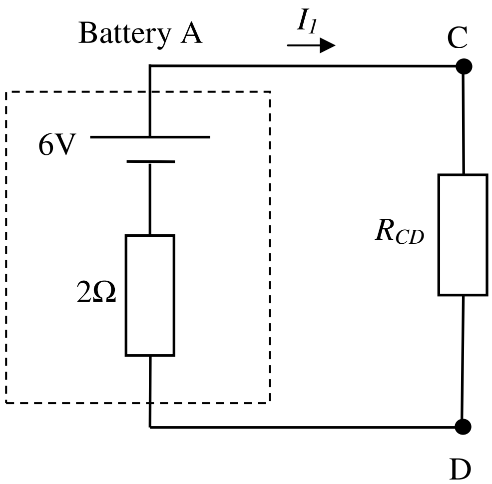
Battery B is represented by its internal resistance alone. Let the currents in the branches be I_1, I_2 and I_3 respectively. (Note it is usual to chose the direction of the current to flow out of the positive terminal of the battery). The circuit then becomes:



It can be seen that the 10Ω and 3Ω resistor are in parallel so the resistance between points C and D is:

$$R_{CD} = \frac{1}{\frac{1}{10} + \frac{1}{3}} = \frac{30}{13} = 2.31\Omega$$

The circuit can then be drawn as:



R_{CD} is then in series with the 2Ω resistance of battery A which gives a total circuit resistance of:

$$R_T = R_{CD} + 2 = 4.31\Omega$$

and the current flowing out of battery A, I_1, may be found:

$$I_1 = \frac{V_A}{R_T} = \frac{6}{4.31} = 1.393A$$

Once we have found I_1, the current that flows through battery A and our equivalent resistor $R_{CD,}$ we can then work back to find the currents, I_2 and I_3:

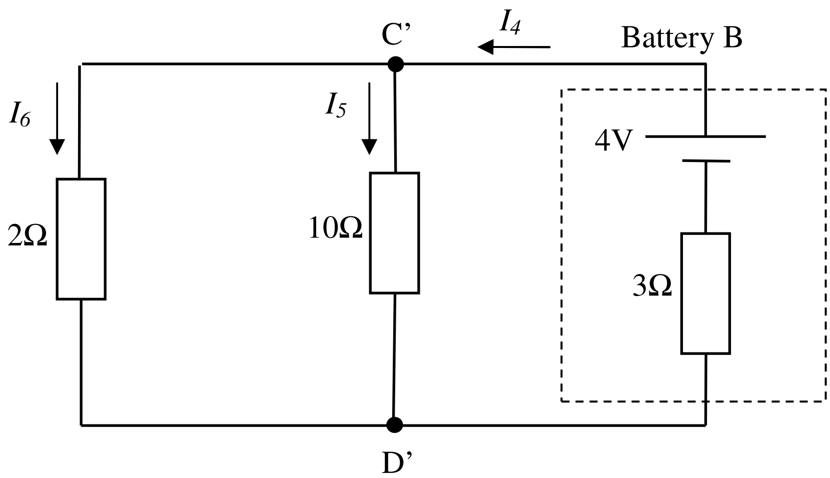
$$I_2 = I_1 \times \frac{3}{10+3} = 0.321A$$

$$I_3 = I_1 \times \frac{10}{10+3} = 1.071A$$

(Note the direction of the currents as indicated by the arrows in the diagram. We need to know this when we come to sum them with the currents from battery B).

Now consider the effect of battery B alone.

Battery A is represented by its internal resistance alone. Let the currents in the branches be I_4, I_5 and I_6, so the circuit becomes:



It can be seen that now it is the 10Ω and 2Ω resistor which are in parallel so the resistance between points C' and D' is:

$$R_{C'D'} = \frac{1}{\frac{1}{10}+\frac{1}{2}} = \frac{20}{12} = 1.667\Omega$$

$R_{C'D'}$ is then in series with the 3Ω resistance of battery B which gives a total circuit resistance of:

$$R_{T'} = R_{C'D'} + 3 = 4.667\Omega$$

and the current flowing out of battery B, I_4, may be found:

$$I_4 = \frac{V_B}{R_{T'}} = \frac{4}{4.667} = 0.857A$$

Once we have found I_4, the current that flows through battery B and our equivalent resistor $R_{C'D'}$, we can then work back to find the currents, I_5 and I_6:

$$I_5 = I_4 \times \frac{2}{10+2} = 0.143A$$

$$I_6 = I_4 \times \frac{10}{10+2} = 0.714A$$

We can now superimpose these two sets of results to find the resultant current in each branch of the circuit. (Take care with the directions).

For the 10Ω resistor in the central branch, battery A gave the contribution of I_2 = 0.321A(↓) flowing downwards and battery B gave the contribution of I_5 = 0.143A(↓) again flowing downwards. Hence the total current is:

$$I_{10\Omega}(\downarrow) = I_2 + I_5 = 0.321 + 0.143 = 0.464\text{A}(\downarrow) \text{ flowing downwards.}$$

For the branch containing the battery A we have I_1 (↑)flowing upwards and I_6(↓) flowing downwards (i.e. in the opposite direction), hence:

$$I_A(\downarrow) = I_6 - I_1 = 0.714 - 1.393 = -0.678\text{A flowing downwards (or +0.678A flowing upwards)}$$

Likewise if we follow the same procedure for battery B we have I_3 = 1.072(↓) flowing downwards and I_4 = 0.857A(↑) flowing upwards. Hence the current in the branch containing battery B is:

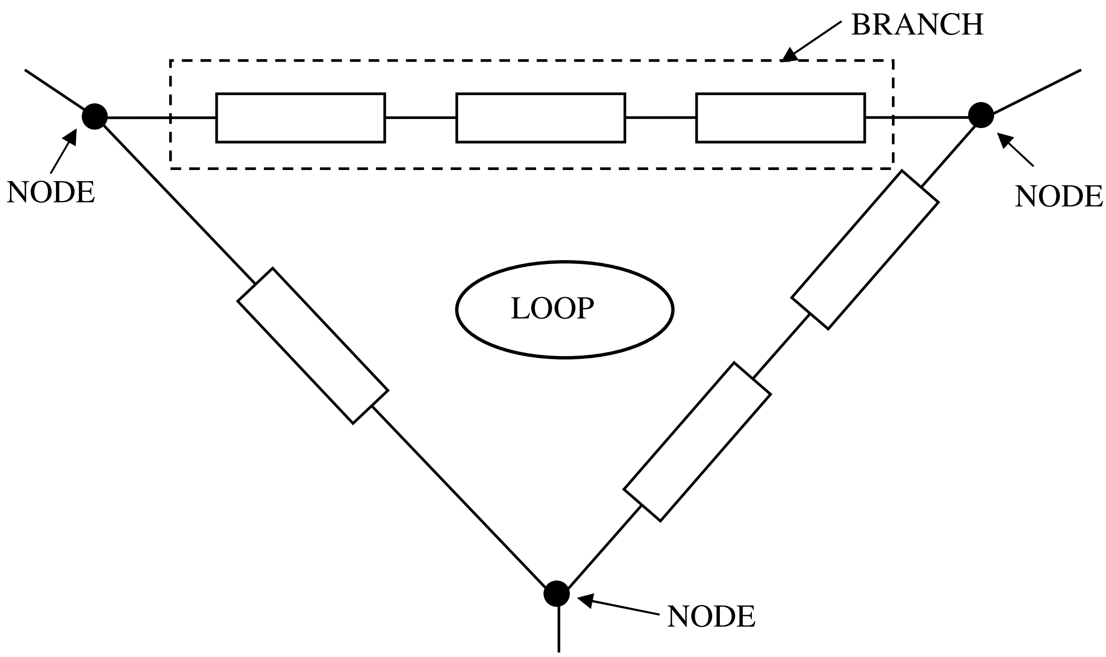
$I_B(\downarrow) = I_3 - I_4 = 1.071 - 0.857 = 0.214$A(↓) flowing downwards (or –0.214A flowing upwards)

What this indicates is that we have a positive current flowing into the positive terminal of battery B (I_B is in the downward direction) and battery B is in fact being charged by battery A.

The general method for solving circuits by superposition is to consider the effect of each source in turn. (Other voltage sources are short circuited, current sources are open-circuited) and then to sum the components taking care with the directions of the currents.

Kirchhoff's Laws

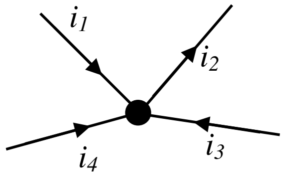
Before embarking on Kirchhoff's laws it is first necessary to define some terms. Consider the network shown below:



BRANCH	–	Part of a circuit connecting two nodes
NODE	–	The point where two or more branches meet
LOOP	-	Closed path formed by connecting branches

Kirchhoff's 1st Law

The algebraic sum of all the instantaneous currents entering any node is zero at all times. For example:



$$i_1 + i_3 + i_4 - i_2 = 0$$

Note in this example i_1, i_3 and i_4 are flowing into the node and i_2 is flowing out of the node, hence the opposite sign. In general, at any node:

$$\sum i = 0$$

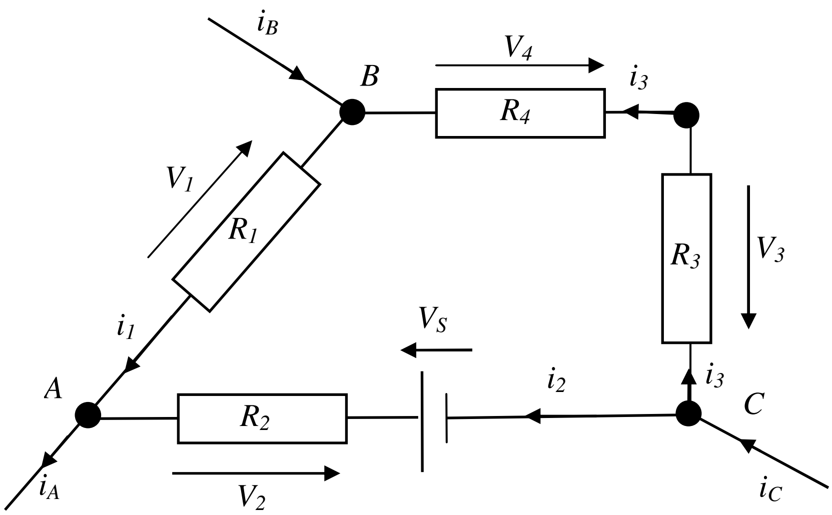
(small i is usually used for currents that may vary with time and capital I is used for d.c. currents. So far we have only considered d.c. currents but Kirchhoff's laws are equally applicable to a.c. circuits).

Kirchhoff's 2nd Law

The algebraic sum of all the instantaneous voltages around a loop is zero at all times.

$$\sum v = 0$$

For example:



Note – very important! – the choice of the direction of currents is largely arbitrary, however resistors are energy sinks, therefore there is a decrease in potential in the direction of the current flow or an increase in potential in the opposite direction to the current flow. Choose the direction of the current then draw the voltages in the opposite directions. For sources there is an increase in potential from the negative to positive terminals.

Starting at node C and working round the loop in a clockwise direction summing the voltages gives:

$$V_S - V_2 + V_1 + V_4 + V_3 = 0$$

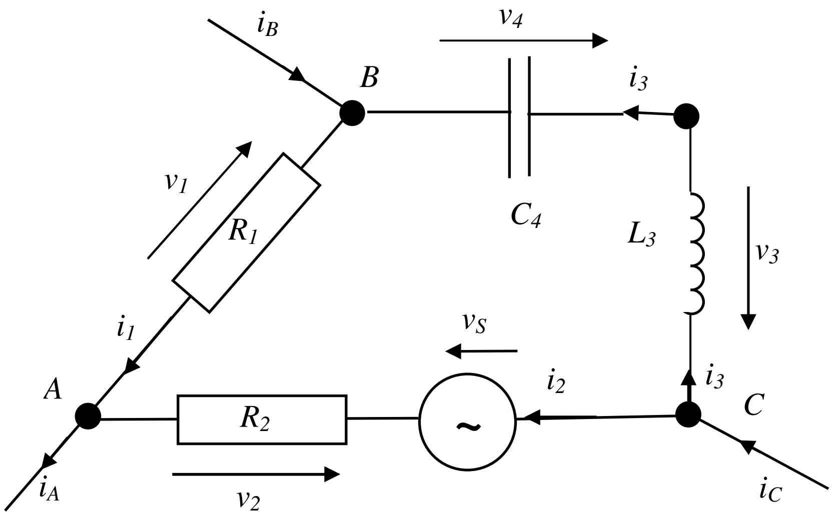
We could also apply Kirchhoff's 1st law at the nodes, A, B, and C.

At node A: $i_1 + i_2 - i_A = 0$ or $i_A = i_1 + i_2$

At node B: $i_3 + i_B - i_1 = 0$ or $i_B = i_1 - i_3$

At node C: $i_C - i_2 - i_3 = 0$ or $i_C = i_2 + i_3$

Kirchhoff's laws are equally applicable to much more complex circuits as we shall see later in the course.

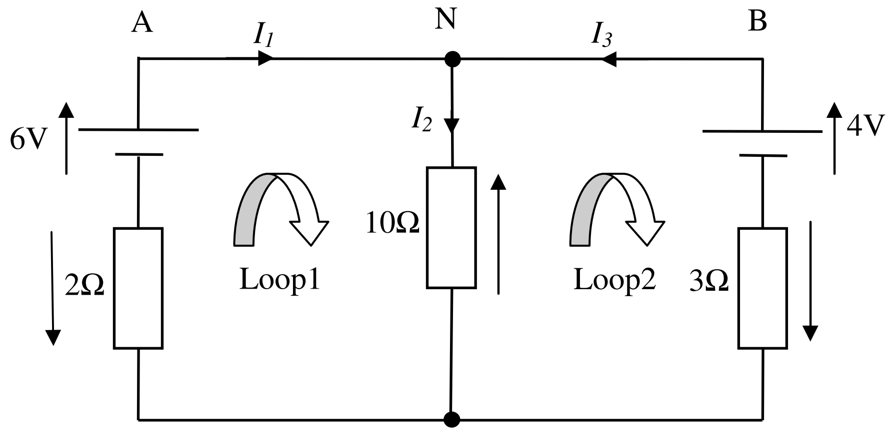


$$v_S - v_2 + v_1 + v_4 + v_3 = 0$$

$$\left(v_s - R_2 i_2 + R_1 i_1 + \frac{1}{C_4}\int i_3 dt + L_3 \frac{di_3}{dt} = 0 \right)$$

<u>Example</u>

Use Kirchhoff's laws to find the currents in our previous example which was solved using the superposition theorem.



Choose direction of the currents and indicate these on the circuit diagram. This is arbitrary, but it is normal to choose the direction as emanating from the positive side of a source. Then indicate the direction of increasing potential (voltage) on the diagram. For sources this is always in the direction negative to positive. For sinks (resistors) it is always in the opposite direction to the defined current. For example, looking at branch A on the left, the voltage across the 6V source is in the direction negative to positive. The current in this branch, I_1, has been defined as upwards and the direction of increasing potential across the 2Ω resistor is in the opposite direction to the defined current (i.e. downwards in this case).

Apply Kirchhoff's 1st law to node N:

$$I_1 + I_3 - I_2 = 0 \qquad \text{or} \qquad I_1 + I_3 = I_2$$

Now apply Kirchhoff's 2nd law to loop1, the loop formed by branch A and the central branch. Work round the loop summing the voltages, taking account of the direction of the voltage arrows we have included on the diagram:

$$6 - (10 \times I_2) - (2 \times I_1) = 0$$

Using the equation for the currents at node N to eliminate I_2 gives:

$$6 - 10(I_3 + I_1) - 2I_1 = 0 \qquad \text{or} \qquad 10I_3 + 12I_1 = 6$$

Now choose another loop. In this case let us choose loop 2, the loop formed by branch B and the central branch:

$$-4 + (3 \times I_3) + (10 \times I_2) = 0$$

Once again eliminating I_2 gives:

$$-4 + 3I_3 + 10(I_1 + I_3) = 0 \qquad \text{or} \qquad 10I_1 + 13I_3 = 4$$

We now have two equations and two unknown currents which can be solved. Multiplying loop1 equation by 5 and the loop2 equation by 6 gives:

$$50I_3 + 60I_1 = 30$$

$$78I_3 + 60I_1 = 24$$

Subtracting these equations gives:

$$-28I_3 = 6 \qquad \text{or} \qquad I_3 = -0.214A$$

And back substituting gives:

$$50 \times (-0.214) + 60I_1 = 30 \qquad \text{or} \qquad I_1 = 0.678A$$

and

$$I_2 = I_1 + I_3 = 0.678 - 0.214 = 0.464A$$

Which are the same values as found by the superposition method.

Note: In the above example we chose loops 1 and 2 to form the two simultaneous equations. We could just as easily have chosen the loop around the outside branches, A and B, as one of our equations. In this case we would have obtained:

$$6 - 4 + 3I_3 - 2I_1 = 0$$

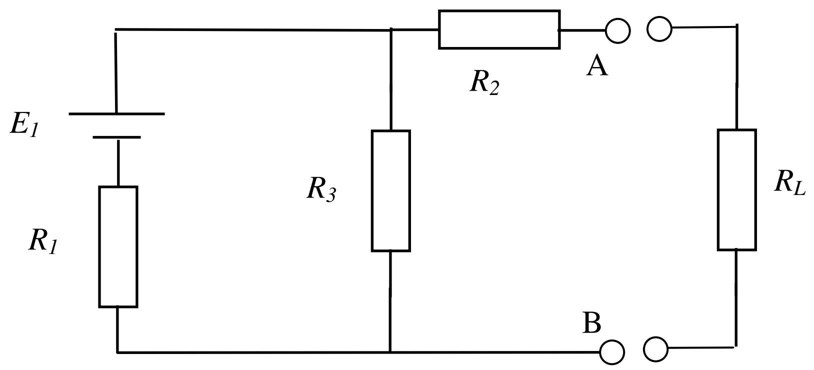
and proceeded to solve the problem in a similar manner.

Thévenin's Theorem

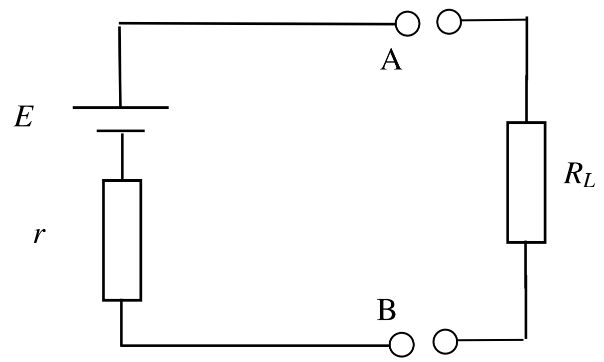
This is an extension of the Superposition Theorem. It is particularly useful when we are looking at circuits whose 'load' resistance is changing. This saves having to analyse the full network when different loads are connected.

An active network having 2 terminals (A and B) can be replaced by a constant voltage source, ***E***, the magnitude of which is equal to the open-circuit voltage between A and B, and an internal resistance, ***r***. ***r*** is the resistance between A and B with the load disconnected and emf sources replaced by their internal resistance.

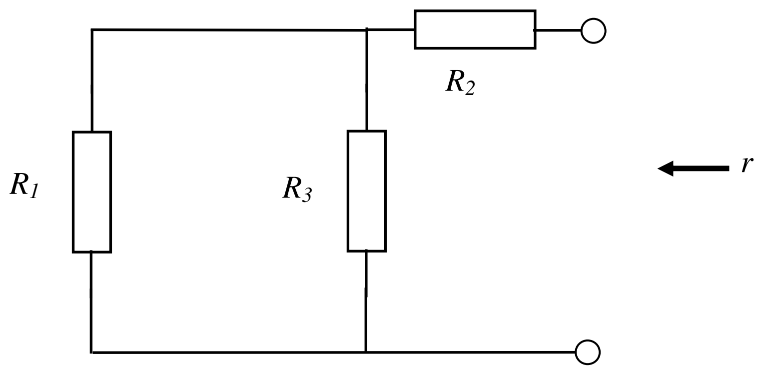
For example consider the following circuit:



This can be represented by the Thévenin equivalent circuit:



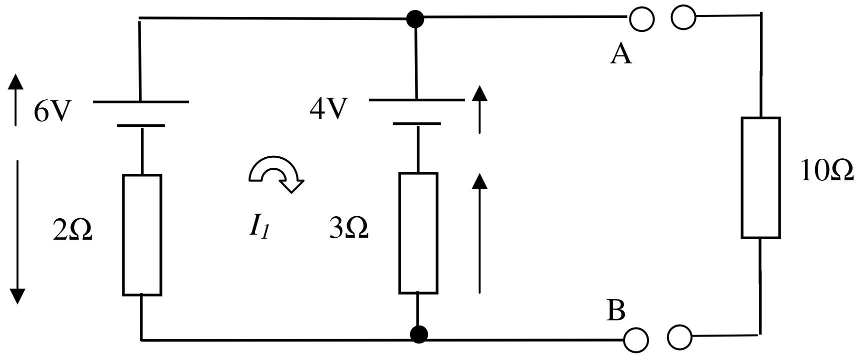
Where *E* is the open-circuit (load disconnected) voltage between A and B and the equivalent resistance, 'r' is obtained from the following circuit:



(Note: The above network of resistors is only true for the particular circuit shown here).

Example

Using the example that was used previously with Kirchhoff's laws, but rearranging the layout slightly. Move the 10Ω resistor to the right-hand side. This circuit is exactly the same as before.



With the 10Ω load disconnected, sum the voltages around the loop. Since no current can flow in the load branch (load disconnected) then current I_1 only flows round the loop formed by the two batteries as shown in the diagram. Once again indicating the direction of voltages reduces the chance of making errors (For sources – negative to positive; for sinks – in the opposite direction to the defined current). Hence summing around the loop:

$$6-4-3I_1-2I_1=0$$

therefore:

$$I_1=0.4\ \text{A}$$

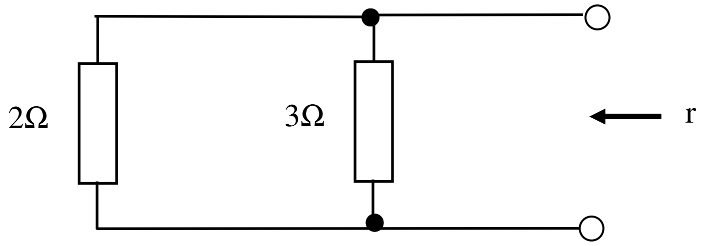
The potential difference across AB is:

$$4+3I_1=5.2\ \text{V}$$

Alternatively:

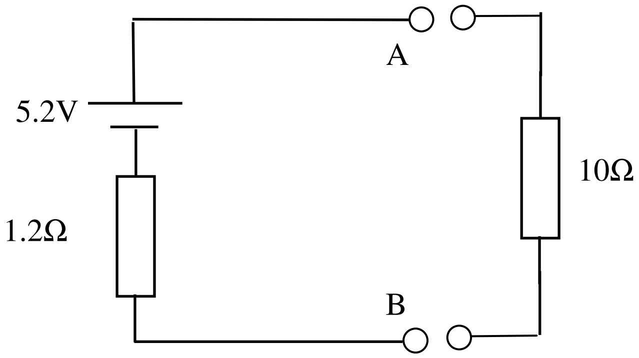
$$6-2I_1=5.2\ \text{V}$$

Now find 'r' with the voltage source short-circuited removed:



$$r=\frac{2\times 3}{2+3}=1.2\ \Omega$$

Therefore the Thévenin equivalent circuit becomes:



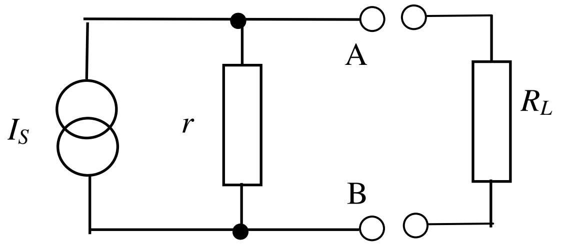
The current through the 10Ω resistor is then:

$$I_L=\frac{V}{r+R_L}=\frac{5.2}{11.2}=0.464\ \text{A}$$

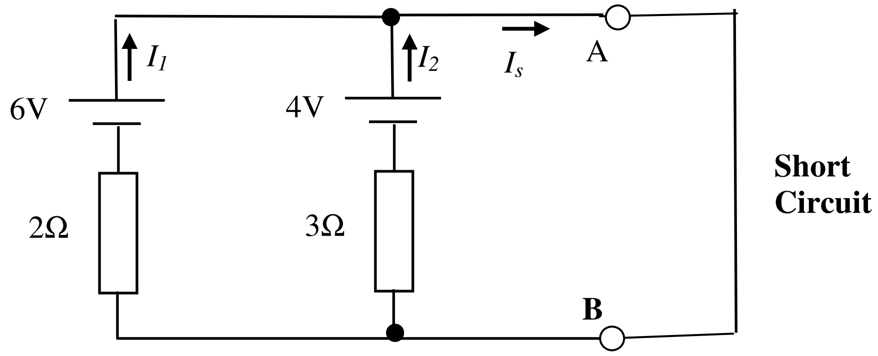
Norton's Theorem

This follows on from Thévenin's theorem.

An active network having 2 terminals (A and B) can be replaced by a constant current source, $\boldsymbol{I_S}$, where $\boldsymbol{I_S}$ is equal to the short circuit current at the terminals, and a shunt resistance $\boldsymbol{r}$, which is equal to the resistance between (A and B) with the sources removed (same as for Thévenin).



Considering our previous example again:



Placing the short-circuit across terminals A and B means that they are at the same potential, i.e. the potential difference between A and B is 0V, hence:

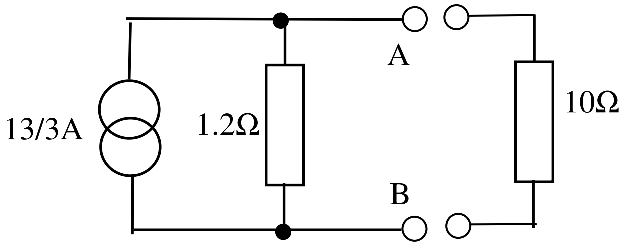
$$6 - 2I_1 = 0 \;\Rightarrow\; I_1 = 3 \text{ A}$$

$$4 - 3I_2 = 0 \;\Rightarrow\; I_2 = \frac{4}{3} \text{ A}$$

therefore:

$$I_S = I_1 + I_2 = \frac{13}{3} \text{ A}$$

and $r = 1.2\,\Omega$ as before. The Norton equivalent circuit thus becomes:



$$I_L = \frac{13}{3} \times \frac{1.2}{10 + 1.2} = 0.464 \text{ A}$$

as before.

Note: since our Thévenin circuit represents our original network we could have simply shorted out the terminals of our Thévenin circuit to obtain the short-circuit current for use in the Norton equivalent circuit. $I_S = \frac{5.2}{1.2} = 4.333A \quad or \quad \frac{13}{3}A$