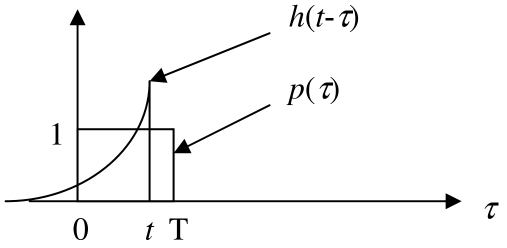
Solution EEE201 Jan 2008

Q1)

a) The response is $y(t) = p(t) * h(t) = \int_{-\infty}^{\infty} p(\tau)h(t-\tau)d\tau$.

Alternatively evaluate $y(t) = h(t) * p(t) = \int_{-\infty}^{\infty} h(\tau)p(t-\tau)d\tau$.

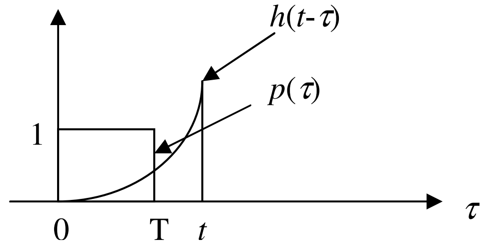
For $t < 0$, $p(\tau)h(t-\tau) = 0$.
For $0 < t < T$,



We need to integrate from 0 to t.

$$y(t) = \int_{-\infty}^{\infty} p(\tau)h(t-\tau)d\tau = \int_0^t \frac{1}{RC} e^{-(t-\tau)/RC} d\tau = \frac{RC}{RC}\left[e^{-(t-\tau)/RC}\right]_0^t = 1 - e^{-t/RC}$$

For $t \geq T$,

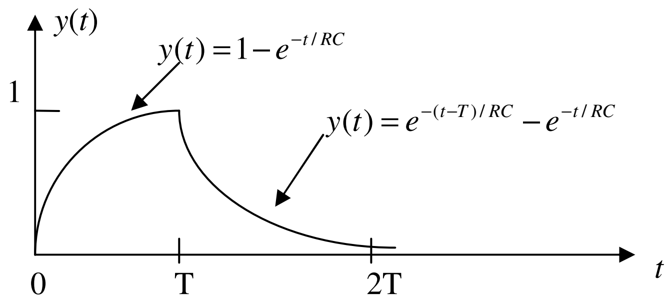


We need to integrate from 0 to T.

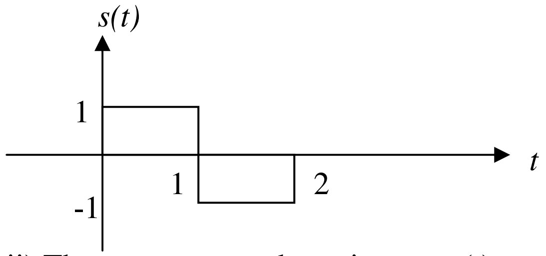
$$y(t) = \int_0^T \frac{1}{RC} e^{-(t-\tau)/RC} d\tau = \frac{RC}{RC}\left[e^{-(t-\tau)/RC}\right]_0^T = e^{-(t-T)/RC} - e^{-t/RC}$$

Therefore we have

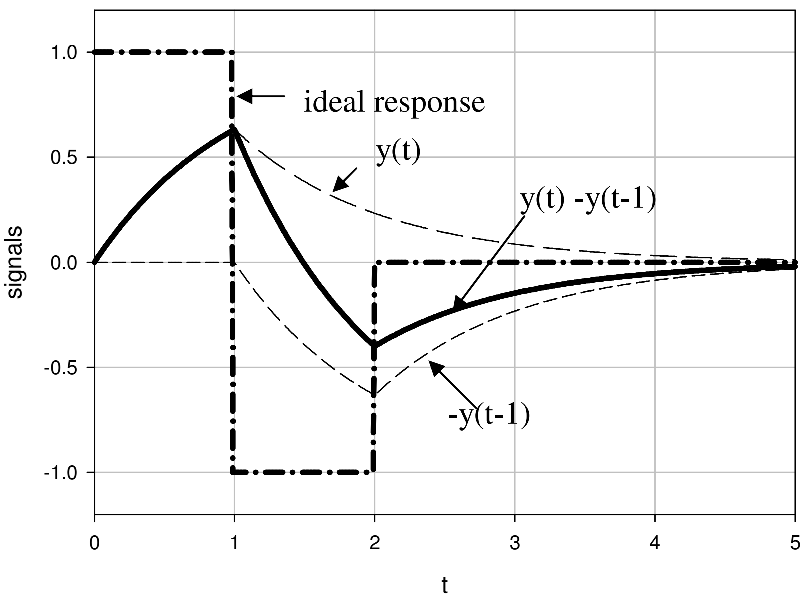
$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t/RC} & 0 < t < T \\ e^{-(t-T)/RC} - e^{-t/RC} & t \geq T \end{cases}$$



Q1bi) The sequence “1 0” looks like



ii) The sequence can be written as s(t) = p(t) – p(t-1). Since the system is LTI, the response is given by r(t) = y(t) – y(t-1). Therefore we have



iii) The response r(t) showed a severely distorted pulse shape due to very significant ISI. When the time constant is small, i.e RC is << 1, the circuit has high bandwidth and the response r(t) will approach that of the ideal response. When RC ~ 1 the circuit behaves like a low pass filter causing the distortion in the pulse shape and produces ISI. The peak at t = 2s is significantly lower than –1, potentially causing an error during decoding.

Q2)

a) i)

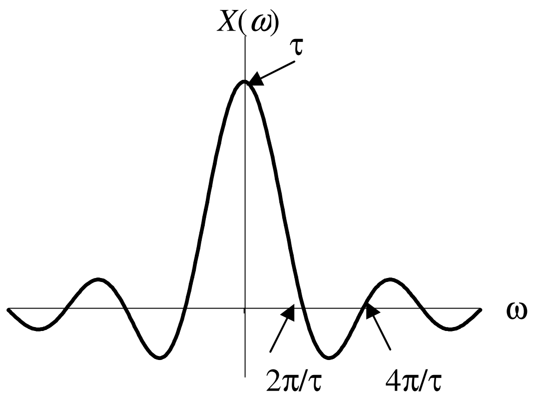
$$X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=\int_{-\tau/2}^{\tau/2}e^{-j\omega t}dt=\frac{1}{j\omega}\left[-e^{-j\omega t}\right]_{-\tau/2}^{\tau/2}=\frac{1}{j\omega}\left[e^{j\omega\tau/2}-e^{-j\omega\tau/2}\right]=2\frac{\sin(\omega\tau/2)}{\omega}$$

$$=\tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}.$$

ii) The maximum value of $X(\omega)$ is τ when $\omega = 0$.

Using L'Hopital rule $\left.\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right|_{\omega=0}=1$.

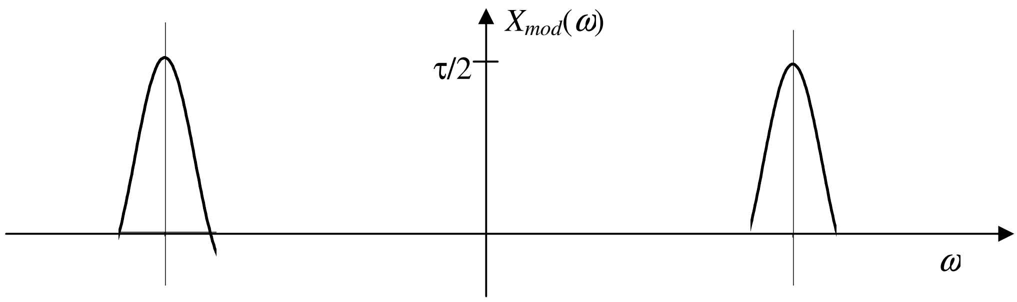
The nulls occur when $\sin(\omega\tau/2) = 0$ at $\omega = 2n\pi/\tau$, where $n = 1, 2, 3\ldots$.



b) The modulated signal is given by $x_{mod}(t) = x(t)\cos(\omega_c t)$. Using Frequency shift property of FT

$$X_{\text{mod}}(\omega)=\frac{1}{2}\left[X(\omega+\omega_c)+X(\omega-\omega_c)\right]==\frac{1}{2}\left[\tau\frac{\sin((\omega+\omega_c)\tau/2)}{((\omega+\omega_c)\tau/2)}-\tau\frac{\sin((\omega-\omega_c)\tau/2)}{((\omega-\omega_c)\tau/2)}\right]$$

c) Ignoring spectrum above $2\pi/\tau$, we have

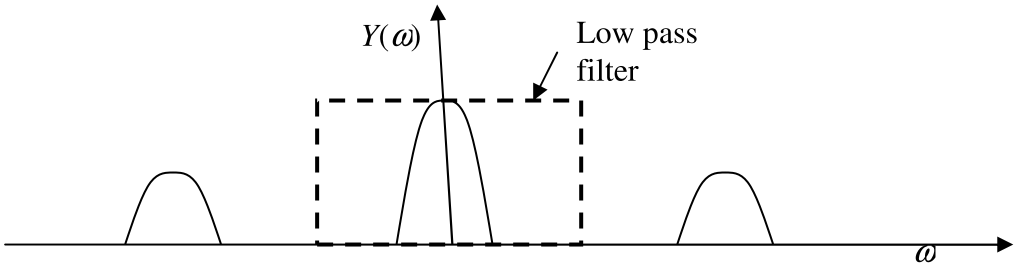


To demodulate using a synchronous demodulator, we multiply $x_{mod}(t)$ with $x_c(t)$.

$$y(t)=x_{\text{mod}}(t)x_c(t)=x_{\text{mod}}(t)\cos(\omega_c t)$$

$$Y(\omega)=\frac{1}{2}\left[X_{\text{mod}}(\omega+\omega_c)+X_{\text{mod}}(\omega-\omega_c)\right]=\frac{1}{2}\left[\frac{1}{2}X(\omega+2\omega_c)+X(\omega)+\frac{1}{2}X(\omega-2\omega_c)\right]$$

$$=\frac{1}{2}X(\omega)+\frac{1}{4}X(\omega+2\omega_c)+\frac{1}{4}X(\omega-2\omega_c)$$



The low pass filter bandwidth needs to be $\frac{2\pi}{\tau} < BW < 2\omega_c - \frac{2\pi}{\tau}$.

Q3) a) Poles = $-\zeta\omega_n \pm j\omega_d = -2 \pm j2$. Zero = -4.

$$H(s) = \frac{(s+4)}{(s+2+j2)(s+2-j2)} = \frac{(s+4)}{(s^2+2s-j2s+2s+4-j4+j2s+j4+4)} = \frac{(s+4)}{(s^2+4s+8)}$$

or

$$H(s) = \frac{(s+4)}{(s+2)^2+2^2}.$$

Therefore $N(s) = s + 4$ and $D(s) = s^2 + 4s + 8$.

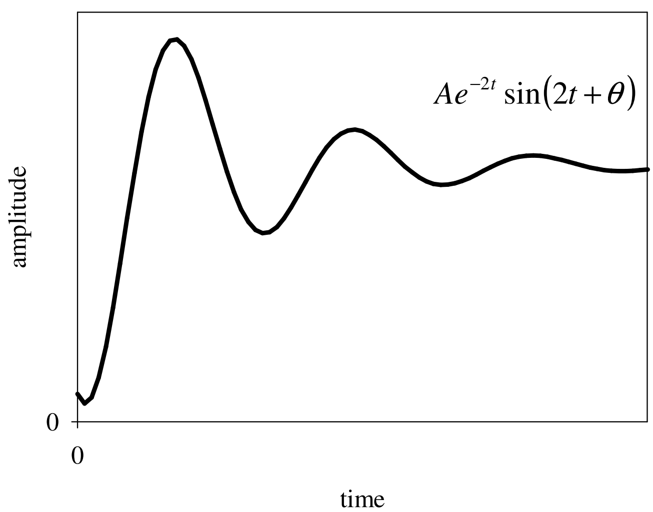
b) $H(s) = \frac{(s+4)}{s^2+4s+8}$, therefore we have, $2\zeta\omega_n = 4$ and $\omega_n = \sqrt{8}$.

$$\zeta = \frac{2}{\omega_n} = \frac{2}{\sqrt{8}} = \frac{1}{\sqrt{2}}.$$

The unit step response is $Ae^{-\zeta\omega_n t}\sin(\omega_d t + \theta)$ where

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{8}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{8}.\sqrt{\frac{1}{2}} = \sqrt{4} = 2 \text{ rad/s.}$$

c) The step response of the system is a sinusoidal oscillation with a frequency of $\omega_d = 2$ rad/s and amplitude modulated by a decaying exponential with a time constant of $\tau = \frac{1}{\zeta\omega_n} = 1/2$ s.



When ω_n is doubled, the oscillation frequency increases to $2\omega_d$ and the time constant reduces to $\tau = \frac{1}{4}s$. Therefore the system response oscillates at a higher frequency and the amplitude of the oscillation decays faster.

d) If $x(t) = e^{-4t}.u(t)$, $X(s) = \frac{1}{s+4}$. The Laplace Transform of the response is

$$Y(s) = X(s).H(s)$$

$$= \frac{s+4}{\left((s+2)^2+2^2\right)(s+4)} \frac{1}{} = \frac{1}{\left((s+2)^2+2^2\right)} = \frac{1}{2}\frac{2}{\left((s+2)^2+2^2\right)}.$$

Therefore the system response when $x(t) = e^{-4t}.u(t)$ is $y(t) = \frac{1}{2}e^{-2t}.\sin(2t).u(t)$.

Q4 a) The signal has an odd symmetry. Therefore d.c and even terms are zero.

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin(n\omega_0 t)dt = \frac{2}{T}\left[\int_{-T/2}^{0} -A\sin(n\omega_0 t)dt + \int_{0}^{T/2} A\sin(n\omega_0 t)dt\right]$$

$$= \frac{2}{T}\left[\frac{A\cos(n\omega_0 t)}{n\omega_0}\right]_{-T/2}^{0} + \frac{2}{T}\left[\frac{-A\cos(n\omega_0 t)}{n\omega_0}\right]_{0}^{T/2}$$

$$= \frac{2A}{n\omega_0 T}\left[1-\cos(n\omega_0 T/2) - \cos(n\omega_0 T/2)+1\right] = \frac{4A}{n\omega_0 T}\left[1-\cos(n\omega_0 T/2)\right]$$

$$= \frac{4A}{n\left(\frac{2\pi}{T}\right)T}\left[1-\cos\left(n\left(\frac{2\pi}{T}\right)\frac{T}{2}\right)\right] = \begin{cases} \frac{4A}{n\pi} & n = 1,3,5,......odd \\ 0 & n = 2,4,6,......even \end{cases}$$

b)

$$p(t) = \frac{4A}{\pi}\left[\sin(\omega_0 t) + \frac{1}{3}\sin(3\omega_0 t) + \frac{1}{5}\sin(5\omega_0 t)\right] = \frac{4}{\pi}\left[\sin(2\pi t) + \frac{1}{3}\sin(6\pi t) + \frac{1}{5}\sin(10\pi t)\right]$$

c) The transfer function is $\frac{V_o(\omega)}{V_i(\omega)} = \frac{1/j\omega C}{1/j\omega C + R} = \frac{1}{1+j\omega RC} = \frac{1}{1+j\omega/\omega_c}$

where $\omega_c = 1/RC$.

d) i)To make sure that the magnitude of the 5th harmonic is > 69% of its value before filtering, $|H(10\pi)| > 0.69$.

If $\frac{1}{\sqrt{1+\left(\frac{10\pi}{\omega_c}\right)^2}} = 0.69$, we have $1+\frac{100\pi^2}{\omega_c^2} = \frac{1}{0.69^2}$. $\omega_c = 30$ RC=1/30, C=10μF.

After filtering the amplitude of the 5th harmonic = $0.69 \times 4/5\pi = 0.176$.

ii) $|H(2\pi)| \frac{1}{\sqrt{1+\left(\frac{2\pi}{30}\right)^2}} = 0.979$, $|H(6\pi)| \frac{1}{\sqrt{1+\left(\frac{6\pi}{30}\right)^2}} = 0.847$

The amplitudes of the 1st and 3rd harmonics are $0.979 \times 4/\pi = 1.247$ and $0.847 \times 4/3\pi = 0.359$ respectively.

Therefore the RMS value = $\dfrac{1}{\sqrt{2}}\sqrt{1.247^2 + 0.359^2 + 0.176^2} = \dfrac{1.309}{\sqrt{2}} = 0.926$