EEE6081 (EEE421)
Visual Information Engineering (VIE)
Topic 4 – Filter Banks and Wavelet Transforms

- Filter Banks
  - Orthogonal filter banks
  - Perfect reconstruction condition
  - Filter bank design
  - Dyadic decomposition
- Wavelet Transforms
  - What is a wavelet?
  - Wavelet implementation
  - Wavelet decomposition schemes
  - Wavelet transforming of images
  - Multi-resolution analysis (MRA)

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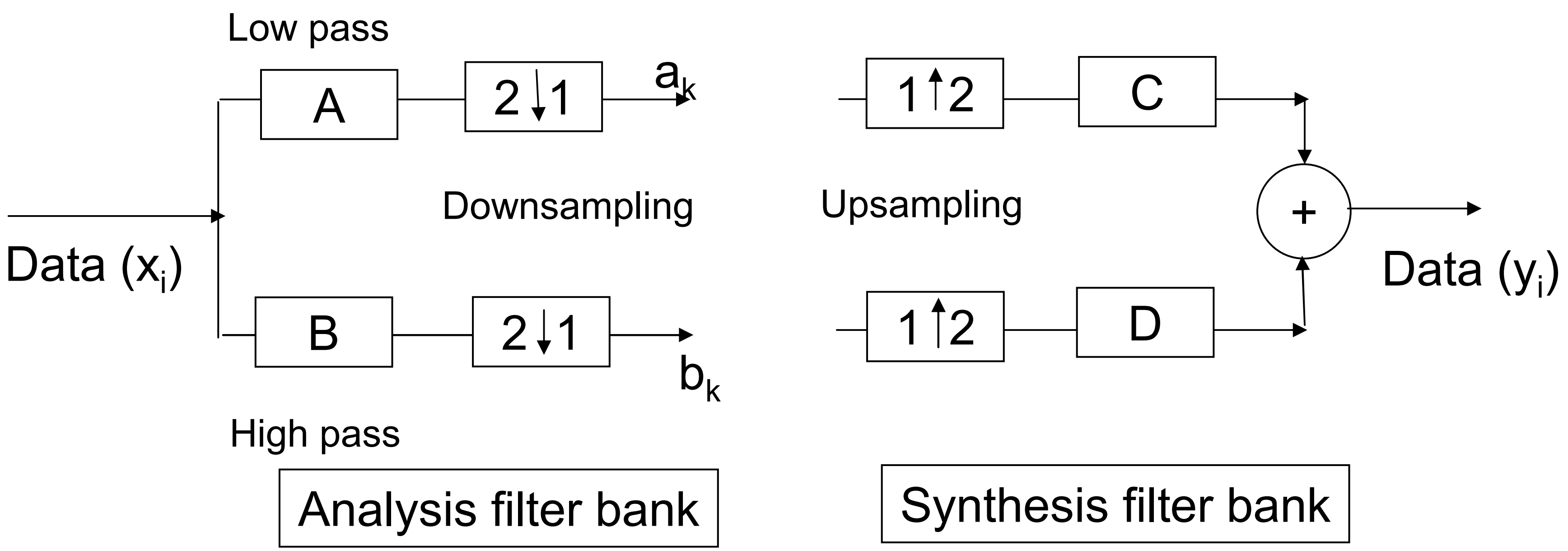
Filter Banks

- Filters are generally not perfect reconstructing (not invertible or not lossless).
- However, using filter banks (i.e., a bank of filters), results in low complexity transforms giving perfect reconstruction.
- The forward transform is obtained by the "analysis filter bank"
- The inverse transform is realised by the "synthesis filter bank"



- Derive the conditions for perfect reconstruction (x_i = y_i)

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Filter Banks

- Consider the z-transform representation
  - Input signal $X(z)$
  - Outout signal $Y(z)$
  - The filters $A(z)$, $B(z)$, $C(z)$ and $D(z)$
  - The Downsampling operator $\boxed{2 \downarrow 1}$ (2:1 downsampling) on an input signal $F(z)$ is $F(z^2)$.
  - The Interpolation operator $\boxed{1 \uparrow 2}$ (1:2 upsampling) of an input signal $F(z)$ is $\frac{1}{2}[F(z^{1/2})+F(-z^{1/2})]$

- Now for the filter bank: For the upper banch:
  - After the low pass filter A: $A(z)X(z)$
  - After downsampling: $A(z^2)X(z^2)$
  - After Upsampling: $\frac{1}{2}[A(z)X(z) + A(-z)X(-z)]$
  - After the filter C: $\frac{1}{2}C(z)[A(z)X(z) + A(-z)X(-z)]$ -----(1)

- Similarly for the lower Branch
  - We can write: $\frac{1}{2}D(z)[B(z)X(z) + B(-z)X(-z)]$ ------(2)

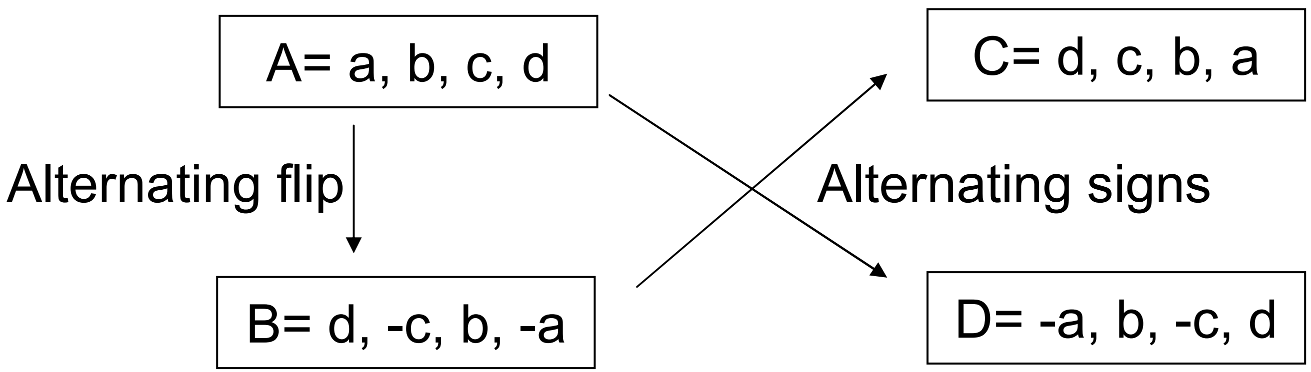
- Now by (1)+(2) we can get $Y(z)$

Filter Banks

- The output of the filter bank
  $Y(z) = ½ C(z)[A(z)X(z) + A(-z)X(-z)] + ½ D(z)[B(z)X(z) + B(-z)X(-z)]$
  $½ [A(z)C(z)+B(z)D(z)] X(z) + ½ [A(-z)C(z)+B(-z)D(z)] X(-z)$ ----------(3)

- For the Perfect Reconstruction (PR)
  $A(z)C(z) + B(z)D(z) = 2z^{-l}$ For no distortion (i.e., the Coefficient of X(z)=1)
  $A(-z)C(z) + B(-z)D(z) = 0$ For no aliasing (i.e, the Coefficient of X(-z)=0)

- To satisfy the PR conditions, choose C(z)=B(-z), D(z)=-A(-z)
- and choose B(z) as the corresponding high pass filter of the low pass filter A(z)



- That means if we know A(z), we can find the other 4 filters.

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Filter Banks

- Filter Bank design Criteria:
- Let's say the low pass filter A has the coefficients: $\{h_0, h_1, h_2, \ldots\}$

- (1) Orthogonality condition for the filter bank:

$$\sum_i h_i h_{i+2k} = \delta_{0k}$$

  - We only require to retain the orthogonality only for double shifts of the filter (why?)

- (2) Regularity condition for the filter bank:
  - B is a high pass filter. So its coefficients add up to zero. This requirement and the Perfect Reconstruction condition mean,

$$\sum_i h_i = \sqrt{2}$$

- We can use these two conditions to design filter banks:
  - Exercise: Design length N=2, N=3 and N=4 two-channel filter banks

Filter Banks

- Length N=2 filter bank

$$A = \{h_0, h_1\}$$

(1)

$$\sum_i h_i h_{i+2k} = \delta_{0k}$$

$$-> k = 0: \quad h_0^{\,2} + h_1^{\,2} = 1$$

(2)

$$\sum_i h_i = \sqrt{2}: \quad h_0 + h_1 = \sqrt{2}$$

$$h_0 = h_1 = \frac{1}{\sqrt{2}}$$

Length N=3 filter bank

$$A = \{h_0, h_1, h_2\}$$

(1)

$$\sum_i h_i h_{i+2k} = \delta_{0k}$$

$$-> k = 0: \quad h_0^{\,2} + h_1^{\,2} + h_2^{\,2} = 1$$

$$-> k = 1: \quad h_0 h_2 = 0$$

(2)

$$\sum_i h_i = \sqrt{2}: \quad h_0 + h_1 + h_2 = \sqrt{2}$$

$$h_2 = 0; \quad h_0 = h_1 = \frac{1}{\sqrt{2}};$$

- Not possible to design odd length filter banks.

- Homework: For N=2, filter bank, verify the perfect reconstruction for the input data sequence X={ 0 1 2 3 4 0}

Filter Banks

- Length N=4 filter bank

$$A = \{h_0, h_1, h_2, h_3\}$$

$$(1) \qquad \sum_i h_i h_{i+2k} = \delta_{0k}$$

$$-> k = 0: \quad h_0^{\ 2} + h_1^{\ 2} + h_2^{\ 2} + h_3^{\ 2} = 1$$

$$-> k = 1: \quad h_0 h_2 + h_1 h_3 = 0$$

$$(2) \qquad \sum_i h_i = \sqrt{2}: \quad h_0 + h_1 + h_2 + h_3 = \sqrt{2}$$

- 3 equations & 4 unknowns. So there is one free choice. We can use this to optimise the filter bank performance. (Daubechies - 4 filter bank is one example).
- Can we make the filter symmetric?
  - i.e., $h_0=h_3$ and $h_1=h_2$ in N=4

$$-> k = 1: \quad h_0 h_2 + h_1 h_3 = 0$$

$$h_0 h_1 + h_1 h_0 = 0$$

$$2 h_1 h_0 = 0$$

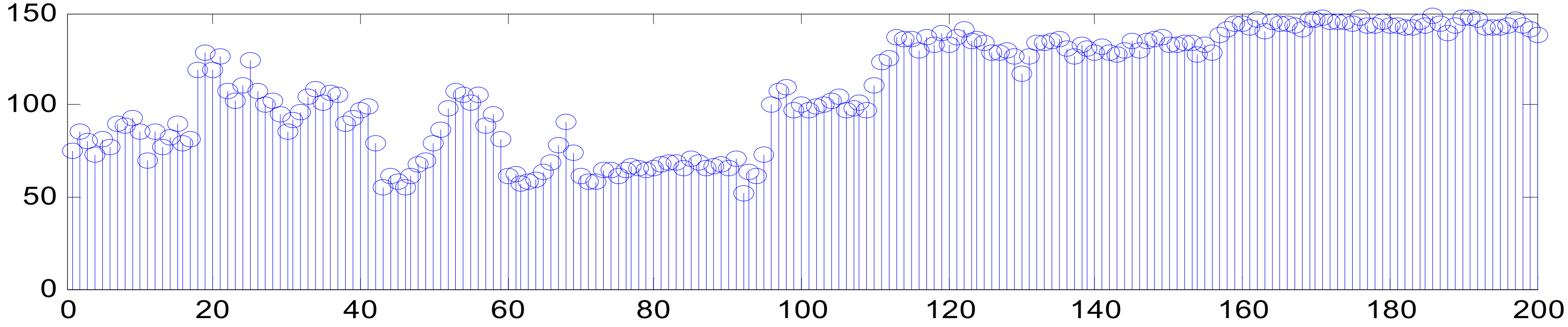
- Filters in orthogonal filter banks can't be symmetric. Therefore, they have phase distortion (no linear phase response). A solution to this is coming soon in couple of lectures (Biorthogonal filter banks – Topic 6)

Filter Banks

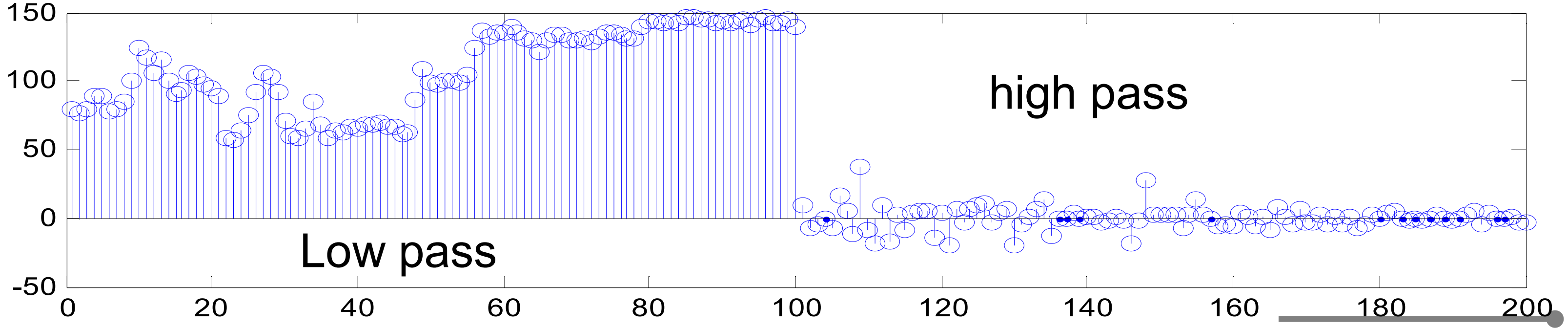
- A 2- channel filter bank decomposes data into two sub bands low pass an high pass.
- Non-expanding --- i.e., The length of output data = The length of input data
- Remember for filters, The length of output data= The length of input data + The length of filter -1.

- The low pass signal looks the same as the original (only smoothed)
- The filter bank can be applied repeatedly on the low pass signal. This is called Dyadic decomposition

Input Data



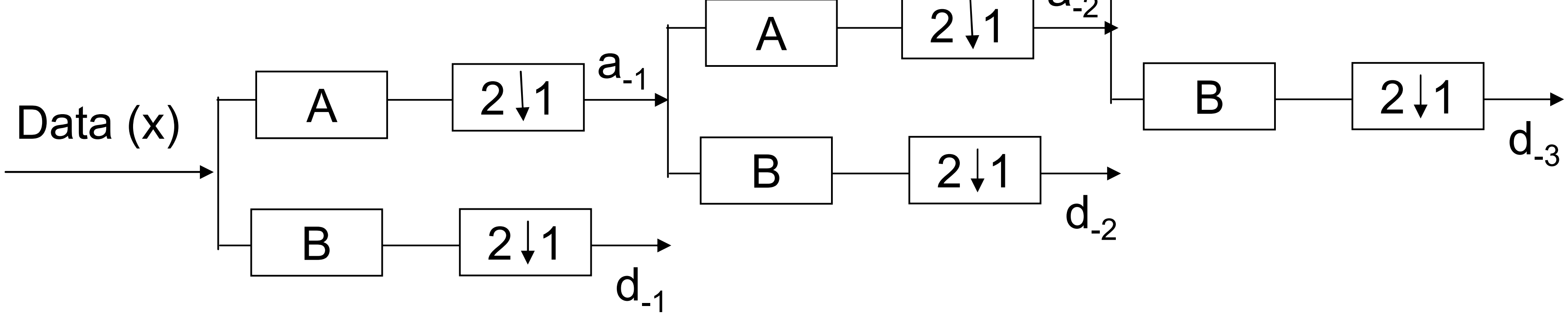
Output Data



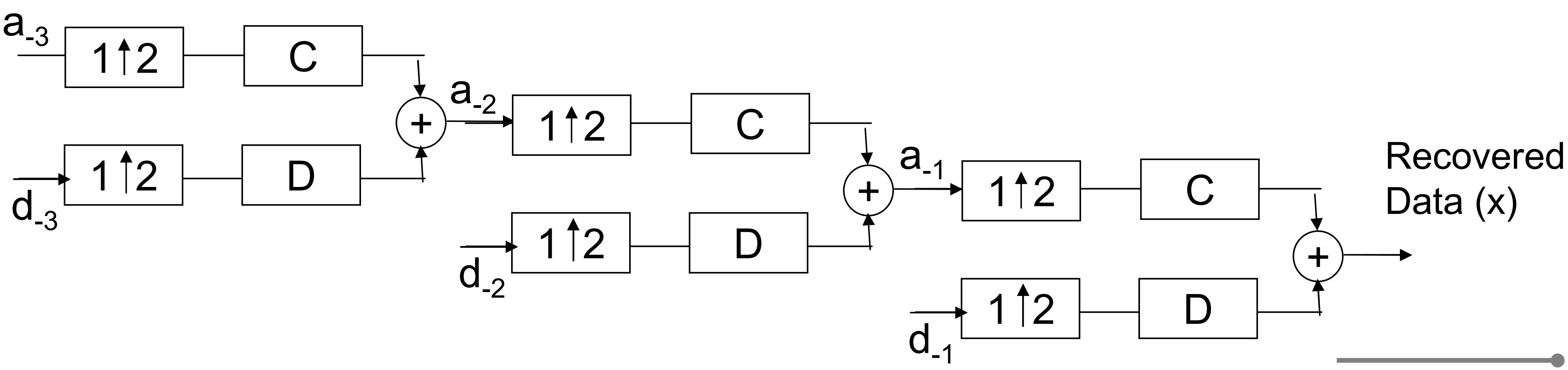
Filter Banks

- Dyadic Decomposition (Draw a diagram for a 3-level dyadic decomposition) and its corresponding reconstruction.

Decomposition

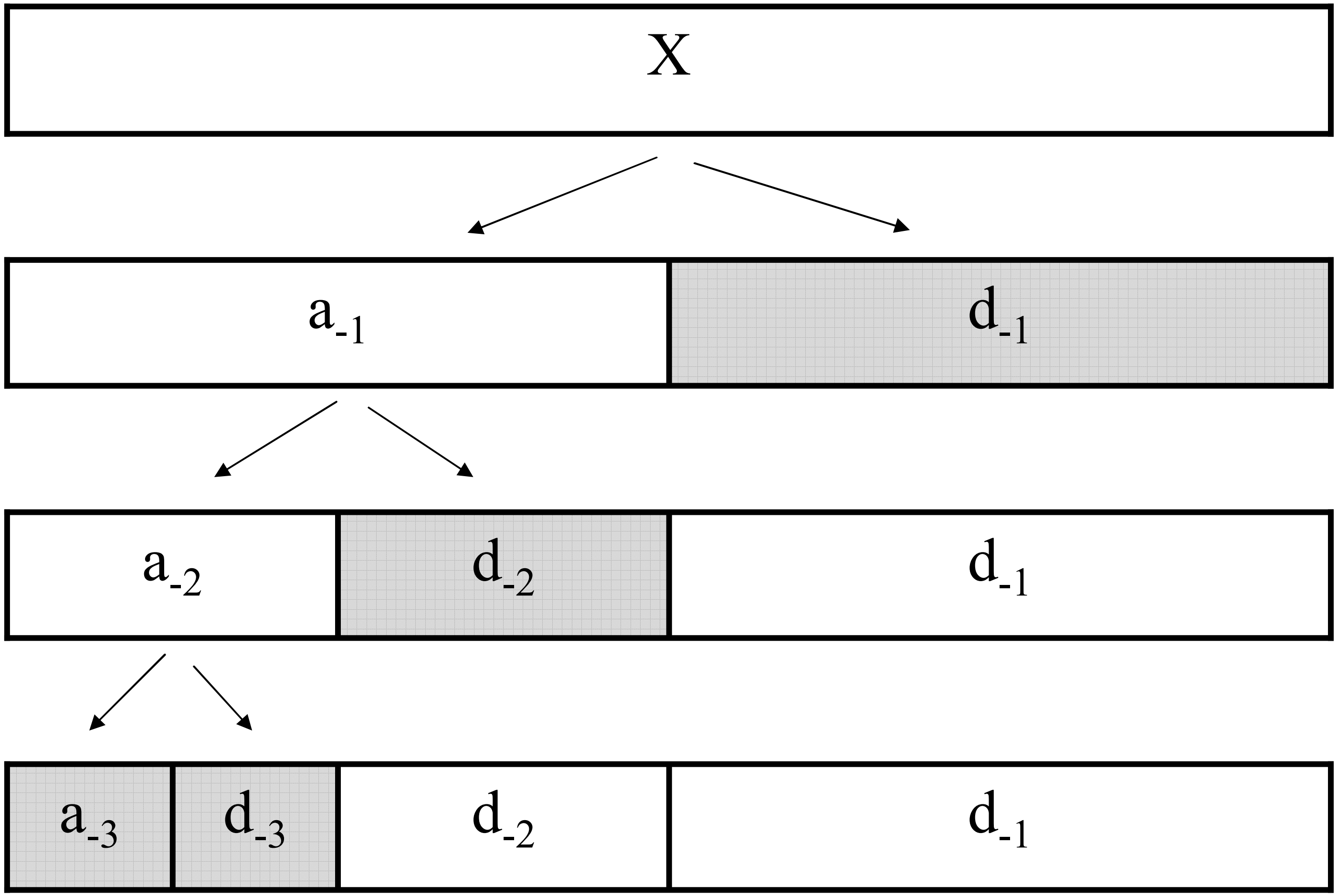


Reconstruction



Filter Banks

- Dyadic Decomposition



a_n = Low pass filtered data
d_n = High pass filtered data

Wavelet Transforms

- A wavelet is a short localised wave used as a basis function in a wavelet transform.
- What are the basis functions used in the Fourier transform?
- In a wavelet transform the main wavelet, usually called mother wavelet (w(n)), is defined first.
- Then in the transformation, the mother wavelet is scaled (by a factor s) or translated by k points to obtain the other wavelets $w_{(s,k)}(n)$ as basis functions:

$$w_{(s,k)}(n) = 2^{s/2} w(2^s n - k)$$

- Wavelet transforms can be implemented using two different methods:
  - Filter banks (In the current topic)
  - Lifting (Topic 6)

- It can be shown that a wavelet is the impulse response of the high pass filter of the inverse filter bank iterated to infinity. ☹

[Beyond the scope of this module ☺]

Wavelet Transforms

- Wavelet basis functions: $$w_{(s,k)}(n) = 2^{s/2} w(2^s n - k)$$

- The translation by a factor k
  - corresponds to the location of the wavelet
  - the high pass filter operation in the filter bank (Convolution) corresponds to this.

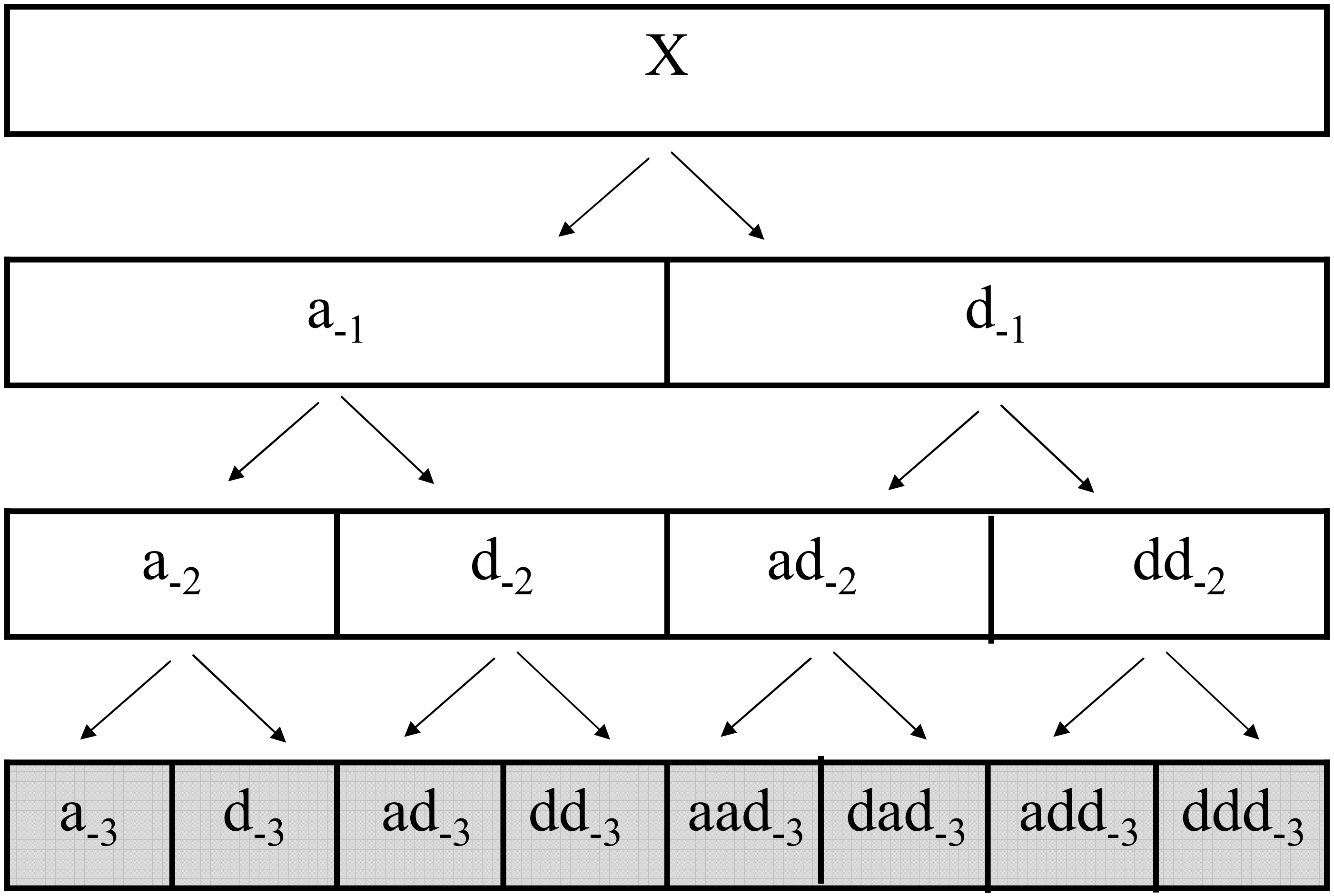
- The scaling of the mother wavelet by a factor s is represented in the filter bank
  - when the high pass filter is applied on the output of one level of decomposition.
  - And corresponds to wavelet operation on the down-sampled low passed signal.

- Why do we need a low pass filter in the filter bank?

- Different forms of wavelet decomposition schemes
  - Dyadic wavelet transform (Using the dyadic filter bank decomposition in slides 9 and 10)
  - Wavelet Packet transform (either as a full tree or an optimum tree decomposition)

Wavelet Transforms

- Full tree wavelet packet transform:
  - Both the low pass and high pass sub bands are decomposed further following a complete binary tree:

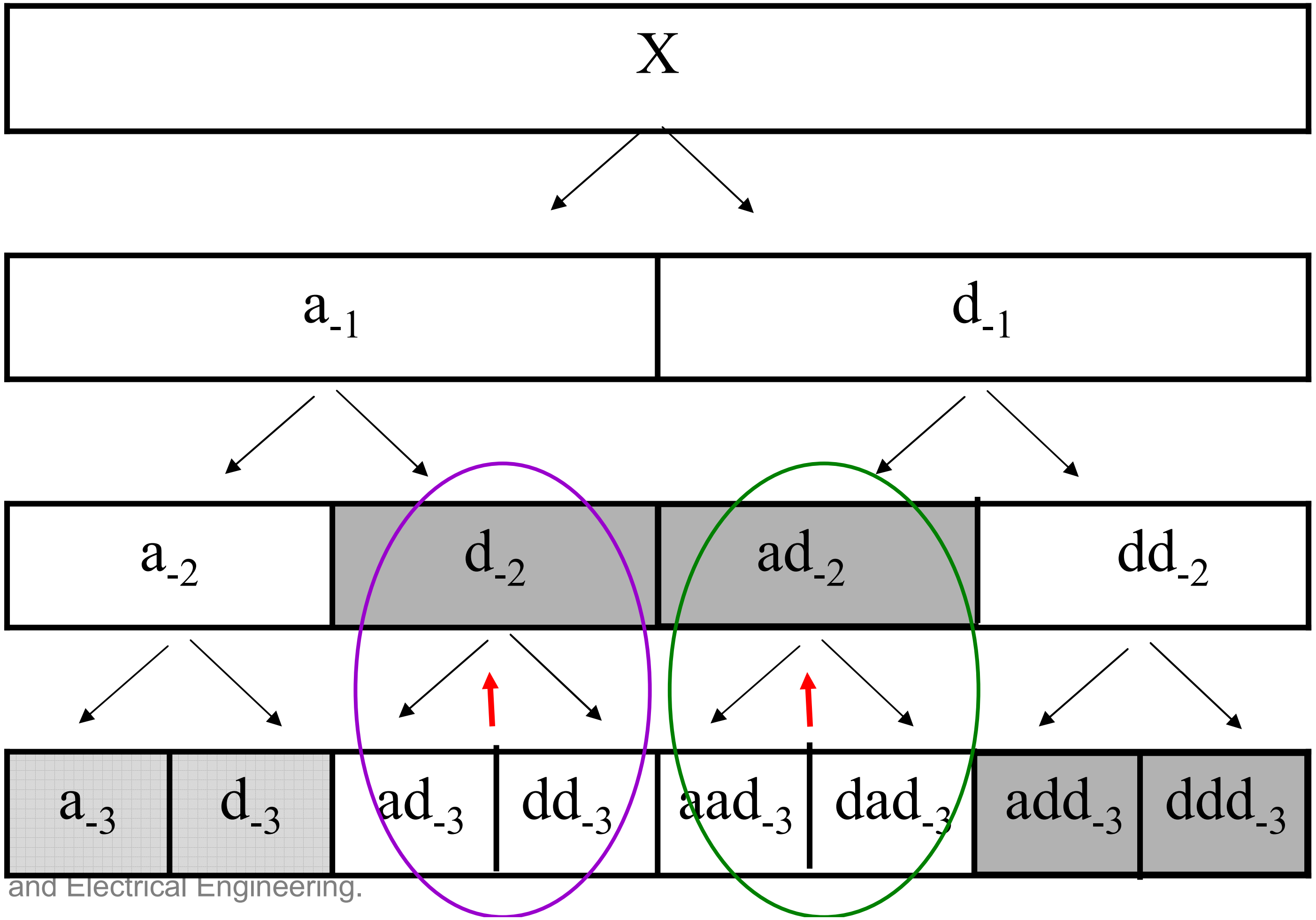


Wavelet Transforms

- Full tree wavelet packet transform:
  - Draw the corresponding filter bank based realisation for the forward transform (decomposition) and the inverse transform (reconstruction)

Wavelet Transforms

- Wavelet packet transform (With the optimum tree):
  - Both the low pass and high pass sub bands are decomposed further following a complete binary tree:
  - Then the tree nodes are merged together to obtain the optimum tree decomposition.
  - An example:



Wavelet Transforms

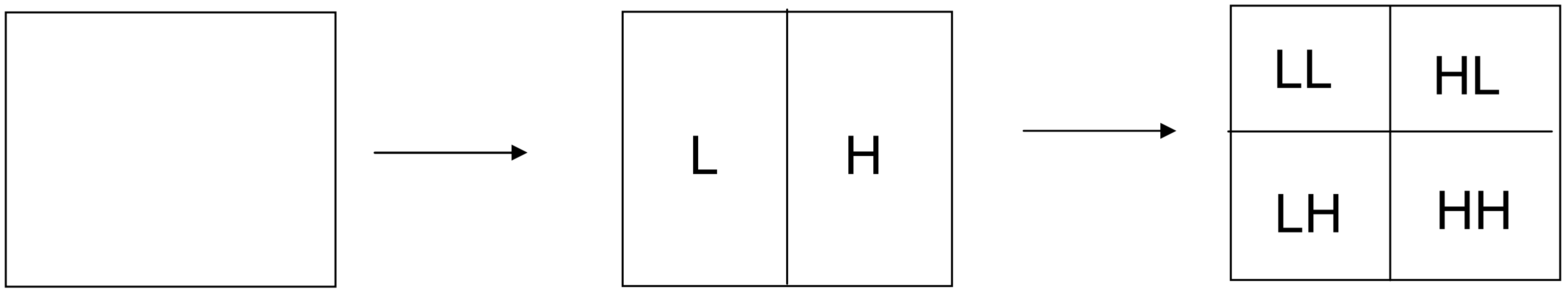
- Wavelet packet transform (With the optimum tree):
  - Draw the corresponding filter bank based realisation for the forward transform (decomposition) and the inverse transform (reconstruction)

Wavelet Transforms

- Frequencies shown in each sub band
  - For the Fourier transform, we know a signal in time domain representation is transformed and shown in frequency domain.
    - e.g. The axes are "time" and "frequency" in the 2 domains.
- But for the Wavelet transform,
  - It shows a joint time-frequency (or space-frequency) representation.
    - The original signal (a_0) represents the full resolution signal with all normalised frequencies 0 – π. High spatial resolution and low frequency resolution.
    - The First level decomposition:
      - ½ spatial resolution
      - (a_{-1}) represents frequencies 0 – π/2.
      - (d_{-1}) represents frequencies π/2 – π.
      - i.e, low spatial resolution, but high frequency resolution.
    - Similarly for level 2 of the decomposition.
      - ¼ spatial resolution
      - (a_{-2}) represents frequencies 0 – π/4.
      - (d_{-2}) represents frequencies π/4 – π/2.
    - These define the bandwidths of the filters A and B.

Wavelet Transforms

- Wavelet transformation (wt) of images:
  - So far we have learned about 1D filter banks
  - We use separable transformation approach: first on rows and then on columns for images.



  - Image wt on rows wt on columns
  - L=low pass H=high pass
  - LL – low-low pass LH- Low-high pass HL – High-low pass HH- High-high pass
  - LH - Horizontal edges
  - HL – Vertical Edges
  - HH – diagonal edges
  - LL – Half resolution image

Wavelet Transforms

- Wavelet transformation (wt) of images:

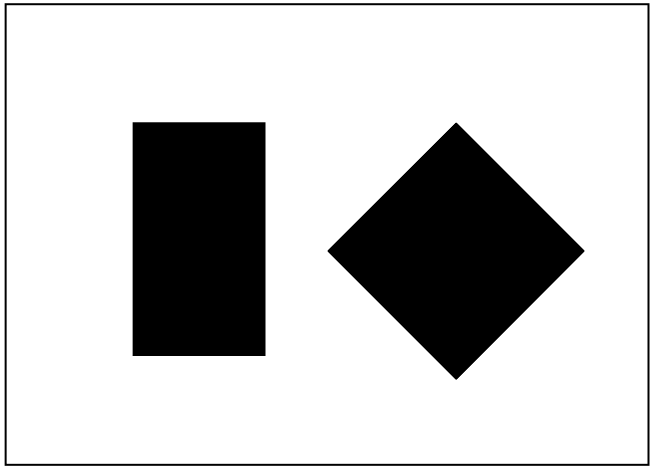
1 level 2D decompo sition

2 level 2D decompo sition

Wavelet Transforms

- Wavelet transformation (wt) of images:

- Draw the LL, LH, HL and HH sub bands when the following image is transformed to a single level wavelet decomposition



Wavelet Transforms

Using wavelet transforms we can analyse images in multiple resolutions. Some features are best viewed on some scale spaces (resolutions).

Applications of Multi-resolution analysis and wavelet transformation of images:

1. Image compression
2. Image denoising
3. Image fusion – fusing images from multiple sources
4. Low complexity operations using multiple low resolution scales.
5. Information hiding - e.g., watermarking
6. Feature extraction – texture features and colour features

(The coursework will be on one of the above applications)

Multi-resolution analysis (MRA)

- At each level of decomposition, the low pass sub band represents a half resolution approximation of the low pass signal of the previous level.

- Draw the filter bank operation for the dyadic decomposition:

- We can represent the Multi-resolution analysis using wavelets as below.
  - Let the starting resolution as a_0. After one level of decomposition we have

    $$\begin{array}{ccc} a_0 & \rightarrow & a_{-1} \\ & \searrow & d_{-1} \end{array}$$

    where a_{-1} is the half resolution approximation and d_{-1} the details seen at that resolution.

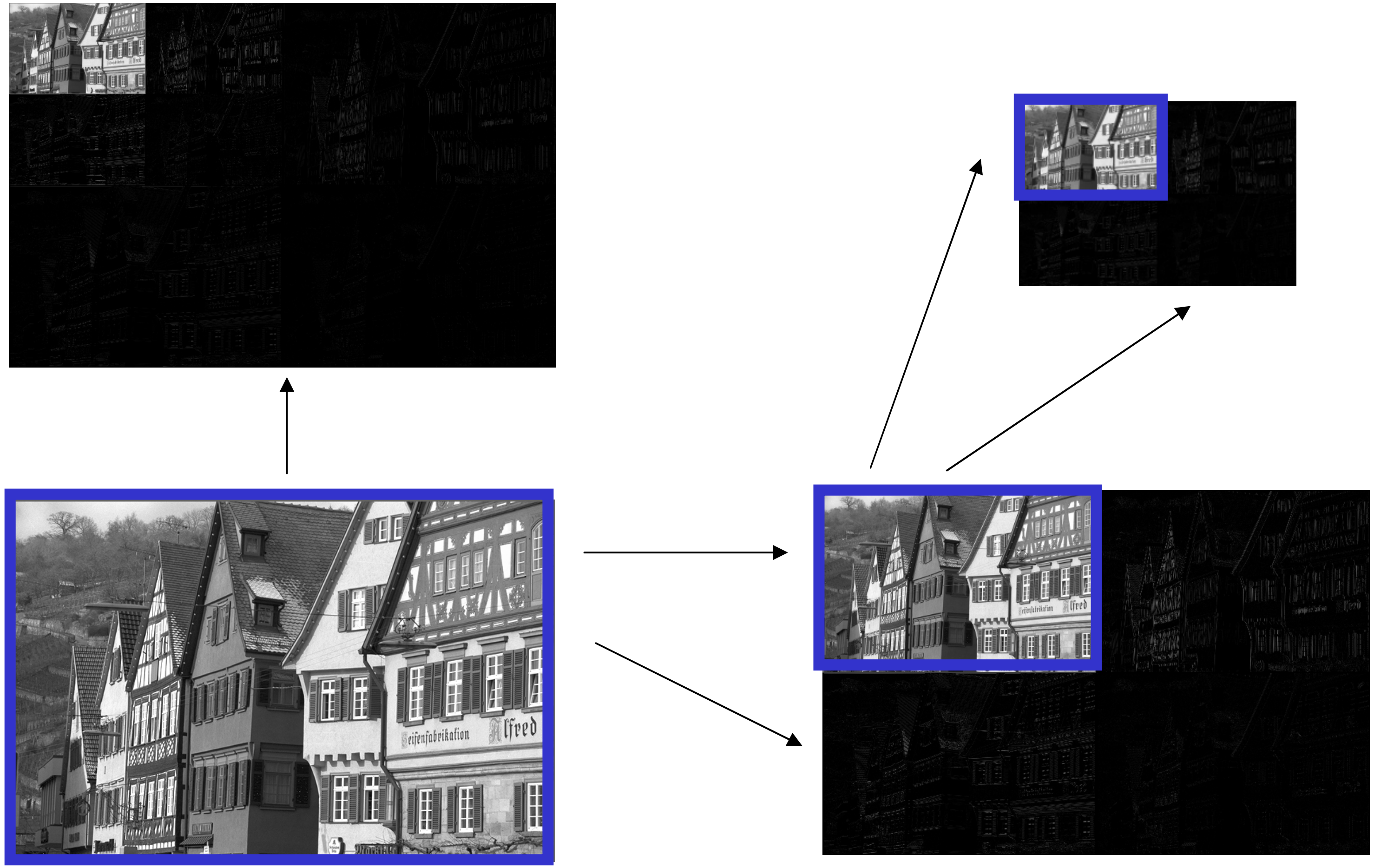
  - For n levels

    $$\begin{array}{ccccccccc} a_0 & \rightarrow & a_{-1} & \rightarrow & a_{-2} & \rightarrow & a_{-3} & \quad\quad \rightarrow & a_{-n} \\ & \searrow & d_{-1} & \searrow & d_{-2} & \searrow & d_{-3} & \quad\quad \searrow & d_{-n} \end{array}$$

  - Which resolution-bands are resulted in a 2 level decomposition?
  - Specify the memory requirement (in terms of the original signal size) for a 1-D wavelet based MR representation.
  - How are the resolution bands combined to get back the original resolution data?

Multi-resolution analysis (MRA)

- What are a_{-n} and d_{-n} when a 2D wavelet transform (dyadic) is used?



Multi-resolution analysis (MRA)

Tutorial Questions:

Now you can attempt Q2 of the problem set 1.