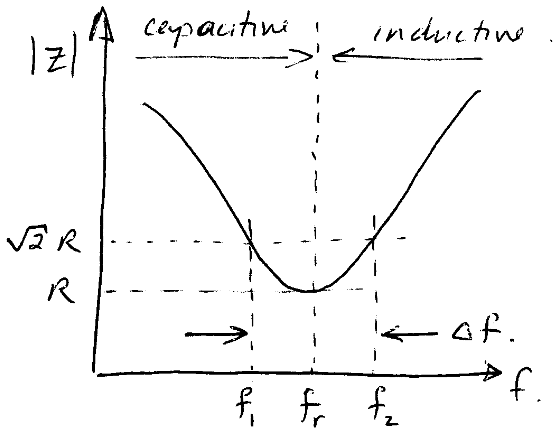
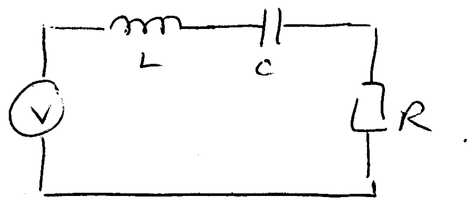
Resonance – proof of $Q = \frac{f_r}{\Delta f}$

The current is proportional to $1/|Z|$ for a series circuit driven by a constant V_0. Thus the current will have fallen by a factor of $\sqrt{2}$ from is maximum value when $|Z|$ has increased by a factor of $\sqrt{2}$ from its minimum value.



$$Z = j\omega L + \frac{1}{j\omega C} + R$$

$$= R + j\left(\omega L - \frac{1}{\omega C}\right)$$



and

$$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

The minimum $|Z| = R$ and this occurs at $\omega = \frac{1}{\sqrt{LC}}$

For ω_2 $(= 2\pi f_2)$, cct is inductive so $\omega L > \frac{1}{\omega C}$ we want to know when $|Z|$ increases to $\sqrt{2}|Z|_{min}$.

ie $\sqrt{2}|Z|_{min} = \sqrt{R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2} = \sqrt{2}R$.

or $2R^2 = R^2 + \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2$.

or $R^2 = \left(\omega_2 L - \frac{1}{\omega_2 C}\right)^2$

or $R = \omega_2 L - \frac{1}{\omega_2 C}$

or $0 = \omega_2^2 LC - \omega_2 CR - 1$

$$\omega_2 = \frac{CR \pm \sqrt{C^2R^2 + 4LC}}{2LC}$$

2

since the -ve root would give a negative frequency, take the +ve root...

$$\omega_2 = \frac{CR + \sqrt{C^2R^2 + 4LC}}{2LC} = 2\pi f_2 .$$

For ω_1 $(=2\pi f_1)$ the cct is capacitive so $\frac{1}{\omega C} > \omega L$

so $$\sqrt{2}R = \sqrt{R^2 + \left(\frac{1}{\omega_1 C} - \omega_1 L\right)^2}$$

or $$R^2 = \left(\frac{1}{\omega_1 C} - \omega_1 L\right)^2 .$$

or $$R = \frac{1}{\omega_1 C} - \omega_1 L$$

following the same process as for ω_2 leads to...

$$\omega_1 = \frac{-CR + \sqrt{C^2R^2 + 4LC}}{2LC} = 2\pi f_1$$

$$\omega_2 - \omega_1 = \frac{CR + \sqrt{(CR)^2 + 4LC}}{2LC} - \frac{-CR + \sqrt{(CR)^2 + 4LC}}{2LC}$$

$$= \frac{CR}{LC} = \frac{R}{L}$$

$$\frac{f_r}{f_2 - f_1} = \frac{\omega_r}{\omega_2 - \omega_1} = \frac{1}{\sqrt{LC}} \cdot \frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} = \underline{\underline{Q}} .$$

also

$$\sqrt{\omega_1 \omega_2} = \sqrt{\frac{CR\sqrt{(CR)^2 + 4LC} + C^2R^2 + 4LC - C^2R^2 - CR\sqrt{(CR)^2 + 4LC}}{4L^2C^2}}$$

$$= \sqrt{\frac{4LC}{4L^2C^2}} = \frac{1}{\sqrt{LC}} = \underline{\underline{\omega_r}}$$

Admittance, Conductance + Susceptance

These are the inverse of impedance, resistance and reactance.

$$\text{Conductance}, \; G = \frac{1}{R} = \frac{1}{\text{resistance}}$$

$$\text{Susceptance}, \; B = \frac{1}{X} = \frac{1}{\text{reactance}}$$

$$\text{Admittance}, \; Y = \frac{1}{Z} = \frac{1}{\text{impedance}} = \frac{I}{V}$$

Y, G + B have units of Ω^{-1}, sometimes written $\mho$, mhos, or Siemens.

"Siemens" is the standard SI unit for $\frac{1}{\text{Ohms}}$.

Y, G and B are particulary useful for parallel networks which often occur naturally at very high frequencies. If one writes down the impedance of a parallel network it is always of the form

$$Z = \frac{1}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3}}$$

or $$\frac{1}{Z} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3}$$

Using Y, G + B this would become

$$Y = G_1 + G_2 + G_3 + B_1 + B_2 + B_3$$

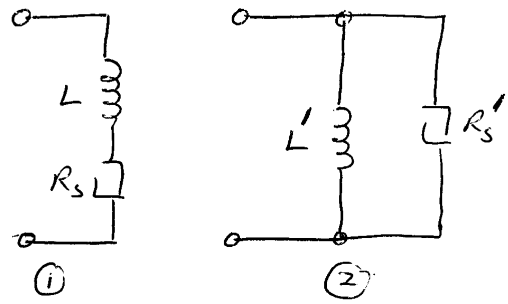
An inductive susceptance $= \frac{1}{j\omega L}$.

A capacitive susceptance $= j\omega C$

The significance of "j" is the same as for Z but the phase of Y (ie $\tan^{-1}\left(\frac{\text{im}\,Y}{\text{re}\,Y}\right)$) is the phase of I with respect to V.

Series to Parallel Transformation

What values of L' and R_s' will make circuit (2) have the same impedance as circuit (1) ?



We must have

$$Z_{(1)} = Z_{(2)}$$

or $$Y_{(1)} = Y_{(2)}$$

$$Z_{(1)} = R_s + j\omega L \qquad Y_{(2)} = \frac{1}{R_s'} + \frac{1}{j\omega L'}$$

$$Y_{(1)} = \frac{1}{Z_{(1)}} = \frac{1}{R_s + j\omega L} = \frac{R_s - j\omega L}{R_s^2 + \omega^2 L^2}$$

$$= \frac{R_s}{R_s^2 + \omega^2 L^2} - \frac{j}{\omega\left(\frac{R_s^2 + \omega^2 L^2}{\omega L}\right)}$$

$$= \frac{1}{R_s\left(\frac{R_s^2 + \omega^2 L^2}{R_s^2}\right)} - \frac{j}{\omega L\left(\frac{R_s^2 + \omega^2 L^2}{\omega^2 L^2}\right)}$$

Compare with $Y_{(2)} = \frac{1}{R_s'} - \frac{j}{\omega L'}$, for equivalence

$$R_s' = R_s\left(\frac{R_s^2 + \omega^2 L^2}{R_s^2}\right)$$

$$L' = L\left(\frac{R_s^2 + \omega^2 L^2}{\omega^2 L^2}\right)$$

Similar processes can be used to find equivalent series elements that will represent a given parallel combination.

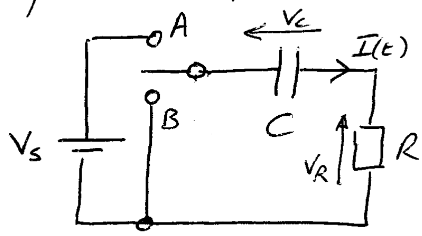
A similar process can be used to transform a series CR circuit to a parallel equivalent

If there is more than one reactive element in the original circuit it will usually transform to a single susceptance.

Transient Responses.

Analysis performed using the "$j\omega$" or phasor diagram approach assumes a sinusoidal source and allows the calculation of circuit parameters as a function of frequency. A transient analysis works out the response of a circuit to a source as a function of time. The approach to a transient analysis is as follows ...

Assume that switch has been in position B for a long time so that there is no charge in C.



At $t=0$, the switch is switched to position A ....

$$V_s = V_c + V_R = \frac{1}{C}\int I(t)\,dt + I(t)R.$$

First get rid of the integral by differentiating both sides

$$\frac{dV_s}{dt} = \frac{1}{C}I(t) + R\frac{dI(t)}{dt} = 0 \quad \text{since } V_s = \text{const.}$$

Then rearrange the equation ...

$$\frac{dI(t)}{I(t)} = -\frac{dt}{CR}$$

Then integrate to get an expression for $I(t)$ ...

$$\int \frac{dI(t)}{I(t)} = -\int \frac{dt}{CR} + \text{const.}$$

$$\ln I(t) = -\frac{t}{CR} + \text{const.}$$

$$\text{or} \quad I(t) = e^{-t/CR + \text{const}} = Ae^{-t/CR}$$

We now need to use boundary conditions to define A. When $t = 0 + \delta t$ where $\delta t \to 0$, $V_c = 0$ because there has been no time for charge to flow in the circuit. Therefore $V_R = V_s$ and $I(t)$ at $t = 0 + \delta t = V_s/R$

Thus $I(0) = \frac{V_s}{R} = A e^{-0} = A$

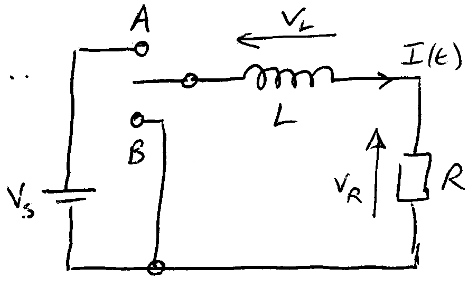
$$\therefore \quad I(t) = \frac{V_s}{R} e^{-t/RC}$$

RC is called the time constant, τ.

$$\left[R \times C = \frac{V}{A} \times \frac{Q}{V} = \frac{V}{Q/s} \times \frac{Q}{V} = s \right]$$

An inductor example ...

This time assume that the switch has been in position A for a long time.



The equation describing $I(t)$ is

$$0 = L \frac{dI(t)}{dt} + I(t)R$$ immediately after the switch has been moved to B.

Following the same process as for the previous case ...

$$\frac{dI(t)}{I(t)} = -\frac{R}{L}\,dt$$

$$\int \frac{dI(t)}{I(t)} = -\frac{R}{L}\int dt + c.$$

or $\ln I(t) = -\frac{R}{L}t + C.$

or $I(t) = A e^{-\frac{R}{L}t}$

The boundary condition here is that $I(t)$ immediately after switching is the same as the $I(t)$ immediately before switching – ie $I(t)$ at $t = 0 - \delta t$ where $\delta t \to 0$ $= V_s/R$.

$$\therefore A = \frac{V_s}{R} \quad \text{and} \quad I(t) = \frac{V_s}{R} e^{-t/\tau} \quad \text{where } \tau = L/R$$

dB and Filter Responses

The dB (deciBel) is a logarithmic unit used to express ratios of quantities such as current, voltage, power. It was originally devised as a measure of sound level in the context of loss of intensity in early telephone systems. Used in many applications today.

(i) As a power ratio $dB = 10 \log \frac{P_1}{P_2}$

(ii) As a voltage ratio $dB = 10 \log \frac{V_1^2}{V_2^2} = 20 \log \frac{V_1}{V_2}$.

(iii) As a current ratio $dB = 10 \log \frac{I_1^2}{I_2^2} = 20 \log \frac{I_1}{I_2}$

Often the lower part of the ratio (P_2, V_2, I_2) is a fixed reference - eg

dBV $\longrightarrow$ reference level is 1V rms.
dBu $\longrightarrow$ reference level is 1mW in 600Ω · (0.775Vrms)
dBm $\longrightarrow$ reference level is 1mW in 50Ω (0.223V rms)
dB(SPL) $\longrightarrow$ reference level is 20μPa. } threshold of human
dB(SWL) $\longrightarrow$ reference level is 10^{-12}W } hearing

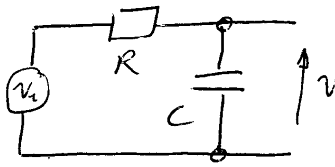
If $V_1 = \frac{V_{Rm}}{\sqrt{2}}$ and $V_2 = V_{Rm}$, the change in dB involved in that factor of $\frac{1}{\sqrt{2}}$ is $20 \log \frac{V_R/\sqrt{2}}{V_R} = -3.01$ dB ≈ -3dB.

This is why the width of a resonant peak when V_R is $\frac{1}{\sqrt{2}} \times V_{Rmax}$ is called the −3dB bandwidth.

The dBm is used as an absolute power measure in 50Ω impedance matched systems such as satellite, radar and other microwave systems.
The dBu is similarly used in audio systems which use impedance matched 600Ω signal transfer methods.

dB are used extensively in their ratio form in the plotting of the magnitude responses of frequency dependent circuits ....

Consider the circuit consisting of C + R.



$$\frac{v_o}{v_i} = \frac{1/j\omega C}{R + 1/j\omega C} = \frac{1}{1 + j\omega CR}.$$

To work out the response of this circuit the first step would normally be to let $CR = 1/\omega_o$. This is an arbitrary substitution and is done simply because if one wants to evaluate behaviour as a function of frequency, it makes sense to have constants defined in terms of frequency rather than time...

so $\frac{v_o}{v_i} = \frac{1}{1 + j\,\omega/\omega_o}$ and $\left|\frac{v_o}{v_i}\right| = \left[\frac{1}{1+(\omega/\omega_o)^2}\right]^{1/2}$.

when $\omega \ll \omega_o$, $1+\left(\frac{\omega}{\omega_o}\right)^2 \approx 1$ so $\left|\frac{v_o}{v_i}\right| = 1 \equiv 0\text{dB}$

when $\omega = \omega_o$, $1+\left(\frac{\omega}{\omega_o}\right)^2 \approx 2$ so $\left|\frac{v_o}{v_i}\right| = \frac{1}{\sqrt{2}} \equiv -3\text{dB}$

when $\omega \gg \omega_o$, $1+\left(\frac{\omega}{\omega_o}\right)^2 \approx \left(\frac{\omega}{\omega_o}\right)^2$ so $\left|\frac{v_o}{v_i}\right| = \frac{\omega_o}{\omega}$.

ie if ω increases by a factor of 10 (a decade)
$\left|\frac{v_o}{v_i}\right|$ decreases by a factor of 10 (−20dB)

so in the region $\omega \gg \omega_o$, gain "rolls off" at −20dB/decade.