EEE345 Worked solutions [redacted]

Question 1

a. *Show that the current flowing in a lossless transmission line obeys the one dimensional wave equation.*

Book work lecture 2.

b. *Derive the expression for the propagation constant β of a transmission line that has small but finite resistance and conductance per unit length.*

Bookwork lecture 3.

$$\beta \approx \omega(LC)^{1/2}\left\{1-\frac{j}{2}\left(\frac{G}{\omega C}+\frac{R}{\omega L}\right)\right\}$$

A 1 GHz sine wave with peak amplitude of 5 V is launched into the lossy transmission line. Draw the time dependence of the signal at a distance of 100 m along the line. Assume the following values for the transmission line: G = 0 S/m; R = 1 Ω/m; C = 80 pF/m; L = 20 μH/m

$V(x) = V0 \exp(-\beta_{imaginary}\, x)$

Hence $V(100) = 5 \exp(-((20\times10^{-6} \times 80 \times 10^{-12})^{0.5}/2 \times (0 + 1/20\times10^{-6})) \times 100 = 5 \exp(-1\times10^{-3} \times 100)$

$= 5\, e^{-0.1} = \underline{4.5\ V}$

Sketch (qualitatively) the time dependence at 100 m for a square pulse launched into the same transmission line.

c. Figure 1 (below) represents the cross-section of a microstrip transmission line. Suggest what suitable materials would be used for A1, A2 and A3 and their approximate sizes and their purposes.

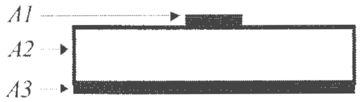


Figure 1

A1 - metal (copper); 5 x 20 μm upto 30 x 500 μm; conductor for signal

A2 - dielectric (glass/epoxy); ~0.2-3 mm thickness; insulator to separate signal and return electrical paths; ε effects signal propagation speed.

A3 – as A1; ground plane for signal return path

d. *Assuming that the transmission line in Figure 1 is lossless and has characteristic impedance $Z_0 = 50\ \Omega$ and is connected to a load with impedance $Z_L = 75\ \Omega$. A square pulse of amplitude 5V is launched into the transmission line from a source with impedance $Z_s = 100\ \Omega$. What is the amplitude of the signal received by the load?*

Gamma (reflection coef) = V-/V+ = (ZL-Z0)/(ZL+Z0) = 25/125 = 0.2

V+ = 5V hence V- = 5x0.2 = 1V

VL = Vs-V- = 5-1 = 4V

e. *The apparent impedance at the source end of a transmission line ZA is given by the following equation, where the terms have their usual meanings:*

$$Z_A = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{jZ_L \tan(\beta d) + Z_0}$$

i) For what length/lengths of transmission line does the apparent impedance match the load impedance?

ii) If the condition in 1.e.i. (above) is not met, what can be done in practise to match ZA to ZL?

i) ZA = ZL when βd = nπ (n=0,1,2,...) Also β = 2π/λ

Hence: d = nπλ/2π = nλ/2

ii) Tuning stubs – short unterminated lengths of transmission line positioned to alter the apparent impedance seen by the source.

Question 2

a. *A 1 MHz TEM00-mode electromagnetic wave propagates in free space in the positive x direction. Write down mathematical expressions that describe the magnetics and electric fields. Sketch the spatial relationship between the magnetic and electric fields at time t. Include an indication of the direction of propagation of the wave on your sketch.*

Electric field: $\boldsymbol{E}(x,t) = E_0 \hat{\boldsymbol{y}} e^{j(\omega(-\beta x)}$ Magnetic field: $\boldsymbol{H}(x,t) = H_0 \hat{\boldsymbol{z}} e^{j(\omega(-\beta x)}$

b. *Define the Poynting vector S.*

S = ExH – vector defining the direction and magnitude of energy flow (power/m^2)

c. *The wave in 2.a. has an electric field strength E0 = 1x10^6 V m-1 and is incident at normal incidence on a non-magnetic material with relative permittivity ε_r = 2. What is the electrical field strength of the wave immediately inside the interface?*

See hand written sheet…

… $E_t / E_0 = 2/(1+\varepsilon_r^{0.5})$

Hence E_t = <u>8.3 x 10^5 V/m</u>

d. *For the situation described in 2.c. what is the power density of the wave within the material?*

$S = E_t H_t = \ldots = 2.5\text{x}10^9$ W/m^2

e. *The material in 2.c. is weakly absorbing, hence the propagation constant β can be described by the following equation:*

$$\beta = \omega / v - \frac{j\sigma}{2}\left(\frac{\mu\mu_0}{\varepsilon\varepsilon_0}\right)^{1/2}$$

- where the symbols have their usual meanings and the AC conductivity of the material at 1 MHz is σ = 1x10^{-5} Ω/m. At what distance into the material does the power density fall to 50 % of its value immediately inside the interface?

See hand written sheet…

<u>x = 258 m</u>

Givn $E_i = 1\times10^6$ V/m $\mu_r = 1$ $\varepsilon_r = 2$ 2b 2c

$H_i - H_r = H_t$ (1) ← so that Poynting vector is reversed.

$E_i + E_r = E_t$ (2)

$$\frac{E_i}{H_i} \; (3) = \frac{E_r}{H_r} \; (4) = \sqrt{\frac{\mu_0}{\varepsilon_0}} \qquad \frac{E_t}{H_t} = \sqrt{\frac{\mu_0}{\varepsilon_r\varepsilon_0}} \; (5)$$

from (1): $H_r = H_i - H_t$

using (3-5): $E_r\sqrt{\frac{\varepsilon_0}{\mu_0}} = E_i\sqrt{\frac{\varepsilon_0}{\mu_0}} - E_t\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}$

using (2): $(E_t - E_i)\sqrt{\frac{\varepsilon_0}{\mu_0}} = E_i\sqrt{\frac{\varepsilon_0}{\mu_0}} - E_t\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}$

simplify: $E_t\left(\sqrt{\frac{\varepsilon_0}{\mu_0}} + \sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}\right) = E_i\left(2\sqrt{\frac{\varepsilon_0}{\mu_0}}\right)$

$$\therefore \frac{E_t}{E_i} = \cancel{\frac{2\sqrt{\frac{\varepsilon_0}{\mu_0}}}{\sqrt{\ }}} \quad \frac{2\sqrt{\frac{\varepsilon_0}{\mu_0}}}{\sqrt{\frac{\varepsilon_0}{\mu_0}} + \sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0}}}$$

$$= \cancel{\sqrt{\frac{\varepsilon_0}{\mu_0}}\left(\frac{1}{1}\right)}$$

(*) pro!
$$\boxed{\frac{E_t}{E_i} = \frac{2}{1+\sqrt{\varepsilon_r}}}$$ ✓✓ (6)

$\Rightarrow E_t = \cancel{\left(\frac{1+\sqrt{\varepsilon_r}}{2}\right)} E_i$

$\Rightarrow E_t = \frac{2}{1+\sqrt{2}} \cancel{\left(\frac{1+\sqrt{2}}{2}\right)} \cdot 10^6$

$\boxed{E_t = 0.83 E_i}$

$\boxed{E_t = 8.3\times10^5 \text{ V/m}}$ ← $\cancel{4.2\times10^6}$??

using (5) & (6) :

(2.a)

$$H_t = E_t \sqrt{\frac{\cancel{\mu_o}\varepsilon_o \varepsilon_r}{\mu_o}}$$

$$= \left(\frac{2}{1+\sqrt{\varepsilon_r}}\right) E_i \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o}} \quad (7)$$

$$\boxed{E_t = \frac{2}{1+\sqrt{\varepsilon_r}} E_i} \quad (8)$$

$$S_t = E_t H_t$$

$$= (7) \times (6a)$$

$$S = \left(\frac{2}{1+\sqrt{\varepsilon_r}}\right) E_i \cdot \frac{2}{1+\sqrt{\varepsilon_r}} E_i \sqrt{\frac{\varepsilon_o \varepsilon_r}{\mu_o}}$$

$$= 0.83 \times 0.83 \times 2.6\times10^{-3} \times \sqrt{2} \times E_i^2$$

$$\cancel{= 2.5\times10^{-3} E_i^2} = \cancel{2.5\times10^{9} 6\times b}$$

$$= \boxed{2.5\times10^{9} \, W/m^2}$$

(2.b)

(*) $$\frac{E_t}{E_i} = \frac{2a}{a+ab}$$

let $a = \sqrt{\frac{\varepsilon_o}{\mu_o}}$

$b = \sqrt{\varepsilon_r}$

$$\therefore \frac{E_i}{E_t} = \frac{a+ab}{2a} = \frac{1}{2} + \frac{b}{2}$$

$$= \cancel{\tfrac{1}{2}(1+b)} = \frac{1}{2}(1+b)$$

$$= \frac{(1+b)}{2}$$

$$\therefore \boxed{\frac{E_t}{E_i} = \frac{2}{1+b} = \frac{2}{1+\sqrt{\varepsilon_r}}}$$

(2c) $\cancel{S}$ $\underline{S} = \underline{E} \times \underline{H}$ = power/m^2

Direction & magnitude of energy flow

EEE345/6084 2013 solutions

Q3

a.

Since the divergence of the curl of any vector $\overline{F}$ is zero, i.e.

$$\nabla \bullet (\nabla \times \overline{F}) = 0 \qquad (1)$$

and

$$\nabla \bullet \overline{B} = 0 \qquad (2)$$

where $\overline{B}$ is the magnetic flux density (Wb/m^2 or T), then we may assign $\overline{B}$ a relation thus

$$\overline{B} = \nabla \times \overline{A} \qquad (3)$$

where $\overline{A}$ is the magnetic vector potential (Vs/m). Since

$$\nabla \times \overline{H} = \overline{J} \qquad (4)$$

where $\overline{H}$ is the magnetic field strength (A/m) and $\overline{J}$ the volume current density (A/m^2)

and

$$\overline{B} = \mu \overline{H} \qquad (5)$$

then from (3), (4), (5)

$$\nabla \times \overline{B} = \mu \overline{J} = \nabla \times \nabla \times \overline{A} \qquad (6).$$

Since

$$\nabla \times (\nabla \times \overline{A}) = \nabla(\nabla \bullet \overline{A}) - \nabla^2 \overline{A} \qquad (7)$$

and defining

$$\nabla \bullet \overline{A} = 0 \qquad (8)$$

then we have from (6), (7), (8)

$$\nabla^2 \overline{A} = -\mu \overline{J} \quad (9).$$

b.

(i) We only have a $\hat{z}$ component of vector potential since only a $\hat{z}$ component of current density, so

$$\nabla^2 A_z = \frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2} \qquad (10).$$

Thus

$$\frac{\partial^2 A_z}{\partial x^2} = \mu_o \beta J \cos(\beta y) e^{-\beta x} \quad (11)$$

and

$$\frac{\partial^2 A_z}{\partial y^2} = -\mu_o \beta J \cos(\beta y) e^{-\beta x} \qquad (12)$$

and

$$\frac{\partial^2 A_z}{\partial z^2} = 0 \qquad (13)$$

so from (10, (11), (12), (13)

$$\nabla^2 \overline{A} = 0 \qquad (14).$$

(ii)

We only have a $\hat{z}$ component of vector potential, so from (3)

$$\overline{H} = \frac{1}{\mu_o} \nabla \times A_z \qquad (15)$$

so

$$H_x = \frac{1}{\mu_o} \frac{\partial A_z}{\partial y} = -J \sin(\beta y) e^{-\beta x} \qquad (16)$$

and

$$H_y = -\frac{1}{\mu_o} \frac{\partial A_z}{\partial x} = J \cos(\beta y) e^{-\beta x} \qquad (17).$$

c.

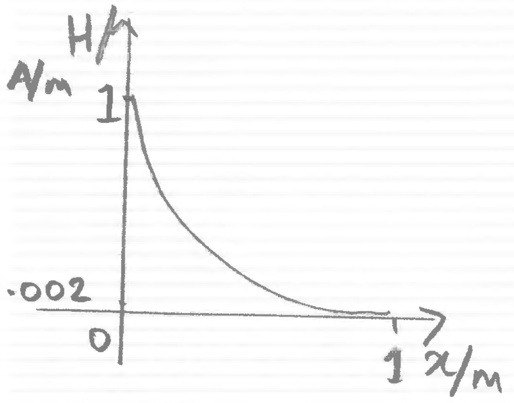
Modulus of magnetic field is

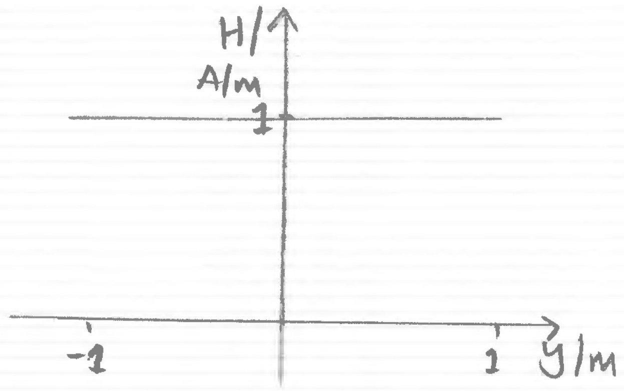
$$H = \sqrt{H_x{}^2 + H_y{}^2} = \sqrt{J^2 e^{-2\beta x} (\sin^2(\beta y) + \cos^2(\beta y))} = J e^{-\beta x} \qquad (18)$$

and it is not a function of y. Therefore

$$H(x = 0) = 1 A/m \qquad (19)$$
$$H(x = 1m) = 1.867 mA/m \qquad (20)$$

Hence the sketches:





Q4

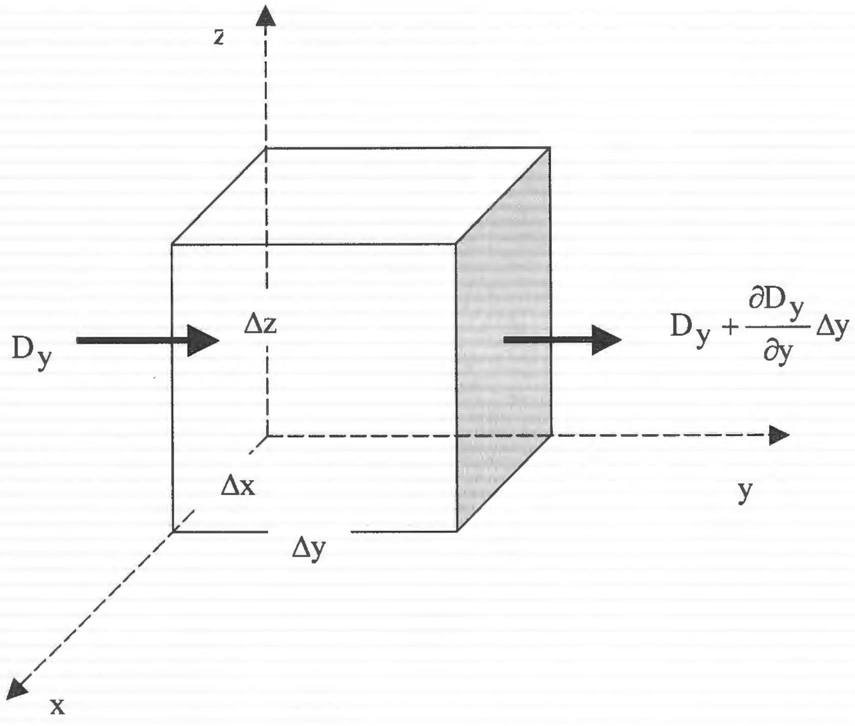
a.

Gauss Law:

$$\oint\!\!\oint_S \bar{D} \bullet d\bar{S} = \iiint_V \rho dv = Q \qquad (1)$$

The total electric flux flowing through a closed surface (LHS (1)) is equal to the total charge enclosed by that surface (RHS (1)), or a volume integral of charge density bounded by the surface (middle (1)).

b.



Consider the above cube with electric flux density

$$\bar{D} = D_x\hat{x} + D_y\hat{y} + D_z\hat{z} \qquad (1)$$

flowing through it. Taking the example of the sides normal to the y axis, the net flux leaving the cube through these two sides is

$$\left(D_y + \frac{\partial D_y}{\partial y}\Delta y\right)\Delta x\Delta z - D_y\Delta x\Delta z = \frac{\partial D_y}{\partial y}\Delta x\Delta y\Delta z \qquad (2)$$

and similarly for the sides normal to the x axis and z axis respectively

$$\left(D_x + \frac{\partial D_x}{\partial x}\Delta x\right)\Delta y \Delta z - D_x \Delta y \Delta z = \frac{\partial D_x}{\partial x}\Delta x \Delta y \Delta z \quad (3)$$

$$\left(D_z + \frac{\partial D_z}{\partial z}\Delta z\right)\Delta x \Delta y - D_z \Delta x \Delta y = \frac{\partial D_z}{\partial z}\Delta x \Delta y \Delta z \quad (4).$$

Summing (2), (3), (4) gives the total flux leaving the cube as

$$\oiint_S \overline{D} \bullet d\overline{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z \quad (5).$$

From (1)

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z = Q \quad (6).$$

Hence

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \frac{Q}{\Delta x \Delta y \Delta z} = \rho \qquad (7)$$

and since

$$\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) = \nabla \bullet \overline{D} \qquad (8)$$

then the relation is shown,

$$\nabla \bullet \overline{D} = \rho \qquad (9).$$

From (1) and (9) therefore

$$\oiint_S \overline{D} \bullet d\overline{S} = \iiint_V \nabla \bullet \overline{D} dv \quad (10)$$

which is the divergence theorem.

c.

(i) We need to evaluate (9) for the charge density. Thus from (8)

$$\frac{\partial D_x}{\partial x} = 16xyz, \frac{\partial D_y}{\partial y} = 8xyz \; \frac{\partial D_z}{\partial z} = 4xyz \quad (11)$$

from the question,

$$xyz = 1 \times 2 \times 3 = 6 \qquad (12)$$

thus

$$\nabla \bullet \overline{D} = \rho = (16 + 8 + 4) \times 6 = 168 \text{ C/m}^3 \qquad (13).$$

(ii) Here, since $\overline{D} = \varepsilon\overline{E}$, then

$$\frac{\partial D_x}{\partial x} = 0, \frac{\partial D_y}{\partial y} = 0, \frac{\partial D_z}{\partial z} = 0 \qquad (14)$$

so there is zero charge density at the point.