Tutorial 5: Solutions

1. Prove mathematically that convolution is
(i) a commutative operation, i.e, $x(t)*h(t) = h(t)*x(t)$.

Start with the convolution integral $x(t)*h(t)=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau$.

Let $\lambda = t-\tau$, $d\lambda = -d\tau$. We have

$$x(t)*h(t)=\int_{\infty}^{-\infty}x(t-\lambda)h(\lambda)(-d\lambda)=\int_{-\infty}^{\infty}h(\lambda)x(t-\lambda)d\lambda = h(t)*x(t).$$

Note that $\int_{b}^{a}x(t)dt=-\int_{a}^{b}x(t)dt$

Therefore convolution is a commutative operation.

(ii) an associative operation, i.e, $(x(t)*h(t))*g(t) = x(t)*(h(t)*g(t))$.

LHS $=(x(t)*h(t))*g(t)=\left[\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau\right]*g(t)$.

$$= g(t)*\left[\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau\right]$$

We need to introduce another constant to replace t to perform the second convolution step

$$(x(t)*h(t))*g(t)=\int_{-\infty}^{\infty}g(\sigma)\left[\int_{-\infty}^{\infty}x(\tau)h(t-\sigma-\tau)d\tau\right]d\sigma$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\sigma)x(\tau)h(t-\sigma-\tau)d\tau d\sigma$$

RHS $=x(t)*(h(t)*g(t))=x(t)*(g(t)*h(t))$.

$$=x(t)*\int_{-\infty}^{\infty}g(\sigma)h(t-\sigma)d\sigma=\int_{-\infty}^{\infty}x(\tau)\left[\int_{-\infty}^{\infty}g(\sigma)h(t-\tau-\sigma)d\sigma\right]d\tau$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\sigma)x(\tau)h(t-\sigma-\tau)d\tau d\sigma = \text{LHS}.$$

Therefore convolution is an associative operation.

(iii) a distributive operation, i.e, $x(t)*(h(t) + g(t)) = x(t)*h(t) + x(t)*g(t)$.

$$x(t)*(h(t)+g(t))=\int_{-\infty}^{\infty}x(\tau)[h(t-\tau)+g(t-\tau)]d\tau$$

$$=\int_{-\infty}^{\infty}x(\tau)h(t-\tau)d\tau+\int_{-\infty}^{\infty}x(\tau)g(t-\tau)d\tau = x(t)*h(t)+x(t)*g(t).$$

2. An *RC* high-pass circuit has a step response $g(t)=u(t)\exp(-t/RC)$. Sketch and derive an equation for the impulse response.

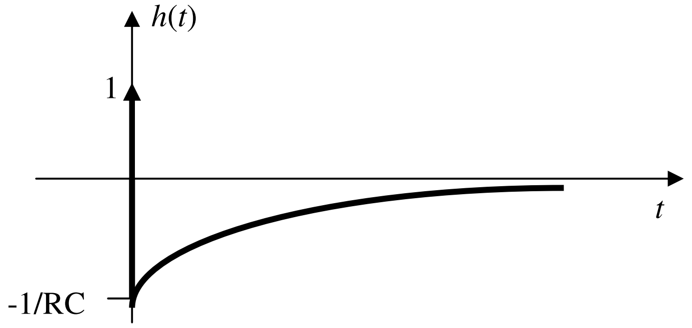
We know that impulse response $=\frac{d}{dt}$(step response).

Therefore the impulse response

$$h(t)=\frac{d}{dt}[g(t)]=\frac{d}{dt}[u(t)\exp(-t/RC)]$$

$$=\exp(-t/RC)\frac{d}{dt}[u(t)]+u(t)\frac{d}{dt}[\exp(-t/RC)]$$

$$=\exp(-t/RC)\delta(t)+u(t)\left[-\frac{1}{RC}\exp(-t/RC)\right]=\delta(t)\exp(-t/RC)-\frac{u(t)}{RC}\exp(-t/RC).$$



3. A system has an impulse response $h(t)=\exp(-t)u(t)$. Find the step response of this system.

The step response is

$$s(t)=\int_{-\infty}^{t}h(\tau)d\tau=\int_{-\infty}^{t}\exp(-\tau)u(\tau)d\tau=\int_{0}^{t}\exp(-\tau)d\tau=-\exp(-\tau)\Big|_{o}^{t}=1-\exp(-t).$$

Alternatively we can also use the convolution technique to compute the step response as follows

$$s(t)=h(t)*u(t)=\int_{-\infty}^{\infty}h(\tau)u(t-\tau)d\tau=\int_{-\infty}^{\infty}\exp(-\tau)u(\tau)u(t-\tau)d\tau.$$

Since $u(\tau)u(t-\tau)$ only has value between 0 and t we have

$$s(t)=\int_{0}^{t}\exp(-\tau)d\tau=-\exp(-\tau)\Big|_{0}^{t}=1-\exp(-t)$$

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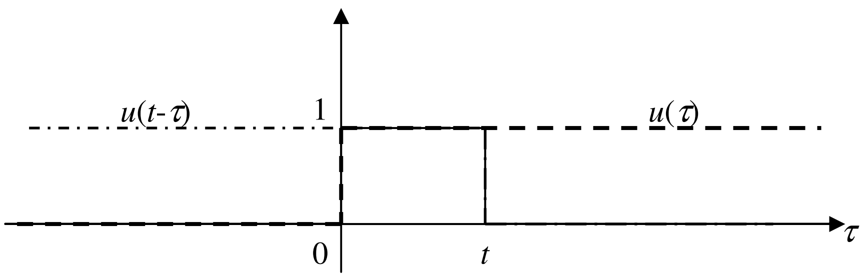


4. Compute and sketch $y[n]=x[n]*z[n]$ where:

$x[n]$ =1,-1,2 for n =0,1,2

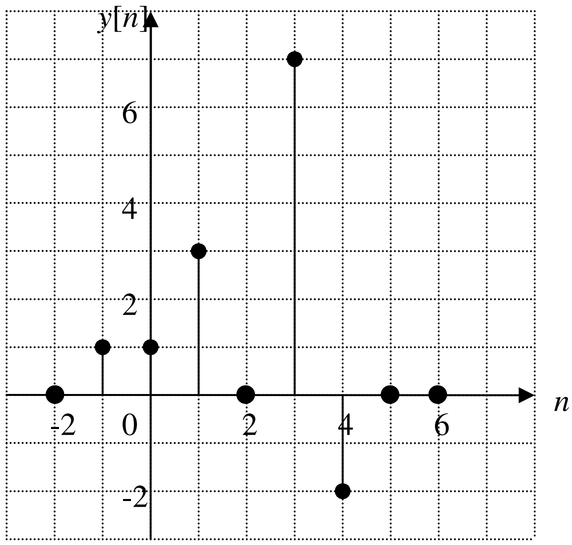
$z[n]$ = 1,2,3,-1 for n = -1,0,1,2

assume that each signal is zero elsewhere.

We can compute $y[n]$ using a table as follows

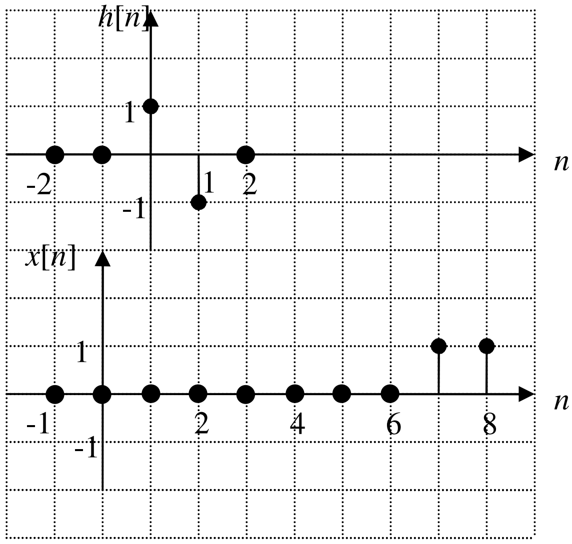
	k	-3	-2	-1	0	1	2	3	4	5
	$x[k]$	0	0	0	1	-1	2	0	0	0
n =-1	$z[-1-k]$	-1	3	2	1	0	0	0	0	0
n = 0	$z[-k]$	0	-1	3	2	1	0	0	0	0
n = 1	$z[1-k]$	0	0	-1	3	2	1	0	0	0
n = 2	$z[2-k]$	0	0		-1	3	2	1	0	0
n = 3	$z[3-k]$	0	0	0	0	-1	3	2	1	0
n = 4	$z[4-k]$	0	0	0	0	0	-1	3	2	1
n = 5	$z[5-k]$	0	0	0	0	0	0	-1	3	2

	$y[n] = \sum x[k]z[n-k]$
n = -1	$1\times1=1$
n = 0	$(2\times1)+(1\times(-1))=1$
n = 1	$(3\times1)+(2\times(-1))+(1\times2)=3$
n = 2	$((-1)\times1)+(3\times(-1))+(2\times2)=0$
n = 3	$((-1)\times(-1))+(3\times2)=7$
n = 4	$((-1)\times2=-2$
n = 5	0



$y[n] = x[n] * z[n]$.

5. The impulse response of a system is given by $h[n]=-\delta[n-1]+\delta[n]$. By considering the input signal $x[n] = u[n-7]$, show that the system acts as an edge detector.

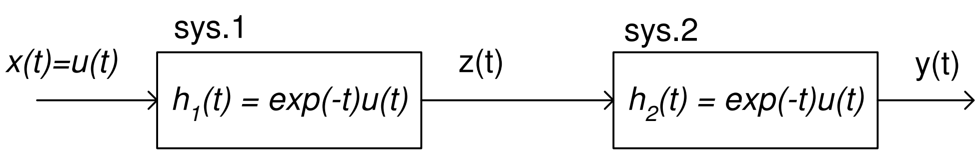


The response of the system can be obtained by performing a convolution between $x[n]$ and $h[n]$ as below:

	k	3	4	5	6	7	8	9	$y[n] = \sum x[k]h[n-k]$
	$x[k]$	0	0	0	0	1	1	1	
$n = 6$	$h[6-k]$	0	0	-1	1	0	0	0	0
$n = 7$	$h[7-k]$	0	0	0	-1	1	0	0	$1\times1 = 1$
$n = 8$	$h[8-k]$	0	0	0	0	-1	1	0	$(-1\times1) + (1\times1) = 0$
$n = 9$	$h[9-k]$	0	0	0	0	0	-1	1	$(-1\times1) + (1\times1) = 0$

$y[n] = x[n]*h[n]$ is zero everywhere except when $n = 7$. This shows that the system acts as an edge detector as it only has value at $n = 7$.

6. Find the output $y(t)$ for the system shown below when a unit-step input, $u(t)$ is applied.



$$z(t) = h_1(t) * x(t) = \int_{-\infty}^{\infty} h_1(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} \exp(-\tau)u(\tau)u(t-\tau)d\tau$$

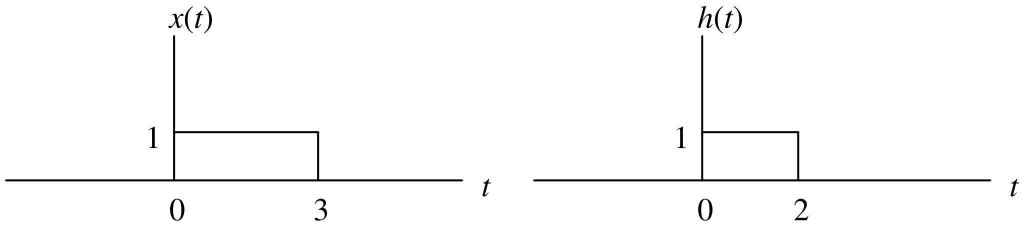
$$= \int_{0}^{t} \exp(-\tau)d\tau = 1-\exp(-t)\text{, for } t \geq 0 \text{ or } [1-\exp(-t)]u(t).$$

$$y(t) = h_2(t) * z(t) = \int_{-\infty}^{\infty} h_2(\tau)z(t-\tau)d\tau = \int_{-\infty}^{\infty} \exp(-\tau)u(\tau)[1-\exp(-(t-\tau))]u(t-\tau)d\tau$$

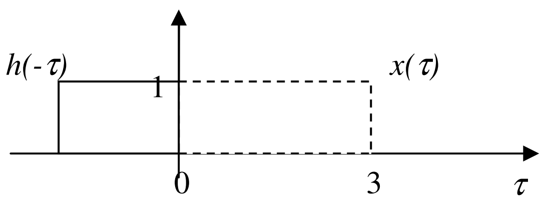
$$= \int_0^t \exp(-\tau)[1-\exp(-(t-\tau)]d\tau = \int_0^t [\exp(-\tau)-\exp(-t)]d\tau$$

$$= -\exp(-\tau)\Big|_0^t - \tau\exp(-t)\Big|_0^t = 1-\exp(-t)-t\exp(-t) = 1-\exp(-t)(1+t).$$

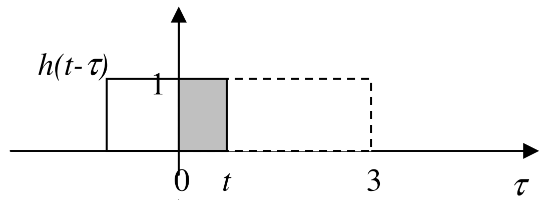
7. Consider the signals $x(t)$ and $h(t)$ shown below. Compute $y(t) = x(t)*h(t)$ using (i) the graphical method (ii) the analytical method and write down the analytical expressions for $y(t)$.



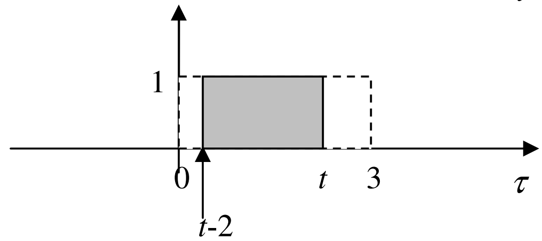
(i) Graphical method



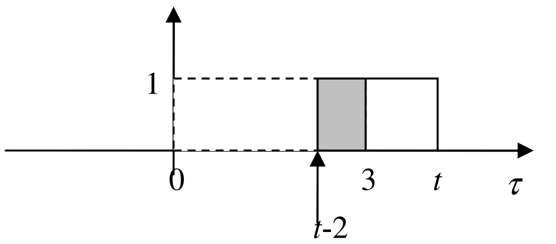
Interval I: For $t \le 0$, no area overlap, $y(t) = 0$.



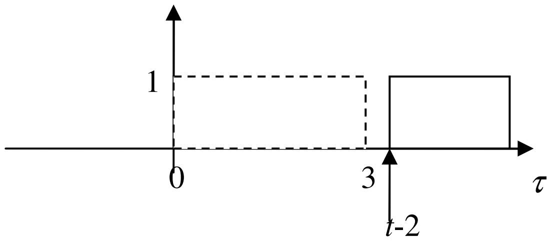
Interval II: For $0 < t \le 2$ shaded area = $1 \times t = t$, $y(t) = t$.



Interval III: For $2 < t \le 3$, shaded area = $1 \times 2 = 2$, $y(t) = 2$.



Interval IV: For $3 < t \le 5$, shaded area = $1\times(3-(t-2)) = 5-t$, $y(t) = 5-t$.



Interval V: For $t > 5$, no area overlap, $y(t) = 0$.

In summary $y(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t \le 2 \\ 2 & 2 < t \le 3 \\ 5-t & 3 < t \le 5 \\ 0 & t > 5 \end{cases}$.

(ii) Analytical method

Consider the following intervals:

Interval I: For $t \le 0$, $x(\tau)h(t-\tau) = 0$, $y(t) = 0$.

Interval II: For $0 < t \le 2$, $x(\tau)h(t-\tau) = 1$, $y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t 1 d\tau = t$.

Interval III: For $2 < t \le 3$, $x(\tau)h(t-\tau) = 1$,

$$y(t) = \int_{t-2}^{t} x(\tau)h(t-\tau)d\tau = \int_{t-2}^{t} 1 d\tau = t-(t-2) = 2 \ .$$

Interval IV: For $3 < t \le 5$, $x(\tau)h(t-\tau) = 1$,

$$y(t) = \int_{t-2}^{3} x(\tau)h(t-\tau)d\tau = \int_{t-2}^{3} 1 d\tau = 3-(t-2) = 5-t \ .$$

Note that the upper integration limit is 3 as shown in the diagram above.

Interval V: For $3 < t \le 5$, $x(\tau)h(t-\tau) = 0$, $y(t) = 0$.

In summary $y(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t \le 2 \\ 2 & 2 < t \le 3 \\ 5-t & 3 < t \le 5 \\ 0 & t > 5 \end{cases}$.

8. Consider a signal $y[n] = 3x[n] + x[n\text{-}2]$. Obtain the impulse response and evaluate the response of the system to an input

$$x_1[n] = \begin{cases} 1 & n = 0 \\ 1 & n = 1 \\ 2 & n = 2 \\ 0 & otherwise \end{cases}.$$

To obtain the impulse response $h[n]$ substituting $x[n] = \delta[n]$ gives
$h[n] = 3\delta[n] + \delta[n-2]$ or

$$h[n] = \begin{cases} 3 & n = 0 \\ 0 & n = 1 \\ 1 & n = 2 \\ 0 & otherwise \end{cases}.$$

To compute the response due to $x_1[n]$, express $x_1[n]$ as a sum of weighted impulses, i.e
$x_1[n] = \delta[n] + \delta[n-1] + 2\delta[n-2]$.

Now the response is $y_1[n] = h[n] + h[n-1] + 2h[n-2]$

$n = 0$: $y_1[0] = h[0] + h[-1] + h[-2] = 3$

$n = 1$: $y_1[1] = h[1] + h[0] + 2h[-1] = 3$

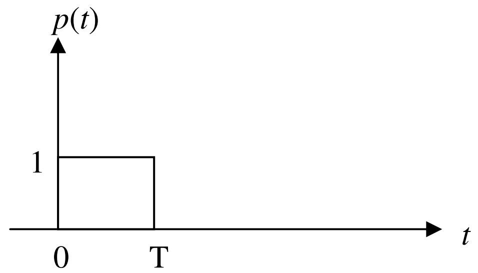
$n = 2$: $y_1[2] = h[2] + h[1] + 2h[0] = 1 + 6 = 7$

$n = 3$: $y_1[3] = h[3] + h[2] + 2h[1] = 1$

$n = 4$: $y_1[4] = h[4] + h[3] + 2h[2] = 2$

$y_1[n]$ can also be obtained using technique in Q8 and Q9.

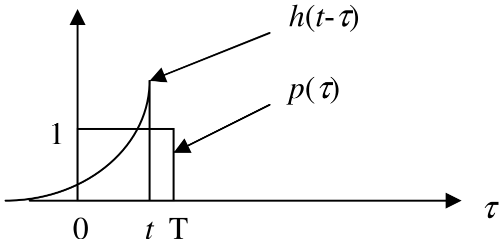
9. The impulse response of the RC circuit shown below is given by
$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$. Derive the expression for the response of the circuit to the signal $p(t)$ shown below. Sketch and label the response signal.



The response is $y(t) = p(t) * h(t) = \int_{-\infty}^{\infty} p(\tau)h(t-\tau)d\tau$.

For $t < 0$, $p(\tau)h(t\text{-}\tau) = 0$.

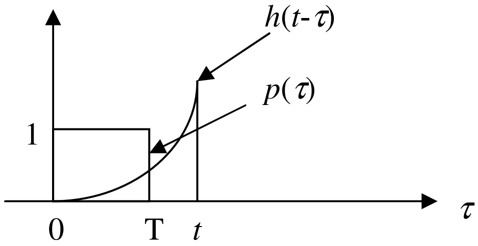
For $0 < t < T$,



We need to integrate from 0 to t.

$$y(t)=\int_{-\infty}^{\infty}p(\tau)h(t-\tau)d\tau=\int_{0}^{t}\frac{1}{RC}e^{-(t-\tau)/RC}d\tau=\frac{RC}{RC}\left[e^{-(t-\tau)/RC}\right]_{0}^{t}=1-e^{-t/RC}$$

For $t \geq T$,

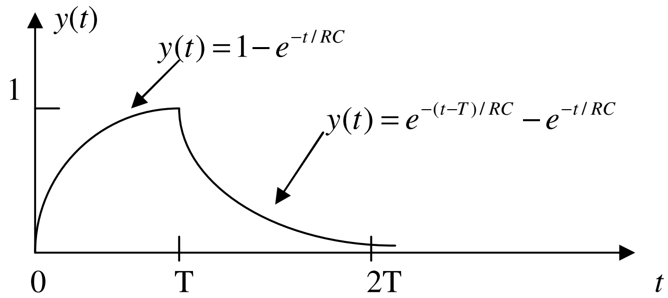


We need to integrate from 0 to T.

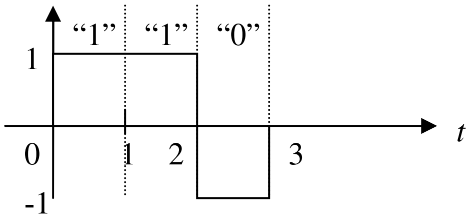
$$y(t)=\int_{0}^{T}\frac{1}{RC}e^{-(t-\tau)/RC}d\tau=\frac{RC}{RC}\left[e^{-(t-\tau)/RC}\right]_{0}^{T}=e^{-(t-T)/RC}-e^{-t/RC}$$

Therefore we have

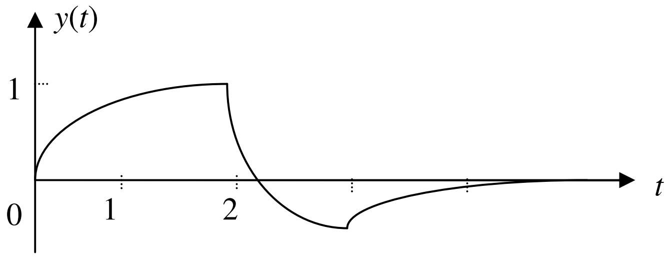
$$y(t)=\begin{cases}0 & t<0\\ 1-e^{-t/RC} & 0<t<T\\ e^{-(t-T)/RC}-e^{-t/RC} & t\geq T\end{cases}$$



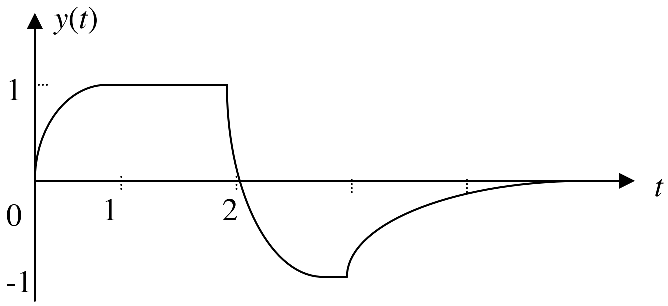
10. Consider an LTI digital communication system, in which a bit "1" is represented by $p(t)$ in Q.13 and a bit "0" is represented by $-p(t)$. Evaluate the response of the circuit for a sequence "110" for cases where $T = 1/RC$ and $T = 1/(5RC)$. Hence comment how the intersymbol interference (ISI) of this digital communication system is affected by T.
[You may assume $T = 1$s]
The sequence "110" is represented by $p(t) + p(t\text{-}1) - p(t\text{-}2)$ as shown below.



Therefore the response is $y(t) + y(t\text{-}1) - y(t\text{-}2)$.
For $T = 1$ and $RC = 1$, we have



For $T = 1$ and $RC = 1/5$, we have



Therefore we can see that the inter-symbol interference (ISI) is more severe when the pulse width, T is comparable to RC. To minimise ISI it is important to make sure that $T >> RC$, i.e $h(t)$ is much narrower than $p(t)$. If $RC > T$, the bits will overlap making it difficult to differentiate between 1 and 0.