The University Of Sheffield.

Lecture content

Nyquist Sampling Theorem

- Stroboscopic effect
- Sampling
- Aliasing

Nyquist Sampling theorem

- http://www.tinafad.com/strobe.php

WARNING: IF YOU ARE EPILEPTIC PLEASE RAISE YOUR HAND.

Nyquist Sampling theorem

Consider a CT signal $x(t)$ with frequency spectrum

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

If we sample the signal $x(t)$ every T seconds we have a sampled version of $x(t)$,

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT)$$

The discrete Fourier Transform is

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\omega t}\,dt = \sum_{k=-\infty}^{\infty} x(kT)e^{-j\omega kT}$$

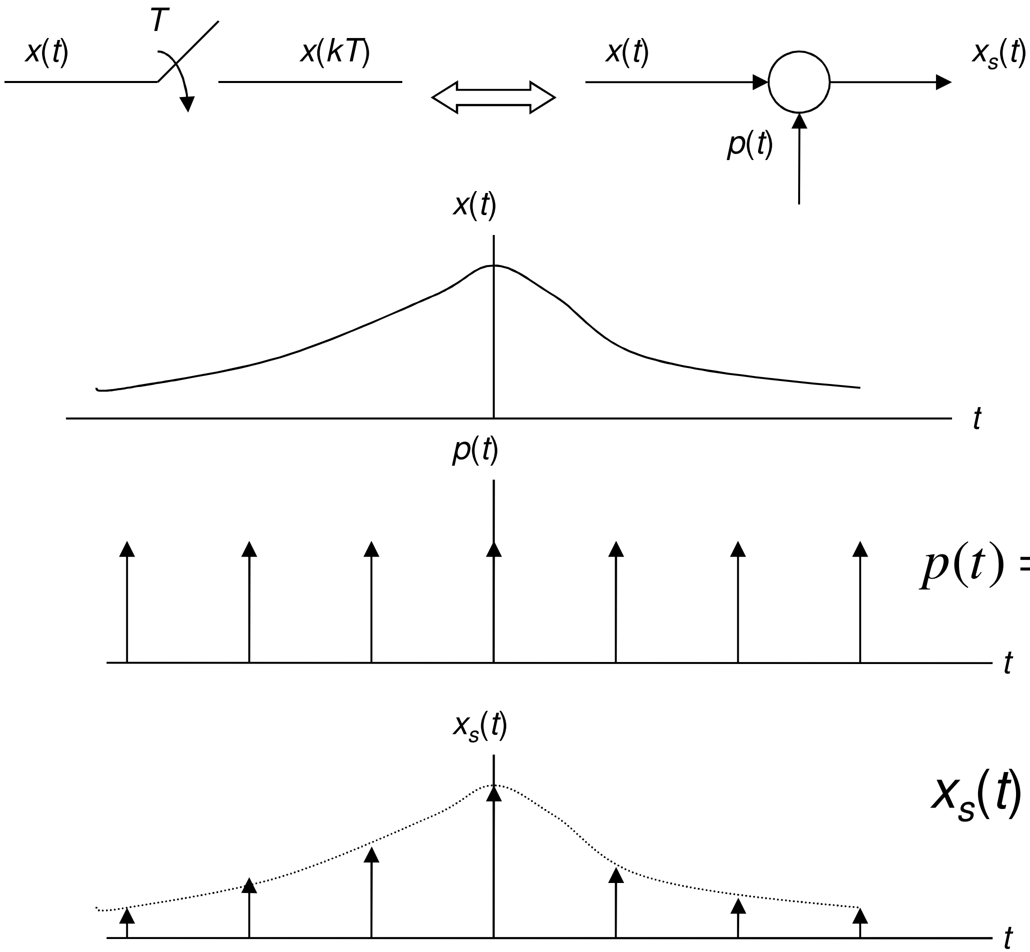
Nyquist Sampling theorem

Recall that $x(kT)\delta(t\text{-}kT) = x(t)\delta(t\text{-}kT)$. We have,

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = x(t)\,p(t)$$

where $p(t)$ is the sampling function so that the process of sampling can be considered as a modulation process.

Nyquist Sampling theorem



$$p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$$

$$x_s(t) = x(t)p(t)$$

Nyquist Sampling theorem

The Fourier Series coefficient of $p(t)$ is

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jn\omega_s t}dt = \frac{1}{T}$$

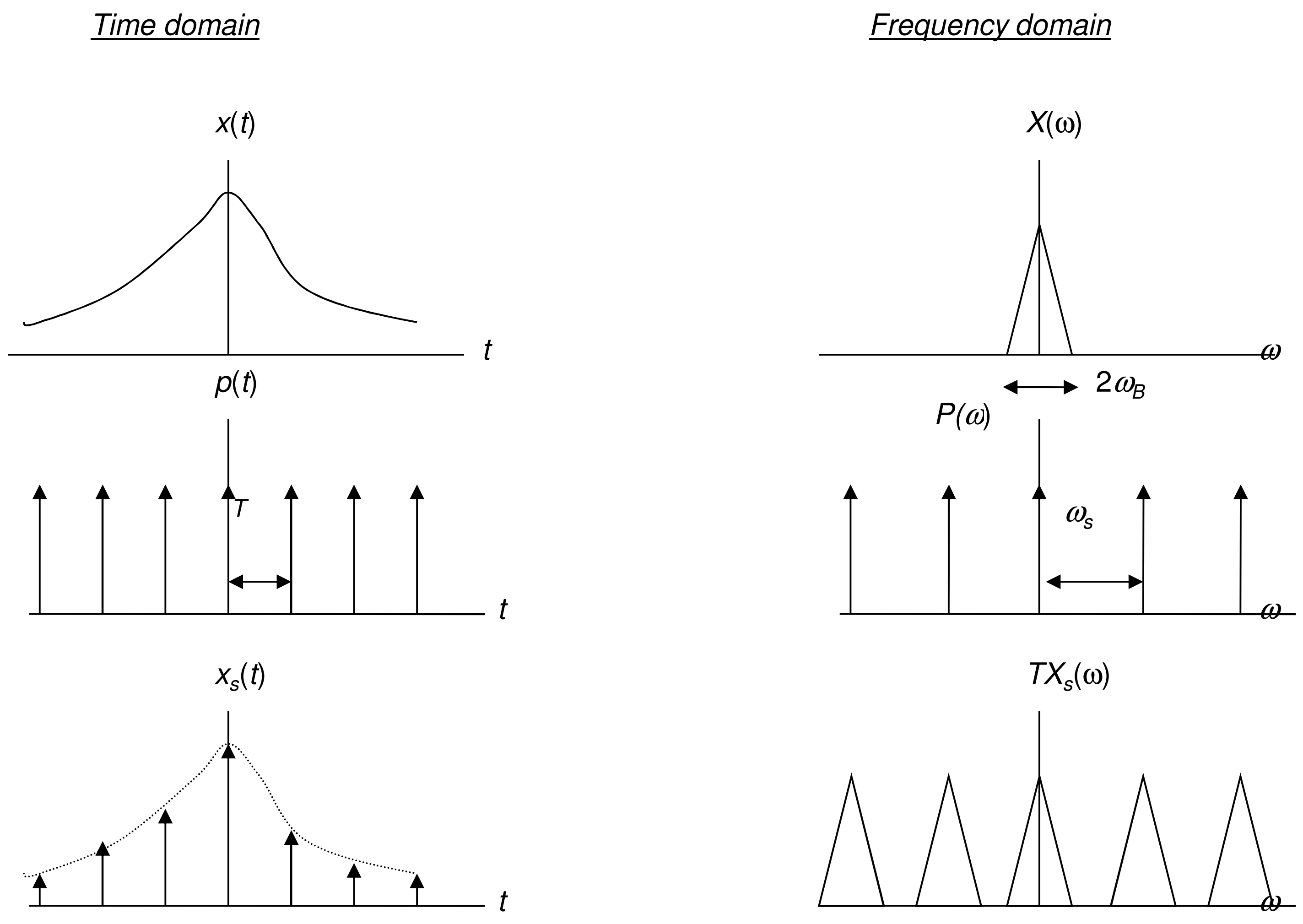
so that we can write $p(t)$ as

$$p(t) = \sum_{n=-\infty}^{\infty}\frac{1}{T}e^{jn\omega_s t}$$

where $\omega_s = 2\pi/T$ is the sampling frequency. The Fourier Transform of the sampling function is

$$P(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)$$

Nyquist Sampling theorem



Nyquist Sampling theorem

The signal $x(t)$ is said to be band-limited to ω_B if $[-\omega_B, \omega_B]$ is the smallest frequency band that contains all the nonzero spectrum $X(\omega)$ of $x(t)$, i.e $X(\omega) = 0$, for $|\omega| > \omega_B$. The spectrum $X_s(\omega)$ is given by

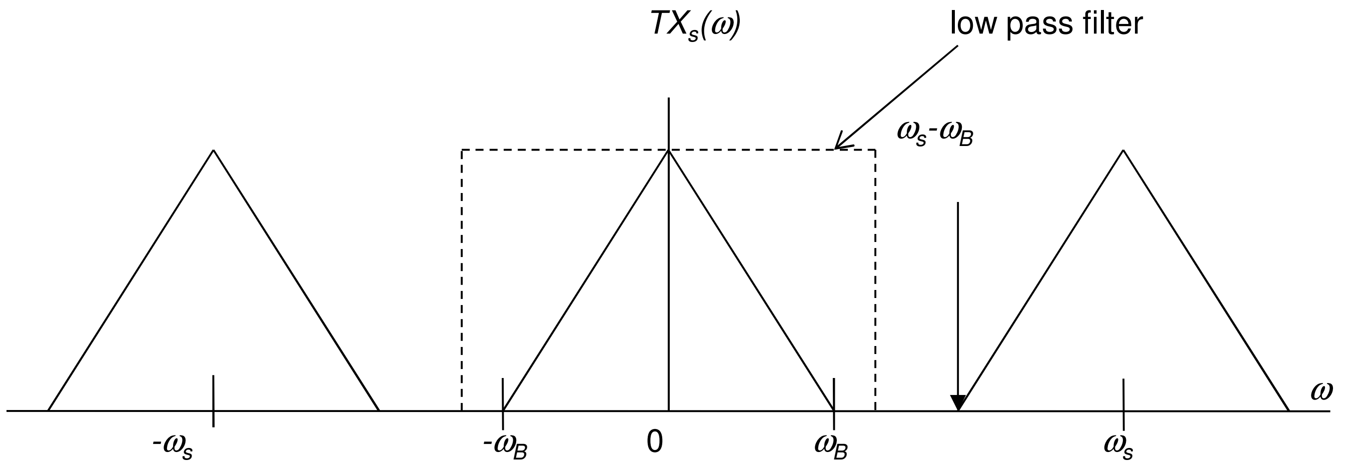
$$X_s(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) P(\omega - \lambda) d\lambda$$

$$X_s(\omega) = \frac{1}{T} \int_{-\infty}^{\infty} X(\lambda) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s - \lambda) d\lambda = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$TX_s(\omega) = \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

Nyquist Sampling theorem

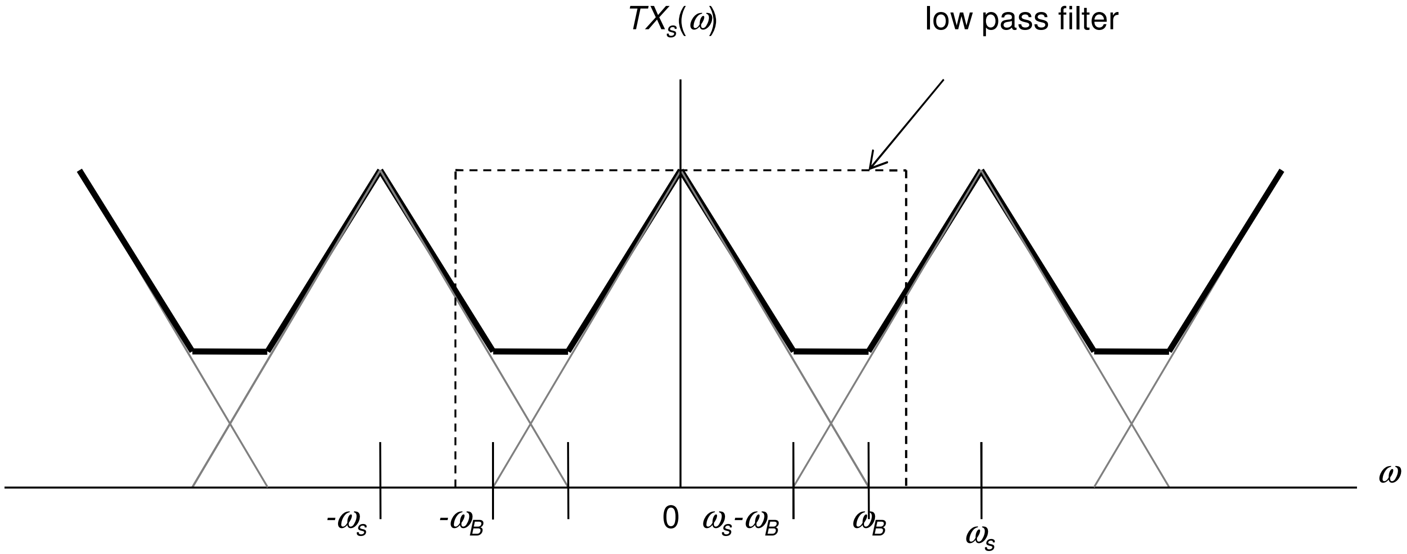
This shows that a replica of $X(\omega)$ has been produced at $\pm\omega_s$, $\pm 2\omega_s$,… $\pm n\omega_s$. If ω_s - ω_B > ω_B we have,



In this case we can recover the signal by low pass filtering.

Nyquist Sampling theorem

If ω_s - ω_B < ω_B we have,



In this case we cannot recover the signal by low pass filtering. The spectrum of $x(t)$ is no longer replicated in $TX_s(\omega)$ and therefore is no longer recoverable by low pass filtering. It is clear that different components of $TX_s(\omega)$ overlap and this effect is referred to as ***aliasing***.

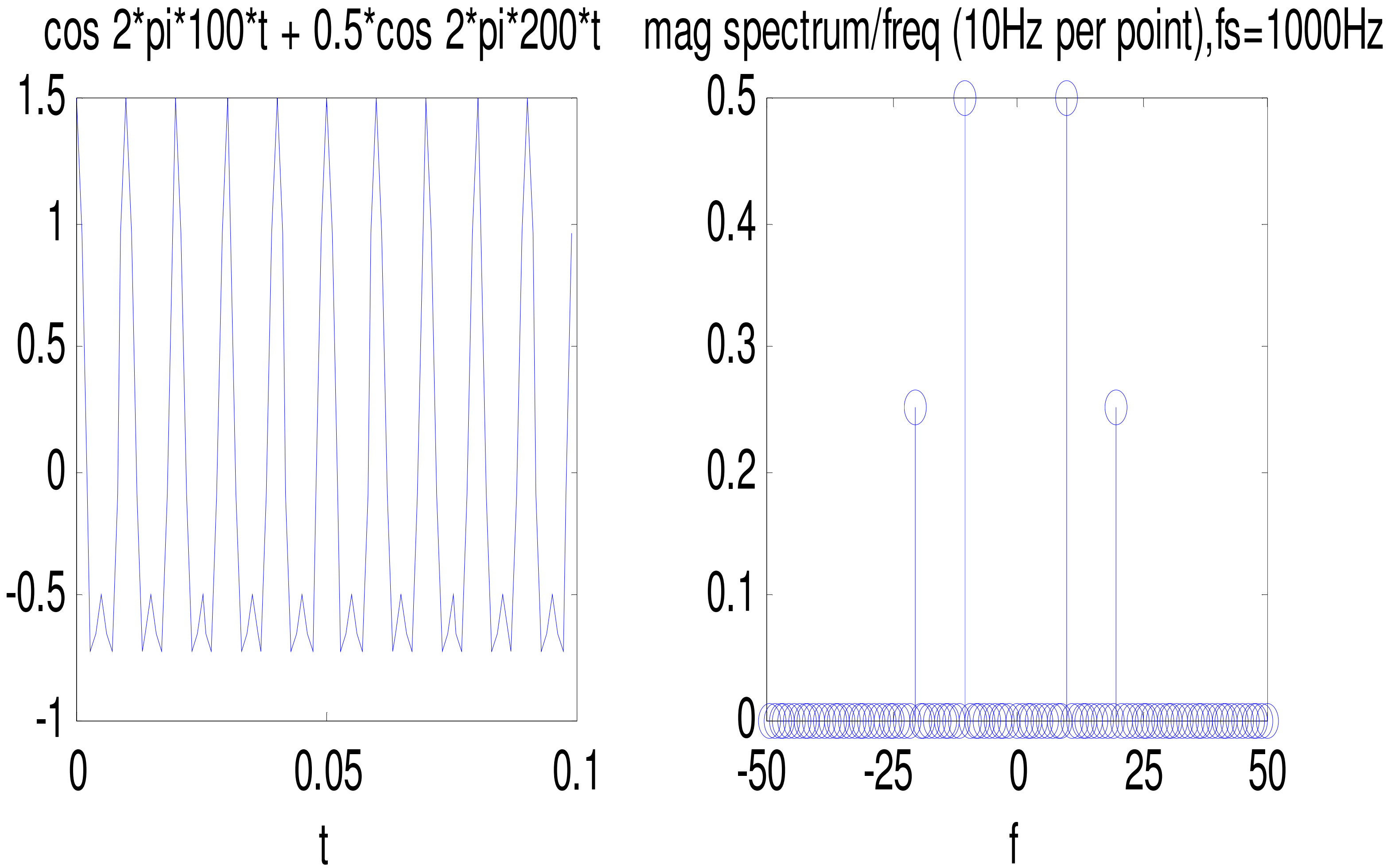
Nyquist Sampling theorem

Consider a signal $x(t) = \cos 2\pi f_1 t + 0.5\cos 4\pi f_1 t$. We have,

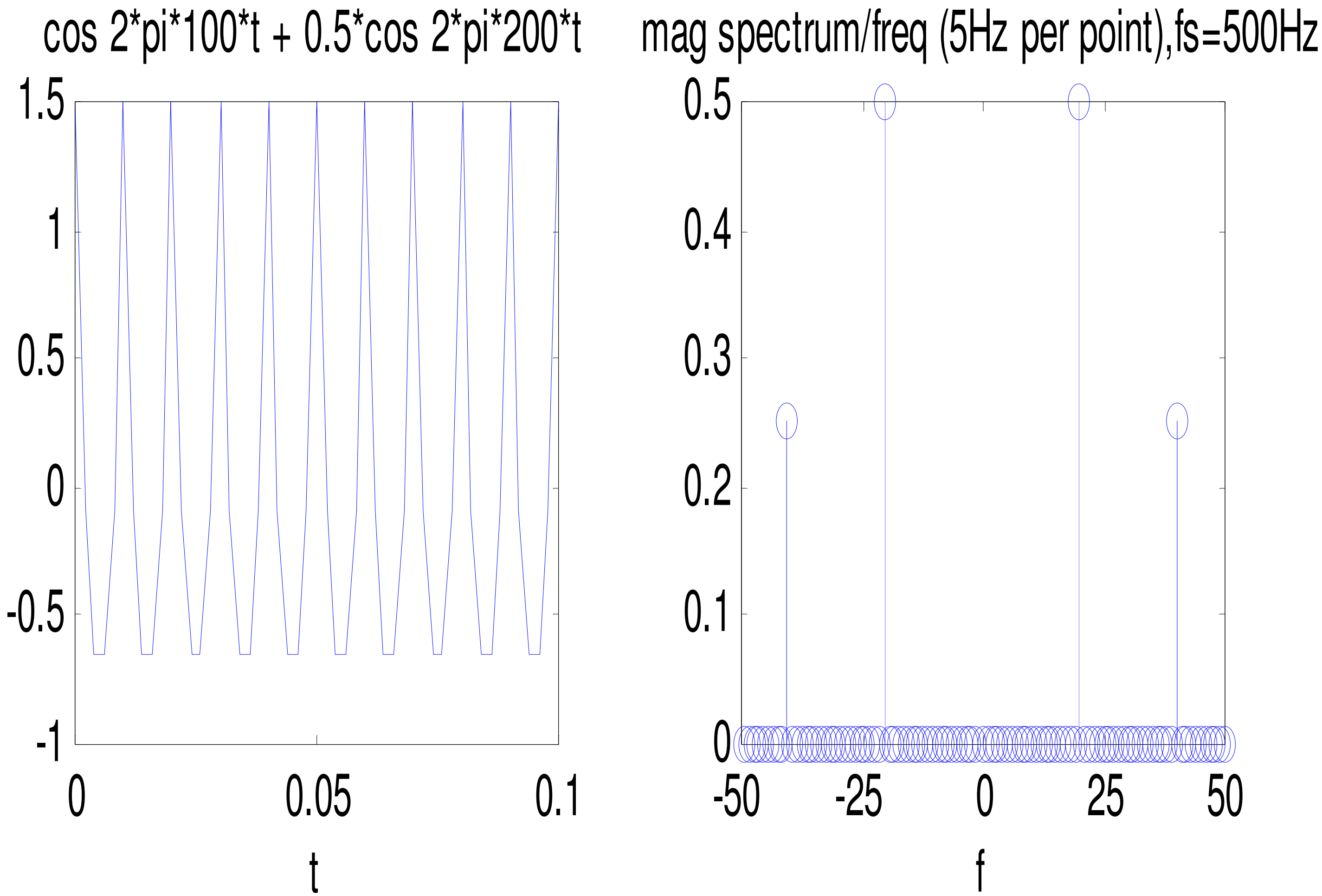
$$x(t) = \frac{1}{2}\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right) + \frac{1}{4}\left(e^{j2\omega_1 t} + e^{-j2\omega_1 t}\right)$$

so that the Fourier Series coefficients are $c_1 = 1/2$, $c_{-1} = 1/2$, $c_2 = 1/4$ and $c_{-2} = 1/4$. Figure 6.1 shows the effect of changing the sampling frequency f_s on the spectrum of $x(t)$ when $f1 = 100$Hz.

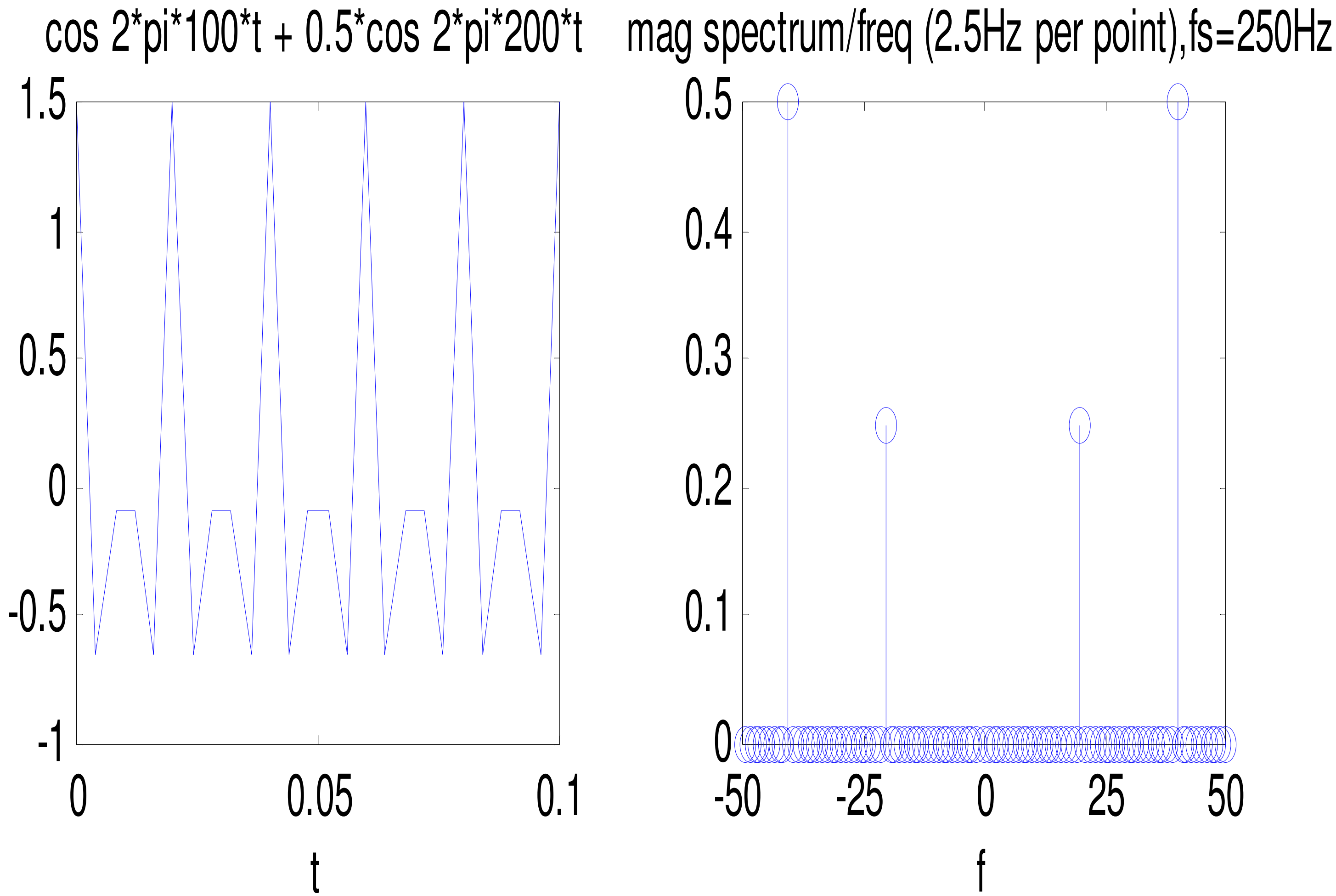
Nyquist Sampling theorem



Nyquist Sampling theorem



Nyquist Sampling theorem



Sampling