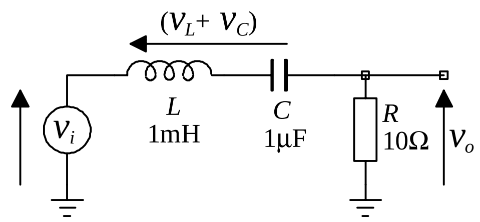
The University of Sheffield, Department EEE

Electronic Devices in Circuits Tutorial Solutions: Second Order Circuits and Noise

1 (i) The circuit is a series resonant circuit with the output measured across R. The transfer function can be worked out by potential division:



$$\frac{v_o}{v_i} = \frac{R}{R + \frac{1}{sC} + sL} = \frac{sCR}{1 + sCR + s^2LC}$$

(ii) The response is a band-pass response with a standard form

$$\frac{v_o}{v_i} = k\,\frac{\frac{s}{\omega_0 q}}{1 + \frac{s}{\omega_0 q} + \frac{s^2}{\omega_0^2}}. \qquad \textbf{(1.1)}$$

(iii) By comparing the transfer function with the standard form,

$$\omega_0 = \frac{1}{(LC)^{½}} \text{ or } f_0 = \frac{1}{2\pi\,(LC)^{½}} = 5\text{ kHz}$$

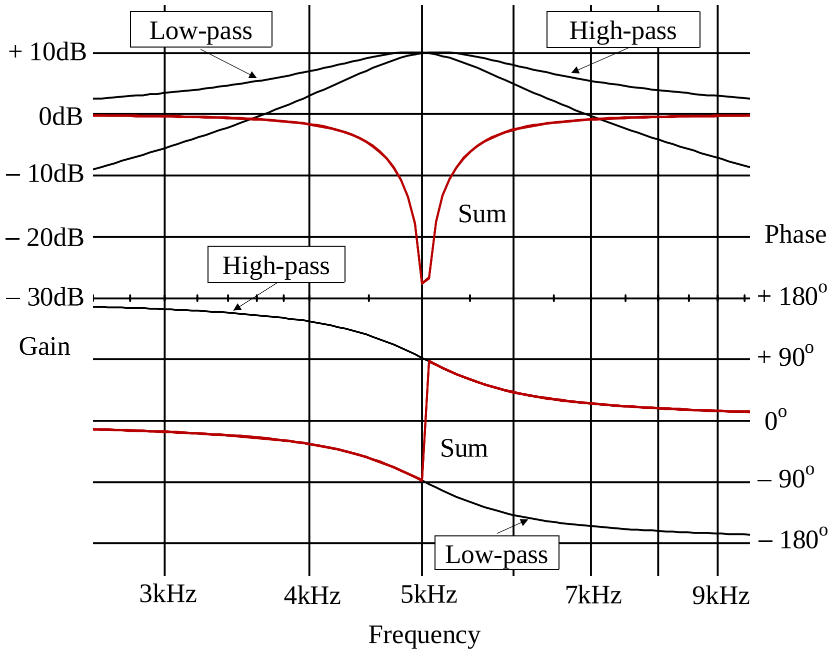
(iv) By comparing the transfer function with the standard form,

$$\frac{s}{\omega_0 q} = sCR \text{ or } q = \frac{1}{\omega_0 CR} = \frac{L^{½}C^{½}}{CR} = \frac{1}{R}\sqrt{\frac{L}{C}} = 3.16$$

(v) Again, by potential division,

$$\frac{v_o}{v_i} = \frac{\frac{1}{sC} + sL}{R + \frac{1}{sC} + sL} = \frac{1 + s^2LC}{1 + sCR + s^2LC}$$

(vi) The transfer function of part **(v)** is the linear sum of a high-pass (h-p) response and a low-pass (l-p) response. To see what the overall response will be, it is helpful to sketch the responses of each part of the sum as has been done for about an octave either side of the undamped natural frequency, f_0, in the diagram below. Note that the phase of the h-p response is always 180^o greater than the phase of the l-p response. This means that the phase of the linear sum of the responses at a particular frequency will be that of whichever response has the largest magnitude at that frequency. Thus for $f < f_0$, the phase follows the l-p phase while for $f > f_0$, the phase follows the high-pass phase. At f_0, the magnitudes of l-p and h-p responses are equal; the overall amplitude is therefore zero since two signals of equal amplitude and opposite phase are being added. In the diagram the amplitude response of the sum does not fall to zero ($-\infty$ dB) at f_0 and the phase does not change instantaneously from -90^o to $+90^o$ at f_0. This is because the numerical process that produced the graph evaluates phase and gain at discrete points in frequency - about every 70Hz in the vicinity of 5kHz.



The behaviour could also be worked out by taking the modulus and phase angle of the transfer function:

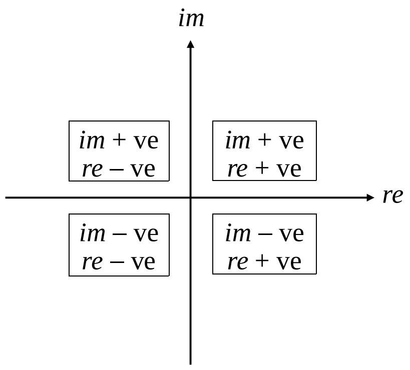
$$\left|\frac{v_o}{v_i}\right| = \left|\frac{1+s^2LC}{1+sCR+s^2LC}\right| = \frac{1-\omega^2LC}{(1-\omega^2LC)^2+\omega CR^2} \tag{1.2}$$

$$\angle\left|\frac{v_o}{v_i}\right| = -\tan^{-1}\frac{\left(\dfrac{\omega CR}{1-\omega^2LC}\right)}{\left(\dfrac{1-\omega^2LC}{1-\omega^2LC}\right)} = -\tan^{-1}\frac{\left(\dfrac{\omega CR}{1-\omega^2LC}\right)}{1} \tag{1.3}$$

It is easy to see from **(1.2)** why the modulus is taken to zero by the numerator when $\omega = 1/LC$. It is important not to confuse the phase of **(1.2)**, given by **(1.3)**, with the apparently identical expression for the phase of a second order low pass system given by **(1.4)** below:

$$\angle\left|\frac{v_o}{v_i}\right| = -\tan^{-1}\left(\frac{\omega CR}{1-\omega^2LC}\right) \tag{1.4}$$

In **(1.3)** the ***imaginary*** part can be positive or negative depending on the value of ω; the ***real*** part is always positive. In **(1.4)** the ***imaginary*** part is always positive whereas the ***real*** part can be positive or negative depending on the value of ω. This subtle difference affects the quadrant into which the calculated phase angle falls, as shown in the adjacent diagram. For an unambiguous determination of angle, knowledge of the sign of *im*/*re* is not sufficient; the signs of both *im* and *re* are necessary.

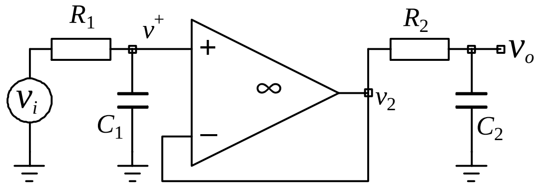


$$\varphi = \tan^{-1}\frac{im}{re}$$

The sign of *im* and *re* provide the information that tells you in which quadrant the angle φ lies.

(vii) For obvious reasons most people regard "band-stop filter" or "notch filter" as suitable descriptions of this response shape.

2 **(i)** Since the op-amp gain is ∞, the op-amp circuit is a unity gain buffer and $v_2 = v^+$. $v^+/v_i = 1/(1 + sC_1R_1)$ and $v_o/v_2 = 1/(1 + sC_2R_2)$, and since $v^+ = v_2$,



$$\frac{v_o}{v_i} = \frac{1}{1+ s\,C_1R_1}\;\frac{1}{1+ s\,C_2R_2} = \frac{1}{1+ s\,(C_1R_1+ C_2R_2) + s^2C_1C_2R_1R_2}$$

(ii) The standard low-pass form is:

$$\frac{v_o}{v_i} = k\,\frac{1}{1+\dfrac{s}{\omega_0 q}+\dfrac{s^2}{\omega_0^2}} \qquad \textbf{(2.1)}$$

so by comparison the undamped natural frequency is $f_o = 1/(2\pi\,(C_1C_2R_1R_2)^{1/2})$

(iii) By comparison with **(2.1)**, $1/\omega_0 q = (C_1R_1+ C_2R_2)$ or,

$$\frac{1}{q} = \frac{C_1R_1+ C_2R_2}{\sqrt{(C_1C_2R_1R_2)}} = \sqrt{\frac{C_1R_1}{C_2R_2}} + \sqrt{\frac{C_2R_2}{C_1R_1}} \qquad \textbf{(2.2)}$$

$$\text{or } q = \frac{\sqrt{(C_1C_2R_1R_2)}}{C_1R_1+ C_2R_2}$$

(iv) To find the maximum q, the usual approach of finding the minimum $1/q$ is used. From **(2.2)**,

$$\frac{1}{q} = \sqrt{\frac{C_1R_1}{C_2R_2}} + \sqrt{\frac{C_2R_2}{C_1R_1}} = x + \frac{1}{x} \quad \text{where } x = \sqrt{\frac{C_1R_1}{C_2R_2}}$$

Differentiating $1/q$ with respect to x and equating the result to zero gives the condition for minimum $1/q$ as, $d(1/q)/dx = 1 - 1/x^2 = 0$ or $x = 1$ for minimum $1/q$. Minimum $1/q$ is therefore $1 + 1 = 2$ so maximum $q = 0.5$

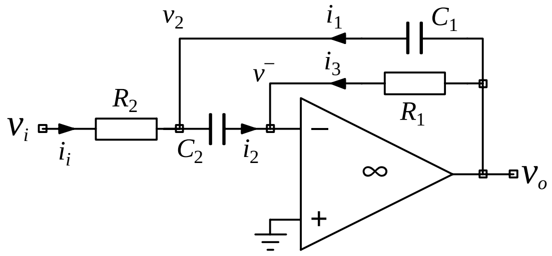
(v) From the working of part **(iv)**, q is maximum when $C_1R_1 = C_2R_2$

3 **(i)** Begin by summing currents at the v_2 node:

$$i_i + i_1 = i_2 \text{ or,}$$

$$\frac{v_i - v_2}{R_2} + (v_o - v_2)sC_1 = v_2 sC_2$$

since v^- is an ideal virtual earth and therefore $= 0$. Summing currents at the v^- node then gives:



$$i_2 + i_3 = 0 = v_2 sC_2 + \frac{v_o}{R_1}$$

v_2 can be eliminated from these two equations to give,

$$v_2 = \frac{v_i + v_o\, s\, C_1 R_2}{1 + s\,(C_1 + C_2)R_2} = -\frac{v_o}{s\, C_2 R_1}$$ which can be developed as follows:

$$v_i\, s\, C_2 R_1 + v_o\, s^2\, C_1 R_2 C_2 R_1 = -\, v_o\,(1 + s\,(C_1 + C_2)R_2\,)$$

$$v_i\, s\, C_2 R_1 = -\, v_o\,(1 + s\,(C_1 + C_2)R_2 + s^2 C_1 R_2 C_2 R_1\,)$$

$$\frac{v_o}{v_i} = \frac{-\, s\, C_2 R_1}{1 + s\,(C_1 + C_2)R_2 + s^2 C_1 R_2 C_2 R_1} = \frac{C_2 R_1}{(C_1 + C_2)R_2}\ \frac{-\, s\,(C_1 + C_2)R_2}{1 + s\,(C_1 + C_2)R_2 + s^2 C_1 R_2 C_2 R_1}$$

(ii) The response is a bandpass response.

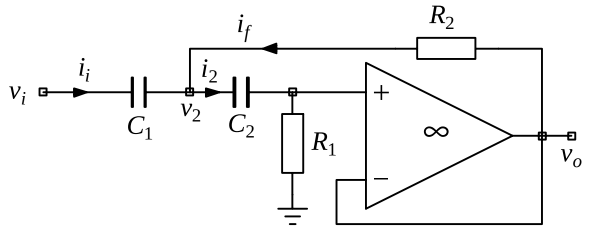
(iii) If $C_1 = C_2 = C$, comparison with the standard form **(1.1)** gives:

$$\omega_0^2 = \frac{1}{C^2 R_1 R_2}\,,\ \ \frac{1}{\omega_0 q} = 2CR_2 \ \text{ and therefore }\ q = \frac{1}{2}\sqrt{\frac{R_1}{R_2}}$$

For a q of 3, $R_1/R_2 = 36$

(iv) For a standard bandpass form such as **(1.1)**, the gain at the undamped natural frequency is given by k. Thus at ω_0,

$$\frac{v_o}{v_i} = -\frac{C_2 R_1}{(C_1 + C_2)R_2} = -\frac{R_1}{2R_2} = -\frac{36}{2} = -18$$



4 The analysis follows the usual line. Sum currents at the v_2 node to find v_2 in terms of v_i and v_o:

$i_i + i_f = i_2$ or, in terms of node voltages,

$$(v_i - v_2)sC_1 + \frac{v_o - v_2}{R_2} = (v_2 - v^+)sC_2 \qquad \textbf{(4.1)}$$

A second relationship between v_2 and v^+ is defined by the potential division caused by C_2 and R_1:

$$v^+ = v_2\,\frac{sC_2 R_1}{1 + sC_2 R_1} \qquad \textbf{(4.2)}$$

v^+ is equal to v_o because the op-amp is configured as an ideal unity gain buffer amplifier so **(4.1)** and **(4.2)** can be rewritten to express v_2 in terms of v_i and v_o. These relationshipsare then manipulated by collecting v_i and v_o terms together and forcing to a standard form as follows;

$$\frac{v_i\, sC_1 R_2 + v_o\,(1 + sC_2 R_2)}{1 + s\,(C_1 + C_2)R_2} = v_2 = \frac{v_o(1 + sC_2 R_1)}{sC_2 R_1}$$

$$v_i\, s^2C_1C_2R_1R_2 = v_o\,(1+sC_2R_1+s\,(C_1+C_2)R_2+s^2C_1C_2R_1R_2+s^2C_2^2R_1R_2-sC_2R_1-s^2C_2^2R_1R_2)$$

$$v_i\, s^2C_1C_2R_1R_2 = v_o\,(1+s\,(C_1+C_2)R_2+s^2C_1C_2R_1R_2), \text{ or,}$$

$$\frac{v_o}{v_i} = \frac{s^2C_1C_2R_1R_2}{1+s\,(C_1+C_2)R_2+s^2C_1C_2R_1R_2}$$

5 The frequency normalised fifth order Chebychev filter will have a cascade of three sections with the transfer functions:

$$\frac{v_o}{v_i} = \frac{1}{0.362+s} = \frac{1}{0.362}\left(\frac{1}{1+s\,2.76}\right) \quad \textbf{(5.1)}$$

$$\frac{v_o}{v_i} = \frac{1}{1.036+s\,0.224+s^2} = \frac{1}{1.036}\left(\frac{1}{1+s\,0.216+s^2 0.965}\right) \quad \textbf{(5.2)}$$

$$\frac{v_o}{v_i} = \frac{1}{0.477+s\,0.586+s^2} = \frac{1}{0.477}\left(\frac{1}{1+s\,1.229+s^2 2.096}\right) \quad \textbf{(5.3)}$$

These standard forms are then used to find the relative values of corner frequency (for the first order section) and undamped natural frequency, UDNF, (for the second order sections). The frequency independent gains do not affect the shape of the frequency response so they can be ignored.

(5.1) is a first order section with a corner frequency of 1/2.76 = 0.362 rad s^{-1}

(5.2) is a second order section with a UDNF of $(1/0.965)^{1/2}$ = 1.018 rad s^{-1}

(5.3) is a second order section with a UDNF of $(1/2.096)^{1/2}$ = 0.691 rad s^{-1}

These numbers give the relative values of corner frequency or UDNF for a cut off of 1 rad s-1. In other words they give the ratio ω_0/ω_c or f_0/f_c for each of the sections, where ω_0 and f_0 are the angular and cyclic corner frequency or UDNFs required of the sections to give an overall filter cut off frequency of ω_c or f_c .

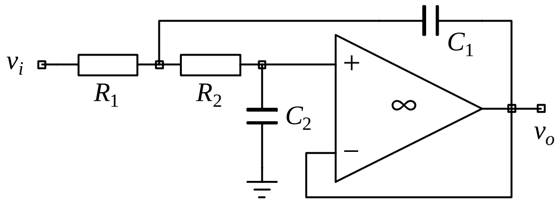
(i) Thus the corner frequency of the first order section = 1kHz x 0.362 = 362Hz.

(ii) The second order section UDNFs are, 1kHz x 1.018 = 1018Hz for **(5.2)** and 1kHz x 0.691 = 691Hz for **(5.3)**.

(iii) Comparing **(5.2)** with the standard form **(2.1)**, 1/(1.018 q) = 0.216 or q = 4.55

Comparing **(5.3)** with the standard form **(2.1)**, 1/(0.691 q) = 1.229 or q = 1.18

6 The analysis of a low-pass Sallen and Key circuit follows the procedure used in question 4 for the high-pass Sallen and Key. The result is:



$$\frac{v_o}{v_i} = \frac{1}{1 + sC_2(R_1 + R_2) + s^2 C_1 C_2 R_1 R_2}$$

which with equal Rs reduces to,

$$\frac{v_o}{v_i} = \frac{1}{1 + s\,2C_2R + s^2 C_1 C_2 R^2} \qquad \textbf{(6.1)}$$

For the 4th order Butterworth, the filter will consist of two second order factors in series:

$$\frac{v_o}{v_i} = \frac{1}{1 + 0.765s + s^2} \qquad \textbf{(6.2)}$$

$$\frac{v_o}{v_i} = \frac{1}{1 + 1.848s + s^2} \qquad \textbf{(6.3)}$$

By comparison with the low-pass standard form **(2.1)**, the undamped natural frequency and q s of the factors **(6.2)** and **(6.3)** of the frequency normalised transfer are 1 and 1.31 for **(6.2)** and 1 and 0.541 for **(6.3)**. Comparison of the Sallen and Key transfer function (6.1) with the standard form of (2.1) gives undamped natural frequency and q:

$$f_0 = \frac{1}{2\pi R\sqrt{C_1 C_2}} \quad \text{and} \quad q = \sqrt{\frac{C_1}{4C_2}}$$

For factor **(6.2)** where $q = 1.31$ and $f_0 = 10$kHz; $C_1 = 6.76C_2$ $RC_1 = 41.4\mu$s

For factor **(6.3)** where $q = 0.541$ and $f_0 = 10$kHz; $C_1 = 1.17C_2$ $RC_1 = 17.2\mu$s

It is usually a good idea to start by finding suitable capacitor values since the range of resistor values readily available is usually greater than that for capacitors. The AD711 data sheet specifies an input impedance of $3\text{x}10^{12}\Omega$ in parallel with 5.5pF between each input and ground and between the two inputs. Since the inverting input is connected to the low impedance op-amp output, its capacitance to ground is irrelevant. The feedback will increase the apparent impedance between the input terminals by a factor approximately equal to open loop gain - ie the capacitance will appear to be many orders of magnitude smaller than 5.5pF and can be neglected. The only elements which matter will thus be the parallel combination of $3\text{x}10^{12}\Omega$ and 5.5pF between the non-inverting input and ground and since the resistive part of this impedance is extremely high, only the capacitance needs to be considered. The 5.5pF effectively adds to C_2 and will therefore modify its value so the amount by which the value of C_2 exceeds the 5.5pF will determine the magnitude of the error caused by the 5.5pF input capacitance. For example for an error due to the 5.5pF of < 5%, C_2 > 110pF; for an error of < 1%, C_2 > 550pF. Using the 1% criterion (you might have chosen a different one) suitable values would be:

For factor **(6.2)**	$C_1 = 6.8$nF	$C_2 = 1$nF	$R = 6.2\text{k}\Omega$
For factor **(6.3)**	$C_1 = 1.3$nF	$C_2 = 1.1$nF	$R = 13\text{k}\Omega$

This is one set of suitable values out of a very large range. As a rough guide, if your Cs lie between 100pF and 100nF, your R lies between 1kΩ and 100kΩ and you have followed the thinking outlined above, your answer will be OK. You should have stated the condition you have used to define minimum C_2 in terms of 5.5pF.

7 (i) There are several ways in which a problem like this can be solved, all based on standard circuit analysis methods. One approach, illustrated in figure 7, uses Thevenin to Norton transformations to simplify the circuit until the problem becomes trivial. The circuit of figure

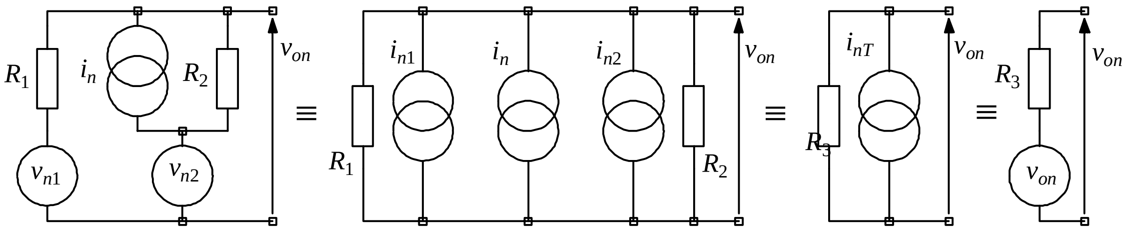


Figure 7

7a can be transformed into figure 7b by performing Thevenin to Norton transformations:

v_{n1} in series with $R_1 \Rightarrow i_{n1}$ in parallel with R_1, where $i_{n1} = v_{n1}/R_1$, and

v_{n2} in series with $R_2 \Rightarrow i_{n2}$ in parallel with R_2, where $i_{n2} = v_{n2}/R_2$.

The second of these is not quite as obvious as the first unless it is appreciated that it makes no difference to the circuit whether the bottom end of i_n is connected to the top or the bottom of v_{n2}. This can be verified easily by confirming that the component of output voltage due to i_n is independent of v_{n2}. The three current sources and two resistors of figure 7b can be combined into the single current source i_{nT} in parallel with the single resistor R_3, shown in figure 7c, where $\overline{i_{nT}^2} = \overline{i_{n1}^2} + \overline{i_{n2}^2} + \overline{i_n^2}$ and $R_3 = R_1//R_2$, and this can in turn be transformed into the Thevenin form of figure 7d where:

$$R_3 = R_1//R_2 = 12\text{k}\Omega//22\text{k}\Omega = 7.76\text{k}\Omega \text{ and}$$

$$\overline{v_{on}^2} = \overline{i_{nT}^2}\, R_3^2 = R_3^2\left(\frac{\overline{v_{n1}^2}}{R_1^2} + \frac{\overline{v_{n2}^2}}{R_2^2} + \overline{i_n^2}\right) = (7.76\text{ k}\Omega)^2\left(\frac{(40\text{nV})^2}{(22\text{k}\Omega)^2} + \frac{(10\text{nV})^2}{(12\text{k}\Omega)^2} + (1.5\text{pA})^2\right)$$

$$= 376.8 \times 10^{-18}\text{ V}^2\text{ Hz}^{-1} = 19.4\text{ nV Hz}^{-1/2}$$

A second approach is based on superposition. Figures 7e, 7f and 7g show the three partial circuits that define the component of output voltage due to each of the three generators.

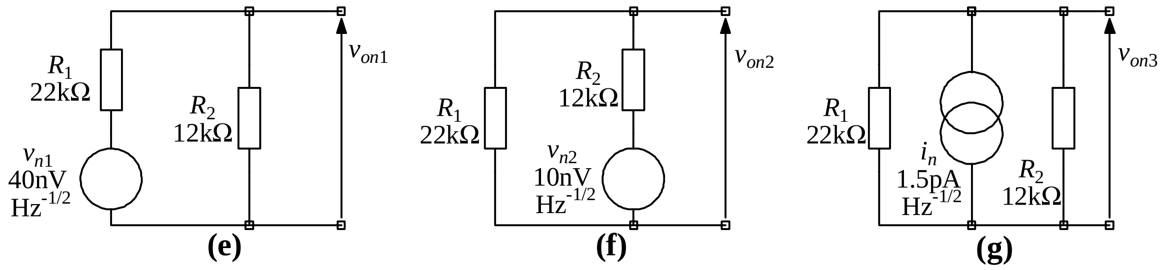


Figure 7

$$\overline{v_{on1}^2} = \overline{v_{n1}^2}\,\frac{R_2^2}{(R_1 + R_2)^2}, \quad \overline{v_{on2}^2} = \overline{v_{n2}^2}\,\frac{R_1^2}{(R_1 + R_2)^2} \text{ and } \overline{v_{on3}^2} = \overline{i_n^2}\,\frac{(R_1 R_2)^2}{(R_1 + R_2)^2}$$

$\overline{v_{on}^2} = \overline{v_{on1}^2} + \overline{v_{on2}^2} + \overline{v_{on3}^2}$ which gives the same answer as the Thevenin approach above.

(ii) To find the total rms noise voltage over a defined bandwidth, the spectral density must be integrated over the bandwidth of interest, ie

$$\overline{v_{nT}^2} = \overline{v_{on}^2}\,\Delta f = 376.8 \times 10^{-18} \times 20\text{kHz} = 7.54 \times 10^{-12}$$

or $v_{onT} = 2.75\mu\text{V}$

(iii) The temperature to which R_3 must be notionally raised in order to generate the same mean squared noise voltage as the source v_{on} is given by:

$$\overline{v_{on}^2} = 4kT_E R_3 \text{ or } T_E = \text{Noise temp.} = \frac{\overline{v_{on}^2}}{4kR_3} = \frac{376.8 \times 10^{-18}}{4 \times 1.38 \times 10^{-23} \times 7760} = 880\text{K.}$$

(iv) If the resistors in the circuit were themselves noisy, a new value of v_{on} that includes the noise contribution due to the resistors must be worked out. Since the contributions due to the explicit sources in the circuit was worked out in part **(i)**, the contributions due to each resistor need to be added to that result. The contributions from the two resistors in figure 7a can be considered separately or alternatively the contribution from the Thevenin equivalent of figure 7d can be added to v_{on}.

Using the first approach, the noise voltage source associated with the 22kΩ resistor is in the same position in the circuit as the 40nV source so its contribution to the output can be worked out with the help of figure 7e with the 40nV source replaced by the resistor noise voltage. Similarly figure 7f is appropriate for dealing with the effects of the noise associated with the 12kΩ resistor if the 10nV source is replaced by the 12kΩ resistor noise. Thus,

$$\overline{v_{o(R_1)}^2} = 4kTR_1 \frac{R_2^2}{(R_1 + R_2)^2} \text{ and } \overline{v_{o(R_2)}^2} = 4kTR_2 \frac{R_1^2}{(R_1 + R_2)^2}, \text{ giving a total extra}$$

contribution to mean square noise voltage of,

$$\overline{v_{o(R_1)}^2} + \overline{v_{o(R_2)}^2} = 4kT \frac{R_1R_2^2 + R_2R_1^2}{(R_1 + R_2)^2} = 4kT \frac{R_1R_2}{(R_1 + R_2)} = 1.286 \times 10^{-16}\ \text{V}^2$$

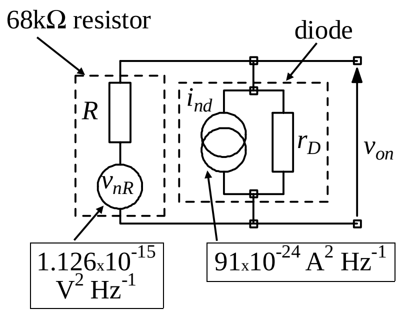
Note that since the Thevenin resistance of figure 7d is the parallel combination of R_1 and R_2 the answer for a calculation based on figure 7d will be as above. This contribution to mean squared noise must be added to the $\overline{v_{on}^2}$ calculated in part **(i)** to give:

$$\overline{v_{o(R_1)}^2} + \overline{v_{o(R_2)}^2} + \overline{v_{on}^2} = 1.286 \times 10^{-16}\ \text{V}^2 + 3.768 \times 10^{-16}\ \text{V}^2 = 5.054 \times 10^{-16}\ \text{V}^2$$

To find the effective noise temperature of the Thevenin equivalent resistance of figure 7d, the temperature at which the Thevenin resistance will generate the total mean squared noise voltage above must be found:

$$\overline{v_{on}^2} = 4kT_E R_3 \text{ or } T_E = \frac{\overline{v_{on}^2}}{4kR_3} = \frac{505.4 \times 10^{-18}}{4 \times 1.38 \times 10^{-23} \times 7760} = 1180\text{K.}$$

8 (i) In order to work out the magnitude of the shot noise generated by the diode, the dc current flow through it must be evaluated. Assuming a diode voltage drop of 0.7V, I_D = (20 - 0.7)/68kΩ = 284µA and the mean squared shot noise current is $2eI_D$ = 91.0 x 10^{-24} A^2 Hz^{-1}. The thermal noise associated with the resistor is $4kTR$ = 1.126 x 10^{-15} V^2 Hz^{-1}.



The noise equivalent circuit contains only noise sources so the 20V dc source is not included. The resistor is represented by its thermal noise voltage generator in series with an ideal resistor and the diode is represented by the diode incremental (or "slope" or "differential" or "small signal") resistance in parallel with the shot noise current generator. Diode incremental resistance is given by kT/eI_D = 91Ω in this case.

Evaluation of output noise is simply a matter of dealing with this circuit:

$$\overline{v_{on}^2} = \overline{i_{nd}^2}\frac{R^2 r_D^2}{(R+r_D)^2} + \overline{v_{nR}^2}\frac{r_D^2}{(R+r_D)^2} = \frac{r_D^2}{(R+r_D)^2}(\overline{i_{nd}^2}R^2 + \overline{v_{nR}^2}) = 754 \text{ x } 10^{-21} \text{ V}^2 \text{ Hz}^{-1}$$

or v_{on} = 868 pV $Hz^{-1/2}$

(ii) By inspection, the Thevenin equivalent resistance from which this noise voltage appears to come is $R \,//\, r_D$ = 68kΩ // 91Ω ≈ 91Ω.

(iii) The temperature at which a 91W resistor would have to be maintained in order to generate the noise voltage calculated in **(i)** is given by;

754 x 10^{-21} = $4kT_E$ 91 or T_E = 150K

(iv) If the output is loaded by a capacitor, the total noise voltage at the output will be given by the kT/C noise at the noise temperature calculated in (iii), ie

$$\overline{v_{onT}^2} = \frac{kT_E}{C} = 207 \text{ x } 10^{-12} \text{ V}^2 \text{ or } v_{onT} = 14.4\ \mu\text{V}$$

9 (i) The noise equivalent circuit of the amplifier is as shown in figure 9. The resistance r_i is noise free because as part of the amplifier its effects are included in v_n and i_n. To work out the noise factor of the circuit, the mean squared noise output voltage from the real amplifier must be divided by that from an ideal version of the amplifier.

The mean sqare noise voltage at the output of the real amplifier is given by:

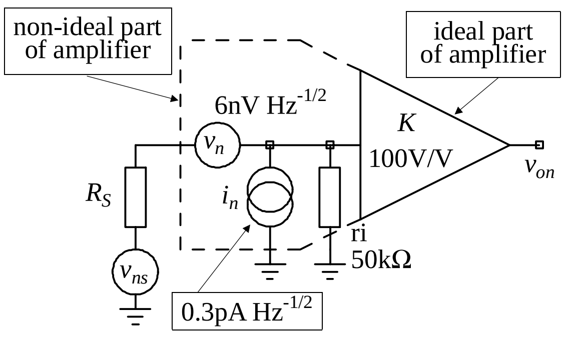


Figure 9

$$\overline{v_{onr}^2} = K^2\left(\frac{\overline{v_n^2}\,r_i^2}{(r_i+R_S)^2} + \frac{\overline{v_{nS}^2}\,r_i^2}{(r_i+R_S)^2} + \frac{\overline{i_n^2}\,r_i^2R_S^2}{(r_i+R_S)^2}\right) = K^2\frac{r_i^2}{(r_i+R_S)^2}\left(\overline{v_n^2} + \overline{v_{nS}^2} + \overline{i_n^2}\,R_S^2\right)$$

while for the ideal amplifier, v_n and i_n are zero giving:

$$\overline{v_{oni}^2} = K^2\frac{r_i^2}{(r_i+R_S)^2}\,\overline{v_{nS}^2}$$

The noise factor is thus:

$$F = \frac{\overline{v_{onr}^2}}{\overline{v_{oni}^2}} = \frac{K^2\dfrac{r_i^2}{(r_i+R_S)^2}\left(\overline{v_n^2} + \overline{v_{nS}^2} + \overline{i_n^2}\,R_S^2\right)}{K^2\dfrac{r_i^2}{(r_i+R_S)^2}\,\overline{v_{nS}^2}} = \frac{\overline{v_n^2}}{4kTR_S} + 1 + \frac{\overline{i_n^2}\,R_S}{4kT}$$

Since F depends both on R_S and $1/R_S$, F will be large for R_S very small and R_S very large and there will be some intermediate value at which F is a minimum. To find ths condition, the usual approach of differentiating and equating the differential to zero is used:

$$\frac{\mathrm{d}F}{\mathrm{d}R_S} = -\frac{\overline{v_n^2}}{4kTR_S^2} + 0 + \frac{\overline{i_n^2}}{4kT} = 0 \text{ or } F \text{ is minimum when } R_S = \frac{v_n}{i_n} = 20\text{k}\Omega$$

The result $R_S = v_n/i_n$ for minimum F is an easy one to remember but you should also know how to derive it.

(ii) This is simply a matter of putting numerical values into the real amplifier output noise expression,

$$\overline{v_{onr}^2} = K^2\frac{r_i^2}{(r_i+R_S)^2}\left(\overline{v_n^2} + \overline{v_{nS}^2} + \overline{i_n^2}\,R_S^2\right)$$

$$= 10^4 \times \frac{25}{49} \times (3.312\times10^{-16} + 0.36\times10^{-16} + 0.36\times10^{-16}) = 2.057\times10^{-12}\ \text{V}^2\ \text{Hz}^{-1}$$

or $v_{onr} = 1.43\ \mu\text{V Hz}^{-1/2}$

(iii) When a transformer with a turns ratio 1:n, where n is on the amplifier (secondary) side of the transformer, is used to couple a source with Thevenin equivalent resistance R_S to the amplifier, the amplifier sees a source rsistance of n^2R_S. The problem is simply to find the n that will make $n^2 \times 50 = 20\text{k}\Omega$. Thus $n^2 = 20{,}000/50 = 400$ so $n = 20$.

(iv) On the amplifier side of the transformer, the source will appear to the amplifier as a voltage source of $n \times 1\text{mV} = 20$ mV in series with a resistance of $n^2 \times 50 = 20\text{k}\Omega$. The effective source voltage of 20 mV will be potentially divided by the interaction between $n^2 \times R_S$ and r_i before being amplified by the amplifier voltage gain of 100. Thus,

$$v_o = 20\text{mV} \times \frac{50\text{k}\Omega}{50\text{k}\Omega + 20\text{k}\Omega} \times 100 = 1.43\text{V}$$

(v) The signal to noise ratio at the output is

$$\frac{S_o}{N_o} = \frac{\text{output signal power}}{\text{output noise power}} = \frac{\text{mean squared output signal voltage}}{\text{mean squared output noise voltage}}$$

The mean squared output voltages are related to powers by the load resistance seen from the amplifier output according to $P_o = V_o^2 / R_L$. Since both voltages are measured at the same node of the circuit, R_L is the same for both and cancels to leave the ratio of mean squared voltages as shown. The mean squared signal voltage at the output can be found by squaring the answer to part **(iv)**, ie $S_o = 2.045\ \mathrm{V}^2$. The mean squared noise voltage at the output over the bandwidth of interest can be found by taking the spectral density figure calculated in part **(ii)**, $2.057 \times 10^{-12}\ \mathrm{V}^2\ \mathrm{Hz}^{-1}$, and multiplying it by the system bandwidth of 20kHz to find the overall noise output. Thus,

$N_o = 2.057 \times 10^{-12} \times 20\text{kHz} = 4.114 \times 10^{-8}\ \mathrm{V}^2$ and S_o / N_o is 4.971×10^7 or 77dB.

10 (i) The gains of matched amplifiers is specified assuming that the amplifier is fed from an impedance matched source and feeds an impedance matched load. If the two amplifiers have power gains A_{P1} and A_{P2}, the overall power gain is the product $A_{P1}A_{P2}$. If the amplifier gains are expressed in dB, a logarithmic form, the overall gain in dB is the sum of the two individual gains in dB.

The overall gain here is 25dB + 15 dB = 40dB, or in linear terms $316.2 \times 31.62 = 10{,}000$

(ii) The noise factor of each module can be found by remembering that

noise figure, $NF = 10 \log$ (noise factor, F) or $F = 10^{NF/10}$

The noise factors corresponding to noise figures of 4.5dB and 7dB are therefore 2.82 and 5.01 respectively.

(iii) The noise factor of a cascade of two matched amplifiers with individual gains and noise factors A_{P1}, F_1 and A_{P2}, F_2 respectively is:

$F_{overall} = F_1 + (F_2 - 1) / A_{P1} = 2.82 + 4.01 / 316.2 = 2.83$ in this case. The noise figure is therefore $10 \log 2.83 = 4.52$ dB.

Note how small an effect the second stage has on the overall system noise performance.

(iv) Treating the cascade as a single amplifier, $F_{overall} = 1 + N_{Aoverall} / A_{Poverall} \times N_i$. Using $F_{overall} = 2.83$ (from part **(iii)**), $A_{Poverall} = 10{,}000$ and $N_i = kT\Delta f$, $N_{Aoverall} = 75.8\text{nW}$

(v) The definition of noise factor is the ratio of input to output signal to noise ratios. Thus if input signal to noise ratio and noise factor are known, it is easy to calculate output signal to noise ratio.

Available noise power at the input is $kT\Delta f = 1.38 \times 10^{-23} \times 300 \times 10^9 = 4.14\text{pW}$ so input signal to noise ratio is 10pW / 4.14pW = 2.42.

The noise factor, $F = (S_i/N_i) / (S_o/N_o) = 2.42 / (S_o/N_o) = 2.83$ (from part **(iii)**) which gives an output signal to noise ratio of 0.855 (or – 0.68dB).

(vi) The noise factor of an impedance matched system can be written $F = 1 + T_E / T_A$ where T_E is the noise temperature of the amplifier and T_A is the ambient temperature.

Thus $2.83 = 1 + T_E / 300$ or $T_E = 549\text{K}$