EEE331/6037 exam 2010: exam questions and model solutions

1. single BJT circuits

5 points

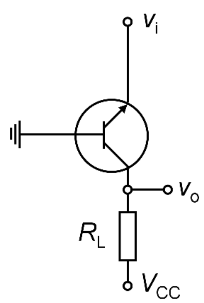
a. The fundamental relationship for the small signal collector current of a bipolar junction transistor (BJT) is given by $g_m v_{BE}=\beta i_B$. Describe what all parameters in this equation denote, interpret in words what the equation means for amplification and derive the equation, starting from the definition of g_m as a derivative.

Solution:
g_m= mutual conductance or transconductance,
v_{BE}= voltage between base and emitter,
β= small signal current gain,
i_B=base current
Interpretation: the BJT can be considered as a voltage amplifier (v_{BE} -> i_C) or a current amplifier (i_B -> i_C).
Derivation of the relationship:
$g_m = dI_C/dV_{BE} = dI_C/dI_B \times dI_B/dV_{BE} = i_C/i_B \times i_B/v_{BE} = \beta / r_{BE}$,
then multiply both sides by v_{BE} to get $g_m v_{BE}=\beta i_B$.
Alternative derivation of the relationship:
$g_m = dI_C/dV_{BE}$ by definition, where $dI_C=i_C=\beta i_B$ and $dV_{BE}=v_{BE}$

4 points

b. Draw a simple circuit diagram for a bipolar junction transistor (BJT) in common base configuration. Show input and output voltage connections. Show that for a mutual conductance g_m and load resistor R_L the voltage gain is approximately given by the product $g_m R_L$ and that the current gain is almost unity. State all approximations you need to derive this expression.

Solution:



input voltage: $V_i = V_{BE}$
output voltage: $V_o = i_C (R_L || r_{CB}) \approx i_C R_L$ (if $r_{CB} >> R_L$)
$= \beta i_B R_L$
$= g_m v_{BE} R_L$ ($g_m V_{Be} = \beta i_B$)
voltage gain: $V_o/V_i \approx g_m R_L$
current gain: $i_o/i_i = i_C/i_E = \beta i_B / [(\beta+1) i_B] = \beta/(\beta+1) \approx 1$ ($\beta >> 1$)
where V_{BE}=base-emitter voltage, β=current gain, i_B=base current, r_{XY}=resistance between X and Y terminals of the BJT
Alternative derivation of voltage gain:
$v_i = r_{BE}\, i_E = r_{BE} (\beta+1) i_b$ and

$v_o=(R_L||r_{CB})i_C=(R_L||r_{CB})\beta i_B$, therefore
$v_o/v_i=(R_L||r_{CB})\beta / [r_{BE}(\beta+1)] =(R_L||r_{CB}) / [r_{BE}+r_{BE}/\beta] \approx g_m R_L$
if $R_L \ll r_{CB}$ and $r_{BE} \ll r_{BE}/\beta = g_m^{-1}$

4 points

c. Name the type of circuit configuration shown in Figure 1 and explain in words what the functions of the resistors R_B and R_E and the capacitors C_1, C_2 and C_3 are.

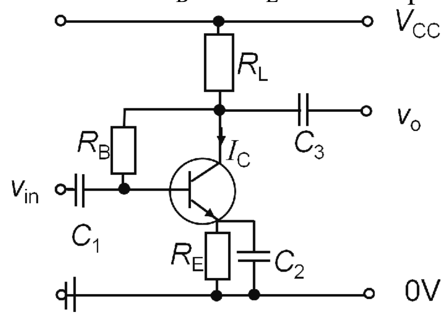


Figure 1

Solution:
This is a BJT in common emitter configuration with feedback. R_B provides negative feedback between collector and base, stabilising the collector current; R_E provides emitter degeneration, thereby enhancing the bias of emitter and base; C_1 and C_3 couple the input and output nodes to base and collector, respectively, thereby eliminating DC signals and further improving the bias; C_2 decouples the emitter to ensure that R_E has the stabilising effect only for DC but not for high-frequency AC signals.

7 points

d. Explain the Miller transform and what its physical origin is. Then apply it to the base-collector capacitance C_{CB} for a BJT in common emitter configuration, derive the transformation of C_{CB} at both input and output nodes and approximate what happens for large voltage gains $|G| \gg 1$ (i.e. $G \ll -1$ for an inverting amplifier). Which will be the frequency limiting side of the circuit, input or output, and why?

Solution:
The Miller transform explains how a feedback impedance Z_f that connects input and output nodes in a system with gain and feedback is effectively modified by the feedback. For this, input and output are considered to be separated. As the input sees a smaller voltage than the output, for given current through the bridging impedance this corresponds to an apparently reduced input impedance Z_f', while the output impedance Z_f'' is only slightly reduced. The transforms of the impedances can be derived as follows:
Derivation:

definition of voltage gain:	$G=v_o/v_i$	(i)

note: $G \ll -1$ means that v_o and v_i are opposed in sign (inversion) but $|v_o| \gg |v_i|$.

Ohm's Law for impedance:	$Z_f=(v_i-v_o)/i_f$	(ii)
apparent impedance at input:	$Z_f'=v_i/i_f'$	(iii)
apparent impedance at output:	$Z_f''=-v_o/i_f''$	(iv)
same feedback current everywhere:	$i_f=i_f'=i_f''$	(v and vi)

from (i) and (ii): $Z_f=(v_i-v_o)/i_f=v_i(1-G)/i_f$
from (iii) and (v): $Z_f'=v_i/i_f$
from (i), (iv) and (vi): $Z_f''=-Gv_i/i_f$

This yields as results: $Z_f' = Z_f/(1-G)$ and $Z_f'' = Z_f\, G/(G-1)$
Considering the impedance due to a capacitance C at $\omega=2\pi f$ is $Z=1/(j\omega C)$ it is clear that capacitances must transform exactly reciprocal to impedances, i.e.
at input: $C_{CB}' = C_{CB}(1-G) \approx -GC_{CB} = |G|\, C_{CB}$ is enlarged significantly for CE,
at output: $C_{CB}'' = C_{CB}(G-1)/G \approx C_{CB}$ is not changed significantly.

As for large voltage gain $|G| >> 1$the capacitance is enhanced at the input side and the corner frequency is proportional to $f=[2\pi\, R_{eff} C_{eff}]^{-1}$ the larger the capacitance is the lower the corner frequency, i.e. the input side limits the frequency transfer.

2. Multiple BJT circuits

5 points

a. Explain what Early effect and Early voltage are. A modified Ebers-Moll equation for a BJT that takes into account the Early effect can be written as $i_C' = I_S (1+v_{CE}/V_A) \exp(v_{BE}/V_t)$. Explain all variables in this equation and show that the Early voltage is linearly related to the new output resistance. How would common emitter output curves look like in the active region for $V_A \to \infty$?

Solution:
With increasing V_{BE} the base-collector voltage V_{CB} also increases so that the effective base width is reduced. Then more carriers can transit the base per time, because of
a) reduced recombination probability and
b) increased diffusion gradient across thee base, thereby increasing the collector current I_C.
In the output characteristic the result is a finite slope of the set of $I_C(V_{CE})$ curves that increases with V_{BE} so that all curves extend to a common point on the negative x-axis. This is the Early voltage.
For i_c=collector current, I_S=constant, v_{BE}= base-emitter voltage, $V_t=kT/q$= thermal voltage, V_A = Early voltage:
If the 'normal' collector current without Early effect is $i_C = I_S \exp(v_{BE}/V_t)$ and the one with Early effect ('modified') is $i_C' = I_S \exp(v_{BE}/V_t)\,(1+v_{CE}/V_A)$ then
$\partial i_C'/\partial v_{CE} = I_S \exp(v_{BE}/V_t)\,(1/V_A) = i_C/V_A$
The modified output resistance then is given by
$r_o' = (\partial i_C'/\partial v_{CE})^{-1} = V_A/i_C = V_A/[i'_C / (1+v_{CE}/V_A)] = (V_A+v_{CE})/i'_C$. This is linear in V_A.
Without the Early effect ($V_A \to \infty$) the $I_C(V_{CE})$ curves would be horizontally flat in the active region, hence the output resistance (i.e. inverse of slope) would be ∞.

6 points

b. Calculate for the current mirror with base current compensation shown in Figure 2 the ratio of output to input current for the general case that all three transistors have different small signal current gains of β_i, i=1,2,3. Assume the base currents to transistors T_1 and T_2 at point B are equal. Neglect the Early effect. Which transistor has the least effect on the current ratio and why? Show that for that case that all transistors are identical ($\beta_1=\beta_2=\beta_3=\beta$) and β>>1 the result approximates to $1/(1+2/\beta^2)$.

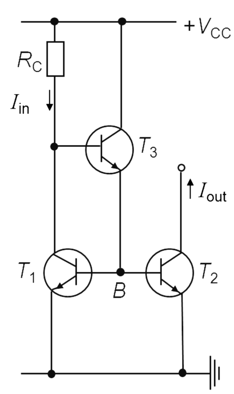


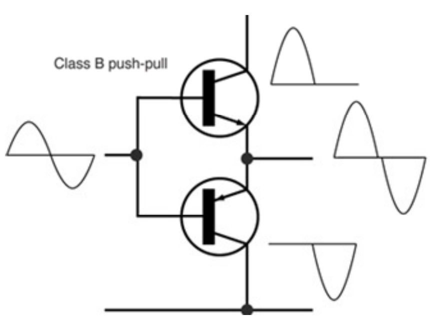
Figure 2

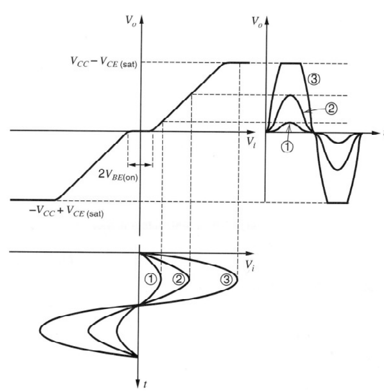
Solution:
Starting at point B, the base currents into T_1 and T_2 are I_B. Hence, $2I_B$ flows into the emitter of T_3. This yields at the base of T_3 a current of $2I_B/(\beta_3+1)$. The sum of this plus the collector current of T_1, which is $\beta_1 I_B$, must flow through R_C, so
(i) $I_{in}=\beta_1 I_B + 2I_B/(\beta_3+1)$
The output current is the collector current of T2, which is
(ii) $I_{out}=\beta_2 I_B$.
The ratio is thus
(iii) $I_{out}/I_{in}=\beta_2/[\beta_1+2/(\beta_3+1)]=(\beta_2\beta_3+\beta_2)/(\beta_1\beta_3+\beta_1+2)$
From this we can conclude:
iv) β_3 is the only parameter that shows up with a linear term in the products in both numerator and denominator; so that a change of β_3 almost cancels out. This can also be seen from the first expression in (iii). Hence, T3 is least critical.
v) This is clear as T_1 is directly connected to the input, so a change of β_1 by some % reduces I_{in} via the first term in (i) by an almost similar fraction. T2 is directly connected to the output, so a change of β_2 reduces I_{out} by a similar fraction in (ii). T3 is not directly connected to input or output; it influences the input only marginally via its small base current (second term in (i)).
vi) For $\beta_1=\beta_2=\beta_3=\beta$ we get $I_{out}/I_{in}=(\beta^2+\beta)/(\beta^2+\beta+2)=1/[1+2/(\beta(\beta+1))]\approx 1/(1+2/\beta^2)$ where the approximation is for $\beta>>1$ ($\beta\approx\beta+1$).

5 points

c. Sketch a circuit diagram of a class B output stage. Explain briefly how it works and sketch the V_{out}/V_{in} voltage characteristic of your circuit. Compare power consumption and distortion generated qualitatively to that of class A and class AB output stages.

Solution:





The principle of the class B output stage is to use a complementary pair (npn & pnp) in push-pull configuration so each BJT transfers only a half wave of the signal and is switched off otherwise. There is no bias so without input both transistors are off and do not consume any power; the power efficiency is thus much better than that of a class A stage. In the transfer characteristic severe cross-over distortion occurs in the class B stage, because for voltages below the switch-on voltage V_{BEon}, the BJTs remain off. This strongly distorts small signals, as shown by (1) in the sketch. A class AB stage operates with a bias of $2V_{BEon}$ between the BJTs, which can be provided by resistors, diodes or other transistors.

This minimises the cross-over distortion as the two slopes in the right sketch are moved towards each other, eliminating them if the bias is perfect (and temperature stable) but consuming some power even without input signals.

4 points

d. Explain the various transistor pairs and their function in the operational amplifier shown in Figure 3 below.

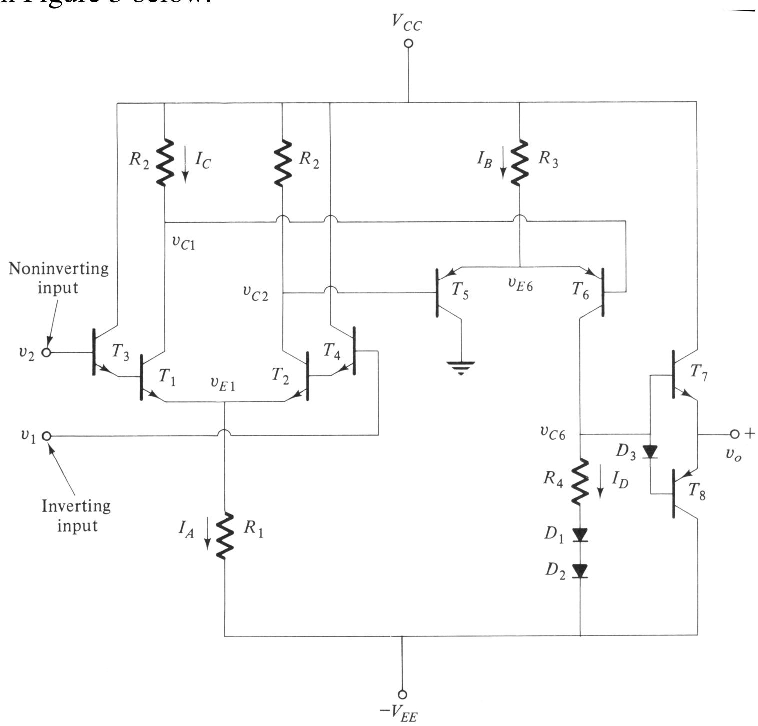


Figure 3

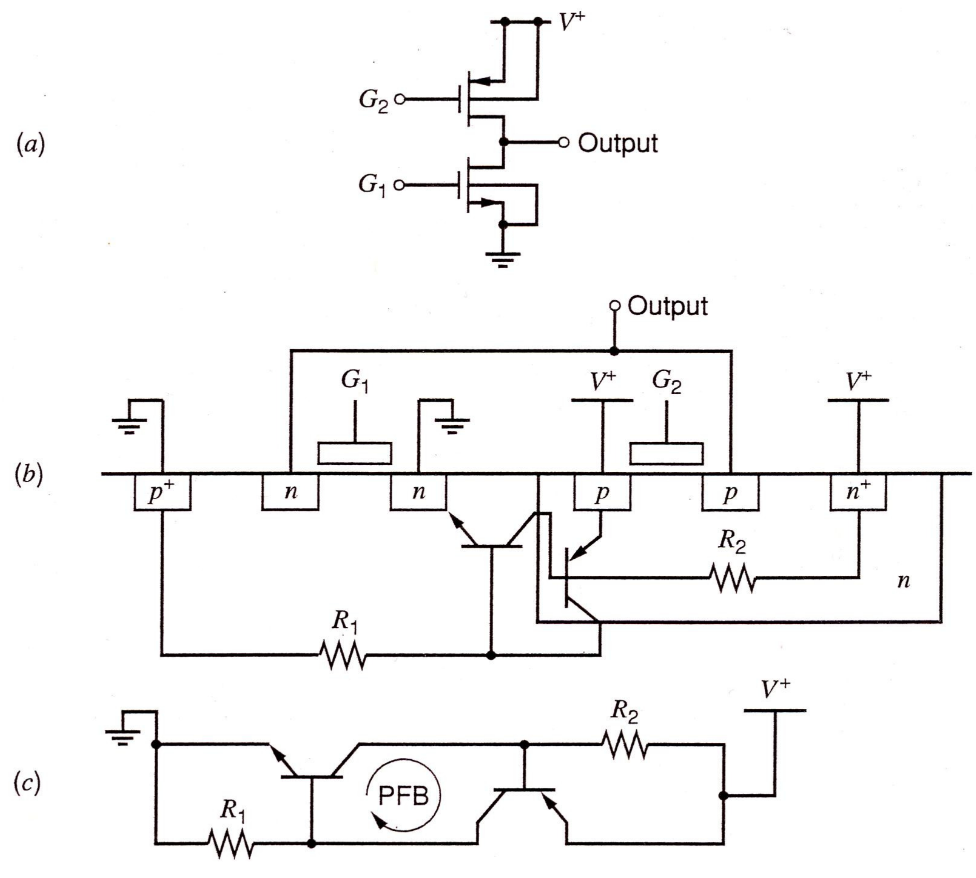
Solution:
T_1+T_3 and T_2+T_4 are Darlington pairs (emitter of T_3 feeding into the base of T_1 etc.) for initial current amplification.
The two Darlington pairs form a differential amplifier as the first stage of voltage amplification.
T_5+T_6 are pnp transistors forming a second differential pair to boost the voltage gain.
T_7+T_8 from a class AB push-pull power amplifier as output stage, where cross-over distortion is minimised by the bias provided by diode D_3.
Resistor R_4 and diodes D_1, D_2 provide level shifting prior to power amplification to suppress output voltage when there is no differential input signal.

3. MOSFETs

5 points

a. Using as an example a complementary pair of an n-channel and a p-channel metal-oxide-semiconductor field effect transistor (MOSFET) fabricated on the same p-doped substrate, sketch the phenomenon of latchup, explain what it is, why it occurs and what the consequences can be.

Solution:
To make the p-channel MOSFET an n-doped region needs to be created first into which the p-doped source and drain regions can be fabricated. This means a number of p- and n-doped regions co-exist next to each other that can interact. In particular, two parasitic BJTs will be formed at the boundary of the two MOSFETs, a lateral npn BJT (between drain of n-channel, p-substrate and n-region encapsulating the p-channel MOSFET) and a vertical pnp (between the source of the p-channel MOSFET, its n-channel surrounding and the p-substrate). These can interact and create a positive feedback where one drives the other BJT. If they enter the active region and start to conduct, the resulting high current will heat up and destroy the circuit. NB: Sketch (a) is not part of the answer.



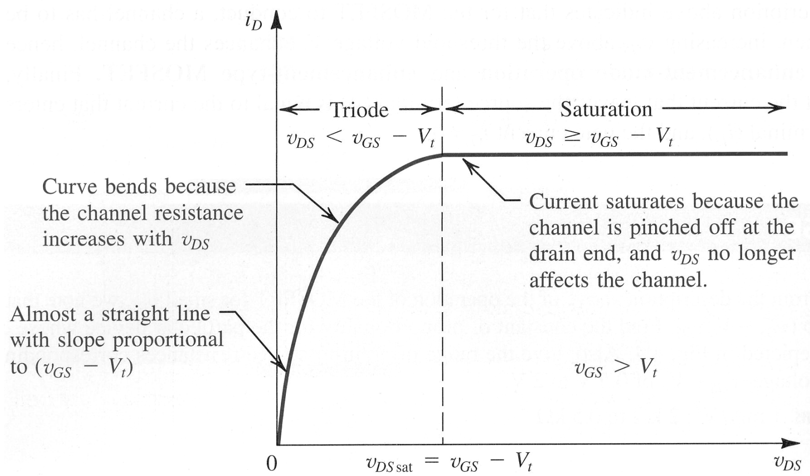
4 points

b. The output characteristic of a MOSFET in the linear region may be described by the equation $I_D = k\,[2(V_{GS}-V_{to})\,V_{DS} - V_{DS}^2]$ where I_D is the drain current, k is some material constant, V_{GS} the gate-source voltage, V_{to} the turn-on voltage and

V_{DS} the drain-source voltage. State up to which value of V_{DS} this equation is valid, above which the saturation region starts, derive the corresponding equation for the saturation region and sketch the output characteristic of both regions. Neglect the Early effect.

Solution:

The linear or triode region extends up to $V_{DS} \leq V_{GS} - V_{to}$. This voltage difference is called the over-voltage V_{ov}. For $V_D = V_{GS} - V_{to} = V_{ov}$ we get $I_D = k\,(V_{GS} - V_{to})^2$ and this will stay constant even for larger values of V_{DS}, if the Early effect is negligible. The MOSFET is then fully switched on with maximum drain current.



6 points

c. Define what is called the overdrive voltage V_{ov} of a MOSFET and what it means for the operation mode. Compare qualitatively the typical turn-on voltage of a MOSFET to the turn-on voltage of a BJT. Which is larger and why? Using the quadratic relationship between drain current I_D and overdrive voltage derive an expression for the mutual conductance g_m of a MOSFET for large signals and compare this to a BJT using the Ebers-Moll equation.

Solution:

$V_{ov} = V_{GS} - V_{to}$ decribes the transition from linear ($V_{ov}<0$) to saturation mode ($V_{ov} \geq 0$) operation. A MOSFET turns on in the region of V_{to}=2-3V, i.e. V_{GS} must be at least that large for the MOSFET to turn on. This is much larger than the V_{BEon} of a BJT, which for a Si BJT will be ~0.7V and for a Ge BJT ~0.2V. This is due to the MOSFET operating with an electrostatic voltage that has to create a high electric field at the gate to induce or deplete a conductive channel, whereas in a BJT the base-emitter voltage just influences carrier diffusion.
From $I_D = k\,(V_{GS} - V_{to})^2$ for the MOSFET we get for small signals $i_D = k\,(v_{GS} - V_{to})^2$ and by differentiation: $g_{m,MOSFET} = \partial i_D / \partial v_{GS} = k\,(v_{GS} - V_{to}) = 2 i_D / V_{ov}$
For i_D of 1-5A and V_{ov}=0.2-0.5V this yields $g_{m,MOSFET}$=10-50.
For a BJT the Ebers-Moll equation given in Question 2.a is $i_C = I_S \exp(v_{BE}/V_t)\,(1+v_{CE}/V_A)$, therefore $g_m = \partial i_C / \partial v_{BE} = i_C / V_{to}$ (independent of Early effect!) where the collector current can similarly be up to 1-5A but the voltage V_t now is the thermal voltage of the order of only kT/q=25mV. Note: one cannot use $g_m v_{BE} = \beta i_B$ because this is a small signal equation. This then yields $g_{m,\,BJT}$=40-200>$g_{m,MOSFET}$

5 points

d. Use the square law model for the drain current of a MOSFET, $I_D=k\ (V_{GS}-V_{to})^2$ with $k=½\ \mu\ C_{ox}\ W/L$ where μ is the carrier mobility, C_{ox} the specific oxide capacity per oxide area and W and L are the width and the length of the transistor channel, and consider the MOSFET as a plate capacitor to explain the modern trends towards materials with higher dielectric constants, further gate oxide thickness reduction and general miniaturisation. Consider as aims both high current and high frequency transfer.

Solution:
The MOSFET gate works like a plate capacitor whose capacitance is given by $C=\varepsilon_0\varepsilon\,A/t$ where ε_0 = electric field constant, ε = dielectric constant, A = area and t = distance between electrodes = oxide thickness. This means $C_{ox}=C/A=\varepsilon_0\varepsilon/t$ and hence $I_D \propto \mu\ \varepsilon\ W/(tL)$
Thus for large drain currents materials with high carrier mobility, high dielectric constant, thinner oxides and improved aspect ratios will be needed. In order to enable higher frequency operation (faster switching), the total capacitance C needs to be reduced as well, which only agrees with the previous aim if the area $A=W\times L$ is reduced (as ε/t is to be increased); i.e. the MOSFET will have to became much smaller in footprint. If both W and L are reduced by factors of n, then W/D does not change so I_D stays constant but C is reduced by $1/n^2$ and thus f increased by n^2. Hence both ε/t increase (C/A increase) and $W\times L$ decrease (C decrease) will be necessary.
Alternative explanation of need for smaller transistor gate lengths:
The transition frequency of the MOSFET is $f_t=g_m/(2\pi C)$ where $C=C_{GS}+C_{GD}$. As $g_m \propto i_D \propto \mu\ C_{ox}\ W/L$ and $C \propto C_{ox}\ W\cdot L$ we get $f_t \propto \mu/L^2$ which indicates the need for higher mobility materials and much shorter transistor gates.

4. Filters

6 points

a. Name the order and type of filter shown in Figure 4. Justify qualitatively its frequency behaviour. Derive its corresponding leap-frog structure. Write down the equations for all components.

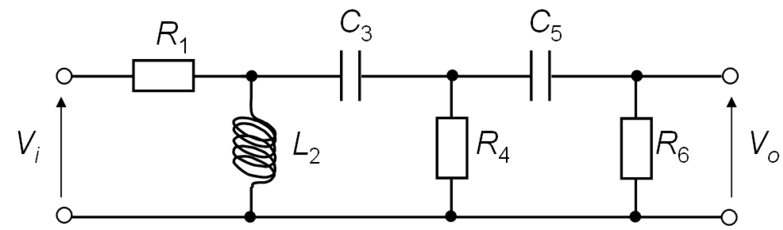
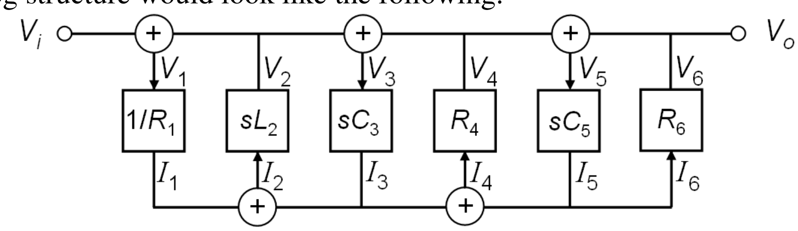


Figure 4

Solution:
The filter is a 3rd order LC high-pass filter. It acts as high-pass because C_3 effectively blocks DC signals which L_2 then shortens to ground, while there is no capacitance between the lines to shorten high-frequency signals to ground. The leap-frog structure would look like the following:



With input and output resistors there are 6 components. For each of them we can write down two equations, one from Ohm's Law and one from Kirchhoff's Law. Hence, there are 12 equations to describe the interdependence of 6 voltages $V_{1\text{-}6}$ (V_i and $s=j\omega$ are given, $V_\mathrm{o}=V_6$) and six currents $I_{1\text{-}6}$. So it is possible to calculate step by step the output voltage V_o for any given set of V_i and ω.

(i) $I_1=V_1/R_1$; $V_1=V_2-V_\mathrm{i}$
(ii) $V_2=I_2\,sL_2$; $I_2=I_3-I_1$
(iii) $I_3=V_3\,sC_3$; $V_3=V_4-V_2$
(iv) $V_4=I_4R_4$; $I_4=I_5-I_3$
(v) $I_5=V_5\,sC_5$; $V_5=V_6-V_4$
(vi) $V_6=I_6R_6$; $I_6=I_5$

8 points

b. Determine the transfer functions of the two circuits A and B shown in Figures 5A and 5B. Name and compare the type of the filter functions they describe and their order. Sketch and describe how the behaviour of both circuits differs around the frequency $\omega=(LC)^{-1/2}$.

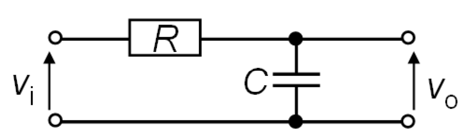


Figure 5A

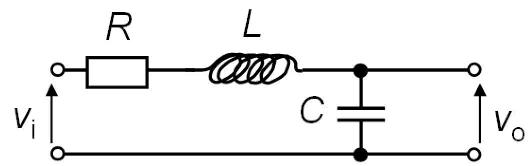
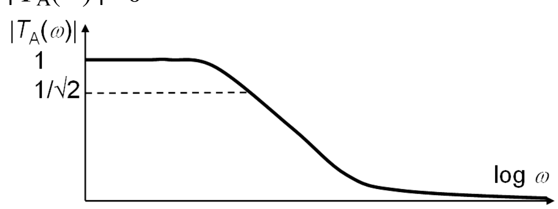
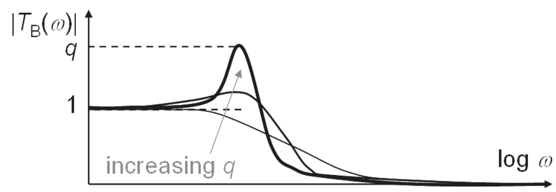


Figure 5B

Solution:
The transmission function is defined as the ratio of output to input voltage, i.e.
$T(s) = V_o(s)/V_i(s)$
With complex impedances Z and admittances Y=1/Z we get for current I:
$T(s) = Z_o(s)\ I\ /\ [Z_i(s)\ I] = Y_i(s)/\ Y_o(s)$
For circuit A:
The impedance for open input is simply: $Z_o = 1/(sC)$
The impedance for open output is: $Z_i = R+1/(sC)$
Thus we get: $T_A(s) = [1/(sC)]\ /[R+1/(sC)] = 1\ /\ (1+sCR)$
Alternative derivation:
The output admittance for short circuited input is: $Y_o = 1/R+sC$
The input admittance for short circuited output is: $Y_i = 1/R$
Thus we get: $T_A(s) = (1/R)\ /(1/R+sC) = 1\ /\ (1+sCR)$, as above
This is a classical 1st order low-pass filter as C effectively shortens high-f signals to ground. The transmission amplitude $|T_A(\omega)|$ decreases monotonically from
$|T_A(0)|=1$ over
$|T_A(1/\sqrt{(LC)})|=|1/[1+jR\sqrt{(C/L)}]|=1/[1+R^2C/L]<1$ and
$|T_A(1/(CR))|=1/\sqrt{2}$ (not asked for) to
$|T_A(\infty)|=0$



For circuit B:
The impedance for open input is: $Z_o = 1/(sC)$
The impedance for open output is: $Z_i = R+sL+1/(sC)$
Thus we get: $T_B(s) = [1/(sC)]\ /[R+sL+1/(sC)] = 1\ /\ (1+sCR+s^2LC)$
This is a 2nd order low-pass filter. C effectively shortens high-f signals to ground as above, while the additional L compensates some of the phase shift introduced by the capacitance.
The transmission amplitude $|T_B(\omega)|$ does <u>not</u> decrease monotonically from $|T_B(0)|=1$ to $|T_B(\infty)|=0$, as inserting $s=j\omega$ shows
$T_B(\omega)=1\ /\ (1-\omega^2LC+j\omega CR)$, so
$|T_B(\omega)|=1/\sqrt{[(1-\omega^2LC)^2+(\omega CR)^2]}$, which for $\omega=1/\sqrt{(LC)}$ reaches a maximum of
$|T_B(1/\sqrt{(LC)})|=1/R\ \sqrt{(L/C)}$,
which can be >>1 and describes the quality factor q at the resonance frequency.



6 points

c. Find the zeros and poles and sketch the Bode plot of the magnitude of the transfer function $T(s) = s^2/[(1+s/10^3)(1+s/10^5)(1+s/10^7)]$. Name the order and the type of the filter.

Solution:
zeros: s=0 and s=∞
poles: $s= -10^3$, $s= -10^5$ and $s= -10^7$
The Bode plot can be obtained from the multiplicative superposition of four curves for
(i) $T(s)= s^2$, which is a straight line through (1rad/s, 0db) with a steep slope of +40dB/decade.
(ii) $T(s)=1/(1+s/10^3)$ gives a line of slope −20dB/decade intersecting at $\omega=10^3$, thereby reducing the slope from $\omega=10^3$ onwards to +20dB/decade.
(iii)$T(s)=1/(1+s/10^5)$ gives a line of slope −20dB/decade intersecting at $\omega=10^5$, thereby eliminating the slope from $\omega=10^5$ onwards.
(iv)$T(s)=1/(1+s/10^7)$ gives a line of slope −20dB/decade intersecting at $\omega=10^7$, thereby yielding a negative slope of −20dB/decade from $\omega=10^7$ onwards. The gain thus approaches unity at $\omega=10^{15}$. This is a 3rd order band-pass filter.