THE UNIVERSITY OF SHEFFIELD

GUIDE SOLUTIONS FOR EXTERNAL EXAMINER

SETTER: Paul A Trodden

Data Provided:
Laplace and z-transforms
Compensator design formulae
Performance criteria mappings
Ziegler-Nichols tuning rules

DEPARTMENT OF AUTOMATIC CONTROL & SYSTEMS ENGINEERING
Autumn Semester 2015–2016

ACS342 FEEDBACK SYSTEMS DESIGN **2 hours**

Answer THREE questions.

No marks will be awarded for solutions to a fourth question.

Solutions will be considered in the order that they are presented in the answer book. Trial answers will be ignored if they are clearly crossed out.

If more than the required number of questions are attempted, DRAW A LINE THROUGH THE ANSWERS THAT YOU DO NOT WISH TO BE MARKED.

All questions are marked out of 20. The breakdown on the right-hand side of the paper is meant as a guide to the marks that can be obtained from each part.

1. a) Design a PI controller for a plant with the **open-loop** unit step response shown in Figure 1.1. **[6 marks]**

Answer:

The PI controller is

$$C(s) = K_P\left(1 + \frac{1}{sT_I}\right)$$

where

$$K_P = 0.9\frac{T}{KL} \qquad T_I = \frac{L}{0.3}$$

and K, L and T are obtained by fitting the first-order-plus delay system

$$\frac{K}{sT+1}e^{-sL}$$

to the provide step response. A procedure for this fit is the following.

1. Find K: steady-state value of step response is KA
2. Mark steepest tangent line, $y = mt + c$
3. Find L: $mt + c$ intersects $y = 0$ at $t = L$
4. Find T: $mt + c$ intersects $y = y_{ss}$ at $t = L + T$

Following this procedure (marked on the step response), we obtain

$$\boxed{K_P = 0.9\frac{0.7}{0.11 \times 0.25} = 23} \qquad \boxed{T_I = \frac{0.25}{0.3} = 0.8}$$

Candidates may also give the controller in the form

$$C(s) = K_P + \frac{K_I}{s}$$

with

$$\boxed{K_P = 23} \qquad \boxed{K_I = \frac{K_P}{T_I} = 29}$$

The numeric values are sensitive to the accuracy of the curve fitting, so allowances can be made; the procedure is the most important part. **[6 marks]**

b) Show that a PI controller is a special case of a phase-lag compensator. Express the locations of the pole and zero of the compensator in terms of the gains K_P and K_I. **[4 marks]**

Answer:

The PI controller has the transfer function

$$C(s) = \frac{U(s)}{E(s)} = K_P + \frac{K_I}{s}$$

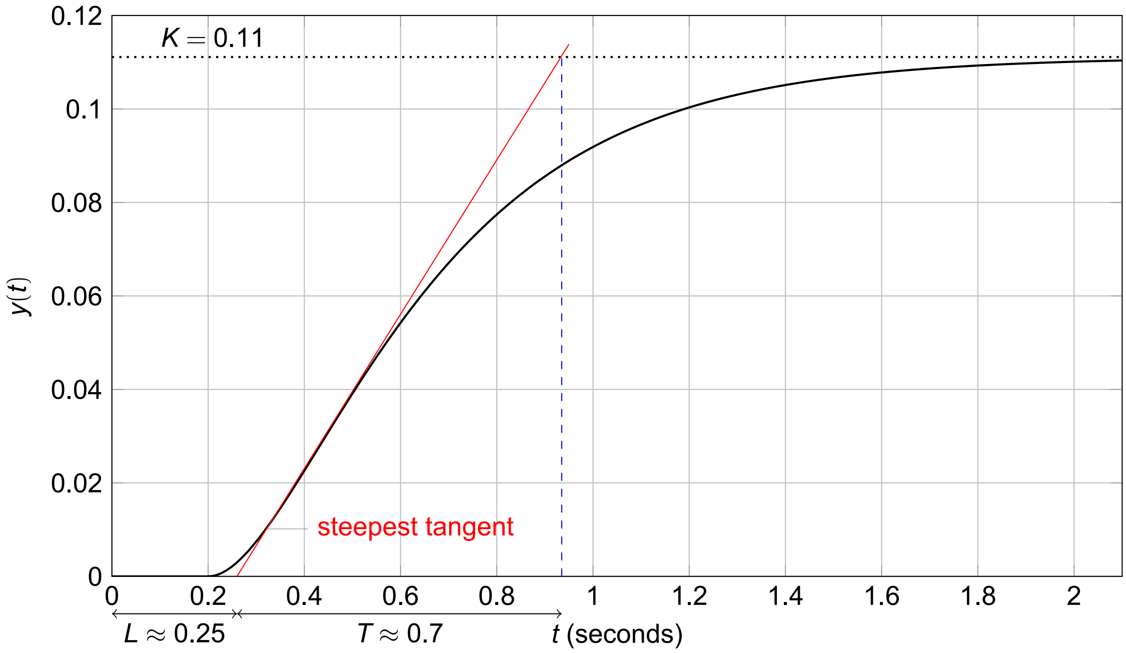


Figure 1.1

Thus,

$$\boxed{C(s) = \frac{K_P s + K_I}{s} = K_P \frac{s + \frac{K_I}{K_P}}{s}}$$

This is in the form of the compensator $K\frac{s+z}{s+p}$, with $K = K_P$, $z = \frac{K_I}{K_P}$ and $p = 0$. Because $|p| < |z|$, then it is phase lag. **[4 marks]**

c) The plant in part (a) is to be controlled by a digital control system, using a discrete-time (sampled data) version of the PI controller.

(i) Explain why the continuous-time PI controller

$$u(t) = K_P e(t) + K_I \int_0^t e(\tau)\,d\tau \tag{1.1}$$

must be converted to a discrete-time representation in order to be suitable for use in a digital control system. **[2 marks]**

Answer:

Examining the continuous-time PI controller, the control input is a continuous function of the current error and the entire history of the error. A digital (sampled data) control system takes samples of continuous-time inputs, errors and/or outputs at discrete (usually periodic) intervals. Therefore, a difference, rather than differential, equation is needed to account for the discrete sampling of signals. The difference equation gives an expression for the output of the controller in terms of previous samples of the error. **[2 marks]**

(ii) Derive the difference equation of the discrete-time PI controller (in terms of K_P and K_I) by discretizing the controller in equation (1.1). Assume a sampling period of T seconds. **[5 marks]**

Answer:

Can be achieved by a number of different discretization methods. Candidates are taught zero-order hold (ZOH) and, in the context of PID, direct (Euler) discretization by numerical integration and differentiation.
Using ZOH, which has impulse response $g_o(t)$ and transfer function $G_o(s)$, on the controller (impulse response $c(t)$, transfer function $C(s)$), (showing as many steps as possible)

$$\begin{aligned}
D(z) &= \mathcal{Z}\{(g_o * c)(t)\} \\
&= \mathcal{Z}\{G_0(s)C(s)\} \\
&= \mathcal{Z}\left\{\frac{1-e^{-sT}}{s}\left(K_P + \frac{K_I}{s}\right)\right\} \\
&= (1-z^{-1})\mathcal{Z}\left\{\frac{K_P}{s} + \frac{K_I}{s^2}\right\} \\
&= (1-z^{-1})\left(K_P\frac{z}{z-1} + K_I\frac{zT}{(z-1)^2}\right) \\
&= \frac{z-1}{z}\left(K_P\frac{z}{z-1} + K_I\frac{zT}{(z-1)^2}\right) \\
&= K_P + K_I\frac{T}{z-1}
\end{aligned}$$

Therefore,

$$\begin{aligned}
U(z) &= D(z)E(z) \\
&= K_P E(z) + K_I\frac{T}{z-1}E(z)
\end{aligned}$$

Inverse transforming, noting that the second term represents integration,

$$\boxed{u[k] = K_P e[k] + K_I T\sum_{i=0}^{k} e[i]}$$

The direct discretization is faster. Approximate the integral:

$$\int_0^t e(\tau)\,d\tau \approx \sum_{i=0}^{k} e[i]T$$

Therefore

$$\boxed{u[k] = K_P e[k] + K_I T\sum_{i=0}^{k} e[i]}$$

[5 marks]

(iii) Using the step response in Figure 1.1, select an appropriate sampling period, T, for the digital control system. Justify your answer. **[3 marks]**

Answer:

The "optimal" rate is the slowest one that still meets performance spec. We are not given a specification here, but it is known that under-sampling (e.g. at frequencies just above the Nyquist frequency) leads to unpredicted overshoot and even instability. Some rules of thumb for sampling in a control system are

- Sampling rate $f = 1/T$ equal to 5–30 (ideally 20–30) times closed-loop bandwidth, f_B
- Sample period T between $0.05T_r$ and $0.5T_r$, where T_r is the 10–90% rise time
- Sample period T such that $0.15 \leqslant T\omega_c \leqslant 0.5$, where ω_c is the gain crossover frequency

Here, the second "rule" is the most appropriate, for we are given the (open-loop step response). From the provided step response, we see that the rise-time of the open-loop system is approximately 0.9 seconds. Therefore, a sample period of

$$\boxed{0.04 \leqslant T \leqslant 0.45}$$

would be appropriate. **[3 marks]**

2. A plant has transfer function

$$G(s) = \frac{20}{s(s+1)(s+10)}$$

a) Sketch the Bode diagram of $G(s)$. **[8 marks]**

Answer:

Re-arranging $G(s)$ into the time-constant form, and letting $s = \jmath\omega$

$$G(\jmath\omega) = \frac{20}{10}\frac{1}{\jmath\omega(\jmath\omega+1)(0.1\jmath\omega+1)}.$$

Therefore, $G(\jmath\omega)$ comprises four factors:

1. Constant gain $K = 2$, with

$$20\log_{10}|2| = 6 \text{ dB}$$
$$\arg 2 = 0$$

2. Integrator $1/\jmath\omega$, with

$$20\log_{10}\left|\frac{1}{\jmath\omega}\right| = -20\log_{10}\omega \text{ dB}$$
$$\arg\left(\frac{1}{\jmath\omega}\right) = -90^\circ$$

3. Pole on the real axis, corner frequency $1/\tau_1 = 1\,\text{rad s}^{-1}$, with

$$20\log_{10}\left|\frac{1}{\jmath\omega\tau_1+1}\right| = -10\log_{10}(\omega^2+1) \text{ dB}$$
$$\arg\left(\frac{1}{\jmath\omega\tau_1+1}\right) = -\tan^{-1}\omega$$

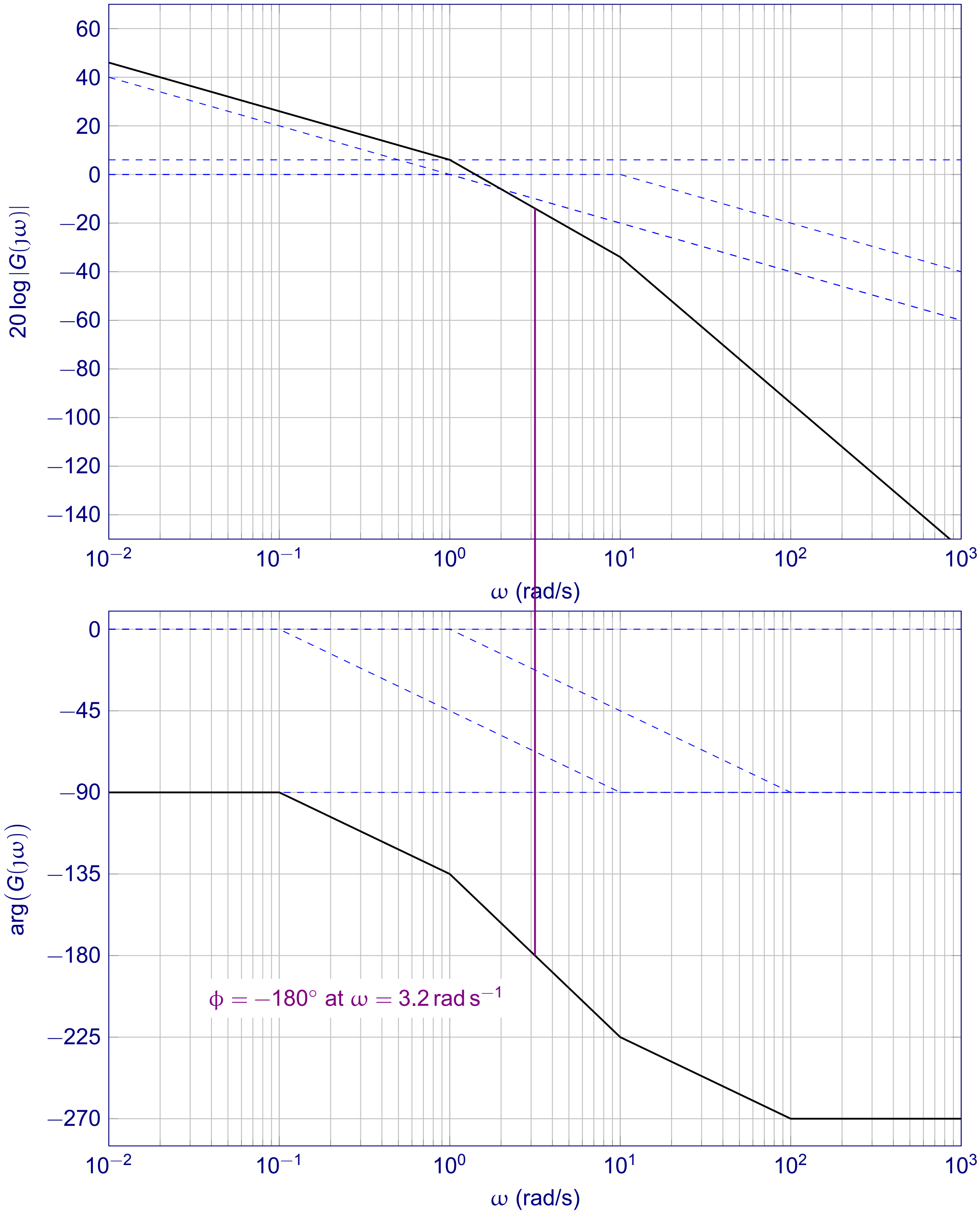
4. Pole on the real axis, corner frequency $1/\tau_2 = 10\,\text{rad s}^{-1}$, with

$$20\log_{10}\left|\frac{1}{\jmath\omega\tau_2+1}\right| = -10\log_{10}(0.01\omega^2+1) \text{ dB}$$
$$\arg\left(\frac{1}{\jmath\omega\tau_2+1}\right) = -\tan^{-1}0.1\omega$$

The individual factors and overall Bode plot are shown in Figure 2.1. **[8 marks]**

b) Explain why the Bode diagram is usually drawn for the open-loop system rather than the closed-loop one, even though it is the stability and performance of the closed-loop system that is most important. **[2 marks]**

Answer:



60
40
20
0
−20
−40
−60
−80
−100
−120
−140
20 log |G(jω)|
10^−2
10^−1
10^0
10^1
10^2
10^3
ω (rad/s)
0
−45
−90
−135
−180
−225
−270
arg(G(jω))
φ = −180° at ω = 3.2 rad s^−1
10^−2
10^−1
10^0
10^1
10^2
10^3
ω (rad/s)

Figure 2.1

The Bode plot is usually of the open-loop system in order to isolate the inherent or physical characteristics of the system to be controlled from the effects of feedback. In other words, a closed-loop Bode diagram will include the effects of feedback, which may mask important characteristics of the system. Moreover, closing the loop can be dangerous (feedback can be destabilizing), while obtaining open-loop frequency response measurements is relatively simple and safe (for a stable loop transfer function). Therefore, it is customary to design the feedback controller by using the open-loop Bode diagram. (Many closed-loop characteristics can be inferred from this, including stability margins, steady state performance, etc.) **[2 marks]**

c) Calculate or estimate (from your Bode diagram) the gain margin of $G(s)$. **[4 marks]**

Answer:

The gain margin is measured at the phase crossover frequency, which is the frequency at which the phase of $G(\jmath\omega) = -180^\circ$. This is estimated in Figure 2.1 as 3.2 rad s^{-1}, or calculated as

$$\begin{aligned}
G(\jmath\omega_p) &= -180^\circ \\
\implies -90^\circ - \tan^{-1}(\omega_p) - \tan^{-1}(0.1\omega_p) &= -180^\circ \\
\implies \tan^{-1}(\omega_p) + \tan^{-1}(0.1\omega_p) &= 90^\circ \\
\implies \tan^{-1}\left(\frac{\omega_p + 0.1\omega_p}{1 - 0.1\omega_p^2}\right) &= 90^\circ \\
\implies 1 - 0.1\omega_p^2 &= 0 \\
\implies & \boxed{\omega_p = \sqrt{10} \text{ or } 3.16}
\end{aligned}$$

At this frequency, the magnitude of $G(\jmath\omega)$ is estimated from the Bode plot as -15 dB, or calculated as

$$\begin{aligned}
|G(\jmath\omega_p)| &= \left|\frac{2}{\jmath\omega(\jmath\omega+1)(0.1\jmath\omega+1)}\right|_{\omega=\omega_p} \\
&= \frac{2}{|\jmath\omega_p||\jmath\omega_p+1||0.1\jmath\omega_p+1|} \\
&= \frac{2}{\sqrt{10}\times\sqrt{10+1}\times\sqrt{1+1}} \\
&= 0.1348 \\
&\boxed{= -17.40 \text{ dB}}
\end{aligned}$$

The gain margin is 0 dB minus this magnitude, so a 17.4 dB, although anything in the range 10–20 dB is acceptable (allowing for drawing precision). **[4 marks]**

d) Complete the design of a phase-lead compensator

$$C(s) = K\frac{s\alpha\tau + 1}{s\tau + 1} \text{ where } K = 1$$

in order to increase the system phase margin by 30 degrees. You may use the values in Table 2.1 to guide your design. **[6 marks]**

Table 2.1

ω [rad s^{-1}]	0.85	1.00	1.24	1.53	1.74
$20\log_{10}\lvert G(\jmath\omega)\rvert$ [dB]	$+5.0$	$+3.0$	0.0	-3.0	-5.0

Answer:

Two main points to consider:

1. 30° of phase advance is required. This fixes the parameter α, via

$$\sin 30^\circ = \frac{\alpha - 1}{\alpha + 1} \implies \alpha = \frac{1 + \sin 30^\circ}{1 - \sin 30^\circ} = 3$$

(It is acceptable to add a safety margin of up to 10%, which gives an α in the range)

$$\boxed{3 \leqslant \alpha \leqslant 3.4}$$

2. The maximum phase advance should occur at the gain crossover frequency, accounting for the fact that the phase lead compensator introduces $10\log_{10}\alpha$ extra gain at the maximum phase advance frequency.
That is, the new gain crossover frequency, ω_p', is the frequency at which $\lvert G(\jmath\omega)\rvert = 1/\sqrt{\alpha}$, or $-10\log_{10}\alpha$ in dB. Using the range of acceptable α, that means we are looking for $20\log_{10}\lvert G(\jmath\omega)\rvert$ in the range -4.8 dB to -5.3 dB. From Table 2.1, choose

$$\boxed{\omega_p' = 1.74\,\text{rad s}^{-1}}$$

Then,

$$\tau = \frac{1}{\omega_m\sqrt{\alpha}}$$

which gives τ in the range

$$\boxed{0.31 \leqslant \tau \leqslant 0.33}$$

[6 marks]

3. A vehicle's active suspension system is modelled by the *quarter-car* model

$$m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + c\frac{\mathrm{d}y}{\mathrm{d}t} + ky = f$$

where y is the vertical displacement of the car body from its equilibrium position (m), m is the mass of the quarter-car body (kg), c is the damping coefficient ($\mathrm{N\,m^{-1}\,s^{-1}}$), and k is the spring stiffness ($\mathrm{N\,m^{-1}}$). The system includes a hydraulic actuator, which exerts a force f (N) in order to actively control the suspension.

a) (i) Derive the open-loop transfer function, $G(s)$, between control input f and displacement y. **[3 marks]**

Answer:

Taking Laplace transforms,

$$m[s^2Y(s) - sy(0) - \dot{y}(0)] + c[sY(s) - y(0)] + kY(s) = F(s)$$

Neglecting all initial conditions,

$$\boxed{G(s) = \frac{Y(s)}{F(s)} = \frac{1}{ms^2 + cs + k}}$$

[3 marks]

(ii) Obtain expressions for the damping ratio ζ and natural frequency ω_n in terms of the model parameters m, c and k. **[3 marks]**

Answer:

Comparing the denominator (characteristic function) of $G(s)$ with that of a standard second order linear system

$$s^2 + 2\zeta\omega_n s + \omega_n^2$$

we obtain

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}} \qquad \boxed{\zeta = \frac{1}{2\omega_n}\frac{c}{m} = \frac{c}{2\sqrt{mk}}}$$

(Note that the original characteristic function, ms^2+cs+k, must be divided through by m.) **[3 marks]**

(iii) Given $m = 250\,\mathrm{kg}$, $c = 1000\,\mathrm{N\,m^{-1}\,s^{-1}}$, $k = 15\,000\,\mathrm{N\,m^{-1}}$, show that the open-loop suspension system is under-damped and estimate the 2% settling time. **[3 marks]**

Answer:

With the values given,

$$\boxed{\zeta = \frac{c}{2\sqrt{mk}} = \frac{1000}{2\sqrt{250 \times 15000}} = 0.258}$$

As this is less than 1, the system is under-damped.
The settling time is

$$\boxed{T_\mathrm{s} = \frac{4}{\zeta\omega_\mathrm{n}} = \frac{4}{0.5(c/m)} = \frac{8m}{c} = 2\,\mathrm{s}}$$

[3 marks]

(iv) The suspension system has an adjustable damper. For what value of damping coefficient, c, would the system be critically damped ($\zeta = 1$)? What happens to the settling time in this case? **[3 marks]**

Answer:

Since $\zeta = \dfrac{c}{2\sqrt{mk}}$ then, for $\zeta = 1$,

$$\boxed{c = 2\sqrt{mk} = 2\sqrt{250 \times 15000} = 3873\,\mathrm{N\,m^{-1}\,s^{-1}}}$$

In that case, the settling time is reduced to

$$\boxed{T_\mathrm{s} = \frac{8m}{c} = \frac{2000}{3873} = 0.52\,\mathrm{s}}$$

[3 marks]

b) The closed-loop active suspension system is shown in Figure 3.1. $D(s)$ represents a disturbance force caused by the road surface. The controller provides a hydraulic force proportional to the measured displacement $y(t)$, with the aim of smoothly rejecting disturbances and keeping the displacement small.

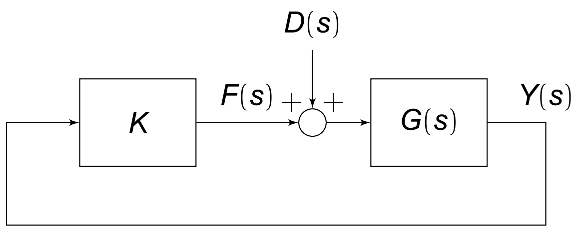


Figure 3.1

(i) Why is a reference input unnecessary for this system? **[1 mark]**

Answer:

The aim is to reject the disturbance rather than for the output to track a reference input or setpoint. The displacement should be kept small but there is no requirement for it to settle to a particular value. **[1 mark]**

(ii) Derive the closed-loop transfer function between disturbance $D(s)$ and displacement $Y(s)$, expressed in terms of the parameters m, c and k. **[4 marks]**

Answer:

$$\begin{aligned} Y(s) &= G(s)[D(s) + F(s)] \\ &= G(s)[D(s) + KY(s)] \\ &= G(s)D(s) + KG(s)Y(s) \\ \implies Y(s)[1 - KG(s)] &= G(s)D(s) \\ \implies \frac{Y(s)}{D(s)} &= \frac{G(s)}{1 - KG(s)} \end{aligned}$$

Now, $G(s) = 1/d(s)$ where $d(s) = ms^2 + cs + k$, so

$$\begin{aligned} \frac{Y(s)}{D(s)} &= \frac{1/d(s)}{1 - K/d(s)} \\ &= \frac{1}{d(s) - K} \\ &\boxed{= \frac{1}{ms^2 + cs + k - K}} \end{aligned}$$

[4 marks]

(iii) Determine the combination of damping coefficient c and controller gain K that makes the closed-loop system critically damped ($\zeta = 1$) with a settling time of 2 seconds. **[3 marks]**

Answer:

To achieve a settling time of 2 s,

$$T_s = \frac{8m}{c} = 2 \implies \boxed{c = 4m = 1000\,\mathrm{N\,m^{-1}\,s^{-1}}}$$

(Note that T_s is independent of K, so the result from part (a)(iii) stands.) Comparing the closed-loop transfer function with the standard second-order linear transfer function, we obtain

$$\omega_n = \sqrt{\frac{k - K}{m}} \qquad \zeta = \frac{1}{2\omega_n}\frac{c}{m} = \frac{c}{2\sqrt{m(k - K)}}$$

To make the system critically damped,

$$\zeta = \frac{c}{2\sqrt{m(k - K)}} = 1 \implies \sqrt{m(k - K)} = \frac{c}{2}$$

$$\boxed{K = k - \frac{c^2}{4m} = 15000 - \frac{1000^2}{1000} = 14\,000\,\mathrm{N\,m^{-1}}}$$

[3 marks]

4. A unity feedback system has open-loop transfer function

$$KG(s) = \frac{K}{(s-0.5)(s^2+4s+4)}$$

a) Explain why it is open-loop unstable. **[1 mark]**

Answer:

It has a pole at $s = +0.5$, *i.e.*, in the right-hand side of the s-plane. **[1 mark]**

b) Find the range of K for which the closed-loop system is stable. **[5 marks]**

Answer:

First, obtain the closed-loop transfer function as

$$\frac{KG(s)}{1+KG(s)} = \frac{K}{(s-0.5)(s^2+4s+4)+K} = \frac{K}{s^3+3.5s^2+2s-2+K}$$

As this is third order, stability cannot be assessed by inspecting the coefficients of the characteristic function, but the Routh array can be used.

$$\begin{array}{lrr} s^3: & 1 & 2 \\ s^2: & 3.5 & K-2 \\ s^1: \quad \frac{3.5\times 2-1\times(K-2)}{3.5} = & 2-\frac{K-2}{3.5} & 0 \\ s^0: & K-2 & 0 \end{array}$$

For stability, require all same signs in the first column. Hence,

$$2-\frac{K-2}{3.5} > 0 \implies K < 7$$
$$K-2 > 0 \implies K > 2$$

Therefore, $\boxed{2 < K < 7}$.

[5 marks]

c) Determine what is the largest possible value of position error constant K_p – and the corresponding steady-state error e_{ss}^{step} (%) – that maintains closed-loop stability. **[3 marks]**

Answer:

$$K_P = \lim_{s\to 0} KG(s) = -\frac{K}{2}$$

Then

$$e_{ss}^{step} = \frac{A}{1+K_P} = \frac{A}{1-\frac{K}{2}}$$

Considering the range of stable K, $|K_P|$ is maximized, and $|e_{ss}^{step}|$ is minimized, for $K \to 7$.

$$\boxed{|K_P| < 3.5}$$

$$\boxed{|e_{ss}^{step}| < 40\%}$$

[3 marks]

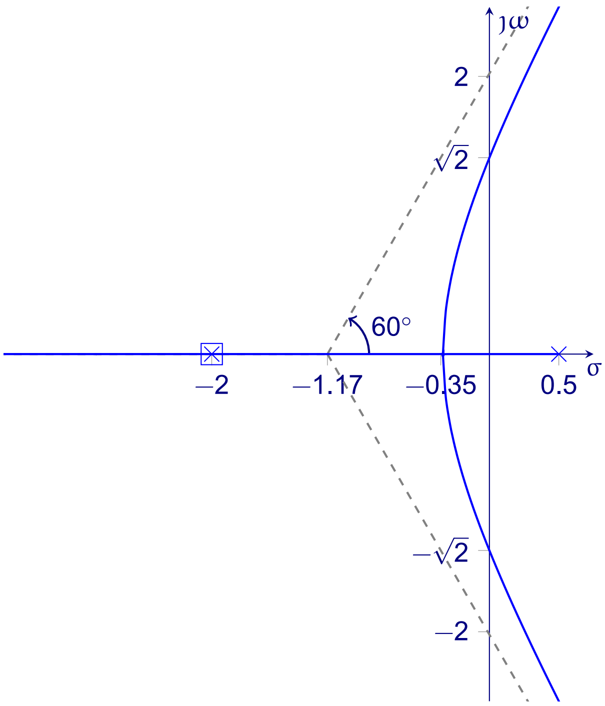
d) Sketch the root locus diagram. (You may use the fact that there is a single break-away point, located at $s \approx -0.35$.) **[8 marks]**

Answer:

The root locus is constructed as follows.

Prepare sketch	OL poles at $\boxed{s = +0.5, -2, -2}$ (repeated). No zeros. $n = 3, m = 0, \therefore$ three branches, all terminating at infinity.
Segments on real axis	$\boxed{s \in (-\infty, -2] \cup [-2, +0.5]}$.
Asymptotes	$\sigma_A = \frac{[0.5 + (-2) + (-2)] - [0]}{3}$ $= -1.1667$ $\phi_A = \frac{2k+1}{3} 180°, k = 0, 1, 2$ $= 60°, 180°, 300°$
Break-away/-in point	$s = -0.35$ (given)
Departure from complex poles	No complex poles.
Intersection with $\jmath\omega$-axis	$3.5(\jmath\omega)^2 + K - 2 = 0 \implies \begin{cases} \omega = 0, & K = 2 \\ \omega = \pm\sqrt{2}, & K = 7 \end{cases}$

The sketch is shown below.



[8 marks]

e) Explain, with reference to your root locus sketch, why using a phase-lag compensator to improve steady-state performance is problematic here. How else might steady-state performance be improved? **[3 marks]**

Answer:

Since the locus covers the entire negative real axis, then placing the pole and zero of a phase-lag compensator near the origin will significant change the locus and, hence, closed-loop transient performance and stability. Recall that the idea of phase-lag compensation is to improve steady-state performance while leaving the transient performance (primarily governed by the locus) unchanged.

A good answer will include more technical details. For example, if the lag compensator pole and zero are placed at $s = -p$ and $s = -z$, where $|z| > |p|$ and p is near the origin, then the locus lies on the real axis as

$$s \in (-\infty, -2] \cup [-2, -z] \cup [-p, +0.5]$$

The latter segment, which comprises two poles, will induce a break-away point somewhere in the interval $[-p, +0.5]$, very possibly in the positive part of that domain. The outcome will be that this part of the locus breaks away and remains in the right-hand side of the complex plane; thus, the closed-loop system is unstable.

Steady-state performance might be improved by using a lead-lag compensator: using the lead part to deliver the desired transient performance, and the lag part in improve steady-state performance. **[3 marks]**

Laplace and z-transforms

Time domain	s-domain	z-domain
$f(t)$	$F(s)$	$F(z)$
$f(t-T)$	$e^{-sT}F(s)$	$z^{-1}F(z)$
$\delta(t)$	1	–
1	$\frac{1}{s}$	$\frac{z}{z-1}$
t	$\frac{1}{s^2}$	$\frac{zT}{(z-1)^2}$
e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$
te^{-at}	$\frac{1}{(s+a)^2}$	$\frac{zTe^{-aT}}{(z-e^{-aT})^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z\sin(\omega T)}{z^2-2z\cos(\omega T)+1}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	$\frac{z^2-z\cos(\omega T)}{z^2-2z\cos(\omega T)+1}$
$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2-ze^{-aT}\cos(\omega T)}{z^2-2ze^{-aT}\cos(\omega T)+e^{-2aT}}$
$f^{(n)}(t)=\frac{d^n}{dt^n}f(t)$	$s^nF(s)-s^{n-1}f(0)-\ldots-f^{n-1}(0)$	Various forms

Compensator design formulae

Transfer function	$\frac{s\alpha\tau+1}{s\tau+1}$ (lead) $\frac{s\tau+1}{s\alpha\tau+1}$ (lag)
Maximum phase lead/lag, ϕ_m	$\sin^{-1}\frac{\alpha-1}{\alpha+1}$
Centre frequency, ω_m	$\frac{1}{\tau\sqrt{\alpha}}$

Performance criteria mappings

2% settling time, T_s	$\dfrac{4}{\zeta\omega_n}$
10–90% rise time, T_r	$\dfrac{2.16\zeta + 0.6}{\omega_n}$ for $0.3 \leqslant \zeta \leqslant 0.8$
Percentage overshoot, P.O.	$100 \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$
Peak time, T_p	$\dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ for $0 < \zeta < 1$
Peak response, M_p	$1 + \exp\left(\dfrac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$ for $0 < \zeta < 1$
Resonant frequency, ω_r	$\omega_n\sqrt{1-2\zeta^2}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Resonant peak magnitude, $M_{p\omega}$	$\dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Phase margin, ϕ_{pm}	100ζ for $0 < \zeta < \dfrac{1}{\sqrt{2}}$
Phase margin, ϕ_{pm}	$\tan^{-1}\left(\dfrac{2\zeta\omega_n}{\omega_c}\right)$
Phase margin, ϕ_{pm}	$\tan^{-1}\left(\dfrac{8}{T_s\omega_c}\right)$
Bandwidth, ω_B	$(1.85 - 1.19\zeta)\omega_n$ for $0.3 \leqslant \zeta \leqslant 0.8$

Ziegler-Nichols tuning rules

First method (*T* time constant; *L* delay time; *K* process gain)

	K_P	T_I	T_D
P	T/KL	∞	0
PI	$0.9T/KL$	$L/0.3$	0
PID	$1.2T/KL$	$2L$	$0.5L$

Second method (*K* critical gain; *P* critical period of oscillation)

	K_P	T_I	T_D
P	$0.5K$	∞	0
PI	$0.45K$	$P/1.2$	0
PID	$0.6K$	$0.5P$	$0.125P$

END OF QUESTION PAPER