Worked Solutions to Problems in Tutorial Sheet 1

1. First convert vector ***B*** in cylindrical co-ordinate to Cartesian co-ordinate:

$$B_x = r\cos\theta = 4\cos(75^0) = 1.04$$
$$B_y = r\sin\theta = 4\sin(75^0) = 3.86$$
$$B_z = z = 5$$

Thus

(a) $\boldsymbol{A} + \boldsymbol{B} = 5.04\ \boldsymbol{e_x} + 8.86\ \boldsymbol{e_y} + 7\ \boldsymbol{e_z}$

(b) $\boldsymbol{A} \bullet \boldsymbol{B} = 4\times1.04 + 5\times3.86 + 2\times5 = 33.46$

(c) Since $\vec{A}\bullet\vec{B} = |\vec{A}|\cdot|\vec{B}|\cos\theta$, thus

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|} = \frac{33.46}{6.71\times 6.40} = 0.78 \quad ; \quad \theta = 38.84^0$$

(d) $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)e_x + (A_zB_x - A_xB_z)e_y + (A_xB_y - A_yB_x)e_z$

$= (5\times5-2\times3.86)\ \boldsymbol{e_x} + (2\times1.04 - 4\times5)\ \boldsymbol{e_y} + (4\times3.86 - 5\times1.04)\ \boldsymbol{e_z}$

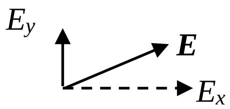
$= 17.28\ \boldsymbol{e_x} - 17.92\ \boldsymbol{e_y} + 10.24\ \boldsymbol{e_z}$

Comments: Could you verify if $\vec{A}\times\vec{B}$ is perpendicular to both ***A*** and ***B***?

2. $\boldsymbol{E} = -\nabla(V)$

$= -(-2\ \boldsymbol{e_x} - 1\ \boldsymbol{e_y})$

$= 2.24\ \angle 26.6^\circ$ (V/m)



3. A straightforward partial differentiation gives:

$$\vec{E} = -\frac{\partial V}{\partial x}e_x - \frac{\partial V}{\partial y}e_y - \frac{\partial V}{\partial z}e_y = -2x\ e_x - z^3\ e_y - 3yz^2 e_z$$

4. Similarly, we have:

$$V = 10xz + 15yz^2$$

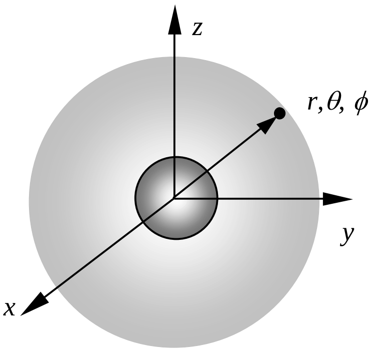
$$\vec{E} = -\frac{\partial V}{\partial x}e_x - \frac{\partial V}{\partial y}e_y - \frac{\partial V}{\partial z}e_y = -10z\ e_x - 15z^2\ e_y - (10x + 30yz)\ e_z$$

and at the point of $(x, y, z) = (5, 4, 3)$

$V = 10\times5\times3 + 15\times4\times3^2 = 690$ (V)

$\boldsymbol{E} = -10\times3\ \boldsymbol{e_x} - 15\times3^2\ \boldsymbol{e_y} - (10\times5 + 30\times4\times3)\ \boldsymbol{e_z} = -30\ \boldsymbol{e_x} - 135\ \boldsymbol{e_y} - 410\ \boldsymbol{e_z}$ (V/m)

5. Select a Cartesian co-ordinate whose origin coincides with the point charge as shown below:



From the Gauss's law, the electric flux density is given by:

$$\vec{D} == \frac{Q}{4\pi r_o^2}\vec{r}$$

Thus the electric flux over the designated surface is given by the following integration:

$$\iint_S \vec{D}\bullet d\vec{S} = \iint_S \frac{Q}{4\pi r_o^2}(\vec{r}\bullet\vec{r})dS = \frac{Q}{4\pi r_0^2}\int_0^{\pi} d\theta\ r_0^2\int_0^{\pi/4}\sin\phi d\phi = \frac{Q}{4}\left(1-\frac{\sqrt{2}}{2}\right) = 0.073\,(\text{C/m}^2)$$

Note in calculating the above integration, $dS = r_0^2\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$ is used.

6.

$$\left|\vec{D}\right| = \frac{Q}{4\pi r_o^2} \quad ; \quad Thus \quad r_0 = \sqrt{\frac{Q}{4\pi\left|\vec{D}\right|}} = \sqrt{\frac{0.3}{4\pi 1.25}} = 0.14(m)$$

7. Assume that the problem concerned is in free space

(a) $\boldsymbol{E} = \boldsymbol{D}/\varepsilon_0 = (9yx^3\ \boldsymbol{e_x} + 6y\ \boldsymbol{e_y} + 7z\ xy^2\ \boldsymbol{e_z})/\ \varepsilon_0$
(b) $\nabla\bullet\boldsymbol{D} = (\ 27yx^2 + 6 + 7\ xy^2)$

Thus
$\boldsymbol{E} = (9\times1\times2^3\ \boldsymbol{e_x} + 6\times1\ \boldsymbol{e_y} + 7\times3\times2\times1^2\ \boldsymbol{e_z})/(8.85\times10^{-12})$
$= (8.14\ \boldsymbol{e_x} + 0.68\ \boldsymbol{e_y} + 4.75\ \boldsymbol{e_z})\times10^{12}$ (V/m)

$\nabla\bullet\boldsymbol{D} = 27\times1\times2^2 + 6 + 7\times2\times1^2 = 128$ (C/m^3)

8.
$\boldsymbol{D} = r\ \boldsymbol{e_r} + r\sin\theta\ \boldsymbol{e_\theta}$

$$\nabla \bullet \vec{D} = \frac{1}{r}\frac{\partial}{\partial r}\left(rD_r\right) + \frac{1}{r}\frac{\partial D_\theta}{\partial \theta} \;\; = \frac{1}{r}\frac{\partial}{\partial r}\left(r^2\right) + \frac{1}{r} r\cos\theta = 2 + \cos\theta$$

9. The electric charge density is given by:

$$\rho = \nabla \bullet \vec{D} = 27x^2 + 10y + 2 = 27 + 10\times 5 + 2 = 79\,(\text{C/m}^3)$$

10. $\boldsymbol{E} = 3y\,\boldsymbol{e_x} + 5xz^4\,\boldsymbol{e_y} + 2xy^3\,\boldsymbol{e_z}$

$\boldsymbol{D} = \varepsilon\boldsymbol{E}$

$$\nabla\bullet\boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho = 0 + 0 + 0$$

$\therefore$ ρ = 0, i.e., no charge is present at any point

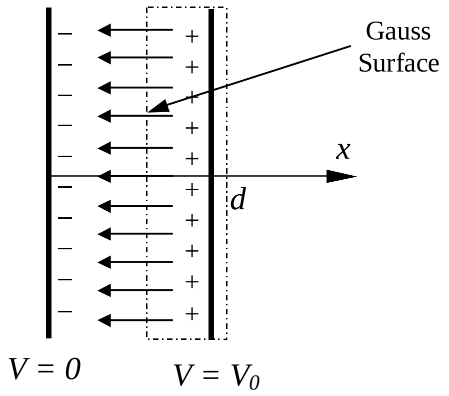
11. Neglecting end-effects, there will be no variation in electric potential with respect to y and z. The Laplace's equation is reduced to:

$$\frac{d^2V}{dx^2} = 0$$

Assume that at $x = 0$, V = 0, and $x = d$, V = V_0

The solution to the above equation under the boundary condition is given by:

$$V = V_0 x / d$$



The electric field strength E is therefore:

$E = -\dfrac{dV}{dx} = -V_0 / d$ Note the negative sign indicates that the direction of E is opposite to that of the x axis.

From the Gauss's law, the charge contained on the capacitor plate is:

$$Q = \oiint_S \vec{D} \bullet d\vec{S} = \varepsilon ES = \frac{\varepsilon V_0}{d} S\,,$$ where S is the surface of the plate.

The capacitance can be found by:

$$C = \frac{Q}{V_0} = \frac{\varepsilon S}{d} = \frac{8.85\times 10^{-12}\times 400\times 10^{-6}}{50\times 10^{-6}} = 70.8\times 10^{-12}\,(\text{F})$$

The energy stored in the capacitor is related to the capacitance and the voltage by:

$$J = \frac{1}{2}CV_0^2 \qquad thus \quad V_0 = \sqrt{\frac{2J}{C}} = \sqrt{\frac{2\times 0.345\times 10^{-6}}{70.8\times 10^{-12}}} = 98.7\,(\text{V})$$

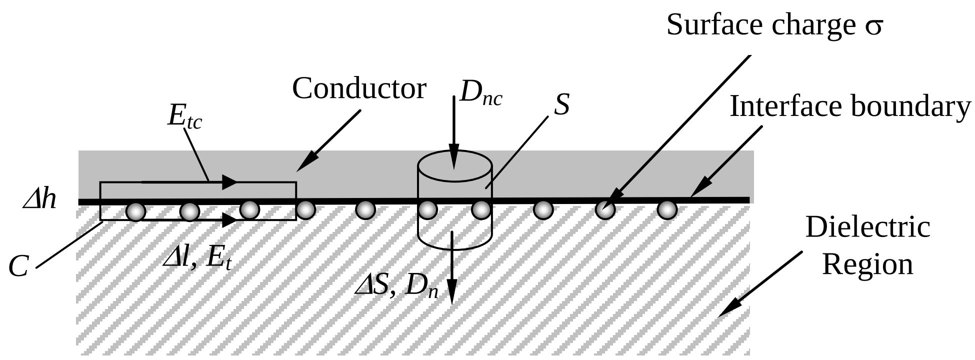
Finally,

$$E = -V_0 / d = -1.974\times 10^6\,(\text{V/m})$$

12. **This question is essentially a repeat of example 2 in section 2.5. Please carefully digest the example, and work out the solution. You should be able to make the derivation without referring the lecture note.**

13. **Derive interface conditions at the boundary between a conductor and dielectric material** *(hint conductor is an equi-potential body)* **.**

At the interface boundary between the conductor and dielectric, consider the contour C that has depth Δh and length Δl, as shown in the figure below:



Applying the conservative property of $\vec{E}$ along the contour C yields:

$$\oint_C \vec{E}\cdot d\vec{l} = E_t\Delta l + \frac{1}{2}(E_n + E_{nc})\Delta h - E_{tc}\Delta l - \frac{1}{2}(E_n + \frac{\partial E_n}{\partial l}\Delta l + E_{nc} + \frac{\partial E_{nc}}{\partial l}\Delta l)\Delta h$$

$$= E_t\Delta l - E_{tc}\Delta l - \frac{1}{2}(\frac{\partial E_n}{\partial l} + \frac{\partial E_{nc}}{\partial l})\Delta l\Delta h = 0$$

Dividing the both sides by Δl

$$E_t - E_{tc} - \frac{1}{2}(\frac{\partial E_n}{\partial l} + \frac{\partial E_{nc}}{\partial l})\Delta h = 0$$

In the limit as $\Delta h \to 0$, the above equation becomes

$$E_t - E_{tc} = 0$$

Since the conductor is an equipotential body and E_{tc}= 0, $E_t = 0$, i.e. the tangential component of electric field strength at the conductor surface is zero.

Similarly, consider the closed a "pillbox" surface S which straddles the boundary between the two regions, and has the height Δh and end-cap area ΔS. However, *it is essential to recognise that at the conductor surface which is in contact with the dielectrics, there will be surface charge density distribution* σ. Application of Gauss's law to the surface S gives:

$$\oiint_S \vec{D}\cdot d\vec{S} = D_n\Delta S - D_{nc}\Delta S + \iint_{S1}\vec{D}\cdot d\vec{S} = \sigma\Delta S(\text{charge enclosed})$$

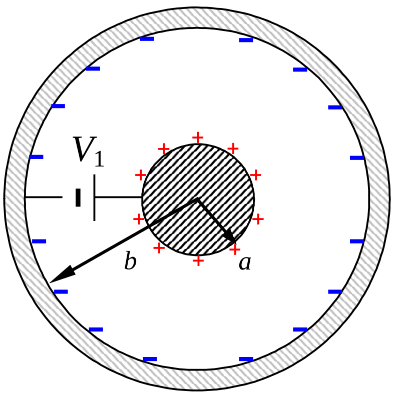
where S_1 denotes the cylindrical surface of S. If $\Delta h \to 0$, the area of the cylindrical portion of the surface becomes zero, so the contributions to the integral come from the bottom and top end-cap surfaces. Thus:

$D_n\Delta S - D_{nc}\Delta S = \sigma\Delta S \quad or \quad D_n = \sigma$, since the electric flux density, D_{nc}, inside the conductor is zero.

In summary the interface boundary conditions are given by:

$$E_t = 0$$
$$D_n = \sigma$$

14. Derive expressions for potential *V* and electric field strength *E* in cylindrical coordinate system for the coaxial cable as shown below:



The Laplace equation of the potential V in the cylindrical co-ordinate system is given by:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\left(\frac{\partial^2 V}{\partial \theta^2}\right) + \frac{\partial^2 V}{\partial z^2} = 0$$

Due to the symmetry of the field distribution in the two co-axial conductors, *V* will be independent of θ and *z*. Thus the Laplace equation is reduced to

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dV}{dr}\right) = 0$$

Integrating with respect to r gives:

$$\left(r\frac{dV}{dr} \right) = C_1$$

where C_1 is a constant. Integrating again yield:

$$V = C_1 \ln r + C_2$$

Boundary conditions: $V\big|_{r=a} = V_0 \ \ : \ \ V\big|_{r=b} = 0$

$$V_0 = C_1 \ln a + C_2$$
$$0 = C_1 \ln b + C_2$$

Solving for C_1 and C_2 results in

$$C_1 = \frac{V_0}{\ln(a/b)}$$
$$C_2 = -C_1 \ln b = -\frac{V_0 \ln b}{\ln(a/b)}$$

and

$$V = \frac{V_0 \ln(r/b)}{\ln(a/b)}$$

The electric field strength is given by:

$$E_r = -\frac{dV}{dr} = -\frac{V_0}{\ln(a/b)}\frac{1}{r}$$

Apparently the absolute value of E_r reaches the maximum at $r = a$

Can you find out the total charge per unit length on the surface of the inner conductor, and hence determine the per unit length capacitance of the cable? Compare the results obtained using Gauss's law (page 2-3 of the lecture note).