Third and Higher Order Circuits

Introduction

First and higher order circuits occur in many areas of electronics such as instrumentation systems, modern power supply circuitry, feedback control systems and many communication systems. The key behavioural characteristics of first and second order systems are relatively easily identified by the methods described already in the First Order Circuits and Second Order Circuits handouts. The behaviour of third and higher order circuits is, however, generally more difficult to identify than that of first and second order systems. As the order of a system increases the number possible ways in which it can behave also increases.

High order circuits can be designed to exhibit frequency dependent behaviour that is particularly attractive in frequency selective, or *filtering*, applications and it is higher order circuits for filtering applications that are of interest here.

Filters

A filter is a circuit designed to exhibit a well defined frequency dependent gain so that one range (or band) of frequencies is treated differently from another. This kind of behaviour is also known as frequency selective behaviour and filters can be described as frequency selective circuits. One of the primary determinants of a filter's behaviour is its order, which is simply the order of the polynomial that describes the denominator of its transfer function. A first order low pass filter has a transfer function of the form

$$\frac{v_o}{v_i} = k \frac{1}{1 + s\tau_0} = k \frac{1}{1 + \dfrac{j\omega}{\omega_0}} \qquad \textbf{(1)}$$

This filter has a roll-off of 20dB per decade as frequency increases above the corner frequency ω_0. Most filter applications require more rapid roll-off rates than 20dB per decade and rapid roll-off can be achieved by increasing the filter order. In general, roll-off rate increases by 20dB per decade per order so, for example, a 3rd order filter could achieve 60 dB per decade and a 10th order filter could achieve 200 dB per decade.

The problem with higher order filters is obtaining a well defined response shape. For example, how should the coefficients of the transfer function denominator be chosen to obtain a response that is, say, as flat as possible in the pass band while cutting off as quickly as possible in the stop (or attenuation) band? Fortunately, over the last century some oustanding mathematical minds have worked on this problem and the result of their efforts is a range of *filter polynomials*.

Filter Polynomials

Much of the work on filters was done in the early to mid part of the twentieth century by mathematicians working in the telecommunications and radar areas where frequency selective circuits were essential system components. More recently, biomedical instrumentation and image processing have provided other demanding frequency selective problems. What came out of the work of these mathematicians was a collection of polynomials in s that were all special in some way - ie optimised to achieve certain specific characteristics - and usually bore the name of their inventor (eg Butterworth, Chebyshev, Bessel, etc) or the type of function on which they were based (eg Gaussian, Elliptical, etc).

The filter polynomials of interest in this module form the denominator of the transfer function of a low-pass version of the filter type concerned. There are standard methods for transforming a low-pass transfer function into either a high-pass one or a band-pass one but for this module only the low-pass case will be considered.

Filter polynomials are usually given in frequency normalised form - ie, the polynomial coefficients are appropriate to give a cut-off frequency of unity. Cut-off frequency in the context of a filter polynomial has a similar meaning to corner frequency in the context of a first order low-pass circuit. Design of a filter circuit based on one of these polynomials is essentially a problem of synthesising a circuit that can execute the mathematical operation described by the polynomial. The polynomials of interest in this module are Butterworth polynomials and Chebyshev polynomials.

Butterworth and Chebyshev Filter Polynomials

Butterworth

Butterworth filters are used extensively. They do not have roll-off rates as rapid as those of Chebyshev filters but they are well behaved and quite tolerant of small component value errors. Butterworth's optimisation objective was to seek a set of polynomials that had the maximum flatness of pass-band without incurring any gain peaking and thus Butterworth responses are sometimes described as *maximally flat*.

Table 1 gives the factored forms of frequency normalised Butterworth polynomials from first to eighth order. The factored form is the most useful form of these polynomials from the point of view of the design strategy presented later and it is easier to work out the unfactored form from the factored form than vice versa.

TABLE 1

Order	**Factored Butterworth polynomials**
1	$(s + 1)$
2	$(s^2 + 1.414s + 1)$
3	$(s + 1)(s^2 + s + 1)$
4	$(s^2 + 0.765s + 1)(s^2 + 1.848s + 1)$
5	$(s + 1)(s^2 + 0.618s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.518s + 1)(s^2 + 1.414s + 1)(s^2 + 1.932s + 1)$
7	$(s + 1)(s^2 + 0.445s + 1)(s^2 + 1.247s + 1)(s^2 + 1.802s + 1)$
8	$(s^2 + 0.390s + 1)(s^2 + 1.111s + 1)(s^2 + 1.663s + 1)(s^2 + 1.962s + 1)$

Chebyshev

Chebyshev responses are used less frequently than Butterworth ones. In the vicinity of the cut-off frequency and for third order and greater, Chebyshev roll-off rates are significantly more rapid than those of Butterworth responses of the same order. These rapid roll-off rates, however, create a number of practical difficulties related to a high sensitivity to component errors - a problem that becomes more severe as filter order increases. Chebyshev's optimisation process was to find the set of polynomials that gave the fastest roll-off rates possible with a fixed level of passband gain ripple.

Table 2 gives the factored forms of frequency normalised Chebyshev polynomials for a constant ripple of 1dB. Chebyshev polynomials can be created for any pass band gain ripple specification; values like 0.5 dB, 2 dB, 3 dB etc are readily available as tabulated functions in filter text books. Note that the initial cut-off rate increases as the allowed constant ripple magnitude increases.

TABLE 2

Order	**Factored Chebyshev polynomials (1dB ripple)**
1	$(s + 1.965)$
2	$(s^2 + 1.098s + 1.103)$
3	$(s + 0.4942)(s^2 + 0.4942s + 0.9942)$
4	$(s^2 + 0.2790s + 0.9865)(s^2 + 0.6738s + 0.2794)$
5	$(s + 0.2895)(s^2 + 0.1790s + 0.9883)(s^2 + 0.4684s + 0.4293)$
6	$(s^2 + 0.1244s + 0.9907)(s^2 + 0.3398s + 0.5577)(s^2 + 0.4642s + 0.1247)$
7	$(s + 0.2054)(s^2 + 0.0914s + 0.9927)(s^2 + 0.2562s + 0.6535)(s^2 + 0.3702s + 0.2304)$
8	$(s^2 + 0.0700s + 0.9942)(s^2 + 0.1994s + 0.7236)(s^2 + 0.2994s + 0.3408)(s^2 + 0.3518s + 0.0702)$

Butterworth and Chebyshev characteristics

Figures 1a and 1b (on page 4) show the roll-off rates for the filter functions given in Tables 1 and 2. The roll-off rate advantages of the Chebyshev response become clear at the higher orders. Consider the eighth order case. At a normalised frequency of 2, the Chebyshev response has fallen by 80dB whilst that of the Butterworth response has fallen by only 50dB. Notice, though, that the increased roll-off rate offered by the Chebyshev filter function arises because of a more rapid initial roll-off rate; for a given order the roll-off slopes of the two filters approach the same value for frequencies well above the cut-off frequency.

TABLE 3 **f_0 and q for frequency normalised Butterworth polynomials.**
(The factors are in the same order as those in the polynomials of table 1)

Order	**factor 1**		**factor 2**		**factor 3**		**factor 4**	
	f_0	q	f_0	q	f_0	q	f_0	q
1	1.000	------						
2	1.000	0.707						
3	1.000	------	1.000	1.000				
4	1.000	1.307	1.000	0.541				
5	1.000	------	1.000	1.618	1.000	0.618		
6	1.000	1.932	1.000	0.707	1.000	0.518		
7	1.000	------	1.000	2.247	1.000	0.802	1.000	0.555
8	1.000	2.564	1.000	0.900	1.000	0.601	1.000	0.590

An insight into the way in which the two polynomials achieve their goals can be gained from figures 2a and 2b and tables 3 and 4. Look first at tables 3 and 4. The most striking difference between the two is that **all the factors of the Butterworth polynomials have corner frequencies or undamped natural frequencies of unity** - ie, the same as the overall filter cut-off frequency - whereas the factors of the Chebyshev polynomials have corner frequencies and undamped natural frequencies that are different from the overall filter cut-off frequency. It is also clear that for a given filter order, the Chebyshev factors have higher *q* factors than the Butterworth ones.

TABLE 4 **f_0 and *q* for frequency normalised Chebyshev polynomials (1dB ripple)**
(The factors are in the same order as those in the polynomials of table 2)

Order	factor 1		factor 2		factor 3		factor 4	
	f_0	q	f_0	q	f_0	q	f_0	q
1	1.965	------						
2	1.050	0.957						
3	0.494	------	0.997	2.018				
4	0.993	3.561	0.528	0.784				
5	0.289	------	0.994	5.553	0.655	1.400		
6	0.995	8.001	0.747	2.198	0.353	0.761		
7	0.205	------	0.996	10.90	0.808	3.155	0.480	1.297
8	0.997	14.24	0.851	4.266	0.584	1.950	0.265	0.753

Figures 2a and 2b show how the factors of a seventh order polynomial combine to produce the overall response for a Butterworth (figure 2a) and for a Chebyshev (figure 2b) seventh order low-pass filter. In the Butterworth filter, all the normalised undamped natural frequencies of the factors are unity and the shape is constructed by careful selection of the *q* factors of the second order factors. The largest *q* factor is just over 2.5 - a relatively small value. The

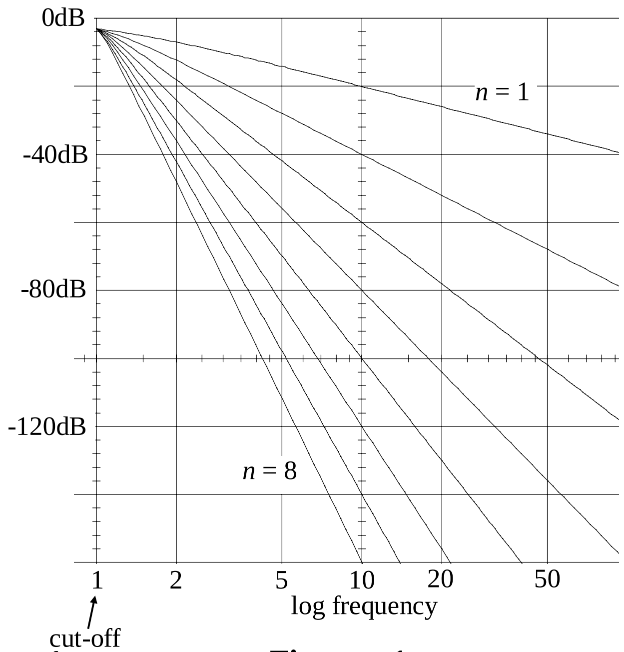


Figure 1a

Butterworth roll-off for 1st to 8th order responses. Cut-off frequency = unity.

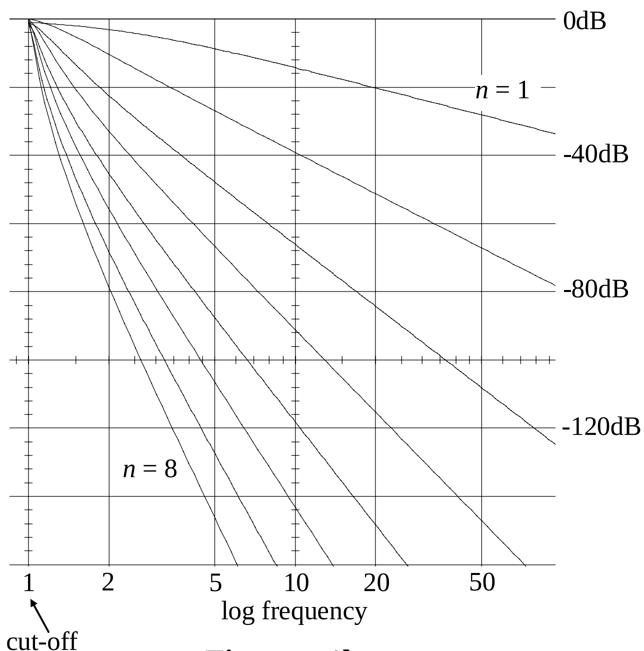
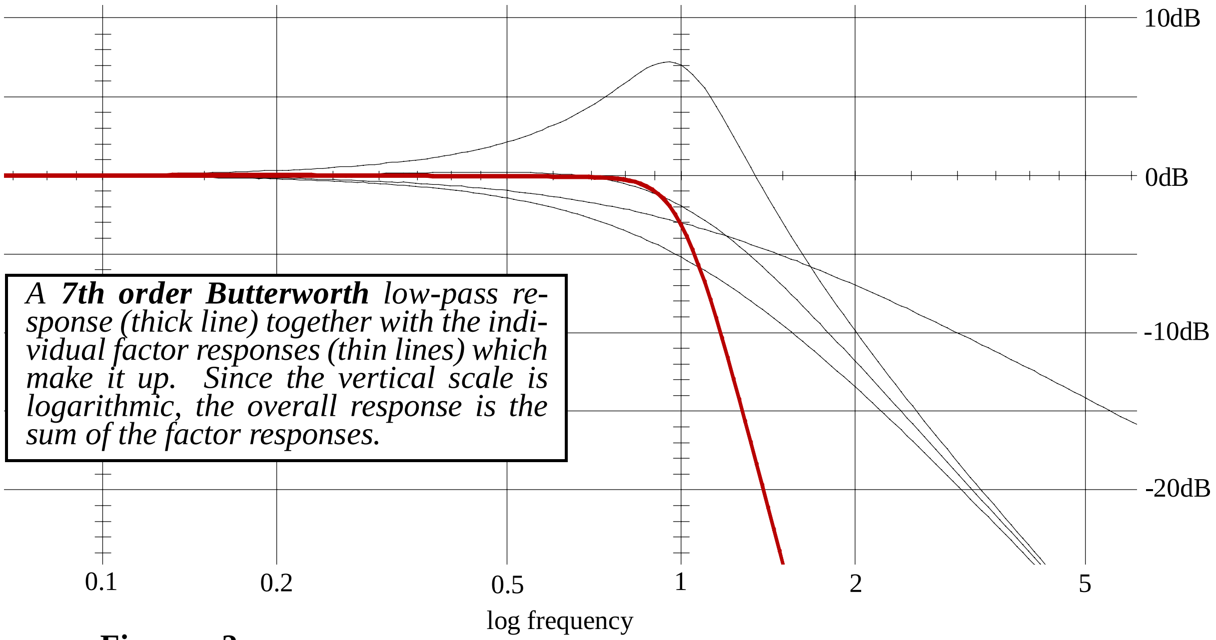


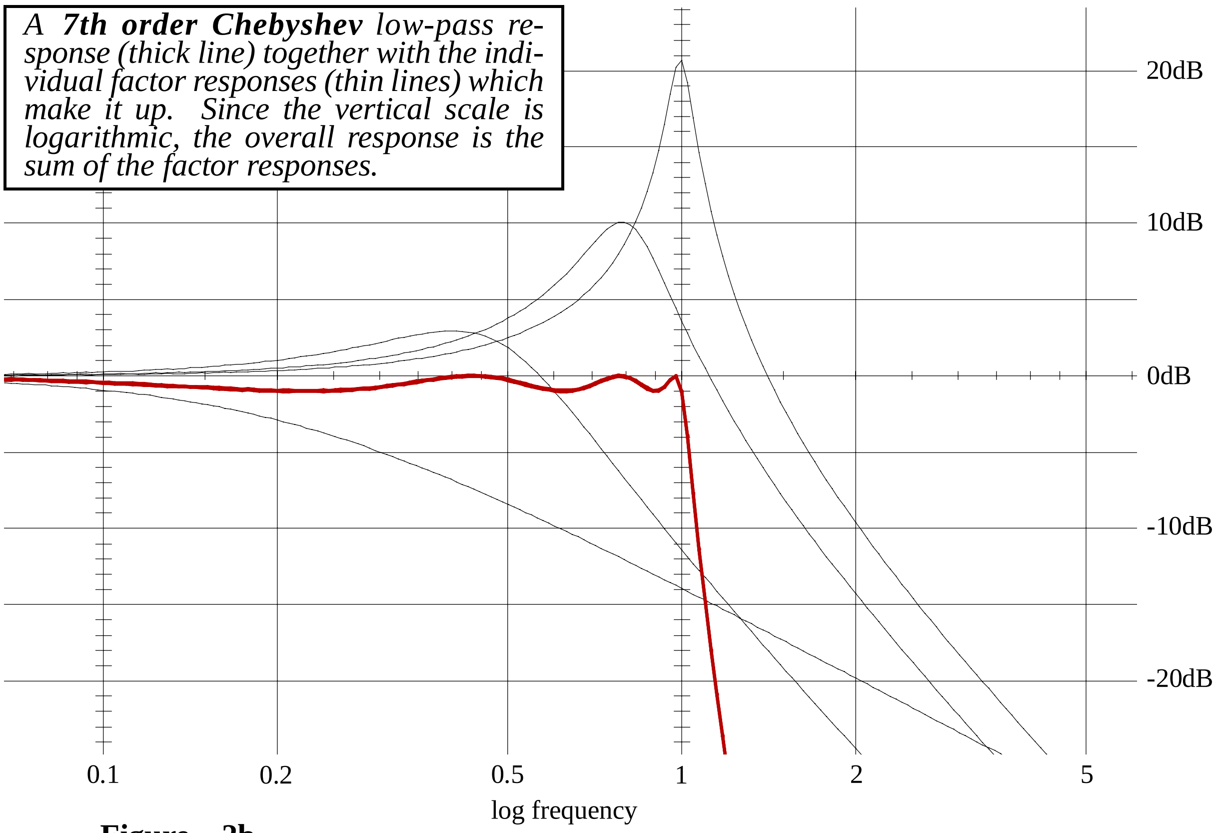
Figure 1b

Chebyshev roll-off for 1st to 8th order responses. Cut-off frequency = unity.



A ***7th order Butterworth*** *low-pass response (thick line) together with the individual factor responses (thin lines) which make it up. Since the vertical scale is logarithmic, the overall response is the sum of the factor responses.*

Figure 2a



A ***7th order Chebyshev*** *low-pass response (thick line) together with the individual factor responses (thin lines) which make it up. Since the vertical scale is logarithmic, the overall response is the sum of the factor responses.*

Figure 2b

Chebyshev filter uses high q sections in the region of the cut-off frequency to achieve rapid rates of change of gain with frequency. The normalised undamped natural frequencies and q factors of the polynomial factors are chosen to give the constant pass band ripple specified whilst achieving the fatest roll-off rate possible for that level of ripple.

In both cases the vertical scales are logarithmic so the overall response is the sum of those due to the individual factors. The Butterworth factor responses vary relatively slowly with frequency, even in the region of the cut off, so an error in q or undamped natural frequency in a particular factor will have a relatively minor effect on the overall response. The Chebyshev factors have higher q factors and this makes the overall response significantly more sensitive to errors in q and undamped natural frequency.

Realisation

The polynomial filter functions can be realised in a number of ways. In high frequency applications (eg radio and radar filters) and in high current applications at low frequencies (eg, loudspeaker crossover circuits and class D amplifier output filters), *L–C* ladder implementations of filters, such as the one shown in figure 3 are commonly used. There are well defined synthesis procedures for finding the values of L and C required to achieve a particular polynomial function.

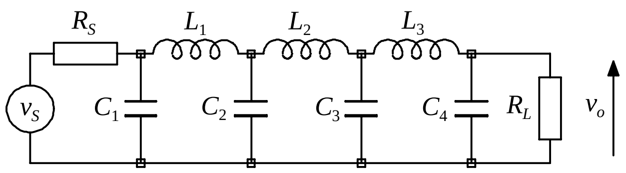


Figure 3

A seventh order L–C ladder topology

Inductors, however, often deviate significantly from their ideal behaviour. Air cored inductors can be very good quality but inductance values are small and the magnetic field extends well outside the space occupied by the inductor. Low frequency low power filters need inductances of the order of mH to H which cannot sensibly be realised without the use of a magnetic core material. Although a magnetic core will contain the magnetic field, it will also introduce non-linearities.

For almost all low power (signal) applications between dc and around 10MHz, so called *active filters* are used.

Active Filters

Active filters are based on op-amps, resistors and capacitors. Using resistors, capacitors and op-amps is attractive because

- values of R and C can be relied on - tolerances are good and parasitic effects are usually negiligible.
- op-amps automatically give the filter a very low output resistance - ie, the load applied to the filter doesn't affect its performance.
- R, C and the op-amp are physically small. This means that designs can be compact and hence can be easily screened.
- linearity is excellent.

There are several different ways in which active filters can be realised. Some of these approaches are based upon imitating an *L-C* ladder topology as closely as possible. This approach is attractive because it can be shown that the ladder realisation of a polynomial filter function is the least sensitive to component errors of all realisations.

Another approach is to create a cascade of first and second order active blocks (or sub-systems) that together will have the same transfer function as that required by the polynomial. This approach requires that the filter polynomials are available in factored form - ie, as a product of first and second order factors with real coefficients as in tables 1 and 2. Realisation by cascading factors does not have the sensitivity advantages of a ladder realisation but it is a practically and conceptually convenient approach to realising a filter function and is the one adopted here. The design approach is illustrated by a couple of examples.

Design Examples

Example 1: Design a 5th order Butterworth low-pass filter with a cut-off frequency of 20kHz.

The 5th order Butterworth polynomial (from table 1) is

$$(s+1)(s^2+0.618s+1)(s^2+1.618s+1) \qquad \textbf{(2)}$$

The factored polynomial **(2)** is the denominator of a frequency normalised version of the filter, ie, the cut-off frequency is unity. So the required transfer function is

$$\frac{v_o}{v_i} = \frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)} \qquad \textbf{(3)}$$

This transfer function **(3)** can be realised by the system represented in block form in figure 4. The problem is thus reduced to one of designing one first order and two second order circuits

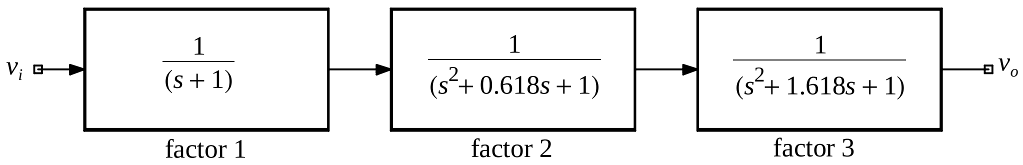


Figure 4

Block diagram of the system necessary to realise the 5th order Butterworth low-pass filter.

with specified q factors and undamped natural frequencies. Since the filter polynomials are given in frequency normalised form, the frequency variable is effectively ω/ω_c and the corner frequency of the first order, and the undamped natural frequencies of the second order, frequency normalised factors are effectively ω_0/ω_c and ω_n/ω_c respectively, where ω_c is the desired filter cut off frequency. In the case of frequency normalised Butterworth polynomials, the corner frequencies and undamped natural frequencies of the factors are unity - ie the same as the overall filter cut-off frequency.

The q factor and undamped natural frequency are determined by comparing the factors with the appropriate standard forms given in **(4)** and **(5)**.

$$\frac{v_o}{v_i} = k \,.\, \frac{1}{1+\dfrac{s}{\omega_0}} \qquad \textbf{(4)}$$

$$\frac{v_o}{v_i} = k \, \frac{1}{1+\dfrac{s}{\omega_n q}+\dfrac{s^2}{\omega_n^2}} \qquad \textbf{(5)}$$

This comparison gives:

For factor 1, $\omega_0 / \omega_c = 1$
For factor 2, $\omega_n / \omega_c = 1$ and $q = 1.62$
For factor 3, $\omega_n / \omega_c = 1$ and $q = 0.62$

Notice that $k = 1$ for all three factors. k can be ignored in a cascaded system because it affects only the overall gain of the filter; it has no effect on response shape. If gain is required it is usually better to deal with this separately by using a pre-filter or post-filter amplifier.

Having identified the key parameters of the factors, suitable circuit shapes must be chosen for them. Here the Sallen and Key low-pass circuit of figure 5 will be used for the second order factors and a buffered *R-C* circuit of figure 6 used for the first order factor. The transfer function of the circuit of figure 5 is

$$\frac{v_o}{v_i} = \frac{1}{1 + sC_2(R_1 + R_2) + s^2 C_1 C_2 R_1 R_2} \quad \textbf{(6)}$$

and the key parameters with equal R are

$$\omega_0 = \frac{1}{R\,(C_1 C_2)^{1/2}} \quad \textbf{(7)}$$

$$q = \frac{1}{2}\left(\frac{C_1}{C_2}\right)^{1/2} \quad \textbf{(8)}$$

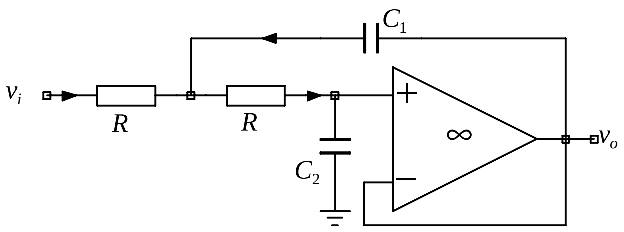


Figure 5
The low-pass Sallen and Key circuit

The derivation of relationships **(6)**, **(7)** and **(8)** can be found on pages 7 and 8 of the Second Order Circuits handout. Equations **(7)** and **(8)** can be deduced from **(6)** by comparing **(6)** with the standard form **(5)**. The buffered first order low-pass circuit of figure 6 has the transfer function

$$\frac{v_o}{v_i} = \frac{1}{1 + sC_3 R_1} \quad \textbf{(9)}$$

and the corner frequency is

$$f_0 = \frac{1}{2\pi C_3 R_1} \quad \textbf{(10)}$$

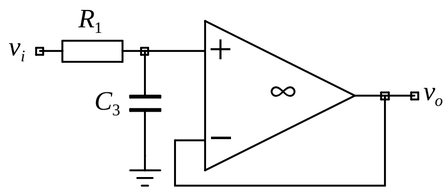


Figure 6
A low-pass R-C curcuit followed by a unity gain buffer.

Components are then chosen to achieve the required parameter values. It is usually a good policy to choose the capacitors first because capacitors are not available in as large a range of values as resistors. Capacitor values should be large compared to parasitic circuit capacitances; >100pF is adequate for most purposes. The table below gives one of many possible sets of

	required τ	R_1	C_3	required q	$\frac{C_1}{C_2} = 4q^2$ from **(8)**	C_1	C_2	$R = \frac{1}{2\pi f_0 (C_1 C_2)^{1/2}}$ from **(7)**
factor 1	$\frac{1}{2\pi 20\text{kHz}}$	22kΩ	360pF	-----	-----	-----	-----	-----
factor 2	-----	-----	-----	1.62	10.5	1.6nF	150pF	16.24kΩ ≈ 16kΩ
factor 3	-----	-----	-----	0.62	1.54	200pF	130pF	49.35kΩ ≈ 51kΩ

values, eg, R_1 and C_3 can be any sensible values that give a product of 7.96µs and C_1 and C_2 can be any sensible values that give the required ratio.

The only remaining task is to build the filter circuit. The order of the factors is important. The factor with a q of 1.62 will give rise to gain peaking in the vicinity of the cut-off frequency but the overall filter is known to be maximally flat and thus have no gain peaking. In order to be sure that the peak signal voltage within the filter is never larger than the peak input voltage, put the first order section at the input end of the cascade and then add factors in order of ascending q factor. The complete design of this filter is shown in figure 7.

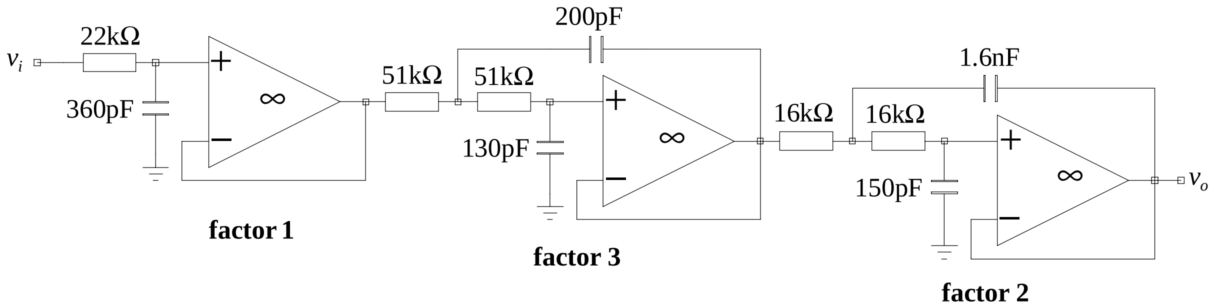


Figure 7

The complete 5th order Butterworth low-pass design

Example 2: Design a 5th order Chebyshev 1dB ripple low-pass filter with a cut-off frequency of 10kHz.

The fifth order 1dB ripple Chebyshev frequency normalised polynomial (from table 2) is

$$(s + 0.2895)(s^2 + 0.1790s + 0.9883)(s^2 + 0.4684s + 0.4293)$$

so the transfer function of the corresponding frequency normalised filter is

$$\frac{v_o}{v_i} = \frac{1}{(s + 0.2895)(s^2 + 0.1790s + 0.9883)(s^2 + 0.4684s + 0.4293)}$$

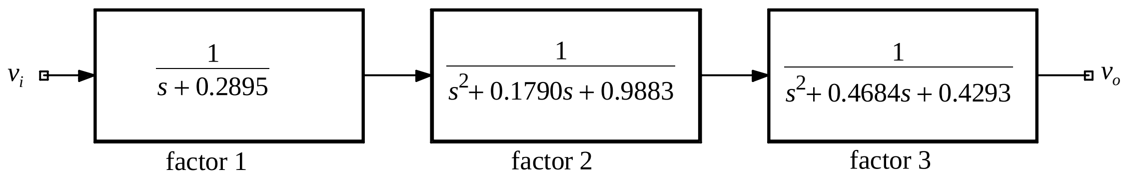


Figure 8

Block diagram of the system necessary to realise the 5th order Chebyshev low-pass filter.

The filter can be realised in the form of the cascade shown in figure 8.

The first obvious difference between this example and the Butterworth one is that none of the factors are in the same standard form as **(4)** and **(5)** because the real parts of their denominators are not unity. The first step, as before, is to identify the frequency normalised corner and undamped natural frequencies and the qs of the second order factors.

Factor 1

$$\frac{v_o}{v_i} = \frac{1}{s + 0.2895} = \frac{1}{0.2895\,(\frac{s}{0.2895} + 1)}$$

Comparison with the standard first order form, **(4)**, gives

$k = 1 / 0.2895$ and

$\omega_0 / \omega_c = 0.2895$.

If the first order section is realised using the circuit of figure 6, the required corner frequency is $\omega_0 = 0.2895\omega_c = 0.2895 \times 2 \times \pi \times 10\text{kHz} = 18.19\text{x}10^3$ rad s^{-1} = $1/R_1C_3$. The gain k can be ignored because it operates on the whole cascade and does not affect the response shape.

Factor 2

$$\frac{v_o}{v_i} = \frac{1}{s^2 + 0.1790s + 0.9883} = \frac{1}{0.9883\,(\frac{s^2}{0.9883} + \frac{0.1790}{0.9883}s + 1)}$$

Comparison with the standard second order form, **(5)**, gives

$k = 1 / 0.9883$ and

$\omega_n / \omega_c = (0.9883)^{1/2} = 0.9941$ and

$$\frac{1}{(\omega_n / \omega_c)\, q} = \frac{0.1790}{0.9883} \text{ or } q = \frac{0.9883^{1/2}}{0.1790} = 5.55$$

If factor 2 is realised using the circuit of figure 5, the required undamped natural frequency is $\omega_n = 0.9941\omega_c = 0.9941 \times 2 \times \pi \times 10 \times 10^3 = 62.46 \times 10^3$ rad s^{-1}. Using **(8)** the q of 5.55 will require a C_1 / C_2 ratio of 123.2 which is about the limit of what is sensible in the context of a Sallen and Key circuit. Again, k can be ignored.

Factor 3

$$\frac{v_o}{v_i} = \frac{1}{s^2 + 0.4684s + 0.4293} = \frac{1}{0.4293\,(\frac{s^2}{0.4293} + \frac{0.4684}{0.4293}s + 1)}$$

Comparison with the standard second order form, **(5)**, gives

$k = 1 / 0.4293$ and

$\omega_n / \omega_c = (0.4293)^{1/2} = 0.6552$ and

$$\frac{1}{(\omega_n / \omega_c)\, q} = \frac{0.4684}{0.4293} \text{ or } q = \frac{0.4293^{1/2}}{0.4684} = 1.40$$

If factor 3 is realised using the circuit of figure 5, the required undamped natural frequency is $\omega_n = 0.6552\omega_c = 0.6552 \times 2 \times \pi \times 10 \times 10^3 = 41.17 \times 10^3$ rad s^{-1}. Using **(8)** the q of 1.40 will require a C_1 / C_2 ratio of 7.84 which is easily achieved with a Sallen and Key circuit. Again, k can be ignored.

Having identified the key parameters of the factors, components can be chosen as for the Butterworth example but that process will not be done here. Starting from the input signal end, the realisation order should be factor 1, factor 3 and lastly factor 2.

Conclusions

This has been a very brief introduction to high order filter design. Nothing has been said about how the polynomials were created or how they can be used to make band-pass and high-pass designs. There are also many more polynomial types besides the two that have been mentioned here - they all have some aspect of their behaviour that is special by design. More information can be found in text books. Three examples are

"Analog Filter Design", M.E. Van Valkenburg, H.R.W., 1982

"Micrelectronics", second ed., J. Millman and A. Grabel, McGraw-Hill, 1987

"Handbook of Filter Synthesis", A.I. Zverev, Wiley, 1967

Millman contains basic information about Butterworth and Chebyshev filter polynomials and describes concisely how they may be used with op-amps to realise filter circuits.

Van Valkenburg offers a very good introduction to analogue filters. Explained in a careful way with a minimum of jargon it goes far beyond the contents of this handout - a good reference if you end up in a filter design environment.

Zverev is a mine of information. It has pages and pages of graphs of filter characteristics, design tables and a concise, in depth analytical description of whats going on. A valuable reference source but very hard going - much more technical detail than Van Valkenburg but written in a less sympathetic way.