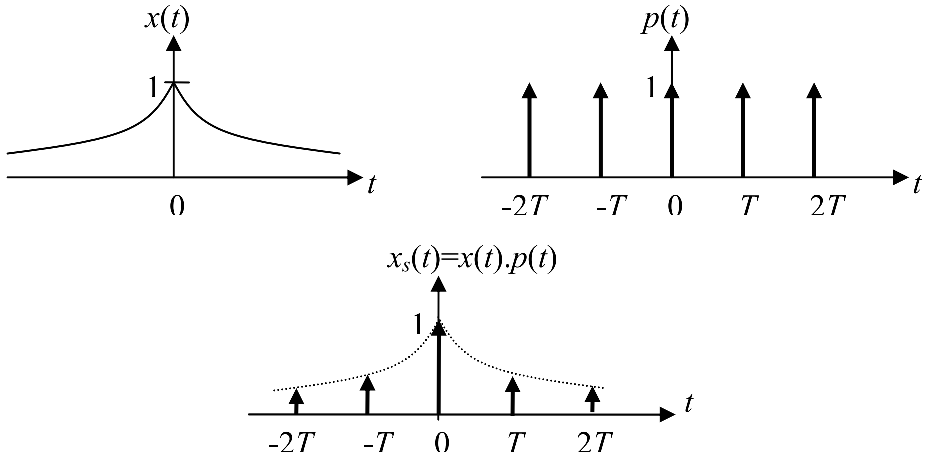
1. a) The complex Fourier Series coefficient of $p(t)$ is $c_n = \frac{1}{T}\int_{<T>} p(t)e^{-jn\omega_s t}dt$,

where ω_s is the sampling frequency.

$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)e^{-jn_s\omega t}dt = \frac{1}{T}$, since $\delta(t) = 0$ everywhere, except at $t = 0$.

Therefore we have $p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} = \sum_{n=-\infty}^{\infty} \frac{1}{T}e^{jn\omega_s t}$. Using the Fourier Transform pair $e^{j\omega_s t} \leftrightarrow 2\pi\delta(\omega-\omega_s)$ the Fourier Transform of $p(t)$ is $P(\omega) = \sum_{n=-\infty}^{\infty} \frac{1}{T}2\pi\delta(\omega - n\omega_s) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)$.

b)



c) Using the multiplication property of Fourier Transform we have
$x(t).p(t) \leftrightarrow X(\omega) * P(\omega)$.

Therefore the Fourier Transform of $x_s(t)$ is $X_s(\omega) = X(\omega) * P(\omega)$.

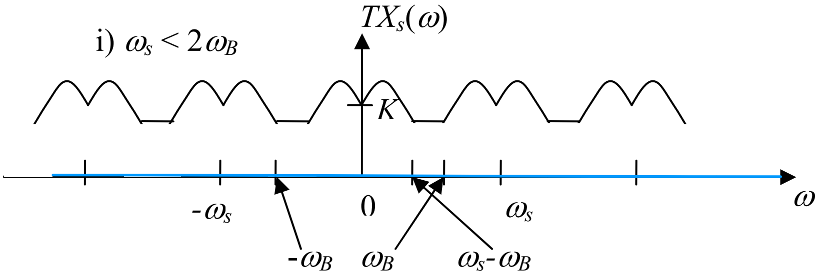
$$X_s(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)P(\omega-\lambda)d\lambda$$

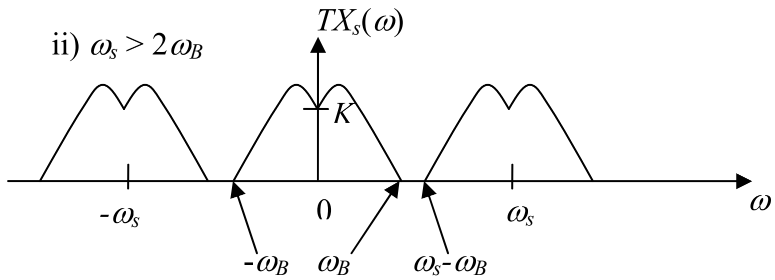
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s - \lambda)d\lambda .$$

Since $\int_{-\infty}^{\infty} X(\lambda)\delta(\omega - n\omega_s - \lambda)d\lambda = X(\omega - n\omega_s)$ we have

$$X_s(\omega) = \frac{1}{2\pi}\left(\frac{2\pi}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)\right) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X(\omega - n\omega_s).$$

d)





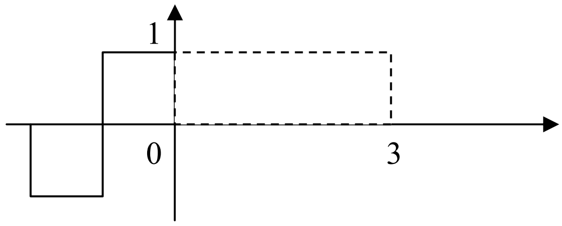
Spectrum of $x(t)$ can be recovered by low pass filtering only when $\omega_s > 2\omega_B$. this is the Nyquist sampling theorem. When $\omega_s < 2\omega_B$ the repetitions of $X(\omega-n\omega_s)$ will overlap as shown in (ii). This effect is known as aliasing.

2. a) A signal $x(t)$ can be represented by $x(t)=\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$. For an LTI system with an impulse response $h(t)$ we have

input	response	
$\delta(t)$	$h(t)$	(definition)
$\delta(t-\tau)$	$h(t-\tau)$	(time shifting)
$x(\tau)\delta(t-\tau)$	$x(\tau)\mathrm{h}(t-\tau)$	(homogeneity)
$x(t)=\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$	$y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$.	

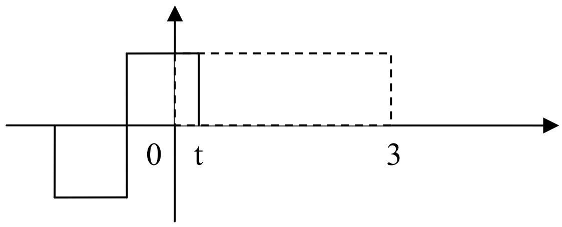
Therefore $y(t)=\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ is the response of the LTI system to an input $x(t)$.

b)



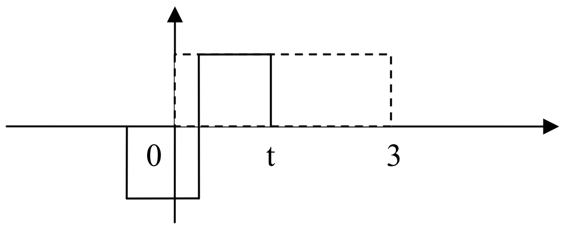
$t \leq 0$

$y(t) = 0$



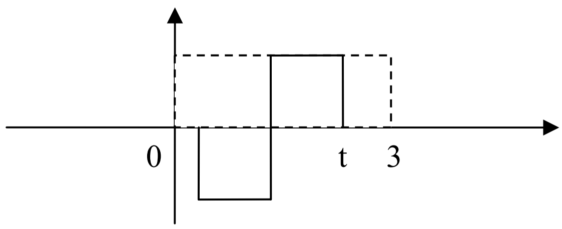
$0 < t \leq 1$

$y(t) = t$



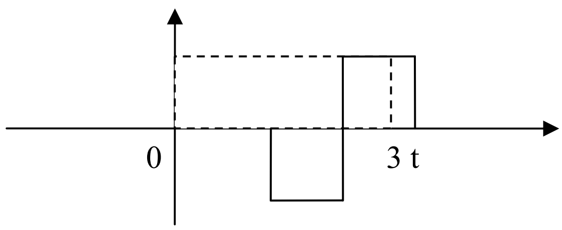
$1 < t \leq 2$

$y(t) = 1-(t-1) = 2-t$



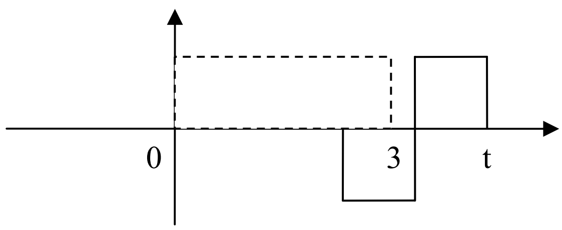
$2 < t \leq 3$

$y(t) = 1-1 = 0$



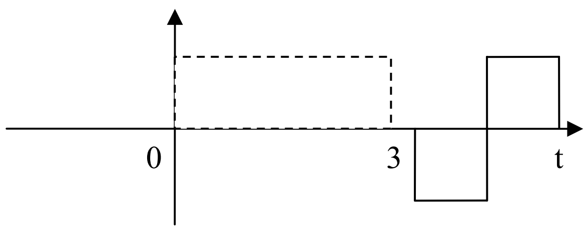
$3 < t \leq 4$

$y(t) = 3-(t-1)-1 = 3-t$



$4 < t \leq 5$

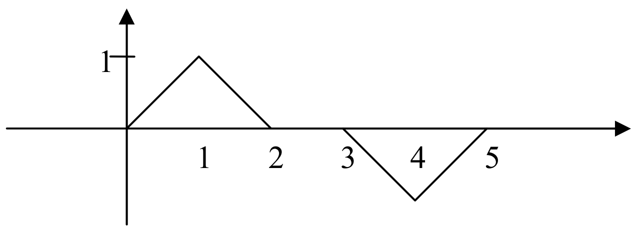
$y(t) = -[3-(t-2)] = t-5$



$t > 5$

$y(t) = 0$

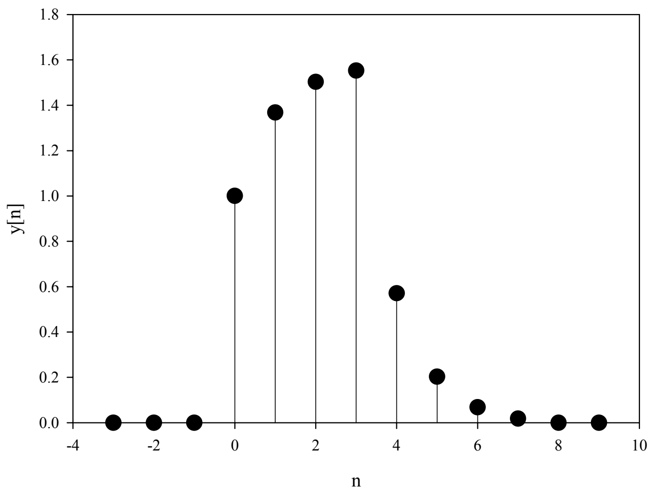
In summary,



$$y(t)=\begin{cases}0, & t\le 0\\ t, & 0<t\le 1\\ 2-t, & 1<t\le 2\\ 0, & 2<t\le 3\\ 3-t, & 3<t\le 4\\ t-5, & 4<t\le 5\\ 0, & t>5.\end{cases}$$

c) $x[n]=\begin{cases}1, & 0\le n\le 3\\ 0, & otherwise\end{cases}$ and $h[n]=\begin{cases}e^{-n}, & 0\le n\le 4\\ 0, & otherwise\end{cases}$

k	-4	-3	-2	-1	0	1	2	3	4	5	$\sum x[k]h[n-k]$
$x[k]$	0	0	0	0	1	1	1	1	0	0	
$h[0\text{-}k]$	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	0	0	0	1
$h[1\text{-}k]$	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	0	0	1.368
$h[2\text{-}k]$	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	0	1.503
$h[3\text{-}k]$	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	1.553
$h[4\text{-}k]$	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0.571
$h[5\text{-}k]$	0	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0.203
$h[6\text{-}k]$	0	0	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	0.068
$h[7\text{-}k]$	0	0	0	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	0.018
$h[8\text{-}k]$	0	0	0	0	0	0	0	0	e^{-4}	e^{-3}	0



3. a) The d.c term, $a_0=\int_{1/2}^{3/2}-1dt+\int_{3/2}^{5/2}1dt$

$$=\left[-t\right]_{1/2}^{3/2}+\left[t\right]_{3/2}^{5/2}$$

$$=-\frac{3}{2}+\frac{1}{2}+\frac{5}{2}-\frac{3}{2}=0$$

Since $v(t)$ is an even function $b_n = 0$ (i.e no sine components in the Fourier Series).

We have $T = 2$ and $\omega_o = 2\pi/T = \pi$.

$$a_n = \int_{1/2}^{3/2} -\cos n\pi t dt + \int_{3/2}^{5/2} \cos n\pi t dt$$

$$= \frac{1}{n\pi}[-\sin n\pi t]_{1/2}^{3/2} + \frac{1}{n\pi}[\sin n\pi t]_{3/2}^{5/2}$$

$$= \frac{1}{n\pi}\left[-\sin\frac{3}{2}n\pi + \sin\frac{1}{2}n\pi + \sin\frac{5}{2}n\pi - \sin\frac{3}{2}n\pi\right]$$

$$= \frac{4}{n\pi} \quad if \quad n = 1,5,9......$$

$$= -\frac{4}{n\pi} \quad if \quad n = 3,7,11......$$

$$= 0 \quad if \quad n = even.$$

Therefore the trigonometric Fourier Series representation for v(t) is

$$v(t) = \frac{4}{\pi}\left(\cos\pi t - \frac{1}{3}\cos 3\pi t + \frac{1}{5}\cos 5\pi t - \frac{1}{7}\cos 7\pi t + ...\right).$$

b) Parseval's theorem states that $P_{ave} = \sum_{n=-\infty}^{\infty} |c_n|^2$.

We know that $a_n = 2\mathrm{Re}[c_n]$ and $\omega_o = \pi$. Therefore only c_n with $n = \pm1$, $n = \pm3$ and $n = \pm5$ exist within the frequency range [-6π rad/s, 6π rad/s].

$$P_{ave} = |c_{-5}|^2 + |c_{-3}|^2 + |c_{-1}|^2 + |c_1|^2 + |c_3|^2 + |c_5|^2 .$$

$|c_{-5}| = |c_5| = \frac{1}{2}\left(\frac{4}{5\pi}\right) = \frac{2}{5\pi}$, $\quad |c_{-3}| = |c_3| = \frac{1}{2}\left(\frac{4}{3\pi}\right) = \frac{2}{3\pi}$ and

$$|c_{-1}| = |c_1| = \frac{1}{2}\left(\frac{4}{\pi}\right) = \frac{2}{\pi} .$$

Therefore $P_{ave} = 2\left(\frac{2}{5\pi}\right)^2 + 2\left(\frac{2}{3\pi}\right)^2 + 2\left(\frac{2}{\pi}\right)^2 = 0.933$.

c) The amplitude of the fundamental after filtering is given by

$$\frac{4}{\pi} \times |H(\omega_o)| = \frac{3.2}{\pi} .$$

Hence we have $|H(\omega_o)| = \frac{3.2}{4} = 0.8$ and

$$\frac{1}{\sqrt{1 + (\omega_o/\omega_c)^2}} = 0.8$$

$$(\omega_o/\omega_c)^2 = 0.5625$$

$$\omega_o/\omega_c = 0.75$$

$$\pi \times 200 \times 10^3 \times C = 0.75$$

$$C = 1.19 \times 10^{-6} F.$$

4. a)

$$H(s)=\frac{(s+2)}{(s+1+j2)(s+1-j2)}=\frac{(s+2)}{\left(s^2+s-j2s+s+1-j2+j2s+j2+4\right)}=\frac{(s+2)}{\left(s^2+2s+5\right)}$$

or

$$H(s)=\frac{(s+2)}{(s+1)^2+2^2}.$$

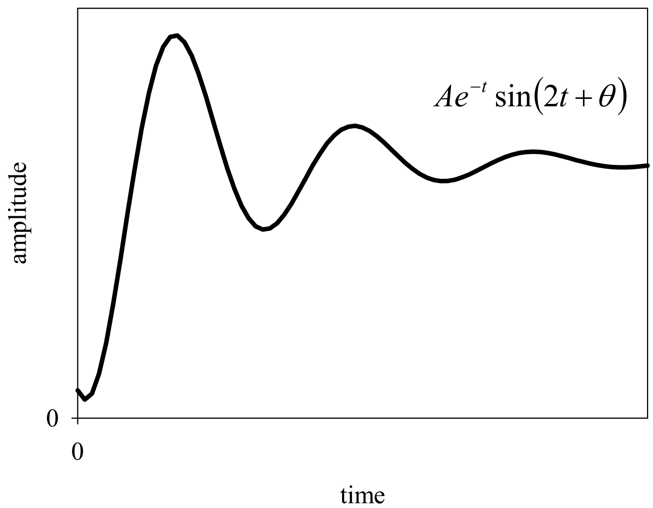
Therefore $N(s) = s + 2$ and $D(s) = s^2 + 2s + 5$.

b) $H(s)=\frac{(s+2)}{s^2+2s+5}$, therefore we have, $2\zeta\omega_n = 2$ and $\omega_n = \sqrt{5}$.

$$\zeta=\frac{2}{2\omega_n}=\frac{1}{\sqrt{5}}.$$

Poles = $-\zeta\omega_n \pm j\omega_d = -1 \pm j2$.

c) The step response of the system is a sinusoidal oscillation with a frequency of 2 rad/s and amplitude modulated by a decaying exponential with a time constant of $\tau=\frac{1}{\zeta\omega_n}=1$ s.



d) If $x(t)=e^{-2t}.u(t)$, $X(s)=\frac{1}{s+2}$. The Laplace Transform of the response is

$$Y(s)=X(s).H(s)$$

$$=\frac{s+2}{\left((s+1)^2+2^2\right)}\frac{1}{(s+2)}=\frac{1}{\left((s+1)^2+2^2\right)}=\frac{1}{2}\frac{2}{\left((s+1)^2+2^2\right)}.$$

Therefore the system response when $x(t)=e^{-2t}.u(t)$ is $y(t)=\frac{1}{2}e^{-t}.\sin(2t).u(t)$.