GUIDE SOLUTIONS FOR EXTERNAL EXAMINER

SETTER: P. Trodden

Data Provided:
Laplace and z-transforms
Compensator design formulae
Performance criteria mappings

DEPARTMENT OF AUTOMATIC CONTROL & SYSTEMS ENGINEERING
Spring Semester 2017–2018

ACS342 FEEDBACK SYSTEMS DESIGN 2 hours

Answer ALL THREE questions.

Trial answers will be ignored if they are clearly crossed out.

All questions are marked out of 20. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

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1. A feedback control system is shown in Figure 1.1.

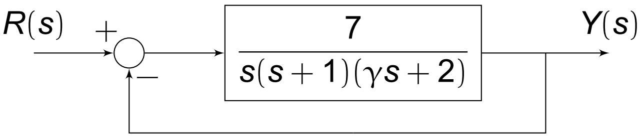


Figure 1.1

a) Write down the open-loop pole locations, and hence identify the range of γ for which the open-loop system is stable.

Answer:

$s = 0, -1$ and $-2/\gamma$; the open-loop system is stable iff $-2/\gamma < 0 \iff \gamma > 0$.

[2 marks]

[2 marks]

b) Show that the closed-loop transfer function of the system is

$$\frac{Y(s)}{R(s)} = \frac{7}{\gamma s^3 + (\gamma+2)s^2 + 2s + 7}$$

Hence, determine the range of γ for which the closed-loop system is stable.

Answer:

Let $G(s) = \dfrac{7}{s(s+1)(\gamma s+2)}$. Then

$$\frac{Y(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{7}{s(s+1)(\gamma s+2)+7} = \frac{7}{\gamma s^3 + (\gamma+2)s^2 + 2s + 7}$$

The stability is ascertained from the Routh array:

$$\begin{array}{lrl} s^3: & \gamma & 2 \\ s^2: & \gamma+2 & 7 \\ s^1: & \frac{2(\gamma+2)-7\gamma}{\gamma+2} = 2 - \frac{7\gamma}{\gamma+2} & 0 \\ s^0: & 7 & 0 \end{array}$$

For stability, require all same signs in the first column. Therefore

$$\gamma > 0 \because 7 > 0$$
$$\gamma + 2 > 0 \implies \gamma > -2$$
$$2 - \frac{7\gamma}{\gamma+2} > 0 \implies 2/\gamma > 2.5 \implies \gamma < 0.8$$

Therefore,

$$\boxed{0 < \gamma < 0.8}$$

[6 marks]

[6 marks]

The next two parts of this question use the Bode diagram of the open-loop system (for a particular, but unknown, value of $\gamma > 0$) provided overleaf in Figure 1.2.

c) (i) Estimate the gain margin and phase margin of the system. Is the closed-loop system stable or unstable for this particular value of γ?

Answer:

The guidelines are annotated on the plot.

Gain margin: approximately 15 dB at 4.5 rad/s.
Phase margin: approximately 25 degrees at 1.7 rad/s.

Both are positive, so the closed-loop system is stable.

[2 marks]

(ii) Estimate the rise time and overshoot of the closed-loop system.

Answer:

Several ways to estimate the rise time. The easiest is to estimate the closed-loop bandwidth, ω_B, as being between 1 and 2 times the gain crossover frequency, ω_c (which is, from the plot, approximately 1.7 rad/s):

$$1.7 \lesssim \omega_B \lesssim 3.4$$

(In reality, the closed-loop bandwidth of this system is 2.8 rad/s, to 1 d.p., so this is a reasonable assumption.)
The rise-time–bandwidth product then gives

$$T_r \approx \frac{2.2}{\omega_B} \implies 0.65 \lesssim T_r \lesssim 1.3$$

(The actual rise time is 0.67 seconds, so again the approximation is reasonable.)
For overshoot, first find the damping ratio

$$\zeta \approx 0.01\phi_{pm} \approx 0.25$$

The percentage overshoot is then

$$100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) = 45$$

Depending on phase margin estimates, an overshoot in the range 35% to 55% is acceptable. (The actual overshoot is 49% – again showing the approximation is reasonable.)

[4 marks]

[6 marks]

d) Design a phase-lead compensator

$$C(s) = \frac{s\alpha\tau + 1}{s\tau + 1}$$

in order to achieve a phase margin of 45° for the system. Use a safety margin of 5°, and do not attempt to use the provided transfer function of the system to perform exact calculations—your design should be done using readings from the Bode diagram in Figure 1.2.

Answer:

The transfer function of the lead compensator is

$$C(s) = \frac{s\alpha\tau + 1}{s\tau + 1}$$

We need to determine α and τ, using the known and provided data.

1. 45° of phase margin is required. The existing phase margin is around 25°, and we are told to add a safety margin of 5°. Hence we are looking for

$$\phi_m = 25^\circ$$

of phase advance from $C(s)$. This fixes the parameter α, via

$$\sin 25^\circ = \frac{\alpha - 1}{\alpha + 1} \implies \alpha = \frac{1 + \sin 25^\circ}{1 - \sin 25^\circ} = 2.46$$

Because estimated phase margins in the range 20° to 30° are acceptable, a range of answer α is expected and acceptable:

$$\boxed{2.0 \lesssim \alpha \lesssim 3.0}$$

2. The maximum phase advance should occur at the gain crossover frequency, accounting for the fact that the phase lead compensator introduces $10\log_{10}\alpha$ extra gain at the maximum phase advance frequency.
That is, the new gain crossover frequency, ω_c', is the frequency at which $|G(\jmath\omega)| = 1/\sqrt{\alpha}$, or $-10\log_{10}\alpha$ in dB. Using the range of acceptable α, that means we are looking to identify the frequencies corresponding to the range -3 dB to -5 dB. From the figure,

$$\boxed{2.0 \lesssim \omega_c' \lesssim 2.5}$$

Anything close to this range is acceptable. We set

$$\omega_m = \omega_c'$$

Then,

$$\tau = \frac{1}{\omega_m\sqrt{\alpha}}$$

which gives τ in the range

$$\boxed{0.23 \leqslant \tau \leqslant 0.35}$$

The lower bound on τ corresponds to $\alpha = 3.0$ and the upper bound to $\alpha = 2.0$. Therefore, we expect

$$\boxed{0.69 \lesssim \alpha\tau \lesssim 0.70}$$

Taking $C(s) = \dfrac{0.70s + 1}{0.35s + 1}$ gives a phase margin of 39°, while $C(s) = \dfrac{0.69s + 1}{0.23s + 1}$ gives 47°. Both short, but the important thing is to follow the correct process.

[6 marks]

[6 marks]

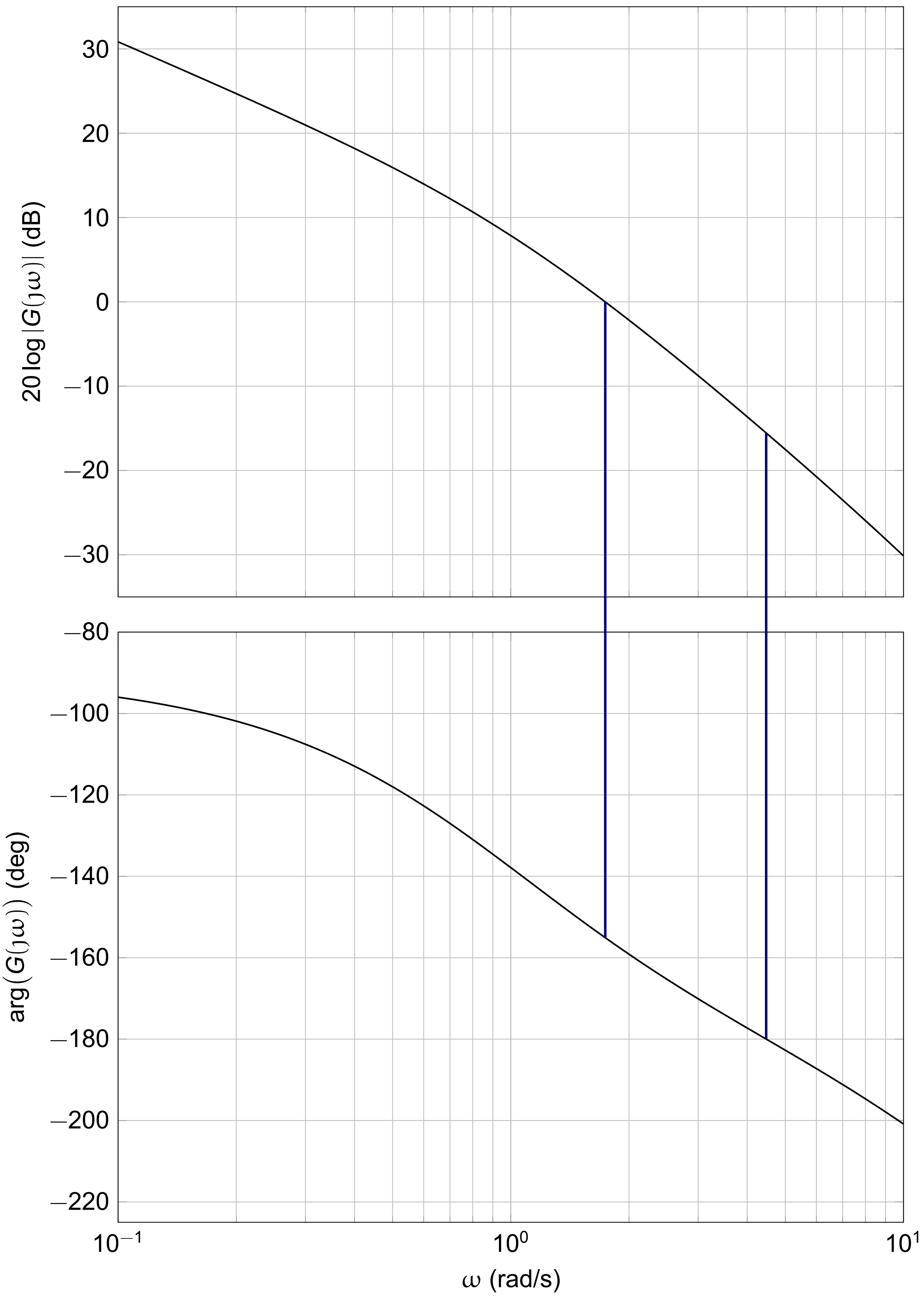


Figure 1.2: Bode diagram for Q1.

2. A unity-feedback system has the open-loop transfer function

$$KG(s) = \frac{K}{s^2 + 4s + 4}$$

a) Find the closed-loop transfer function, and hence determine the damping ratio and natural frequency of the closed-loop system as functions of K. Show that the settling time of the closed-loop step response is constant (*i.e.*, independent of K).

Answer:

The closed-loop transfer function is

$$T(s) = \frac{KG(s)}{1 + KG(s)} = \frac{K}{s^2 + 4s + (4 + K)}$$

The damping ratio ζ and natural frequency ω_n are obtained by comparing the denominator to the canonical charactistic function:

$$s^2 + 4s + (4 + K) = s^2 + 2\zeta\omega_n s + \omega_n^2$$

Therefore, $\omega_n = \sqrt{4 + K}$ and

$$\zeta = \frac{4}{2\omega_n} = \frac{2}{\omega_n} = \frac{2}{\sqrt{4 + K}}$$

The 2% settling time is

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\frac{2}{\omega_n}\omega_n} = 2$$

which is constant and independent of K.

[5 marks]

[5 marks]

b) Find an expression for the position error constant of $KG(s)$ in terms of K, and hence calculate the percentage steady-state tracking error (in response to a step) when K is chosen to provide an overshoot of 5%.

Answer:

The position error constant is

$$K_p = \lim_{s \to 0} KG(s) = \lim_{s \to 0} \frac{K}{s^2 + 4s + 4} = K/4$$

For 5% overshoot, we require

$$\begin{aligned}
\zeta &= \frac{-\ln(\text{O.S. } (\%)/100)}{\sqrt{\pi^2 + [\ln(\text{O.S. } (\%)/100)]^2}} \\
&= \frac{-\ln 0.05}{\sqrt{\pi^2 + [\ln 0.05]^2}} \\
&= 0.69
\end{aligned}$$

This requires a gain of

$$\sqrt{4+K} = \frac{2}{0.69} \implies K = 4.4$$

The corresponding steady-state error is

$$e_{ss}(\%) = 100\frac{1}{1+K_p} = \frac{100}{1+4.4/4} = 47.6$$

[5 marks]

[5 marks]

c) Design a phase-lag compensator in order that the closed-loop system meets the following specification.

Overshoot (%)	$\leqslant 5$
Position error constant	$\geqslant 20$

You are given that the desired dominant pole location is $s^{\star} = -2 \pm \jmath 3.5$.

Answer:

As we have seen, a gain $K = 4.4$ achieves the right overshoot but excessive steady-state error.
The uncompensated error constant is

$$K_p^{\text{uncomp}} = 4.4/4 = 1.1$$

The phase-lead compensator is

$$C(s) = K\frac{s+z}{s+p}$$

leading to a compensated error constant

$$K_p^{\text{comp}} = \frac{z}{p}K_p^{\text{uncomp}}$$

Hence, for $K_p^{\text{comp}} = 20$, require

$$z/p = \frac{20}{1.1} = 18.2$$

This fixes the ratio of z to p. Next, p has to be positioned such that $z = 18.2p$ does not affect pole dominance. The dominant poles have real parts of $\sigma = 2$, so a reasonable first placement is

$$p = 0.01$$
$$z = 18.2p = 0.182$$

In practice, round up to $z = 0.19$.

The final step is to calculate the angles from the dominant pole location, s^* to the pole and zero of the compensator.

$$\theta_{s^* \to z} = \arctan \frac{3.5}{2 - 0.19} = 62.7^\circ$$
$$\theta_{s^* \to p} = \arctan \frac{3.5}{2 - 0.01} = 60.4^\circ$$

The difference is around 2°, so zero and pole are apparently close together and pole dominance is not affected. The design is acceptable.

$$C(s) = 4.4 \frac{s + 0.19}{s + 0.01}$$

[5 marks]

[5 marks]

d) The following continuous-time compensator is to be implemented on a digital platform.

$$C(s) = 5 \frac{s + 1}{s + 0.1}$$

Derive a z-transform representation of the compensator's transfer function. Use a sampling time of $T = 0.1$ seconds and zero-order hold for sampling of the continuous-time input signal to the compensator.

Answer:

Need to find the equivalent z-domain expression for

$$\frac{1 - e^{-sT}}{s} C(s) = (1 - e^{-sT}) K \frac{s + z}{s(s + p)}$$

The mapping is $z = e^{sT}$. Then (with abuse of notation)

$$D(z) = K(1 - z) \mathcal{Z} \left\{ \frac{s + n}{s(s + p)} \right\}$$

where n is being used as the zero location, to avoid confusion with z. The rational fraction requires partial fraction expansion:

$$\frac{s + n}{s(s + p)} = \frac{A}{s} + \frac{B}{s + p}$$
$$\implies s + n = A(s + p) + Bs$$
$$\implies A + B = 1, Ap = n$$
$$\implies A = n/p, B = 1 - n/p$$

Then

$$\mathcal{Z}\left\{\frac{s+n}{s(s+p)}\right\} = \mathcal{Z}\left\{\frac{n/p}{s} + \frac{1-n/p}{s+p}\right\}$$
$$= (n/p)\frac{z}{z-1} + (1-n/p)\frac{z}{z-e^{-pT}}$$

Finally,

$$D(z) = K\frac{z-1}{z}\left[(n/p)\frac{z}{z-1} + (1-n/p)\frac{z}{z-e^{-pT}}\right]$$
$$= K\left[(n/p) + (1-n/p)\frac{z-1}{z-e^{-pT}}\right]$$
$$= K\left[\frac{(n/p)(z-e^{-pT}) + (1-n/p)(z-1) - 1}{z-e^{-pT}}\right]$$
$$= K\left[\frac{z-(n/p)e^{-pT} + (n/p) - 1}{z-e^{-pT}}\right]$$

Substituting in the numbers, $K = 5$, $n/p = 10$, $e^{-pT} = e^{-0.1\times 0.1} = 0.9900$, so

$$D(z) = 5\frac{z-0.9005}{z-0.9900} = \frac{5z-4.5025}{z-0.9900}$$

[5 marks]

[5 marks]

3. The attitude dynamics of a satellite are modelled by the ordinary differential equation

$$50\frac{d^2\theta(t)}{dt^2} = \tau(t)$$

where θ is the attitude (orientation) of the satellite with respect to a particular coordinate frame, and $\tau(t)$ is the torque applied to the satellite (by reaction wheels).

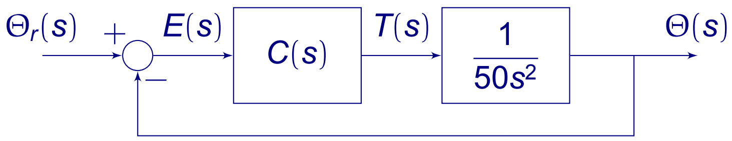
The aim is to design an automatic control system that can re-orient the satellite smoothly and exactly to a desired reference attitude, θ_r. In particular, the specification is as follows:

Overshoot (%)	$\leqslant 5$
Settling time (s)	$\leqslant 10$

To achieve this aim, feedback control is proposed, and a controller $C(s)$, acting on the error between θ_r and θ, is to be designed.

a) Draw a block diagram of the feedback control system.

Answer:



[3 marks]

[3 marks]

b) For the case of $C(s) = K$, determine the following for the closed-loop system (in terms of K where appropriate):

(i) the transfer function, $\Theta(s)/\Theta_r(s)$;

Answer:

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{KG(s)}{1+KG(s)} = \frac{K}{50s^2+K}$$

[1 mark]

(ii) the pole locations;

Answer:

$$50s^2 + K = 0 \implies s = \pm\sqrt{-K/50} = \pm\jmath\sqrt{K/50}$$

[1 mark]

(iii) the damping ratio and natural frequency;

Answer:

By comparison with the standard second order form

$$s^2 + K/50 = s^2 + 2\zeta\omega_n s + \omega_n^2 \implies \zeta = 0, \omega_n = \sqrt{K/50}$$

[2 marks]

(iv) the impulse response.

Answer:

The impulse response is the inverse Laplace transform of the closed-loop transfer function (because $\mathcal{L}\{\delta(t)\} = 1$).

$$\theta(t) = \mathcal{L}^{-1}\left\{\frac{K/50}{s^2 + K/50}\right\} = \mathcal{L}^{-1}\left\{\frac{\omega_n \times \omega_n}{s^2 + \omega_n^2}\right\} = \omega_n \sin \omega_n t$$

[3 marks]

Hence, explain why the feedback control system is unable to meet the specification with $C(s) = K$.

Answer:

The response to a step or impulsive change in the reference angle is an undamped oscillation.

[1 mark]

[8 marks]

c) Design a PD controller

$$C(s) = K_P + sK_D,$$

by finding suitable gains K_P, K_D in order that the closed-loop poles lead to satisfaction of the specification.

Answer:

With the PD controller, the new closed-loop transfer function is

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{K_P + K_D s}{50s^2 + K_D s + K_P} = \frac{(K_P + K_D s)/50}{s^2 + \frac{K_D}{50}s + \frac{K_P}{50}}$$

Hence $\omega_n = \sqrt{K_P/50}$ and $2\zeta\omega_n = K_D/50$. The specification calls for $T_s = \frac{4}{\zeta\omega_n} = 10 \implies \zeta\omega_n = 0.4 \implies K_D = 40$.

Overshoot of 5% requires $\zeta = 0.69$. Since $\zeta\omega_n = 0.4$, $\omega_n = 0.4/0.69 = 0.58$. Finally, since $\omega_n = \sqrt{K_P/50}$ then $K_P = 50\omega_n^2 = 16.8$.
The PD controller is

$$C(s) = 16.8 + 40s$$

[5 marks]

[5 marks]

d) In the real system, the following PD controller is implemented:

$$C(s) = 16 + 40s$$

Experiments with the PD-controlled closed-loop system reveal that the overshoot is significantly more (close to 20%) than that predicted from the closed-loop poles. By analysing the transfer function of the closed-loop system, identify a possible cause of this (aside from modelling errors). Explain how the excessive overshoot might be eliminated.

Answer:

The cause is the zero that the PD controller introduces. Inspecting the closed-loop transfer function,

$$\frac{\Theta(s)}{\Theta_r(s)} = \frac{(K_P + K_D s)/50}{s^2 + \frac{K_D}{50}s + \frac{K_P}{50}} = \frac{40s + 16}{50s^2 + 40s + 16}$$

and letting $S(s) = \dfrac{16}{50s^2 + 40s + 16}$, the step response is the inverse of

$$\Theta(s) = S(s)R(s) + (40/16)sS(s)R(s)$$

In other words, the step response is equal to the second-order step response that delivers 5% overshoot **plus** 2.5 times its derivative (which is positive up until the peak time).
The problem can be minimize by including a pre-filter $F(s)$. Then

$$\frac{\Theta(s)}{\Theta_r(s)} = F(s)\frac{40s + 16}{50s^2 + 40s + 16}$$

If $F(s) = \frac{16}{40s+16}$, then the problematic zero is cancelled and the desired standard second-order transfer function is recovered.

[4 marks]

[4 marks]

Laplace and z-transforms

Time domain	s-domain	z-domain
$f(t)$	$F(s)$	$F(z)$
$f(t-T)$	$e^{-sT}F(s)$	$z^{-1}F(z)$
$\delta(t)$	1	–
1	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{zT}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$f^{(n)}(t)=\frac{d^n}{dt^n}f(t)$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{n-1}(0)$	Various forms

Compensator design formulae

Transfer function	$\frac{s\alpha\tau+1}{s\tau+1}$ (lead) $\frac{s\tau+1}{s\alpha\tau+1}$ (lag)
Maximum phase lead/lag, ϕ_m	$\sin^{-1}\frac{\alpha-1}{\alpha+1}$
Centre frequency, ω_m	$\frac{1}{\tau\sqrt{\alpha}}$

Performance criteria mappings

2% settling time, T_s	$\dfrac{4}{\zeta\omega_n}$
10–90% rise time, T_r	$\dfrac{2.16\zeta + 0.6}{\omega_n}$ for $0.3 \leqslant \zeta \leqslant 0.8$
Percentage overshoot, O.S. (%)	$100 \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$
Damping ratio from overshoot	$\zeta = \dfrac{-\ln(\text{O.S. }(\%)/100)}{\sqrt{\pi^2 + [\ln(\text{O.S. }(\%)/100)]^2}}$
Peak time, T_p	$\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ for $0 < \zeta < 1$
Peak response, M_p	$1 + \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$
Resonant frequency, ω_r	$\omega_n\sqrt{1-2\zeta^2}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Resonant peak magnitude, $M_{p\omega}$	$\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Phase margin, ϕ_{pm}	100ζ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Bandwidth–Rise time	$T_r = \dfrac{2.2}{\omega_B}$

END OF QUESTION PAPER