Data Provided: $\varepsilon_0 = 8.854\times 10^{-12}$ *F/m*, $\mu_0 = 4\pi\times 10^{-7}$ *H/m*

Formulae for Vector Differential Operations

The University Of Sheffield.

DEPARTMENT OF ELECTRONIC AND ELECTRICAL ENGINEERING

Spring Semester 2010-2011 (2 hours)

EEE 6084 Applied Electromagnetics 6

Answer **THREE** questions. **No marks will be awarded for solutions to a fourth question.** Solutions will be considered in the order that they are presented in the answer book. Trial answers will be ignored if they are clearly crossed out. **The numbers given after each section of a question indicate the relative weighting of that section.**

1. a. Ground-penetrating radar, or GPR, is a method of imaging where pulses of electro-magnetic radiation at microwave wavelengths are used to reveal features underground. It relies on the fact that different materials have different impedances, so that they reflect the incident electromagnetic pulses. GPR is used, for example, to locate buried pipes or to determine different types of rock.

In this question, $E(x,t)$ represents the electric field emitted by a ground penetrating radar, where x is the distance into the ground and t is time. It can be written in phasor notation as:

$$E(x,t) = E_0 \exp\left(j(\omega t - \beta x)\right)$$

Where

$$\beta^2 = \omega^2 \mu\mu_0\varepsilon\varepsilon_0 - j\omega\sigma\mu\mu_0$$

and where μ, ε and σ are the permeability, permittivity and conductivity of the ground.

Show that if $\sigma >> \omega\varepsilon\varepsilon_0$ then β can be approximated by:

$$\beta \approx (1-j)\sqrt{\frac{\omega\sigma\mu\mu_0}{2}}$$

(6)

b. The field generated by the radar has a frequency of 10MHz. It travels through sandy ground of relative permittivity ε=10, relative permeability μ=1 and conductivity $\sigma = 0.02(\Omega\text{m})^{-1}$.

Show that the field decays as $\exp(-x/\delta)$ and provide an approximate expression for δ.

How far into the ground will the wave reach before its amplitude has decayed by 90% of its original value?

(5)

c. At a depth of 2m the sandy ground interfaces with a layer of granite with a relative permittivity of ε=4, relative permeability μ=1 and conductivity $\sigma = 0.001(\Omega\text{m})^{-1}$.

Show that the approximation derived in part *a* will not accurately model the passage of the radar wave through this granite layer. Will the radar wave decay more or less rapidly as it passes through this less conductive medium? Explain your reasoning.

(3)

d. Assuming that the radar wave is normally incident upon the granite layer, calculate the proportion of the wave that is reflected from the interface and the proportion that is transmitted.

Use this result and the equation from part ***a*** and the result from part ***b*** to determine the amplitude of the wave reflected from the granite when it reaches the surface as a percentage of E_0. Why is it reasonable to ignore secondary reflections between the surface and the granite in your answer?

(6)

2.

a. Which ONE of the three functions listed below represents a travelling wave? Explain your answer.

i. $h(x,t) = (ax - bt)^2$

ii. $h(x,t) = \left(at^2 + x^2\right)$

iii. $h(x,t) = \sin(axt)$

What is the velocity and what is the direction of travel of the function you have identified ?

(3)

b.

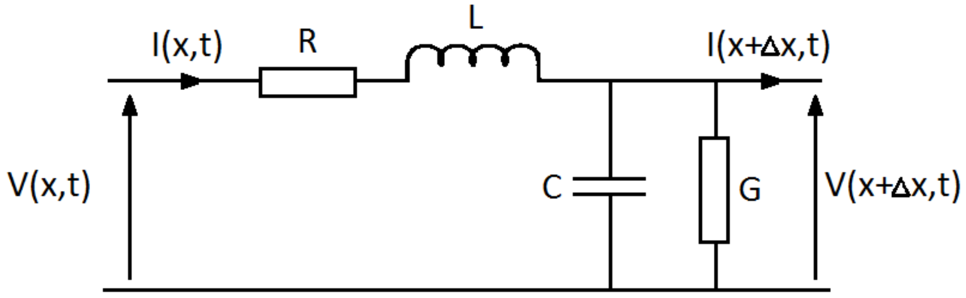

Figure 1

Figure 1 shows a model of a short section Δx of a transmission line, where R, L, C and G represent its resistance per unit length (in units of Ω/m), inductance per unit length (in H/m), capacitance per unit length (in F/m) and conductance per unit length (in $(\Omega m)^{-1}$) respectively.

The transmission line, whose length is x_0 metres, is driven from the left by a purely sinusoidal source represented by the phasor:

$$V(x_0,t) = V_0 \exp(j\omega t)$$

where t is time and ω is the angular frequency of the source. You may assume the source is impedance-matched to the transmission line.

Derive the following expression for the gradient of the voltage along the line as the length Δx tends to zero:

$$-\frac{\partial V(x,t)}{\partial x} = (R + j\omega L)I(x,t) \qquad \text{(equation 1)}$$

(3)

c. The equation for the gradient of the current along the line is:

$$-\frac{\partial I(x,t)}{\partial x} = (G + j\omega C)V(x,t) \qquad \text{(equation 2)}$$

Use equations 1 and 2 to show that

$$-\frac{\partial^2 V(x,t)}{\partial x^2} = -\beta^2 V(x,t)$$

and provide an expression for β. What do the real and imaginary parts of β represent? **(4)**

d. Assume that the transmission line is lossless, so that R = G = 0, if L=1.5μH/m and C=1.4nF/m calculate:

i. the characteristic impedance Z_0 of the transmission line,

ii. the phase velocity of a signal on the line.

Explain the advantages of analysing the response of the transmission line in terms of pure sinusoids, rather than a more general signal.

(4)

e. The *apparent impedance* of a second lossless transmission line, whose characteristic impedance is Z_0=30Ω, at a distance *d* from a load impedance of Z_L is given by:

$$Z_A = Z_0 \frac{Z_L + jZ_0 \tan(\beta d)}{jZ_L \tan(\beta d) + Z_0}$$

if Z_L=50Ω, carefully sketch the magnitude of the apparent impedance of the transmission line as a function of *d*, where *d* varies from *d*=0m to *d*=-λ metres, and λ is the wavelength of the wave on the line. Label the x-axis in terms of λ and the y axis in ohms.

(6)

3.

3. **a.** The electric potential V in a region of free space (ε_0 = 8.85×10^{-12} F/m) is given by:

$$V(x,y,z) = (0.5x^4 + y^3 + 2z^2)\times 10^{-9} / \varepsilon_0$$

(i) Determine whether it satisfies the Laplace equation.

(ii) Calculate the electric field strength $\boldsymbol{E}$ and charge density at the point (x, y, z) = (1, 0, 2) m. **(6)**

b. The electric field strength $\boldsymbol{E}$ and electric flux density $\boldsymbol{D}$ in electrostatic fields are governed by:

$$\begin{cases} \nabla \cdot \boldsymbol{D} = \rho \\ \nabla \times \boldsymbol{E} = 0 \end{cases}$$

where ρ is the electric charge density. Show that by introducing an electric potential V defined as $\boldsymbol{E} = -\nabla V$, it satisfies Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

where ε is the permittivity in the region of concern. **(4)**

c. A p-n semiconductor junction has a depletion layer width of 200nm, a volume charge density of 5x10^3 C/m^3, a relative permittivity of 10 and a cross-sectional area of 10^{-9} m^2, as shown in Fig. 3. Starting from Poisson's equation, and listing any assumptions you make, calculate the maximum value of electric field strength, the voltage across the junction and the total energy stored in the junction.

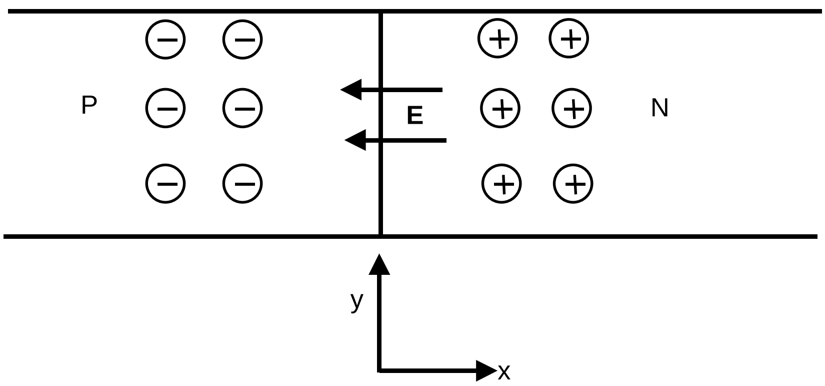

Figure 3 Schematic of a p-n junction **(10)**

4. **a.** The vector magnetic potential $\boldsymbol{A}$ in a region of free space ($\mu_0 = 4\pi\times10^{-7}$ H/m) is given by:

$$\boldsymbol{A} = (0.3xyz)\,\boldsymbol{e_x} + (0.4xy)\,\boldsymbol{e_y} + (0.3)\,\boldsymbol{e_z} \quad \text{(Wb/m)}$$

Calculate the magnetic flux density $\boldsymbol{B}$ and current density $\boldsymbol{J}$ at the point (1, 3, 9)m. **(6)**

b. Figure 4 (a) shows a semi-infinite plate. The x and y components of the vector magnetic potential $\boldsymbol{A}$ are zero in the free space above the plate , i.e.,

$$A_x = A_y = 0 \quad \text{for} \quad y > y_0$$

(i) show that the z component, A_z, of the vector magnetic potential is governed by:

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = -\mu_0 J_z$$

where J_z is the z component of the current density in the region $y > y_0$.

(ii) If A_z is given by

(1) $A_z = 4x^3y^3$

(2) $A_z = 4x^2$

derive expressions for the flux density components and indicate what type of material the plate is likely made from.

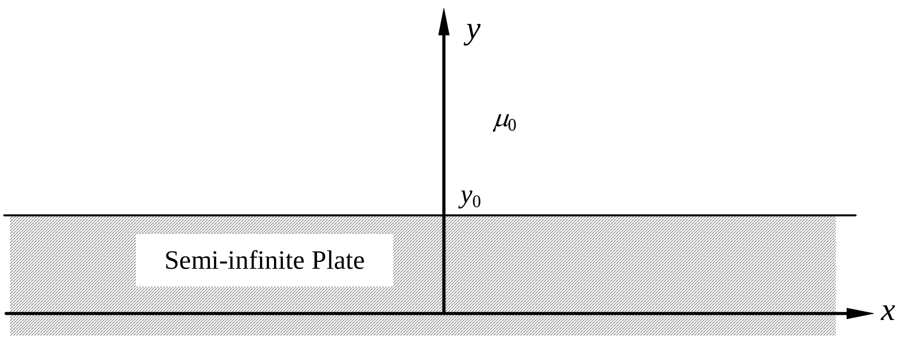

Fig. 4 (a) Magnetic field region above a semi-infinite plate

(6)

c. Figure 4 (b) shows a simplified model for one-dimensional eddy current flow in a thick plate. The plate, having conductivity σ and permeability μ, is exposed to a sinusoidally time-varying field in the z direction perpendicular to the paper plane, whose magnitude is H_s and angular frequency is ω.

(i) Calculate the skin depth of the eddy current at 50 Hz given $\mu = 8\pi\text{x}10^{-4}$ (H/m), $\sigma = 0.45\text{x}10^6$ ($\Omega^{-1}m^{-1}$).

(ii) Show that the magnetic field strength in steady state inside the plate is governed by:

$$\frac{\partial^2 H_z}{\partial y^2} = \alpha^2 H_z$$

(8)

where $\alpha = \sqrt{j}\,\sqrt{\omega\sigma\mu}$

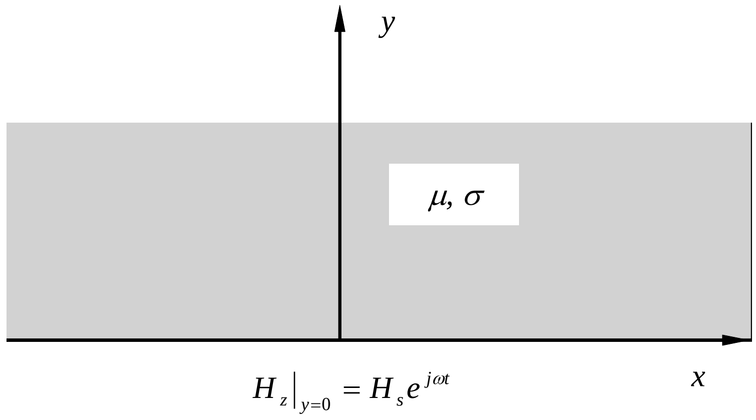

Figure 4 (b) Simplified model for one-dimensional eddy current flow in thick plate

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Vector differential operations

Let Φ be a scalar function and $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{A}$ be vector functions.

<u>Cartesian Co-ordinates (x, y, z)</u>

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\boldsymbol{e_x} + \frac{\partial\Phi}{\partial y}\boldsymbol{e_y} + \frac{\partial\Phi}{\partial z}\boldsymbol{e_z}$$

$$\nabla \bullet \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \boldsymbol{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\boldsymbol{e_x} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\boldsymbol{e_y} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\boldsymbol{e_z}$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\boldsymbol{A} = \nabla^2 A_x\boldsymbol{e_x} + \nabla^2 A_y\boldsymbol{e_y} + \nabla^2 A_z\boldsymbol{e_z}$$

<u>Cylindrical Co-ordinates (r, θ, z)</u>

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\boldsymbol{e_r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\boldsymbol{e_\theta} + \frac{\partial\Phi}{\partial z}\boldsymbol{e_z}$$

$$\nabla \bullet \boldsymbol{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial\theta} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \boldsymbol{H} = \left[\frac{1}{r}\frac{\partial H_z}{\partial\theta} - \frac{\partial H_\theta}{\partial z}\right]\boldsymbol{e_r} + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\boldsymbol{e_\theta} + \left[\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial\theta}\right]\boldsymbol{e_z}$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\boldsymbol{A} = \left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{A_r}{r^2}\right)\boldsymbol{e_r} + \left(\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2}\right)\boldsymbol{e_\theta} + \left(\nabla^2 A_z\right)\boldsymbol{e_z}$$