summary of transmission lines so far

voltage drop along lines: $-\frac{\partial V}{\partial x} = R^* I + L^* \frac{\partial I}{\partial t}$

current flowing between lines: $-\frac{\partial I}{\partial x} = G^* V + C^* \frac{\partial V}{\partial t}$

$\Rightarrow$ wave equations with

$v = \lambda \cdot f = \frac{\omega}{\tilde{k}}$ where $\tilde{k}^2 = \underbrace{\omega^2 L^* C^*}_{= k^2 \text{ in lossless case}} - R^* G^* \underbrace{- j\omega (R^* C^* + L^* G^*)}_{\text{attenuation}}$

$$= -\underbrace{(R^* + j\omega L^*)}_{Z^*} \cdot \underbrace{(G^* + j\omega C^*)}_{Y^*}$$

telegrapher's equations:

$$V = V(x,t) = \underset{\longrightarrow}{V_0^+ e^{-j\tilde{k}x}} + \underset{\longleftarrow}{V_0^- e^{+j\tilde{k}x}} \quad (\text{analogous } I(x,t))$$

$\Rightarrow$ characteristic impedance

$Z_0 = \frac{Z^*}{j\tilde{k}}$ $\left(= \frac{j\omega L^*}{j\omega \sqrt{L^* C^*}} = \sqrt{\frac{L^*}{C^*}} \text{ in lossless line with } R^* = 0 = G^* \right)$

$\rightarrow$ and $v = \frac{1}{\sqrt{L^* C^*}}$

$\Rightarrow V_0^{\pm} = \pm Z_0 I_0$

$\Rightarrow \Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L - Z_0}$ is voltage reflection coefficient

$$\Gamma = \begin{cases} 0 & \text{if } Z_L = Z_0 \quad (\text{matched}) \\ +1 & \quad Z_L = \infty \quad (\text{open line}) \\ -1 & \quad Z_L = 0 \quad (\text{short-cut}) \end{cases}$$

In general, Γ is complex.

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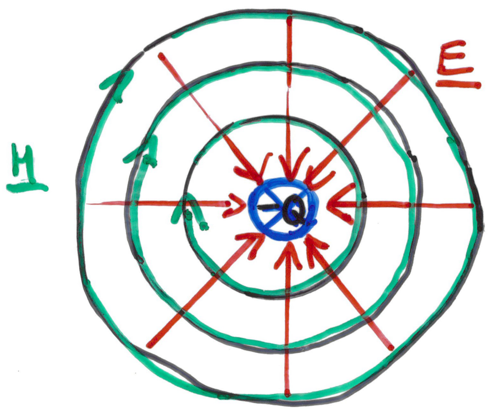
continuation: how to calculate Z_0 of a transmission line

consider current flowing into page, I $\otimes$

and charge $-Q$ at time t on inner wire ($+Q$ on outer sheath)



direction of $\underline{E}$: from $+Q$ to $-Q$

direction of $\underline{H}$: along curved fingers if thumb of right hand points along current flow

Calculation of $Z_0 = \sqrt{\frac{L^*}{C^*}}$ of lossless transmission line

get C^* from coaxial cable of length l with charge $-Q$ on inner wire of radius r_i, outer wire radius of R_a

$$-Q = -\iiint_V \rho \, dV = \iiint_V \operatorname{div} \underline{D} \, d\underline{V} \underset{\text{Gauß}}{=} \oiint_S \underline{D} \, d\underline{S}$$

with $\underline{D} = D_r \cdot \underline{e_r}$ along radius

and $d\underline{S} = 2\pi r l \cdot \underline{e_r}$ "

$$-Q = 2\pi r l D_r$$

$$\Rightarrow E_r = \frac{D_r}{\varepsilon_0 \varepsilon_r} = \frac{-Q}{2\pi \varepsilon_0 \varepsilon_r r l} \overset{!}{=} -\frac{\partial V}{\partial r}$$

$$\Rightarrow dV = \frac{Q}{2\pi \varepsilon_0 \varepsilon_r l} \frac{1}{r} dr$$

$$\Rightarrow V = \frac{Q}{2\pi \varepsilon_0 \varepsilon_r l} \underbrace{\int_{r_i}^{R_a} \frac{1}{r} dr}_{\ln \frac{R_a}{r_i}}$$

$$\Rightarrow C^* = \frac{C}{l} = \frac{Q}{Vl} = \frac{2\pi \varepsilon_0 \varepsilon_r}{\ln \frac{R_a}{r_i}}$$

get L^* from same geometry

$$I = \iint_S \underline{j} \, d\underline{S} = \int_S \operatorname{rot} \underline{H} \, d\underline{S} \underset{\text{Stokes}}{=} \oint_{\text{border of } S} \underline{H} \, d\underline{r} = 2\pi r H_r$$

$$\Rightarrow H(r) \equiv H_r = \frac{I}{2\pi r}$$

$$\Rightarrow B_r = \frac{\mu_0 \mu_r I}{2\pi r}$$

$$\Rightarrow \text{flux: } \Phi = l \int_{r_i}^{R_a} B_r \, dr = \frac{\mu_0 \mu_r I}{2\pi} \underbrace{\int_{r_i}^{R_a} \frac{1}{r} dr}_{\ln \frac{R_a}{r_i}}$$

$$\Rightarrow L \cdot \frac{\partial I}{\partial t} = V = \frac{\partial \Phi}{\partial t} = \frac{\mu_0 \mu_r}{2\pi} \ln \frac{R_a}{r_i} \frac{\partial I}{\partial t} \cdot l$$

$$\Rightarrow L^* = \frac{L}{l} = \frac{\mu_0 \mu_r}{2\pi} \ln \frac{R_a}{r_i}$$

$$\Rightarrow \quad Z_0 \approx \sqrt{\frac{L^*}{C^*}}$$

$$= \sqrt{\frac{\frac{\mu_0\mu_r}{2\pi}\cancel{\ln\frac{R_a}{r_i}}}{\frac{2\pi\varepsilon_0\varepsilon_r}{\cancel{\ln\frac{R_a}{r_i}}}}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}\ \ln\frac{R_a}{r_i}$$

and

$$v = \frac{1}{\sqrt{L^* C^*}}$$

$$= \frac{1}{\sqrt{\frac{\mu_0\mu_r}{\cancel{2\pi}}\cancel{\ln\frac{R_a}{r_i}} \cdot \frac{\cancel{2\pi}\varepsilon_0\varepsilon_r}{\cancel{\ln\frac{R_a}{r_i}}}}}$$

$$= \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$$

$$= \frac{c}{\sqrt{\mu_r\varepsilon_r}}$$

example: coax cable RG-55/U

$$\left.\begin{array}{l} r_i = 0.5\ \text{mm} \\ R_a = 2.95\ \text{mm} \end{array}\right\} \ln\frac{R_a}{r_i} = 1.77$$

$\varepsilon_r = 2.25$ (polymer)

$\mu_r \approx 1$ (non-magnetic)

$$\Rightarrow \quad C^* = 0.71\ \text{pF/m}$$

$$L^* = 0.36\ \mu\text{H/m}$$

$$\Rightarrow \quad Z_0 \approx \sqrt{\frac{L^*}{C^*}} = 71\ \Omega$$

$$v = \frac{c}{\sqrt{\mu_r\varepsilon_r}} \approx 2/3\ c \approx 2\cdot 10^8\ \text{m/s}$$