4. the electrodynamic dipole

If the dipole $\mathbf{p} = \mathbf{p}(t)$ becomes a function of time, then the time derivates come into play as well.
In the following we write

$$\dot{\mathbf{p}} = \frac{\partial \mathbf{p}}{\partial t}$$

$$\text{and} \quad \ddot{\mathbf{p}} = \frac{\partial^2 \mathbf{p}}{\partial t^2}$$

for the first and second derivates with respect to time.
We do not derive the full field equations (see e.g. Jackson: Classical Electrodynamics; or: Kraus: Electromagnetics for full derivations).

$$V_{dipole}(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\left(\frac{\dot{\mathbf{p}}\cdot\mathbf{r}}{c r^2} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\right) \qquad \text{with } \mathbf{p} = \mathbf{p}(t)$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}\,\dot{\mathbf{p}} \qquad \text{with } \dot{\mathbf{p}} = \dot{\mathbf{p}}\left(t - \frac{r}{c}\right)$$

$\Rightarrow$

$$\mathbf{E} = -\operatorname{grad} V - \dot{\mathbf{A}}$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{(\ddot{\mathbf{p}}\mathbf{r})\mathbf{r}}{c^2 r^3} - \frac{\ddot{\mathbf{p}}}{c^2 r} + \frac{3(\dot{\mathbf{p}}\mathbf{r})\mathbf{r}}{c r^4} - \frac{\dot{\mathbf{p}}}{c r^2} + \frac{3(\mathbf{p}\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right]$$

$$\mathbf{B} = \operatorname{rot} \mathbf{A}$$

$$= \frac{\mu_0}{4\pi}\left[\frac{\ddot{\mathbf{p}}\times\mathbf{r}}{c r^2} + \frac{\dot{\mathbf{p}}\times\mathbf{r}}{r^3}\right] \qquad \text{with } \mathbf{p} = \mathbf{p}\left(t - \frac{r}{c}\right)$$

Note:

a) only terms with $\dot{\mathbf{p}}$ and $\ddot{\mathbf{p}}$ are new – the terms with $\mathbf{p}$ are identical to the case for the electrostatic dipole (stationary solution for $\frac{\partial}{\partial t} = 0$ and $\frac{\partial^2}{\partial t^2} = 0$).

b) the terms with $\ddot{\mathbf{p}}$, $\dot{\mathbf{p}}$ and $\mathbf{p}$ fall off with $\frac{1}{r}$, $\frac{1}{r^2}$ and $\frac{1}{r^3}$, respectively. We therefore need to distinguish near-field conditions and far-field conditions!

c) near-field solution:

neglect terms proportional to $\frac{1}{r}$ or $\frac{1}{r^2}$ and retain only those proportional to $\frac{1}{r^3}$.

$$\Rightarrow \underline{E}_{\text{near-field, dipole}} \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\underline{p}\,\underline{r})\underline{r}}{r^5} - \frac{\underline{p}}{r^3}\right)$$

is the same as for the electrostatic case, $\underline{E}_{\text{dipole, stat.}}$!

$$\underline{B} \approx \frac{\mu_0}{4\pi}\,\frac{\dot{\underline{p}}\times\underline{r}}{r^3} \neq 0$$ depends on <u>velocity</u> of dipole movement.

d) far-field solution:

neglect terms proportional to $\frac{1}{r^2}$ and $\frac{1}{r^3}$ and retain only those proportional to $\frac{1}{r}$ that decay the slowest with distance.

$$\Rightarrow \underline{E}_{\text{far-field, dipole}} \approx \frac{1}{4\pi\varepsilon_0 c^2}\left(\frac{(\ddot{\underline{p}}\,\underline{r})\underline{r}}{r^3} - \frac{\ddot{\underline{p}}}{r}\right)$$

$$\underline{B}_{\text{far-field, dipole}} \approx \frac{\mu_0}{4\pi c}\,\frac{\ddot{\underline{p}}\times\underline{r}}{r^2}$$

both depend on the <u>acceleration</u> of the dipole's charges.

In particular, using the vector rule $\underline{A}\times(\underline{B}\times\underline{C}) = \underline{B}(\underline{A}\cdot\underline{C}) - \underline{C}(\underline{A}\cdot\underline{B})$, we can show that

$$\boxed{\underline{E}_{\text{far-field, dipole}} \equiv \underline{E}_{\text{far}} = -c\,\underline{B}_{\text{far}}\times\underline{e}_r}$$

and

$$\boxed{\underline{B}_{\text{far}} = \frac{1}{c}\,\underline{e}_r\times\underline{E}_{\text{far}}}$$

where again $\underline{e}_r = \frac{\underline{r}}{r}$ is the radial unity vector.

Define angle $\theta = \sphericalangle(\underline{p}, \underline{r})$.

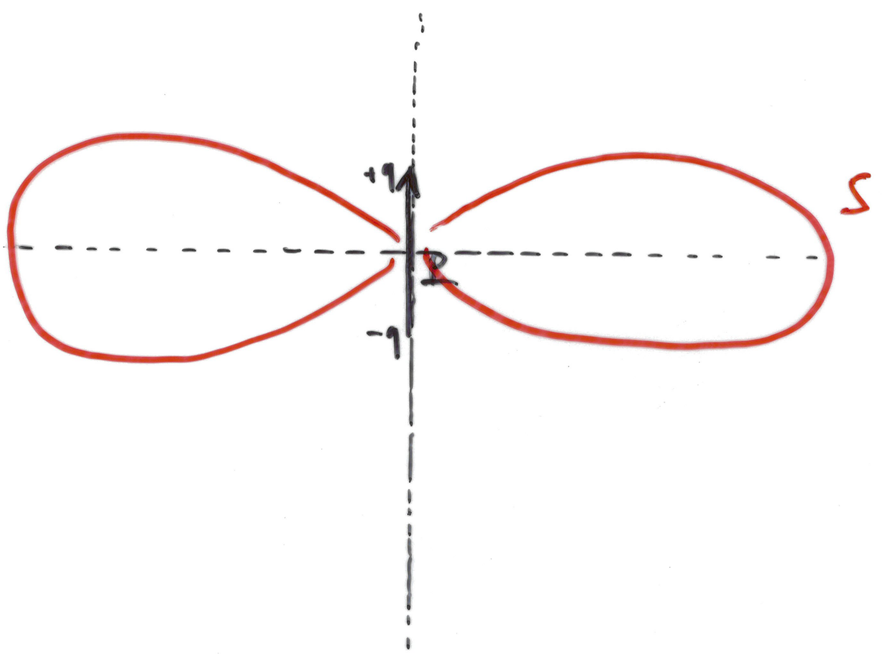
$$\Rightarrow \boxed{\begin{aligned} \underline{S} &= \underline{E}_{\text{far}}\times\underline{H}_{\text{far}} \\ &= \frac{c}{\mu_0}\left(\underline{B}_{\text{far}}\times\underline{e}_r\right)\times\underline{B}_{\text{far}} \\ &= \frac{c}{\mu_0}\,B_{\text{far}}^2\,\underline{e}_r \\ &= \frac{\mu_0}{16\pi^2 c}\,\underbrace{\frac{(\ddot{\underline{p}}\times\underline{r})^2}{r^4}}_{\frac{\ddot{p}^2\sin^2\theta}{r^2}}\,\underline{e}_r \end{aligned}}$$

Note that $|\underline{S}| = \frac{\mu_0}{16\pi^2 c} \frac{\ddot{p}^2 \sin^2\theta}{r^2} \propto \sin^2\theta$

strongly depends on the angle θ between $\underline{p}$ and $\underline{r}$:

$S = 0$ for $\theta = 0°$

$S = \frac{\mu_0}{16\pi^2 c} \frac{\ddot{p}^2}{r^2}$ for $\theta = 90°, 270°$



special case of harmonic oscillator: $\underline{p} = \underline{p_0} \sin\omega t$

$\Rightarrow \ddot{\underline{p}}(t - \frac{r}{c}) = -\omega^2 \underline{p_0} \sin[\omega(t - \frac{r}{c})]$

As the time average over $\sin^2[\omega(t - \frac{r}{c})] = \frac{1}{2}$

$\Rightarrow \langle \underline{S} \rangle = \frac{\mu_0}{32\pi^2 c} \underline{p_0}^2 \omega^4 \frac{1}{r^2} \underline{e_r}$

$\hookrightarrow$ decays like a spherical wave

$\propto p_0^2$; i.e. $\propto q^2 ds_{max}^2$

$\propto f^4$; i.e. explains Rayleigh-scattering „why the sky is blue"