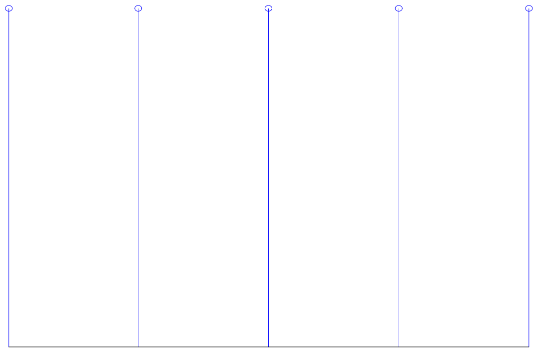
EEE6209 Advanced Signal Processing 2014-15
Exam Solutions

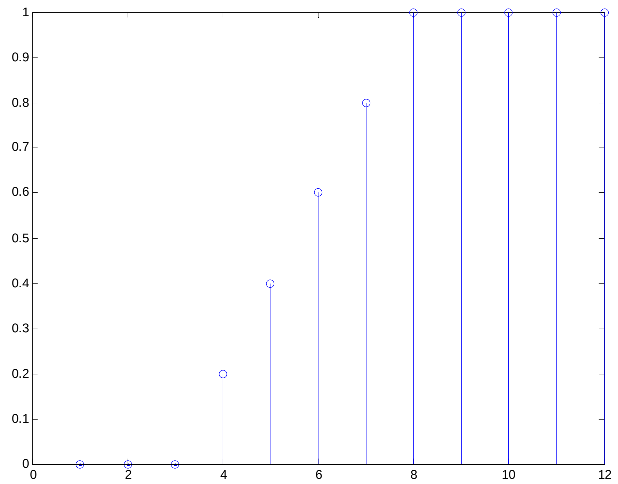
1.

a.

h(n) = { 1/5, 1/5, 1/5, 1/5, 1/5}   the third element is at n=0.



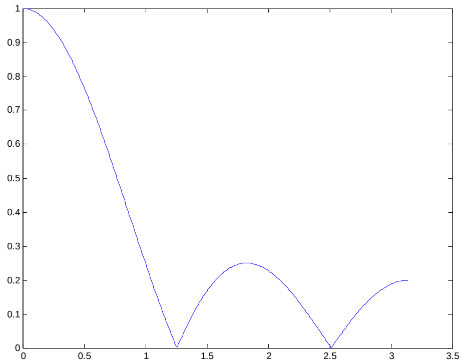
Time domain Performance:
Step response:
Convolve the h(n) with step function u(n). In other words, taking the discrete integral of h(n). Results in { ... 0, 1/5, 2/5, 3/5, 4/5, 1, ...}



(1 mark)

Frequency response:
$y(n) = 1/5(x[n-2]+x[n-1]+x[n]+x[n+1]+x[n+2])$
$h(n) = 1/5, 1/5, 1/5, 1/5, 1/5$
$h(z) = 1/5\ (z^{-2} + z^{-1} + 1 + z^{1} + z^{2})$
$z= e^{-j\omega}$,
$H(j\omega) = 1/5\ (e^{2j\omega} + e^{j\omega} + 1 + e^{j\omega} + e^{-2j\omega}) = (1+2\cos\omega+2\cos 2\omega)/5$ **(3)**

$|H(j\omega)| = |(1+2\cos\omega+2\cos 2\omega)/5|$
(1 mark)



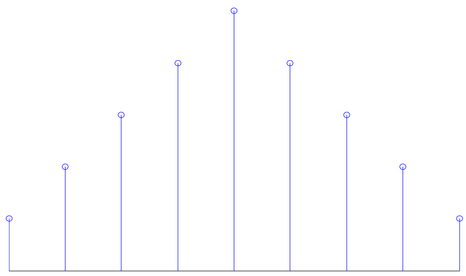
(1 mark)

b. Convolve h(n{ 1/5, 1/5, 1/5, 1/5, 1/5} with itself.

New impulse response: p(n) =h(n)*h(n)
={ 1/5, 1/5, 1/5, 1/5, 1/5} *{ 1/5, 1/5, 1/5, 1/5, 1/5}
= { 1/25, 2/25, 3/25, 4/25, 5/25, 4/25, 3/25, 2/25, 1/25}

(1.5 marks)

1/5
4/25
3/25
2/25
1/25



-4 -3 -2 -1 0 1 2 3 4

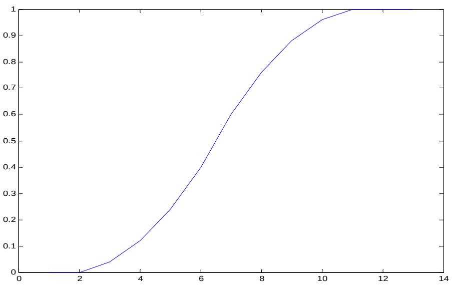
(0.5 marks)

(2)

c.

Time domain properties:
The step response is as follows:

(3)



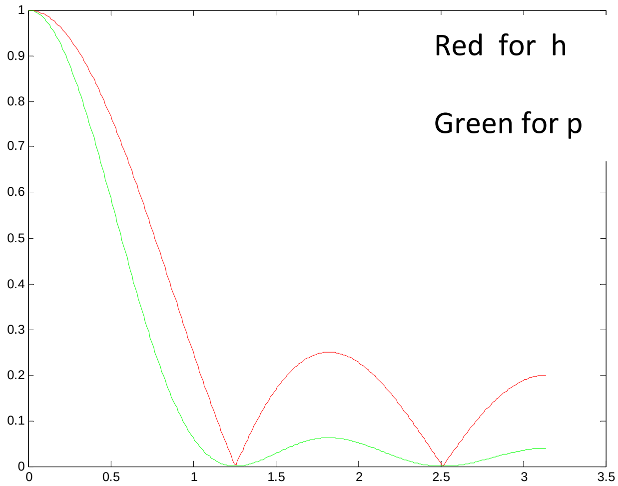
Smooth rise. Since the kernel is larger, more emphasis is on centre data points in the filter kernel. Therefore sharp changes are preserved, while smoothing out noise compared to h(n).

(1 mark)

For h(n), $|H(j\omega)| = |(1+2\cos(\omega) + 2\cos(2\omega))/5|$

For p(n)=h(n)*h(n)

$P(j\omega) = H(j\omega).\ H(j\omega)$



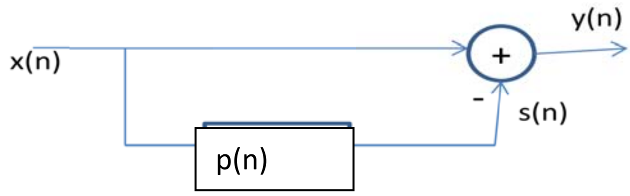
(1 mark)

(Only an estimated sketch is required to explain the performance difference.)

P(n) provides a slower time-domain response with smooth transition compared to h(n). The stop-band attenuation performance is better in p(n) leading to good suppression of high frequencies compared to h(n).

(1 mark)

d.

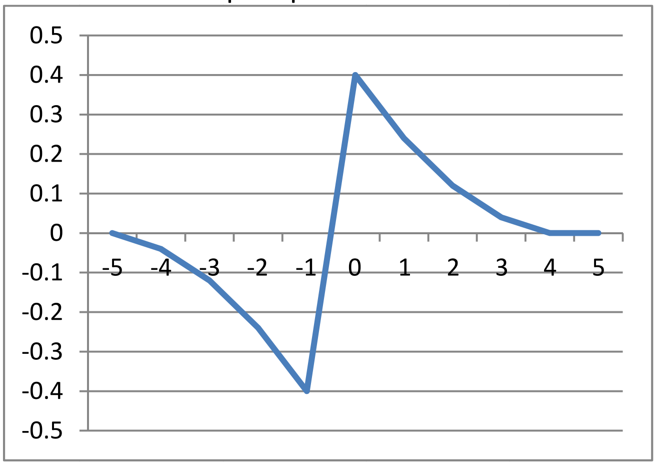


(1 mark)

y(n) = x(n) – p(n)* x(n)
  = (δ(n) –p(n)) * x(n)
Therefore r(n) = (δ(n) –p(n))
= δ(n) -{ 1/25, 2/25, 3/25, 4/25, 5/25, 4/25, 3/25, 2/25, 1/25}
= {- 1/25, -2/25, -3/25, -4/25, 4/5, -4/25, -3/25, -2/25, -1/25}
(1 mark)

(2)

e. Time domain- step response



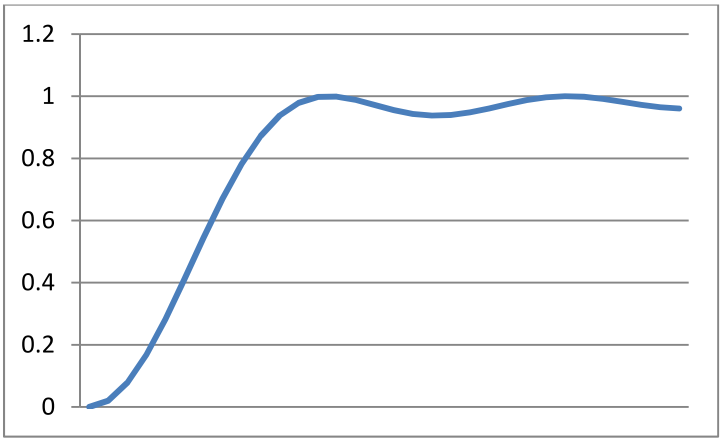
(1 mark)

Frequency domain response

r(n) = (δ(n) –p(n))
in freq domain
R(jω)= 1- P(jω)

A rough sketch would look like

(2)



(1 mark)

f.

The output y(n) is computed as follws
y(n) = (x[n-2]+x[n-1]+x[n]+x[n+1]+x[n+2])/5

consider two consecutive points i and i+1
y[i] = (x[i-2]+x[i-1]+x[i]+x[i+1]+x[i+2])
y[i-1] = (x[i-3]+x[i-2]+x[i-1]+x[i]+x[i+1])

y[i] = y[i-1]+(x[i+2]-x[i-3])

(1 mark)

That means if y[0] is computed all the following points can be computed recursively using the above expression.

all y(n) values are multiplied by 1/5 at the end

Number of multiplications: L (L is the length of y(n)) -----------------(A)
Number of additions:
4 for y[0] and 2(L-1) for the rest ---------------(B)

(1 mark)

For non-recursive implementation
Number of multiplications: 5L ----------------(C)
Number of additions: 4L ---------------------------(D)

(A) and (B) are much smaller than (C) and (D).

(1 mark)

(3)

2. **a.** (i)

$y(n) = \{0, b, 0, d, 0, f, 0, h, 0, j, 0\}$

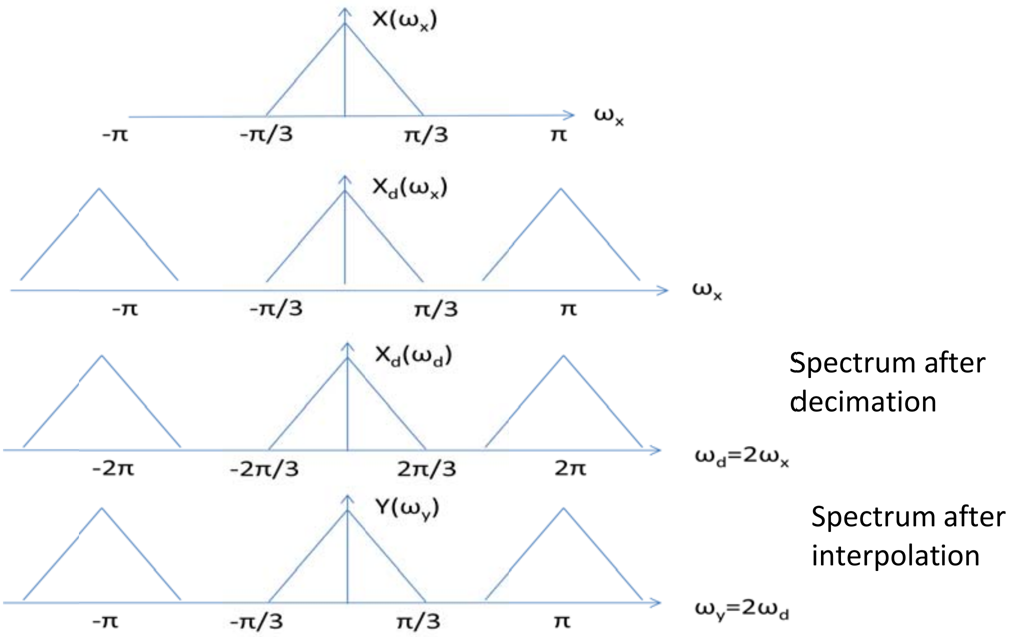
(1 mark)

(ii) This represents a situation where x(n) is decimated by a factor of 2 followed by interpolation by a factor of 2.
Starting from:

$$Y(j\omega_x) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(j(\omega_x - 2\pi k / M)\right)$$

For M=2,

$$Y(j\omega_x) = \frac{1}{2}\sum_{k=0}^{1} X\left(j(\omega_x - \pi k)\right)$$

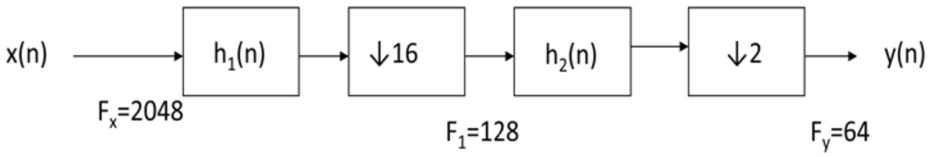


(2 marks)

(iii) The maximum frequency is smaller than half of the new sampling frequency so anti-aliasing filter is not needed
But for interpolation an anti-imaging filter required with
Pass band edge at π/3 and
Stop band edge at 2π/3.
(2 marks)

(5)

b.



(1 mark)

Passband deviation: 0.01dB → 0.00115
Stopband atteneuation: 80dB → 0.0001
For both filters we choose
δ_p=0.00115/2 =0.00058
δ_s=0.0001

(1 mark)

Filter length given by $N \approx \frac{-10\log(\delta_p\,\delta_s)-13}{14.6(\Delta f)}+1$

$$N \approx \frac{-10\log(0.0005 \text{ x } 0.0001)-13}{14.6(\Delta f)}+1$$

$$N \approx \frac{4.066}{(\Delta f)}+1$$

For h_2:
Passband 0 - 20 Hz
Stopband 32-64 Hz
Transition band 30Hz – 32Hz
Normalised transition bandwidth (32-20)/128 = 12/128
(2 marks)

Therefore $N_2 \approx \frac{4.066}{\left(\frac{12}{128}\right)}+1 = 45$ **(0.5 marks)**

For h_1:
Passband 0 - 20 Hz
Stopband (128-64/2) -1024 Hz = 96-1024
Transition band 20Hz – 96Hz
Normalised transition bandwidth (96-20)/2048 = 76/2048
(2 marks)

Therefore $N_1 \approx \frac{4.066}{\left(\frac{76}{2048}\right)}+1 = 111$ **(0.5 marks)**

(7)

c. MPS = $\sum_{i=1}^{2} F_i N_i$ = 128x111 + 64x45 = 17088 multiplications /second

(1.5 marks)

(3)

N is inversely proportion to Δf. If a single-stage was used Δf would have been (32-20)/2048. To make this value larger, we need to make the numerator bigger and the denominator smaller. This can be achieved by factoring F into a product of several smaller sampling rates. Each of the early stage filetrs the transition bandwidth is large because the correson=ding sampling rates are closer to F.
(1.5 marks)

3.

a. $[p, q]$ is the low pass filter
and $[r, s]$ is the high pass filter

for filter [p q]
For orthogonality:
$p^2 + q^2 = 1$ (1)
For regularity:
$p + q = \sqrt{2}$ (2)
(2 marks)

from (1) and (2)
$(\sqrt{2}-q)^2 + q^2 = 1$
$1 - 2\sqrt{2}q + 2q^2 = 0$
$(1 - \sqrt{2}b)^2 = 0$
$q = 1/\sqrt{2}$
from (4) $p = 1/\sqrt{2}$ **(0.5 marks)**

For the orthogonality of the transform matrix
$r^2 + s^2 = 1$ (3)
$pr + qs = 0$ (4)
(1 mark)

from (4) $r + s = 0$ therefore, $r = -s$
from(2) $2r^2 = 1$ $r = \pm 1/\sqrt{2}$
choose $r = 1/\sqrt{2}$ then $c = -1/\sqrt{2}$
(0.5 marks)

(4)

b.
The top half corresponds to the low pass filtering while the bottom half corresponds to the high pass filtering.

$$T1=\begin{bmatrix} p & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & p & q \\ r & s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & s & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & r & s \end{bmatrix}\begin{bmatrix} a1 \\ a2 \\ a3 \\ a4 \\ a5 \\ a6 \\ a7 \\ a8 \end{bmatrix}$$

(2)

c. [p q] and [r s] form an orthogonal basis
All the rows in T1 are obtained using the double translations. Therefore
All rows in T1 are orthogonal. **(1 mark)**

(3)

Therefore the inverse is the transpose

$$\begin{bmatrix} p & 0 & 0 & 0 & r & 0 & 0 & 0 \\ q & 0 & 0 & 0 & s & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & r & 0 & 0 \\ 0 & q & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & p & 0 & 0 & 0 & r & 0 \\ 0 & 0 & q & 0 & 0 & 0 & s & 0 \\ 0 & 0 & 0 & p & 0 & 0 & 0 & r \\ 0 & 0 & 0 & q & 0 & 0 & 0 & s \end{bmatrix}$$

(2 marks)

d. The transform is applied only on the top half. The bottom half (high pass) doesn't change.

$$T2=\begin{bmatrix} p & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p & q & 0 & 0 & 0 & 0 \\ r & s & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y1 \\ y2 \\ y3 \\ y4 \\ y5 \\ y6 \\ y7 \\ y8 \end{bmatrix}$$

(2 marks)

T2 . T1 gives the overall transform **(1 mark)**

(3)

e. Derive the transform matrix for N number of dyadic decompositions. N depends on the length of the signal

T= (TN).....(T2)(T1)

(1 mark)

Choose appropriate thresholds for each of the high frequency subabnds

All coeffiecints with magnitudes smalle than the relevant threshold set to zero.

(1 mark)

Then compute the inverse transform (computed using the transpose of T)

(1 mark)

(3)

Part B

Q4 a. (6 marks)

i)
Mean: (1+2+4+3+5)/5=3
(**1 mark**)
Variance: ((1-3)^2+(2-3)^2+(4-3)^2+(3-3)^2+(5-3)^2)/5=2
(**1 mark**)
Mean-square: ((1)^2+(2)^2+(4)^2+(3)^2+(5)^2)/5=11
(**1 mark**)

ii)
The variance $\sigma_x^2(n)$, mean-square $E[x^2(n)]$ and the mean $m_x(n)$:

$$\sigma_x^2(n) = E[(x(n) - m_x(n))^2]$$
$$= E[x^2(n) - x(n)m_x(n) - x(n)m_x(n) + m_x^2(n)]$$
$$= E[x^2(n)] - 2E[x(n)]m_x(n) + m_x^2(n)$$
$$= E[x^2(n)] - 2m_x^2(n) + m_x^2(n) = E[x^2(n)] - m_x^2(n)$$

(**2 marks**)
11-3^2=2, which verifies the above general result.
(**1 mark**)

Q4 b. (**3 marks**)

For cosine wave input, the dynamic range R_D of the quantiser can be calculated from the equation in Section 7.5.2 since sine wave and cosine wave have the same power given the same amplitude.
Then, for a 10-bit A/D converter (M=10):

R_D=1.76+6M dB=1.76+6*10=61.76dB,

(**3 marks**)

Q4 c. (**6 marks**)

Consider an impinging complex plane wave $e^{j\omega t}$ with a frequency ω and direction of arrival (DOA) angle θ, where the angle θ is measured with respect to the broadside of the linear array.
(**1 mark**)

For convenience, we assume the phase of the signal is zero at the first sensor. Then the signal received by the first sensor is $x_0(t) = e^{j\omega t}$ and $x_m(t) = e^{j\omega(t-m\Delta)}$), m = 1, 2, …, M-1, where mΔ is the propagation delay for the signal from sensor 0 to sensor m and it is a function of θ, with Δ=dsinθ /c, where c is the speed of the signal.
(**2 marks**)
Then the beamformer output is

$$y(t) = \sum_{m=0}^{M-1} w_m^* e^{j\omega(t-m\Delta)} = e^{j\omega t} \sum_{m=0}^{M-1} w_m^* e^{-jm\omega\Delta}$$

(**1 mark**)

Therefore, the response of the beamformer is given by

$$p(\omega,\theta) = \sum_{m=0}^{M-1} w_m^* e^{-jm\omega\Delta} = \sum_{m=0}^{M-1} w_m^* e^{-jm\omega\frac{d\sin\theta}{c}} = \sum_{m=0}^{M-1} w_m^* e^{-j2m\pi\frac{d}{\lambda}\sin\theta}$$

(**1 mark**)

For d=λ/2, we have

$$p(\omega,\theta) = \sum_{m=0}^{M-1} w_m^* e^{-jm\pi\sin\theta}$$

(**1 mark**)

Q5 a. (6 marks)

There are four zeros for $S_{yy}(z)$: ½, 3, 2, 1/3. So $S_{yy}(z)$ can be formed by passing a zero-mean white signal through four possible filters:

$H_0(z)=(z-1/2)(z-3)$

$H_1(z)=(z-1/2)(z-1/3)$

$H_2(z)=(z-2)(z-3)$

$H_3(z)=(z-2)(z-1/3)$

The inverse of any of the above four filters will whiten the signal y(n).

(**4 marks, 1 mark for each result**)

The inverse of $H_1(z)$ will be the one with minimum phase since all of its zeros are inside the unit circle.

(**2 marks**)

Q5 b. (4 marks)

i) $H_1(z)=2-3z^{-1}$

z-transform of the autocorrelation at the output

$$S_{y_1y_1}(z) = H_1(z)\ H_1(z^{-1})\ \sigma_x^2$$

$=(2-3z^{-1})(2-3z)*1=4-6z^{-1}-6z+9=-6z+13-6z^{-1}$

(**2 marks**)

Inverse z-transform by inspection to give autocorrelation sequence:

$$\phi_{y_1y_1}(m) = Z^{-1}[\ S_{y_1y_1}(z)\ ]$$

Autocorrelation sequence: -6 for m=-1, 13 for m=0, -6 for m=1 and zero for other values of m

(**2 marks**)

Q5 c. (5 marks)

The updated equation of the LMS algorithm is given by

$\mathbf{h}(n)=\mathbf{h}(n-1)+2\mu\mathbf{y}(n)e(n)$

where μ is the stepsize.
(**1 mark**)

e(11)=x(11)-$\mathbf{h}^T(10)\mathbf{y}(11)$=-0.2-[1 6][0.3 0.25]^T
=-2
(**2 marks**)

The impulse response is then updated by
$\mathbf{h}(15)=\mathbf{h}(14)+2\mu\mathbf{y}(15)e(15)$
=[1 6]^T+0.2*(-2)*[0.3 0.25]^T
=[0.88   5.9]^T
(**2 marks**)

Q6 a. (4 marks)

Two random processes are uncorrelated if
E[x(n)y(k)]=E[x(n)]E[y(k)],
where E is the expectation operation.
(**1 mark**)

Two random processes are independent if
p(x(n),y(k))=p(x(n))p(y(k)),
where p(x(n),y(k)) is the joint probability density function.
(**1 mark**)

Since the two random processes are independent, then from p(x(n),y(k))=p(x(n))p(y(k)), we have

$$E[x(n)y(k)] = \iint x(n)y(k)p(x(n),y(k))dx(n)dy(k)$$
$$= \iint x(n)y(k)p(x(n))p(y(k))dx(n)dy(k)$$
$$= \int x(n)p(x(n))dx(n)\int y(k)p(y(k))dy(k) = E[x(n)]E[y(k)]$$

(**2 marks**)

Q6 b. (11 marks)

i)

$e(n) = x(n) - \hat{x}(n)$
The mean-square error (MSE) cost function
$$\xi(n) = E[\ e^2(n)\ ]$$
(**1 mark**)

$$\hat{x}(n) = \sum_{i=0}^{N-1} h_i \, y(n-i)$$

$$= [\, h_0 \; h_1 \cdots h_{N-1} \,] \begin{bmatrix} y(n) \\ y(n-1) \\ . \\ . \\ . \\ y(n-N+1) \end{bmatrix}$$

$$= \mathbf{h}^T \, \mathbf{y}(n) = \mathbf{y}^T(n) \, \mathbf{h}$$

(2 marks)

Differentiate

$$\frac{\partial \xi}{\partial h_j} = \frac{\partial}{\partial h_j} E[\, \{ e^2(n) \} \,]$$

$$= E[\, \frac{\partial}{\partial h_j} \{ e^2(n) \} \,]$$

$$= E[\, 2\, e(n) \, \frac{\partial e(n)}{\partial h_j} \,]$$

$$= E[\, 2\, e(n) \, \frac{\partial}{\partial h_j} \{ x(n) - \mathbf{h}^T \, \mathbf{y}(n) \} \,]$$

$$= E[\, 2\, e(n) \, \frac{\partial}{\partial h_j} \{ -h_j \, y(n-j) \} \,]$$

$$= E[\, 2\, e(n) \, y(n-j) \} \,]$$

$$= 0$$

for j=0, 1, …, N-1.
(**2 marks**)

In vector form, the gradient is given by

$$\underline{\nabla} = -2\ E[\ \mathbf{y}(n)\ e(n)\ ]$$

$$= -2\ E[\ \mathbf{y}(n)\ (\ x(n) - \mathbf{y}^T(n)\ \mathbf{h}\ )\ ]$$

$$= -2\ E[\ \mathbf{y}(n)\ x(n)\ ] + 2\ E[\ \mathbf{y}(n)\mathbf{y}^T(n)\ ]\ \mathbf{h}$$

$$= -2\ \Phi_{yx} + 2\ \Phi_{yy}\ \mathbf{h}$$

$$= \underline{0}$$

(**2 marks**)
where

Autocorrelation matrix

$$\Phi_{yy} = E[\ \mathbf{y}(n)\ \mathbf{y}^T(n)\ ]$$

Cross-correlation vector

$$\Phi_{yx} = E[\ \mathbf{y}(n)\ x(n)\ ]$$

(1 mark)

Optimal Solution

$$\Phi_{yy}\ \mathbf{h}_{opt} = \Phi_{yx}$$

Alternative formulation

$$\mathbf{h}_{opt} = \Phi_{yy}^{-1}\ \Phi_{yx}$$

(**1 mark**)

ii)

Method of Steepest Descent

$$\mathbf{h}_{i+1} = \mathbf{h}_i - \mu\ \underline{\nabla}_i$$

$$\underline{\nabla}_i = \left[\ \frac{\partial \xi}{\partial h_0}\ \frac{\partial \xi}{\partial h_1}\ \cdot\ \cdot\ \frac{\partial \xi}{\partial h_{N-1}}\ \right]^T_{\underline{h} = \mathbf{h}_i}$$

$$= 2\ \Phi_{yy}\ \mathbf{h}_i - 2\ \Phi_{yx}$$

where μ is the step size for this update.
(**1 mark**)

The Method of Steepest Descent is a way to calculate the Wiener solution without the need of matrix inversion. It starts from an arbitrary initial guess of the weight vector. First we calculate the gradient of the cost function at this point and then move to the negative direction of this gradient by a small amount, which will be closer to the Wiener solution. Repeat this process, and as long as the step size is small enough, we will be able to reach the optimum point.
(**1 mark**)