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Lecture content

- Laplace Transform
  - Laplace Transform pairs and properties
  - Transfer Function
  - Transform Impedance
  - First order systems
  - Second order systems

Transfer Function Example 1

$$Y(s) = X(s)H(s)$$

The function $H(s) = Y(s)/X(s)$ is known as the **transfer function**

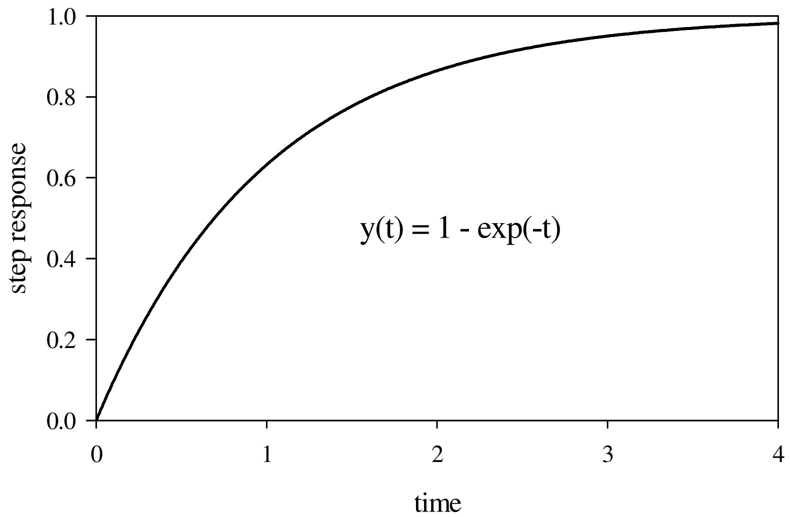
1. Consider a system with a step response given by

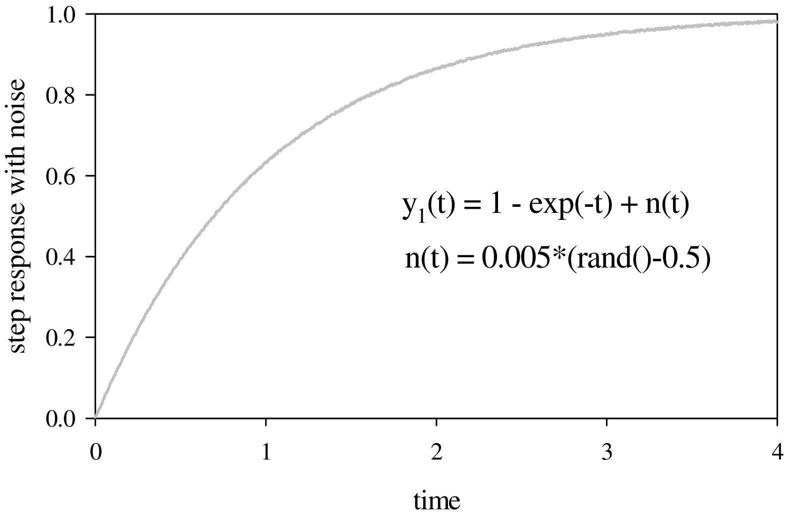
$$y(t) = 1 - e^{-t}u(t).$$

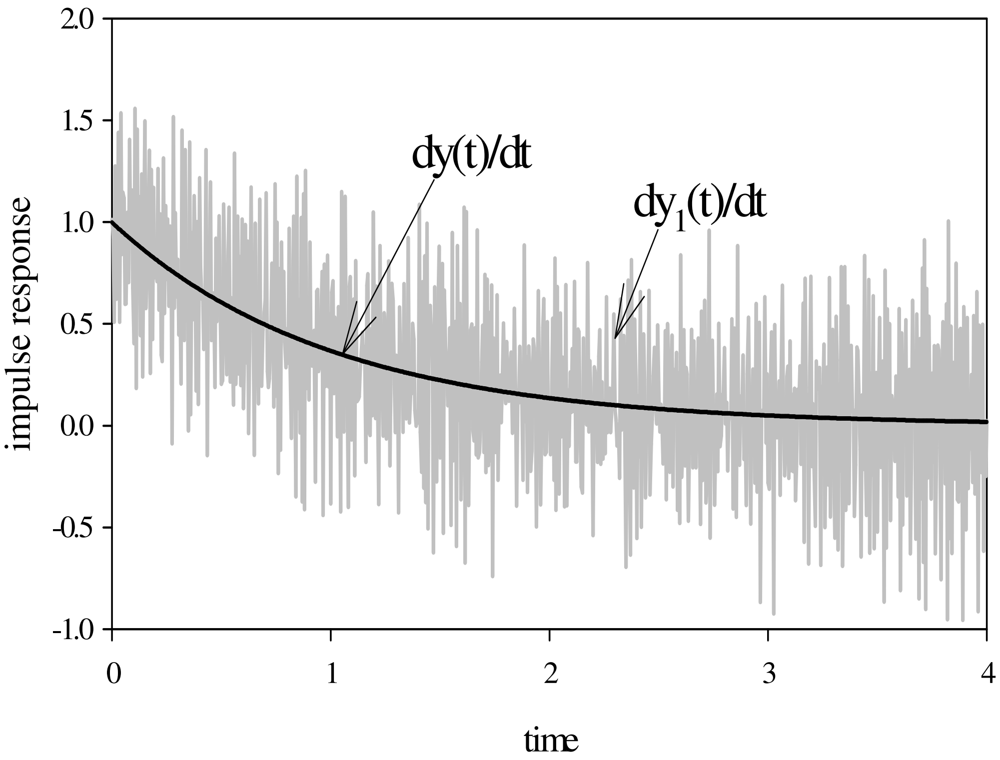
In theory the impulse response can be obtained as $h(t) = dy(t)/dt = e^{-t}u(t)$.

However the differentiation process is not desirable in practice because of high frequency noise as illustrated in figure 3

Transfer Function Example 1







Transfer Function Example 1

Alternatively we can use Laplace Transform to evaluate the impulse response. The Laplace Transform of *y*(*t*) is

$$y(t) = 1 - e^{-t}u(t)$$

$$Y(s) = \frac{1}{s} - \frac{1}{s+1} = \frac{(s+1)-s}{s(s+1)} = \frac{1}{s(s+1)}$$

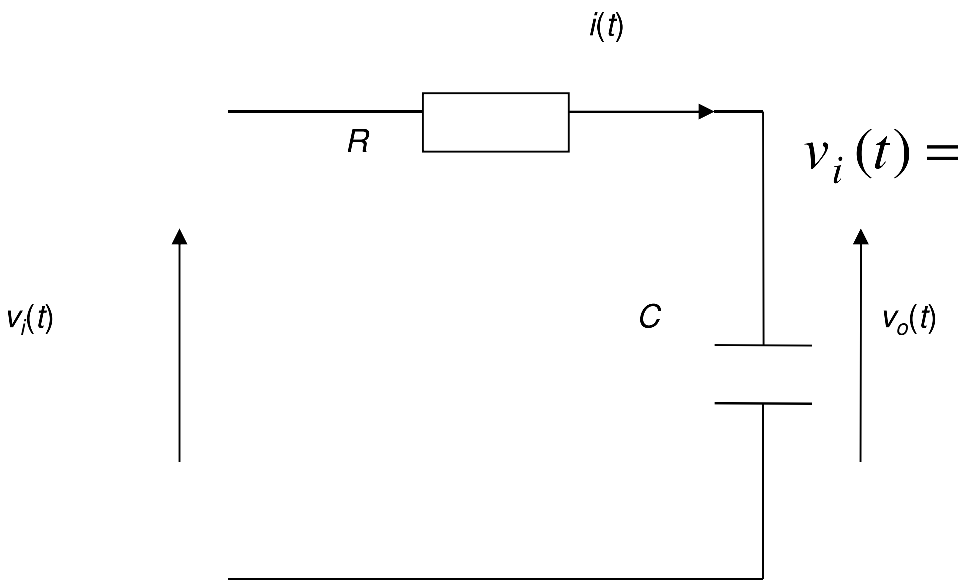
The Laplace Transform of x(t) is $X(s) = \frac{1}{s}$

Therefore we have

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\frac{1}{s(s+1)}}{\frac{1}{s}} = \frac{1}{s+1} \leftrightarrow h(t) = e^{-t}u(t)$$

Transfer Function Example 2

2. Consider an RC circuit shown in figure below with **zero initial condition**.



$$v_i(t) = i(t)R + v_o(t)$$

$$i(t) = C\frac{dv_o(t)}{dt}$$

$$v_i(t) = RC\frac{dv_o(t)}{dt} + v_o(t) = \tau\frac{dv_o(t)}{dt} + v_o(t)$$

$$\frac{dv_o(t)}{dt} \leftrightarrow sV_o(s) - v_o(0)$$

We can solve the differential equation to find $v_o(t)$ when $v_i(t) = u(t)$.

Transfer Function Example 2

Instead we will apply the Laplace Transfrom to the differential equation.

$$V_i(s) = \tau s V_o(s) + V_o(s) = (1 + s\tau)V_o(s)$$

$$V_o(s) = \frac{1}{1 + s\tau} V_i(s) = H(s)V_i(s)$$

If $v_i(t) = u(t)$, $V_i(s) = 1/s$ and we have,

$$V_o(s) = \frac{1}{s(1 + s\tau)} = \frac{a}{s(s + a)}$$

where $a = 1/\tau$.

$$V_o(s) = \frac{1}{s} - \frac{1}{(s + a)}$$

$v_o(t) = 1 - e^{-at}u(t) = 1 - e^{-t/\tau}u(t)$

Forced response

Natural system response

Transfer Function Example 2

The first order differential equation can be solved using the Laplace Transform.

The algebraic operations (addition and multiplication) used in the Laplace Transform are much simpler than the calculus operations (differentiation and integration) required in solving the differential equation.

$$v_i(t) = \tau \frac{dv_o(t)}{dt} + v_o(t)$$

$$V_i(s) = \tau s V_o(s) + V_o(s)$$

$$V_i(s) = (1 + s\tau) V_o(s)$$

Transform Impedance

$$v(t) = L\frac{di(t)}{dt} \qquad i(t) = C\frac{dv(t)}{dt} \qquad v(t) = i(t)R$$

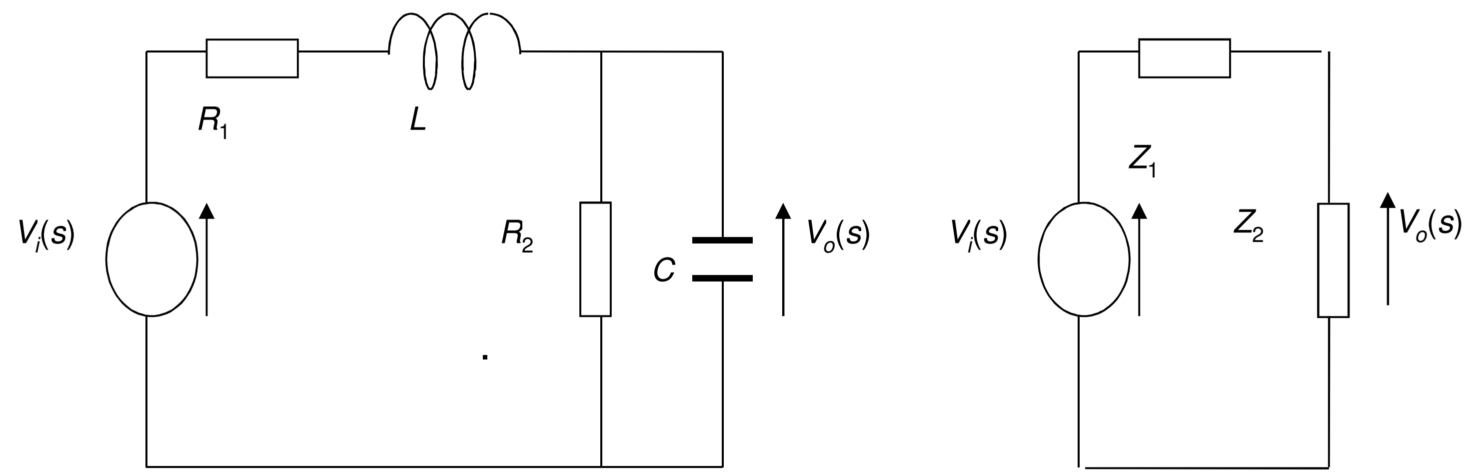
$$V(s) = LsI(s) \qquad I(s) = CsV(s) \qquad V(s) = I(s)R$$

$$\frac{V(s)}{I(s)} = sL \qquad \frac{V(s)}{I(s)} = \frac{1}{sC} \qquad \frac{V(s)}{I(s)} = R$$

$Z(s) = sL$, $Z(s) = 1/sC$ and $Z(s) = R$

Transform Impedance example

Compute the transfer function of the circuit shown below



$$Z_1 = R_1 + sL \text{ and } Z_2 = R_2 || 1/sC = \frac{R_2 / sC}{R_2 + 1/sC} = \frac{R_2}{1 + sR_2C}$$

Transform Impedance

The transfer function is given by

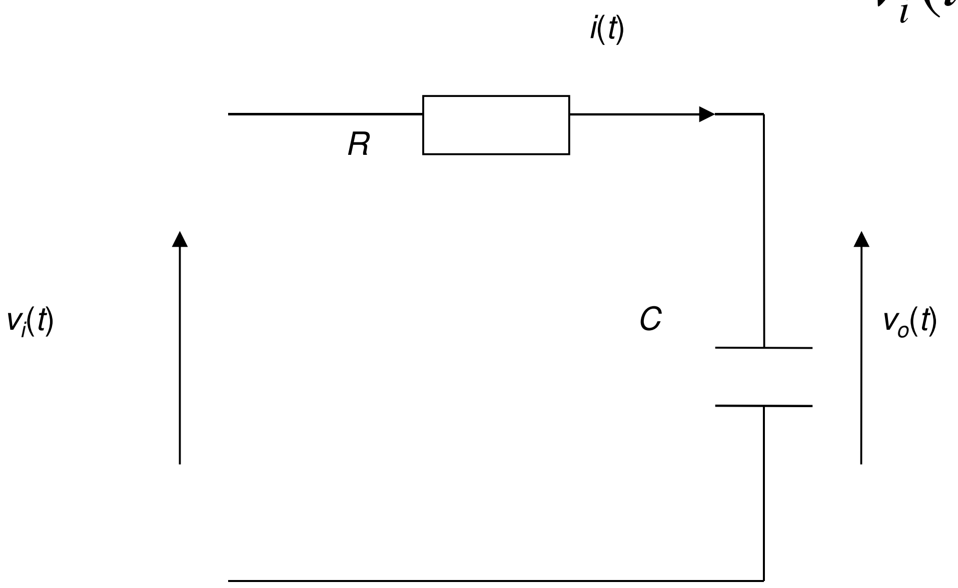
$$\frac{V_o(s)}{V_i(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{\dfrac{R_2}{1+sR_2C}}{R_1 + sL + \dfrac{R_2}{1+sR_2C}} = \frac{R_2}{R_2LCs^2 + (L + R_1R_2C)s + (R_1 + R_2)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{1/LC}{s^2 + \left((L + R_1R_2C)/R_2LC\right)s + (R_1 + R_2)/R_2LC}$$

$$H(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1^{st} order systems

The RC circuit shown below is an example of a 1^{st} order system.



$$v_i(t) = \tau \frac{dv_o(t)}{dt} + v_o(t)$$

The transfer function of a 1^{st} order system can be expressed as

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

Re{s} >-1/τ and the impulse response is

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

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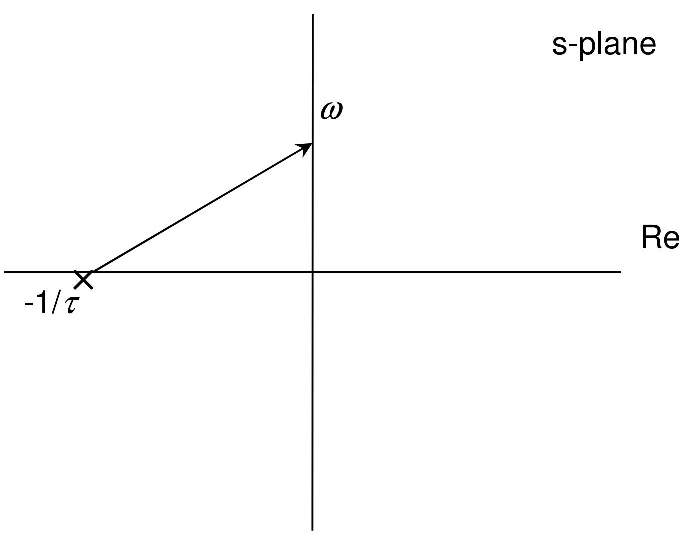
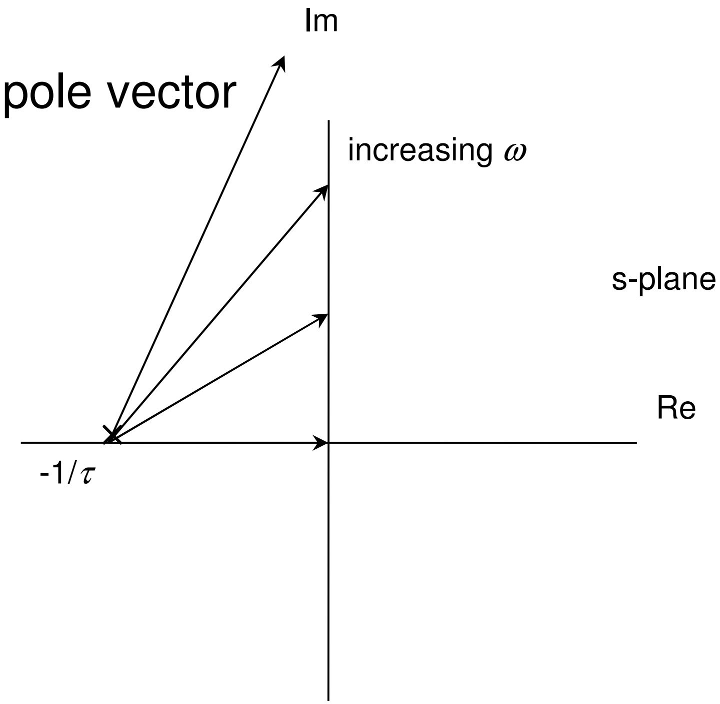
1^{st} order systems

The pole-zero plot is

$|s+1/\tau|$ = length of pole vector





The length of the pole vector corresponds to the magnitude of the denominator polynomial of $H(s)$ which is minimal for $\omega = 0$ and increases with ω. The angle of the pole increases from 0 to $\pi/2$ as ω increases from 0 to ∞.

1st order systems

Assuming that $s = j\omega$ we have,

$$H(j\omega) = H(\omega) = \frac{1/\tau}{j\omega + 1/\tau} = \frac{1}{1 + j\omega/\omega_c}$$

where $\omega_c = 1/\tau$.

For $\omega << \omega_c$, $|H(\omega)| \approx 1$ and

$$\angle H(\omega) \approx -\tan^{-1}(0) = 0$$

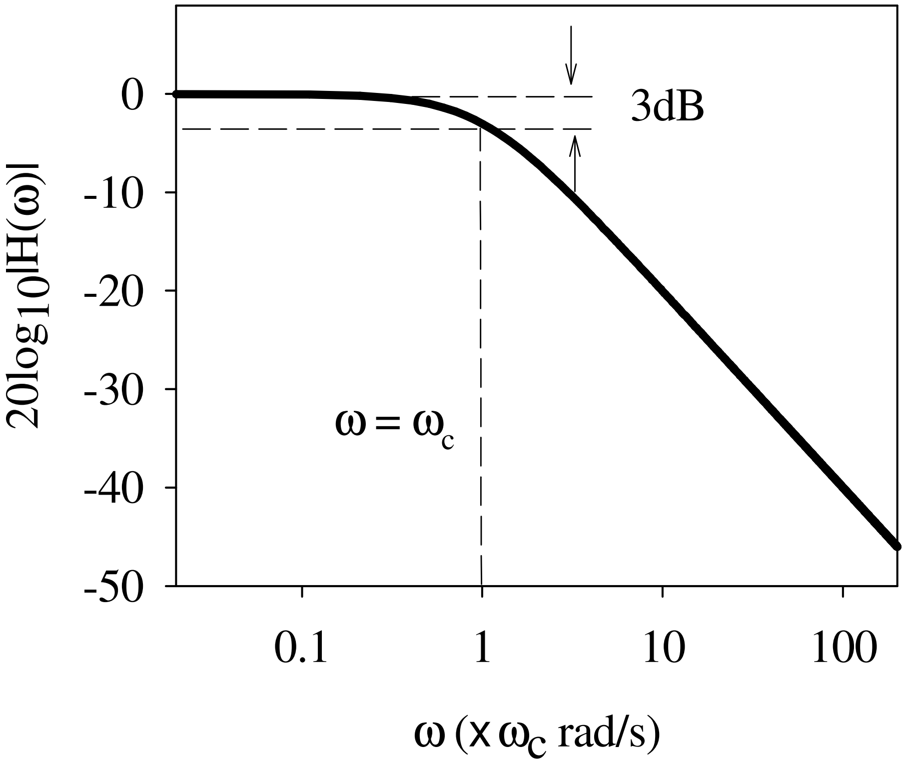
For $\omega = \omega_c$, $|H(\omega)| = \frac{1}{\sqrt{2}}$ and

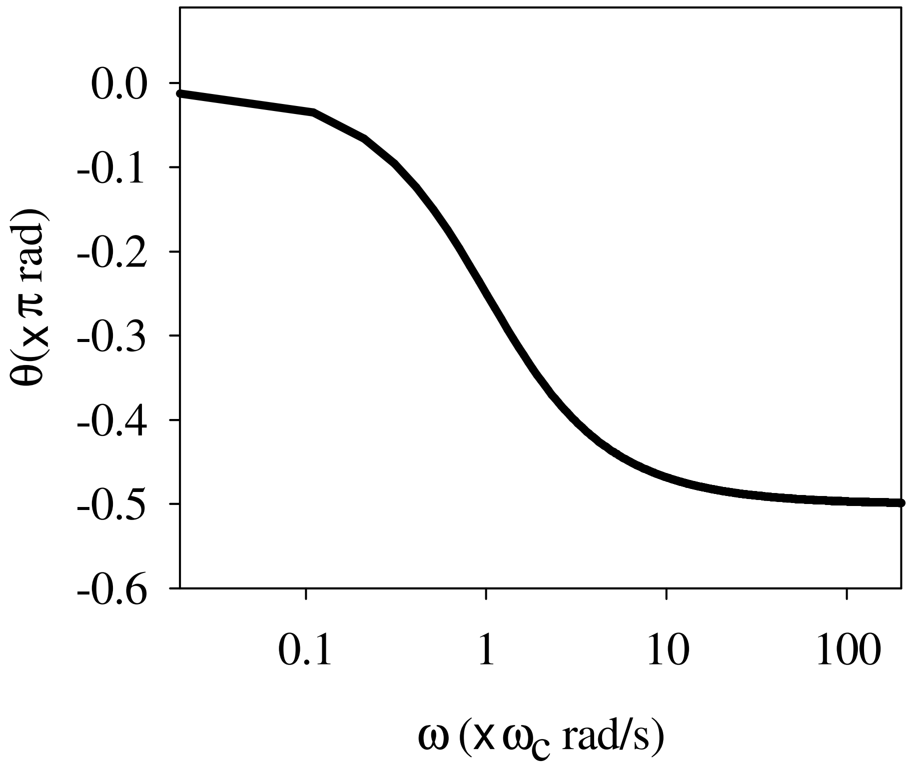
$$\angle H(\omega) = -\tan^{-1}(1) = -\pi/4$$

For $\omega >> \omega_c$, $|H(\omega)| \approx \frac{1}{\omega}$ and

$$\angle H(\omega) \approx -\tan^{-1}(\infty) = -\pi/2$$

1^{st} order systems





1^{st} order systems

Changing the time constant τ or equivalently changing the position of the pole $s = -1/\tau$ changes the characteristics of $H(s)$.

When τ is reduced the pole moves farther to the left hand plane corresponding to a larger cut-off frequency ω_c and a faster decay in the impulse response $h(t)$.

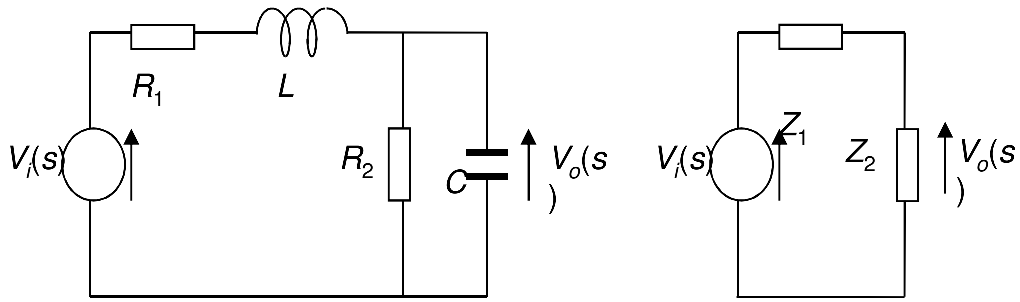
In general, if the poles are farther away from the $j\omega$-axis, the cut-off frequency is higher and the impulse response decays faster.

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

$$H(s) = \frac{1/\tau}{s + 1/\tau}$$

High pass filter has transfer function $H(s) = \dfrac{s + C}{s + B}$

2nd order systems



$$\frac{V_o(s)}{V_i(s)} = \frac{1/LC}{s^2 + \left((L + R_1 R_2 C)/R_2 LC\right)s + (R_1 + R_2)/R_2 LC}$$

The circuit above is an example of a second order system. The transfer function has a general form

$$H(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

2^{nd} order systems

$$H(s) = \frac{N(s)}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}}$$

$$Q = \frac{1}{2\zeta}$$

ω_n is the natural frequency of the system, ζ is the damping factor and $N(s)$ is the numerator polynomial with order less than or equal to that of the denominator polynomial.

2^{nd} order systems

Assuming that $N(s) = k$, $\omega_n > 0$ and $\zeta > 0$

$$H(s) = \frac{k}{s^2 + 2\zeta\omega_n s + \omega_n^{\ 2}} = \frac{k}{(s - p_1)(s - p_2)}$$

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$ are the poles

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

If $\zeta > 1$, the system will be non-oscillatory and is said to be overdamped. The poles are real but unequal.

If $\zeta = 0$, the system has no losses and the oscillation is undamped. The poles are imaginary but unequal and are given by

$$p_{1,2} = \pm j\omega_n$$

If $\zeta = 1$, the system is said to be critically damped with real and equal poles,

$$p_1 = p_2 = -\omega_n$$

If $0 < \zeta < 1$, the system will be oscillatory and is said to be underdamped. The poles cause $H(s) = \infty$, are complex conjugates and are given by

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$$