EEE224/227

2012/13 exam solutions

Q1a

A system is linear if

1) The response to x1(t) + x2(t) is y1(t) + y2(t) (additivity property). **1 mark**

2) The response to ax1(t) is ay1(t) where a is a constant (homogeneity property). **1 mark**

Consider the equation below, which is a differentiator

$$y(t) = K\frac{dx(t)}{dt}$$

Step 1: consider the input $x_1(t)$

$$y_1(t) = K\frac{dx_1(t)}{dt}$$

Step 2: consider the input $ax_1(t)$ **0.5 marks**

$$y(t) = K\frac{d(ax_1(t))}{dt} = aK\frac{dx_1(t)}{dt} = ay_1(t)$$

So what happens for the input $ax_1(t)+bx_2(t)$ **0.5 marks**

$$y(t) = K\frac{d\left(ax_1(t)+bx_2(t)\right)}{dt} = aK\frac{dx_1(t)}{dt} + bK\frac{dx_2(t)}{dt} = ay_1(t)+by_2(t)$$

Hence, the equation is linear **1 mark**

Consider the second equation

$$y(t) = 4x(t) + 7$$

Using the additivity rule for x1 and x2 gives

$$y1(t) = 4x1(t) + 7$$

$$y2(t) = 4x2(t) + 7$$

Hence,

$$y1(t) + y2(t) = 4x1(t) + 4x2(t) + 14$$ **0.5 marks**

If the input is x1(t)+x2(t) then the output is **0.5 marks**

$$y(t) = 4\left(x1(t) + x2(t)\right) + 7 = 4x1(t) + 4x2(t) + 7$$

The 2 outputs are not the same so the equation is not linear **1 mark**

Q1b

The equation which describes the saw tooth wave form is:-

$$v(t) = t \qquad 0 \le t \le T$$

The Fourier coefficient a0, which is the average value, is simply

$$a_0 = \frac{T}{2}$$ **1 mark**

a_n is given by

$$a_n = \frac{2}{T}\int_0^T t\cos\left(\frac{2n\pi t}{T}\right)dt$$

This can be solved using integration by parts

$$\int u\frac{dv}{dt}dt = uv - \int v\frac{du}{dt}dt$$

Hence,

$$u = t$$
$$\frac{dv}{dt} = \cos\left(\frac{2n\pi t}{T}\right)$$
$$v = \frac{T}{2n\pi}\sin\left(\frac{2n\pi t}{T}\right)$$
$$\frac{du}{dt} = 1$$

Solving for a_n gives

$$a_n = \frac{2}{T}\left[\frac{tT}{2n\pi}\sin\left(\frac{2n\pi t}{T}\right)\Big|_0^T - \int_0^T \frac{T}{2n\pi}\sin\left(\frac{2n\pi t}{T}\right)dt\right]$$

$$a_n = \frac{2}{T}\left[\frac{tT}{2n\pi}\sin\left(\frac{2n\pi t}{T}\right)\Big|_0^T - \left(\frac{T}{2n\pi}\right)^2\left(-\cos\left(\frac{2n\pi t}{T}\right)\Big|_0^T\right)\right]$$

$$a_n = \frac{2}{T}\left[\frac{T^2}{2n\pi}\left(\sin\left(\frac{2n\pi T}{T}\right) - \sin(0)\right) + \left(\frac{T}{2n\pi}\right)^2\left(\cos\left(\frac{2n\pi T}{T}\right) - \cos(0)\right)\right]$$ **4 marks**

$$a_n = \frac{2}{T}\left[\frac{T^2}{2n\pi}(\sin(2n\pi)) + \left(\frac{T}{2n\pi}\right)^2(\cos(2n\pi) - 1)\right]$$

$$a_n = \frac{2}{T}\left[\frac{T^2}{2n\pi}(0) + \left(\frac{T}{2n\pi}\right)^2(1-1)\right] = 0$$

Solving for b_n gives

$$b_n = \frac{2}{T}\left[-\frac{tT}{2n\pi}\cos\left(\frac{2n\pi t}{T}\right)\Big|_0^T + \int_0^T \frac{T}{2n\pi}\cos\left(\frac{2n\pi t}{T}\right)dt\right]$$

$$b_n = \frac{2}{T}\left[-\frac{tT}{2n\pi}\cos\left(\frac{2n\pi t}{T}\right)\Big|_0^T + \left(\frac{T}{2n\pi}\right)^2\left(\sin\left(\frac{2n\pi t}{T}\right)\Big|_0^T\right)\right]$$

$$b_n = \frac{2}{T}\left[\left(\frac{T^2}{2n\pi}\cos\left(\frac{2n\pi T}{T}\right) - \frac{0T}{2n\pi}\cos(0)\right) + \left(\frac{T}{2n\pi}\right)^2\left(\sin\left(\frac{2n\pi T}{T}\right) - \sin(0)\right)\right]$$ **4 marks**

$$b_n = \frac{2}{T}\left[\frac{T^2}{2n\pi}(\cos(2n\pi) - 0) + \left(\frac{T}{2n\pi}\right)^2(\sin(2n\pi) - 0)\right]$$

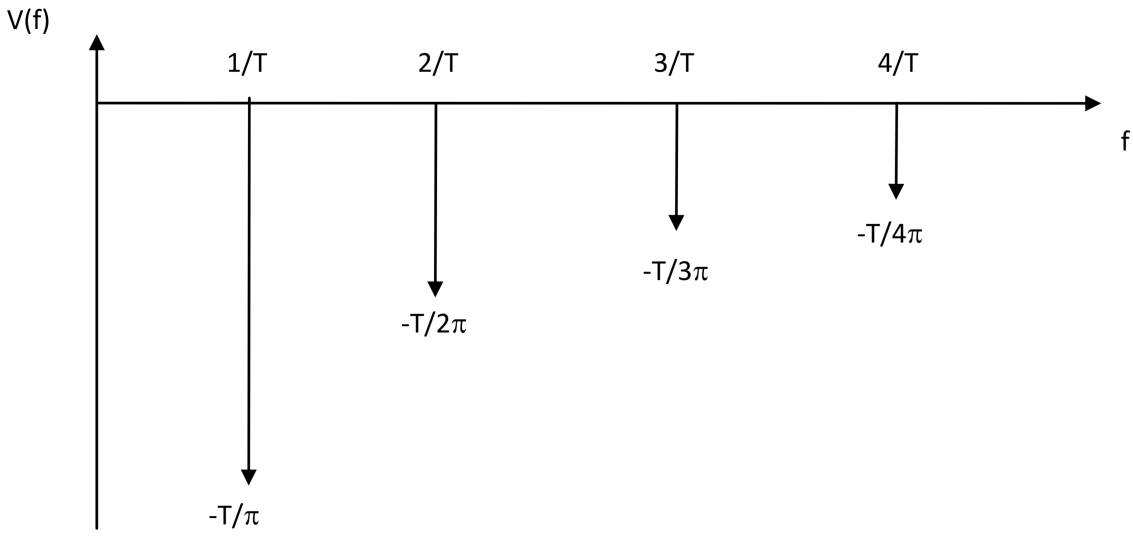
$$b_n = \frac{2}{T}\left[\frac{T^2}{2n\pi}(-1) + \left(\frac{T}{2n\pi}\right)^2(0-0)\right]$$

$$b_n = \frac{-T}{n\pi}$$

Hence, the Fourier series is given by

$$v(t) = \frac{T}{2} - \sum_{n=1}^{\infty}\frac{T}{n\pi}sin\left(\frac{2n\pi t}{T}\right)$$ **1 mark**

Q1c

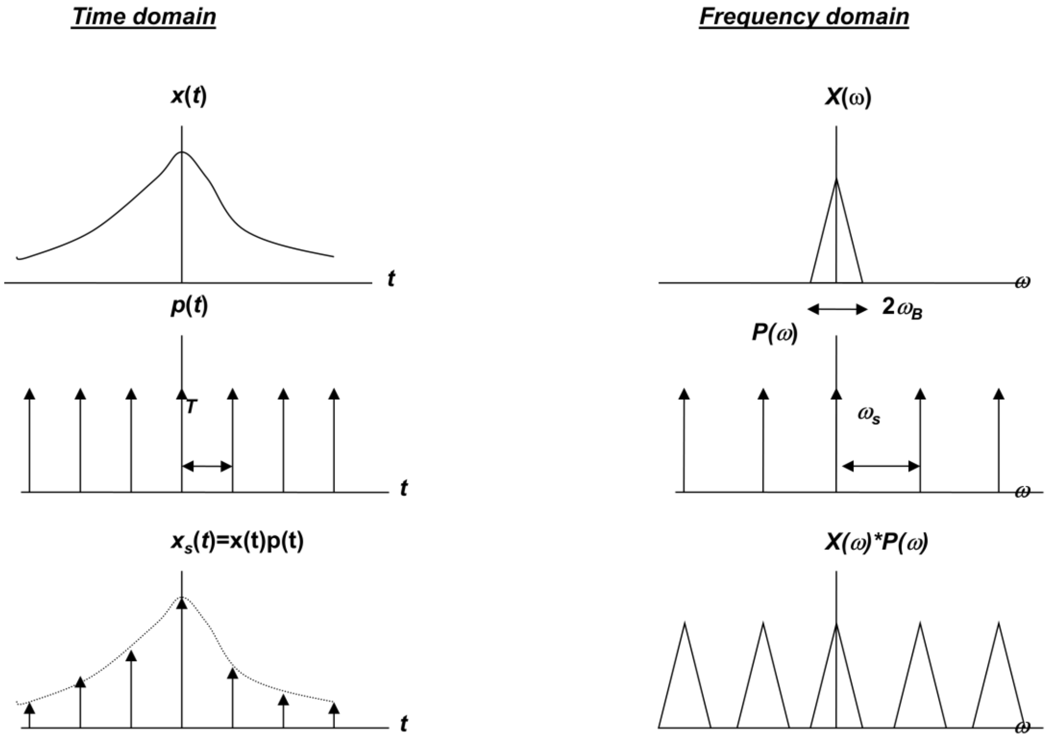


1 mark awarded for each harmonic having the correct amplitude and frequency position

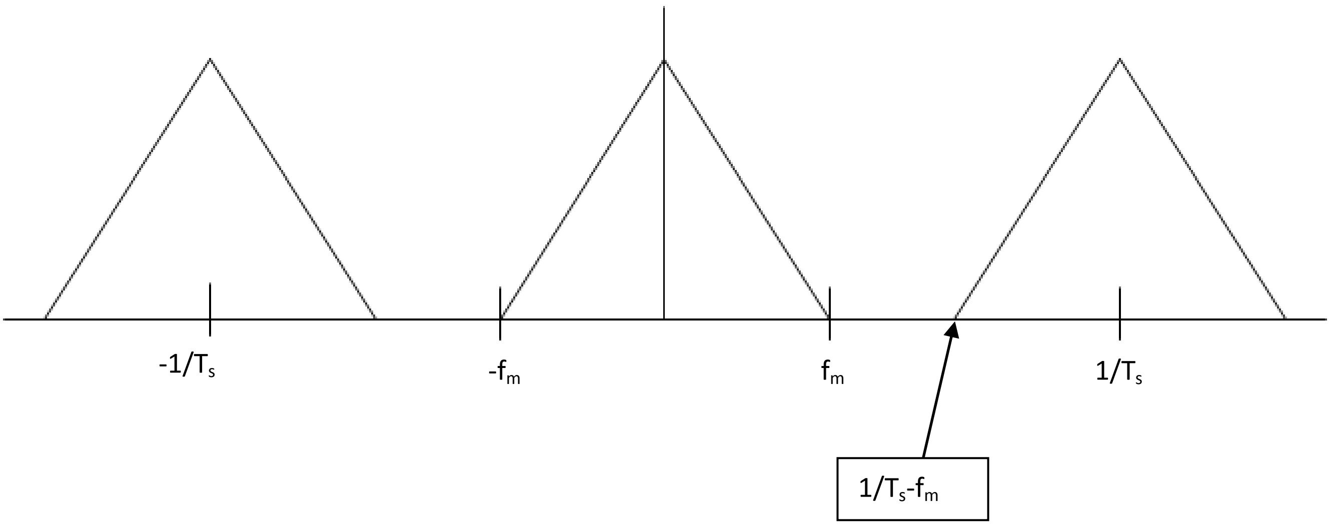
Subtract 1 mark for not including axis labels **(4 mark max)**

Q2a

4 marks awarded for diagrams illustrating the sampling process



Below shows the spectrum of the sampled signal, which needs to be filtered to obtain original signal.



In order to reconstruct original baseband signal the following must be met

$$\frac{1}{Ts} - f_m \geq f_m$$ **2 marks**

Hence $T_s \leq \frac{1}{2f_m}$ **2 marks**

Aliasing occurs if the Nyquist rate is not met and we get overlapping of the original signal. **2 marks**

Q2b

Maximum frequency of audio signal is 20kHz, hence the minimum sampling frequency is 40kHz. The sampling time period, **Ts = 1/40000 =25μs**

2 marks

If the maximum input voltage is 10V the maximum change in pulse width is **1 μs=T_{max}**

2 marks

The maximum pulse width **= 0.2+0.4+T_{max} μs =0.2+0.4+1=1.6 μs** **2 marks**

The maximum number of channels in this case is N=25/1.6=15.625, hence **15** channels can be used.

2 marks

The maximum bandwidth occurs when the pulse width is smallest i.e. T=0.4 μs

The estimate of the bandwidth is **1/T=2.5MHz** **2 marks**

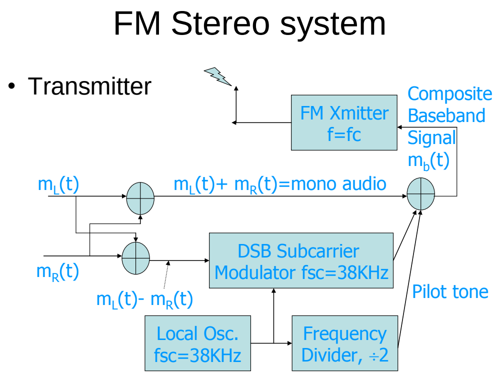
Q3a

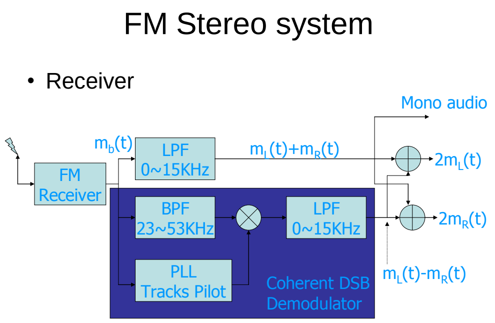
Many radio stations are simultaneously transmitting signals containing the same band of frequencies, e.g. audio/music. If everyone sent those signals as directly transmitted radio waves, interference would cause them to be inefficient and incomprehensible.

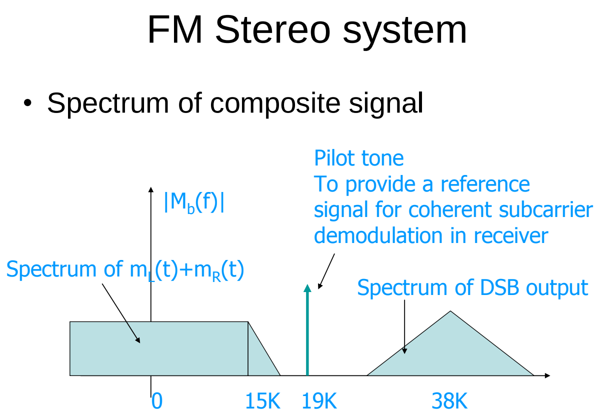
- These signals can be received separately if a different <u>carrier</u> frequency is used for each station.
- Improvement of signal-to-noise ratio
- Ease of radiation in radio systems - to overcome hardware limitations -
- Efficient radiation from antenna requires length to be ≥λ/4
    - e.g. at 10kHz: λ/4 = 750 km, at 100MHz: λ/4 = 75 cm
- Enables multiplexing **4 marks**

Q3b

3 marks for the transmit diagram and 3 marks for the receive diagram **6 marks**







2 marks

Q3c

Need f_c=100MHz, minimum Δf = 75kHz

Given: m(t) : f_s = 100Hz to 15kHz audio

f_1 = 0.1MHz, β_1 = 0.2 radians.

$\therefore$ At 100Hz$\rightarrow$ Δf_1 = 20Hz ($\Delta f_1 = \beta_1 f_s$) **2 marks**

At 15kHz$\rightarrow$ Δf_1=3kHz

To make minimum Δf=75kHz

$$n_1 n_2 = \frac{\Delta f}{(\min \Delta f_1 =)20Hz} = \frac{75000}{20} = 3750$$

$$f_c = (f_2 - n_1 f_1) n_2$$ **4 marks**

$$100 = (9.5 - 0.1 n_1) n_2$$

By solving above n_1=75 & n_2=50 **2 marks**

Question 4

a. (2 marks)

h.f. gain = 1 (since $X_{c_1} \ll R_1 + R_2$)

l.f. gain = $\frac{R_2}{R_1+R_2}$ (since $X_{c_1} \gg R_1 + R_2$)

b. (5 marks)

$$\frac{v_o}{v_i} = \frac{R_2}{R_2 + R_1 /\!/ X_{c_1}} = \frac{R_2}{R_2 + \dfrac{R_1/j\omega C_1}{R_1 + 1/j\omega C_1}}$$

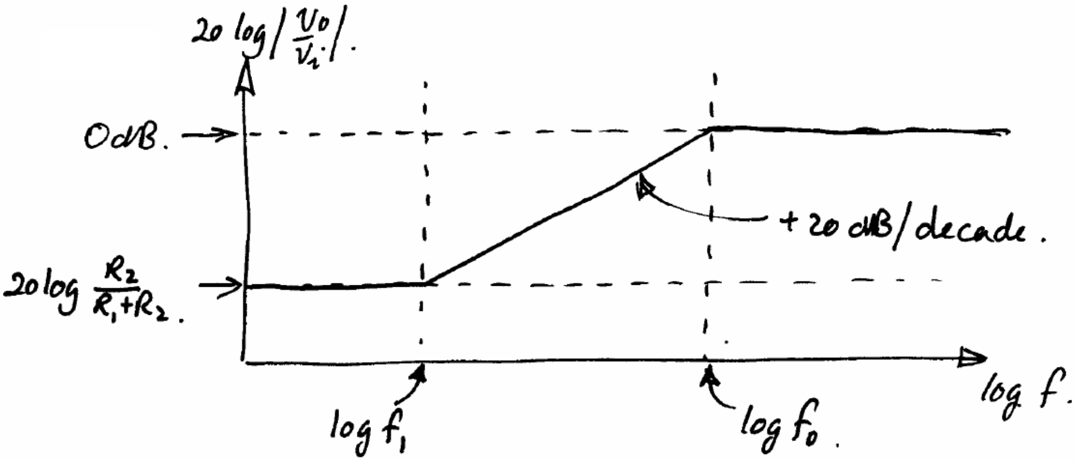
$$= \frac{R_2}{R_2 + \dfrac{R_1}{1+j\omega C_1 R_1}}$$

$$= \frac{R_2(1+j\omega C_1 R_1)}{R_2(1+j\omega C_1 R_1) + R_1} = \frac{R_2(1+j\omega C_1 R_1)}{R_2 + R_1 + j\omega C_1 R_1 R_2}$$

$$= \frac{R_2}{R_1+R_2} \cdot \frac{1+j\omega C_1 R_1}{1+j\omega C_1 \dfrac{R_1 R_2}{R_1+R_2}} \equiv k \cdot \frac{1+j\,f/f_1}{1+j\,f/f_0}.$$

$$k = \frac{R_2}{R_1+R_2}, \quad f_1 = \frac{1}{2\pi C_1 R_1}, \quad f_0 = \frac{R_1+R_2}{2\pi C_1 R_1 R_2}.$$

c. (5 marks)



d. (8 marks)

at h.f. $X_c \ll R_1$ and $X_c \ll R_2$ $\therefore$ $\dfrac{v_o}{v_i} \approx 1$

at l.f. $X_c \gg R_1$ and $X_c \gg R_2$ $\therefore$ $\dfrac{v_o}{v_i} \approx \dfrac{R_1+R_2}{R_1}$ =6

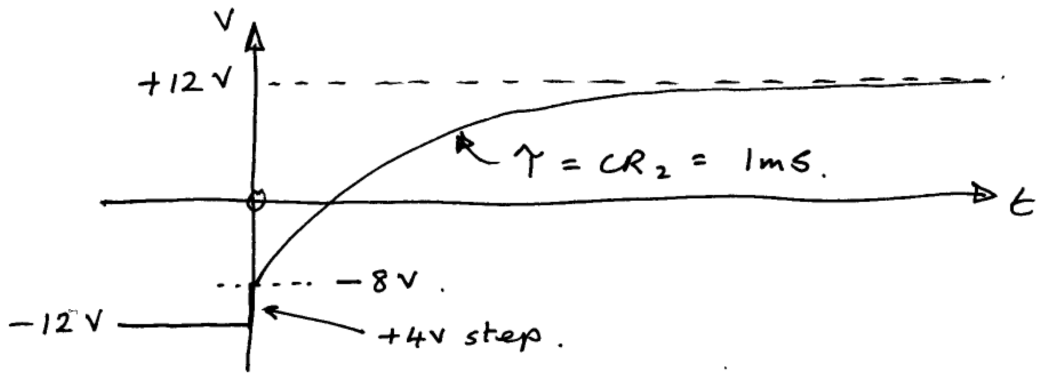
immediately before the step, l.f gain dominates

so $v_o|_{t=0^-} = -2 \times \text{l.f gain} = -12\,V$

$v_o|_{t=0^+} = -12 + \Delta v_o$ due to step.

$= -12 + 4 \times \text{h.f gain} = -12+4 = -8\,V$

$v_o|_{t\to\infty} = +2 \times \text{l.f gain} = +12\,V.$



Question 5

a. (6 marks)

$v^- = \dfrac{v_o}{2}$,

$v^+ = v_C$,

$$i_C = i_i + i_f = \frac{v_i - v_C}{R} + \frac{v_o - v_C}{R}$$

since $A_v = \infty$, $v^+ (= v_C) = v^- = v_o/2$,

$$i_C = \frac{v_i - v_C}{R} + \frac{v_o - v_C}{R} = \frac{v_i}{R} - \frac{v_o}{2R} + \frac{v_o}{R} - \frac{v_o}{2R} = \frac{v_i}{R},$$

$$\frac{v_o}{2} = v_C = \frac{i_C}{sC} = \frac{v_i}{sCR},$$

$$v_o = \frac{2v_i}{sCR}.$$

b. (5 marks)

$$G(s) = \frac{1}{s^2 + 6s + 13} = \frac{1}{(s^2 + 6s + 9) + 4} = \frac{1}{(s+3)^2 + (2)^2}.$$

Compare $G(s) = \dfrac{1}{s^2 + 6s + 13}$ with $\dfrac{k}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ and

$G(s) = \dfrac{1}{(s+3)^2 + (2)^2}$ with $\dfrac{k}{(s + \zeta\omega_n)^2 + \omega_d^2}$.

Natural oscillating frequency is $\omega_n = \sqrt{13}$ rad/s.

Damping factor is $\zeta = \dfrac{6}{2\omega_n} = \dfrac{3}{\sqrt{13}} = 0.832$

The system is underdamped. The poles are complex and are given by

$p_{1,2} = -\zeta\omega_n \pm j\omega_d$, where $\omega_d = 2$.

$$p_{1,2} = -\frac{3}{\sqrt{13}}\sqrt{13} \pm j2 = -3 \pm j2.$$

c. (7 marks)

$$x(t) = e^{-3|t|} = e^{-3t}u(t) + e^{3t}u(-t)$$

The Laplace Transform of $e^{-3t}u(t)$ is

$$\int_{-\infty}^{\infty} e^{-3t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-3t}e^{-(\sigma+j\omega)t}dt = \int_{0}^{\infty} e^{-(3+\sigma)t}e^{-j\omega t}dt$$

where the region of convergence is described by $\sigma = \mathrm{Re}\{s\} > -3$.

$$\int_0^{\infty} e^{-(3+s)t}dt = -\frac{1}{3+s}\left[e^{-(3+s)t}\right]_0^{\infty} = \frac{1}{s+3}$$

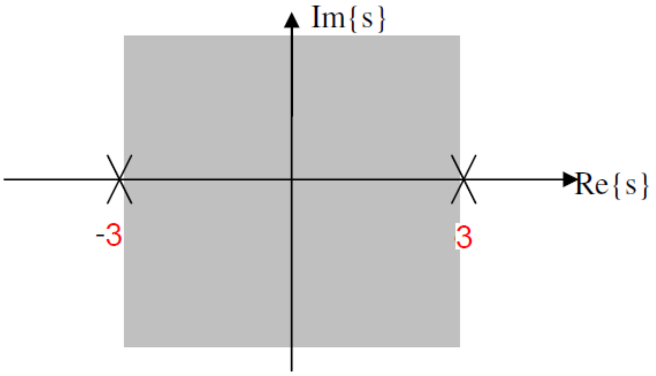
The Laplace transform of $e^{3t}u(-t)$ is given by

$$\int_{-\infty}^{\infty} e^{3t}u(-t)e^{-st}dt = \int_{-\infty}^{0} e^{3t}e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{0} e^{(3-\sigma)t}e^{-j\omega t}dt\ ,$$

Which exists if $3 - \sigma > 0$, i.e., $\sigma = \mathrm{Re}\{s\} < 3$.

$$\int_{-\infty}^{0} e^{(3-s)t}dt = \frac{1}{3-s}\left[e^{(3-s)t}\right]_{-\infty}^{0} = -\frac{1}{s-3}$$

Therefore, $X(s) = \dfrac{1}{s+3} - \dfrac{1}{s-3}$, with region of convergence given by $-3 < \mathrm{Re}\{s\} < 3$.



d. (2 marks)

Gain-bandwidth product (GBP) is the product of the open-loop d.c. gain and the open-loop corner frequency of an operational amplifier, i.e.,

$GBP = A_o\omega_o$ (rads/sec) $= A_o f_o$ (Hz).

Question 6

a. (7 marks)

$i_i + i_1 = i_2$ or,

$$\frac{v_i - v_2}{R_2} + (v_o - v_2)sC_1 = v_2 sC_2$$

$v^- = 0$,

$$i_2 + i_3 = 0 = v_2 sC_2 + \frac{v_o}{R_1}$$

$$v_2 = \frac{v_i + v_o\, s\, C_1 R_2}{1 + s\,(C_1 + C_2)R_2} = -\frac{v_o}{s\, C_2 R_1}$$ which can be developed as follows:

$$v_i\, s\, C_2 R_1 + v_o\, s^2\, C_1 R_2 C_2 R_1 = -\, v_o\,(1 + s\,(C_1 + C_2)R_2\,)$$

$$v_i\, s\, C_2 R_1 = -\, v_o\,(1 + s\,(C_1 + C_2)R_2 + s^2 C_1 R_2 C_2 R_1\,)$$

$$\frac{v_o}{v_i} = \frac{-\, s\, C_2 R_1}{1 + s\,(C_1 + C_2)R_2 + s^2 C_1 R_2 C_2 R_1} = \frac{C_2 R_1}{(C_1 + C_2)R_2}\;\frac{-\, s\,(C_1 + C_2)R_2}{1 + s\,(C_1 + C_2)R_2 + s^2 C_1 R_2 C_2 R_1}$$

The response is a bandpass response.

b. (6 marks)

If $C_1 = C_2 = C$, comparison with the standard-form transfer function gives:

$$\omega_0^2 = \frac{1}{C^2 R_1 R_2},\; \frac{1}{\omega_0 q} = 2CR_2 \text{ and therefore } q = \frac{1}{2}\sqrt{\frac{R_1}{R_2}}$$

For $R_1 = 4R_2$, $q = \frac{\sqrt{4}}{2} = 1$.

For a standard bandpass circuit, the gain at the undamped natural frequency is given by the frequency independent constant k. Thus at ω_0, we have

$$\frac{v_o}{v_i} = -\frac{C_2 R_1}{(C_1 + C_2)R_2} = -\frac{R_1}{2R_2} = -2.$$

c. (7 marks)

$$H(s)=\frac{s}{2s^2+10s+12}=\frac{s}{(2s+4)(s+3)}=\frac{s/2}{(s+2)(s+3)}=k_0+\frac{k_1}{s+2}+\frac{k_2}{s+3}$$

Using partial fraction expansion,

$$k_o=\frac{s}{2s^2+10s+12}\bigg|_{s=\infty}=\frac{1/s}{2+10/s+12/s^2}\bigg|_{s=\infty}=0,$$

$$k_1=\frac{s/2}{(s+2)(s+3)}(s+2)\bigg|_{s=-2}=\frac{s/2}{(s+3)}\bigg|_{s=-2}=\frac{-1}{1}=-1,$$

$$k_2=\frac{s/2}{(s+2)(s+3)}(s+3)\bigg|_{s=-3}=\frac{s/2}{(s+2)}\bigg|_{s=-3}=\frac{-3/2}{-1}=\frac{3}{2}.$$

Alternatively,

$$\frac{s/2}{(s+2)(s+3)}=\frac{k_0(s+2)(s+3)+k_1(s+3)+k_2(s+2)}{(s+2)(s+3)}$$

$$s/2=k_0(s^2+5s+6)+(k_1+k_2)s+3k_1+2k_2=k_0s^2+(5k_0+k_1+k_2)s+(6k_0+3k_1+2k_2)$$

Comparing the coefficients for s^2 gives $k_0 = 0$.

Comparing the coefficients for s gives $k_1 + k_2=1/2$.

We also have $3k_1+2k_2=0$ and hence $k_2=-3k_1/2$.

Substituting k_2 gives $k_1 - 3k_1/2=1/2$ and hence $k_1 = -1$ and $k_2 = 3/2$

The transfer function is

$$H(s)=-\frac{1}{s+2}+\frac{3/2}{s+3}=\frac{3}{2}\frac{1}{s+3}-\frac{1}{s+2}$$

Therefore the impulse response is described by $h(t)=\frac{3}{2}e^{-3t}u(t)-\ e^{-2t}u(t)$ in the time domain.

If the input is a unit step, The output is given by

$$Y(s)=H(s)U(s)=\frac{s/2}{(s+2)(s+3)}\frac{1}{s}=\frac{1/2}{(s+2)(s+3)}=\frac{k_1}{(s+2)}+\frac{k_2}{(s+3)}.$$

Using partial fraction expansion,

$$k_1=\frac{1/2}{(s+2)(s+3)}(s+2)\bigg|_{s=-2}=\frac{1/2}{(s+3)}\bigg|_{s=-2}=\frac{1/2}{1}=\frac{1}{2},$$

$$k_2=\frac{1/2}{(s+2)(s+3)}(s+3)\bigg|_{s=-3}=\frac{1/2}{(s+2)}\bigg|_{s=-3}=\frac{1/2}{-1}=-\frac{1}{2}.$$

Alternatively,

$$\frac{1/2}{(s+2)(s+3)} = \frac{k_1(s+3)+k_2(s+2)}{(s+2)(s+3)}$$

$$1/2 = (k_1 + k_2)s + (3k_1 + 2k_2)$$

Comparing the coefficients for s, $k_1 = -k_2$.

$k_1 = 1/2$ *and* $k_2 = -1/2$.

$$Y(s) = \frac{1}{2(s+2)} - \frac{1}{2(s+3)}.$$

Therefore the unit step response in time domain is

$$y(t) = \frac{1}{2}e^{-2t}u(t) - \frac{1}{2}e^{-3t}u(t) = \frac{1}{2}u(t)(e^{-2t} - e^{-3t}).$$