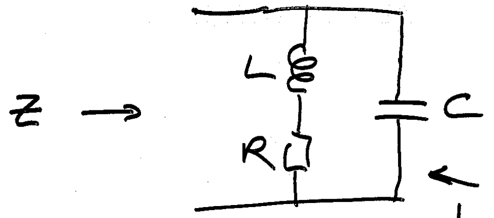


typical radio set tuning cct.

$$Z = \frac{\frac{1}{jwC}(R+jwL)}{\frac{1}{jwC} + R + jwL}$$

$$= \frac{R+jwL}{1 + jwCR + (jw)^2LC}$$

$$= \frac{R+jwL}{(1-w^2LC) + jwCR}$$

$$= \frac{(R+jwL)(1-w^2LC - jwCR)}{(1-w^2LC)^2 + w^2C^2R^2}$$

taking just the j-term products from the top line (ie only the j×real terms)

$$\frac{jwL(1-w^2LC) - jwCR^2}{\sim} = 0$$

for resonance

2 2 2

ie $L - w^2L^2C - CR^2 = 0$

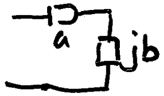
$$L - CR^2 = w^2L^2C$$

$$\frac{L-CR^2}{L^2C} = w^2 = \frac{1}{LC} - \frac{R^2}{L^2}$$

$$\therefore \quad w = \sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}$$

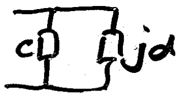
Series to parallel transformations allow an impedance

$Z = a + jb$



to be expressed as an admittance

$Y = c + jd$.



eg $\frac{1}{Z} = \frac{1}{a+jb} = Y = \frac{a-jb}{a^2+b^2}$

$$\therefore \quad c \equiv \frac{a}{a^2+b^2}$$

$$d \equiv \frac{-jb}{a^2+b^2}$$

Transient behaviour of circuits

— what is the response of the circuit to a sudden step change in its input conditions ?

for C $I = C\frac{dV}{dt}$ or $V = \frac{1}{C}\int I\,dt$.

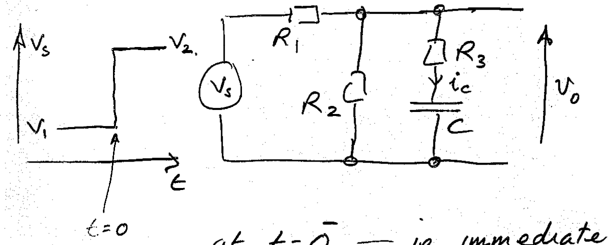
for L $V = L\frac{dI}{dt}$ or $I = \frac{1}{L}\int V\,dt$

Initial Conditions

— what is the state of the circuit before the step ?

— what will be the conditions just after the step.

consider the cct



at $t=0^-$ — ie immediately before the step.

$i_c = 0$ (because no d.c. current flows through a capacitor).

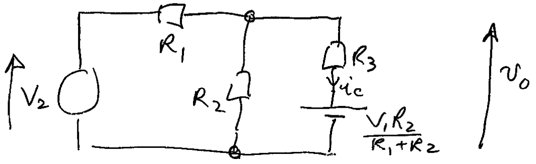
$$v_0 = V_s \frac{R_2}{R_1+R_2} = V_1 \frac{R_2}{R_1+R_2}.$$

$$\left[\; v_c = v_0 = \frac{V_1 R_2}{R_1+R_2}. \;\right]$$

at $t=0^+$ — ie immediately after the step.

$\left[\; v_c = \frac{V_1 R_2}{R_1+R_2} \Rightarrow \right.$ between $t=0^-$ + $t=0^+$ C behaves like a voltage source. $\left.\right]$

We can re-draw the circuit at $t=0^+$ with C replaced by a voltage source of value $\frac{V_1 R_2}{R_1+R_2}$

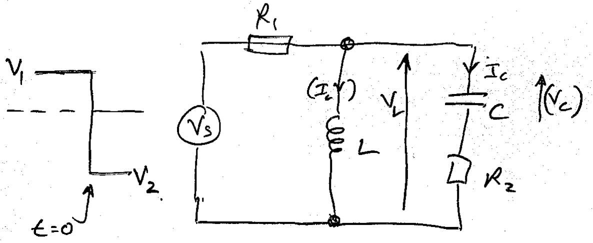


the $t=0^+$ equivalent cct.

$$v_0 = V_2 \frac{R_2 /\!/ R_3}{R_1 + R_2 /\!/ R_3} + \frac{V_1 R_2}{R_1+R_2} \cdot \frac{R_1 /\!/ R_2}{R_3 + R_1 /\!/ R_2}.$$

$$i_c = \frac{v_0 - \frac{V_1 R_2}{R_1+R_2}}{R_3}.$$

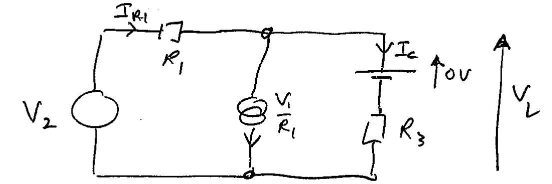
Consider another cct



conditions at $t=0^-$

$$V_L = 0$$
$$I_c = 0$$
$$\left[V_c = 0 \,, \quad I_L = \frac{V_1}{R_1} \right].$$

at $t = 0^+$

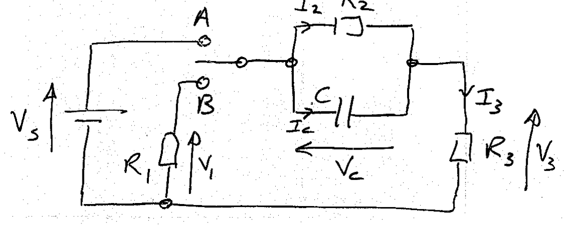


$$V_L = V_2 \frac{R_3}{R_1+R3} + \left(-\frac{V_1}{R_1} . \; R_1 \| R_3 \right)$$

$$I_{R1} = \frac{V_2 - V_L}{R_1}$$

$$I_c = I_{R1} - \frac{V_1}{R_1}$$

An example transient analysis



assume switch has been in position A for a long time

- aim is to form a differential equation with V_c as the subject

- first sum voltages at $t = 0^+$

$$V_1 = V_c + V_3$$
$$\swarrow \qquad \downarrow \qquad \searrow$$
$$-I_3 R_1 = V_c + I_3 R_3 .$$

or
$$0 = V_c + I_3 (R_1 + R_3).$$

Summing currents at the R_3, R_2 node.

$$I_3 = I_c + I_2 = C\frac{dV_c}{dt} + \frac{V_c}{R_2}$$

$$0 = V_c + \left(C\frac{dV_c}{dt} + \frac{V_c}{R_2} \right)(R_3 + R_1)$$

$$-V_c \left[1 + \frac{R_3+R_1}{R_2} \right] = C\frac{dV_c}{dt}(R_1 + R_3)$$

$$-\frac{1 + \frac{R_3+R_1}{R_2}}{C(R_1+R_3)} \, dt = \frac{dV_c}{V_c}$$

let this = k

Integrate both sides

$$-kt + Const = \ln V_c$$

$$e^{(-kt + Const)} = V_c = e^{-kt} e^{Const} = Ae^{-kt}$$

where A = const. (another one)

The conditions at $t = 0^+$ will give the value of A

when $t = 0^+$ $\quad V_c = V_s \frac{R_2}{R_3+R_2}$.

$$\therefore \quad V_c(t) = V_s \frac{R_2}{R_3+R_2} e^{-kt}$$

$$k = \frac{R_2+R_3+R_1}{C R_2(R_1+R_3)} = \frac{1}{\tau}$$

$$V_c(t) = V_s \frac{R_2}{R_3+R_2} e^{-t/\tau}$$

$$\tau = \frac{1}{k} = \frac{C R_2(R_1+R_3)}{R_2+R_3+R_1}$$

If at the outset one had gone for an equation with I_c as the subject.

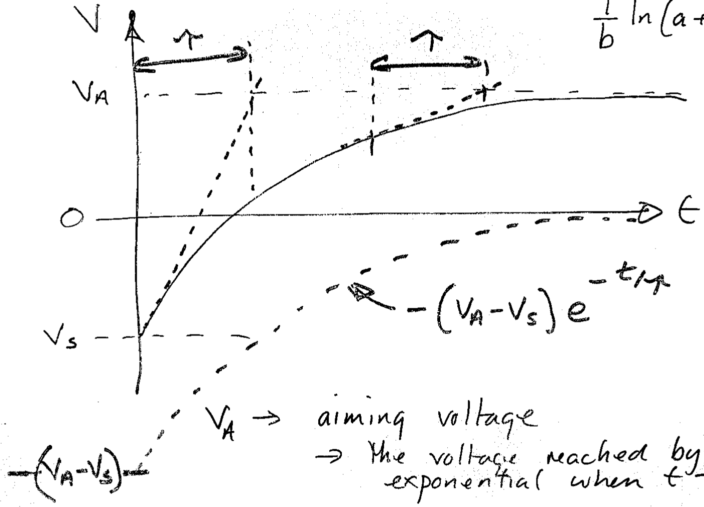
$$-I_c = \frac{dI_c}{dt} \cdot \frac{C(R_1+R_3)R_2}{R_1+R_2+R_3}$$

— This is the same equation as the one just derived except that Vc is replaced by Ic. This because all voltages + currents in the circuit arising as a response to the step will have the same exponential form.

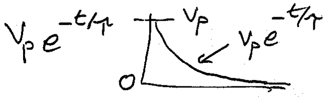
Exponential Geometry.

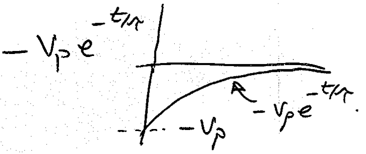
$$\int \frac{dx}{a+bx} =$$

$$\int \frac{dx}{a+bx} = \frac{1}{b} \ln(a+bx)$$



$V_A \rightarrow$ aiming voltage
$\rightarrow$ the voltage reached by the exponential when $t \rightarrow \infty$





need to add V_A to the basic shape

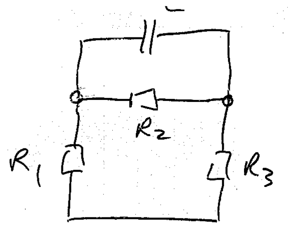
$$-(V_A-V_s)\,e^{-t/\tau}$$

to give $\quad V(t) = V_A - (V_A-V_s)\,e^{-t/\tau}$

For any simple exponential

$$V(t) = (V_{start} - V_{aiming})\,e^{-t/\tau} + V_{aiming}$$

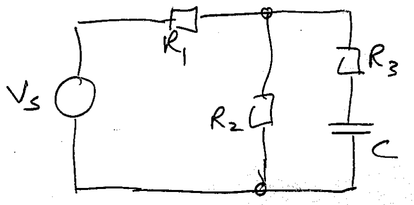
To find τ



look at cct from point of view of C

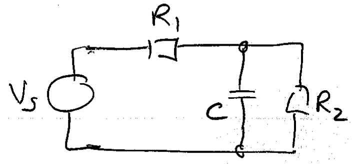
$$\tau = C\, R_2 \| (R_1 + R_3)$$

look at another circuit

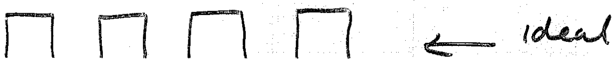


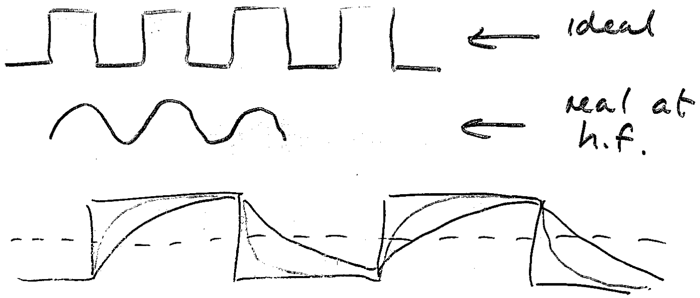
$$\tau = C\left(R_3 + (R_1 \| R_2)\right)$$

look at another cct



$$\tau = C(R_1 \| R_2)$$





Feedback on Homework 4.

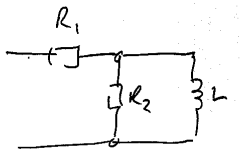
$$V\angle\theta \Rightarrow a + jb.$$

$$a = V\cos\theta$$
$$b = V\sin\theta$$

$$\underset{\text{magnitude}}{\underset{\uparrow}{3}}\cos(wt + \underset{\text{phase.}}{\underset{\uparrow}{45}}) \qquad \cos(wt + \underset{\nearrow}{0})$$

$$3\angle(wt+45) \rightarrow \times \quad \text{should be } 3\angle 45$$

$$4\sin wt = 4\cos(wt - 90)$$
$$4\angle -90.$$



$$Z = R_1 + R_2 \| \underset{\text{write as } Z_L}{\underset{|}{L}}$$

$$R_1 + \frac{R_2 L}{R_2 + L}$$

$$= R_1 + \frac{R_2 j\omega L}{R_2 + j\omega L}$$

Decibels

— originally a power ratio

P in dB $= 10 \log \frac{P_1}{P_2}$

often power (acoustically) is specified in terms of dB

eg 90 dB

— only makes sense if a reference level is defined

— usually human hearing threshold $\approx 10^{-12}$ W.

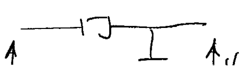
In terms of voltage

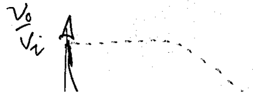
$$\text{dB} = 20 \log \frac{V_2}{V_1}$$

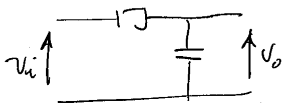
$$\text{dBV} = 20 \log \frac{V}{1\text{Vrms}}.$$

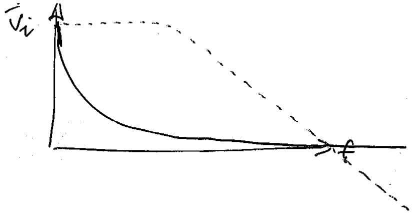
In terms of electrical

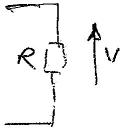
$$\text{dB}_m = 10 \log \frac{P}{1\text{mW into } 50\Omega}$$







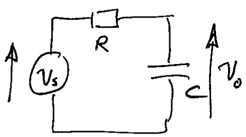


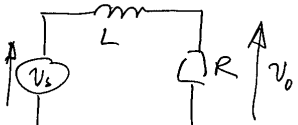


Voltage in dB $20 \log \frac{V}{\text{ref.}}$

Power in dB $10 \log \frac{V^2}{(\text{ref})^2}$

Standard forms of 1st order ccts.





$$\frac{v_o}{v_s} = \frac{1/j\omega C}{R + \frac{1}{j\omega C}}$$

$$= \frac{1}{1 + j\omega CR}$$

define constant ω_0 where $\omega_0 = \frac{1}{CR}$

$$= \frac{1}{1 + j\,\omega/\omega_0}$$

$$\frac{v_o}{v_s} = \frac{R}{R + j\omega L}$$

$$= \frac{1}{1 + j\omega\, L/R}.$$

define a constant ω_1 where $\omega_1 = \frac{1}{L/R}$

$$= \frac{1}{1 + j\,\omega/\omega_1}$$

This function has a well defined response

— look at 3 frequencies

$f \ll f_0$ $(\omega \ll \omega_0)$
$f \gg f_0$ $(\omega \gg \omega_0)$
$f = f_0$ $(\omega = \omega_0)$

$$\frac{v_o}{v_s} = \frac{1}{1 + j\omega/\omega_0}$$

$$\left|\frac{v_0}{v_s}\right| = \left[\frac{1}{1+(\omega/\omega_0)^2}\right]^{1/2}$$

$$\left|\frac{v_0}{v_s}\right|_{f \ll f_0} = \left[\frac{1}{1}\right]^{1/2} \begin{bmatrix} \text{because} \left(\frac{\omega}{\omega_0}\right)^2 \\ \ll 1 \end{bmatrix} = 1 \Rightarrow 0\text{dB}.$$

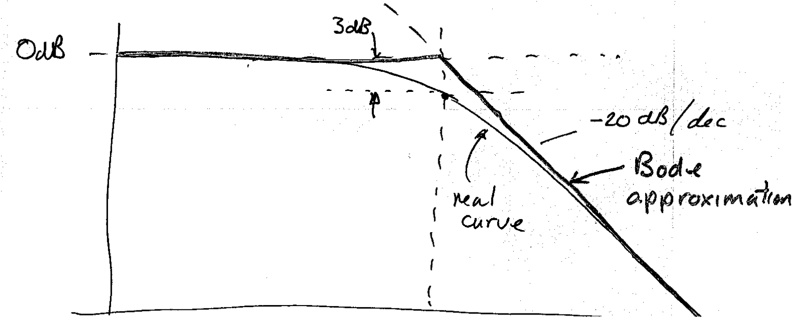
$$\left|\frac{v_0}{v_s}\right|_{f=0} = \left[\frac{1}{1+1}\right]^{1/2} = \frac{1}{\sqrt{2}} \quad \Rightarrow -3\text{dB}.$$

$$\left|\frac{v_0}{v_s}\right|_{f \gg f_0} = \left[\frac{1}{(\omega/\omega_0)^2}\right]^{1/2} \begin{bmatrix} \text{because} \left(\frac{\omega}{\omega_0}\right)^2 \\ \gg 1 \end{bmatrix} = \frac{\omega_0}{\omega}$$

⇒ slope of − 20 dB/decade on a logarithmic plot.

or − 6dB/octave

(an octave = factor of 2 change in f.)



ω_0 ← − 3dB frequency.
− corner frequency.
− band width

What about phase.

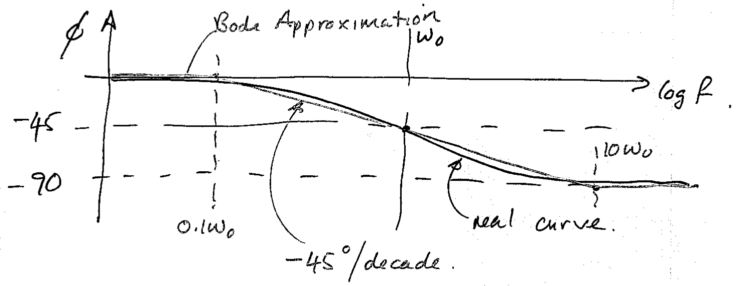
$$\frac{v_0}{v_s} = \frac{1}{1 + j\omega/\omega_0}$$

$$\angle\left(\frac{v_0}{v_s}\right) = -\tan^{-1} \frac{\omega/\omega_0}{1}$$

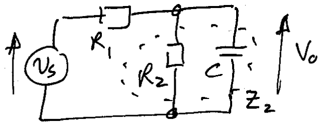
when $\omega \ll \omega_0$ $\phi \Rightarrow -\tan^{-1}$(a small fraction)
$\Rightarrow$ 0 from below.

when $\omega = \omega_0$ $\phi = -\tan^{-1} 1 = -45°$

when $\omega \gg \omega_0$ $\phi = -\tan^{-1}$(a large number)
$\Rightarrow$ −90 from above.



The most common complication is the introduction of a gain term.



$$\frac{V_o}{V_s} = \frac{Z_2}{R_1 + Z_2} \qquad Z_2 = R_2 /\!/ Z_c = \frac{R_2 / j\omega C}{R_2 + 1/j\omega C}$$

$$= \frac{\frac{R_2}{1 + j\omega C R_2}}{R_1 + \frac{R_2}{1 + j\omega C R_2}} \qquad = \frac{R_2}{1 + j\omega C R_2}$$

$$= \frac{R_2}{R_1(1 + j\omega C R_2) + R_2} = \frac{R_2}{(R_1 + R_2) + j\omega C R_1 R_2}$$

$$= \frac{R_2}{(R_1 + R_2)\left(1 + j\omega C \frac{R_1 R_2}{R_1 + R_2}\right)}$$

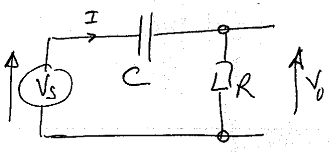
$$= k \cdot \frac{1}{1 + j\omega/\omega_0} \qquad \text{where } k = \frac{R_2}{R_1 + R_2}$$

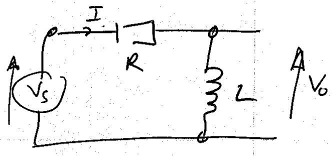
$$\omega_0 = \frac{1}{C \frac{R_1 R_2}{R_1 + R_2}}$$

$$20 \log \left| k \cdot \frac{1}{1 + j\omega/\omega_0} \right|$$

$$= 20 \log k + 20 \log \left| \frac{1}{1 + j\omega/\omega_0} \right|$$

High pass circuits





$$I = \frac{V_s}{Z} \qquad\qquad I = \frac{V_s}{Z}$$

$$= \frac{V_s}{R + 1/j\omega C} \qquad\qquad = \frac{V_s}{R + j\omega L}$$

$$V = I j\omega L$$

$$V_o = IR$$

$$= \frac{V_s R}{R + 1/j\omega C}$$

$$\frac{V_o}{V_s} = \frac{j\omega C R}{1 + j\omega C R}$$

$$\text{let } CR = \frac{1}{\omega_0}$$

$$\frac{V_o}{V_s} = \frac{j\omega/\omega_0}{1 + j\omega/\omega_0}$$

$$V_o = I j\omega L$$

$$= \frac{V_s j\omega L}{R + j\omega L}$$

$$\frac{V_o}{V_s} = \frac{j\omega L/R}{1 + j\omega L/R}$$

$$\text{let } \frac{L}{R} = \frac{1}{\omega_1}$$

$$\frac{V_o}{V_s} = \frac{j\omega/\omega_1}{1 + j\omega/\omega_1}$$

$$20 \log \left| \frac{V_o}{V_s} \right| = 20 \log \left| \frac{j\omega/\omega_0}{1 + j\omega/\omega_0} \right|$$

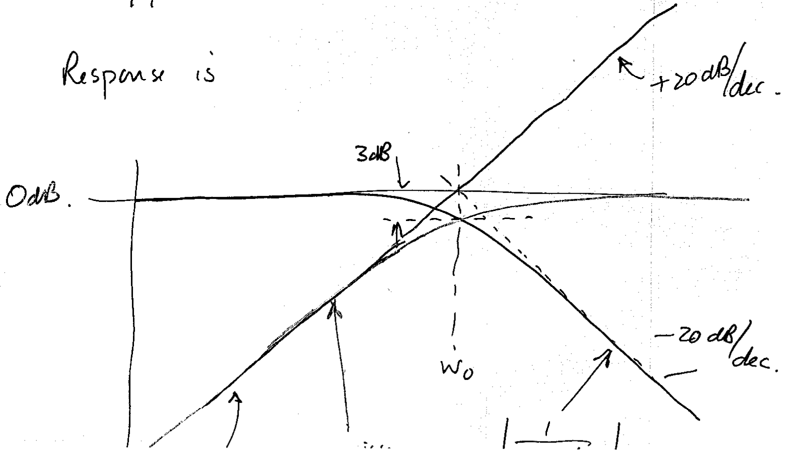
$$= 20 \log \left| j\omega/\omega_0 \right| + 20 \log \left| \frac{1}{1 + j\omega/\omega_0} \right|$$

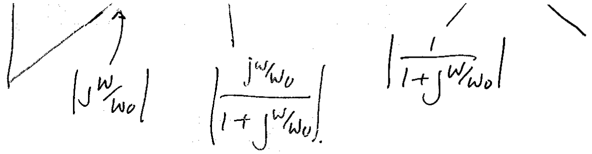
This is the low pass response shape.

$$\left| j \frac{\omega}{\omega_0} \right| = \frac{\omega}{\omega_0}$$

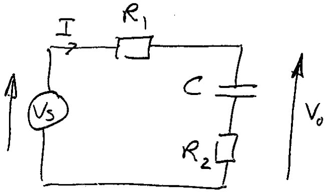
$$- \text{ie } |\,| \propto \omega$$

Response is





Linear sum of high pass + low pass .....



$$\frac{V_s}{Z} = I$$

$$I = \frac{V_s}{R_1 + \frac{1}{jwC} + R_2}$$

$$V_o = I\left(R_2 + \frac{1}{jwC}\right)$$

$$= \frac{V_s\left(R_2 + \frac{1}{jwC}\right)}{R_1 + R_2 + \frac{1}{jwC}}$$

$$\frac{V_o}{V_s} = \frac{1 + jwCR_2}{1 + jwC(R_1 + R_2)}$$

let $\frac{1}{CR_2} = w_0$ let $\frac{1}{C(R_1+R_2)} = w_1$

$$\frac{V_o}{V_s} = \frac{1 + j\,w/w_0}{1 + j\,w/w_1}$$

ASIDE

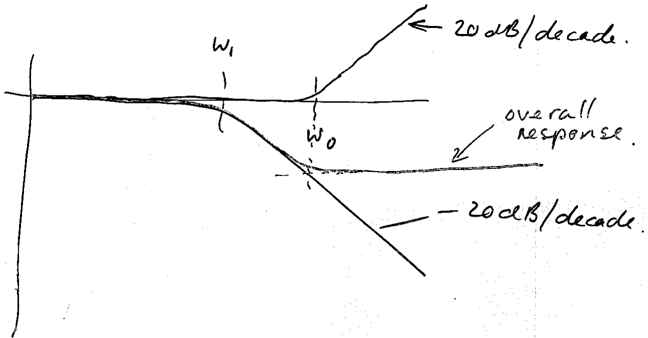
$$\frac{1 + j\,w/w_0}{1 + j\,w/w_1} = \frac{1}{1 + j\,w/w_1} + \frac{j\,w/w_0}{1 + j\,w/w_1}$$

$$= \frac{1}{1 + j\,w/w_1} + \frac{\frac{1 \times w_1}{w_0}\, jw/w_1}{1 + j\,w/w_1}$$

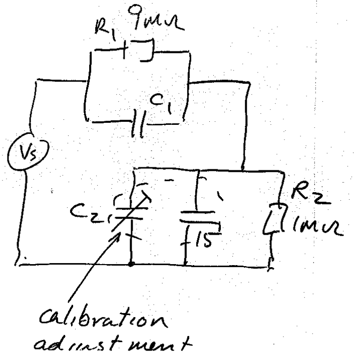
$$= \frac{1}{1 + j\,w/w_1} + \frac{w_1}{w_0} \cdot \frac{j\,w/w_1}{1 + j\,w/w_1}$$

$$20\log\left|\frac{V_o}{V_s}\right| = 20\log\left|\frac{1 + j\,w/w_0}{1 + j\,w/w_1}\right|$$

$$= 20\log\left|1 + j\,w/w_0\right| + 20\log\left|\frac{1}{1 + j\,w/w_1}\right|$$



Scope probe



$$\frac{V_o}{V_i} = \frac{\dfrac{R_2/j\omega C_2}{R_2+1/j\omega C_2}}{\dfrac{R_1/j\omega C_1}{R_1+1/j\omega C_1}+\dfrac{R_2/j\omega C_2}{R_2+1/j\omega C_2}} = \frac{\dfrac{R_2}{1+j\omega C_2R_2}}{\dfrac{R_1}{1+j\omega C_1R_1}+\dfrac{R_2}{1+j\omega C_2R_2}}$$

$$= \frac{\left(\dfrac{R_2}{\cancel{1+j\omega C_2R_2}}\right)}{\left(\dfrac{R_1(1+j\omega C_2R_2)+R_2(1+j\omega C_1R_1)}{(1+j\omega C_1R_1)(\cancel{1+j\omega C_2R_2})}\right)}$$

$$= \frac{R_2\,(1+j\omega C_1R_1)}{R_1+R_2 + j\omega(C_1+C_2)R_1R_2}$$

$$= \frac{R_2}{R_1+R_2}\cdot\frac{1+j\omega C_1R_1}{1+j\omega(C_1+C_2)\dfrac{R_1R_2}{R_1+R_2}}$$

To make the frequency dependent terms disappear, need to make time constants in the numerator and denominator the same

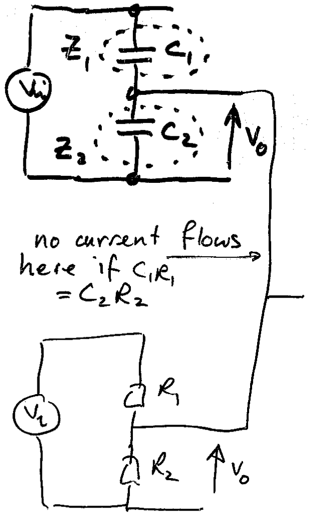
ie $C_1R_1 = (C_1+C_2)\dfrac{R_1R_2}{R_1+R_2}$

or $C_1R_1(R_1+R_2) = (C_1+C_2)R_1R_2$

or $C_1R_1^2 + \cancel{C_1R_1R_2} = \cancel{C_1R_1R_2} + C_2R_1R_2$

$C_1R_1^2 = C_2R_1R_2$

or $C_1R_1 = C_2R_2$



no current flows here if $C_1R_1 = C_2R_2$

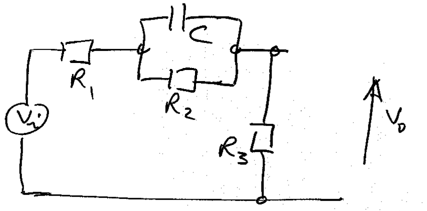
$$\frac{V_o}{V_i} = \frac{Z_2}{Z_1+Z_2}$$

$$= \frac{1/\cancel{j\omega}C_2}{\dfrac{1}{\cancel{j\omega}C_1}+\dfrac{1}{\cancel{j\omega}C_2}} = \frac{\dfrac{1}{C_2}}{\dfrac{1}{C_1}+\dfrac{1}{C_2}}$$

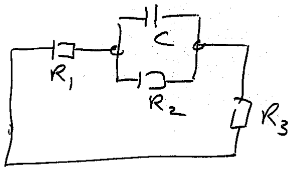
$$= \frac{\dfrac{1}{C_2}}{\dfrac{C_1+C_2}{C_1C_2}} = \frac{\dfrac{C_1C_2}{C_2}}{C_1+C_2} = \frac{C_1}{C_1+C_2}$$

$$\frac{V_o}{V_i} = \frac{R_2}{R_1+R_2}$$

Time constant by inspection



(1) replace source by its internal impedance



from C's point of view there are two possible discharge paths

— one through R_2

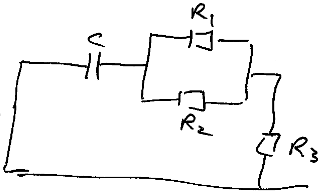
— one through the R_1 R_3 combination

the two paths are in parallel

Net resistance seen by C is $R_2 || (R_1+R_3)$

so $\tau = C\ R_2 || (R_1+R_3)$

If the cct was

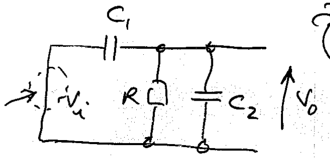


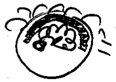
V_i replaced by its internal resistance.

Here there is only one path. out of C through $R_1 || R_2$ and through R_3

$$\tau = C\left(R_3 + (R_1 || R_2)\right)$$

If the cct was





V_i replaced by its internal resistance

$$\tau = R\left(C_1+C_2\right)$$

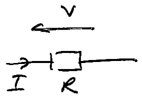
$$\frac{V_o}{V_i} = \frac{\dfrac{R/j\omega C_2}{R + 1/j\omega C_2}}{\dfrac{1}{j\omega C_1} + \dfrac{R/j\omega C_2}{R + 1/j\omega C_2}} = \frac{\dfrac{R}{1+j\omega C_2 R}}{\dfrac{1}{j\omega C_1} + \dfrac{R}{1+j\omega C_2 R}}$$

$$= \frac{\dfrac{R}{\cancel{1+j\omega C_2 R}}}{\left(\dfrac{1+j\omega C_2 R + j\omega C_1 R}{j\omega C_1 \cancel{(1+j\omega C_2 R)}}\right)} = \frac{j\omega C_1 R}{1+j\omega R(C_1+C_2)}$$

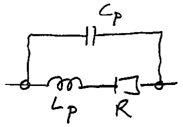
$$= \frac{C_1 \cancel{R}}{\cancel{R}(C_1+C_2)} \quad \frac{j\omega R(C_1+C_2)}{1+j\omega R(C_1+C_2)}$$

$$= \quad k \, . \quad \frac{j\omega/\omega_0}{1+j\omega/\omega_0}$$

Some practical issues with inductors resistors and capacitors.



$V = IR$



← a more realistic model of a resistor

$L_p \approx 10nH.$

$C_p \approx 0.5$ to $1pF$

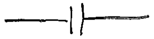
Z of $10nH$ @ $100MHz$ $\approx 2\pi f L$

$6.10^8 . 10^{-8}$

$\approx 6\Omega.$

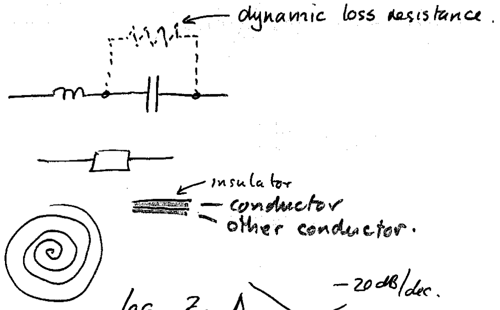
Z of $1pF$ at $1MHz$

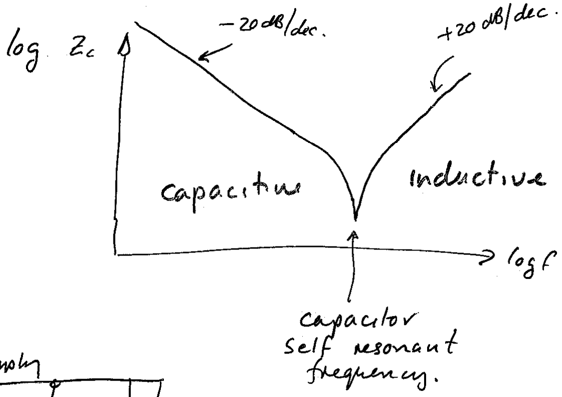
$$Z_c = \frac{1}{2\pi f C} \approx \frac{1}{6 \times 10^{-12} \times 10^6} \approx 0.16 \times 10^6 \approx 160 k\Omega .$$

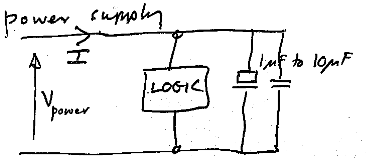


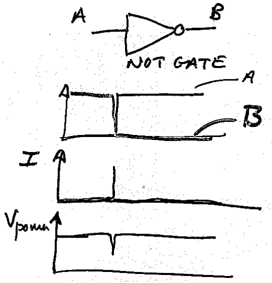
ideal model

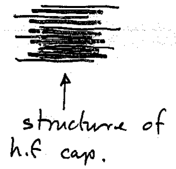
— dynamic loss resistance

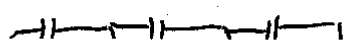


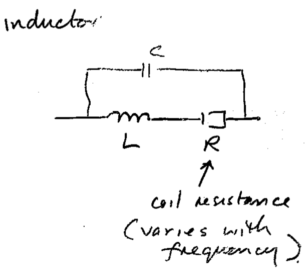


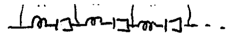


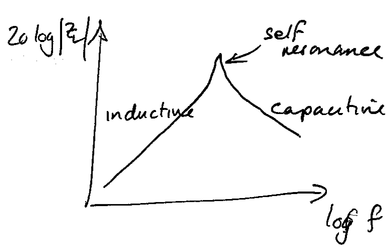






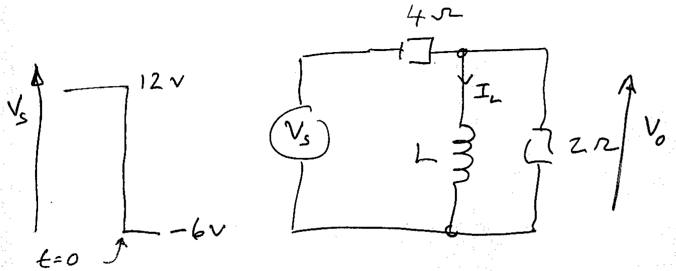






inductor $Q = \frac{\omega L}{R}$

specified by manufacturers at a particular frequency



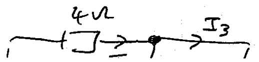
Find I_L + V_o at $t=0^-$ $t=0^+$ and as $t \Rightarrow \infty$

at $t=0^-$ $V_o = V_L = \underline{0\,V}$ since $\frac{dI_L}{dt}$ must be zero at dc.

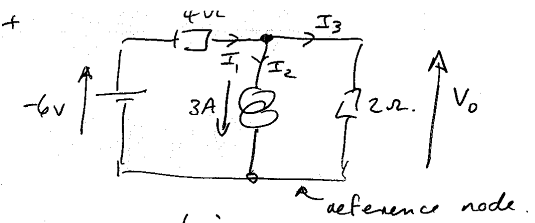
$I_L = \frac{V_s - V_o}{4\Omega} = \frac{V_s - 0}{4\Omega}$. since $V_o = 0\,V$ so $I_{2\Omega} = 0$

$= \frac{12}{4} = \underline{3A}$.

at $t = 0^+$



at $t = 0^+$



Could use superposition

$$V_0\Big|_{-6V} = -6 \times \frac{2}{2+4} = -2V$$

$$V_0\Big|_{3A} = -3A \times \frac{2\times4}{2+4} = -\frac{8\times3}{6} V$$

$$= -4V.$$

$$V_0\Big|_{TOT} = (-2 + -4)V = -6V.$$

A nodal approach $I_1 = I_2 + I_3$

$$\frac{-6-V_0}{4} = 3 + \frac{V_0}{2}$$

$$-\frac{6}{4} - 3 = \frac{V_0}{2} + \frac{V_0}{4}$$

$$-6 - 12 = 2V_0 + V_0$$

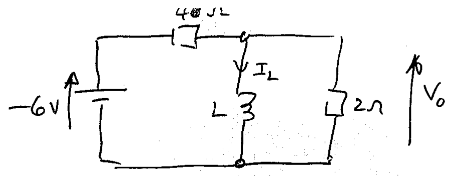
$$-18 = 3V_0$$

$$V_0 = -6V.$$

As $t \Rightarrow \infty$

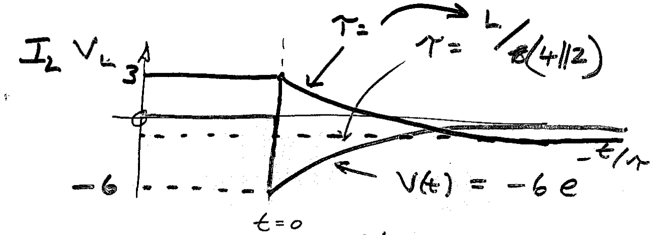
- all the changes that happened at the transient will have settled down
  - all quantities will be constant — ie dc.

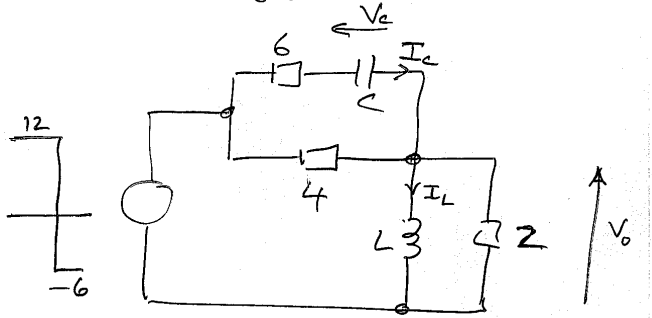
4Ω



since this is a dc problem $V_L = 0$
(because $\frac{dI_L}{dt} = 0$)

$$I_L = \frac{-6V - 0}{4Ω} = -1.5A.$$





at $t = 0^-$

$I_c = 0$ c doesn't conduct at dc

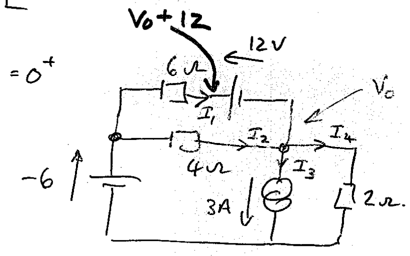
$V_o = 0$ L looks like o/c at d.c.

$$I_L = \frac{12 - V_o}{4} = \frac{12}{4} = 3A.$$

$V_o = V$

$V_c = V_{across\ 4\Omega} = 12\ V$

at $t = 0^+$



$I_1 + I_2 = I_3 + I_4$

$3 + (-4.09)$

$= -1.64 + 0.55$

$$\frac{-6-(V_0+12)}{6} + \frac{-6-V_0}{4} = 3 + \frac{V_0}{2}$$

$$-1 - \frac{V_0}{6} - 2 + (-1.5) - \frac{V_0}{4} = 3 + \frac{V_0}{2}$$

$$-4.5 - 3 = \frac{V_o}{2} + \frac{V_o}{4} + \frac{V_o}{6}$$

$$-27 - 18 = 3V_o + 1.5V_o + V_o$$

$$-45 = 5.5V_o$$

$$V_o = \frac{45}{5.5} - \frac{9}{1.1} = -8.18\ V.$$

$$\frac{-6-(-8.18)}{4}$$

$\frac{2.18}{4}$ 0.55

$$I_c = I_1 = \frac{-6-(-8.18+12)}{6.}$$

$$= -1 - \frac{3.82}{6}$$

$$= -1.64\ A$$