wave reflection and voltage standing wave ratio (VSWR)

$$V = V(x,t)$$
$$= V_0^+ e^{j\omega t} e^{-j\tilde{k}x} + V_0^- e^{j\omega t} e^{j\tilde{k}x}$$

(forwards travelling) (backwards travelling)

where

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}$$ is voltage reflection coefficient

$\Rightarrow$ $V = V_0^+ e^{j\omega t}\left(e^{-j\tilde{k}x} + \Gamma e^{+j\tilde{k}x}\right)$ and analogous for current

$\rightarrow$ need to distinguish several cases:

① Line matched case: $Z_L = Z_0$

$\Rightarrow$ $\Gamma = 0$

$\Rightarrow$ $V(x,t) = V_0^+ e^{j\omega t} e^{-j\tilde{k}x}$ (no reflected wave)

Of both oscillations (time and space) we can only measure the real component. It is common to use as boundary conditions $V(x=0, t=0) = 0$, which reduces to the sine (imaginary) component; i.e.

$$\mathrm{Im}\, V(x,t) = V_0^+ \sin(\omega t - \tilde{k}x)$$

describes a freely propagating wave.

② open circuit : $Z_L = \infty$

$\rightarrow$ $\Gamma = +1$

$\rightarrow$ $V(x,t) = V_0^+ e^{j\omega t}\Big(\underbrace{e^{-j\tilde{k}x}}_{\cos\tilde{k}x - \cancel{j\sin\tilde{k}x}} + \underbrace{e^{+j\tilde{k}x}}_{\cos\tilde{k}x + \cancel{j\sin\tilde{k}x}}\Big)$

$$= 2V_0^+ \cdot e^{j\omega t} \cos\tilde{k}x$$

$\rightarrow$ $\mathrm{Im}\, V(x,t) = 2V_0^+ \sin\omega t \cos\tilde{k}x$

describes a wave with zero amplitude at positions $\tilde{k}x$ = odd multiple of $\pi/2$

3. short-circuited transmission line : $Z_L = 0$

$\Rightarrow \Gamma = -1$

$$\Rightarrow V(x,t) = V_0^+ e^{j\omega t} \left(\underbrace{e^{-j\tilde{k}x}}_{\cancel{\cos \tilde{k}x} - j \sin \tilde{k}x} - \underbrace{e^{+j\tilde{k}x}}_{\cancel{\cos \tilde{k}x} + j \sin \tilde{k}x} \right)$$

$$= -2V_0^+ \underbrace{e^{j\omega t}}_{\cos \omega t + j \sin \omega t} j \sin \tilde{k}x$$

$$\Rightarrow \operatorname{Im} V(x,t) = -2V_0^+ \cos \omega t \sin \tilde{k}x$$

describes a wave with zero amplitude if $\tilde{k}x = n \cdot \pi$

4. general case

set $\Gamma = \gamma e^{j\varphi}$ as complex wave reflection coefficient with amplitude γ and phase angle φ

$$\Rightarrow V(x,t) = V_0^+ e^{j\omega t} \left(e^{-j\tilde{k}x} + \gamma e^{j\varphi} e^{j\tilde{k}x} \right)$$

$$= V_0^+ e^{j\omega t} e^{j\frac{\varphi}{2}} \left[e^{-j(\tilde{k}x + \frac{\varphi}{2})} + \gamma e^{j(\tilde{k}x + \frac{\varphi}{2})} \right]$$

define new angle by $\theta := \tilde{k}x + \frac{\varphi}{2}$

$$\Rightarrow V(x,t) = V_0^+ e^{j\omega t} e^{j\frac{\varphi}{2}} \underbrace{\left(e^{-j\theta} + \gamma e^{+j\theta} \right)}_{\cos\theta - j \sin\theta + \gamma \cos\theta + j\gamma \sin\theta}$$

$$= (1+\gamma) \cos\theta - j (1-\gamma) \sin\theta$$

$$\Rightarrow |V(x,\cancel{t})| = V_0^+ \left[(1+\gamma)^2 \cos^2\theta + (1-\gamma)^2 \sin^2\theta \right)^{1/2}$$

expand bracketts and use $\cos^2\theta + \sin^2\theta = 1$ and also $\cos^2\theta - \sin^2\theta = \cos(2\theta)$

$$\Rightarrow |V(x,\cancel{t})| = V_0^+ \left[\cos^2\theta + 2\gamma \cos^2\theta + \gamma^2 \cos^2\theta + \sin^2\theta - 2\gamma \sin^2\theta + \gamma^2 \sin^2\theta \right]^{1/2}$$

$$= V_0^+ \left[1 + \gamma^2 + 2\gamma \underbrace{\cos(2\theta)}_{= -1, \dots, +1} \right]^{1/2}$$

$|V(x)|$ is max., if $\cos(2\theta)=1$.

Then:

$$|V(x)|^{max} = V_0^+ \left(\underbrace{1+\gamma^2+2\gamma}_{(1+\gamma)^2}\right)^{1/2} = (1+\gamma)\, V_0^+$$

$|V(x)|$ is min., if $\cos(2\theta)=-1$.

Then.

$$|V(x)|_{min} = V_0^+ \left(\underbrace{1+\gamma^2-2\gamma}_{(1-\gamma)^2}\right)^{1/2} = (1-\gamma)\, V_0^+$$

Define voltage standing wave ratio as

$$\boxed{VSRW = \frac{|V(x)|^{max}}{|V(x)|_{min}} = \frac{1+\gamma}{1-\gamma} = \frac{1+|\Gamma|}{1-|\Gamma|}}$$

→ again, distinguish special cases:

1. line matched case: $Z_L = Z_0$

   $\Rightarrow \Gamma = 0 \Rightarrow VSRW = 1$, as $|V(x,t)| = V_0^+ = \text{const.}$

2. open circuit: $Z_L = \infty$

   $\Rightarrow \Gamma = 1 \Rightarrow VSRW = +\infty$, as $|V(x,t)|_{min} = 0$

3. short-circuited case: $Z_L = 0$

   $\Rightarrow \Gamma = -1 \Rightarrow VSRW = +\infty$ as above for 2.

Summary:

When transmission line and load are mismatched, part of the voltage (and current) waves are reflected so that

$$V(x,t) = V_0^+ e^{j\omega t}\left(e^{-j\tilde{k}x} + \Gamma e^{+j\tilde{k}x}\right)$$

with

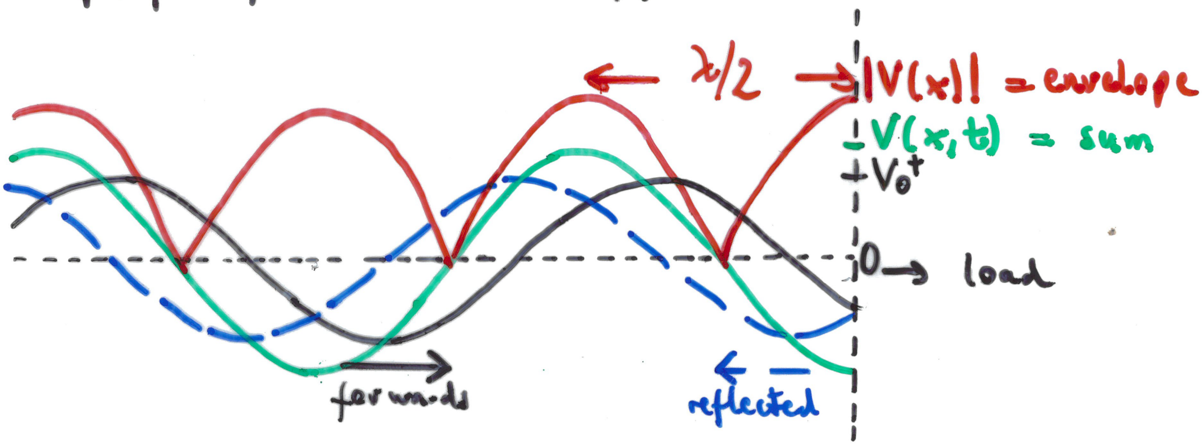
$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \gamma e^{i\varphi}$$

This yields a standing wave as solution to the time-independent wave equation where the actual voltage (or current) varies sinusoidally within the standing wave envelope

$$\boxed{|V(x)| = V_0^+ \left[\gamma^2 + 1 + 2\gamma \cos(2\tilde{k}x + \varphi)\right]^{1/2}}$$

Note that the oscillation period is $2\theta = 2\tilde{k}x + \varphi$, i.e. peaks are separated by $\lambda/2$.

example for open circuit ($\Gamma = 1$):



example for short circuit ($\Gamma = -1$):