8. Finite element methods

Finite element techniques are now a well-established series of numerical methods for the routine solution of spatial field problems in many branches of engineering, mechanics, electromagnetics, heat transfer etc. The pioneering work in FE methods was done in Civil Engineering, with arguably the pioneering work and the classical textbook being that of Zienciewiz (Swansea University). Although the physical phenomena in these fields seem very different, they are governed by the same underlying type of differential equations.

The finite element method is based on sub-dividing the problem **domain** up into a large number of individual **elements** (usually many thousands). The array of elements is referred to as a mesh. Each element within the **mesh** is defined by a series of nodes at which the potential is regarded as an unknown, with these **nodes** in most cases (i.e other than at the boundary) being shared with adjacent elements. By way of example, Figure 1 shows a close up of a typical finite element mesh. In this particular case, the elements are of regular size and shape. In many practical meshes based on triangular elements, the shape and size of the individual elements within the mesh can vary greatly.

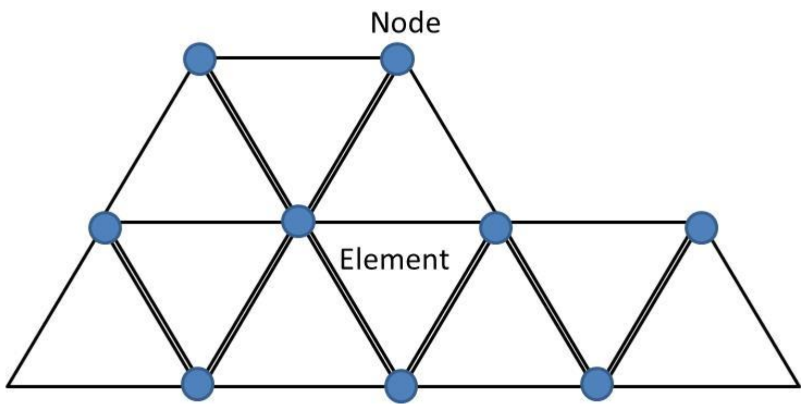

Figure 1 General form of a finite element mesh

There are many different types of element used in finite element analysis in terms of the geometry and number of nodes. Although triangular elements are widely used in electromagnetics, it is also possible to have quadrilateral elements (4 nodes) or even higher in 2D models. In 3D models, the range of possible geometries is even wider, e.g. tetrahedra, bricks, prisms and pyramids as shown Figure 2.

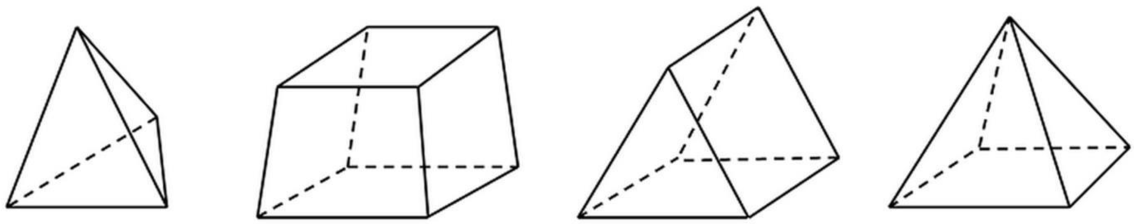

Figure 2 Different element types for 3D finite element analysis

Within each individual element, the variation in potential within the element itself is approximated by some simplified expression. This is referred as the order of the element. The most straightforward elements, known as first-order, assume a linear variation in potential throughout the element. Higher order elements in which quadratic or higher order polynomials are assumed, although more complex, tend to give more accurate predictions for the same discretisation.

The other key features of a finite element model are the sources and boundaries. We will explore these in more detail when we consider some examples, but it important to recognise that the imposition of correct boundary conditions (whether this is to represent a source, a remote known potential condition or to exploit symmetry) is essential in terms of generating meaningful results.

Underpinning mathematics

Although modern finite element packages tend to obscure the intricacies of the mathematical foundations from the user, it is still useful prior to using packages to understand something of the principles upon which the analysis is built. Consider a first-order triangular element shown in Figure 3.

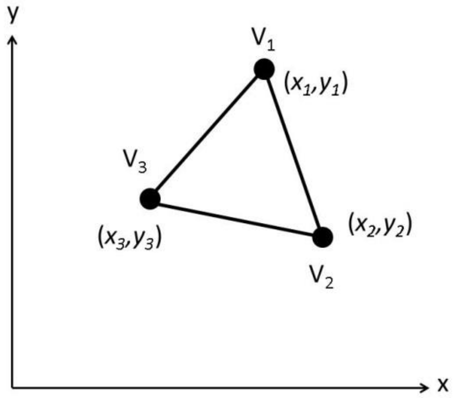

Figure 3 Triangular first-order element

If the electrostatic potential is assumed to vary linearly with displacement within the element, the point at any point (x,y) is given by:

$$V_e = a + bx + cy$$

where *a*,*b* and *c* are coefficients that take a particular value for that element.

The consequence of this simplifying assumption for the variation potential is that the electric field within a given element (which it will be recalled is a vector) is constant in magnitude and direction since:

$$\vec{E} = -\nabla V_e = \vec{e}_x \frac{\partial V}{\partial x} + \vec{e}_y \frac{\partial V}{\partial y} = \vec{e}_x b + \vec{e}_y c$$

If we consider the potential at each of the three nodes of the triangular element:

$V_1 = a + bx_1 + cy_1$

$V_2 = a + bx_2 + cy_2$

$V_3 = a + bx_3 + cy_3$

This can be written in matrix form as:

$$\begin{vmatrix} V_1 \\ V_2 \\ V_3 \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} \begin{vmatrix} a \\ b \\ c \end{vmatrix}$$

Establishing the values of the coefficients *a*, *b* and *c* for this particular element would enable us to calculate the localised variation of the potential. These coefficients can be determined from:

$$\begin{vmatrix} a \\ b \\ c \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}^{-1} \begin{vmatrix} V_1 \\ V_2 \\ V_3 \end{vmatrix}$$

The inverse of the matrix (using the standard expression contained Appendix 1) in the above by:

$$\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}^{-1} = \frac{1}{Determinent} \begin{vmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix}$$

The Determinent is given by:

$$Determinent = 1(x_2y_3 - x_1y_2) - x_1(1y_3 - 1y_2) + y_1(1x_1 - 1x_2)$$

$$= (x_2y_3 - x_1y_2) - (x_1y_3 - x_1y_2) + (y_1x_1 - y_1x_2)$$

This expression can be rearranged to give:

$$= (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

By careful inspection the above is seen to be equal to **twice** the area of the triangle. If we designate the area of the triangle as A and substituting back into the expression for the inverse gives:

$$\begin{vmatrix} a \\ b \\ c \end{vmatrix} = \frac{1}{2A} \begin{vmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \\ V_3 \end{vmatrix}$$

Substituting for a,b and c back into the equation for the potential V_e at any point (x,y) in the element gives:

$$V_e = |1 \quad x \quad y| \begin{vmatrix} a \\ b \\ c \end{vmatrix} = |1 \quad x \quad y| \frac{1}{2A} \begin{vmatrix} (x_2y_3 - x_3y_2) & (x_3y_1 - x_1y_3) & (x_1y_2 - x_2y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{vmatrix} \begin{vmatrix} V_1 \\ V_2 \\ V_3 \end{vmatrix}$$

The expression above allows us to calculate the potential at any point *(x,y)* within the element in terms of the 3 potentials at the nodes of the triangle. This rather complex expression is often re-written in terms of the so-called 'shape functions'.

$$V_e(x, y) = \sum_{i=1}^{3} \alpha_i V_i$$

where the three shape functions are given by:

$$\alpha_1 = \frac{1}{2A}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$

$$\alpha_2 = \frac{1}{2A}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$$

$$\alpha_3 = \frac{1}{2A}[(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]$$

We can clearly apply these expressions to each and every element in the problem domain, establishing a very large number of equations. As will be apparent by inspecting the general finite element mesh shown in Figure 1, individual nodes can feature in several elements within the mesh, i.e. their potential will feature in several individual expressions for the potential within a given element. It is this coupling between adjacent elements that ultimately results in a large number of simultaneous equations, which form a large sparse matrix which can be inverted to find the potentials at each node.

The actual method employed to solve for the potentials is to make use of the fact that Laplace's equation (and indeed Poisson's equation) is satisfied when the total energy in the domain is a minimum. If we establish an expression for partial derivative of the energy with each node potential then solving for a minimum (i.e. partial derivative = 0) gives the unknown node potentials. A full derivation of the procedure for solving Laplace's equation by finite element analysis is given in:

'A simple introduction to finite element analysis of electromagnetic problems', Sadiku, M.N.O., IEEE Transactions on Education, Volume 32, Issue 2, May 1989, pages 85 – 93.

(a copy can be viewed via the University Library subscription to IEEE Explore. Any computer on the University network or a computer connected to the network via a VPN will route through to the IEEE Explore version if you simply enter 'A simple introduction to finite element analysis of electromagnetic' into a search engine).

The full analysis is beyond the scope of this course and will not be examinable, and the paper is intended for background reading.

Appendix A – General analytical expression for inversion of a 3x3 matrix

$$\mathbf{A}^{-1} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A & B & C \\ D & E & F \\ G & H & I \end{bmatrix}^{T} = \frac{1}{\det(\mathbf{A})} \begin{bmatrix} A & D & G \\ B & E & H \\ C & F & I \end{bmatrix}$$

where the determinant of **A** can be computed by applying the rule of Sarrus as follows:

$$\det(\mathbf{A}) = a(ei - fh) - b(id - fg) + c(dh - eg).$$

If the determinant is non-zero, the matrix is invertible, with the elements of the above matrix on the right side given by

$$\begin{matrix} A = (ei - fh) & D = -(bi - ch) & G = (bf - ce) \\ B = -(di - fg) & E = (ai - cg) & H = -(af - cd) \\ C = (dh - eg) & F = -(ah - bg) & I = (ae - bd) \end{matrix}$$