# EEE345 exam 2014/1: exam questions and model solutions

## 1. Maxwell's equations and waves

10 points

**a.** Using Maxwell's equations for the rotation operators of the electrical and magnetic fields, the materials equations relating corresponding fluxes and fields, and the mathematical identity

$$\textbf{rot rot } \underline{\boldsymbol{A}} = \textbf{grad} \text{ div } \underline{\boldsymbol{A}} - \nabla^2 \underline{\boldsymbol{A}} \qquad \text{(equation 1)},$$

show that in vacuum the magnetic vector potential $\underline{\boldsymbol{A}}$ obeys a wave equation.

Solution (similar to lecture where the same was shown for $\underline{\boldsymbol{B}}$):
The definition of the magnetic vector potential is such that $\underline{\boldsymbol{B}}=\text{rot } \underline{\boldsymbol{A}}$ (i).
Applying the rot operator to the right hand side of this and using div $\underline{\boldsymbol{A}}$=0 yields:
$\textbf{rot rot } \underline{\boldsymbol{A}} = \textbf{grad} \text{ div } \underline{\boldsymbol{A}} - \nabla^2 \underline{\boldsymbol{A}} =$ (provided above)
$= -\nabla^2 \underline{\boldsymbol{A}}$ (ii)
Applying the rot operator to the left hand side and using Maxwell's equations for rot $\underline{\boldsymbol{H}}$ and field equations for $\underline{\boldsymbol{B}}=\mu_0\mu_r\underline{\boldsymbol{H}}$ and $\underline{\boldsymbol{D}}=\varepsilon_0\varepsilon_r\underline{\boldsymbol{E}}$ yields:
rot $\underline{\boldsymbol{B}} = \mu_0\mu_r$ rot $\underline{\boldsymbol{H}}$ (iii)
$= \mu_0\mu_r (\underline{\boldsymbol{j}}+\partial\underline{\mathbf{D}}/\partial t)$ (iv)
$= \mu_0\mu_r\varepsilon_0\varepsilon_r(\underline{\boldsymbol{j}}+\partial\underline{\mathbf{E}}/\partial t)$ (v)
In vacuum, $\mu_r=1=\varepsilon_r$ and also without any currents $\underline{\boldsymbol{j}}$=0, hence:
rot $\underline{\boldsymbol{B}} = \mu_0\varepsilon_0\, \partial\underline{\mathbf{E}}/\partial t$ (vi)
From
rot $\underline{\boldsymbol{E}} = -\partial\underline{\boldsymbol{B}}/\partial t$ (vii)
we can get, by inserting again the defintion of $\underline{\boldsymbol{A}}$,
rot $\underline{\boldsymbol{E}} = -\partial\, (\text{rot}\underline{\mathbf{A}})/\partial t = -\text{rot } \partial\underline{\mathbf{A}}/\partial t$, hence
$\underline{\boldsymbol{E}} = -\, \partial\underline{\mathbf{A}}/\partial t$ (modulo some div term) (viii).
Another differentiation according to time yields $\partial\underline{\mathbf{E}}/\partial t = -\, \partial^2\underline{\mathbf{A}}/\partial t^2$ (ix), which can be inserted into (vi) to get
rot $\underline{\boldsymbol{B}} = -\mu_0\varepsilon_0\, \partial^2\underline{\mathbf{A}}/\partial t^2$ (x).
Equating (ii) and (x) and sign reversal yields finally the wave equation for $\underline{\boldsymbol{A}}$:
$\nabla^2 \underline{\mathbf{A}} = \mu_0\varepsilon_0\, \partial^2\underline{\mathbf{A}}/\partial t^2$

5 points

**b.** State for each of the following functions $f(x,t)$ (where $x$= spatial coordinate, $t$= time, $a$,$b$,$c$= constants and $g$= arbitrary functions) whether they represent a travelling wave, a standing wave or no wave at all. Explain your answers.

(i) $f(x,t) = \cos(3xt-a)$
(ii) $f(x,t) = \sin(2at-bx)$
(iii) $f(x,t) = 4\sin(3x)\exp(-10x)$
(iv) $f(x,t) = [g(bt-x)]^2$
(v) $f(x,t) = g(at-x^2)$

Solution:
A wave travelling in $+x$-direction must be of form $f(x,t)=g(vt-x)$ where $v$ is the velocity. A standing wave has no time dependence anymore and is only periodic in $x$. Hence, (ii) and (iv) are travelling waves and (iii) is a damped standing wave. (i) and (v) are neither.

5 points

**c.** Provide a suitable sketch of the electromagnetic fields and the wave vector ***k*** of a wave freely propagating along the *z*-axis. Define the Poynting vector. State its directionality and what its magnitude describes.

Solution:

The sketch of the travelling wave should show that ***E***, ***B*** (or ***H***) and ***k*** form an orthogonal, right-handed system. Electric and magnetic field vectors should be 90º out of phase.

***S***=***E***x***H*** points along the direction of propagation and has the magnitude of power density.

## 2. Transmission Lines

7 points

**a.** The voltage as a function of position, $x$, and time, $t$, along a transmission line can generally be written as

$V(x,t)=V_0^+ \exp[j(\omega t - k'x)] + V_0^- \exp[j(\omega t + k'x)]$ (equation 2)

where $\omega$ is the angular frequency and $k'$ a propagation constant that can be written in complex notation as

$k'=a-jb$ (equation 3)

(i) Explain what both terms on the right-hand side of equation 2 mean physically.

(ii) Assuming $V_0^-=0$ and using equation (3), show that the imaginary part of $k'$ leads to an attenuation of the signal along the line.

(iii) For a lossy transmission line with conductance $G^*$ per unit length, capacitance $C^*$ per unit length, resistance $R^*$ per unit length and inductance $L^*$ per unit length, it can be shown that

$k'^2 = -(G^*+j\omega C^*)(R^* + j\omega L^*)$ (equation 4).

Calculate an approximate solution for $k'$ for the case of weak ohmic losses where $G^* \ll \omega C^*$ and $R^* \ll \omega L^*$.

Solution:

(i) The terms represent forward, i.e. in $+x$-direction (left), and backward, i.e. in $-x$-direction (right) travelling planar waves. (2 points)

(ii) Inserting equation 2 into equation 1 yields for the case of $V_0^-=0$:

$V(x,t)=V_0^+ \exp[j(\omega t - \{a-jb\}x)] = V_0^+ \exp[j(\omega t - ax)] \exp(-bx)$

These three terms describe the amplitude, the propagation and the exponential damping of a planar wave. (1 point)

(iii) Starting from equation 3:

$k'^2 = -(G^*+j\omega C^*)(R^* + j\omega L^*)$, multiplying out the terms in the brackets yields

$k'^2 = -G^*R^*+\omega^2 L^*C^* - j\omega(R^*C^*+L^*G^*)$

$\approx \omega^2 L^*C^* - j\omega(R^*C^*+L^*G^*)$ (neglecting term 1, as $G^*R^* \ll \omega^2 L^*C^*$)

$= \omega^2 L^*C^* [1-j(R^*/L^*+G^*/C^*)/\omega]$

Taking the square root of both sides:

$k' = \omega (L^*C^*)^{1/2} \{1-j[(R^*/(\omega L^*)+G^*/(\omega C^*)]\}^{1/2}$

Using the approximation $\sqrt{(1-x)} \approx 1-x/2$ for small $x$, this gives:

$k' \approx \omega (L^*C^*)^{1/2} [1-j/(2\omega)(R^*/L^*+G^*/C^*)]$ (4 points)

4 points

**b.** A 50kHz signal is fed into a printed circuit board that can be described as a lossy transmission line with the characteristics of $L^*$=1mH/m, $C^*$=1nF/m, $R^*$=1Ω/m, $G^*$=0.001/(Ω m). Use your above approximate solution for $k'$ to calculate over what length the signal can be transferred so that at least 90% of the voltage of the input signal arrives. Is this feasible for implementation?

Solution:

Use $\omega=2\pi f$ where $f$=50kHz (1 point)

and insert numbers in above equation to get

$k' = \omega (L^*C^*)^{1/2} [1-j/(2\omega)(R^*/L^*+G^*/C^*)]$

$=0.3142\ \mathrm{m}^{-1} [1-1.592j\times10^{-6}(10^3+10^6)] = 0.3142\ \mathrm{m}^{-1} [1-1.593j]$ (1 point)

If $k' = a - \mathrm{j}b$ with real components $a$ and $b=0.5\mathrm{m}^{-1}$, then $0.90=|V/V_0|=\exp(-bx)$. (1point)

This yields $x=-(\ln 0.90)/b=2.1\times10^{-4}\mathrm{m}=0.21\mathrm{m}$.
This is of the order of typical printed circuit board sizes and means it will be fine as long as the signal is used to within 21cm from where it is fed in. (1 point)

9 points

**c.** Oliver Heaviside found that if the relationship $G^*/C^*=R^*/L^*$ is obeyed, then a transmission line will show no dispersion at all.

(i) Explain what dispersion means and how it leads to signal distortion on lossy transmission lines.

(ii) Insert the above expression by Heaviside into equation (4) from Question 2**a** and calculate the signal velocity.

(iii) Compare the above result to that from equation (4) for a lossless transmission line with $G^*=R^*=0$.

Solution:

i) Dispersion means the signal velocity depends on the frequency, i.e. $v=v(\omega)$. As every signal in the time domain will have a frequency spectrum (given by its Fourier transform), the signal will be distorted if every frequency component is transferred with a different speed. (2 points)

ii) The (real) signal velocity is given by the ratio $v=\omega/a$ where the denominator is the real part of the wave vector $k'$ as defined in equation 2. (1 point)

Inserting the Heaviside relationship into equation 4 yields:

$$k'^2 = -(G^*+\mathrm{j}\omega C^*)(R^*+\mathrm{j}\omega L^*)$$
$$= -(G^*+\mathrm{j}\omega C^*)(G^*L^*/C^*+\mathrm{j}\omega L^*)$$
$$= -(G^*+\mathrm{j}\omega C^*)(G^*+\mathrm{j}\omega C^*)\,L^*/C^*$$
$$=-(G^*+\mathrm{j}\omega C^*)^2\;L^*/C^* \qquad \text{(2 points)}$$

Calculation of the square root on both sides yields

$$k' = \mathrm{j}\,(G^*+\mathrm{j}\omega C^*)\,(L^*/C^*)^{1/2} \qquad \text{(1 point)}$$

Sorting real and imaginary parts gives

$$k' = (L^*/C^*)^{1/2}\,(-\omega C^*+jG^*) = a+\mathrm{j}b$$

The real part, $a$, is:

$$a=-\omega C^*(L^*/C^*)^{1/2} = -\omega(L^*C^*)^{1/2} \qquad \text{(1 point),}$$

which is linear in $\omega$, hence the ratio

$v=\omega/a=-(L^*C^*)^{-1/2}$ is no longer dependent on $\omega$. (1 point)

iii) For a lossless transmission line with $G^*=R^*=0$, equation (4) yields:

$k'^2 = -(G^*+\mathrm{j}\omega C^*)(R^*+\mathrm{j}\omega L^*) = \omega^2 L^*C^*$, hence

$k'=\omega\,(L^*C^*)^{1/2}$, which is real-valued, so (1 point)

$v=\omega/k'=(L^*C^*)^{-1/2}$

is numerically identical apart from the sign! (1 point)

### 3. Electric potential and electronic devices: capacitors and pn-junctions

10 points

**a.** Consider a plate capacitor of width $w$, length $l$, distance $d$ between the plates that is filled with a dielectric of relative permittivity $\varepsilon_r$.

(i) Using Coulomb's Law and Gauss' Law, calculate the magnitude $E$ of the electric field between the plates as a function of the charge $Q$ on the plates.

(ii) From this, determine the voltage by integration.

(iii) From the above, derive the well-known relationship for the capacitance of a plate capacitor.

Solution:

(i)
$Q = \iiint \rho \, dV$ (definition of charge density)
$= \iiint \text{div}\, \underline{\boldsymbol{D}} \, dV$ (Coulomb's Law: div $\underline{\boldsymbol{D}}=\rho$)
$= \varepsilon_0 \varepsilon_r \iiint \text{div}\, \underline{\boldsymbol{E}} \, dV$ ($\underline{\boldsymbol{D}} = \varepsilon_0 \varepsilon_r \underline{\boldsymbol{E}}$)
$= \varepsilon_0 \varepsilon_r \int_S \underline{\boldsymbol{E}} \, d\underline{\boldsymbol{s}}$ (Gauss' Law, where the surface integral is over the closed surface)
$= \varepsilon_0 \varepsilon_r E \int_S \underline{\boldsymbol{n}} \, d\underline{\boldsymbol{s}}$ (where $\underline{\boldsymbol{n}}$ is the unity normal vector pointing from one to the other plate, and $\int_S \underline{\boldsymbol{n}} \, d\underline{\boldsymbol{s}} = wl$)
$= \varepsilon_0 \varepsilon_r E wl,$
hence $E = Q/(\varepsilon_0 \varepsilon_r wl)$ (6 points)

(ii)
$\underline{\boldsymbol{E}} = -\text{grad}\, V$ (definition of voltage $V$)
$= -\partial V/\partial x$ ($x$ being the direction from one to other plate),
hence
$V = -\int E dx$ (where integration extends from $x$=0 to $x$=$d$)
$= -E \int dx$ (as $E$ is constant between plates)
$= -E d$
$= -Qd/(\varepsilon_0 \varepsilon_r wl)$ (2 points)

(iii)
$C = |Q/V|$ (definition of capacitance $C$)
$= \varepsilon_0 \varepsilon_r wl/d$
$= \varepsilon_0 \varepsilon_r A/d$ with total area $A=wl$ (2 points)

5 points

**b.** The function

$V(x) = (2ax - x^2)\, \rho_{\text{free}}/(2\varepsilon_0 \varepsilon_r)$ (equation 5)

describes the potential profile across a semiconducting pn-junction of total depletion layer width $2a$ along the $x$-direction. Assume $\varepsilon_0=8.8542\times10^{-12}$ F/m. Choose the origin in the middle of the pn-junction to calculate for a depletion layer width of $2a$=100nm, a free charge density of $\rho_{\text{free}}$ =8000C/m$^3$, a dielectric constant of $\varepsilon_r$=9 and a cross-sectional area of $A=10^{-7}$ m$^2$

(i) the voltage drop across the whole junction and

(ii) the junction capacitance.

(iii) Compare the junction capacitance quantitatively to that of a standard plate capacitor.

Solution:

(i) the voltage drop across the whole pn-junction is
$\Delta V = V(a) - V(-a) = (a^2 - 3a^2)\, \rho_{\text{free}}/(2\varepsilon_0 \varepsilon_r) = a^2 \rho_{\text{free}}/(\varepsilon_0 \varepsilon_r)$.
Inserting numbers ($a$=50nm (!)) yields 0.25V. (2 points)

(ii) The charge contained at either side of the pn-junction is

$Q=\rho V=\rho aA=\rho_{\text{free}}\, aA/\varepsilon_r$

The capacitance then is $C=Q/\Delta V=[\rho_{\text{free}}\, aA/\varepsilon_r]/[a^2\rho_{\text{free}}/(\varepsilon_0\,\varepsilon_r)]=\varepsilon_0 A/a=17.7\text{pF}$.

(2 points)

(iii) This is the same capacitance as for a plate capacitor in vacuo with $\varepsilon_r=1$ and effective (average) plate distance $d=a$. (1 point)

5 points

**c**. Consider a dielectric material with complex permittivity $\varepsilon_r=\varepsilon_r'+j\varepsilon_r''$ and complex refractive index $N=n+j\kappa$. Using the relationship $N^2=\varepsilon_r$, derive expressions for real part $\varepsilon_r'$ and imaginary part $\varepsilon_r''$. What happens in the special case of $n=\kappa$ to both, and what does this mean for the relationship between the fields $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{D}}$?

Solution:

$\varepsilon_r'+j\varepsilon_r''=\varepsilon_r=N^2=(n+j\kappa)(n+j\kappa)=n^2-\kappa^2+j2n\kappa$ (1 point)

So, the real part is $\varepsilon_r'=n^2-\kappa^2$ and vanishes if $n=\kappa$. (1 point)

The imaginary part is $\varepsilon_r''=2n\kappa$ and becomes maximal for $n=\kappa$. (1 point)

This implies that $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{D}}=\varepsilon_0\varepsilon_r\underline{\boldsymbol{E}}$ are now 90° out of phase, so when one varies in time like a cosine function, then the other oscillates like a sine function, and vice versa. (2 points)

## 4. Wave Optics

9 points

**a.** Consider two parallel plates of width $w$ and length $l$ that are separated by a distance $d$. Their specific capacitance per length is given by

$$C^*=\varepsilon_0\varepsilon_r w/d \qquad \text{(equation 6).}$$

Their specific inductance per unit length is given by

$$L^*=\mu_0\mu_r d/w \qquad \text{(equation 7).}$$

(i) Calculate the characteristic impedance of the plates.
(ii) Show that the phase velocity of the signal wave travelling on the plates is given by $c\,(\mu_r\varepsilon_r)^{-½}$, where $c$ is the speed of light in vacuum.
(iii) Calculate the ratio of the electric to magnetic field strength, $E/H$.
(iv) From this, show similarly that $E/B=c$ in vacuum.

Solution:

(i) $Z_0=(L^*/C^*)^{½}=[(\mu_0\mu_r d/w)/(\varepsilon_0\varepsilon_r w/d)]^{½}=(d/w)\,[(\mu_0\mu_r)/(\varepsilon_0\varepsilon_r)]^{½}$ (2 points)

(ii) $v=(L^*C^*)^{-½}=[(\mu_0\mu_r d/w)(\varepsilon_0\varepsilon_r w/d)]^{-½}=(\mu_0\mu_r\varepsilon_0\varepsilon_r)^{-½}=c\,(\mu_r\varepsilon_r)^{-½}$
where $c=(\mu_0\varepsilon_0)^{-½}$ (3 points)

(iii) $E/H$
with $E=V/d$ from the definition of $\underline{\mathbf{E}}=-\mathrm{grad}\,V$ (where $E$=const. here)
and $H=I/w$ from Ampere's Law rot $\underline{\mathbf{H}}=\underline{\mathbf{j}}$, or $\int\underline{\mathbf{H}}\,\mathrm{d}\underline{\mathbf{r}}=I$, hence
$E/H=(V/d)/(I/w)$
$=(w/d)\,(V/I)$
where the voltage/current ratio is again the above impedance $Z_0$, so
$=\cancel{(w/d)\,(d/w)}\,[(\mu_0\mu_r)/(\varepsilon_0\varepsilon_r)]^{½}$ (3 points)

(iv) $E/B=1/(\mu_0\mu_r)\,E/H$
$=(\mu_0\mu_r\varepsilon_0\varepsilon_r)^{-½}$ (from above, same as (ii)!)
$=c$ for vacuum, as in vacuum $\mu_r=1=\varepsilon_r$ (1 point)

3 points

**b.** Consider light transversing from a medium with refractive index $n_1$ to another, denser one with refractive index $n_2$. Assume the sine of the angle with respect to the vertical, $\sin\theta$, is proportional to the speed of light in the corresponding medium, which is $v=c\,(\mu_r\varepsilon_r)^{-½}$. From this, derive Snell's Law of refraction, which states

$$n_1\sin\theta_1=n_2\sin\theta_2 \qquad \text{(equation 8)}$$

Solution:

$\sin\theta_1/\sin\theta_2=v_1/v_2=c\,(\mu_{r1}\varepsilon_{r1})^{-½}/\,c\,(\mu_{r2}\varepsilon_{r2})^{-½}=[(\mu_{r2}\varepsilon_{r2})/(\mu_{r1}\varepsilon_{r1})]^{½}=n_2/n_1$
if $n=(\mu_r\varepsilon_r)^{½}$ for each material (for most optical materials, $\mu_r=1$ and so $n=\sqrt{\varepsilon_r}$).

8 points

**c.** Fresnel's formula for the reflectivity of a surface for in-plane polarisation, without any absorption, is given by

$$R_\parallel=(n_2\cos\theta_1-n_1\cos\theta_2)^2/(n_2\cos\theta_1+n_1\cos\theta_2)^2 \qquad \text{(equation 9)}$$

(i) Sketch what happens in the case of $\theta_1+\theta_2=90°$ to the electric field vector $\underline{\mathbf{E}}$ of the reflected wave.

(ii) Applying Snell's Law, derive an expression for the incidence angle $\theta_1$ in the case of $\theta_1+\theta_2=90°$ .

(iii) Describe what happens to $R_\parallel$ in the case of $\theta_1+\theta_2=90°$ .

(iv) From equation 9, derive an expression for $R_\parallel$ for the case of vertical incidence.

Solution:

(i) The sketch should show that the reflected and the transmitted rays are perpendicular to one another. The outgoing reflected wave vector $\underline{\boldsymbol{E}}_r$ has no longer a component in the plane of incidence ($E_r$=0) because that would be aligned parallel to the transmitted ray which already has the total in-plane component required from the continuity of the tangential components of the electric wave vectors: $E_i \cos\theta_1 = E_t \cos\theta_2 = E_t \sin\theta_1$ now)

(2 points)

(ii) Snell's Law states $n_2/n_1 = \sin\theta_1 / \sin\theta_2$.
From $\theta_1+\theta_2=90°$ we get $\sin\theta_2 = \sin(90°-\theta_1) = \cos\theta_1$, which can be substituted into above, giving

$n_2/n_1 = \sin\theta_1 / \cos\theta_1 = \tan\theta_1$, so the result is

$\theta_1 = \arctan n_2/n_1$. (3 points)

NB: This is known as Brewster's angle.

(iii)

In this case, the bracket in the numerator of $R_\parallel$ becomes

$n_2 \cos\theta_1 - n_1 \cos\theta_2$
$= n_2 \sin\theta_2 - n_1 \cos\theta_2$
$= n_1 \sin\theta_1 - n_1 \cos\theta_2$
$= n_1 (\sin\theta_1 - \cos\theta_2)$
$=0$, because also $\sin\theta_1 = \cos\theta_2$ for $\theta_1+\theta_2=90°$.

This means there is <u>no</u> reflected p-polarised wave in reflection. (2 points)

(iv) For vertical incidence, $\theta_1=0=\theta_2$,

hence: $\cos\theta_1=1=\cos\theta_2$,

hence: $R_\parallel = (n_2 - n_1)^2/(n_2 + n_1)^2 = [(n_2 - n_1)/(n_2 + n_1)]^2$ (1 point)