electrostatics with $\frac{\partial B}{\partial t} = 0$

$$\text{div}\, \underline{E} = \frac{\rho}{\varepsilon_0} \quad , \quad \text{rot}\, \underline{E} = -\frac{\partial B}{\partial t} = 0$$

define $\underline{E} = -\text{grad}\, V$ where V = electrical potential

$\Downarrow$

$$\text{div}\,(-\text{grad}\, V) = -\text{div}\,\text{grad}\, V = -\nabla^2 V$$

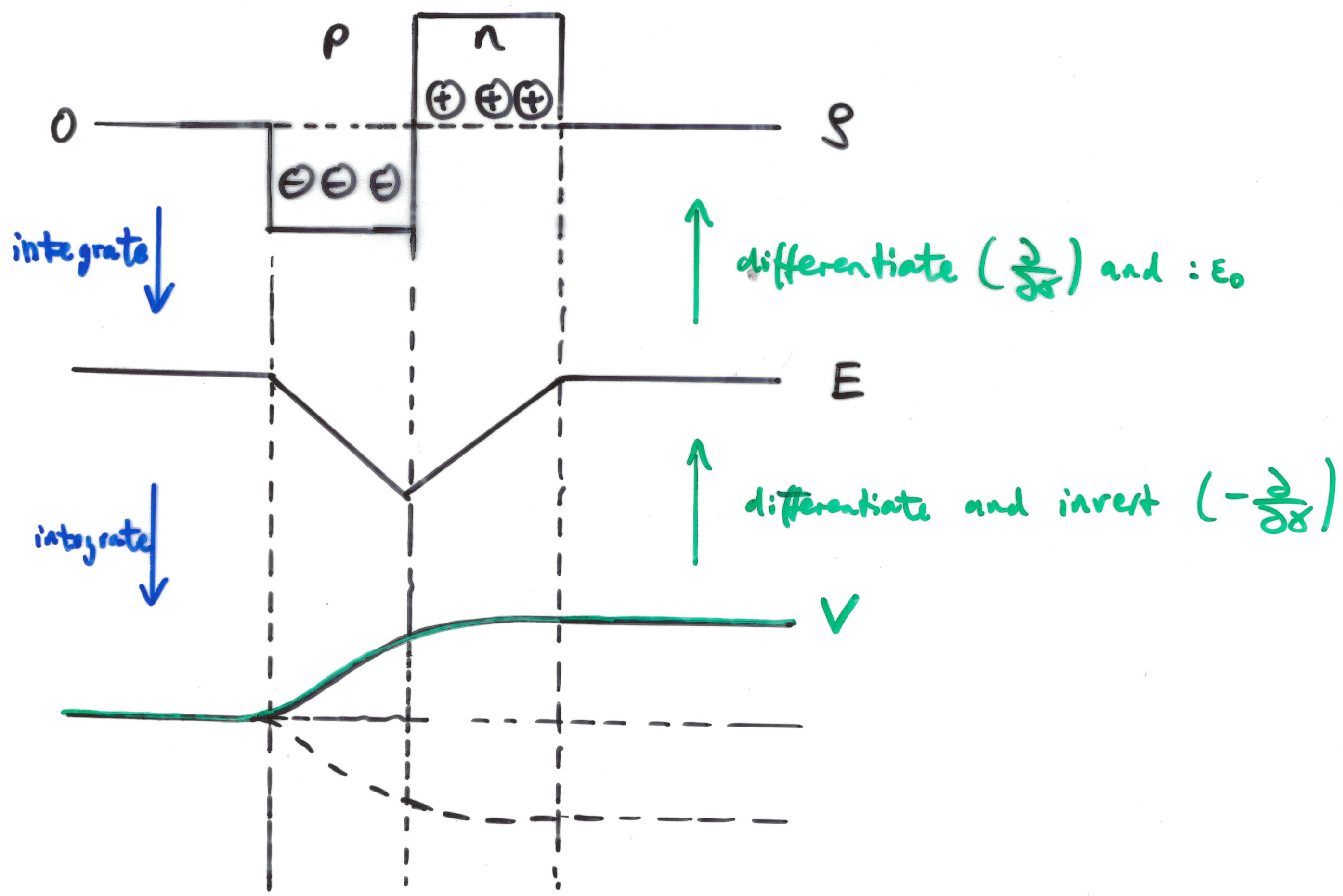
$\Downarrow$

$$\boxed{\nabla^2 V = -\frac{\rho}{\varepsilon_0}}$$ Poisson's equation

without charges, charge density $\rho = 0$, hence $\nabla^2 V = 0$
is called Laplace's equation
Poisson's equation is analytically solvable for simple geometries.

1. example: 1D-section through a p-n-diode (say, x-direction)

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\rho}{\varepsilon_0}$$



relationship

Coulomb's Law $\quad \operatorname{div} \underline{D} = \rho_{free}$

Gauß's Law $\quad \iiint_V \operatorname{div} \underline{D} \, dV = \oint_S \underline{D} \, d\underline{s}$

consider point charge Q at distance r, with radial unity vector $\vec{e_r} = \underline{e_r}$:

$$\underline{E} = \frac{Q}{4\pi \varepsilon_0 r^2} \underline{e_r}$$

Coulomb's Law states: $\operatorname{div} \underline{E} = \frac{\rho}{\varepsilon_0}$

Gauß's Law states:

$$\iiint_V \operatorname{div} \underline{E} \, dV = \oint_S \underline{E} \, d\underline{s} = \oint_S \frac{Q}{4\pi \varepsilon_0 r^2} \underbrace{\underline{e_r} \cdot d\underline{s}}_{=1}$$

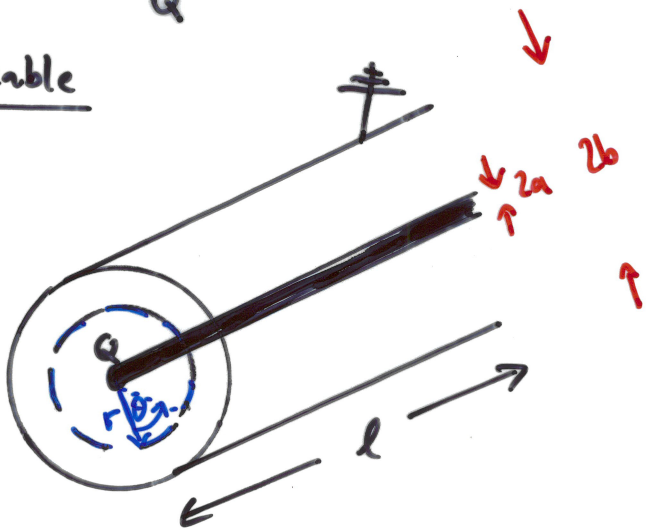
$$= \frac{Q}{4\pi \varepsilon_0 r^2} \cdot 4\pi r^2$$

$$= \frac{Q}{\varepsilon_0}$$

$$= \frac{1}{\varepsilon_0} \iiint_V \underbrace{\rho \, dV}_{Q}$$

is the same!

2 example: coaxial cable



$$\frac{Q}{\varepsilon_0} = \iint_S \underline{E} \, d\underline{s} = \int_0^{2\pi} E_r \, r \, d\theta = E_r \cdot l \cdot 2\pi r$$

$$\Rightarrow E_r = \frac{Q}{2\pi \varepsilon_0 l r} \overset{!}{=} \frac{\partial V}{\partial r} \quad (\underline{E} = -\operatorname{grad} V)$$

$$\Rightarrow V = \int_a^b E_r \, dr = \frac{Q}{2\pi \varepsilon_0 l} \underbrace{\int_a^b \frac{1}{r} \, dr}_{\ln r |_a^b} = \frac{Q}{2\pi \varepsilon_0 l} \ln \frac{b}{a}$$

$$\Rightarrow C = \frac{Q}{V} = \frac{2\pi \varepsilon_0 l}{\ln b/a}$$

proof of

Helmholtz's theorem:

A vector field is completely determined by its div and rot components.

Let the vector field be $\underline{X} = \underline{B} + \underline{E}$ with $\operatorname{div} \underline{B} = 0$ and $\operatorname{rot} \underline{E} = 0$ (as in electrostatics).

Because of the linearity of both operators

$$\operatorname{div} \underline{X} = \operatorname{div} (\underline{B} + \underline{E}) = \cancel{\operatorname{div} \underline{B}} + \operatorname{div} \underline{E} = \operatorname{div} \underline{E}$$

and

$$\operatorname{rot} \underline{X} = \operatorname{rot} (\underline{B} + \underline{E}) = \operatorname{rot} \underline{B} + \cancel{\operatorname{rot} \underline{E}} = \operatorname{rot} \underline{B}$$

$\Rightarrow$ from $\operatorname{div} \underline{E}$ and $\operatorname{rot} \underline{B}$ with boundary conditions, $\underline{X}$ can be calculated.

For electrostatics, we can choose

$\underline{B} = \operatorname{rot} \underline{A}$ (as magnetic field)

$\underline{E} = -\operatorname{grad} V$ (as electrical field)

$$\Rightarrow \quad -\nabla^2 V \equiv -\operatorname{div} \operatorname{grad} V = \operatorname{div} \underline{E} = \operatorname{div} \underline{X}$$

and also

$$\operatorname{rot} \underline{X} = \operatorname{rot} \underline{B} = \operatorname{rot} \operatorname{rot} \underline{A} = \operatorname{grad} \underbrace{\operatorname{div} \underline{A}}_{\text{choose Coulomb gauge with } \operatorname{div} \underline{A} = 0} - \nabla^2 \underline{A} = -\nabla^2 \underline{A}$$

but also

$$\operatorname{rot} \underline{X} = \mu_0 \mu_r \operatorname{rot} \underline{H}$$

$$\Rightarrow \quad \nabla^2 \underline{A} = -\mu_0 \mu_r \left(\underline{j} + \varepsilon_0 \varepsilon_r \frac{\partial \underline{E}}{\partial t} \right)$$

is similar to the Poisson-equation, but now in 3D (as vectors with x, y, z - components!)

Note on choice of gauge for div $\underline{A}$:

Maxwell equations only specify rules for div and rot components. As div rot = 0 = rot grad, any rot-component is specified only ± div $\underline{X}$, and any grad is only specified ± rot $\underline{X}$.

This means div $\underline{A}$ can be chosen ("gauged")

Coulomb gauge (most common): div $\underline{A} = 0$

Lorentz-gauge (for relativistic effects): div $\underline{A} = -\frac{\partial V}{\partial t}$

In both cases:

$$\boxed{\begin{aligned} \underline{B} &= \text{rot } \underline{A} \\ \underline{E} &= -\text{grad} V - \frac{\partial \underline{A}}{\partial t} \end{aligned}}$$

$\Rightarrow$ div $\underline{B} = 0$ ✓

$\Rightarrow$ div $\underline{E}$ - div grad $V = -\nabla^2 V = \frac{\rho}{\varepsilon_0}$

For this we get for the rot-components:

$$\text{rot } \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\frac{\partial}{\partial t} \text{rot } \underline{A} = -\text{rot} \frac{\partial \underline{A}}{\partial t}$$

and

$$\text{rot } \underline{H} = \underline{j} + \varepsilon_0 \varepsilon_r \frac{\partial \underline{E}}{\partial t}$$

$\Rightarrow$ for free space without charges ($\rho = 0$) or currents ($\underline{j} = 0$):

$$\frac{1}{\mu_0 \mu_r} \text{rot } \underline{B} = \varepsilon_0 \varepsilon_r \frac{\partial \underline{E}}{\partial t} \;;$$

insert equation above and use $\mu_r = \varepsilon_r = 1$ for vacuum:

$$\Rightarrow \frac{1}{\mu_0} \underbrace{\text{rot rot } \underline{A}}_{\text{grad } \underbrace{\text{div} \underline{A}}_{=0} - \nabla^2 \underline{A}} = \varepsilon_0 \Big(\underbrace{-\frac{\partial}{\partial t} \text{grad } V}_{=0} - \frac{\partial^2 \underline{A}}{\partial t^2} \Big)$$

$$\Rightarrow -\frac{1}{\mu_0} \nabla^2 \underline{A} = -\varepsilon_0 \frac{\partial^2 \underline{A}}{\partial t^2}$$

$$\Rightarrow \boxed{\frac{\partial^2 \underline{A}}{\partial t^2} = \underbrace{\frac{1}{\varepsilon_0 \mu_0}}_{= c^2} \nabla^2 \underline{A}}$$

is a wave equation for the vector potential $\underline{A}$

If the vector potential $\underline{A}$ obeys a wave equation with $c^2 = \frac{1}{\mu_0 \varepsilon_0}$, then $\underline{B}$ and $\underline{E}$ must also.

Note on wave equations: consider 1D case with propagation along positive x-direction

$$\underline{B} = \underline{B_0}\, e^{i(\omega t - kx)}$$

$$\Rightarrow \frac{\partial^2 \underline{B}}{\partial t^2} = \underline{B_0}\, e^{j(\omega t - kx)} (j\omega)^2 = -\underline{B}\, \omega^2$$

$$\frac{\partial^2 \underline{B}}{\partial x^2} = \underline{B_0}\, e^{j(\omega t - kx)} (-jk)^2 = -\underline{B}\, k^2$$

$$\Rightarrow \frac{\partial^2 \underline{B}}{\partial t^2} = \left(\frac{\omega}{k}\right)^2 \frac{\partial^2 \underline{B}}{\partial x^2}$$

$\hookrightarrow$ propagation speed $\frac{\omega}{k}$

for electromagnetic waves:

$$\left.\begin{aligned} \omega &= 2\pi f \\ k &= \frac{2\pi}{\lambda} \end{aligned}\right\} \quad \frac{\omega}{k} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \lambda f = c$$

- speed of light

Further:

If $\underline{B}(x,t) = \underline{B}(x)\, e^{j\omega t}$ is separable into a product of functions that depend on x and t, then $\frac{\partial^2 \underline{B}}{\partial t^2} = -\underline{B}(x) \cdot \omega^2$ is time-independent and we get a standing wave.

The wave equation above for $\underline{B}$ can be directly obtained from combining Maxwell's equations for free space:

$$\text{rot rot } \underline{B} = \text{rot } \mu_0 \text{ rot } \underline{H}$$

$$\underbrace{\cancel{\text{grad div } \underline{B}}}_{=0} - \nabla^2 \underline{B} = \text{rot } \mu_0 \left(\underbrace{\cancel{\underline{J}}}_{=0} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t} \right)$$

$$-\nabla^2 \underline{B} = \mu_0 \varepsilon_0 \text{ rot } \frac{\partial \underline{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \underbrace{\text{rot } \underline{E}}_{-\frac{\partial \underline{B}}{\partial t}} = -\mu_0 \varepsilon_0 \frac{\partial^2 \underline{B}}{\partial t^2}$$

q.e.d.

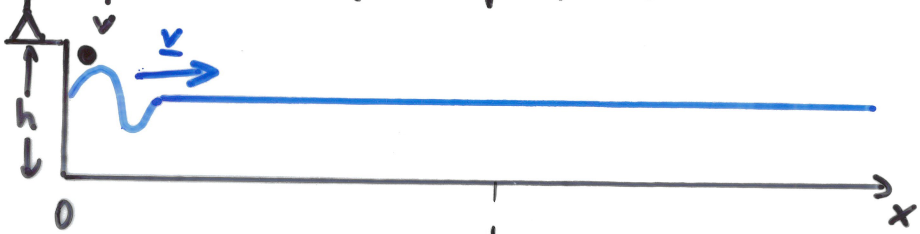
Introduction to waves

general case: wave height is function of position & time:

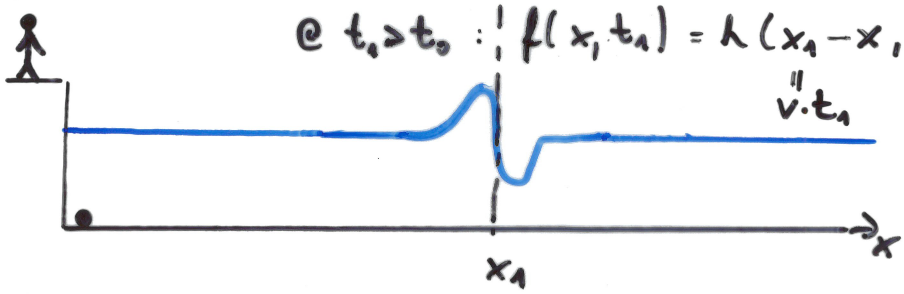
$$f(\underline{x}, t)$$

consider a wave travelling in $\pm$ x-direction

@ $t_0 = 0$: $f(x, t=0) = h(x)$



@ $t_1 > t_0$: $f(x, t_1) = h(x_1 - x, 0)$, $x_1 = v \cdot t_1$



in general: $f(\underline{x}, t) = h(\underline{v}t \pm \underline{x})$

↑ any continuous, differentiable function

$$\rightarrow \frac{\partial h}{\partial \underline{x}} = \pm h'(\underline{v}t \pm \underline{x}) \rightarrow \frac{\partial^2 h}{\partial \underline{x}^2} = h''(\underline{v}t \pm \underline{x})$$

$$\frac{\partial h}{\partial t} = \underline{v}\, h'(\underline{v}t \pm \underline{x}) \rightarrow \frac{\partial^2 h}{\partial t^2} = \underline{v}^2\, h''(\underline{v}t \pm \underline{x})$$

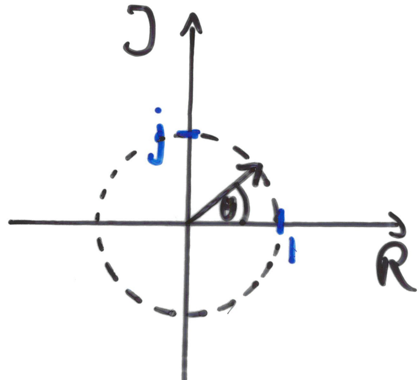
$$\Rightarrow \frac{\partial^2 h}{\partial t^2} = v^2 \frac{\partial^2 h}{\partial x^2}$$ in 1D

$$\boxed{\frac{\partial^2 f}{\partial t^2} = v^2 \nabla^2 f}$$ in 3D : wave equation

↑ propagation speed

phasor notation

$$e^{j\theta} = \cos\theta + j\sin\theta$$



Because of $\frac{\partial}{\partial\theta} e^{j\theta} = e^{j\theta}$, the complex exponential $f(x,t) = e^{j(\omega t - kx)}$ also solves the wave equation.

$e^{j(\omega t - \underline{k}\,\underline{x})}$ describes a planar wave along $-\underline{x}$ direction with $\omega = 2\pi f$ and $k = \frac{2\pi}{\lambda}$

frequency wavelength

standing wave :

time independent, i.e.

$f(\underline{x}, t) = e^{j\omega t} \cdot h(\underline{x})$ separable

Note :

$\underline{k}$ may itself be a phasor (complex); i.e.

$\underline{k} = k_1 + j k_2$ with real part k_1 and imaginary part k_2

$\Rightarrow e^{j[\omega t - (k_1 + j k_2) x]} = e^{j(\omega t - k_1 x)} e^{k_2 x}$

describes an attenuated planar wave if $k_2 < 0$