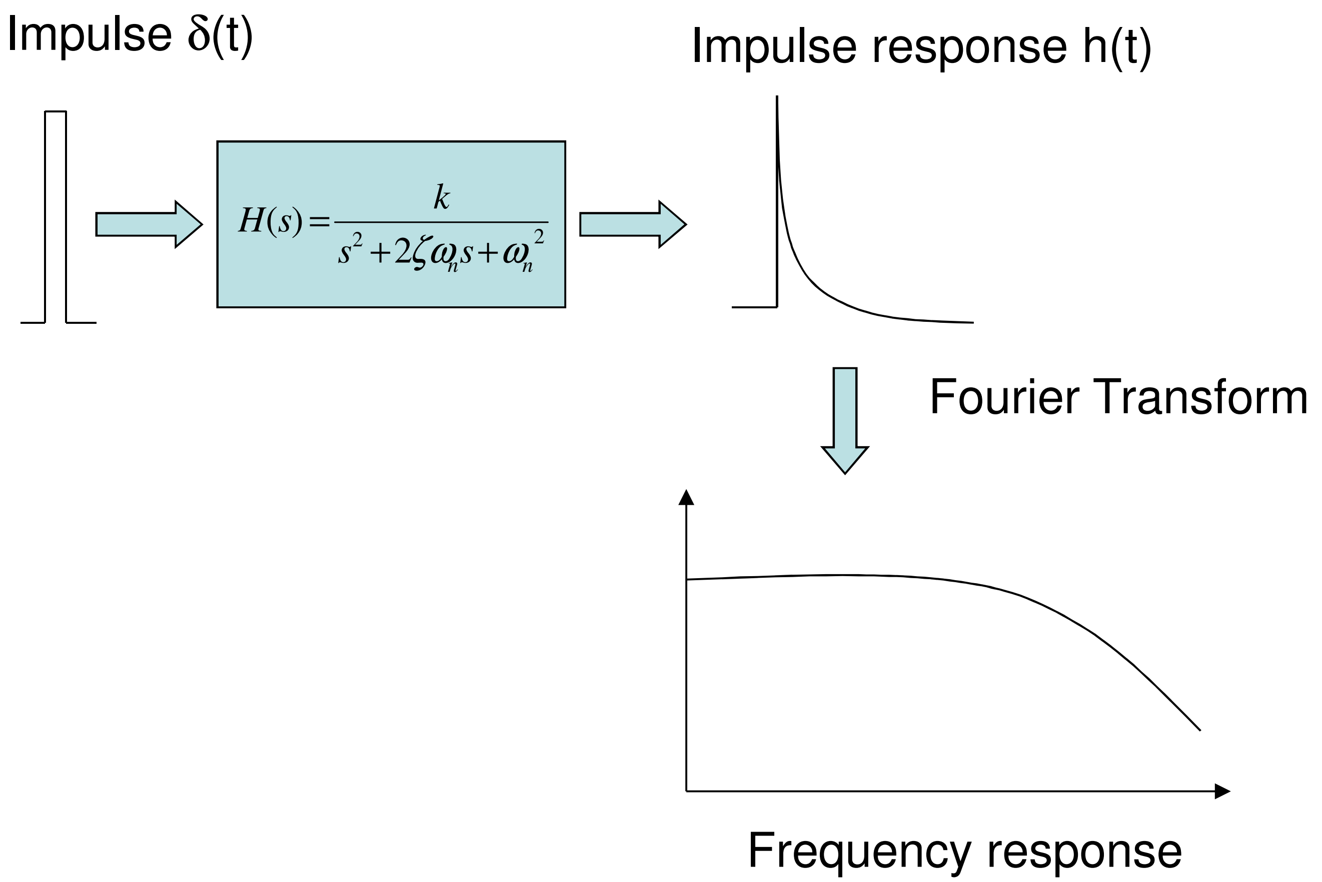
Lecture content

- Fourier Transform
  - Relationship between Fourier Series and Fourier Transform

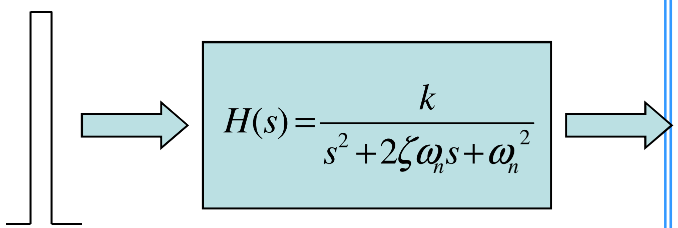
Why Fourier Transform

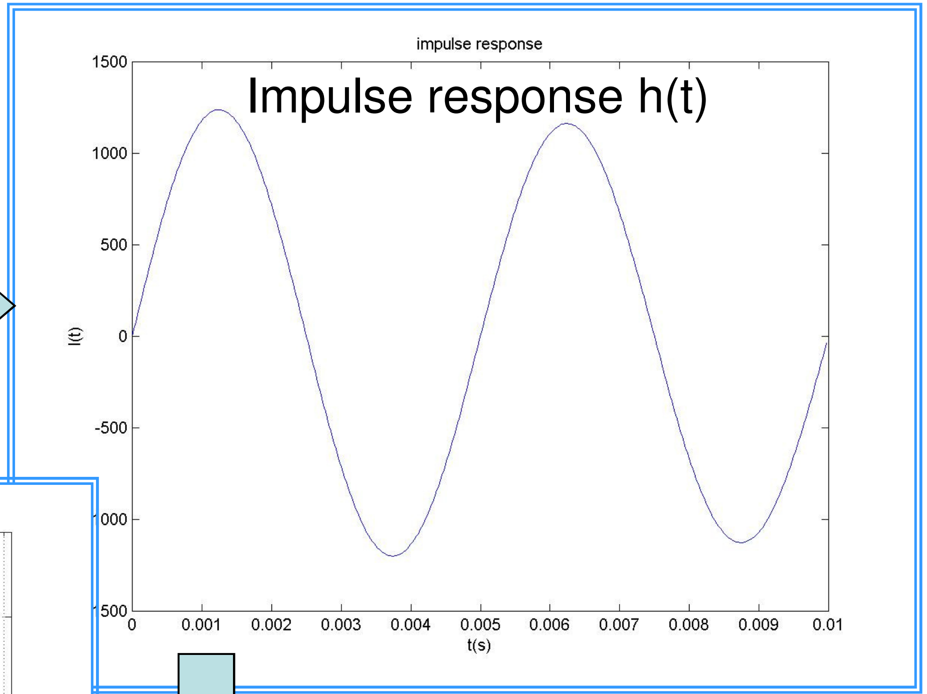


The University Of Sheffield.

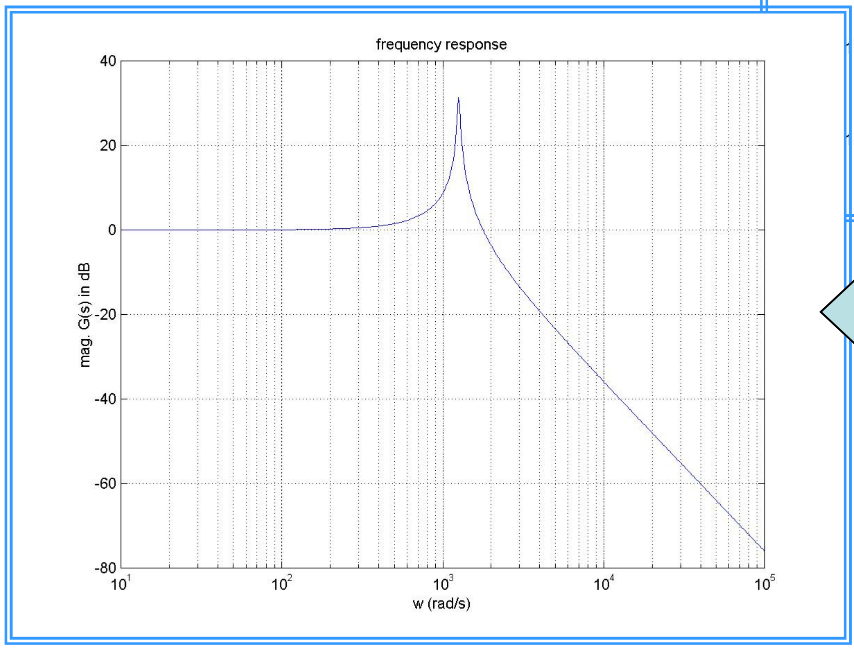
Why Fourier Transform

Impulse $\delta(t)$





Fourier Transform



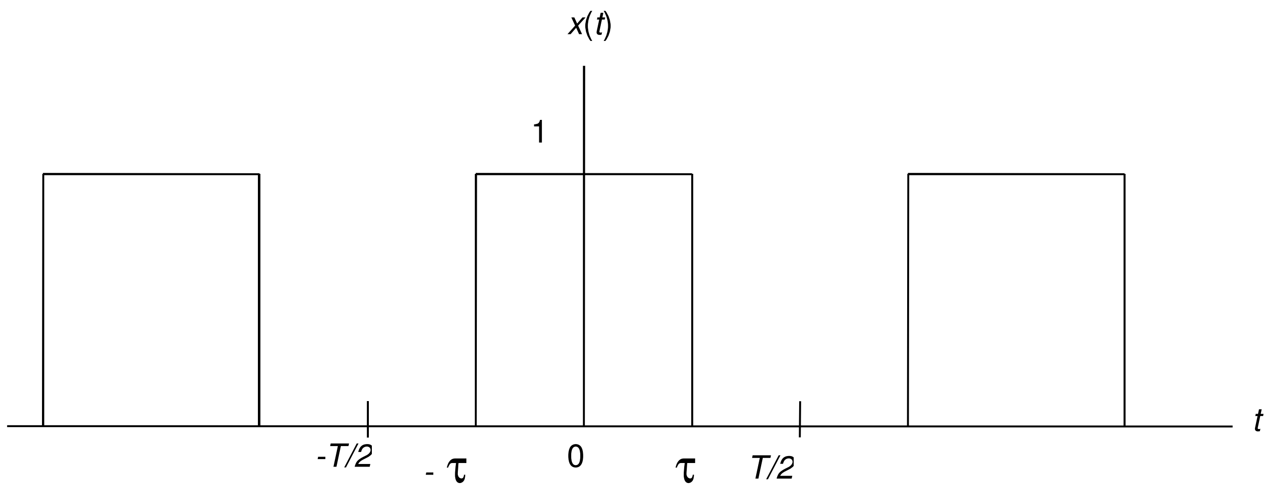
Frequency response

Fourier Transform

The Fourier Series representation is applicable to periodic signals with infinite duration but many practical signals are non-periodic (or aperiodic) and have finite duration. We shall modify the Fourier Series so that it is applicable to aperiodic signals as well. The signal $x(t)$ in figure 1 can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jn\omega_o t} dt$$



$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_o t}\,dt$$

$$c_n = \frac{1}{T}\int_{-\tau}^{\tau}(1)e^{-jn\omega_o t}\,dt = \frac{1}{jn\omega_o T}\left(e^{jn\omega_o\tau} - e^{-jn\omega_o\tau}\right) = \frac{2\tau}{T}\frac{\sin(n\omega_o\tau)}{(n\omega_o\tau)}$$

Fourier Transform

c_n has magnitude with an envelope of $\dfrac{\sin\alpha}{\alpha}$ where $\alpha = n\omega_o\tau$ and a peak magnitude of $2\tau/T$, as shown in figure 2.

Note that the function $\dfrac{\sin\alpha}{\alpha}$ is sampled every ω_o rad/s (i.e the frequency of the harmonics). The function $\dfrac{\sin\alpha}{\alpha}$ is a sinc function and it has a peak magnitude of 1 at $\alpha = 0$.

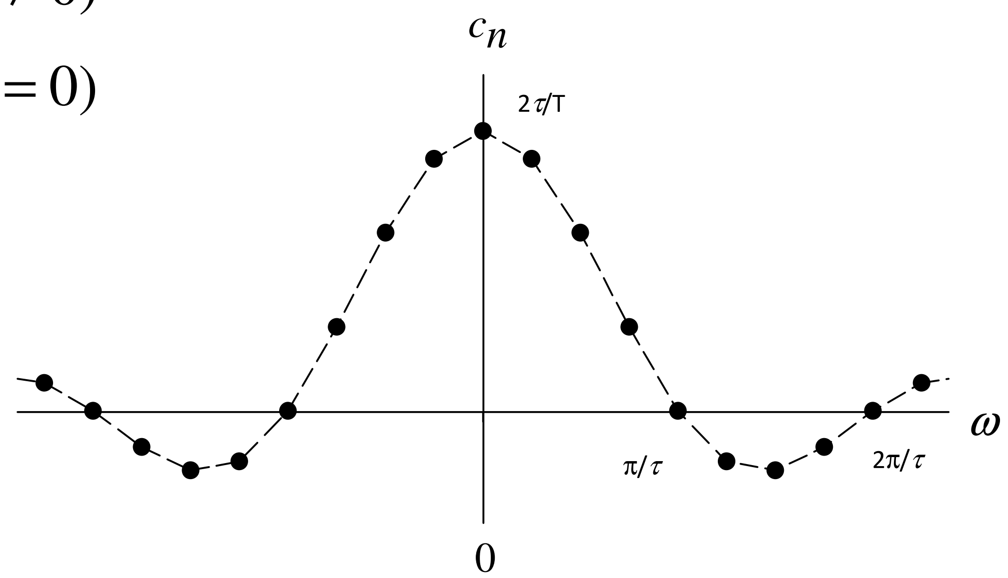
(Use l'Hopital's rule: $\lim\limits_{\alpha\to 0}\dfrac{\sin\alpha}{\alpha} = \lim\limits_{\alpha\to 0}\dfrac{\cos\alpha}{1} = 1$).

Nulls of occur when sin $\alpha = 0$, that is when $\alpha = m\pi$ where m is integer to denote the nulls. Hence, the nulls are at $\omega = m\pi/\tau$ and we have the 1st null at π/τ, the 2nd null at $2\pi/\tau$ and so on.

Fourier Transform

Note: $\text{sinc}(\alpha) = \begin{cases} \dfrac{\sin\alpha}{\alpha} & (\alpha \neq 0) \\ 1 & (\alpha = 0) \end{cases}$

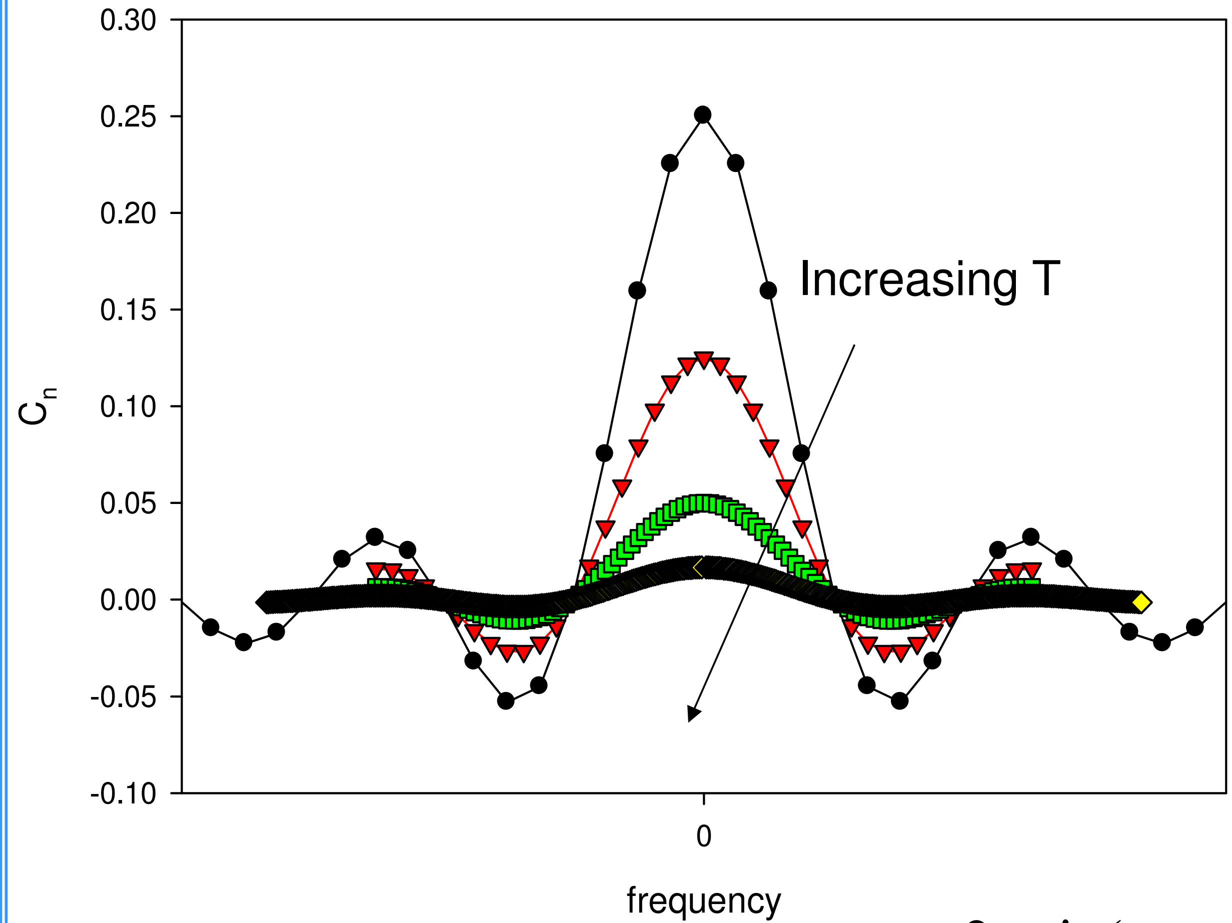
c_n with an envelope of $\dfrac{\sin\alpha}{\alpha}$



	n=0	n=1	n=2	n=3	n=4	n=5
c_n	$2\tau/T$	$(2\tau/T)\text{sinc}(\omega_o\tau)$	$(2\tau/T)\text{sinc}(\omega_o\tau)$	$(2\tau/T)\text{sinc}(3\omega_o\tau)$	$(2\tau/T)\text{sinc}(4\omega_o\tau)$	$(2\tau/T)\text{sinc}(5\omega_o\tau)$
e.g: $\tau/T=1/8$	¼	1/4sin(π/4)	1/4sin(2π/4)	1/4sin(3π/4)	1/4sin(4π/4)	1/4sin(5π/4)

.

Fourier Transform



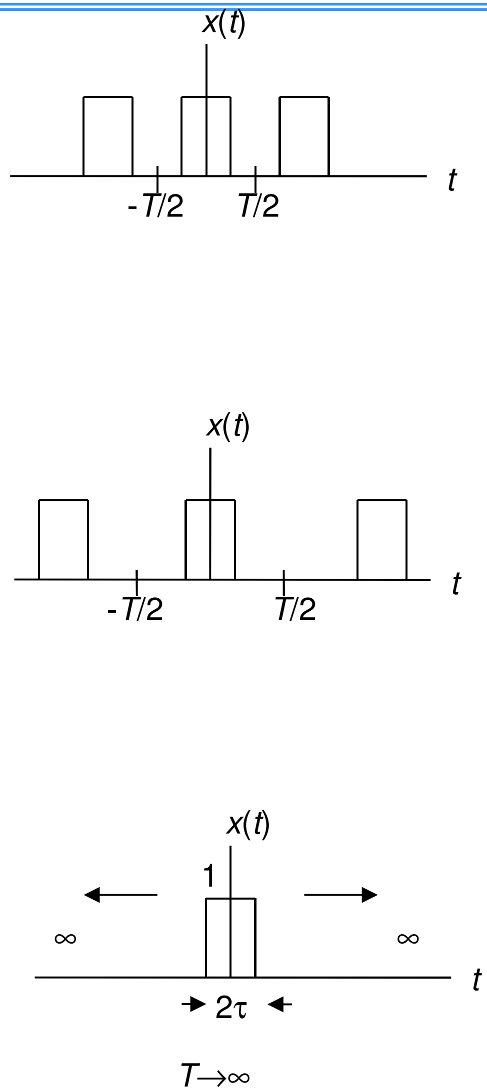
$$c_n = \frac{2\tau}{T}\frac{\sin(n\omega_o\tau)}{(n\omega_o\tau)}$$

Fourier Transform

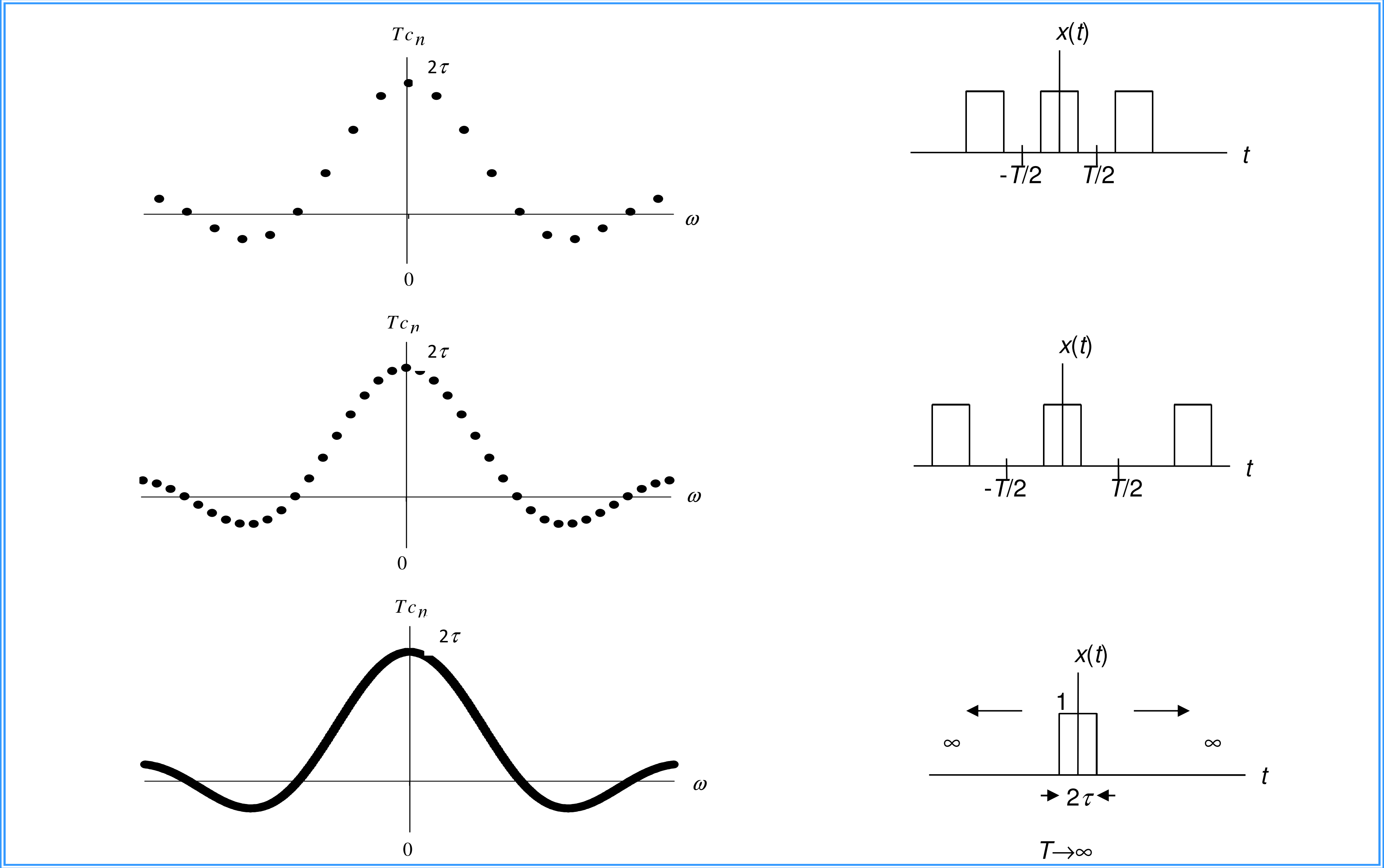
Now, consider the envelope function

$$Tc_n = \left.\frac{2\tau \sin \omega\tau}{\omega\tau}\right|_{\omega = n\omega_o}$$

By changing T we can investigate the changes in the magnitude spectrum.



Fourier Transform



Fourier Transform

We shall now develop the Fourier Transform of a rectangular pulse $x(t)$. Let

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_o t}\,dt$$

$$X(\omega) = Tc_n = \int_{-T/2}^{T/2} x(t)e^{-j\omega t}\,dt$$

We know that

$$x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Tc_n e^{jn\omega_o t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega)e^{jn\omega_o t}\,\omega_o$$

As $T\rightarrow\infty$, $\omega_o\rightarrow 0$ so that ω becomes a continuum and ω_o can be written as $d\omega$. The summation becomes an integration and hence we have

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Inverse FT of X(ω)

and

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

FT of x(t)

Fourier Transform

If the symmetry of the signal $x(t)$ is known we can simplify the Fourier Transform integral to

$$X(\omega) = 2\int_0^\infty x(t)\cos\omega t\,dt$$

if $x(t)$ has an even symmetry and

$$X(\omega) = -j2\int_0^\infty x(t)\sin\omega t\,dt$$

if $x(t)$ has an odd symmetry.

Fourier Transform

1. *Obtain the Fourier Transform of the rectangular window function in figure 4.*

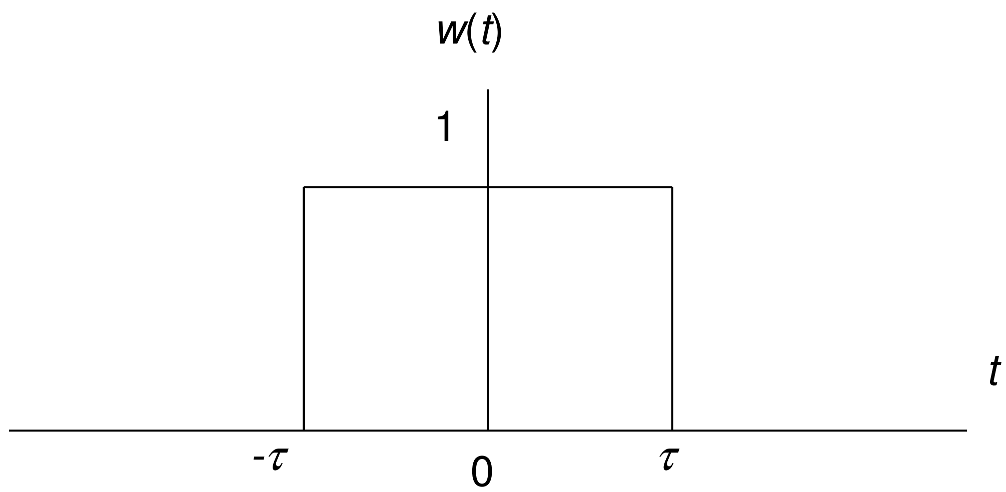
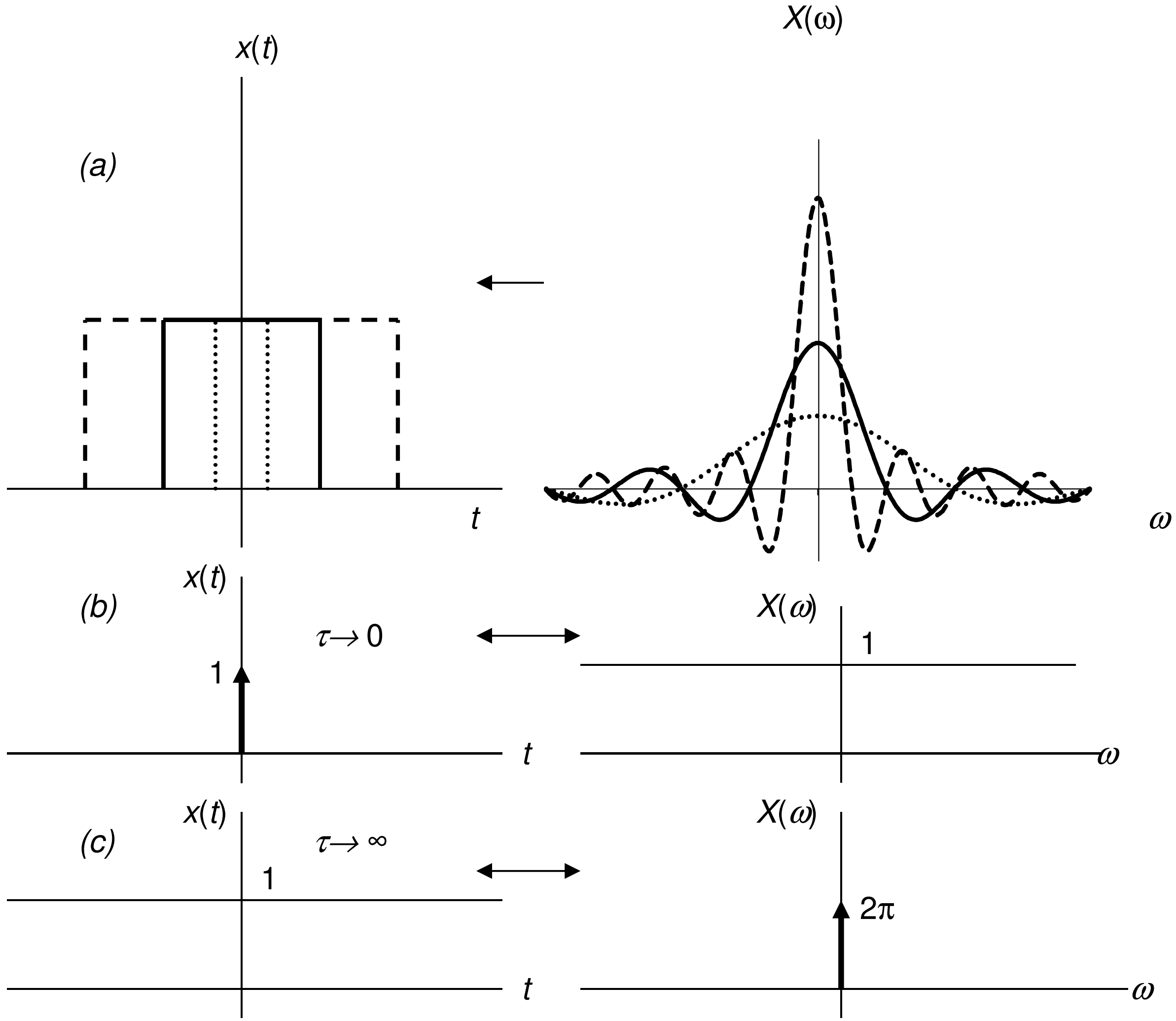
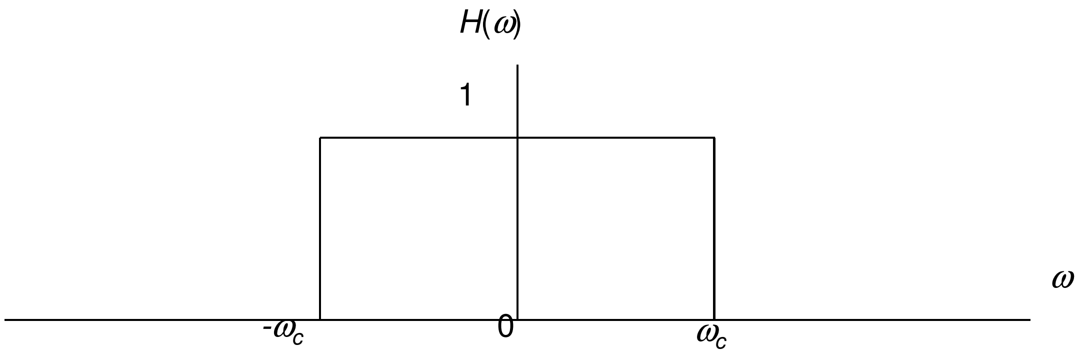


Figure 4: A rectangular window function with a duration of 2τ.

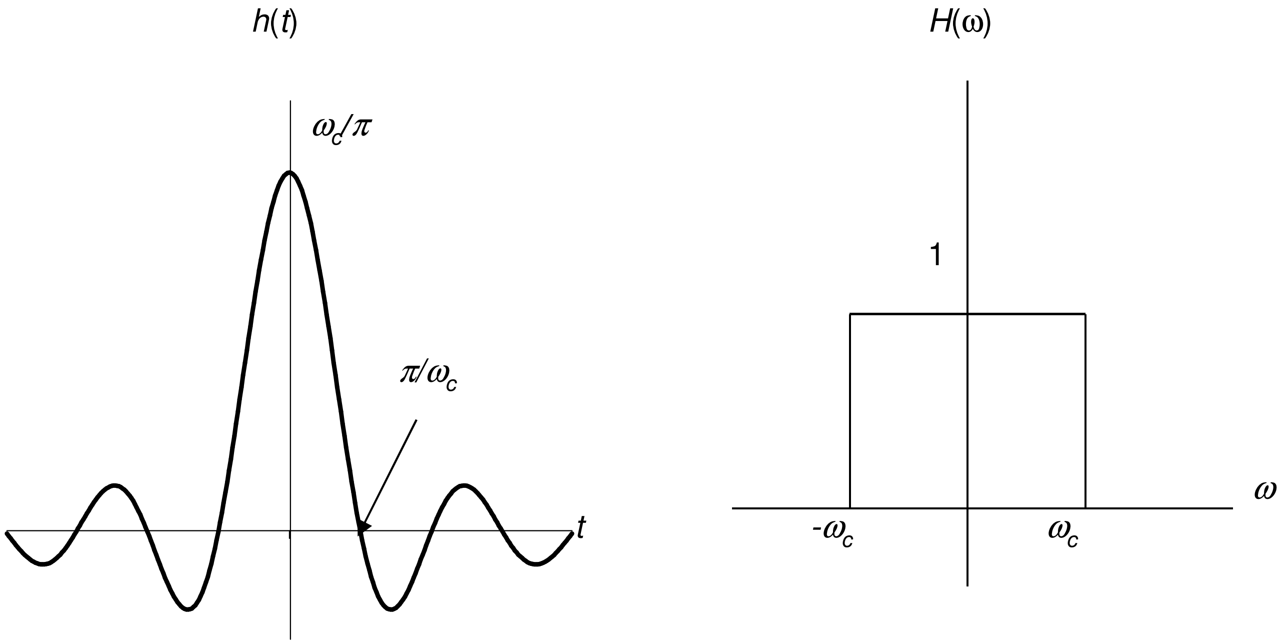
Fourier Transform



2. Compute the time function that has the magnitude spectrum (the positive half of the spectrum, $0 \leq \omega \leq \omega_c$, is an ideal low pass filter) shown in figure 6.



A rectangular spectrum defined by $H(\omega) = 1$ for $|\omega| \leq \omega_c$ and zero otherwise.


$h(t)$
ω_c/π
π/ω_c
t
$H(\omega)$
1
ω
$-\omega_c$
ω_c

The University Of Sheffield.

3. Verify the Fourier Transform pair $x(t) = e^{-at}u(t) \leftrightarrow$, $a > 0$.