3. Applications of electrostatic field calculation methods

Analytical solution of Laplace's and Poisson's in two-dimensions

The geometry and boundary conditions in the examples shown previously (parallel plate capacitor and p-n junction) are such that the field calculations are essentially a one-dimensional problems, in which Laplace's and Poisson's equations reduce the very straightforward integrals in one dimensions. The electrostatic fields in many other practical problems of interest are at least two-dimensional and often three-dimensional. Solving two- and three-dimensional problems in which the source is specified in terms of potential distributions is significantly more challenging and requires the use of some mathematical tricks and tools to establish solution.

The most straightforward solution method for two-dimensional problems (it should be said that even these are mathematically challenging) are based on methods which the variations in two directions are separated into two differential equations in x and y only, exploiting an elegant mathematical trick. The solution to these differential equations are established by specifying a general solution form and then determining the various coefficients of the general solution form by applying the various boundary conditions of the problem domain.

Despite their mathematical complexity, these advanced analytical methods can still only deal with relatively simple geometries and potential distributions. These methods can be used for solving problems such as the trough shown in Figure 2. In this case, the potential of the lid is fixed at V_0 and is insulated from the remaining 3 sides which are grounded.

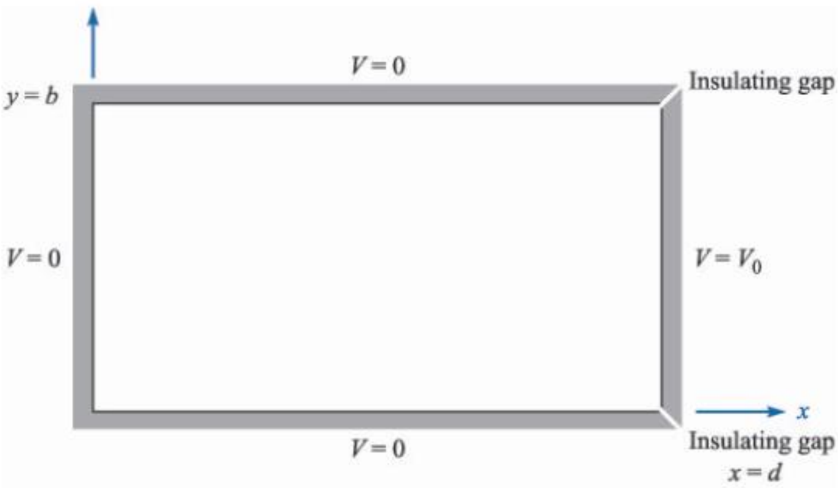

Figure 1 Trough geometry and potential distributions (Source: Engineering Electromagnetics – McGraw Hill)

A full derivation of the field solution, which starts with the underlying assumptions which allow the x and y variations is described in detail in the textbook 'Engineering Electromagnetics' by Hayt and Buck (McGraw Hill – limited number of copies in IC and St. George's Library). [*This material is not examinable as is intended purely as background and give those of you who are interested in seeing the complexity involved in solving even a straightforward problem such as this]*.

The resulting expression for the potential at any point (x,y) within the trough is given by a summation of an infinite series:

$$V(x,y) = \frac{4V_0}{\pi} \sum_{m=1,odd}^{\infty} \frac{1}{m} \frac{\sinh(\frac{m\pi x}{b})}{\sinh(\frac{m\pi d}{b})} \sin\frac{m\pi y}{b}$$

A reasonable approximation to the potential can be obtained by summing the first few terms of this series. The resulting potential distribution (shown with 10V increments in potential contours) is shown in Figure 2, for the particular case of a square trough in which b=d and V_0=100V.

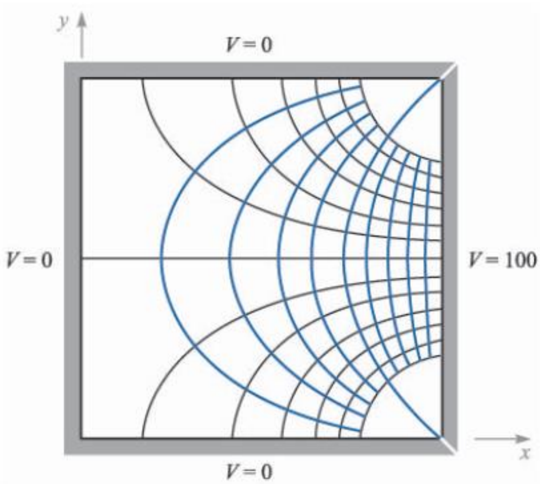

Figure 2 Solution to electric potential distribution with the trough for b=d and V_0 =100V (Source: Engineering Electromagnetics – McGraw Hill)

Whereas analytical calculations of this type are elegant and produce exact solutions at any single point, albeit with an infinite series, their applicability is limited by the relatively simple geometries which can be tackled (there are further examples in 'Engineering Electromagnetics' by Hayt and Buck). Hence, in many applications, engineers resort to the use of numerical methods such as finite element methods to solve electrostatic problems of real components with intricate geometries. We will return to numerical methods later in this module.

Electrostatic field calculations for overhead power transmission lines

Overhead power transmission lines form the back-bone of any large scale power network. Large pylons with a series of conductors at voltages up to 400kV rms (in the UK) are commonplace and the precise calculation of the electric fields is important for designing the pylon and ancillary equipment, and increasingly in quantifying the fields to which adjacent structure are exposed.

In power systems it is usual to specify, and indeed control, the overhead lines in terms to a specific voltage rating rather than a charge density. However, as seen above, the solution of Laplace's and Poisson's equations allow us to calculate the potential distribution throughout a problem domain from known potentials is very challenging in analytical form. Indeed, the calculation of localised and remote fields for the detailed design of high voltage equipment is now routinely performed using numerical models rather than by deriving analytical expressions.

However, if the overhead line conductor is specified in terms of a charge density, then analytical methods based on applying Gauss's Law can provide a wealth of useful practical information on the localised and remote electric fields. It is also worth pointing that although the vast majority of electrical power world-wide is transmitted as AC (there is high voltage DC (~750kV) used in some countries over very long distances or to inter-connect systems) we can use electrostatic field calculations for 50/60Hz power frequencies by taking a 'snapshot' of the instantaneous charge density.

<u>Electric field in the region outside a single conductor of finite cross-section</u> - A useful starting point for calculating electrostatic fields in representative transmission line is to start with a single remote conductor of length *l* and radius R_c which is located in free space as shown in Figure 3.

Define the direction of the radial electric field as being radially outwards for this positive charge

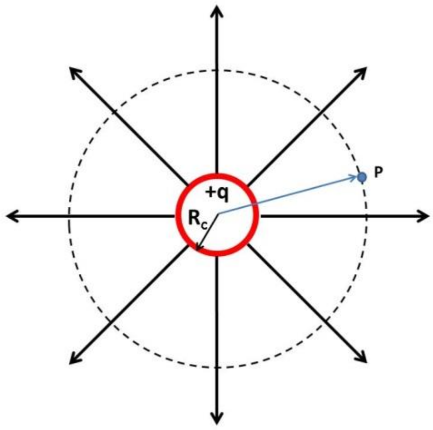

Figure 3 Single line conductor of radius R_c, length l with volume charge density q (C/m^3)

Consider a general point P which is outside the region occupied by the conductor and located at a radius r relative to the centre of the conductor. Applying Gauss's Law over a cylindrical surface at a radius r and noting that the electric field is purely radial yields:

$$\oiint \vec{D}.\vec{ds} = \iiint q\, dv$$

$$2\pi r l \overrightarrow{D_r} = \pi R_c^2 l\, q$$

Hence,

$$\overrightarrow{D_r} = \frac{R_c^2 q}{2r}\overrightarrow{e_r}$$

But $\vec{D} = \varepsilon\vec{E}$ and so:

$$\overrightarrow{E_r} = \frac{R_c^2 q}{2\varepsilon r}\overrightarrow{e_r}$$

This expression allows the electrical field at any point outside the conductor to be calculated by substituting in the appropriate values of r. It is worth noting that different textbooks may start with an alternative definition of the charge on the conductor, specifically they may start with a definition of q as charge per unit length of the conductor (C/m) and hence will end up with an apparently different expression for the electrical field – but once the definition used for q is factored in, the expressions are in fact equivalent.

As described in previous notes, the electric potential difference (or voltage) between any two points in the region outside the conductor can be calculated with a line integral between these two points, recalling that providing the integral is performed between the same start and end points, the net value of the integral is independent of the exact path taken. Furthermore, it should be recalled that potential increases when a work is done against the electric field.

By way of example, consider the path shown between P_1 and P_2 in Figure 4. Any path can be followed between P_1 and P_2, but it is convenient to take a radial path to a point P_{int} (along which $\vec{E}$ is parallel to the path) and a circumferential path from P_{int} to P_2 (along which $\vec{E}$ is perpendicular to the path)

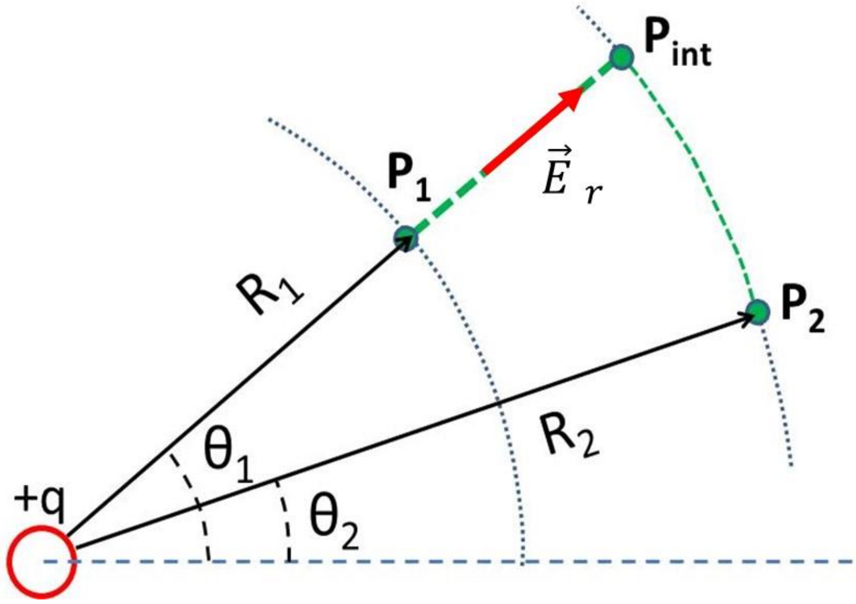

Figure 4 Integration path between P_1 and P_2

$$V = -\int_{R1}^{R2} \overrightarrow{E_r} \cdot \overrightarrow{dr} - \int_{\theta 1}^{\theta 2} \overrightarrow{E_r} \cdot \overrightarrow{d\theta} = -\int_{R1}^{R2} \frac{R_c^2 q}{2\varepsilon r} \overrightarrow{e_r} \cdot \overrightarrow{dr} - \int_{\theta 1}^{\theta 2} 0 \cdot \overrightarrow{e_\theta}\, \overrightarrow{d\theta}$$

$$= -\left[\frac{R_c^2 q}{2\varepsilon} log_e r\right]_{R1}^{R2} + 0 = -\frac{R_c^2 q}{2\varepsilon} log_e \left(\frac{R_2}{R_1}\right)$$

In the case of this positive charge the electric field is radially outward. Hence, potential increases with reduced distance from the conductor.

It is also possible to apply Gauss's Law within the cross-section of the conductor itself – see tutorial sheet 2.

<u>Single-phase, two conductor transmission line</u> - The simplest form of power distribution lines are two parallel lines carrying a single phase as shown in Figure 5. Consider the simplified case in which the separation between the two lines is much smaller than the distance to any surrounding structure or the ground, i.e. they are to all practical purposes a remote pair of conductors. Each conductor has volume charge density of q.

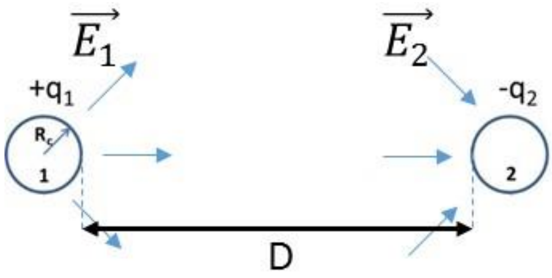

Figure 5 Dual circuit single-phase transmission lines

The presence of the return conductor affects the net electrical field distribution. However, it is still possible to apply the principle of superposition, recalling that the electric field produced by the charge on an individual circular conductor only has a radial component. The contribution to the overall potential difference between conductors 1 and 2 due to conductor 1, noting that angle between the radial component of E and the integration path is 0^o , is given by:

$$V_{1-2,1} = -\int_{R_c}^{D} \overrightarrow{E_{r1}} \cdot \overrightarrow{dr} = -\left[\frac{R_c^2 q_1}{2\varepsilon} log_e r\right]_{R_c}^{D} = -\frac{R_c^2 q_1}{2\varepsilon} log_e\left(\frac{D}{R_c}\right)$$

Similarly the contribution to the overall potential difference between conductors 2 and 1 due to conductor 2, noting in this case that the angle between the radial component of E and the integration path is 180^o is given b:

$$V_{2-1,2} = -\int_{R_c}^{D} \overrightarrow{E_{r2}} \cdot \overrightarrow{dr} = -\left[-\frac{R_c^2 q_2}{2\varepsilon} log_e r\right]_{R_c}^{D} = \frac{R_c^2 q_2}{2\varepsilon} log_e\left(\frac{D}{R_c}\right)$$

Hence, the net potential *difference* between the conductors (taking care to note the subscripts and conventions) is:

$$V_{1-2} = V_{1-2,1} + V_{1-2,2} = V_{1-2,1} - V_{2-1,2} = -\frac{R_c^2 q_2}{2\varepsilon} log_e\left(\frac{D}{R_c}\right) - \frac{R_c^2 q_1}{2\varepsilon} log_e\left(\frac{D}{R_c}\right)$$

But for the simple arrangement shown in Figure 5, the magnitude of the charges q_1 and q_2 are equal and can be regarded as simply q (not always the case in all practical systems where the go and return might be affected differently by the presence of another conductor or earthed structure which is asymmetrically located). It is important to note that the different polarities of q_1 and q_2 have been accounted for already in the definition of the direction of the radial component of electric field.

$$V_{1-2} = -\frac{R_c^2 q}{2\varepsilon}\left(log_e\left(\frac{D}{R_c}\right) + log_e\left(\frac{D}{R_c}\right)\right) = -\frac{R_c^2 q}{\varepsilon} log_e\left(\frac{D}{R_c}\right)$$

This resulting capacitance of this arrangement (for a length of conductor *l*) can be calculated recognising that the total charge of this length of conductor is given by:

$$Q = \pi R_c^2 l q$$

$$C = \frac{Q}{V} = \frac{\pi l \varepsilon}{log_e\left(\frac{D}{R_c}\right)}$$

This can be expressed as a capacitance per unit length, c, simply as:

$$c = \frac{C}{l} = \frac{\pi \varepsilon}{log_e\left(\frac{D}{R_c}\right)}$$

Accounting for the ground plane - method of images - The preceding analysis has considered the cases of charge distributions which are located remotely from any boundaries with defined potentials. In many practical problems, charges may be located near a ground-plane, e.g. an overhead transmission line and the surrounding earth. The method of images provides a useful means of accounting for certain boundary conditions by adding in additional charges to replicate the influence of the boundary. The most straightforward case which illustrates this method is a single, infinitely long, line charge located a distance 'h' from an infinite ground plane as shown in Figure 6. The ground plane in such a way that for all value of x, the potential at y=0 is zero.

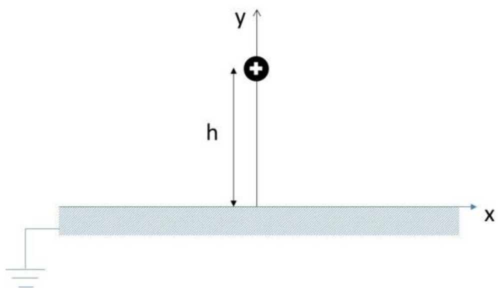

Figure 6 Line charge located a distance 'h' above an infinite ground plane

On inspection of Figure 6, it should be clear the presence of the ground plane will affect the electric field around the point charge as compared to the usual cylindrical symmetry of the field produced by an isolated point charge in free space. The presence of the ground plane introduces a boundary condition that V=0 at y=0 for all values of x, reducing the problem domain to the region to y>0. A convenient means of including the effect of the ground plane into the model, which makes the calculation of the net field in the region y>0 very straightforward is to introduce an equal but opposite polarity charge at a location which is symmetrical about y=0 as shown in Figure 7.

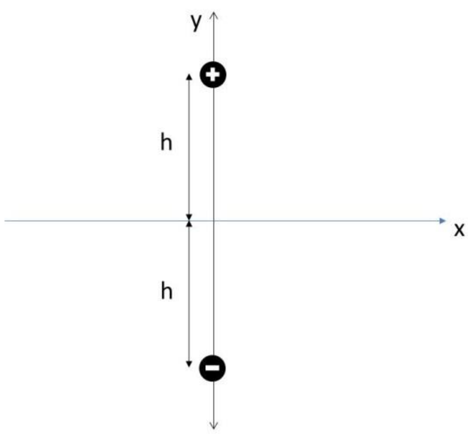

Figure 7 Use of image charge to represent a ground plane

The image charge density located at y=-h allows us to calculate the net electric field at any point above the ground plane. A schematic representation of the electric flux densities at a general point P above the ground plane is shown in Figure 8. It is worth noting that the negative polarity of the charge density of the image relative to the positive physical charge density is already factored into the direction of D_2 with respect to the defintion of the reference angles, i.e. it is defined as being radially inwards relative to the image charge. An equally valid alternative in mathematical terms which is perhaps little less clear when shown graphically would have been to define the electric flux density as being radially outwards and then ascribing a negative charge.

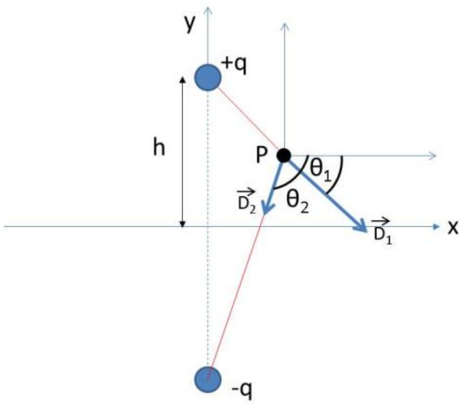

Figure 8 Use of method of images to calculate the electric flux density produced by a line charge above a ground plane

If we apply Gauss's law due to the original physical line charge of length l and radius R_c, then the electric flux density vector at the point P is given by:

$$\overrightarrow{D_1} = \frac{R_c^2 q}{2r_1 l}\left(cos\theta_1\,\overrightarrow{e_x} + sin\,\theta_1\,\overrightarrow{e_y}\right)$$

where $r_1 = \sqrt{{x_p}^2 + \left(h - y_p\right)^2}$

Similarly the electric flux density vector at the point P due to the image line charge is given by:

$$\overrightarrow{D_1} = \frac{R_c^2 q}{2r_2 l}\left(cos\theta_2\,\overrightarrow{e_x} + sin\theta_2\,\overrightarrow{e_y}\right)$$

where $r_2 = \sqrt{{x_p}^2 + \left(h + y_p\right)^2}$

Hence the net electric flux density vector at point P (recalling that the negative polarity of the image charge is already factored into the definition of θ_2) is given by:

$$\overrightarrow{D_{net}} = \frac{R_c^2 q}{2l}\left(\left(\frac{cos\theta_1}{r_1} + \frac{cos\theta_2}{r_2}\right)\overrightarrow{e_x} + \left(\frac{sin\theta_1}{r_1} + \frac{sin\theta_2}{r_2}\right)\overrightarrow{e_y}\right)$$

The net electric field strength vector at point P is given by:

$$\overrightarrow{E_{net}} = \frac{\overrightarrow{D_{net}}}{\varepsilon} = \frac{R_c^2 q}{2l\varepsilon}\left(\left(\frac{cos\theta_1}{r_1} + \frac{cos\theta_2}{r_2}\right)\overrightarrow{e_x} + \left(\frac{sin\theta_1}{r_1} + \frac{sin\theta_2}{r_2}\right)\overrightarrow{e_y}\right)$$

It is possible to apply a quick check at this point to ensure that this expression is consistent with the boundary condition of the ground plane. Since the ground plane has a fixed potential for all values of x, then the electric field strength component E_x will be zero for y=0.

Putting point P at y=0 and x=0 (any value of x will do, but x=0 simplifies the expression) corresponds to $\theta_1 = -90^0$ and $\theta_2 = -90^0$ which indeed yields a zero x-component of $\overrightarrow{E_{net}}$.

Having established the principle of the method of images, we can then calculate the capacitance to ground of the physical charge etc. using the same principles that were used for the pair of line charges above. We have now established a means of accounting for adjacent conductors and ground planes, which in principle, gives us the tools to analyse more complex arrangements of conductors and transmission towers.