THE UNIVERSITY OF SHEFFIELD

GUIDE SOLUTIONS FOR EXTERNAL EXAMINER

SETTER: P. Trodden

Data Provided:
Laplace and z-transforms
Compensator design formulae
Performance criteria mappings
Ziegler-Nichols tuning rules

DEPARTMENT OF AUTOMATIC CONTROL & SYSTEMS ENGINEERING
Autumn Semester 2016–2017

ACS342 FEEDBACK SYSTEMS DESIGN 2 hours

Answer THREE questions.

No marks will be awarded for solutions to a fourth question.

Solutions will be considered in the order that they are presented in the answer book. Trial answers will be ignored if they are clearly crossed out.

If more than the required number of questions are attempted, DRAW A LINE THROUGH THE ANSWERS THAT YOU DO NOT WISH TO BE MARKED.

All questions are marked out of 20. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1. A unity-feedback control system has the open-loop transfer function

$$KG(s) = \frac{K}{s(s+5)(s+15)}$$

a) Sketch the root locus diagram of $KG(s)$. You **do not** need to calculate numerical values for the break-away point and the imaginary axis intersection points.

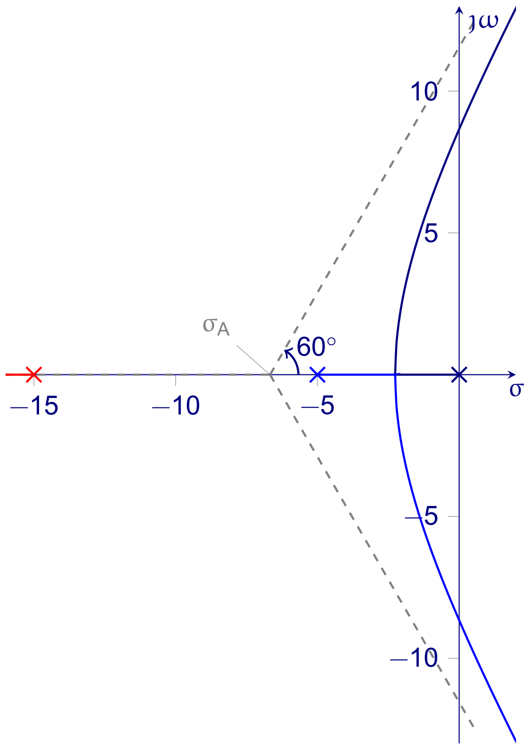
[6 marks]

Answer:

The root locus is constructed as follows.

Prepare sketch	OL poles at $\boxed{s = 0, -5, -15}$. No zeros. $n = 3, m = 0, \therefore$ three branches, all terminating at infinity.
Segments on real axis	$\boxed{s \in (-\infty, -10] \cup [-5, 0]}$.
Asymptotes	$\sigma_A = \frac{[0 + (-5) + (-15)] - [0]}{3}$ $= -6.67$ $\phi_A = \frac{2k+1}{3} 180^\circ, k = 0, 1, 2$ $= 60^\circ, 180^\circ, 300^\circ$

The sketch is shown below.



[6 marks]

b) Find the range of K for which the closed-loop system is stable.

[5 marks]

Answer:

First, obtain the closed-loop transfer function as

$$\frac{KG(s)}{1+KG(s)} = \frac{K}{s(s+5)(s+15)+K} = \frac{K}{s^3+20s^2+75s+K}$$

As this is third order, stability cannot be assessed by inspecting the coefficients of the characteristic function, but the Routh array can be used.

$$\begin{array}{lrr} s^3: & 1 & 75 \\ s^2: & 20 & K \\ s^1: & \frac{20\times 75 - 1\times K}{20} = 75 - \frac{K}{20} & 0 \\ s^0: & K & 0 \end{array}$$

For stability, require all same signs in the first column. Hence,

$$75 - \frac{K}{20} > 0 \implies K < 1500$$
$$K > 0$$

Therefore,

$$\boxed{0 < K < 1500}$$

[5 marks]

c) Show that the dominant pole location corresponding to an overshoot of 15% and a 2% settling time of 1 second is

$$s = -4.0 \pm \jmath 6.6$$

[5 marks]

Answer:

Assuming the closed-loop system may be approximated as second order, with characteristic function

$$s^2 + 2\zeta\omega_n s + \omega_n^2,$$

the dominant poles are at locations

$$s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$$

There are two ways then to approach the problem: one can show that the provided location leads to a pair (ζ, ω_n) corresponding to 15 % overshoot and a settling time of 1 second; alternatively, one can start from the overshoot and settling time specification, and calculate the required pair (ζ, ω_n) and, hence, the desired pole location. Both approaches are outline below.

1. By comparing the provided pole location to the expression $s = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}$, one obtains

$$\zeta\omega_n = 4$$
$$\omega_n\sqrt{1-\zeta^2} = 6.6$$

The relation between ζ, ω_n and settling time, T_s, is

$$T_s = \frac{4}{\zeta\omega_n}$$

Hence, with $\zeta\omega_n = 4$, $T_s = 1$, as required.
To show that the given location corresponds to 15% overshoot, we use the relation

$$\text{O.S. } (\%) \; = 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right).$$

Therefore, ζ needs to be calculated. To obtain this, eliminate $\omega_n = 4/\zeta$ from the simultaneous equations, to yield

$$\frac{4}{\zeta}\sqrt{1-\zeta^2} = 6.6$$

Therefore,

$$16(1-\zeta^2) - 43.56\zeta^2 = 0$$
$$\implies 59.56\zeta^2 - 16 = 0$$
$$\implies \zeta = +\sqrt{16/59.56} = 0.52$$

Substituting $\zeta = 0.52$ into the overshoot expression

$$\text{O.S. } (\%) \; = 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = \exp\left(\frac{-0.52\pi}{\sqrt{0.73}}\right) = 14.8\%$$

as required.

2. The other way uses the relation

$$T_s = \frac{4}{\zeta\omega_n}$$

to show that for $T_s = 1$, $\zeta\omega_n = 4$. This gives us the real part of the dominant pole, but we need to determine ζ and ω_n individually in order to find.
The relationship between ζ and overshoot is

$$\text{O.S. } (\%) \; = 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right),$$

which is inverted as

$$\frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = \ln(\text{O.S. } (\%)/100)$$
$$\implies \frac{\zeta^2}{1-\zeta^2} = \frac{1}{\pi^2}[\ln(\text{O.S. } (\%)/100)]^2$$
$$\implies \zeta = \frac{-\ln(\text{O.S. } (\%)/100)}{\sqrt{\pi^2 + [\ln(\text{O.S. } (\%)/100)]^2}}.$$

Therefore, with 15% overshoot,

$$\zeta = \frac{-\ln 0.15}{\sqrt{\pi^2 + [\ln 0.15]^2}} = 0.52.$$

Now, since $\zeta\omega_n = 4$, $\omega_n = 4/0.52 = 7.69$. Therefore, the imaginary part of the dominant pole is

$$\omega_n\sqrt{1-\zeta^2} = 7.69\sqrt{1-0.52^2} = 6.57.$$

Finally, the dominant pole location is (rounded to 1 d.p.)

$$\boxed{s = -4.0 \pm \jmath 6.6}$$

[5 marks]

d) Design a phase-lead compensator in order that the compensated root locus passes through the location given in part (c). (You do not need to design the gain, K, in the compensator.)

[4 marks]

Answer:

The transfer function of the phase-lead compensator is

$$C(s) = K\frac{s+z}{s+p}.$$

There are three parameters to select: K, z and p. These can be designed as follows.

1. Choose z. Normally, one would place the zero directly beneath the desired dominant pole, *i.e.*, $z = \zeta\omega_n = 4$. However, in this case, pole dominance would be affected. Therefore, the zero should be placed somewhere to the left of the pole at $s = -5$; for example, $z = 5.25$.
2. Calculate p. The location of p follows from applying the angle criterion to the desired dominant pole location, $s^\star = -4 + \jmath 6.6$, given the three poles of $G(s)$ and the pole and zero of $C(s)$. We require

$$\arg\big(C(s)G(s)\big)\Big|_{s=s^\star} = \phi(s^\star) = 180^\circ + k360^\circ$$

Therefore,

$$
\begin{aligned}
\phi(s^\star) &= \arg(s^\star + z) - \arg(s^\star + p) - \arg(s^\star) - \arg(s^\star + 5) - \arg(s^\star + 15) \\
&= \arg(-4 + \jmath 6.6 + 5.25) - \arg(-4 + p + \jmath 6.6) - \arg(-4 + \jmath 6.6) \\
&\quad - \arg(-4 + \jmath 6.6 + 5) + \arg(-4 + \jmath 6.6 + 15) \\
&= \arg(1.25 + \jmath 6.6) - \arg(p - 4 + \jmath 6.6) - \arg(-4 + \jmath 6.6) \\
&\quad - \arg(1 + \jmath 6.6) + \arg(11 + \jmath 6.6) \\
&= \tan^{-1}\left(\frac{6.6}{1.25}\right) - \tan^{-1}\left(\frac{6.6}{p-4}\right) - [180^\circ + \tan^{-1}\left(\frac{6.6}{-4}\right)] \\
&\quad - \tan^{-1}\left(\frac{6.6}{1}\right) - \tan^{-1}\left(\frac{6.6}{11}\right) \\
&= 79.3^\circ - \tan^{-1}\left(\frac{6.6}{p-4}\right) - 121.2^\circ - 81.4^\circ - 31.0^\circ \\
&= -154.3^\circ - \tan^{-1}\left(\frac{6.6}{p-4}\right) \\
&= 180^\circ + k360^\circ
\end{aligned}
$$

So,

$$
\begin{aligned}
\tan^{-1}\left(\frac{6.6}{p-4}\right) &= -154.3^\circ - 180^\circ - k360^\circ \\
\implies \tan^{-1}\left(\frac{6.6}{p-4}\right) &= 25.7^\circ \\
\implies \frac{6.6}{p-4} &= 0.48 \\
\implies p = 4 + \frac{6.6}{0.48} &= 17.8
\end{aligned}
$$

Hence, $\boxed{p = 17.8}$.

3. It remains to calculate K; however, the question asks only to design a compensator that causes the root locus to pass through the desired location, and not the actual K that results in closed-loop poles equal to the desired dominant roots.

[4 marks]

2. A first-order system with input $u(t)$ and output $y(t)$ is modelled by the ordinary differential equation

$$T\frac{\mathrm{d}y(t)}{\mathrm{d}t} = Ku(t) - y(t)$$

where the parameter T is the *time constant*.

a) Show that the transfer function of the system is

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{sT+1}$$

[3 marks]

Answer:

Re-arranging so that dependent variables are all on one side, and taking Laplace transforms

$$\mathcal{L}\left\{T\frac{\mathrm{d}y(t)}{\mathrm{d}t} + y(t)\right\} = \mathcal{L}\left\{Ku(t)\right\}$$
$$\implies sTY(s) - Ty(0) + Y(s) = KU(s).$$

where the latter step uses the linearity and differentiation properties of the transform.

Neglecting initial conditions

$$(sT+1)Y(s) = KU(s).$$

Hence,

$$\boxed{\frac{Y(s)}{U(s)} = \frac{K}{sT+1}}$$

[3 marks]

b) Write down the order, type number, and locations of the poles and zeros of $G(s)$.

Explain what happens to the pole locations, and the speed of the system response, as (i) T is increased and (ii) K is increased.

[6 marks]

Answer:

The system is first order and type zero.

There is a single pole at $s = -1/T$. There are no zeros.

As T increased, the pole location $s = -1/T$ (which is negative and real) moves to the right (closer to the imaginary axis) in the s-plane. In the limit of $T \to \infty$, $s = -1/T \to 0$.

At the same time, the speed of the output decreases as T increases: the output becomes slower and takes longer to settle.

Both the pole location and the speed of response are independent of K.

[6 marks]

c) Show that the output $y(t)$ in response to a step input $u(t) = A, t \geqslant 0$ is

$$y(t) = KA(1 - e^{-t/T})$$

What is the value of $y(t)$ after one time constant (*i.e.*, at $t = T$)?

[6 marks]

Answer:

Given $Y(s) = G(s)U(s)$, insert the transfer function $G(s)$ and the Laplace transform $U(s) = A/s$ of the provided signal $u(t)$:

$$Y(s) = \frac{K}{sT+1}\frac{A}{s} = \frac{KA}{T}\frac{1}{s(s+a)}$$

where $a = 1/T$.

Before inverse transforming, $Y(s)$ needs to be decomposed into constituent factors that can be found in a table of transforms. Hence, employ partial fractions to write

$$\frac{1}{s(s+a)} = \frac{C}{s} + \frac{D}{s+a}.$$

Hence,

$$\begin{aligned} C(s+a) + Ds &= 1 \\ \implies C + D &= 0 \\ Ca &= 1 \\ \therefore C &= \frac{1}{a} \\ D &= -\frac{1}{a}. \end{aligned}$$

So

$$\frac{1}{s(s+a)} = \frac{1/a}{s} - \frac{1/a}{s+a}.$$

and

$$\frac{KA}{T}\frac{1}{s(s+a)} = KA\left(\frac{1}{s} - \frac{1}{s+a}\right)$$

From tables, $\mathcal{L}^{-1}(1/s) = 1$ and $\mathcal{L}^{-1}(1/(s+a)) = e^{-at}$. Therefore,

$$y(t) = \mathcal{L}^{-1}(Y(s)) = KA(1 - e^{-t/T})$$

as required.

Using the provided expression, when $t = T$, $e^{-t/T} = e^{-1} = 0.3679$. Therefore,

$$y(T) = KA(1 - 0.3679) = 0.6321KA$$

[6 marks]

d) Show that the 10%–90% rise time is approximately $2T$ and the 2% settling time is approximately $4T$.

[5 marks]

Answer:

By definition, the 10%–90% rise time, T_r, is the time taken for $y(t)$ to increase from 10% of its final value to 90%. The 2% settling time, T_s, is the time taken for $y(t)$ to increase from 0 to within 98% of final value.

Here, the final value is $\lim_{t\to\infty} y(t) = \lim_{t\to\infty} KA(1 - e^{-t/T}) = KA$. Therefore, to compute the three times needed in the definitions

$$0.1KA = KA(1 - e^{-t_{10\%}/T})$$
$$0.9KA = KA(1 - e^{-t_{90\%}/T})$$
$$0.98KA = KA(1 - e^{-t_{98\%}/T})$$

In general, for $cKA = KA(1 - e^{-t/T})$, $t = -T\ln(1 - c)$. Therefore,

$$t_{10\%} = -T\ln(1 - 0.1) = 0.1054T$$
$$t_{90\%} = -T\ln(1 - 0.9) = 2.3026T$$
$$t_{98\%} = -T\ln(1 - 0.98) = 3.91201T$$

Hence, $T_r = t_{90\%} - t_{10\%} = (2.3026 - 0.1054)T \approx 2T$, and $T_s = t_{98\%} \approx 4T$.

[5 marks]

3. A unity-feedback control system has the open-loop transfer function

$$L(s) = \frac{75}{(s+1)(s+2)(s+10)}$$

a) Sketch the Bode diagram of $L(s)$.

[10 marks]

Answer:

Factor the transfer function as

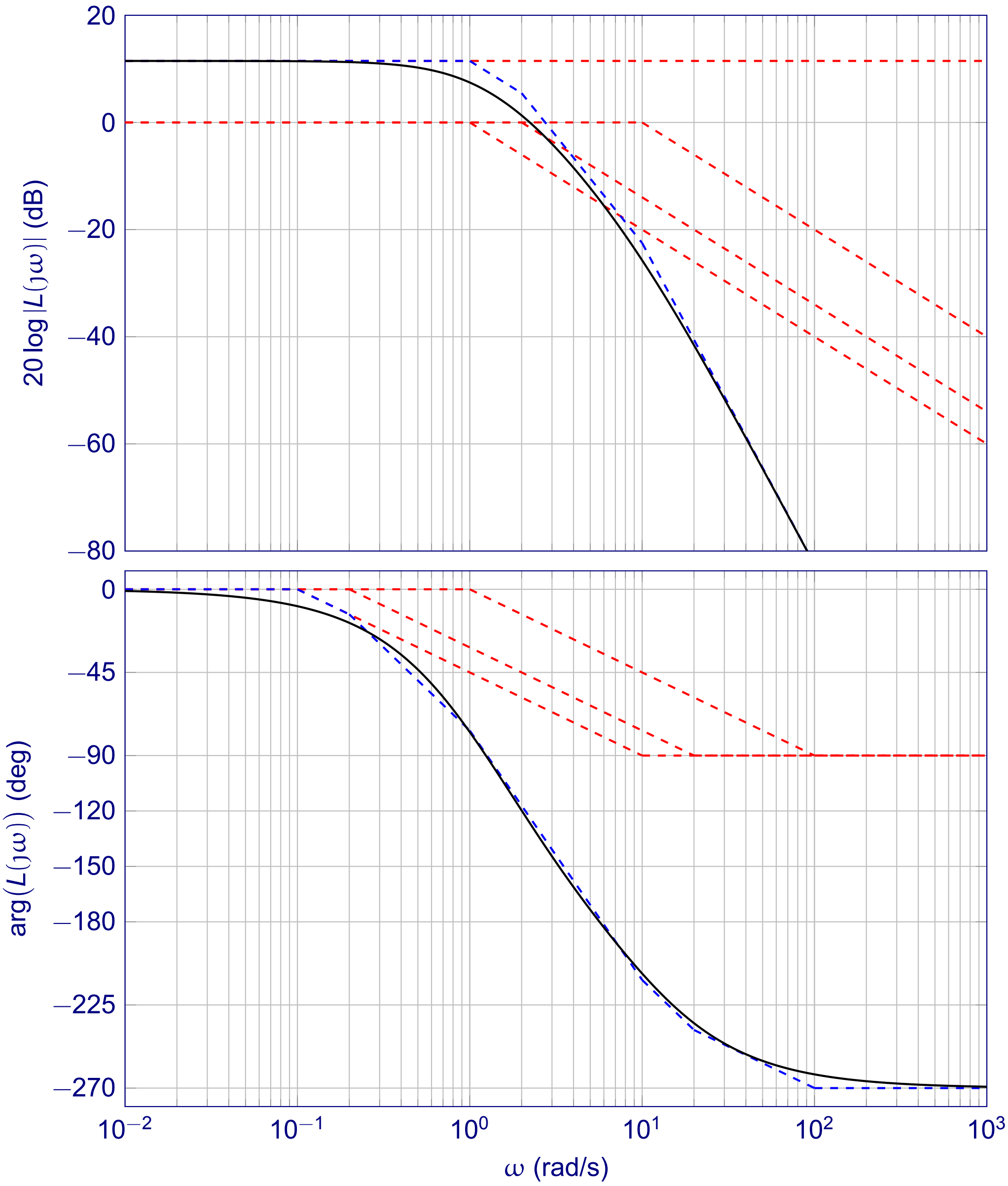
$$L(s) = \frac{75/20}{(s+1)(0.5s+1)(0.1s+1)}.$$

The transfer function comprises the four factors shown in the following tables.

Factor	Log-magnitude (dB)	Corner ω (rad s^{-1})	Slope (dB/dec)
$K = 75/20$	$20\log_{10} 3.75 = 11.48$	–	0
$1/(\jmath\omega + 1)$	$-10\log_{10}(1+\omega^2)$	1	-20
$1/(0.5\jmath\omega + 1)$	$-10\log_{10}(1+0.25\omega^2)$	2	-20
$1/(0.1\jmath\omega + 1)$	$-10\log_{10}(1+0.01\omega^2)$	10	-20

Factor	Phase	Low-ω phase	Lower ω	Higher ω	High-ω phase
$K = 75/20$	0°	0°	–	–	0°
$1/(\jmath\omega + 1)$	$-\tan^{-1}\omega$	0°	0.1	10	-90°
$1/(0.5\jmath\omega + 1)$	$-\tan^{-1} 0.5\omega$	0°	0.2	20	-90°
$1/(0.1\jmath\omega + 1)$	$-\tan^{-1} 0.1\omega$	0°	1	100	-90°

The Bode diagram (asymptotic and exact) is shown in the following plot.



[10 marks]

b) (i) Calculate the position error constant of $L(s)$.

[1 mark]

Answer:

By calculation,

$$K_p = \lim_{s \to 0} L(s) = \lim_{s \to 0} \frac{75}{(s+1)(s+2)(s+10)} = \frac{75}{20} = 3.75.$$

Alternatively, one can read the low-frequency gain from the Bode plot as $\frac{75}{20} = 3.75$ (around 11 dB). **[1 mark]**

(ii) Show that the gain crossover frequency is approximately $2.24\,\mathrm{rad\,s^{-1}}$.

[3 marks]

Answer:

By definition, the gain crossover frequency ω_c is the frequency at which $|L(\jmath\omega_c)| = 1$, or 0 dB. Hence,

$$
\begin{aligned}
1 &= \frac{75}{|\jmath\omega_c + 1||\jmath\omega_c + 2||\jmath\omega_c + 10|} \\
&= \frac{75}{\sqrt{\omega_c^2 + 1}\sqrt{\omega_c^2 + 4}\sqrt{\omega_c^2 + 100}}
\end{aligned}
$$

Substituting $\omega_c = 2.24$ into the denominator

$$
\begin{aligned}
\sqrt{\omega_c^2 + 1}\sqrt{\omega_c^2 + 4}\sqrt{\omega_c^2 + 100} &= \sqrt{5.0 + 1}\sqrt{5.0 + 4}\sqrt{5.0 + 100} \\
&= \sqrt{6}\sqrt{9}\sqrt{105} \\
&= 75.3 \\
&\approx 75
\end{aligned}
$$

Hence, $75/75 = 1$.

[3 marks]

(iii) Estimate, from your Bode diagram, the phase margin of $L(s)$. Is the closed-loop system stable?

[2 marks]

Answer:

By inspection of the Bode plot, at $\omega_c = 2.24$ the value of phase is approximately $-130°$, the the phase margin—which is the difference between $\arg\big(L(\jmath\omega_c)\big)$ and $-180°$—is approximately $50°$. Any answer between $-60°$ and $-40°$ is acceptable. (The wide range is necessary because of the steep phase roll-off.)

The closed-loop system is stable.

[2 marks]

(iv) Using your answers to parts (i)–(iii), estimate the performance characteristics of the closed-loop system, including steady-state error, overshoot and rise time.

[4 marks]

Answer:

We know from parts (i)–(iii) that

- the position error constant is 3.75;
- the phase margin is approximately $50°$;

- the gain crossover frequency is $2.24\,\mathrm{rad\,s^{-1}}$.

The steady-state step error is readily obtained as

$$e_{\mathrm{ss}}^{\mathrm{step}}(\%) = 100\frac{1}{1+K_{\mathrm{p}}} = \frac{100}{4.75} = 21\%$$

The damping ratio of the closed-loop system is related to the phase margin by the approximation $\zeta \approx 0.01\phi_{\mathrm{pm}}$. Thus,

$$\zeta \approx 0.5$$

The corresponding overshoot is

$$\begin{aligned}\text{O.S.} &= 100\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)\\ &= 100\exp\left(\frac{-0.5\pi}{\sqrt{0.75}}\right)\\ &= 16\%\end{aligned}$$

The rise time is related to the closed-loop bandwidth, ω_{B}, via the approximation $T_{\mathrm{r}} \approx 2.2/\omega_{\mathrm{B}}$. The closed-loop bandwidth is typically (for a system that fits the second-order approximation) around 1.6 times the gain crossover frequency. Thus, $\omega_{\mathrm{B}} \approx 3.52\,\mathrm{rad\,s^{-1}}$, and so the rise time is approximately

$$T_{\mathrm{r}} \approx \frac{2.2}{3.52} = 0.6\,\mathrm{s}$$

(Any estimate of closed-loop bandwidth in the range $\omega_c \leqslant \omega_{\mathrm{B}} \leqslant 2\omega_c$ is acceptaible. The resulting range of rise times is $0.5 \leqslant T_{\mathrm{r}} \leqslant 1.0$).

[4 marks]

4. a) Define the terms *characteristic equation*, *open-loop transfer function*, and *type number*.

[3 marks]

Answer:

Characteristic equation: for a transfer function $G(s) = n(s)/d(s)$, the equation formed by equating the denominator (the characteristic function) with zero: $d(s) = 0$.

Open-loop transfer function: in a feedback control system, the transfer function obtained by opening the loop and multiplying the transfer functions between the input and measured output. For example, in a feedback control system with plant $G(s)$, controller $C(s)$ and sensor $H(s)$, the open-loop transfer function is $C(s)G(s)H(s)$.

Type number: for a system with open-loop transfer function $G(s) = n(s)/d(s)$, the type number counts the number of $s = 0$ solutions of $d(s) = 0$; that is, the number of open-loop poles at the origin.

[3 marks]

b) Using block diagram reduction methods, or otherwise, find the transfer function $Y(s)/R(s)$ of the system in Figure 4.1.

[8 marks]

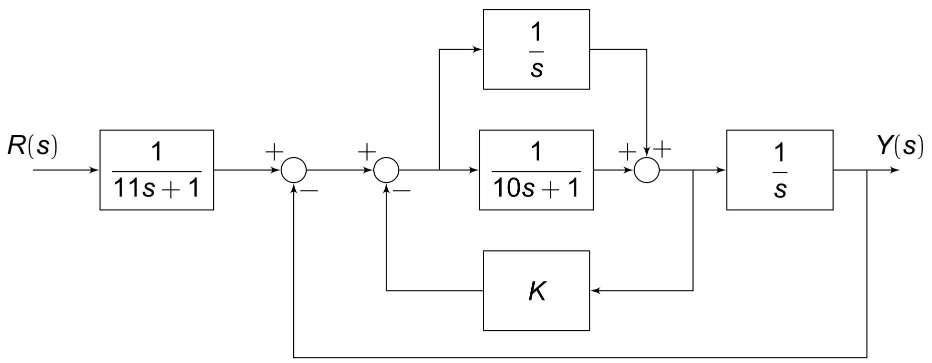
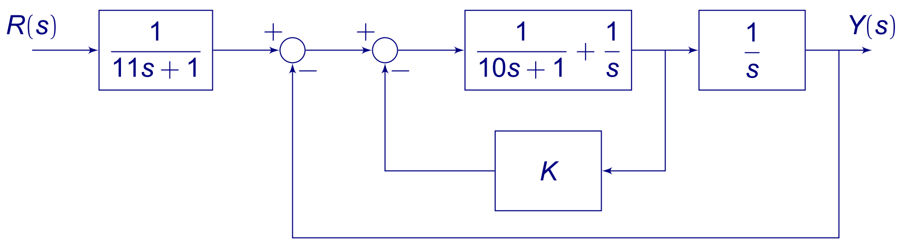


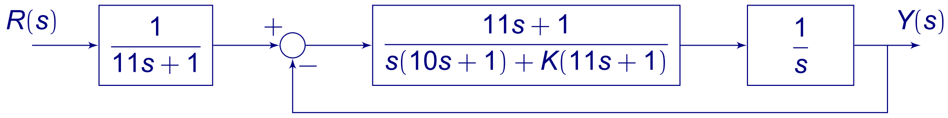
Figure 4.1

Answer:

Starting with the inner-most loop and working outwards, first eliminate the parallel blocks in the forward path:



The central block is simplified to $\dfrac{11s+1}{s(10s+1)}$. Then eliminate the inner feedback loop:



Combine the two blocks in cascade in the forward path, and simplify, so that

$$\frac{11s+1}{s(s(10s+1)+K(11s+1))} = \frac{11s+1}{10s^3+(11K+1)s^2+Ks}$$

Eliminate the feedback loop, so that the closed-loop transfer function after the pre-filter is

$$\begin{aligned}\frac{\frac{11s+1}{10s^3+(11K+1)s^2+Ks}}{1+\frac{11s+1}{10s^3+(11K+1)s^2+Ks}} &= \frac{11s+1}{10s^3+(11K+1)s^2+Ks+11s+1} \\ &= \frac{11s+1}{10s^3+(11K+1)s^2+(11+K)s+1}\end{aligned}$$

Finally, multiply by the pre-filter:

$$\begin{aligned}\frac{Y(s)}{R(s)} &= \frac{1}{11s+1}\frac{11s+1}{10s^3+(11K+1)s^2+(11+K)s+1} \\ &= \frac{1}{10s^3+(11K+1)s^2+(11+K)s+1}\end{aligned}$$

[8 marks]

c) Explain why a type-0 system under proportional feedback control always exhibits a non-zero steady-state error.

[4 marks]

Answer:

A type-0 system, say $G(s) = n(s)/d(s)$, has no integrators (open-loop poles at the origin), and therefore its position error constant (or "DC gain") is finite:

$$|K_\mathrm{p}| = |\lim_{s\to 0} G(s)| < +\infty \because d(0) \neq 0.$$

Given that $G(s)$ is the transfer function between the error $E(s)$ and system output $Y(s)$, the steady-state output is

$$y_{ss} = \lim_{s\to 0} sG(s)E(s) = K_\mathrm{p} e_{ss}$$

Therefore, because K_p is finite, y_{ss} is non-zero if and only if e_{ss} is non-zero: a steady-state error is necessary in order to produce a non-zero output.

[4 marks]

d) Show that, in a digital negative-feedback control system with reference input $R(z)$, output $Y(z)$, plant $G(z)$ and controller $D(z)$, the closed-loop system has a transfer function $Y(z)/R(z) = T(z)$ if

$$D(z) = \frac{1}{G(z)} \frac{T(z)}{1 - T(z)}$$

Hence, design a controller $D(z)$ in order that a system $G(z) = \dfrac{1 - e^{-T}}{z - e^{-T}}$, with sampling time $T = 0.1$ s, has the output

$$y[k] = 0.6065y[k-1] + 0.3935r[k]$$

[5 marks]

Answer:

The system has the open-loop transfer function $D(z)G(z)$ and the closed-loop transfer function

$$\frac{D(z)G(z)}{1 + D(z)G(z)}$$

Suppose this is equal to $T(z)$. Then

$$\begin{aligned} D(z)G(z) &= T(z)[1 + D(z)G(z)] \\ \implies D(z)G(z) - T(z)D(z)G(z) &= T(z) \\ \implies D(z)G(z)[1 - T(z)] &= T(z) \\ \implies D(z)G(z) &= \frac{T(z)}{1 - T(z)} \\ \implies D(z) &= \frac{1}{G(z)} \frac{T(z)}{1 - T(z)} \end{aligned}$$

To design the controller as asked, transform the different equation to a transfer function by using the shift operation $\mathcal{Z}\{x[k-n]\} = z^{-n}X(z)$. So

$$\begin{aligned} Y(z) &= 0.6065z^{-1}Y(z) + 0.3935R(z) \\ \implies (1 - 0.6065z^{-1})Y(z) &= 0.3935R(z) \\ \implies \frac{Y(z)}{R(z)} &= \frac{0.3935}{1 - 0.6065z^{-1}} = T(z). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{T(z)}{1 - T(z)} &= \frac{0.3935}{1 - 0.6065z^{-1} - 0.3935} \\ &= \frac{0.3935}{0.6065(1 - z^{-1})} \\ &= \frac{0.6488}{1 - z^{-1}}. \end{aligned}$$

The controller is then

$$
\begin{aligned}
D(z) &= \frac{z - e^{-T}}{1 - e^{-T}} \frac{0.6488}{1 - z^{-1}} \\
&= \frac{z - 0.9048}{0.0952} \frac{0.6488}{1 - z^{-1}} \\
&= 6.8151 \frac{z - 0.9048}{1 - z^{-1}} \\
&= 6.8151 \frac{z^2 - 0.9048z}{z - 1}
\end{aligned}
$$

[5 marks]

Laplace and z-transforms

Time domain	s-domain	z-domain
$f(t)$	$F(s)$	$F(z)$
$f(t-T)$	$e^{-sT}F(s)$	$z^{-1}F(z)$
$\delta(t)$	1	–
1	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{zT}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$f^{(n)}(t)=\frac{d^n}{dt^n}f(t)$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{n-1}(0)$	Various forms

Compensator design formulae

Transfer function	$\frac{s\alpha\tau+1}{s\tau+1}$ (lead) $\frac{s\tau+1}{s\alpha\tau+1}$ (lag)
Maximum phase lead/lag, ϕ_m	$\sin^{-1}\frac{\alpha-1}{\alpha+1}$
Centre frequency, ω_m	$\frac{1}{\tau\sqrt{\alpha}}$

Performance criteria mappings

2% settling time, T_s	$\dfrac{4}{\zeta\omega_n}$
10–90% rise time, T_r	$\dfrac{2.16\zeta + 0.6}{\omega_n}$ for $0.3 \leqslant \zeta \leqslant 0.8$
Percentage overshoot, P.O.	$100\exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$
Peak time, T_p	$\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ for $0 < \zeta < 1$
Peak response, M_p	$1 + \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$
Resonant frequency, ω_r	$\omega_n\sqrt{1-2\zeta^2}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Resonant peak magnitude, $M_{p\omega}$	$\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Phase margin, ϕ_{pm}	100ζ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Bandwidth–Rise time	$T_r = \dfrac{2.2}{\omega_B}$

Ziegler-Nichols tuning rules

First method (*T* time constant; *L* delay time; *K* process gain)

	K_P	T_I	T_D
P	T/KL	∞	0
PI	$0.9T/KL$	$L/0.3$	0
PID	$1.2T/KL$	$2L$	$0.5L$

Second method (*K* critical gain; *P* critical period of oscillation)

	K_P	T_I	T_D
P	$0.5K$	∞	0
PI	$0.45K$	$P/1.2$	0
PID	$0.6K$	$0.5P$	$0.125P$

END OF QUESTION PAPER