Q1. a) We know that $V_i(t) = V_c(t) + i(t)R$ and $i(t) = C \cdot \frac{dV_c(t)}{dt}$

Therefore $V_i(t) = V_c(t) + RC dV_c(t)/dt$

Taking the Laplace Transform gives

$V_i(s) = V_c(s) + RCsV_c(s) = (1 + RCs)V_c(s)$

Since $V_i(t) = A \cdot u(t)$, we have $V_i(s) = A/s$.

Therefore $A/s = V_c(s)(1 + RCs)$

$$V_c(s) = \frac{A}{s(1+sRC)} = \frac{A}{RC} \cdot \frac{1}{s(s+\frac{1}{RC})} = \frac{A_1}{s} + \frac{A_2}{(s+\frac{1}{RC})}$$

$$A_1 = \left(\frac{A}{RC} \cdot \frac{1}{\left(s+\frac{1}{RC}\right)}\right)|_{s=0} = A$$

$$A_2 = \left(\frac{A}{RC} \cdot \frac{1}{s}\right)|_{s=-1/RC} = -A$$

Therefore $V_c(s) = A \cdot (\frac{1}{s} - \frac{1}{\left(s+\frac{1}{RC}\right)})$

Taking the reverse Laplace Transform

$$V_c(t) = A(1 - e^{-t/RC}) \cdot u(t)$$

b) $i(t) = C \cdot \frac{dV_c(t)}{dt} = C \cdot \frac{d}{dt}\left[A\left(1 - e^{-t/RC}\right)\right] = \frac{AC}{RC} e^{-t/RC}$

Since the signal *u(t)=0 for t<0,* $i(t) = \frac{A}{R} e^{-t/RC} \cdot u(t)$.

Or $i(t) = C \cdot \frac{dV_c(t)}{dt}$ then $V_c(t) = \frac{1}{C}\int_0^{\tau} i(t)dt$

$V_c(s) = \frac{I(s)}{sC}$ assuming zero initial condition.

$$I(s) = sC\left[\frac{A}{RC} \cdot \frac{1}{s\left(s+\frac{1}{RC}\right)}\right] = \frac{A}{R} \cdot \frac{1}{\left(s+\frac{1}{RC}\right)}]$$

$i(t) = \frac{A}{R} e^{-t/RC} \cdot u(t)$.

c) At *t=0,* $i(0) = \frac{A}{R} \cdot e^0 = \frac{A}{R}$.

For *i(t)=0.1A/R*, $i(t) = \frac{A}{R} e^{-t/RC} = 0.1\frac{A}{R}$

$e^{-t/RC} = 0.1$

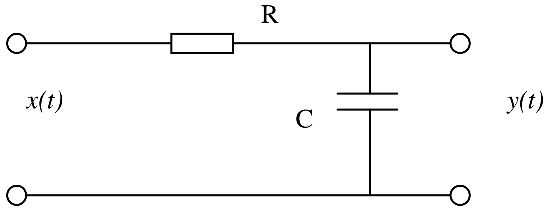
$-t/RC = ln(0.1)$

$-t = RCln(0.1)$. This is a sufficient expression for the time *t*.

d) The cutoff frequency $= \frac{1}{2\pi RC} = \frac{1}{2\pi \cdot 0.01} = 15.9$

the circuit will allow frequencies >16Hz to pass without significant attenuation.

Q2. a)



Assume *x(t)* and *y(t)* are the input and output signals.

Using the transform impedance, we have

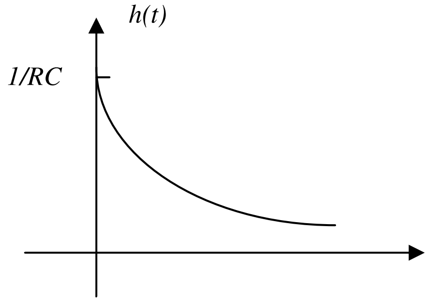
$$\frac{Y(s)}{X(s)} = \frac{1/sC}{\frac{1}{sC}+R}$$

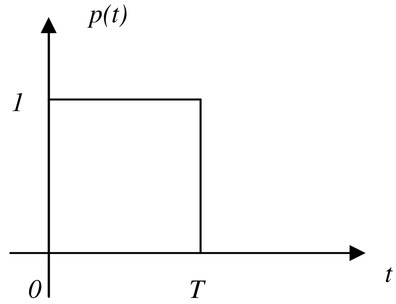
$$H(s) = \frac{1}{1+sRC} = \frac{1/RC}{s+1/RC}$$

Using inverse LT, the impulse response

$$h(t) = \frac{1}{RC} e^{-t/RC} \cdot u(t) \; .$$

b)





The output signal $y(t) = h(t) * p(t) = p(t) * h(t) = \int_0^t p(\tau) h(t-\tau) d\tau$

For $t < 0$, $y(t) = 0$.

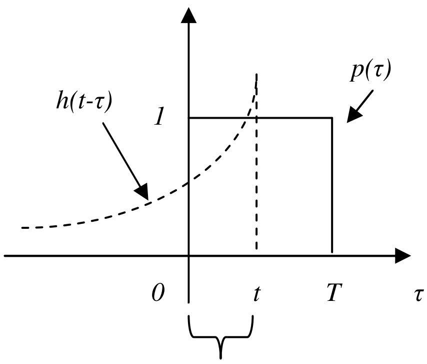
For $t < T$, $0 \le t < T$, $y(t) = \int_0^t \frac{1}{RC} e^{-\frac{t-\tau}{RC}} d\tau$

$$= \frac{1}{RC} \int_0^t e^{-t/RC} e^{\tau/RC} d\tau$$

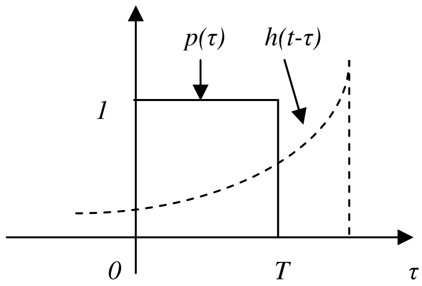
$$= \frac{e^{-t/RC}}{RC} [\frac{1}{1/RC} e^{\frac{\tau}{RC}}]_0^t$$

$$= e^{-t/RC} [e^{\frac{t}{RC}} - 1]$$

$$= 1 - e^{-t/RC}$$



Non-zero integration interval range



For $t \geq T$,

$$y(t) = \frac{e^{-t/RC}}{RC} \int_0^T e^{\frac{\tau}{RC}} d\tau$$

$$= \frac{e^{-t/RC}}{RC} \left[\frac{1}{1/RC} e^{\frac{\tau}{RC}}\right]_0^T$$

$$= e^{-\frac{t}{RC}} \left[e^{\frac{T}{RC}} - 1\right]$$

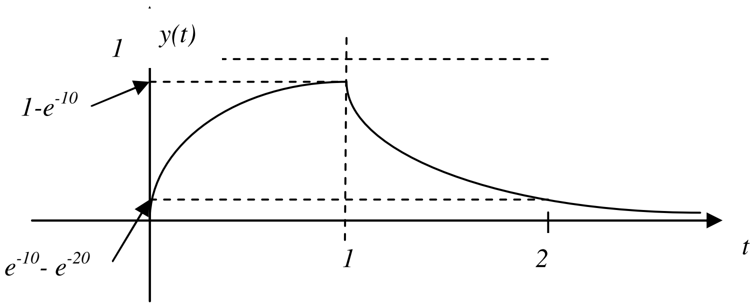
$$= e^{-(t-T)/RC} - e^{-t/RC}$$

c) Response to "1" is given by

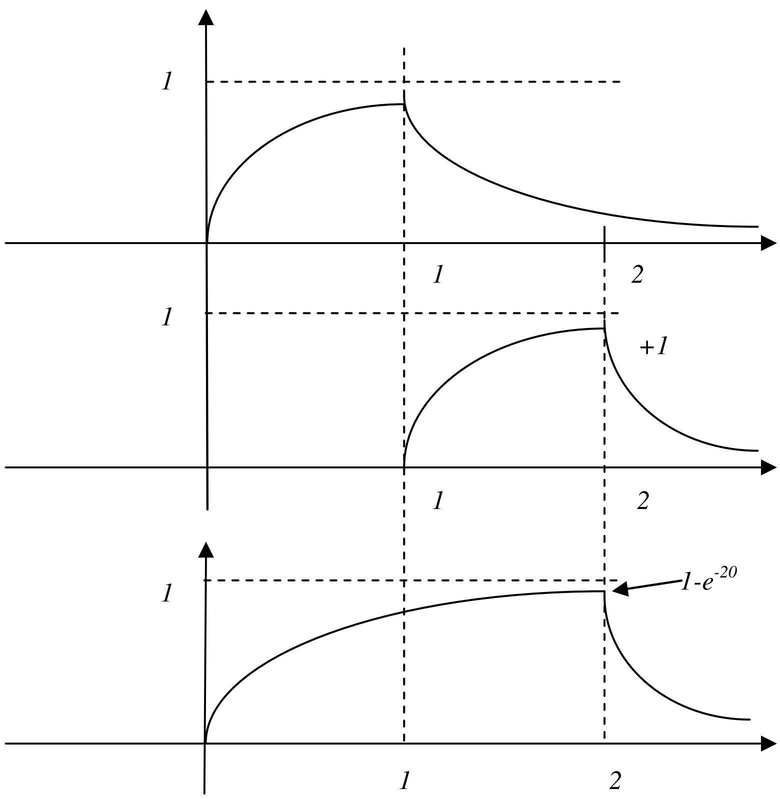
$T = 1s,$ $\quad y(t) = 0 \qquad t < 0$

$= 1 - e^{-t/RC} \qquad 0 \leq t < T \qquad$ when $t = 1, y(1) = 1 - e^{-10}$

$= e^{-(t-T)/RC} - e^{-t/RC} \qquad t \geq T \qquad t = 2, y(2) = e^{-10} - e^{-20}$



d) Response to "1 1" is



Q3. a) The period = T.

The Fourier series coefficient = $C_n = \frac{1}{T}\int_{-T}^{T} \delta(t) e^{-jn\omega_s t} dt$ where $\omega_s = \frac{2\pi}{T}$.

$$= \frac{1}{T} e^{-jn\omega_s(0)} = \frac{1}{T}$$

Therefore the complex Fourier series is

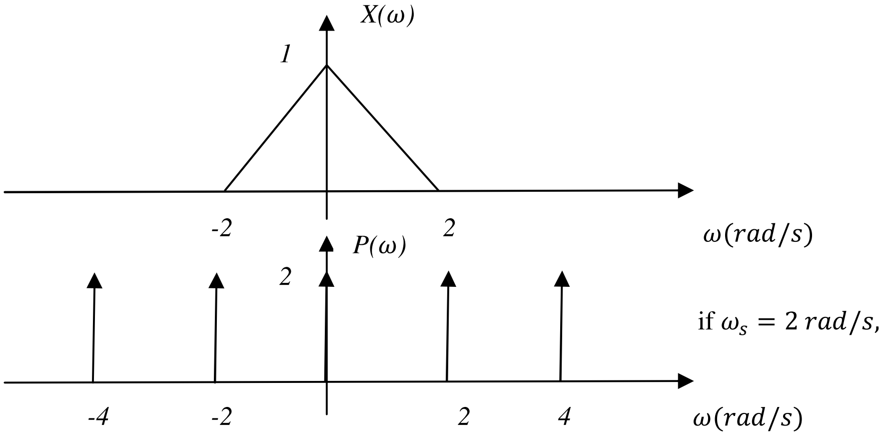
$$p(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_s t} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}$$

b) The FT of $e^{j\omega_s t}$ is $2\pi\delta(\omega - \omega_s)$.

Therefore the FT of *p(t)* is

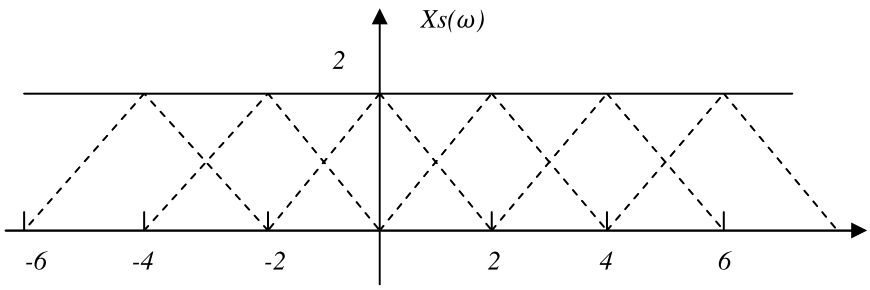
$$P(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

c) We have



Let $X_S(\omega) = X(\omega) * P(\omega)$ since $x_s(t) = x(t) \cdot p(t)$.

Therefore we have



Low pass filtering will not recover the signal.

Since $\omega_s = 2\ rad/s$ equals to the largest frequency present in $X(\omega)$, the Nyquist sampling theorem has not been satisfied. Hence severe aliasing leading to a constant of 2.

d) C must charge rapidly when the diode is conducting. Therefore $R_sC \ll 2\pi/\omega_c$. (1)

C must also discharge slowly through R_l when the diode is not conducting, but not too slow so that it can discharge at a maximum rate determined by the modulating signal.

Therfore $\frac{2\pi}{\omega_c} \ll R_lC \ll \frac{2\pi}{\omega_m}$ (2)

From (1) $R_sC \ll 2\pi/\omega_c$

$R_s \ll 2\pi/C\omega_c$

$R_s \ll 2\pi/0.01 \times 10^{-6} \times 2\pi \times 10^5$

$R_s \ll 1/0.001$

$R_s \ll 1k\Omega$

$R_s \sim 50\Omega.$

From (2) $\frac{2\pi}{\omega_c} \ll R_lC \ll \frac{2\pi}{\omega_m}$

$R_l \ll 2\pi/C\omega_m$

$\ll 2\pi/0.01 \times 10^{-6} \times 2\pi \times 10^3$

$\ll 0.1M\Omega$

$R_l \sim 1k\Omega.$

Q4. a) i) Zeros: $s = -2$.

Poles: $s^2 + 16s + 8 = 0$

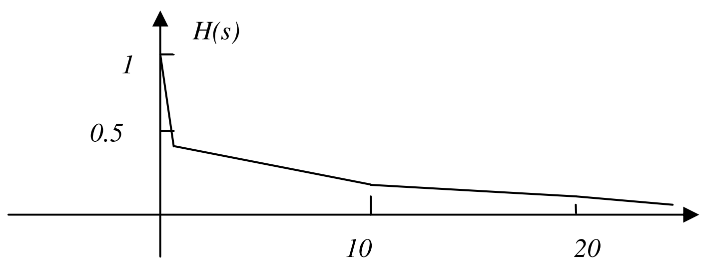
$s = \frac{-16 \pm \sqrt{16^2 - 4(1)(8)}}{2}$

$= -15.48$ and -0.52 .

ii) $H(s) = \frac{4(s+2)}{s^2+16s+8}$

$H(0) = \frac{4(2)}{8} = 1$ $H(1) = \frac{4(3)}{1+16+8} = 0.48$

$H(10) = \frac{4(12)}{100+160+8} = 0.179$ $H(20) = \frac{4(22)}{484+16(22)+8} = 0.104$



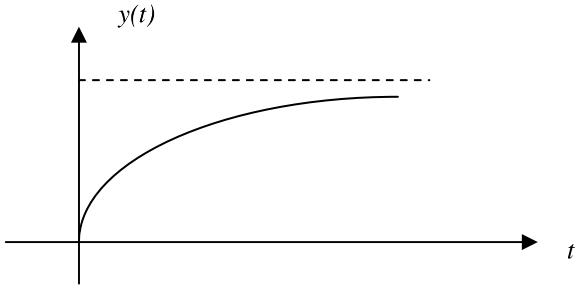
This is a low pass system.

iii) Natured oscillating frequency = $\omega_n = \sqrt{8}\, rad/s$.

Damping factor $=2\xi\omega_n = 16,\ \xi = \frac{16}{2\omega_n} = \frac{8}{\sqrt{8}} = 2.83.$

b) The unit step response consists of 2 exponential terms, $A_1e^{-15.48t}$ and $A_2e^{-0.52t}$ where A_1 and A_2 are constants. ① However the response will be dominated by the dominant pole s = - 0.52.

Note $\xi > 1$, so the step response looks like



The system is stable.

c) Output $Y(s) = H(s)X(s)$

$$= \left(\frac{4(s+2)}{s^2+16s+8}\right)\left(\frac{1}{s+2}\right)$$

$$= \frac{4}{s^2+16s+8}$$

$$= \frac{4}{(s-p_1)(s-p_2)}$$

$$= \frac{k_1}{s-p_1} + \frac{k_2}{s-p_2} \qquad p_1 = -15.48,\ p_2 = -0.52$$

$$k_1 = \frac{4}{s-p_2}\Big|_{s=p_1} = \frac{4}{p_1-p_2} = \frac{4}{-14.96} = -0.27$$

$$k_2 = \frac{4}{p_2-p_1} = \frac{4}{14.96} = 0.27$$

$$y(t) = 0.27(e^{-0.52t} - e^{-15.48t})u(t).$$