Second Order Circuits

1 Introduction

Second order circuits are circuits that have a transient response described by a second order differential equation. In the frequency domain, a second order circuit has a transfer function with a denominator that is second order (ie, contains terms in s up to s^2 where s may be taken as equal to $j\omega$ as far as frequency response is concerned) and a numerator that is either constant, first order or second order. As an example, the circuit of figure 1 has a transient response described by:

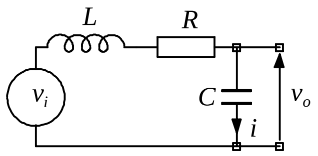


Figure 1
A simple passive second order circuit.

$$v_i(t) = LC\frac{d^2v_o(t)}{dt^2} + RC\frac{dv_o(t)}{dt} + v_o(t) \qquad \textbf{(1.1)}$$

and a transfer function described by:

$$\frac{v_o(s)}{v_i(s)} = \frac{1}{1 + sCR + s^2LC} \qquad \textbf{(1.2)}$$

If L were made zero, the $d^2v_o(t)/dt^2$ term would vanish from equation **(1.1)** and the s^2 term would vanish from equation **(1.2)** - ie, the system would become first order, a result that could be predicted by inspection of the circuit with $L = 0$.

Unlike first order circuits, second order circuits (or systems) are capable of resonating. For example, if $v_i = 0$ and $R = 0$, equation **(1.1)** reduces to the simple harmonic equation, $d^2v_o(t)/dt^2 = -v_o(t)/LC$, and the system will produce an output with no input providing there was some energy stored in either C and /or L at $t = 0$. If $R = 0$ is put into equation **(1.2)**, the s term vanishes and the gain v_o/v_i becomes infinite at a frequency given by $\omega_0^2 = 1/LC$ and infinite gain implies finite output for zero input. ω_0 is called the "undamped natural frequency" of the second order system.

Resonance is a very useful phenomenon. It is put to good use in the design of frequency selective circuits (both analogue and digital) because the gain of the circuit can be made high over a very narrow range of frequencies and this allows the design of circuits in which circuit gain varies rapidly with frequency. The most common application of such circuits is in radio receivers (where radio is used in its widest sense rather than merely the domestic entertainment medium it has commonly been taken to mean). Resonant circuits also form the basis of most very high quality sinusoidal signal generator circuits where harmonic quality is more important than frequency precision.

There are also instances where resonance is a nuisance. Second order effects can arise as a result of unwanted interactions between various parts of a circuit - usually between desired components and some parasitic circuit behaviour such as stray capacitance and/or inductance of circuit wiring or the intrinsic frequency response of an op-amp. Some examples are op-amp based transresistance amplifiers (current-to-voltage converters) and op-amp based differentiator circuits, both of which can exhibit an undesirable resonant behaviour in the form of gain peaking in the frequency domain and "ringing" - a form of damped oscillation caused by the resonant behaviour of the parasitic inductance and capacitance inevitably associated with printed circuit board tracks in digital and analogue circuitry - in the time domain. Understanding second order circuits helps in devising ways of controlling these parasitic effects.

2 Second order standard forms

As with first order circuits, there are a limted number of basic forms for the various possible second order responses. The aim of using a standard form is to make second order transfer functions easily interpretable in terms of two parameters, ω_0 and q. As with the case of first order circuits the aim is to arrive at a transfer function that is the ratio of two polynomials in s with the s^0 coefficient of the denominator equal to unity. There are three standard second order forms,

low-pass:- $$\frac{v_o}{v_i} = k\,\frac{1}{1 + \frac{sT}{q} + s^2T^2} = k\,\frac{1}{1 + \frac{s}{\omega_0 q} + \frac{s^2}{\omega_0^2}} \qquad \textbf{(2.1)}$$

band-pass:- $$\frac{v_o}{v_i} = k\,\frac{\frac{sT}{q}}{1 + \frac{sT}{q} + s^2T^2} = k\,\frac{\frac{s}{\omega_0 q}}{1 + \frac{s}{\omega_0 q} + \frac{s^2}{\omega_0^2}} \qquad \textbf{(2.2)}$$

high-pass:- $$\frac{v_o}{v_i} = k\,\frac{s^2T^2}{1 + \frac{sT}{q} + s^2T^2} = k\,\frac{\frac{s^2}{\omega_0^2}}{1 + \frac{s}{\omega_0 q} + \frac{s^2}{\omega_0^2}} \qquad \textbf{(2.3)}$$

where

k = a frequency independent constant,

ω_0 = the undamped natural frequency of the system,

T = a time defined as $T = 1/\omega_0$, an alternative way of expressing ω_0,

and q = the "quality factor" of the system. q can be defined in a number of ways but is basically a measure of the ratio of energy stored to energy lost per cycle in the circuit.

These standard forms are commonly used by electronic engineers who are usually more interested in the frequency response than in the transient response of the circuit or system. Control engineers are, on the other hand, often more interested in transient response than frequency response and consequently write the standard form in a slightly different way. For example, a low-pass response standard form in a control engineering application would appear as:

$$\frac{v_o}{v_i} = k\,\frac{\omega_n^2}{\omega_n^2 + 2\zeta\omega_n s + s^2} \text{ where } \zeta \text{ is called the "damping factor" and } \zeta = \frac{1}{2q} \qquad \textbf{(2.4)}$$

Note that ω_0 has changed to ω_n. ω_n is commonly used by authors as an alternative to ω_0. It is easy to see that equation **(2.4)** is the same as equation **(2.1)**.

If s is replaced by $j\omega$ in each of the three transfer functions, equations **(2.1)**, **(2.2)** and **(2.3)**, it is easy to see that, for a given q, each response shape depends upon the ratio ω/ω_0 - the normalised frequency - rather than ω alone. Thus, as for first order responses, ω_0 merely defines the position of the response on the frequency axis of a log amplitude versus log frequency plot. Unlike the first order case, where the shapes of the amplitude and phase responses are dependent only on the ratio ω/ω_0, in the second order case they are also dependent on q. Since q can have any value between close to zero and very large (some resonant circuits have $q \approx 10^4$), there exist an infinite number of possible response shapes for the frequency normalised amplitude and phase reponses of a second order system.

In general, a second order circuit or system may have a response which is the weighted sum of the three standard forms of equations **(2.1)**, **(2.2)** and **(2.3)**. A general form of such a transfer function is;

$$\frac{v_o}{v_i} = \frac{k_0 + k_1 s + k_2 s^2}{1 + \frac{s}{\omega_0 q} + \frac{s^2}{\omega_0^2}} \qquad \textbf{(2.5)}$$

Functions of this type, and the circuits which give rise to them, are often described as "biquads" because they are the ratio of two quadratic functions in s. Since they are the sum of three standard reponses it is sometimes not as easy physically to visualise the shapes of the amplitude and phase responses of transfer functions such as **(2.5)** as it is in the case of **(2.1)** to **(2.3)**. It is, however, important to remember that the denominator ***always*** determines the key second order parameters, q and ω_0.

3 Second order response shapes

The shapes of second order responses, in both the time and the frequency domains, are functions both of response type and q. There is a symmetrical relationship between low-pass and high-pass responses in much the same way as for first order responses but there is no first order equivalent of the band-pass circuit.

(i) Low-pass responses

The amplitude, phase and transient responses of a low-pass system, for a range of q values, are shown in figures 2 and 3. The following comments are intended to draw attention to the important features of the responses.

Frequency response

- The gain approaches 20log$k$ dB as frequency decreases below ω_0.
- For $q > 0.7$ there is gain peaking in the response.
- The frequency at which the gain peaking occurs is always less than ω_0 but rapidly gets closer to it as q increases - for example, when $q = 1.4$, $\omega_{peak} = 0.87\omega_0$; when $q = 3$, $\omega_{peak} = 0.97\omega_0$; when $q = 10$, $\omega_{peak} = 0.997\omega_0$.
- The gain at the undamped natural frequency = kq.
- The rate of change of phase with frequency increases as q increases.
- On a linear phase versus log frequency graph, the phase response is symmetrical about the point (ω_0, -90^o) - ie, if rotated about this point by half a revolution, the shape of the phase v log frequency graph will be unchanged.
- As frequency increases above ω_0, the amplitude responses for different qs all approach

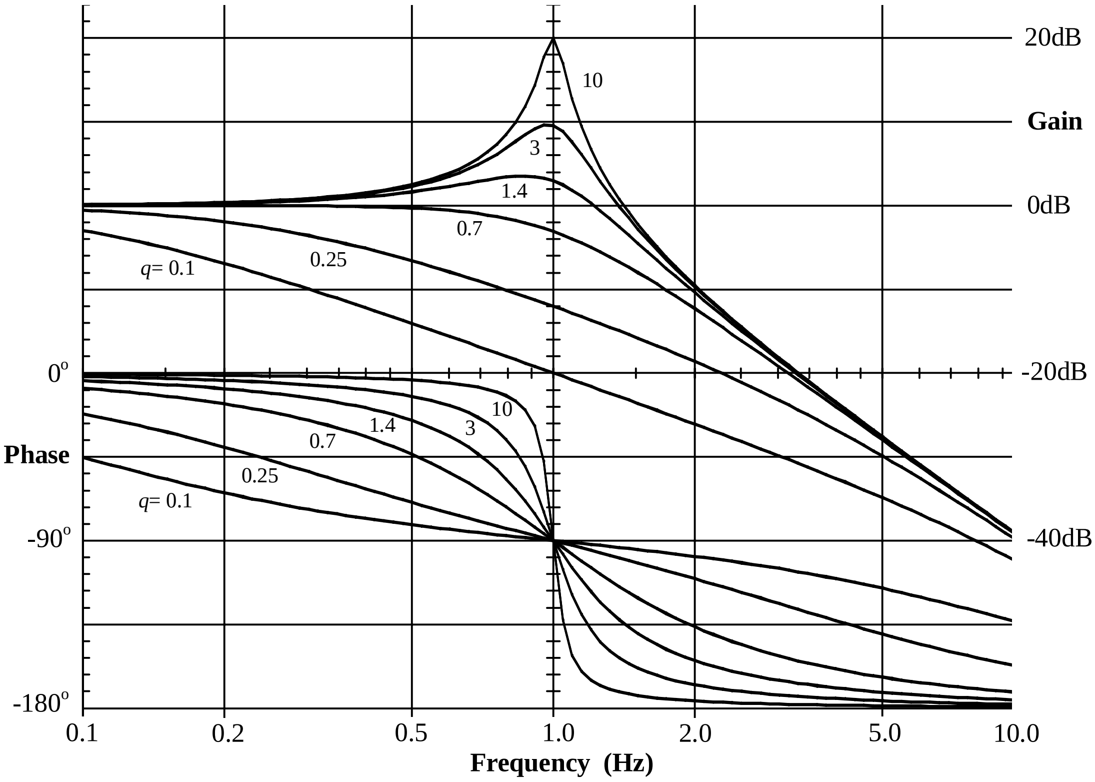


Figure 2

Amplitude and phase responses of a second order low pass circuit with an undamped natural frequency of 1 Hz *for various values of q.*

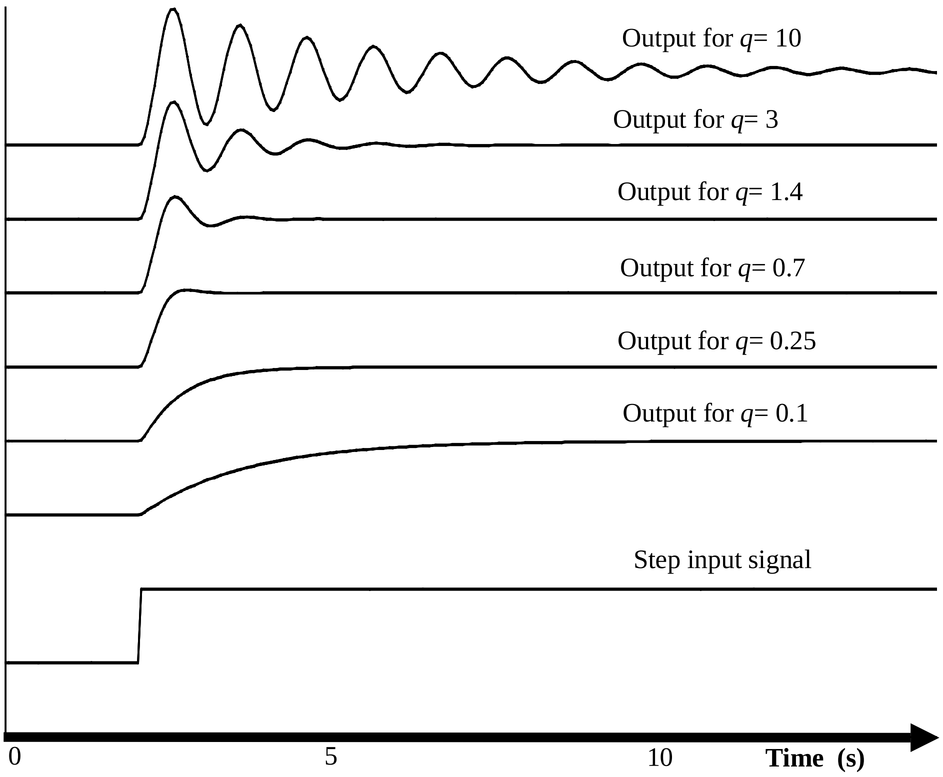


Figure 3

Transient response of a second order low-pass circuit with an undamped natural frequency of 1Hz *for various valyes of q. Each trace is zero until the step occurs at t = 2s.*

an asymptotic roll off of – 40 dB per decade. Responses for qs > 0.7 approach it from above, those for qs < 0.7 approach it from below.

Transient response

- For $q > 0.5$ ($\zeta < 1$), the step response exhibits overshoot in the form of a damped sinusoid - a behaviour often referred to as "ringing". As q increases (damping decreases) the ringing takes longer to die away.
- For q values up to and including 0.5, the step response monotonically approaches a steady final value of kv_i, taking longer to reach its aiming level as q is reduced. $q = 0.5$ is known as the "critically damped" condition (damping factor, $\zeta = 1/2q = 1$) and its response is the most rapid step response that can be achieved without introducing overshoot.

(ii) High-pass responses

The amplitude response of a second order high-pass circuit is a mirror image of the low-pass response of figure 2. For a given q, the phase response of the high-pass circuit is the same as that of the low-pass except that it is shifted up by 180^o by the j^2 term that constitutes the high-pass transfer function numerator. (Remember that for working out frequency response one puts $s = j\omega$.) The phase shift thus starts at 180^o at low frequencies and ends up at 0^o at high frequencies. The following comments apply:

Frequency response

- The gain approaches 20log$k$ dB as frequency increases above ω_0.
- For $q > 0.7$ there is gain peaking in the response.
- The frequency at which the gain peaking occurs is always greater than ω_0 but rapidly gets closer to it as q increases - for example, when $q = 1.4$, $\omega_{peak} = 1.15\omega_0$; when $q = 3$, $\omega_{peak} = 1.03\omega_0$; when $q = 10$, $\omega_{peak} = 1.003\omega_0$.
- The gain at the undamped natural frequency = kq.
- The rate of change of phase with frequency increases as q increases.
- On a linear phase versus log frequency graph, the phase response is symmetrical about the point (ω_0, $+90^o$) - ie, if rotated about this point by half a revolution, the shape of the phase v log frequency graph will be unchanged.
- As frequency falls below ω_0, the amplitude responses for different qs all approach an asymptotic roll off of – 40 dB per decade. Responses for qs > 0.7 approach it from above, those for qs < 0.7 approach it from below.

Transient response

- For all qs the response asymptotically approaches zero as t increases.
- For $q > 0.5$ ($\zeta < 1$), the step response exhibits "ringing" as for the low-pass case. As q increases (damping decreases) the ringing takes longer to die away.
- The detail around $t = 0$ is slightly different from the low-pass behaviour.

(iii) Band-pass responses

The band-pass response is that of a simple resonant circuit.

Frequency response

The response of the standard band-pass form of equation **(2.2)** with $k = 1$, shown for a range of q in figure 4, has unity gain at the undamped natural frequency, ω_0, irrespective of q. Although equation **(2.2)** represents a practically useful circuit, in other commonly encountered circuits, such as a parallel *L-C-R* combination driven by a current source, the gain at ω_0 is equal to q - equivalent to a situation $k = q$ in equation **(2.2)** - and the response is as shown in figure 5. Note that the phase response associated with the amplitude response of figure 5 is identical to that shown in figure 4; phase is not affected by the frequency independent multiplier. The salient features of the amplitude and phase responses are:

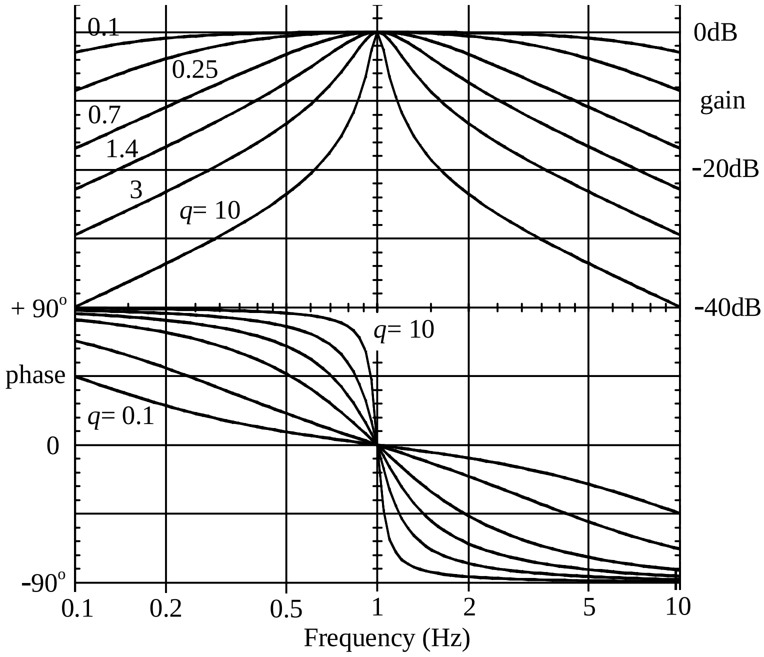


Figure 4

*The amplitude and phase responses of the standard band-pass form of equation **(2.2)**. Note that the phase is the same shape as for the low-pass case of figure 2 except for a +90° shift.*

- The amplitude response has a characteristic peak which always occurs at ω_0.
- The gain at the undamped natural frequency may be a constant independent of q or it may be proportional to q.
- The rate of change of phase with frequency increases as q increases.
- On a linear phase versus log frequency graph, the phase response is symmetrical about the point (ω_0, 0°) - ie, if rotated about this point by half a revolution, the shape of the phase v log frequency graph will be unchanged.
- As frequency falls below or increases above ω_0, the amplitude responses for different qs all approach an asymptotic roll off of – 20 dB per decade. Responses for qs > 0.7 approach it from above, those for qs < 0.7 approach it from below.
- The phase shift at resonance is zero. Resonant frequencies are often identified by identifying the zero phase shift condition; the circuit is sometimes said to be behaving resistively at ω_0.

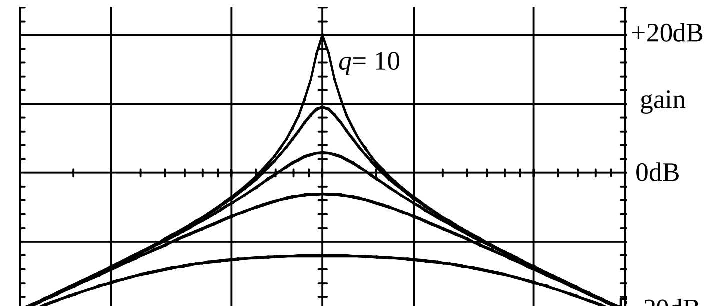


Figure 5

*The more familiar form of a band-pass amplitude response where k= q in equation **(2.2)** and hence the gain at ω_0 is q. The phase response is as in figure 4.*

Transient response

- The transient responses for different qs look very similar to the low-pass case of figure 3 except that they asymptotically approach zero after a long period of time.
- The band-pass response has detail around $t = 0$ which is slightly different from the low-pass and high-pass behaviours.

4 Analysis of second order *R*-*C*-op-amp circuits

There are various op-amp based circuit shapes that are used to achieve second order behaviour, often identified by the name of the person who first proposed the circuit shape. The circuits were usually devised for filtering applications. One example of low-pass, band-pass and high-pass is given. Notice the similarity in the approach to the analysis in each case.

(i) "Sallen and Key" low-pass circuit

Analysis

This circuit is very efficient in its use of components and finds frequent application. Note that the op-amp is wired as a unity gain follower circuit and is assumed to be ideal. All the voltages in figure 6 are measured with respect to ground. The analysis of this type of circuit starts by finding v_x in terms of the desired variables v_i and v_o ....

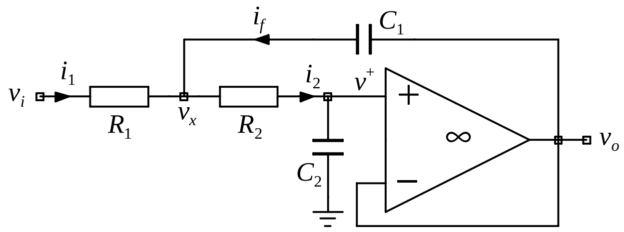


Figure 6

A low-pass Sallen and Key circuit

Summing currents at the v_x node,

$$i_1 + i_f = i_2 \quad \text{or} \quad \frac{v_i - v_x}{R_1} + (v_o - v_x)\, sC_1 = \frac{v_x - v^+}{R_2} \tag{4.1}$$

and since $A_v \Rightarrow \infty$, $v^+ = v^- = v_o$ and equation **(4.1)** becomes

$$\frac{v_i - v_x}{R_1} + (v_o - v_x)\, sC_1 = \frac{v_x - v^+}{R_2} \quad \text{or} \quad v_x = \frac{v_iR_2 + v_oR_1(1 + sC_1R_2)}{R_1 + R_2 + sC_1R_1R_2} \tag{4.2}$$

Another relationship is needed relating v_x to v_o or v_i or both. In this case, since $v^+ = v_o$,

$$v^+ = v_o = v_x \, \frac{\dfrac{1}{sC_2}}{R_2 + \dfrac{1}{sC_2}} = \frac{v_x}{1 + sC_2R_2} \tag{4.3}$$

Eliminating v_x from equations **(4.2)** and **(4.3)** gives

$$v_o\,(1 + sC_2R_2) = \frac{v_iR_2 + v_oR_1(1 + sC_1R_2)}{R_1 + R_2 + sC_1R_1R_2} \quad \text{which can be rearranged as}$$

$$v_o\left((1 + sC_2R_2) - \frac{R_1(1 + sC_1R_2)}{R_1 + R_2 + sC_1R_1R_2}\right) = \frac{v_iR_2}{R_1 + R_2 + sC_1R_1R_2}$$

or $$v_o\,[1 + s\,(C_2R_1 + C_2R_2) + s^2C_1C_2R_1R_2] = v_i$$

so $$\frac{v_o}{v_i} = \frac{1}{1 + sC_2(R_1 + R_2) + s^2 C_1 C_2 R_1 R_2} \qquad \textbf{(4.4)}$$

This is a second order low-pass reponse; second order because the denominator is a second order polynomial in s and low-pass because the numerator contains no s terms. The type of response could also be determined by comparison with the standard forms of equations **(2.1)**, **(2.2)** and **(2.3)**; the transfer function of equation **(4.4)** is of the same type as equation **(2.1)**, the low-pass standard form.

Interpretation

In order to interpret the transfer function, ie, to find k, ω_0 and q, it is necessary to equate coefficients of the appropriate standard form with those of the transfer function of interest. In this case, that means equating the coefficients of equation **(2.1)** to those of equation **(4.4)** to give

$$k = 1 \qquad \textbf{(4.5)}$$

$$\omega_0^2 = \frac{1}{C_1 C_2 R_1 R_2} \text{ or } \omega_0 = \frac{1}{(C_1 C_2 R_1 R_2)^{1/2}} \qquad \textbf{(4.6)}$$

$$\omega_0 q = \frac{1}{C_2(R_1 + R_2)} \text{ so } q = \frac{(C_1 C_2 R_1 R_2)^{1/2}}{C_2(R_1 + R_2)} = \left(\frac{C_1}{C_2}\right)^{1/2} \frac{(R_1 R_2)^{1/2}}{R_1 + R_2} \qquad \textbf{(4.7)}$$

Circuits that are very efficient in terms of component count, like the Sallen and Key, tend to have their range of q limited to modest values by the constraints imposed by real components. It is therefore of interest to identify any conditions which will maximise the circuit q. Inspection of equation **(4.7)** reveals that q is directly proportional to the square root of the capacitor ratio C_1/C_2, but the effects of the resistors is less clear. The nature of the resistive term in equation **(4.7)** - the root of a product over a sum - makes it is easier to deal with the problem of finding a condition that will maximise q by finding a condition that will minimise $1/q$ - effectively the same process. Inverting equation **(4.7)** gives:

$$\frac{1}{q} = \left(\frac{C_2}{C_1}\right)^{1/2} \frac{R_1 + R_2}{(R_1 R_2)^{1/2}} = \left(\frac{C_2}{C_1}\right)^{1/2} \left(\left(\frac{R_1}{R_2}\right)^{1/2} + \left(\frac{R_2}{R_1}\right)^{1/2}\right) \equiv a\left(x + \frac{1}{x}\right) \qquad \textbf{(4.8)}$$

where a is the capacitor ratio, $(C_2/C_1)^{1/2}$, and x is the resistor ratio $(R_1/R_2)^{1/2}$.

Inspection of equation **(4.8)** reveals that $1/q$ is large for both $x \gg 1$ and $x \ll 1$ and so there is likely to be a minimum value of $1/q$ somewhere between these extremes. Differentiating $1/q$ with respect to x and equating the result to zero leads without difficulty to the result that $1/q$ is a minimum, or in other words q is maximum, when $x = 1$ or $R_1 = R_2$. Putting $R_1 = R_2 = R$ into equations **(4.6)** and **(4.7)** gives

$$\omega_0 = \frac{1}{R\,(C_1 C_2)^{1/2}} \text{ and } q = \frac{1}{2}\left(\frac{C_1}{C_2}\right)^{1/2} \qquad \textbf{(4.9)}$$

This result considerably eases the design process because q is now controlled only by the capacitor ratio. Thus if the capacitor values are chosen to achieve the desired q, R can then be chosen to obtain the desired ω_0. If the initial choice of capacitor values leads to an impractical value of R (say less than 500Ω or more than 100kΩ for a typical op-amp), the capacitor values must be reduced or increased as appropriate without changing their ratio. The capacitor ratio gets very large for quite modest values of q (for example, a q of 5 requires a capacitor ratio of

100) and this tends to be the limitation of the Sallen and Key circuit. The circuit is nevertheless used extensively in the audio and signal conditioning application areas.

The ability independently to tune ω_0 and q, known as "orthogonal tuning", is attractive from a design point of view but is especially useful in circuits where adjustment of one or both parameters is necessary. In this circuit ω_0 can be varied independently of q by varying R and/or by varying the product C_1C_2 subject to the condition that C_1/C_2 remain constant. q can be varied independently of ω_0 by varying the ratio C_1/C_2 while keeping the product C_1C_2 constant. This is conditional orthogonal tuning rather than true orthogonal tuning; in a true orthogonal tuning situation there are no conditions to be satisfied to maintain the mutual independence of ω_0 an q.

(ii) "Friend" band-pass circuit

Analysis

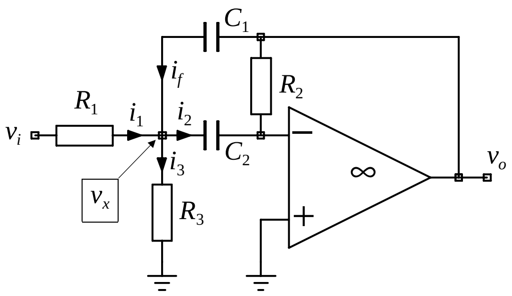


Figure 7

A Friend band-pass circuit

The Friend circuit is shown in figure 7 and the approach to its analysis follows closely the Sallen and Key circuit analysis of section **4 (i)**. As in that analysis, all the voltages referred in this analysis are measured with respect to ground. Start by summing currents at the v_x node:

$$i_1 + i_f = i_2 + i_3$$

or
$$\frac{v_i - v_x}{R_1} + (v_o - v_x)sC_1 = (v_x - v^-)sC_2 + \frac{v_x}{R_3} \qquad \textbf{(4.10)}$$

The inverting input voltage, v^-, is equal to zero since the circuit is a virtual earth circuit and $A_v \Rightarrow \infty$. Using $v^- = 0$, equation **(4.10)** can be rearranged to give v_x in terms of v_o and v_i:

$$v_x = \frac{R_3(v_i + s\,C_1R_1v_o)}{R_3 + R_1 + s\,(C_1 + C_2)R_1R_3} \qquad \textbf{(4.11)}$$

The second relationship between v_x and v_i and/or v_o, necessary for the elimination of v_x, can be found by recognising that i_2, which flows from v_x through C_2 to v^-, must also flow from v^- through R_2 to v_o. Thus,

$$(v_x - v^-)sC_2 = \frac{v^- - v_o}{R_2} \text{ and using } v^- = 0 \text{ and rearranging, } v_x = -\frac{v_o}{sC_2R_2} \qquad \textbf{(4.12)}$$

Eliminating v_x from equations **(4.11)** and **(4.12)**

$$-\frac{v_o}{sC_2R_2} = \frac{R_3(v_i + s\,C_1R_1v_o)}{R_3 + R_1 + s\,(C_1 + C_2)R_1R_3} \text{ which by collecting like terms becomes}$$

$$-v_o\left[\frac{1}{sC_2R_2} + \frac{sC_1R_1R_3}{R_3 + R_1 + s\,(C_1 + C_2)R_1R_3}\right] = \frac{v_i\,R_3}{R_3 + R_1 + s\,(C_1 + C_2)R_1R_3}$$

so
$$-v_o\left[\frac{R_3 + R_1 + s\,(C_1 + C_2)R_1R_3 + s^2C_1C_2R_1R_2R_3}{sC_2R_2}\right] = v_i\,R_3$$

or
$$\frac{v_o}{v_i} = -\frac{R_3}{R_1 + R_3} \times \frac{sC_2R_2}{1 + s\,(C_1 + C_2)\dfrac{R_1R_3}{R_3 + R_1} + s^2C_1C_2\dfrac{R_1R_2R_3}{R_3 + R_1}}$$

$$\text{or} \qquad \frac{v_o}{v_i} = -\frac{C_2}{C_1 + C_2} \times \frac{R_2}{R_1} \times \frac{s\,(C_1 + C_2)R_A}{1 + s\,(C_1 + C_2)R_A + s^2 C_1 C_2 R_2 R_A} \tag{4.13}$$

where R_A is the parallel combination of R_1 and R_3, $R_1R_3/(R_1 + R_3)$. Note that the final form of equation **(4.13)** is a standard band-pass form. The line prior to equation **(4.13)**, although usable with care, is not in the standard band-pass form because the coefficients of s are different in the numerator and denominator. This problem can always be sorted out by appropriate changes to the frequency independent term.

Interpretation

Comparison of equation **(4.13)** with the band-pass standard form of equation **(2.2)** gives

$$k = \frac{C_2 R_2}{(C_1 + C_2)R_1} \tag{4.14}$$

$$\omega_0^2 = \frac{1}{C_1 C_2 R_2 R_A} \quad \text{or} \quad \omega_0 = \left(\frac{1}{C_1 C_2 R_2 R_A}\right)^{1/2} \tag{4.15}$$

$$\omega_0 q = \frac{1}{(C_1 + C_2)R_A} \quad \text{or} \quad q = \left(\frac{R_2}{R_A}\right)^{1/2} \times \frac{(C_1 C_2)^{1/2}}{C_1 + C_2} \tag{4.16}$$

The minus sign in front of the whole transfer function expression means that an extra 180° is added to the phase shift of the circuit. This extra phase shift does not affect the calculation of components in any way and must be ignored in equations **(4.14)** to **(4.16)** if problems of apparent requirements for negative resistances are to be avoided.

The expression for q is similar to equation **(4.7)** for the Sallen and Key circuit except that the places of resistors and capacitors are interposed. A similar argument as was used in the Sallen and Key case can be used with the Friend circuit to show that, for a given value of R_2/R_A, maximum q is obtained when $C_1 = C_2 = C$. Using this condition in equations **(4.14)** to **(4.16)** gives

$$k = \frac{R_2}{2R_1} \qquad \omega_0 = \frac{1}{C\,(R_2 R_A)^{1/2}} \qquad q = \frac{1}{2}\left(\frac{R_2}{R_A}\right)^{1/2} \tag{4.17}$$

and these relationships, with the exception of k, are very similar to those for the Sallen and Key circuit with equal Rs.

The degree of tuning independence between ω_0 and q is the same as for the Sallen and Key case but it is difficult to adjust k without affecting either ω_0 or q or both. There is some scope for altering q independently of k over a limited range since q is affected by R_3 whereas k is not. This is not a particularly useful feature, however, since if ω_0 is to remain unchanged, the product R_2R_A must remain unchanged, a requirement that would necessitate changes of absolute resistor values but maintenance of ratios. These processes are somewhat inconvenient and a more realistic design sequence is:

- set R_2/R_1 using required k value.
- set R_3/R_1 using required q value.
- use required ω_0 value to choose an appropriate value for C and suitable absolute values of R_1, R_2 and R_3.

As an example, suppose that a band-pass filter with $k = 1$, $q = 1$ and $f_0 = 1$kHz is required.

Using the expressions for k, q and ω_0 in equation **(4.17)**:

- $k = 1 = R_2/(2R_1)$ or $R_2 = 2R_1$
- $q = \dfrac{1}{2}\left(\dfrac{R_2}{R_A}\right)^{1/2}$ or $q^2 = \dfrac{R_2}{4R_A} = \dfrac{R_2(R_1 + R_3)}{4R_1R_3} = \dfrac{1}{4} \times \dfrac{R_2}{R_1} \times \left(\dfrac{R_1}{R_3} + 1\right)$ so $R_3/R_1 = 1$

Choose $R_1 = 10\text{k}\Omega$, $R_2 = 20\text{k}\Omega$ and $R_3 = 10\text{k}\Omega$ as sensible values, then

- $\omega_0 = \dfrac{1}{C\,(R_2R_A)^{1/2}}$ or $f_0^2 = \dfrac{(R_1+R_3)}{4\pi^2C^2R_1R_2R_3}$ which gives $C =$ 16nF.

(iii) Sallen and Key high-pass circuit

Analysis

The analysis of this circuit follows that for the low-pass Sallen and Key circuit of section **4 (i)** so it is left as an exercise for the reader and is not repeated here. The result is:

$$\frac{v_o}{v_i} = \frac{s^2C_1C_2R_1R_2}{1 + sR_1(C_1 + C_2) + s^2C_1C_2R_1R_2} \tag{4.18}$$

and by comparison with the high-pass standard form of equation **(2.3)**,

$$\omega_0 = \frac{1}{(C_1C_2R_1R_2)^{1/2}} \tag{4.19}$$

and

$$q = \left(\frac{R_2}{R_1}\right)^{1/2} \times \frac{(C_1C_2)^{1/2}}{C_1 + C_2} \tag{4.20}$$

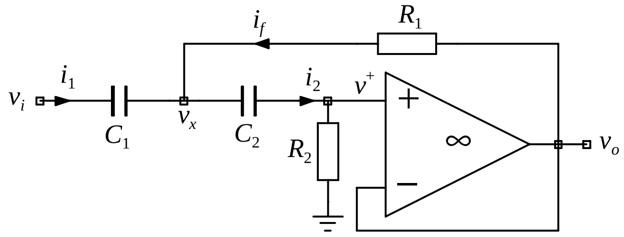


Figure 8

A high-pass Sallen and Key circuit. Note the similarity to the low-pass circuit shape.

Using the same approach as that used in the case of the low-pass Sallen and Key circuit for finding a condition that will maximise q, it can be shown that for the Sallen and Key high-pass circuit, q is a maximum when $C_1 = C_2$. Letting $C_1 = C_2 = C$, ω_0 and q become:

$$\omega_0 = \frac{1}{C\,(R_1R_2)^{1/2}} \qquad q = \frac{1}{2}\left(\frac{R_2}{R_1}\right)^{1/2} \tag{4.21}$$

Note the similarity between equation **(4.21)** and the same parameters for the low-pass circuit, equation **(4.9)**.

(iv) Modifications to Sallen and Key and to Friend circuits

Sallen and Key Circuit

It is possible to modify the Sallen and Key circuit by giving the amplifier a non-inverting gain greater than unity as shown in figure 9. The analysis follows the low-pass Sallen and Key analysis of section **4 (i)** with the exception that here, v_o is related to v_i by $v_o = Av^+$ where A is defined by the resistors R_3 and R_4. The analysis is left as an exercise but it leads to the transfer function:

$$\frac{v_o}{v_i} = \frac{A}{1 + s\,(2C_2 + C_1(1 - A))R + s^2 C_1 C_2 R_1 R_2} \quad \text{where } A = \frac{R_3 + R_4}{R_4} \tag{4.22}$$

This transfer function has an important difference from all those dealt with so far; the damping term can be zero or negative. If the damping term is zero, $q = \infty$; if the damping term is negative, q is negative. For both these conditions the circuit is unstable - that is, an output signal in the form of a sinusoid at the undamped natural frequency will appear at the output, even if there is no input signal.

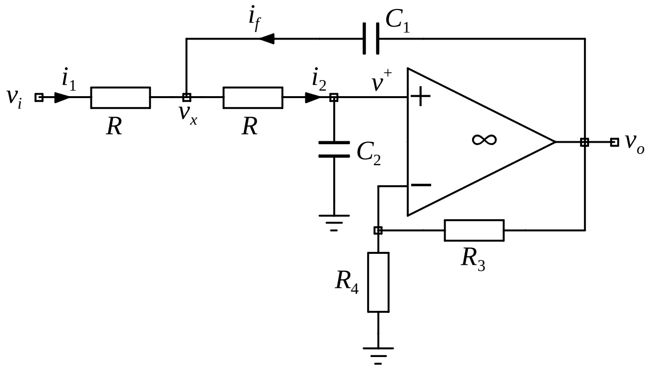


Figure 9

A low-pass Sallen and Key circuit modified by using an amplifier circuit with a gain greater than unity.

Circuits which can be either stable or unstable are described as "conditionally stable". The Sallen and Key circuit of figure 6 is "unconditionally stable" because no combination of circuit parameters can make the damping term zero or negative. The same is true of the Friend circuit of figure 7.

For a conditionally stable second order circuit, the condition that must be satisfied to ensure stability can be identified by finding the condition that keeps the damping term positive. For the transfer function of equation **(4.22)** to be stable,

$$2C_2 + C_1(1 - A) > 0 \quad \text{or } A < \frac{2C_2}{C_1} + 1\,. \tag{4.23}$$

Friend circuit

The modification to the friend circuit involves the inclusion of positive feedback around the amplifier as shown in figure 10. The analysis of this circuit is more algebraically complicated than that of figure 9, mainly because there are more terms involved. The analysis follows that of the Friend circuit of figure 7 with the exception that instead of $v^- = v^+ = 0$, as was the case for figure 7, for figure 10, $v^- = v^+ = Hv_o$ where $H = \dfrac{R_5}{R_4 + R_5}$. This different v^- must be used in equation **(4.10)** to get a modified equation **(4.11)** and in equation **(4.12)**. Assuming $C_1 = C_2 = C$, equations **(4.11)** and **(4.12)** become respectively

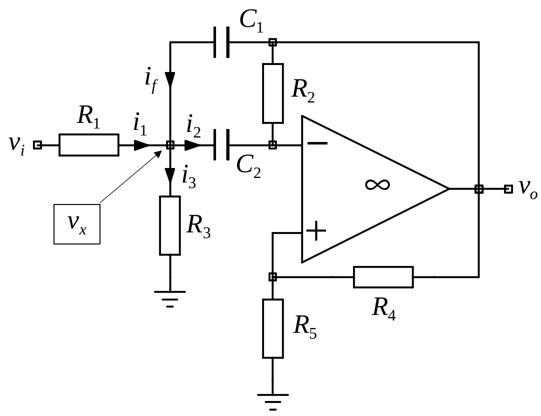


Figure 10

A Friend circuit with positive feedback

$$v_x = \frac{R_3}{R_1 + R_3}\;\frac{(v_i + v_o\, sCR_A\,(H + 1))}{1 + 2sCR_A} \tag{4.24}$$

and $$v_x = \frac{v_o\,(\,H\,(1 + sCR_2) - 1)}{sCR_2} \tag{4.25}$$

and using equation **(4.24)** with equation **(4.25)** to eliminate v_x gives

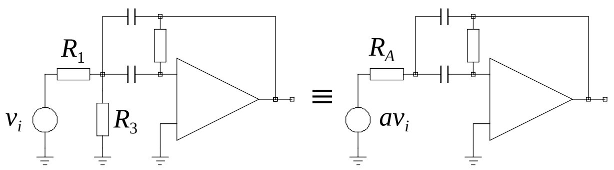
$$\frac{v_o}{v_i} = -\ \frac{1}{1-H} \times \frac{R_3}{R_1 + R_3} \times \frac{sCR_2}{1 + sC\left(\frac{HR_2}{H-1} + 2R_A\right) + s^2C^2R_2R_A} \tag{4.26}$$

The damping term in equation **(4.26)** is $\frac{HR_2}{H-1} + 2R_A = -\,R_2\frac{R_5}{R_4} + 2R_A$ and since this has the potential to be negative, the circuit is once again only conditionally stable.

As a final note on the Friend circuits of figures 7 and 10, their analysis can be simplified by recognising that v_i, R_1 and R_3 can be replaced by a Thevenin equivalent of a modified v_i in series with the parallel combination of R_1 and R_3, a combination that has already been defined as R_A. The process is illustrated in the thumbnail sketch of figure 10a.

Figure 10a

The use of a Thevenin equivalent to simplify the Friend circuit. R_A is the parallel combination of R_1 and R_3 and "a" is the potential division $R_3/(R_1+R_3)$



(v) A multi op-amp circuit.

Multi op-amp circuits are attractive for a number of reasons despite the obvious disadvantage of using an increased number of components. They usually offer better control and more convenient adjustment of q and ω_0, q and ω_0 can usually be adjusted orthogonally and all three basic transfer functions can be obtained simultaneously from a single circuit. The latter feature makes multi op-amp circuits useful as general filter building blocks and many examples of these are available from IC manufacturers, some programmed (adjusted) digitally and some using external resistors.

An example of a multi op-amp circuit is shown in figure 11. Since it is an interconnection of system blocks with well known transfer functions, working out the required overall transfer function is simply a matter of manipulating the transfer functions of the blocks. The equations

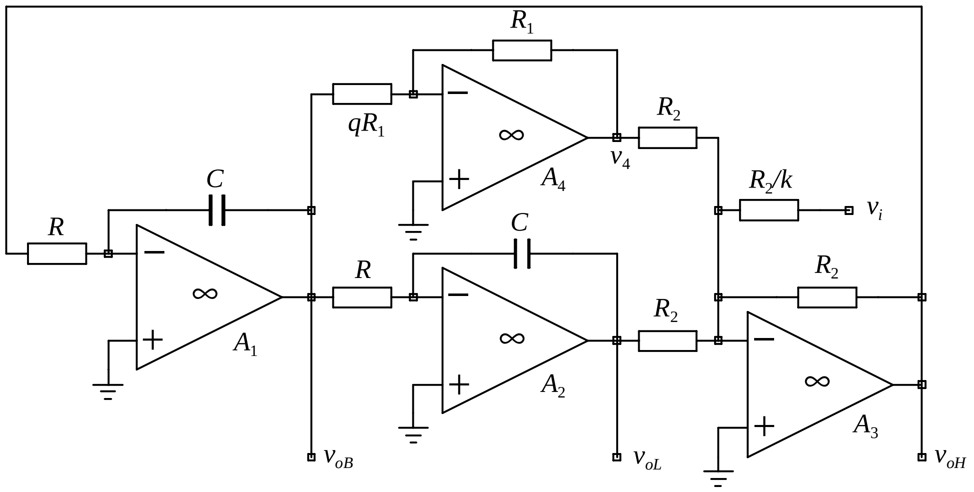


Figure 11

A multi op-amp second order circuit variously known as "universal filter", "two integrator loop", "state variable filter" and "biquad". Note the three simultaneously available outputs. A_1 and A_2 are the two integrators, A_3 is a phase inverter to ensure that the correct phase shift is in place for the feedback at dc to be negative and A_4 controls the damping term.

describing each block are:

$$v_{oB} = -\frac{v_{oH}}{sCR}\ ,\ \ v_{oL} = -\frac{v_{oB}}{sCR},\ \ v_4 = -\frac{v_{oB}}{q}\ \text{ and }\ v_{oH} = -(v_{oL} + v_4 + kv_i)$$

These can be combined to get the output of interest. For example, to get the band-pass transfer function, v_{oB}/v_i;

$$v_{oB} = -\frac{v_{oH}}{sCR} = \frac{(v_{oL} + v_4 + kv_i)}{sCR} = \frac{\left(-\dfrac{v_{oB}}{sCR} - \dfrac{v_{oB}}{q} + kv_i\right)}{sCR}$$

so $$v_{oB}\,[s^2C^2R^2 + 1 + \frac{sCR}{q}\,] = sCR\,kv_i \ \text{ or }\ \frac{v_{oB}}{v_i} = kq\,\frac{\dfrac{sCR}{q}}{1 + \dfrac{sCR}{q} + s^2C^2R^2} \qquad \textbf{(4.27)}$$

A similar process leads to the low-pass and high-pass responses:

$$\frac{v_{oL}}{v_i} = -k\,\frac{1}{1 + \dfrac{sCR}{q} + s^2C^2R^2} \qquad \textbf{(4.28)}$$

$$\frac{v_{oH}}{v_i} = -k\,\frac{s^2C^2R^2}{1 + \dfrac{sCR}{q} + s^2C^2R^2} \qquad \textbf{(4.29)}$$

Note that the frequency independent gain in the band-pass case, which is the gain at resonance, is equal to q. This is the case illustrated in figure 5 and it can be inconvenient if one wants to vary q because the resonant gain will also vary. To get around this difficulty and achieve a response such as figure 4 where the resonant gain is unity, the bandpass output can be taken from the v_4 node. A_4 effectively divides v_{oB} by a factor of q and thus compensates for the resonant gain of q in equation **(4.22)**. A_4 also changes the phase of v_{oB} by 180^{o}. The band-pass transfer function at v_4 is:

$$\frac{v_4}{v_i} = -k\,\frac{\dfrac{sCR}{q}}{1 + \dfrac{sCR}{q} + s^2C^2R^2} \qquad \textbf{(4.30)}$$

In each of the transfer functions of equations **(4.27)**, **(4.28)**, **(4.29)** and **(4.30)** ω_0, q and k can be adjusted completely independently, ω_0 by the product RC, q by the resistor qR_1 and k by the input resistor R_2/k. This circuit therefore exhibits true orthogonal tuning behaviour which makes it easy to tailor its parameters for particular applications and hence makes it attractive as a general purpose second order circuit block from an IC manufacturer's point of view.

5 Concluding comments

This handout has introduced standard second order forms and shown by detailed examination of a number of second order circuits how second order circuits can be interpreted with the help of these standard forms. The main area of interest has been the frequency response so the transient behaviour of the circuits examined has been left largely unexplored.

No analysis of commonly encountered parasitic second order circuits has been included here

- the analysis of parasitic second order effects in circuits is very similar to those presented and offers useful analytical practice.

For those who wish to explore the topic further, two worthwhile sources are

(i) "Analog Filter Design", M.E. Van Valkenburg, HRW, 1982

(reprinted more recently by Oxford)

(ii) "Active Filters", F.E.J. Girling and E.F. Good, Wireless World, Aug 1969 - Dec 1970.

(a series of 16 articles published in (mostly) successive monthly issues)