4. Time-varying Fields

4.1 Diffusion equation

A typical example of time-varying fields is the alternating fields in transformers. According to Faraday's Law:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \rightarrow \textbf{Faraday's Law} \tag{4.1}$$

i.e. a time-changing flux density ***B*** produces a space changing electric field ***E***.

Maxwell hypothesised the converse, viz that a time-changing electric field produces a space changing magnetic field, i.e.

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$

Vector form of Ampere's Law is

$$\nabla \times \vec{H} = \vec{J}$$

but

$$\nabla \cdot (\nabla \times \vec{H}) = 0$$

which implies that $\nabla \cdot \vec{J} = 0$

However, the current (or charge/sec) diverging from a unit volume equals the time rate of the charge per unit volume.

$$i.e. \quad \nabla \cdot \vec{J} = \frac{\partial \rho}{\partial t}$$

This can only be satisfied by adding an extra term to RHS of Ampere's Law.

$$i.e. \quad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\left(\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} + \nabla \cdot \left(\frac{\partial \vec{D}}{\partial t} \right) = \nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} \right)$$

$$\therefore \nabla \times \vec{H} = \underbrace{\sigma \vec{E}}_{\text{Conduction Current}} + \underbrace{\varepsilon_0 \varepsilon_r \frac{\partial \vec{E}}{\partial t}}_{\text{Displacement current}}$$

NOTE: for sinusoidally time-varying fields at angular frequency ω (i.e. $\boldsymbol{E} = E\sin \omega t$), the ratio

$$\frac{displacement\ current}{conduction\ current} = \frac{\varepsilon_0 \varepsilon_r \omega E}{\underset{\substack{\uparrow \\ \text{Conductivity}}}{\sigma E}} = \varepsilon_0 \varepsilon_r \omega \underset{\substack{\uparrow \\ \text{Resistivity}}}{\rho}$$

Consider 50Hz currents flowing in a copper conductor, for which:

$$\varepsilon_r = 1, \quad \rho = 1.7x10^{-8}\ \Omega m,\ \omega = 314 rad/\sec$$

Ratio

$$\frac{displacement\ current}{conduction\ current} = 47.3 \times 10^{-8}$$

i.e. Displacement current is negligible at power frequencies

$$\therefore \nabla \times \vec{H} = \vec{J} = \sigma \vec{E}$$

$$\nabla \times (\nabla \times \vec{H}) = \sigma\ \nabla \times \vec{E} = -\mu\sigma \frac{\partial \vec{H}}{\partial t} = \nabla(\nabla \cdot \vec{H}) - \nabla^2 \vec{H}$$

Since $\nabla \cdot \vec{H} = \dfrac{1}{\mu} \nabla \cdot \vec{B} = 0$

Thus

$$\therefore \nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t}$$

Which is known as the diffusion equation

NOTE: For harmonically time-varying fields

i.e.

$$\vec{H} = \vec{H} e^{-j\omega t}$$

$$\frac{\partial}{\partial t} = j\omega$$

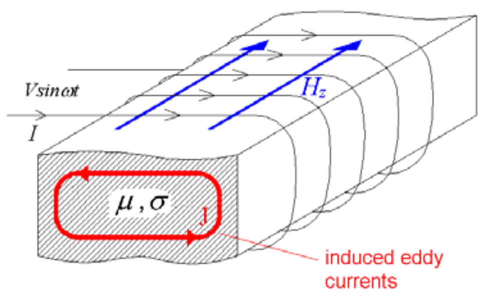
$$\therefore \nabla^2 \vec{H} = j\omega\mu\sigma\vec{H} \rightarrow \text{Complex Diffusion Equation}$$

4.2 Solution of Complex Diffusion Equation

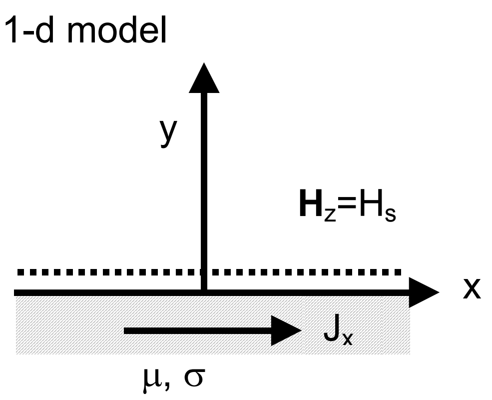
$$\nabla^2 \boldsymbol{H} = j\omega\mu\sigma\boldsymbol{H}$$

Example 1: *One-dimensional eddy current flow in thick plate*.

e.g. Simplified model of an induction heater.



Because depth of penetration of eddy currents (skin depth) is often small compared with other dimensions, many problems can be approximated as 1-dimensional

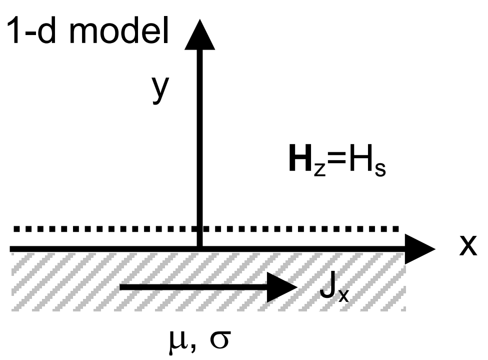


Since the only field component $\boldsymbol{H}_z$ is independent of x and z,   and since

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) e_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) e_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) e_z$$

$$\therefore \nabla \times \vec{H} = a_x \frac{\partial H_z}{\partial y} = J_x$$

ie. only J_x exists



$$\nabla^2 H = \sigma\mu \frac{\partial H}{\partial t}$$

$$\frac{\partial^2 H_z}{\partial y^2} = \sigma\mu \frac{\partial H_z}{\partial t}$$

But $H_z = H_z e^{j\omega t}$

(ie. applied field is sinusoidally time-varying)

$$\therefore \frac{\partial^2 H_z}{\partial y^2} = j\omega\sigma\mu H = \alpha^2 H_z$$

where $\alpha^2 = j\omega\sigma\mu,\qquad ie.\quad \alpha = \sqrt{j}\,\sqrt{\omega\sigma\mu}$

Note: $\sqrt{j} = \dfrac{1+j}{\sqrt{2}}$

$$\therefore \alpha = \frac{(1+j)}{\sqrt{2}}\frac{\sqrt{\omega\sigma\mu}}{} = \frac{1+j}{\delta}$$

where $\delta = \sqrt{\dfrac{2}{\omega\sigma\mu}}$

Note: δ is referred to as the skin depth.

$$\frac{\partial^2 H_z}{\partial y^2} = \alpha^2 H_z$$

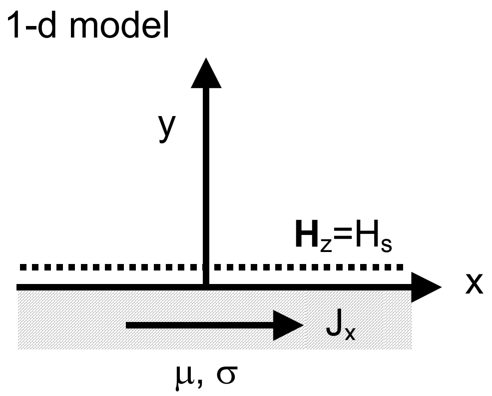
General Solution: $H_z = K_1 e^{\alpha y} + K_2 e^{-\alpha y}$

Boundary Conditions:

(i) At $y = 0$, $\quad H_z = H_s$

(ii) At $y = -\infty$, $\quad H_z = 0$

$\therefore K_2 = 0$
$K_1 = H_s$



$\therefore H_z = H_s\, e^{\alpha y}\, e^{j\omega t}$

From **J** = Curl **H**

$$J_x = \frac{\partial H_z}{\partial y} \quad = \alpha H_s e^{\alpha y} e^{j\omega t} = J_s e^{\alpha y} e^{j\omega t}$$

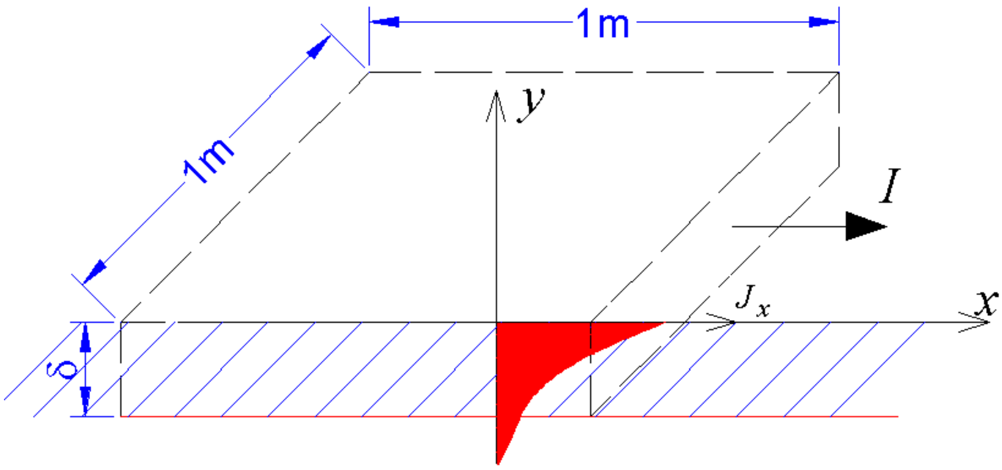
$\therefore$ H_z and J_x decay exponentially with depth into the conducting material.

Total current $I = \int_{-\infty}^{0} J_x dy = \frac{J_s}{\alpha} e^{j\omega t} = \frac{J_s}{\sqrt{2}} \delta e^{-j\pi/4} e^{j\omega t}$

Note: At 50Hz, skin depth $\delta = \sqrt{\frac{2}{\omega\sigma\mu}}$

= 10mm for copper
= 2mm for mild steel ($\mu_r = 2000$)

Total current $I = \frac{J_s}{\alpha} e^{j\omega t} = \frac{J_s}{\sqrt{2}} \delta e^{-j\pi/4} e^{j\omega t}$



This is equivalent to an rms current density $J_s/\sqrt{2}$ flowing uniformly in skin depth δ

$\therefore$ Eddy current loss = $I_{rms}^2 R = ½\, I_{max}^2 R$

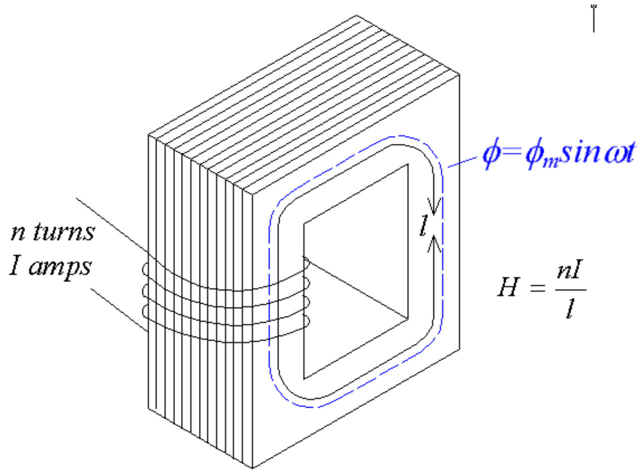
But $R = \frac{\rho\ell}{A} = \frac{\ell}{\sigma A} = \frac{1}{\sigma\delta}$ per unit of surface area

and $I_{max} = \frac{J_s}{\alpha}$

$\therefore$Eddy current loss $P_e = \frac{1}{2}\frac{J_s^2}{\alpha^2\sigma\delta} = \frac{H_s^2}{2\sigma\delta} w/m^2$ of surface area

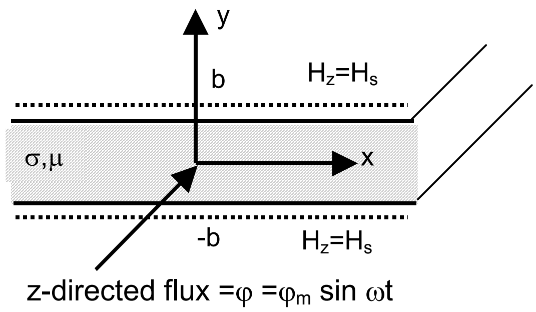
*Example 2: One-dimensional eddy current flow in **thin** plate*

e.g. Simplified model of a transformer lamination



1-d model

Lamination thickness = $2b$



From the previous discussion, we have

$$\frac{\partial^2 H_z}{\partial y^2} = \alpha^2 H$$

$\therefore\ H_z = K_1\, e^{\alpha y} + K_2\, e^{-\alpha y}$

Boundary Conditions:

(i) At $y = b$, $H_z = H_s = K_1\, e^{\alpha b} + K_2\, e^{-\alpha b}$
(ii) At $y = -b$, $H_z = H_s = K_1\, e^{-\alpha b} + K_2\, e^{\alpha b}$

$\therefore\ K_1 = K_2$
$\therefore\ H_s = K_1\,(e^{\alpha b} + e^{-\alpha b})$
$\therefore\ K_1 = \dfrac{H_s}{e^{\alpha b} + e^{-\alpha b}} = K_2$

$H_z = K_1 e^{\alpha y} + K_2 e^{-\alpha y}$

$\varphi = \varphi_m \sin \omega t$

$$\therefore H_z = H_s \frac{\left(e^{\alpha y} + e^{-\alpha y}\right)}{\left(e^{\alpha b} + e^{-\alpha b}\right)} e^{j\omega t} = H_s \frac{\cosh \alpha y}{\cosh \alpha b} e^{j\omega t}$$

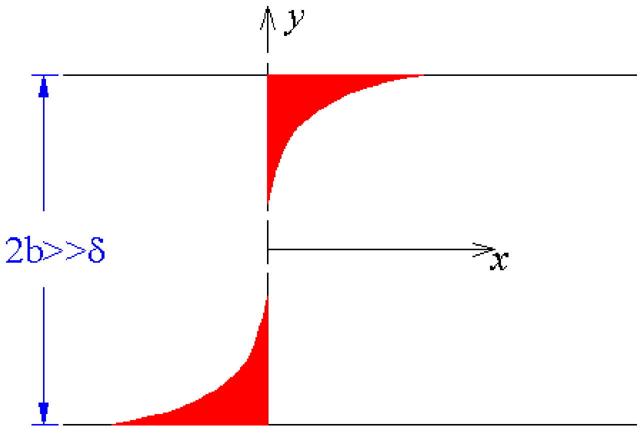
Again, from Curl ***H*** = ***J***

$$J_x = \frac{\partial H_z}{\partial y} = \alpha H_s \frac{\sinh \alpha y}{\cosh \alpha b} e^{j\omega t}$$

Note: $\int_{-b}^{b} J_x \ dy = 0$ ie. Induced eddy currents go and return inside the lamination

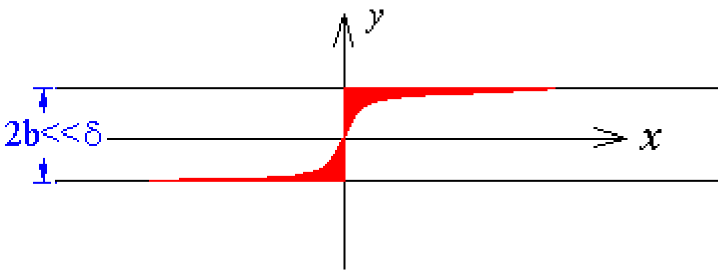
Note: When 2b >>δ, ie. thick plate, the eddy current loss can be shown to be:

$$P_e = \frac{H_s^{\ 2}}{\sigma\delta} \quad \text{w/m}^2 \text{ of surface area}$$



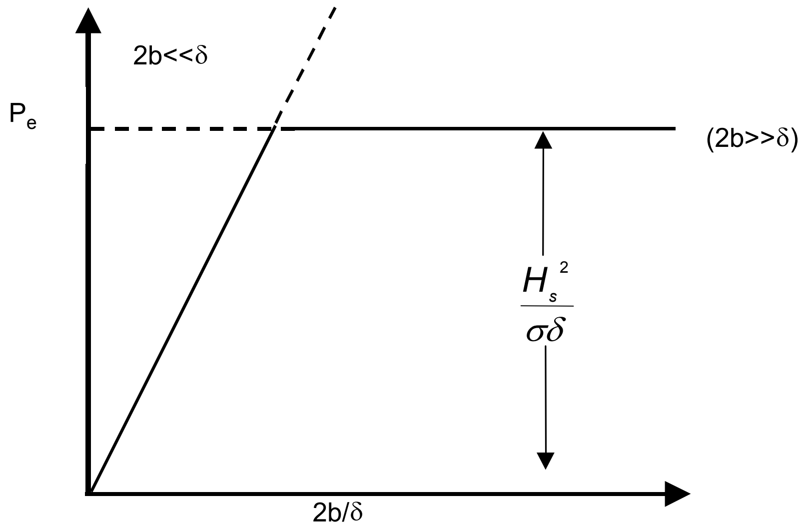
i.e. 2 x loss associated with one surface of plate

When 2b << δ, i.e. thin plate



$$P_e = \frac{1}{3}\omega^2 \sigma \mu_0^2 \mu_r^2 H_s^2 b^3 \ \ w/m^2 \text{ of surface area}$$

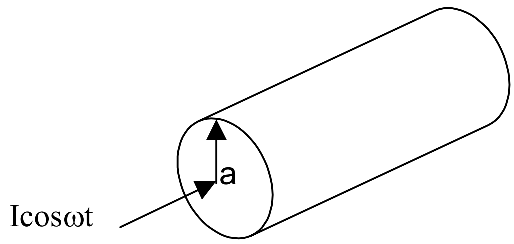
The figure below shows the eddy current loss per surface area as a function of the ratio (2b/δ)



To reduce eddy current loss, thickness of lamination 2b must be significantly less than skin depth $\delta = \sqrt{\frac{2}{\omega\mu\sigma}}$

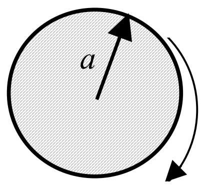
Example 3: AC Resistance of Circular Conductor

- Consider conductor of radius '*a*' carrying load current *I* cosωt.
- Assume δ<<*a*, i.e. there is a pronounced skin effect.

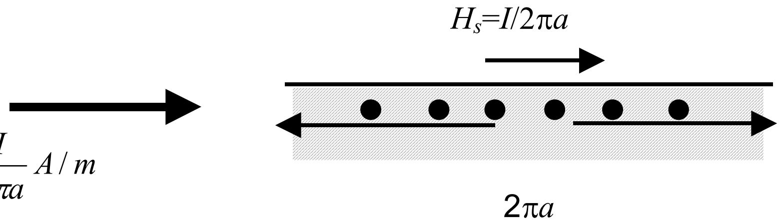


Cautionary note: This is not a good conductor design since interior of conductor will not carry current.

Since δ<<a, curvature of conductor can be neglected



$$H_s = \frac{I}{2\pi a} A/m$$



Loss/unit of conductor surface area $= \dfrac{H_s^2}{2\sigma\delta} \quad W/m^2$

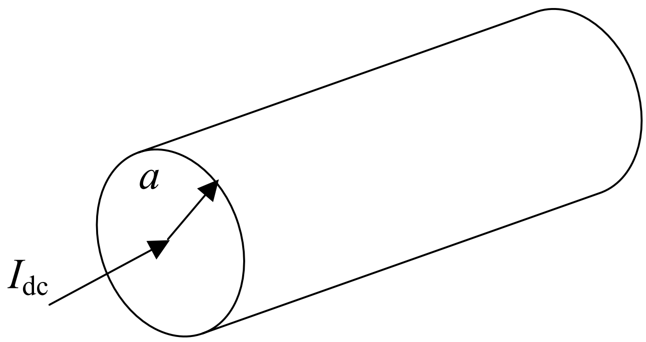
$$= \left(\frac{I}{2\pi a}\right)^2 \frac{1}{2\sigma\delta} \quad W/m^2$$

$\therefore$ Loss/m length of conductor $= \left(\dfrac{I}{2\pi a}\right)^2 \dfrac{1}{2\sigma\delta} 2\pi a \quad W/m$

Loss/m length of conductor $= \dfrac{I^2}{4\pi a\sigma\delta} \quad W/m$

$$= \frac{1}{2} I^2 \ R_{ac}$$

$$\therefore R_{ac} = \frac{1}{2\pi a\sigma\delta} \quad \Omega/m$$



But $R_{dc} = \dfrac{1}{\sigma\pi a^2} \quad \Omega/m$

$$\therefore \frac{R_{ac}}{R_{dc}} \approx \frac{a}{2\delta}$$

ie. As skin depth δ gets smaller the ratio of the ac resistance to the dc resistance increases.

Note: More exact solution for a circular conductor can be derived in terms of Bessel Functions. It approximates to

$$\frac{R_{ac}}{R_{dc}} = \frac{a}{2\delta} - \frac{1}{4} + \frac{3\delta}{32a}$$

Tutorial Sheet 3

1. Calculate the skin depth for the following materials and excitation frequencies:

   (i) Copper at 50 Hz ($\rho = 1.72 \times 10^{-8} \Omega m$)
   (ii) Mild steel at 50 Hz ($\mu_r = 2000$, $\rho = 2.8 \times 10^{-7} \Omega m$)

   (ans (i) $\delta = 9.34$ mm   (ii) $\delta = 0.84$ mm)

2. Starting from the expression for the current density induced in a semi-infinite plate when it is exposed to a sinusoidally time-varying magnetic field, calculate the approximate ac resistance of a copper conductor of diameter 4 mm when it carries a current of 10A at a frequency of 20kHz. The resistivity of copper is $1.72 \times 10^{-8} \Omega m$. Compare the ac resistance with the dc value.

$$\left(\begin{array}{l} \text{ans. } R_{ac} = 2.9 \times 10^{-3} \Omega / m \\ \dfrac{R_{ac}}{R_{dc}} = 2.11 \end{array} \right)$$