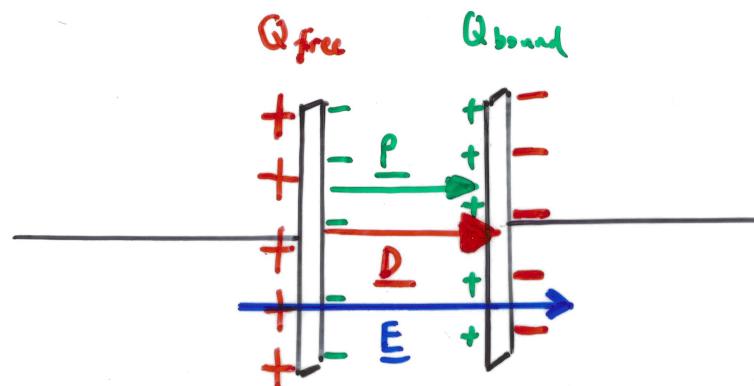


## Polarisation and dipoles

①

### macroscopic interpretation of polarisation



consider plate capacitor with dielectric material between plates

Coulomb's Law for dielectric flux  $\underline{D}$ , which points from  $+Q_{\text{free}}$  to  $-Q_{\text{free}}$

$$\text{div } \underline{D} = S_{\text{free}} = \frac{Q_{\text{free}}}{V}$$

$$\Rightarrow D = \frac{Q_{\text{free}}}{A} \quad \text{is free charge density per unit area}$$

Polarisation vector points from  $-Q_{\text{bound}}$  to  $+Q_{\text{bound}}$  in dielectric

$$\text{div } \underline{P} = -S_{\text{bound}} = -\frac{Q_{\text{bound}}}{V}$$

$$\Rightarrow P = \frac{Q_{\text{bound}}}{A} \quad \text{is bound charge density per unit area}$$

The electric field is the sum:

$$\begin{aligned} \epsilon_0 \text{div } \underline{E} &= S = S_{\text{free}} + S_{\text{bound}} \\ &= \text{div } \underline{D} - \text{div } \underline{P} \\ &= \text{div } (\underline{D} - \underline{P}) \end{aligned}$$

$$\Rightarrow \epsilon_0 \underline{E} = \underline{D} - \underline{P}$$

$$\Rightarrow \boxed{\underline{D} = \epsilon_0 \underline{E} + \underline{P}}$$

$$\text{Also, we have } \underline{D} = \epsilon_0 \epsilon_r \underline{E}$$

$$= \epsilon_0 \underline{E} + \epsilon_0 (\epsilon_r - 1) \underline{E}$$

$$\Rightarrow \boxed{\underline{P} = \underbrace{\epsilon_0 (\epsilon_r - 1)}_{X} \underline{E}}$$

with susceptibility  $X = \epsilon_r - 1$

Polarisation is fraction  $\epsilon_0 X$  of the electric field,  
caused by bound charge density on the surfaces

## ② microscopic interpretation of polarisation

Define microscopic dipoles  $\underline{p} = q \underline{ds}$  due to small pairs of charges  $+q$  and  $-q$  at distance  $ds$ , where  $\underline{ds}$  points from  $-q$  to  $+q$ :



$$\underline{p} = q \underline{ds}$$

Dipole

(sometimes  
"dipole moment")

The Polarisation  $\underline{P}$  is the number density of these microscopic dipoles, i.e.

$$\underline{P} = n \underline{p}$$

For a crystalline solid, this is identical to the macroscopic model because

$$\underline{P} = n \underline{p} = n q \underline{ds} = -n A \underline{ds}$$

$\underbrace{n}_{\text{total number of charges beneath surface } A \text{ within depth } ds}$

$$\frac{q}{A} \underline{e}$$

unit vector from  $+q$  to  $-q$

$\rightarrow$  bound charge density per area



$\downarrow$   $-Q_{\text{bound}}$  material (dielectric) with lots of elementary dipoles, either permanent (large  $\epsilon_r$ ) or induced (small  $\epsilon_r$ ), with density  $n$

$\downarrow$   $+Q_{\text{bound}}$

define polarisability of the dielectric by

$$\alpha = \frac{|\underline{P}|}{|\underline{E}|}$$

to get

$$\underline{P} = n \underline{P} = n \alpha \underline{E} \stackrel{!}{=} \epsilon_0 \chi \underline{E} \quad \text{for gases and liquids}$$

$$\rightarrow \chi_{\text{gas or liquid}} = \frac{n \alpha}{\epsilon_0}$$

In crystalline solids

$$\underline{P} = n \underline{P} = n \alpha \underline{E}_{\text{loc}}$$

where the local electric field is a superposition of externally applied field  $\underline{E}$  and a certain fraction of the polarisation.

For cubic crystals:

$$\underline{E}_{\text{loc}} = \underline{E} + \frac{\underline{P}}{3\epsilon_0}$$

$$\Rightarrow \underline{P} = n \alpha \left( \underline{E} + \frac{\underline{P}}{3\epsilon_0} \right)$$

$$\Rightarrow \underline{P} = \frac{n \alpha}{1 - \frac{n \alpha}{3\epsilon_0}} \underline{E} \stackrel{!}{=} \epsilon \chi \underline{E}$$

$$\Rightarrow \chi_{\text{cubic crystal}} = \frac{\frac{n \alpha}{\epsilon_0}}{1 - \frac{n \alpha}{3\epsilon_0}} \quad \text{is bigger than above value}$$

$$\Rightarrow \underline{E}_{\text{loc}} = \underline{E} + \frac{\underline{P}}{3\epsilon_0} = \underline{E} + \frac{n \alpha}{3\epsilon_0} \underline{E}_{\text{loc}}$$

$$\Rightarrow \underline{E}_{\text{loc}} = \frac{1}{1 - \frac{n \alpha}{3\epsilon_0}} \underline{E}$$

with  $\chi = \epsilon_r - 1$

$$\epsilon_r + 2 = \chi + 3 = \frac{\frac{n \alpha}{\epsilon_0} + 3 - 3 \frac{n \alpha}{3\epsilon_0}}{1 - \frac{n \alpha}{3\epsilon_0}} = \frac{3}{1 - \frac{n \alpha}{3\epsilon_0}}$$

$$\Rightarrow \frac{\epsilon_r - 1}{\epsilon_r + 2} = \frac{n \alpha}{3\epsilon_0} \quad (\text{Clawius-Mosotti equation})$$

$$\Rightarrow \underline{E}_{\text{loc}} = \frac{1}{1 - \frac{\epsilon_r - 1}{\epsilon_r + 2}} \underline{E} = \frac{\epsilon_r + 2}{3} \underline{E} = \underline{E}_{\text{loc}}$$

### ③ electro-static dipole field



at point of origin, both potentials from point charges  $\pm q$  need to be added:

$$V_{\text{dipole, stat}} = V_{+q} + V_{-q}$$

where

$$V_{+q} = \frac{1}{4\pi\epsilon_0} \frac{+q}{|\underline{r}|}$$

$$V_{-q} = \frac{1}{4\pi\epsilon_0} \frac{-q}{|\underline{r} + \underline{ds}|}$$

and we can use the Taylor expansion

$$\frac{1}{|\underline{r} + \underline{ds}|} = \frac{1}{|\underline{r}|} + \underline{ds} \cdot \text{grad} \frac{1}{|\underline{r}|} + O(r^2)$$

$$\Rightarrow V_{\text{dipole, stat}} = \frac{1}{4\pi\epsilon_0} \left( \cancel{\frac{+q}{r}} - \cancel{\frac{q}{r}} - q \underbrace{\underline{ds} \cdot \text{grad} \frac{1}{r}}_{\underline{P} \cdot -\frac{1}{r^2} \underline{e}_r} + O(r^2) \right)$$

$$\Rightarrow V_{\text{dipole, stat}} = \frac{1}{4\pi\epsilon_0} \frac{\underline{P} \cdot \underline{r}}{r^3}$$

Note that  $V_{\text{dipole, stat}} = \frac{1}{4\pi\epsilon_0} \frac{\underline{P} \cdot \underline{e}_r}{r^3} \propto \frac{1}{r^2}$  decays faster as function of  $r = |\underline{r}|$  than the normal Coulomb potential. This is because it is proportional to  $\text{grad} \frac{1}{r} = -\frac{1}{r^2} \underline{e}_r$ .

$$\Rightarrow \underline{E}_{\text{dipole, stat}} = -\text{grad} V_{\text{dipole stat}}$$

$$= -\frac{1}{4\pi\epsilon_0} \underbrace{\text{grad} \frac{\underline{P} \cdot \underline{r}}{r^3}}_{\frac{1}{r^3} \text{grad}(\underline{P} \cdot \underline{r}) + (\underline{P} \cdot \underline{r}) \text{grad} \frac{1}{r^3}}$$

$$- \underline{P} \cdot \text{grad} \frac{\underline{r} \cdot \underline{e}_r}{r^3}$$

$$-3r^{-4} \underline{e}_r = -3 \frac{\underline{r}}{r^5}$$

$$= -\frac{1}{4\pi\epsilon_0} \left[ \frac{1}{r^3} \underline{P} - \frac{3}{r^4} \underline{e}_r \cdot (\underline{P} \underline{r}) \right]$$

$$\underline{E}_{\text{dipole, stat}} = \frac{1}{4\pi\epsilon_0} \left( \underbrace{\frac{3(\underline{P} \cdot \underline{r}) \underline{r}}{r^5}}_{\text{point, along } \underline{r}} - \underbrace{\frac{\underline{P}}{r^3}}_{\text{point, along } \underline{P} = q \underline{ds}} \right)$$

point, along  $\underline{r}$

point, along  $\underline{P} = q \underline{ds}$

The dipole field is not isotropic, and we can distinguish 3 cases:

i)  $\underline{S} \perp \underline{P}$

$$\rightarrow \underline{P} \cdot \underline{r} = 0 \Rightarrow E_{\text{dipole, stat}} = -\frac{1}{4\pi\epsilon_0} \frac{\underline{P}}{r^3} \parallel \underline{P}$$

is anti-parallel to  $\underline{P}$

ii)  $\underline{S} \parallel \underline{P}$

$$\rightarrow \underline{P} \cdot \underline{r} = |\underline{P}| \cdot |\underline{r}| = pr$$

$$\Rightarrow E_{\text{dipole, stat}} = +\frac{1}{4\pi\epsilon_0} \frac{2\underline{P}}{r^3} \parallel \underline{P}$$

is parallel to  $\underline{P}$  and twice as large in amplitude

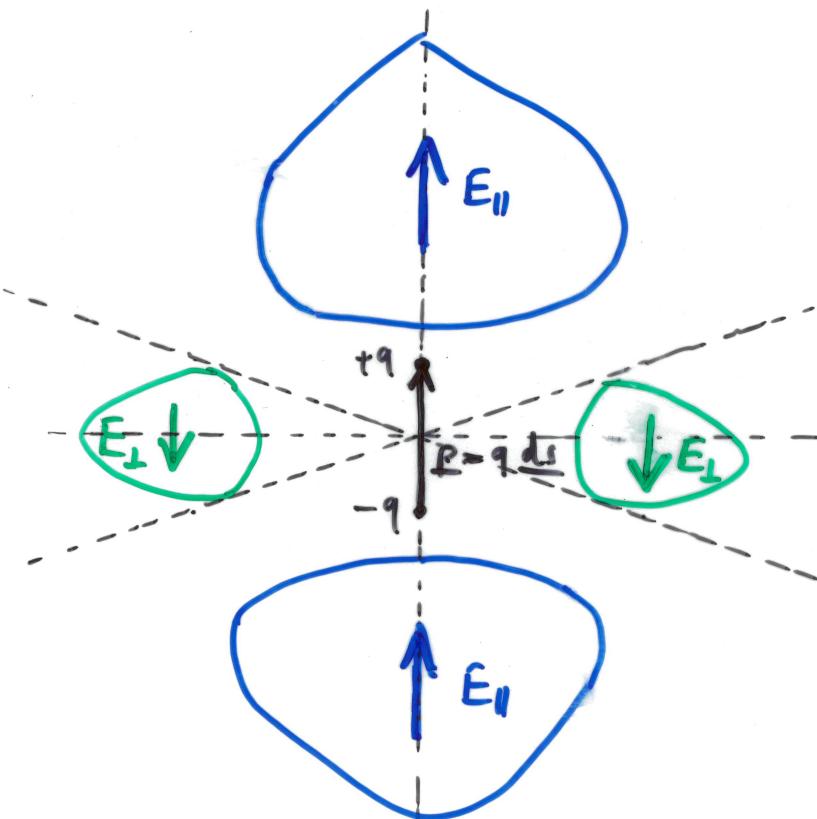
iii)  $\theta = 70.5^\circ$

If  $\theta = \pm 70.5^\circ$ , then

$$= \arccos \gamma_3 = \not{\phi}(\underline{P}, \underline{r}) \text{, hence}$$

$$\frac{(3\underline{P}\cdot\underline{r})\underline{r}}{r^5} = \frac{\underline{P}}{r^3}$$

$$\Rightarrow E_{\text{dipole, stat}} = 0 \text{ if } \not{\phi}(\underline{P}, \underline{r}) = \arccos \gamma_3 = 70.5^\circ$$



Note: as  $B=0$ ,  $H=0 \rightarrow \underline{S} = \underline{E} \times \underline{H} = 0$ ; i.e. no radiated power.