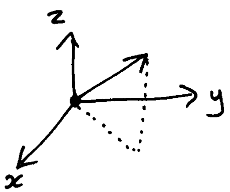
Electric Fields

Vectors

A vector has magnitude and direction.



unit vector, magnitude of...

$$\underline{A} = \hat{\underline{a}}\,|\underline{A}| = \hat{\underline{a}}\,A$$

vector (other notation bold)

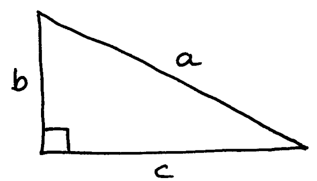
Unit vector $\hat{\underline{a}}$ has magnitude of one

$$\hat{\underline{a}} = \frac{\underline{A}}{|\underline{A}|}$$

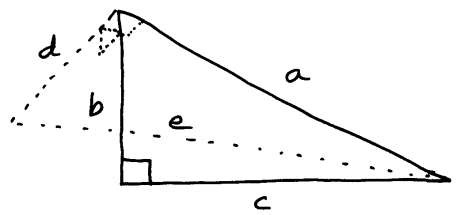
Cartesian system $\underline{A} = \hat{\underline{x}} A_x + \hat{\underline{y}} A_y + \hat{\underline{z}} A_z$

$$\hat{\underline{a}} = \frac{\underline{A}}{|\underline{A}|} = \frac{\hat{\underline{x}} A_x + \hat{\underline{y}} A_y + \hat{\underline{z}} A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Pythagoras's Theorem



$$a^2 = b^2 + c^2$$
$$a = \sqrt{b^2 + c^2}$$



$$e^2 = d^2 + a^2$$
$$= d^2 + \left(\sqrt{b^2+c^2}\right)^2$$
$$= d^2 + b^2 + c^2$$
$$e = \sqrt{b^2 + c^2 + d^2}$$

(still square root not cube root!)

Vector notation

(usually used for position vectors)

$\underline{A} = (A_x, A_y, A_z)$ e.g. $(3, 4, 10)$

$\underline{A} = \hat{\underline{x}} A_x + \hat{\underline{y}} A_y + \hat{\underline{z}} A_z$ e.g. $3\hat{\underline{x}} + 4\hat{\underline{y}} + 10\hat{\underline{z}}$

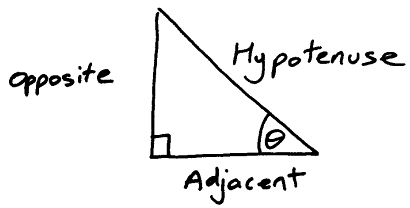
$\underline{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$ e.g. $\begin{pmatrix} 3 \\ 4 \\ 10 \end{pmatrix}$

(usually used for vector not referenced to the origin)

Vector addition

$$\underline{C} = \underline{A} + \underline{B}$$
$$= \left((A_x + B_x), (A_y + B_y), (A_z + B_z) \right)$$

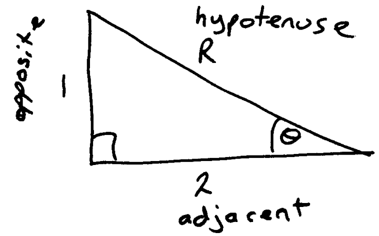
SOH CAH TOA



$$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}}$$

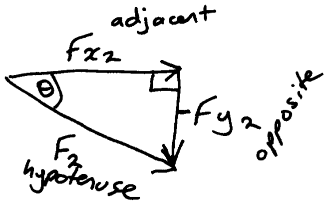
$$\cos\theta = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\tan\theta = \frac{\text{Opposite}}{\text{Adjacent}}$$



$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}} = \frac{1}{R}$$

$$\cos\theta = \frac{\text{Adj}}{\text{hyp.}} = \frac{2}{R}$$



$$\sin\theta = \frac{-F_{y2}}{F_2}$$

$$-F_{y2} = F_2 \sin\theta$$

$$\cos\theta = \frac{F_{x2}}{F_2}$$

$$F_{x2} = F_2 \cos\theta$$

Electric Fields

Coulomb's Law

(Inverse square law, like gravity, etc.)

$$\boxed{F = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2}}$$

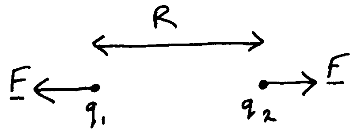
– force between 2 point charges magnitude q_1 and q_2 separation distance R

(4π to make calculations easier)

ε_0 = permittivity of free space

$= 8.854 \times 10^{-12}$ F/m – a constant

Direction of force is along the line joining q_1 & q_2.



(F is a vector)

Opposite charges attract

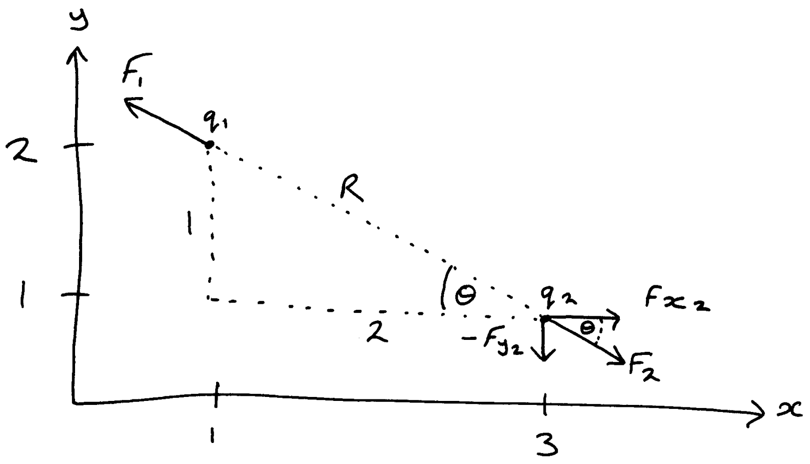
Like charges repel

Example

A charge $q_1 = 10^{-2}$ C at $(1,2,0)$ m

$q_2 = 3\times10^{-2}$ C at $(3,1,0)$ m

What is the force between them?



$$R^2 = 2^2 + 1^2 = 5\,m^2$$

So $$|F| = \frac{10^{-2} \times 3\times10^{-2}}{4\pi \times 8.854\times10^{-12}\times 5} = 5.4\times10^{5}\ N$$

(a very large force
$e = 1.6\times10^{-19}\,C$)

$|\underline{F_1}| = |\underline{F_2}|$ but direction of force is different

– describe using vectors.

From the diagram, we have...

$$F_{x_2} = F_2 \cos\theta = F_2 \times \frac{2}{\sqrt5} = 4.82\times10^5\ N$$

$$-F_{y_2} = F_2 \sin\theta = F_2 \times \frac{1}{\sqrt5} = 2.41\times10^5\ N$$

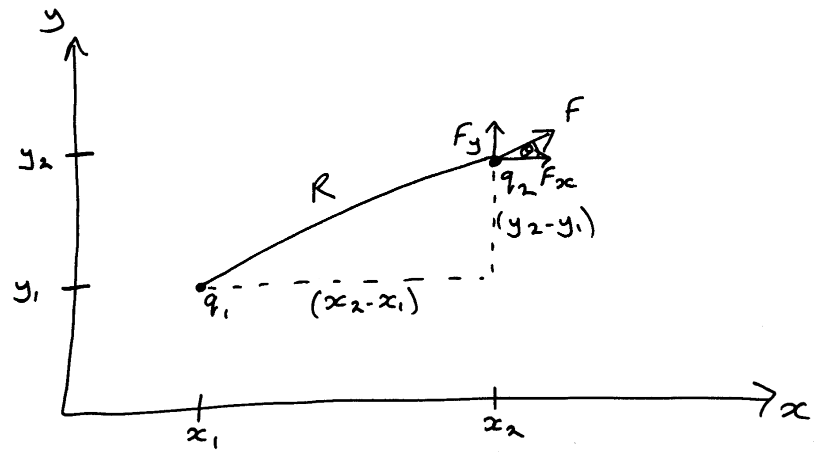
So we can write $\underline{F_2}$ in vector form as

$$\underline{F_2} = (4.82\times10^5,\ -2.41\times10^5,\ 0)$$

No z component

Note $\underline{F_1} = -\underline{F_2} = (-4.82\times10^5,\ 2.41\times10^5,\ 0)$

Now look at general case



We have $$|F| = \frac{q_1 q_2}{4\pi\varepsilon_0 R^2}$$

here $$R^2 = (x_2-x_1)^2 + (y_2-y_1)^2$$

and $$F_x = |F|\cos\theta = |F|\times\left(\frac{x_2-x_1}{R}\right)$$

$$F_y = |F|\sin\theta = |F|\times\left(\frac{y_2-y_1}{R}\right)$$

If we extend to 3-D, then

$$F_z = |F|\times\left(\frac{z_2-z_1}{R}\right)$$

Can therefore write

$$\underline{F} = (F_x, F_y, F_z)$$

$$= \frac{q_1 \cdot q_2}{4\pi \varepsilon_0 R^3} (x_2 - x_1, y_2 - y_1, z_2 - z_1)$$

We note that $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ is also a vector given by

$$\underline{R} = \hat{\underline{x}}(x_2 - x_1) + \hat{\underline{y}}(y_2 - y_1) + \hat{\underline{z}}(z_2 - z_1)$$

unit vectors - can be $\underline{i}, \underline{j}, \underline{k}$

$$\boxed{\underline{F} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^3} \underline{R}}$$

- vector form of Coulomb's Law

(R^3 NOT because of 3-D)

Can also be written as

$$\underline{F} = \underbrace{\frac{q_1 q_2}{4\pi \varepsilon_0 R^2}}_{|\underline{F}|} \cdot \underbrace{\frac{\underline{R}}{R}}_{\text{unit vector } \hat{\underline{R}}}$$

$$\underline{F} = |\underline{F}| \hat{\underline{R}}$$

More generally...

If q_1 is at a location described by the position vector

$$\underline{R}_1 = (x_1, y_1, z_1)$$

and q_2 by $\underline{R}_2 = (x_2, y_2, z_2)$

Force on q_2 due to q_1 is

$$\underline{F}_{2,1} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2} \hat{\underline{R}}_{2,1} \quad \text{where} \quad \hat{\underline{R}}_{2,1} = \frac{\underline{R}_2 - \underline{R}_1}{R}$$

Force on q_1 due to q_2 is

$$\underline{F}_{1,2} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2} \hat{\underline{R}}_{1,2} \quad \text{where} \quad \hat{\underline{R}}_{1,2} = \frac{\underline{R}_1 - \underline{R}_2}{R}$$

Taking the example from earlier

$q_1 = 10^{-2}$ C at $(1, 2, 0)$ m

$q_2 = 3 \times 10^{-2}$ C at $(3, 1, 0)$ m

So...

$$\underline{R}_1 = (1, 2, 0)$$

$$\underline{R}_2 = (3, 1, 0)$$

We need

$\underline{R}_2 - \underline{R}_1$, $\underline{R}_1 - \underline{R}_2$, R

$$\underline{R}_2 - \underline{R}_1 = (3-1,\ 1-2,\ 0-0) = (2, -1, 0)$$

$$\underline{R}_1 - \underline{R}_2 = (-2, 1, 0)$$

$$R = \sqrt{(3-1)^2 + (1-2)^2 + (0-0)^2} = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{5}$$

$$\underline{F}_{2,1} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^2} \cdot \frac{\underline{R}_{2,1}}{R}$$

$$= \frac{10^{-2} \times 3 \times 10^{-2}}{4\pi \times 8.854 \times 10^{-12} \times 5} \cdot \frac{(2,-1,0)}{\sqrt{5}}$$

$$= 5.39 \times 10^5 \cdot \frac{(2,-1,0)}{\sqrt{5}}$$

$$= (4.82 \times 10^5,\ -2.41 \times 10^5,\ 0)$$

same as before!

Note: we have assumed that q_1 and q_2 have the same sign ... ie. repel

In practice it is always easier to work out the direction of force using a diagram and the fact that like charges repel and opposite charges attract.

Electric field

- Introduce concept of electric field because it makes problem solving easier
- Also used when dealing with time-varying fields.

From Coulomb's Law...

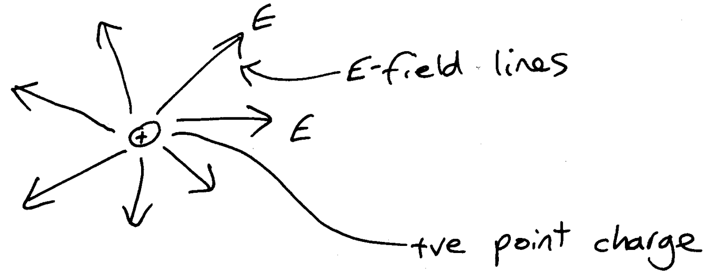
$$\underline{F} = \frac{q_1 q_2}{4\pi \varepsilon_0 R^3} \underline{R}$$

Define electric field due to q_1 as

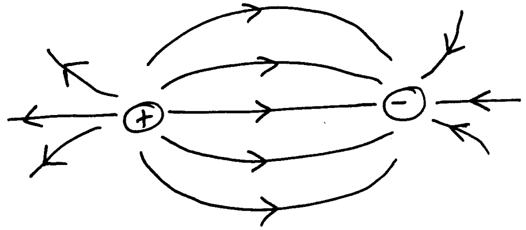
$$\underline{E}_1 = \frac{q_1}{4\pi \varepsilon_0 R^3} \underline{R} \qquad Vm^{-1}$$

so that $\underline{F} = q_2 \underline{E}_1$ [c.f. $F = mg$]

Electric field (or E-field) is a vector and can be described pictorially using Field Lines



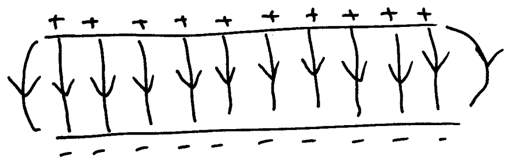
Electric Dipole



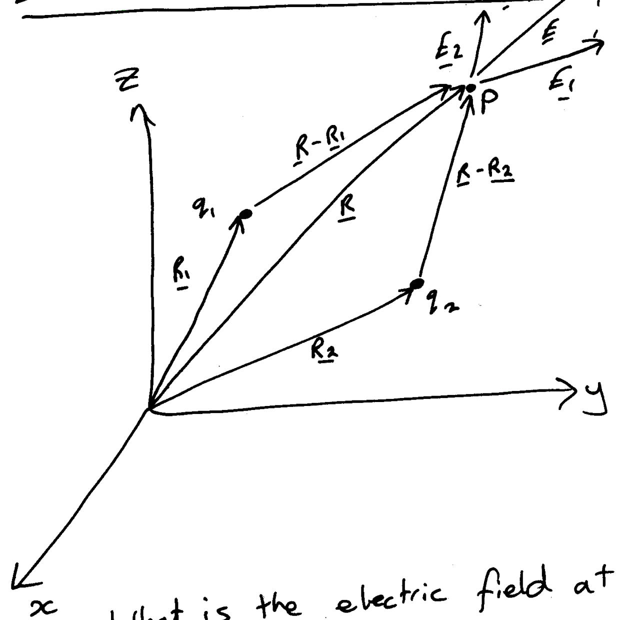
- electric equivalent of a magnetic dipole

Field lines start on +ve charges and end on negative charges

Capacitor



Electric field due to multiple point charges [1.4]



check $\underline{R}_2 + (\underline{R} - \underline{R}_2) = \underline{R}$ ✓

What is the electric field at point P due to the 2 point charges q_1 and q_2.

- Use superposition $\Rightarrow$ do a vector sum

$$\underline{E} = \underline{E}_1 + \underline{E}_2$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1(\underline{R}-\underline{R}_1)}{|\underline{R}-\underline{R}_1|^3} + \frac{q_2(\underline{R}-\underline{R}_2)}{|\underline{R}-\underline{R}_2|^3}\right] \quad Vm^{-1}$$

In the general case, with N point sources, we can write: -

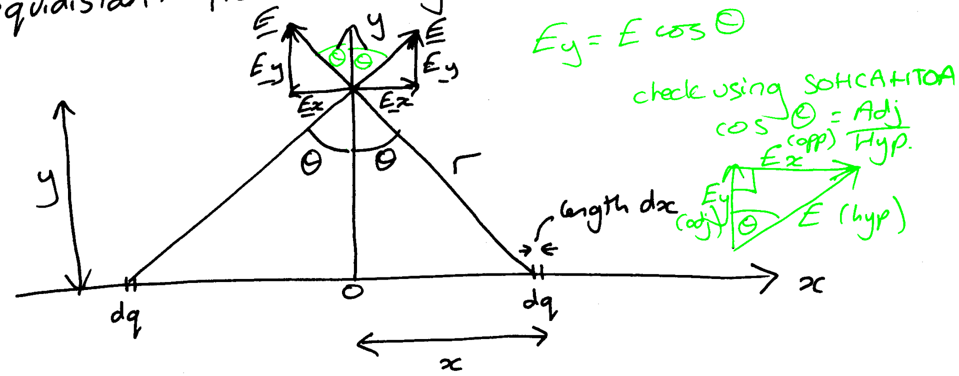
$$\underline{E} = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i(\underline{R}-\underline{R}_i)}{|\underline{R}-\underline{R}_i|^3} \quad Vm^{-1}$$

Electric field due to an infinitely long charged wire

1) Assume wire consists of small point charges and use superposition.

2) To simplify problem put wire along x-axis

3) Consider field due to 2 charges that are equidistant from the origin.



E_x components cancel each other out.

- As the wire is infinitely long, there are as many charges to the right as to the left so E_x is zero everywhere

4) Due to symmetry we can turn the problem into two semi-infinite ones.

5) Let $dq = q_\ell dx$ where q_ℓ = charge per unit length

Now, treating dq as a point source and using the equation for Electric field :-

$$\boxed{\underline{E} = \frac{q}{4\pi\varepsilon_0 R^3}\underline{R}}$$

$|\underline{E}_y| = |\underline{E}|\cos\theta$

remove vector part but only unit vector.

[unit vector: $\frac{\underline{R}}{R}$]

$$|\underline{E}| = \frac{q}{4\pi\varepsilon_0 R^2}$$

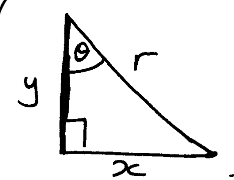
$$|\underline{E}_y| = \frac{q}{4\pi\varepsilon_0 R^2}\cos\theta$$

$$E_y = \frac{q_\ell dx}{4\pi\varepsilon_0 r^2}\cos\theta$$

Integrate between zero and infinity to get E_y

$$E_y = \int_0^\infty \frac{q_\ell \, dx \cos\theta}{4\pi\varepsilon_0 r^2}$$

$$= \frac{q_\ell}{4\pi\varepsilon_0} \int_0^\infty \frac{\cos\theta}{r^2} \, dx$$



But $\cos\theta = \frac{y}{r}$

$$E_y = \frac{y q_\ell}{4\pi\varepsilon_0} \int_0^\infty \frac{1}{r^3} \, dx$$

and $r^2 = x^2 + y^2$ so...

$$E_y = \frac{y q_\ell}{4\pi\varepsilon_0} \int_0^\infty \frac{1}{(x^2+y^2)^{3/2}} \, dx$$

[Use a table of standard integrals to solve]

$$\left[\int \frac{dx}{(ax^2+bx+c)^{3/2}} = \frac{2(2ax+b)}{(4ac-b^2)\sqrt{ax^2+bx+c}} \right]$$

$[a=1, b=0, c=y^2]$

$$= \frac{4x}{4y^2\sqrt{x^2+y^2}}$$

$$E_y = \frac{q_\ell y}{4\pi\varepsilon_0} \left[\frac{1}{y^2} \cdot \frac{x}{\sqrt{x^2+y^2}} \right]_0^\infty$$

$$= \frac{q_\ell}{4\pi\varepsilon_0 y} \left[\underbrace{\frac{\infty}{\sqrt{\infty^2+y^2}}}_{1 \left(\frac{\infty}{\infty}\right)} - \underbrace{\frac{0}{\sqrt{0^2+y^2}}}_{0} \right]$$

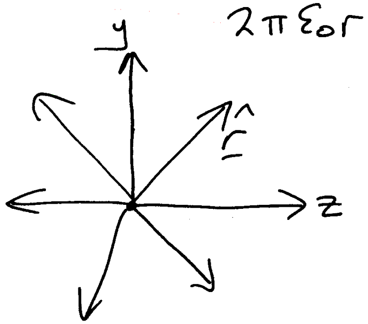
$$E_y = \frac{q_\ell}{4\pi\varepsilon_0 y}$$

This is for a semi-infinite wire.
For full wire we have to double the field.

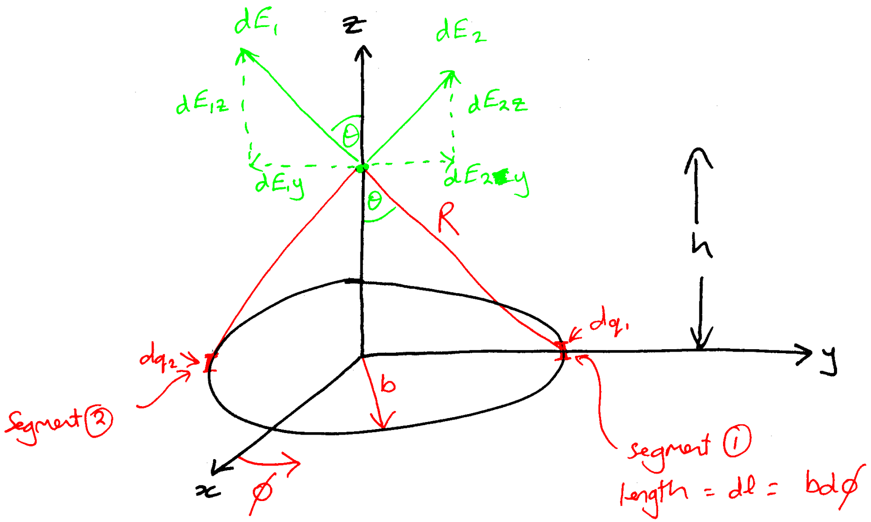
$$\boxed{E = \frac{q_\ell}{2\pi\varepsilon_0 y}}$$

We only solved problem in one plane, but again by symmetry we can write

$$\underline{E} = \frac{q_\ell}{2\pi\varepsilon_0 r}\,\hat{\underline{r}}$$



Electric Field due to a ring of uniform charge



Using the equation for the electric field due to a point charge, which we derived from Coulomb's law, we can find the field on z-axis due to a small segment of charge dq.

Due to symmetry, dE_{1y} will cancel with dE_{2y}

→ Same for any component of E in the x-y plane.

→ Resulting field is along the z-axis

Using Coulomb's Law,

$$dE = \frac{dq}{4\pi\varepsilon_0 R^2} \quad \text{and} \quad dE_z = \frac{dq}{4\pi\varepsilon_0 R^2}\cos\theta$$

Now $\cos\theta = \frac{h}{R}$ and $R^2 = h^2 + b^2$

$$\text{so} \quad dE_z = \frac{dq}{4\pi\varepsilon_0 (h^2+b^2)} \cdot \frac{h}{\sqrt{h^2+b^2}}$$

$$= \frac{h\,dq}{4\pi\varepsilon_0 (h^2+b^2)^{3/2}}$$

Now let q_ℓ = charge / unit length ...

$$dq = q_\ell\, dl$$

but $dl = b\,d\phi$

so $dq = q_\ell\, b\, d\phi$

Hence...

$$dE_z = \frac{h\, q_\ell\, b\, d\phi}{4\pi\varepsilon_0 (h^2+b^2)^{3/2}}$$

To get the total field due to the ring, we integrate from $0 \rightarrow 2\pi$ $(0-360°)$

$$E_z = \frac{q_e b h}{4\pi\varepsilon_0 (h^2+b^2)^{3/2}} \int_0^{2\pi} d\phi$$

$$= \frac{q_e b h}{2\varepsilon_0 (b^2+h^2)^{3/2}}$$

or $$E_z = \frac{h\,Q}{4\pi\varepsilon_0 (b^2+h^2)^{3/2}}$$

where $Q = 2\pi b q_e$ = total charge on ring.

Note: this solution is only valid for points <u>ON</u> the axis of the loop [z-axis in this case]

- Two special cases of this result

1) At the centre of the ring $(h=0)$, the E-field is zero

2) At very large distances away from the ring $(h \gg b)$, the ring looks like a point charge.

ie. $$E = \frac{h\,Q}{4\pi\varepsilon_0 (b^2+h^2)^{3/2}}$$

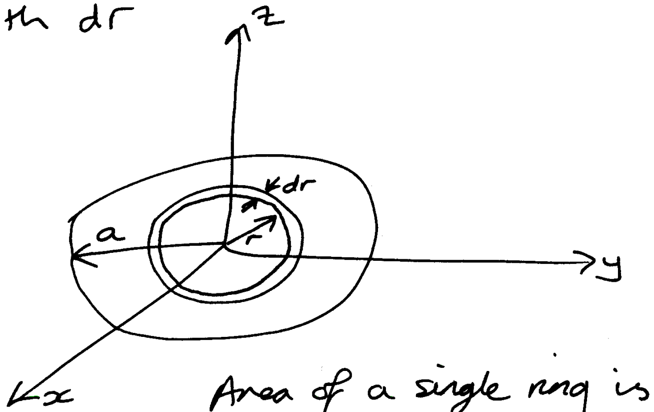
Let $h \gg b$ so $$E \approx \frac{hQ}{4\pi\varepsilon_0 (h^2)^{3/2}} \quad (b \approx 0)$$

$$\approx \frac{hQ}{4\pi\varepsilon_0 h^3}$$

$$\approx \frac{Q}{4\pi\varepsilon_0 h^2}$$

Electric field due to a disk of charge

Let disk consist of concentric rings of radius r and width dr



Area of a single ring is...

$$dS = 2\pi r\,dr$$

Let q_s = charge per unit area

Then total charge on one ring is...

$$Q = q_s dS = q_s 2\pi r\,dr$$

Using this result in the expression for the field from a ring of charge gives

$$dE_z = \frac{h}{4\pi\varepsilon_0 (r^2+h^2)^{3/2}} \cdot \underbrace{(2\pi q_s r\,dr)}_{Q}$$

To find field due to entire disk, we integrate from $0 \rightarrow a$, where a = radius of disk.

$$E_z = \frac{q_s h}{2\varepsilon_0} \int_0^a \frac{r\,dr}{(r^2+h^2)^{3/2}}$$

Standard integral

$$\int \frac{x\,dx}{(ax^2+bx+c)^{3/2}} = \frac{2(bx+2c)}{(b^2-4ac)\sqrt{ax^2+bx+c}}$$

(with $a=1$, $b=0$, $c=h^2$)

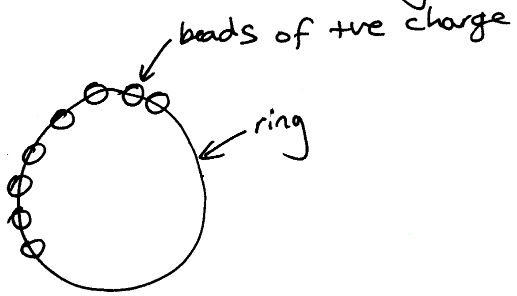
$$E_z = \frac{q_s}{2\varepsilon_0}\left[1 - \frac{h}{\sqrt{a^2+h^2}}\right]$$

To find the electric field due to an infinite sheet of charge we let $a \rightarrow \infty$ to give...

$$\boxed{\underline{E} = \frac{\hat{z}\, q_s}{2\varepsilon_0}}$$

A thought experiment

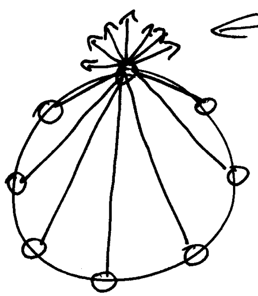
Consider a ring of charge beads which are free to move around a ring.



How do the charges arrange themselves and why?

→ All the charges are of the same sign. Therefore, they repel each other.

Charges will move to a position of equilibrium where they experience no net azimuthal force → a symmetrical distribution

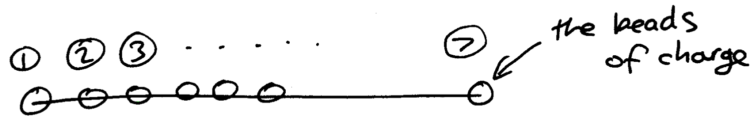


net force is radial (in $\hat{r}$ direction)

no force in azimuth (in $\hat{\theta}$ direction)

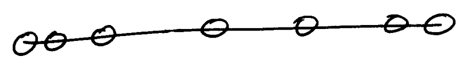
– a uniform charge distribution

Now consider a short line of charge



How do the charges arrange themselves?

→ charges repel
  → will have a bead at each end of the line
  → distribution will have symmetry about centre of line
  → charge in position 2 experiences just one charge pushing it to the right, but has several charges pushing it to the left.
  ∴ charge 2 will move closer to charge 1 $\left[\text{as } F \propto \frac{1}{r^2}\right]$ to balance this force.
  → end up with a non-uniform distribution with charge accumulating at the ends

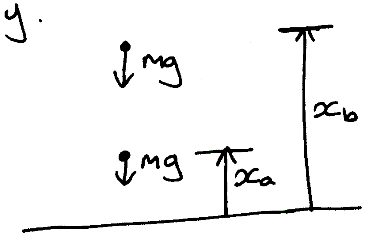


Electric Potential

In circuits the voltage between points represents the amount of work required to move a unit charge between those 2 points.

Voltage is short for voltage potential (sometimes called potential difference, p.d.) and is the same as electric potential.

First, consider potential energy (P.E.) due to gravity.



g = gravitation field strength

By definition,

$$\boxed{\text{Work done} = \text{force} \times \text{Distance moved in the direction of the force}}$$

or $\quad W = \int F(x)\,dx$

Assuming g is independent of x (true over short distances) work done (by gravity) in moving a mass, m, from x_a to x_b is

$$W = \int_{x_a}^{x_b} -mg \; dx$$

$-$ve as force in opposite direction to x.

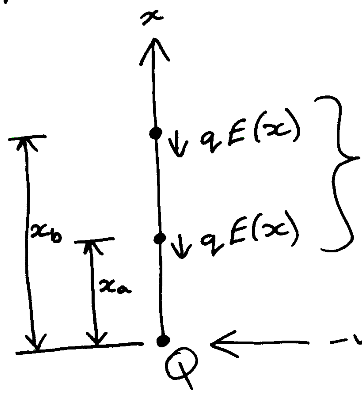
$$W = -mg\left[x\right]_{x_a}^{x_b}$$

$$W = mg\left[x_a - x_b\right]$$

The increase in potential energy (PE) is $-W$

$$= mg\left[x_b - x_a\right]$$

Now consider an electric field



+ve charges

$-$ve point source

$$|\underline{E}| = \frac{Q}{4\pi\varepsilon_0 x^2}$$

Work <u>done</u> in moving a charge, q, from x_a to x_b is...

$$W = \int_{x_a}^{x_b} -qE(x)\,dx$$

$$= \frac{-qQ}{4\pi\varepsilon_0}\int_{x_a}^{x_b}\frac{dx}{x^2} \quad = \quad \frac{+qQ}{4\pi\varepsilon_0}\left[\frac{1}{x}\right]_{x_a}^{x_b}$$

$$= \frac{qQ}{4\pi\varepsilon_0}\left[\frac{1}{x_b} - \frac{1}{x_a}\right]$$

The increase in PE is $-W$

$$= \frac{qQ}{4\pi\varepsilon_0}\left[\frac{1}{x_a} - \frac{1}{x_b}\right]$$

We define the electric potential as

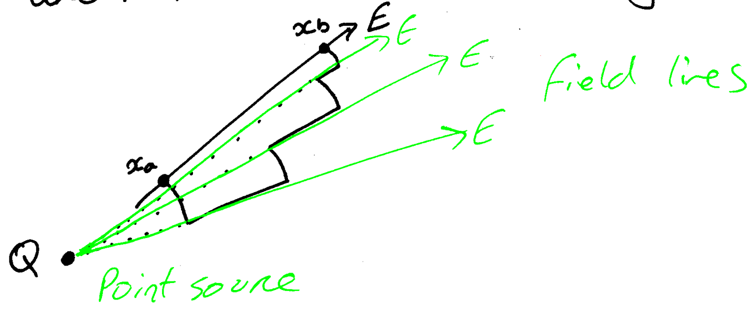
$$\boxed{\phi(x) = \frac{Q}{4\pi\varepsilon_0 x}}$$

So that the difference in electric PE between points a and b is

$$\phi(x_a) - \phi(x_b) = \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{x_a} - \frac{1}{x_b}\right]$$

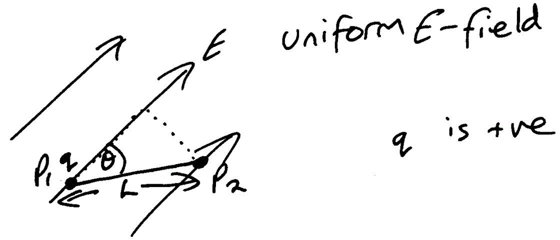
(remember PD in a circuit is the amount of work to move a unit charge between a and b)

The above example is for the special case of a point source which we moved *along* the field line. Now examine a more general case.



move charge from x_a to x_b along blue line.

- no work done when moving perpendicular to the field lines (no increase in potential)
- only do work when moving in the direction of the force.



Work done in moving from P_1 to P_2 is $qE\underbrace{\cos\theta\, L}_{\text{component of path along E-field.}}$

Potential difference is $\phi_2 - \phi_1$

$$= -E\cos\theta\, L$$

If the path from P_1 to P_2 is not a straight line, we can break it up into small straight line vectors of length dl and integrate to give...

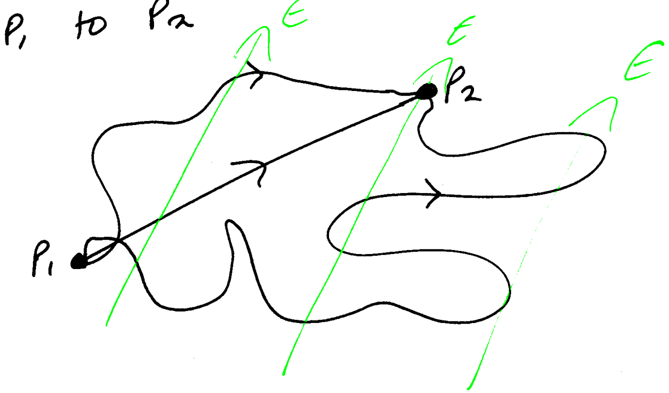
$$\phi_2 - \phi_1 = -\int_{P_1}^{P_2} E \cos\theta \, dl$$

or in terms of vector notation as...

$$\phi_2 - \phi_1 = -\int_{P_1}^{P_2} \underline{E} \cdot \underline{dl}$$

↖ dot product

It does not matter what path we take from P_1 to P_2



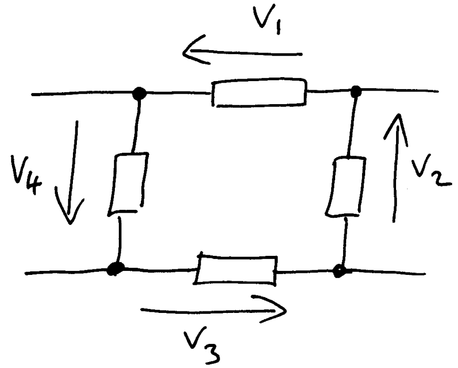
Also note that if we go from P_1 to P_2 and then back to P_1, there is no increase in potential

i.e.

$$\oint_C \underline{E} \cdot \underline{dl} = 0$$

→ Example of a conservative field (energy conserved)

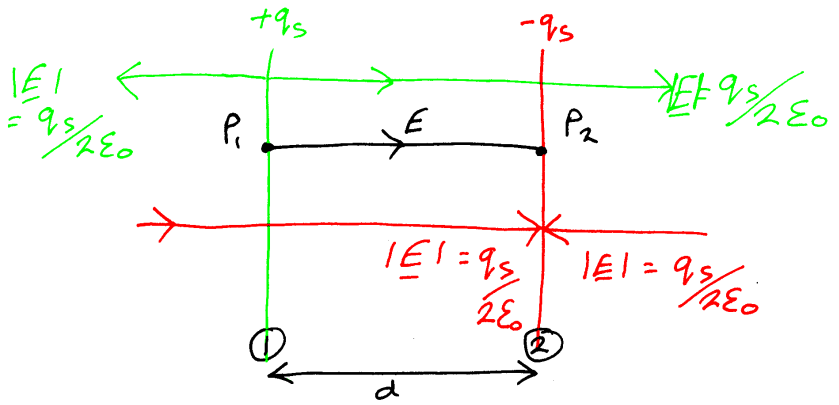
The circuit analogy to this is Kirchoff's Law which states that net voltage drop around a loop is zero.



$$V_1 + V_2 + V_3 + V_4 = 0$$

Example

Two infinite parallel sheets of charge, with charges $+q_s$ and $-q_s$ (per m^2) are a distance d apart.



Fields outside the sheets cancel each other out to become zero.

Fields between sheets is

$$|E| = \frac{q_s}{2\varepsilon_0} + \frac{q_s}{2\varepsilon_0} = \frac{q_s}{\varepsilon_0}$$

To calculate the potential between the 2 sheets we use ...

$$\phi_2 - \phi_1 = -\int_{P_1}^{P_2} \underline{E} \cdot \underline{dx}$$

(choose a straight line between P_1 and P_2 to make the maths easier)

$$\phi_2 - \phi_1 = -\left[E \cdot x\right]_0^d = -E \cdot d = \frac{q_s d}{\varepsilon_0}$$

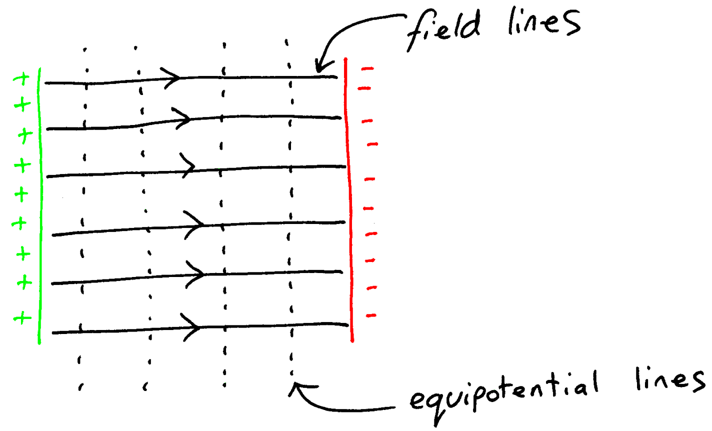
i.e. voltage on plate 2 is $\frac{q_s d}{\varepsilon_0}$ volts lower than on plate 1

If both sheets are of area A and assuming we can use the same expression for field as an infinite sheet

$$q_s = Q/A$$

Q is total charge on plate

and $$V = -\frac{Qd}{A\varepsilon_0}$$



As potential difference is a relative measure, we could say that the −ve plate is at 0 volts and the +ve plate is at $\frac{Qd}{A\varepsilon_0}$ volts

Potential due to several point Charges

- Potential at $\underline{R}$ due to a point charge at $\underline{R}_1$ is

$$\phi(\underline{R}) = \frac{q}{4\pi\varepsilon_0 \underbrace{|\underline{R}-\underline{R}_1|}_{\text{just the distance between } \underline{R} \text{ and } \underline{R}_1}}$$

- If we have N point charges at locations $\underline{R}_i$ $(i = 1, N)$ then total potential at $\underline{R}$ is...

$$\phi(\underline{R}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{|\underline{R}-\underline{R}_i|} \quad V$$

* Note that this is a scalar addition, not a vector addition (as with fields)

[This makes the summation to find ϕ easier than that to find E.]

Electric Potential due to continuous distributions

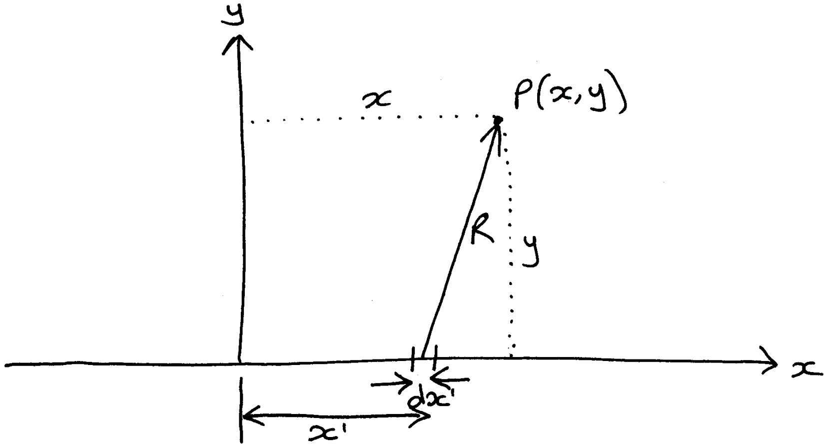
If we have a continuous charge distribution, we replace the above summation by an integration...

$$\phi(R) = \frac{1}{4\pi\varepsilon_0} \int_{l'} \frac{q_l}{R}\, dl' \quad \text{- Line distribution}$$

$$\phi(R) = \frac{1}{4\pi\varepsilon_0} \iint_{S'} \frac{q_s}{R}\, ds' \quad \text{- Surface distribution}$$

$$\phi(R) = \frac{1}{4\pi\varepsilon_0} \iiint_{V'} \frac{q_v}{R}\, dv' \quad \text{- Volume distribution}$$

Electric Potential due to an infinitely long straight wire



Potential at $P(x,y)$ due to a small amount of charge $q_\ell dx'$ is

$$d\phi = \frac{q_\ell dx'}{4\pi \varepsilon_0 R}$$

where $R = \sqrt{(x'-x)^2 + y^2}$

$\therefore$ for whole wire

$$\phi = \frac{q_\ell}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{(x'-x)^2 + y^2}}$$

- This is a very nasty integral to evaluate with limits $-\infty \to \infty$.
- Eventually gives...

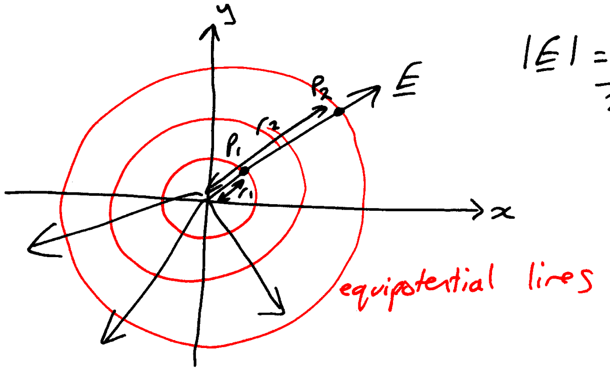
$$\phi = \frac{q_\ell}{2\pi\varepsilon_0} \log_e y$$

- Can we find an easier way to calculate ϕ?
- Remember that the electric field of an infinite line source is: -

$$\underline{E} = \frac{q_\ell}{2\pi\varepsilon_0 r} \cdot \hat{r}$$

And we know that

$$\phi_2 - \phi_1 = -\int_{P_1}^{P_2} \underline{E} \cdot d\underline{l}$$



$$|E| = \frac{q_\ell}{2\pi\varepsilon_0 r}$$

$$\phi_2 - \phi_1 = -\int_{r_1}^{r_2} \frac{q_\ell}{2\pi\varepsilon_0 r}\,dr$$

$$= \frac{q_\ell}{2\pi\varepsilon_0}\Big[\ln(r_2) - \ln(r_1)\Big]$$

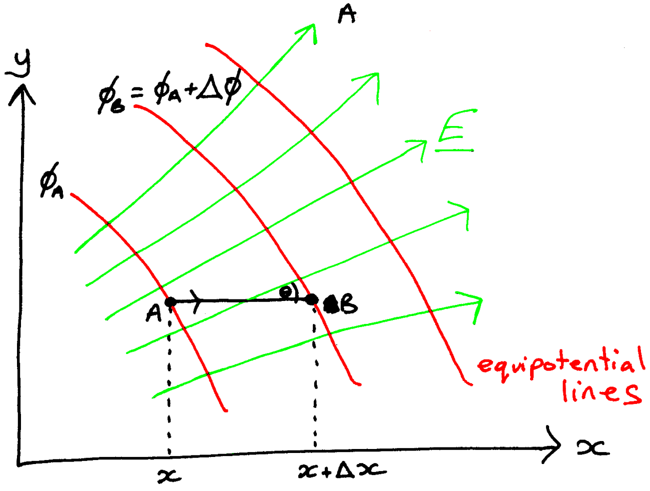
$$= \frac{q_\ell}{2\pi\varepsilon_0}\ln\left(r_1/r_2\right)$$

Note: P_1 and P_2 do not need to be on the same field line.

Can we find $\underline{E}$ from ϕ?

We know that

$$\phi_B - \phi_A = -\int_A^B \underline{E}\cdot d\underline{\ell}$$



- Consider 2 points $A\,(x, y, z)$ and $B\,(x+\Delta x, y, z)$
- If we take a straight line path from A to B, then...

$$\underline{E}\cdot d\underline{\ell} = E\cos\theta\,dx = E_x\,dx$$

so
$$\phi_B - \phi_A = -\int_A^B E_x\,dx$$

$$= -\int_x^{x+\Delta x} E_x\,dx$$

If Δx is very small, we can assume that E_x is constant between A and B.

$$\therefore \quad \phi_B - \phi_A = -E_x \int_x^{x+\Delta x} dx$$

$$= -E_x \Delta x$$

Now let change in potential from A to B be

$$\Delta\phi = \phi_B - \phi_A$$

$$\Delta\phi = -E_x \Delta x$$

$$\text{or} \quad E_x = -\frac{\Delta\phi}{\Delta x}$$

As $\Delta x \to 0$, we write

$$E_x = -\frac{\partial\phi}{\partial x}$$

If we do a similar analysis for the field in the y and z directions, we get

$$E_y = -\frac{\partial\phi}{\partial y}$$

$$E_z = -\frac{\partial\phi}{\partial z}$$

$$\text{or} \quad \underline{E} = \left(-\frac{\partial\phi}{\partial x}, -\frac{\partial\phi}{\partial y}, -\frac{\partial\phi}{\partial z}\right)$$

$$\underline{E} = -\nabla\phi$$

grad

<u>Example</u>

if $\phi = x^2 + y^2 z$

what is the E-field at $(1,-1,2)$

$$\underline{E} = -\nabla\phi = -\left(\hat{\underline{x}}\frac{\partial}{\partial x} + \hat{\underline{y}}\frac{\partial}{\partial y} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)(x^2+y^2z)$$

$$= -\left(\hat{\underline{x}}\,2x + \hat{\underline{y}}\,2yz + \hat{\underline{z}}\,y^2\right)$$

at $(1,-1,2)$ ...

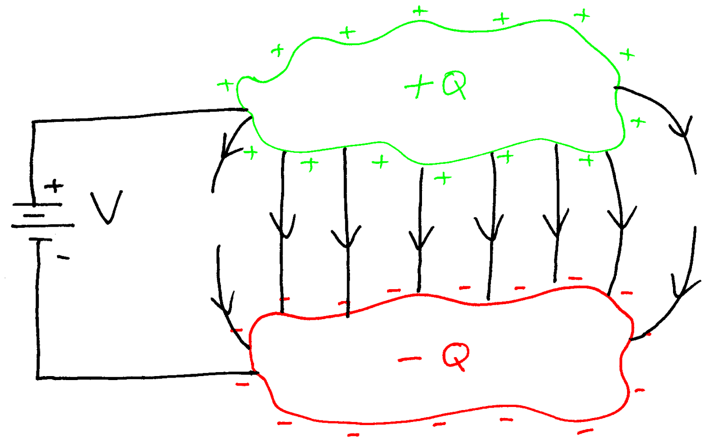
$$\underline{E} = -\left(2\hat{\underline{x}} - 4\hat{\underline{y}} + \hat{\underline{z}}\right)$$

$$\text{or} \quad (-2, 4, -1)$$

Capacitance

Two conducting bodies separated by an insulating material form a capacitor.

If we connect a d-c source between the two conductors, a +ve charge $+Q$ will accumulate on one conductor, and a -ve charge $-Q$ on the other.



Capacitance is defined as $C = \frac{Q}{V}$

and the units are Farads (F)

or Coulombs per Volt.

Points to note:

- The excess charge on a perfect conductor is distributed over the surface of the conductor in such a way as to maintain zero electric field everywhere within that conductor

  → like charges repel

  → potential (voltage) on a perfect conductor is constant everywhere within that conductor

- The voltage between the two conductors of the capacitor is given by...

$$V = \phi_B - \phi_A = -\int_A^B \underline{E} \cdot \underline{dl}$$

and is independent of the path taken between A and B.

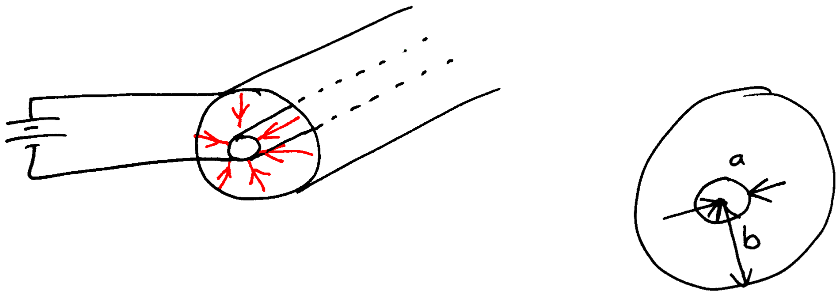
- $\underline{E}$ is always normal to the surface of the conductor
- As the tangential component is zero

---

for a parallel plate capacitor $C = \frac{\varepsilon A}{d}$

(see earlier notes on the electric field between two infinite charged sheets)

Capacitance of a coaxial line



Assume that field due to inner conductor is same as that from an infinite source.

$$\underline{E} = \frac{-q_\ell}{2\pi\varepsilon_0 r}\,\hat{\underline{r}}$$

q_ℓ is charge/unit length

$$= \frac{Q}{\ell} \quad \frac{\text{(total charge)}}{\text{(total length)}}$$

$$\therefore \quad \underline{E} = -\hat{\underline{r}}\cdot\frac{Q}{2\pi\varepsilon_0 r\ell}$$

Voltage between outer and inner conductors is

$$V = -\int_a^b \underline{E}\cdot d\underline{r} = -\int_a^b -\hat{\underline{r}}\,\frac{Q}{2\pi\varepsilon_0 r\ell}\cdot\hat{\underline{r}}\,dr$$

$$= \frac{Q}{2\pi\varepsilon_0\ell}\int_a^b \frac{dr}{r} = \frac{Q}{2\pi\varepsilon_0\ell}\left[\ln b - \ln a\right]$$

$$= \frac{Q}{2\pi\varepsilon_0\ell}\ln\left(b/a\right)$$

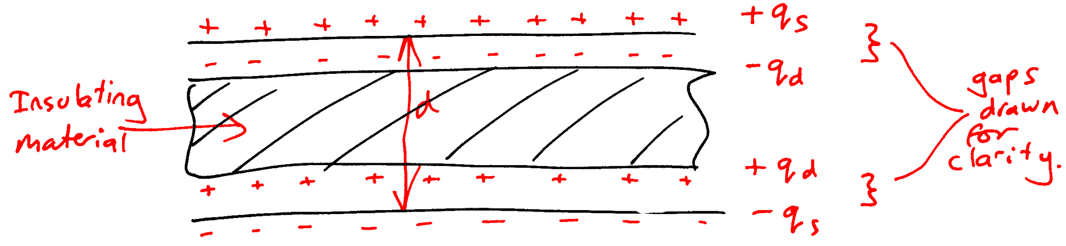
$$C = \frac{Q}{V} = \frac{2\pi\varepsilon_0\ell}{\ln(b/a)} \qquad (F)$$

and the capacitance per unit length

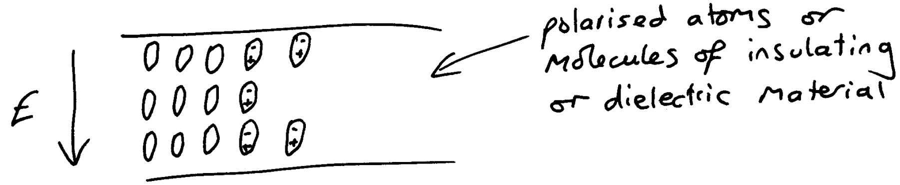
$$C' = \frac{C}{\ell} = \frac{2\pi\varepsilon_0}{\ln(b/a)} \qquad (F/m)$$

Dielectric materials / insulators

In real capacitors the metal plates are separate by an insulating material



The insulating material does not conduct electricity, but its atoms rearrange to produce a surface charge which opposes the applied field.



To calculate the electric field inside the dielectric we use superposition

E field without dielectric material is

$$E_s = \frac{q_s}{\varepsilon_0}$$ (see earlier in notes) field between 2 infinite parallel sheets)

E field due to charge distribution on dielectric is

$$E_d = -\frac{q_d}{\varepsilon_0}$$

-ve as in opposite direction

$\therefore$ Total E field is

$$E = \frac{q_s}{\varepsilon_0} - \frac{q_d}{\varepsilon_0}$$

for linear dielectric materials, the surface charge is proportional to the field so

$$q_d = KE \qquad (K = \text{constant})$$

Hence $$E = \frac{q_s}{\varepsilon_0} - \frac{KE}{\varepsilon_0}$$

$$= \frac{q_s}{\varepsilon_0 (1 + K/\varepsilon_0)} = \frac{q_s}{\varepsilon}$$

... where ε is the permittivity of the material

$$\varepsilon = \left(1 + K/\varepsilon_0\right) \varepsilon_0$$

and the relative permittivity is

$$\varepsilon_r = 1 + K/\varepsilon_0$$

The value for relative permittivity (or dielectric constant) is most often quoted in books etc.

ε_r tells us how much we can polarise a material

Returning to our capacitor problem, the voltage between the plates is

$$V = Ed = \left(\frac{q_s}{\varepsilon_r \varepsilon_0}\right) d$$

and the total charge on a plate of area A is $Q = Aq_s$

So $$V = \frac{Qd}{A\varepsilon_r \varepsilon_0} \quad \text{or} \quad Q = \frac{A\varepsilon_r\varepsilon_0}{d} V$$

And the new capacitance is

$$C = \frac{\varepsilon_r \varepsilon_o A}{d}$$

Hence, we can increase the capacitance by using an insulating material with a high dielectric constant.

Another important characteristic of an insulating material is the Dielectric Strength (V/m). This tells us the highest field we can apply across the material before it breaks down and becomes conducting.

i.e. electrons are ripped free from the molecules

e.g. lightening, spark plugs

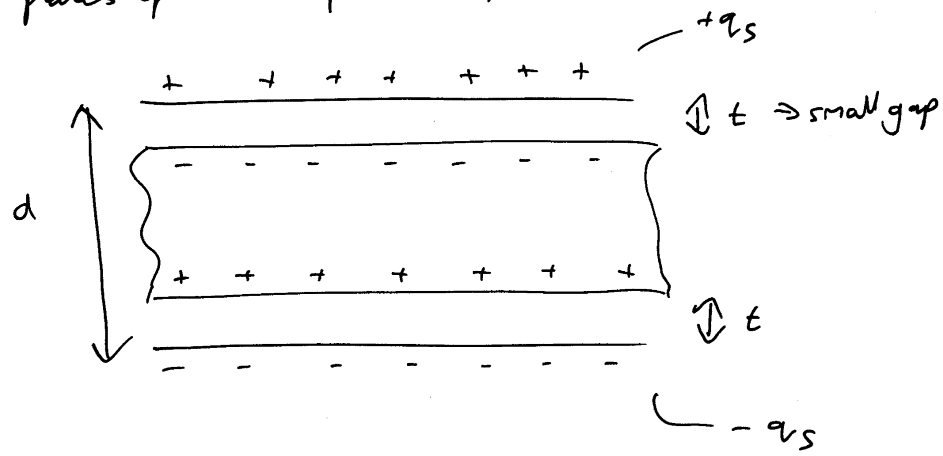
Typical values

	ε_r	Dielectric strength (MV/m)
Air	1.0006	3
Oil	2.1	12
Glass	4.5 → 10	25 - 40
Mica	5.5	200

popular choice for capacitors

Problem

Assume we put a perfect conductor between the plates of a capacitor,



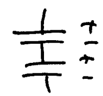
Qs (1) What is the field within the central conductor?

(2) What is the magnitude of the charge distribution on the central conductor?

(3) What is the new value of capacitance.

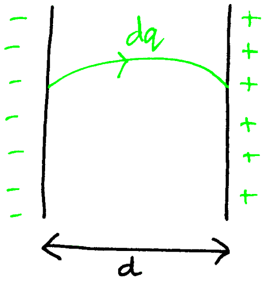
A (1) Field inside perfect conductor is zero.

(2) Charge distribution on conductor opposes that on plates to totally cancel fields

(3) Treat as 2 capacitors in series 

$$C = \frac{\varepsilon_o A}{2t}$$

Energy Stored in a capacitor



Imagine we charge up a parallel plate capacitor by moving small bits of charge, dq, from one plate to the other.

Work done in moving this small charge is :-

$$dW = dq V$$

– where V = voltage difference between plates

[note: V varies with q.]

Now $q = CV$ or $V = \frac{q}{C}$ – where q is the charge stored in the capacitor

so $dW = \frac{q}{C}\, dq$

If we start with zero charge on the capacitor and charge up to Q, the total work done is :-

$$W = \int_0^Q \frac{q}{C}\, dq = \left[\frac{q^2}{2C}\right]_0^Q = \frac{Q^2}{2C}$$

But $Q = CV$ so $\boxed{W = \frac{1}{2} C V^2}$

How much energy is this in real life?

Assume we have a 0.1 F capacitor and charge it up to 12 V...

Energy stored is

$$\frac{1}{2} \times 0.1 \times (12)^2 = 7.2\ \mathrm{J}$$

- compare this with a 12V lead-acid battery that can hold $\approx 4 \times 10^6$ J.

Electric Field Energy Density

The stored energy in a capacitor is...

$$W = \tfrac{1}{2} C V^2$$

for a parallel plate capacitor, we have

$$C = \frac{\varepsilon_0 A}{d} \quad \text{and} \quad V = Ed$$

Hence

$$W = \frac{1}{2} \frac{\varepsilon_0 A}{d} E^2 d^2$$

$$= \frac{1}{2} A d\, \varepsilon_0 E^2$$

Now for a parallel plate capacitor...

Ad = volume between plates

so the energy density is

$$W = \frac{1}{2}\varepsilon_0 E^2$$

Note: if the capacitor had a dielectric spacer then...

$$\boxed{W = \frac{1}{2}\varepsilon_r \varepsilon_0 E^2}$$ → an increase in energy.

Example

A parallel plate capacitor of area $500\,cm^2$ is charged to V and then disconnected.
The plates are moved 4 mm further apart and the voltage is seen to increase by 100 V.
(Why?)

1) What is the charge on the plates?

2) What is the increase in energy stored due to moving the plates?

Soln.

Field between plates is

$$E = \frac{q_s}{\varepsilon_0}$$ – independent of distance

Potential difference between plates is...

$$V = Ed = \frac{q_s}{\varepsilon_0} d$$

so if we increase d and keep q_s fixed, V increases.

As $V = Ed$ is linear, we can write

$$\underbrace{\Delta V}_{\text{change in voltage}} = \underbrace{E}_{\text{constant}}\underbrace{\Delta d}_{\text{change in D}}$$

or $$E = \frac{\Delta V}{\Delta d} = \frac{100}{0.004}\frac{(V)}{(m)} = 25\ kV/m$$

⇒ charge density is $q_s = E\varepsilon_0$
and total charge $Q = q_s A$

$$Q = 25\times10^{3} \times 8.854\times10^{-12} \times 500\times10^{-4}$$
$$= 1.1\times10^{-8}\ C$$

Stored energy is

$$W = \frac{1}{2} C V^2 \qquad \left[C = \frac{Q}{V} \right]$$

$$W = \frac{1}{2} Q V$$

$\therefore$ charge, Q, is constant so

$$\Delta W = \frac{1}{2} Q \Delta V$$

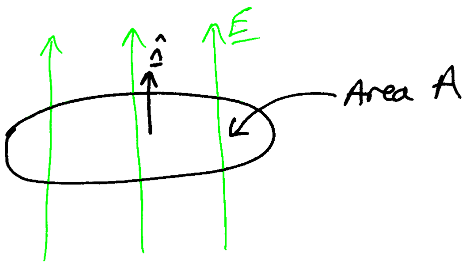
$$= 5.5 \times 10^{-7} \text{ J}$$

Electric Flux and Gauss' Law

Gauss' Law $\rightarrow$ Makes calculating E-fields a lot easier if we know how to use it properly – saves us from tricky line integrals.

What is electric flux?

Electric flux is a measure of the amount of electric field passing through an area



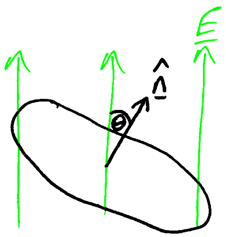
$\hat{n}$ is unit vector perpendicular to the plane of A.

NOTE: $\underline{E}$ perpendicular to plane of A.

We can loosely define electrix flux as

$$\Psi = |\underline{E}|A$$

(Ψ increases if $|\underline{E}|$ increases or A increases)



If we tilt the surface A to an angle θ, less E-field passes through it.

And if $\theta = 90°$, no field passes through it.

Hence we can write

$$\Psi = EA\cos\theta$$

In vector notation this is a dot product

$$\Psi = \underline{E} \cdot \underline{\hat{n}}\, A$$

$\underline{\hat{n}}$ = unit vector normal to A.

In differential form ...

$$d\Psi = \underline{E} \cdot \underline{\hat{n}}\, dA$$

so in general...

$$\Psi = \int_S \underline{E} \cdot \underline{\hat{n}}\, dA$$

integrate over surface S.

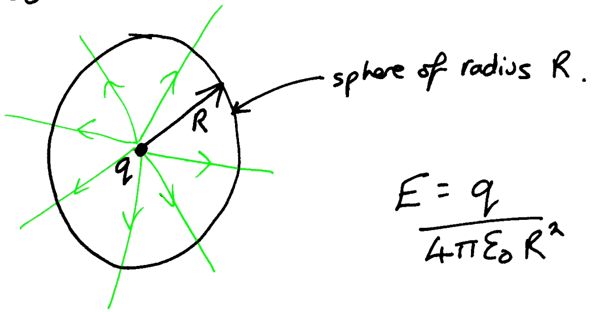
If we understand that $\underline{E} \cdot \underline{\hat{n}}$ means the component of $\underline{E}$ perpendicular to A, we can get rid of it and write...

$$\Psi = \int_S E_{\perp}\, dA$$

where $E_{\perp}$ means the component of $\underline{E}$ perpendicular to A.

[In all the problems we will deal with, we will choose a surface S that is $\perp$ to $\underline{E}$.]

Consider the E-field due to a point charge



$$E = \frac{q}{4\pi\varepsilon_0 R^2}$$ and points radially outwards.

The flux passing through the sphere of radius R centred on q is

$$\Psi = \oint_S E_\perp \, dA \quad = \oint_S \frac{q}{4\pi\varepsilon_0 R^2} \, dA$$

means closed surface

E is always $\perp$ to surface

$$= \frac{q}{4\pi\varepsilon_0 R^2} \underbrace{\oint_S dA}_{\text{surface area of sphere}}$$

$$= \frac{q}{4\pi\varepsilon_0 R^2} \cdot \underbrace{4\pi R^2}_{\text{surface area of sphere}}$$

$$\Psi = \frac{q}{\varepsilon_0}$$

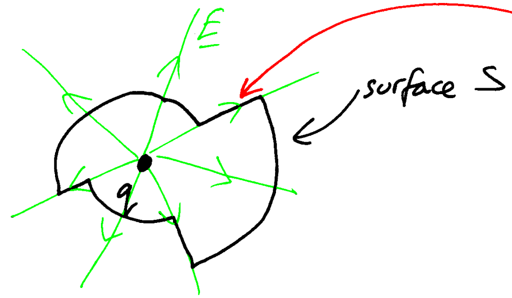
This result is independent of R.

→ As R increases, E drops off as $\frac{1}{R^2}$

→ But area of sphere increases as R^2

→ Balance!

Now change shape of the surface...



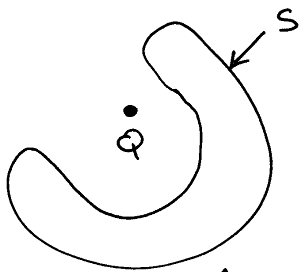
no flux passes through this bit of the surface as it is parallel to E

→ Flux passing through surface S is independent of the shape of S provided it is CLOSED, and is equal to the charge enclosed by the surface divided by ε

$$\boxed{\Psi = \oint_S E_\perp \, dA = \frac{q}{\varepsilon_0}}$$

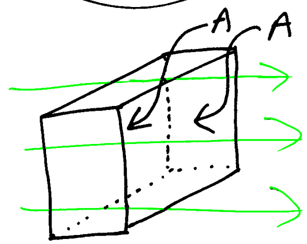
Gauss' Law

egs



In this one there is no charge enclosed in S

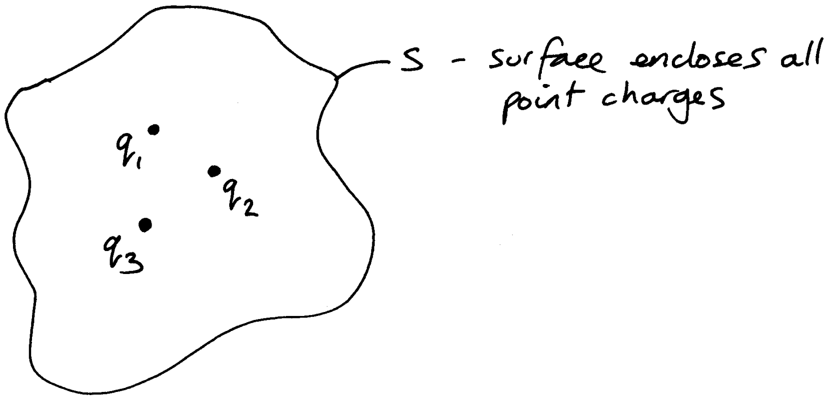
$$\Psi = \oint_S E_\perp \, dA = 0$$



Closed surface does not contain any charge

→ As much flux flows out of surface as flows into it.

The above was from a single point charge. If we have several point charges



By superposition we have

$$\underline{E} = \underline{E}_1 + \underline{E}_2 + \underline{E}_3$$

And

$$\Psi = \oint_S E_\perp \, dA + \oint_S E_\perp \, dA + \ldots$$

$$= \Psi_1 + \Psi_2 + \ldots$$

but $\Psi_1 = \dfrac{q_1}{\varepsilon_0}$, $\Psi_2 = \dfrac{q_2}{\varepsilon_0}$, etc...

$$\Psi = \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \ldots$$

$= \dfrac{Q}{\varepsilon_0}$ where Q = total charge inside S.

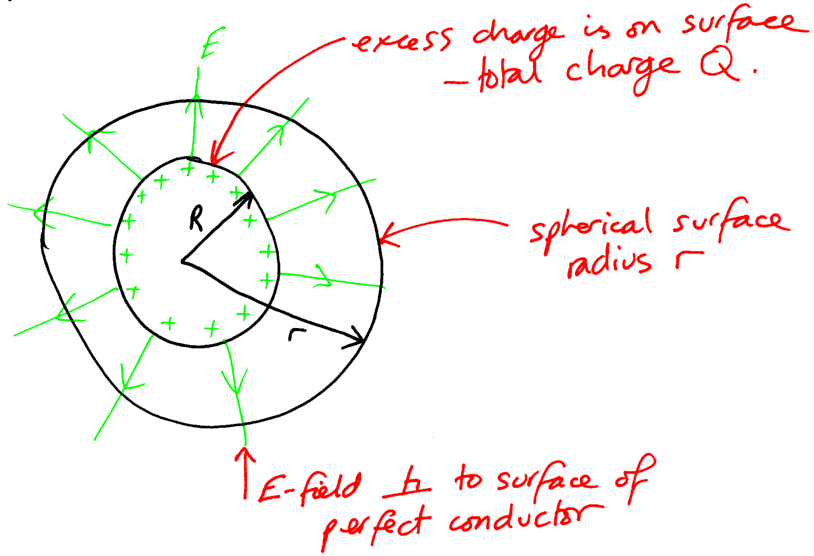
Hence

$$\boxed{\oint_S E_\perp \, dA = \frac{Q}{\varepsilon_0}}$$

Gauss' Law (for multiple charges)

Using Gauss' Law

1) Electric field due to a charged conducting sphere



Using Gauss' Law

$$\oint_S E_\perp \, dA = \frac{Q}{\varepsilon_0}$$

Let S = sphere of radius r $(r > R)$

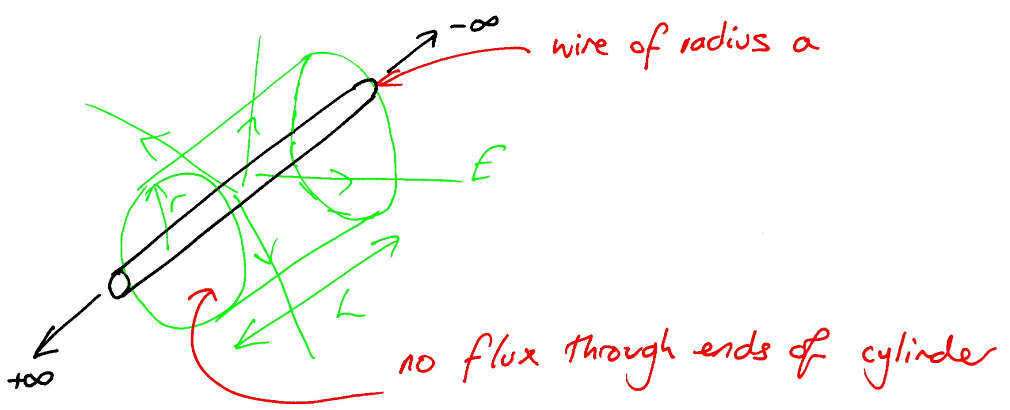
Then E is $\perp$ to S, so

$$\oint_S E_\perp \, dA = E \cdot 4\pi r^2 = \frac{Q}{\varepsilon_0}$$

Hence $E = \dfrac{Q}{4\pi\varepsilon_0 r^2}$ (same as for a point charge)

Note: if we tried to calculate E using Coulomb's law we would have to integrate all over the surface → nasty problem!

2) Electric field due to a long (∞) charged wire



Let q_L = charge / unit length on wire

- Due to symmetry E-field is radially outward and has no variation along length of wire (∞ wire)

- Construct a Gaussian surface that is $\perp$ to E-field $\rightarrow$ a cylinder of radius R and length L

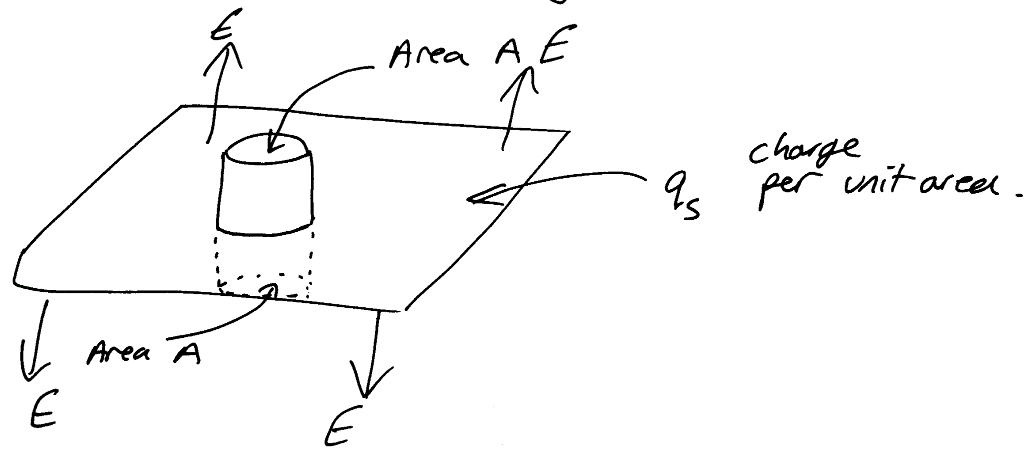
$$\oint_S E_\perp \, dA = \frac{\overset{q_L \cdot L}{Q}}{\varepsilon_0}$$

The ends of the cylinder do not contribute to Ψ as they are parallel to E-field.

Hence $$E \cdot \underbrace{2\pi r L}_{\text{surface area of curved part of cylinder}} = \frac{q_L \cdot L}{\varepsilon_0}$$

$$\Rightarrow E = \frac{q_L}{2\pi \varepsilon_0 r}$$

3) Infinite sheet of charge



Symmetry $E \perp$ to sheet

$$\oint_S E_\perp \, dA = \frac{q_s A}{\varepsilon_0}$$

$$\underbrace{E \cdot 2A}_{\text{top and bottom end surfaces}} = \frac{q_s A}{\varepsilon_0}$$

$$E = \frac{q_s}{2\varepsilon_0}$$