Circuits and Signals

1 Introduction

This module is about **linear passive** components and the analytical tools that can be used to predict their behaviour in terms of the currents that flow through them (or combinations of them) as a result of a potential difference applied across them.

> **"Linear"** means that current, I, is proportional to applied potential difference, V, and the proportionality constant is independent of the magnitude of V or I.
>
> **"Passive"** means that we are dealing with elements that can dissipate energy and with elements that can store energy. They do not create energy.

There are three forms of passive element:

> **Resistor** - resists the flow of current - dissipates energy
>
> **Inductor** - stores energy in a magnetic field - dissipates no energy
>
> **Capacitor** - stores energy in an electric field - dissipates no energy

But something must create the potential difference that drives current through these elements. These somethings are **"sources"**. Sources come in two forms, **voltage** sources and **current** sources and both may be **direct** (dc) or **alternating** (ac) in nature.

> An **ideal voltage source** maintains a given potential difference across its terminals irrespective of the amount of current it is asked to deliver.
>
> An **ideal current source** maintains a given current through it irrespective of the potential difference between its terminals.
>
> A **direct** voltage or current source maintains a constant output level.
>
> An **alternating** voltage or current source changes polarity at regular intervals with a well defined waveshape - often a sinusoidal waveshape.

This module is also about "**signals"**, or rather about the way signals interact with passive circuits. A signal is a way of transmitting information and an electrical signal usually involes a potential difference and/or current that varies with time. It can take the form of relatively straightforward waveshapes such as sinusoids or rectangular pulses, or it can take a more complicated form.

2 Voltage, Current, Sources, Resistance and Circuits

2.1 Voltage (or more correctly, voltage difference)

The voltage difference between any two points in a circuit is the force that drives the current flowing between those two points. Other names given to voltage difference are "potential difference" and "emf" (electro-motive force). Voltage difference has the same significance for electronic circuits as pressure difference has in hydraulic systems.

It is quite common to use the term "voltage" in systems that have some part of their circuit connected to ground (eg, most fixed audio installations) and in that case the voltage is the difference in potential between the node in question and ground.

From a classical physics point of view, the voltage difference $V_2 - V_1$ is equal to the energy expended in moving a unit charge from V_1 to V_2.

Voltage differences exist ***across*** components. A voltage difference sets up an electric field that exerts a force on any charge that lies in its influence. If the charges are mobile, this force will make them move (hence the term "emf") thus forming a current.

2.2 Current

Current is the movement of charge caused by a potential difference. Its magnitude is determined by the rate at which charge crosses an imaginary boundary in the circuit. In other words, the current in a wire is the rate of flow of charge through any cross section of that wire.

$$\text{Thus } I = \frac{\mathrm{d}q}{\mathrm{d}t} \text{ A (or C s}^{-1}\text{)} \qquad \textbf{(1.1)}$$

In reality, charge movement is impeded by the physical properties of the media within which it flows (this current limiting effect is called impedance - resistance is a particular kind of impedance) so I is always finite in a real system. The units of current are Amperes, usually abbreviated to Amps and given the symbol A.

Current must flow in a complete circuit. An analogy is a closed central heating system. The system pump creates the pressure difference (potential) that drives the water (current) through pipes, radiators and boiler (the circuit) back to the pump. Current flows ***through*** components.

2.3 Power and Energy

- **Energy expended equals work done** - it has units of Joules (J). Energy expended in an electrical circuit is given by

$$E = \int_0^{t_1} V(t)I(t)\, dt \ \ \textbf{J.} \qquad \textbf{(1.2)}$$

In general, V and I are functions of time as indicated in equation **(1.2)** but for the special case of direct currents and voltages $E = VIt_1$ **J.**

Physical explanation: The work done on unit charge (1 <u>C</u>oulomb) moving through a potential difference of 1<u>V</u>olt is 1 <u>J</u>oule. (The underlined letters represent the normal symbols for the units concerned.) If a source of potential difference V V is supplying I A (I C s^{-1}) for t_1 seconds, the total energy it supplies is VIt_1 J. (It_1 is the total charge involved and V is the potential difference through which it has been moved.)

- **Power is the rate of dissipation of energy** - it has units of Watts (W) or J s^{-1}. Power dissipated (or sourced in the case of an ideal source) in a circuit element is given by

$$P = \frac{1}{t_1}\int_0^{t_1} V(t)I(t)\, dt \ \ \mathbf{W}. \qquad \mathbf{(1.3)}$$

Again, for dc, V and I are constant and $P = VI$ **W.**

Notes: Equation **(1.3)** is often called the "power integral". It works out the ***average*** value of the product of $V(t)$ and $I(t)$ and hence t_1 is almost always the periodic time of a periodic signal. For dc, time disappears from the consideration. Being the rate of dissipation of energy, power can be described as "the energy dissipated per second" so the expression describing power dissipation is the same as the one describing energy expenditure except that it is divided by the time over which the integral is taken. Very occasionally a term called "instantaneous power" is referred to in the literature. This is simply the product of $V(t)$ and $I(t)$ at some instant in time and should be interpreted with care.

2.4 Voltage Sources

Voltage sources are ideal circuit elements that maintain a constant potential difference between their terminals. Conventionally they deliver current to the circuit connected to them and act as energy sources. Circuit symbols used for voltage sources and a voltage source V-I characteristic are shown in figure 1. If a voltage source is the only source in a circuit, the current always flows out of it into the circuit - i.e., a single source will always deliver energy to a circuit. If there are many sources in a circuit the current through the source can be reversed by the action of the other sources.

Good examples of voltage sources are batteries and main power supply outlets

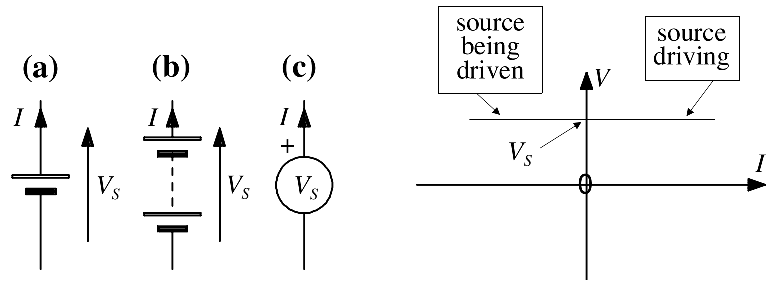

Figure 1

*The V-I characteristic of an ideal voltage source together with three symbols coomonly used to represent voltage sources. **(a)** is a single cell battery, **(b)** is a multi-cell battery and **(c)** is a general source. note that in **(c)**, the "+" end must be marked; in **(a)** and **(b)** it is the longer line that is positive.*

2.5 Current sources

Current sources drive a given current through a circuit irrespective of the voltage across their terminals. They are conceptually more awkward than voltage sources probably because, unlike batteries, one cannot buy them in a shop. A current source symbol and V-I characterisic behaviour are described by figure 2. Current sources are usually labelled using one of the methods shown in figure 2. An arrow in the position shown in figure 2b could also be used to indicate the voltage across the source but the quantity indicated would then have units of volts.

Notes: Current sources are used extensively in applications such as battery chargers and play a key role in the internal circuit biassing of analogue integrated circuits (ICs) such as operational amplifiers. In these practical applications, there is a limit to the voltage that can exist across the current source terminals if it is to behave like a current source. This is no different in principle from real voltage sources like batteries; they will only maintain their terminal voltage for a certain range of current supplied.

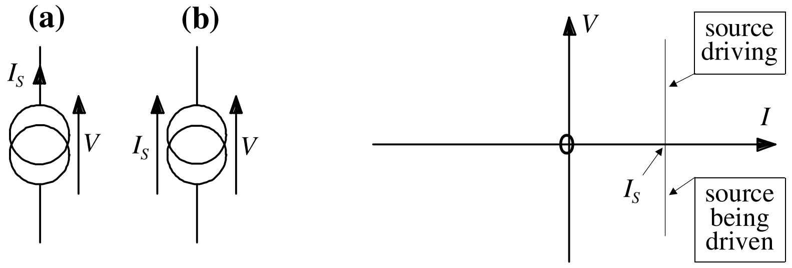

Figure 2

Current source symbols labelled in two different ways together with the V-I charactersic of an ideal current source.

2.6 Resistors

In 1827, Ohm discovered that the potential difference between the ends of a length of wire was proportional to the current passing through it. The constant of proportionality was called "resistance". Resistance is a property intrinsic to all conductive materials. The governing relationship for resistance is

$$R = \frac{V}{I} \text{ or } V = IR \tag{1.4}$$

The unit of resistance is the "Ohm" and is given the unit symbol Ω.

The phenomenon of resistance is very useful in electronic circuits because it allows a circuit designer to create currents proportional to potential differences and vice versa. Devices called resistors are manufactured in huge numbers and are obtainable with a wide range of values*. The symbol used to represent a resistor in circuit diagrams is shown** in figure 3. Note the convention of current and voltage; a current in the direction shown will create a potential difference in the direction shown. Conversely, a potential difference applied in the direction shown will cause a current flow in the direction shown.

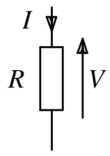

Figure 3

the symbol used for a resistor

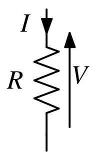

Figure 3a

the symbol commonly used for a resistor outside Europe

*There are many different types of resistor. Some are precision resistors that are very accurate with tolerances of betwee 0.01% and 0.5% and excellent stability as a function of time and temperature. Some are more ordinary with tolerances in the range 1% to 5% with good temperature and time stabilities. Some are thin films, either of carbon or metal, and some are wire wound. Some can dissipate 100s of Watts, some can dissipate only 100s of milli-Watts. Some are in axial wire ended packages, some are in surface mount packages and some are in large metal packages that can be bolted to heatsinks. (Heatsinks are big lumps of aluminium, usually with fins on, designed to remove heat from electronic components that need to dissipate significant power.)

** The symbol shown in figure 3 is the European standard symbol for a resistor. In some other parts of the world such as the USA a different symbol consisting of the zig-zag shape shown in figure 3a is used. The zig-zag shape was used in the UK until around 1970.

2.7 Power and Energy in Resistive Circuits

For direct current and voltage and resistive circuit elements such as that in figure 3,

$$E = \int_0^{t_1} VI\,dt = \int_0^{t_1} V\frac{V}{R}\,dt = \frac{V^2}{R}t_1 = I^2R\,t_1 \text{ W} \tag{1.5}$$

$$P = \frac{1}{t_1}\int_0^{t_1} VI\,dt = \frac{1}{t_1}\int_0^{t_1} V\frac{V}{R}\,dt = \frac{1}{t_1}\frac{V^2}{R}t_1 = \frac{V^2}{R} = I^2R \text{ W} \tag{1.6}$$

Power dissipation is a very important consideration for resistors and for all other electronic components. Energy is usually dissipated as heat and this heat must be removed if component temperature is to be prevented from rising to a level that would cause damage. As a general rule, more surface area is required to get rid of more dissipated heat so components capable of dealing with large power dissipation tend themselves to be physically large. In some cases - for example some high speed CPU and graphics processor integrated circuits - the power generated exceeds the ability of normal conduction and convection processes to remove heat and so forced air cooling (achieved by gluing a fan onto the back of the chip) is employed.

2.8 Circuits and Conventions

A circuit is a combination of components that is usually, but not always, connected to a source. The components must be connected in such a way as to allow current to flow from the source, through the circuit and back to the source - ie, the circuit must form at least one complete loop. The circuit of figure 4a contains one loop consisting of R_2, R_3 and R_4. R_1 and V_S are not part of a complete loop because there is a break between the bottom of R_2 and the negative end of the source. There is thus no circuit involving the source through which current may flow and consequently no current flows in the circuit. Circuit 4b, however has two complete loops, one of which involves the source, and current will flow in all parts of the circuit.

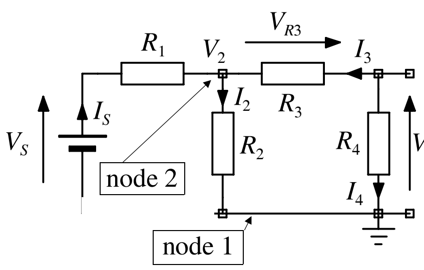

Figure 4a
An incomplete circuit

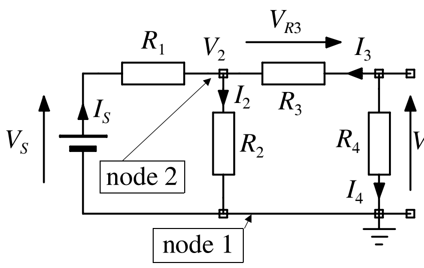

Figure 4b
A complete circuit

The labelling of the various quantities is intended as a guide to convention.

- Notice that the positive end of the voltage across a resistor is the end at which a conventional positive current enters the resistor.
- Notice also that conventional positive current from a voltage source comes out of the positive end of the source, travels through the circuit and returns to the negative end of the source.
- Node 2 is labelled with a voltage V_2. All voltages are in fact voltage differences so V_2 must be measured with respect to some reference node - in this case, node 1. Node 1 is a ground node and ground is often used as an implied reference.

2.9 Series and Parallel Connections

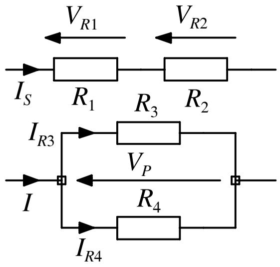

Figure 5

Figure 5 shows two pairs of resistors. R_1 and R_2 are said to be connected in **"series"** (ie, one after the other) whereas R_3 and R_4 are said to be connected in **"parallel".**

- **Elements connected in series have the same current flowing through each of them** (I_S in the case of R_1 and R_2).
- **Elements connected in parallel have the same voltage across them** (V_P in the case of R_3 and R_4).

In both the series and the parallel case, the two (or more) resistors involved can be represented as a single resistor. Consider the series pair, R_1 and R_2. The voltage across the combination is the sum of the voltages across each resistor so,

$$V_{TOTAL} = V_{R1} + V_{R2} = I_S R_1 + I_S R_2 = I_S (R_1 + R_2) \qquad \textbf{(1.7)}$$)

In other words, the resistance of a series combination of two resistors is simply the sum of their values. It is not difficult to extend the argument to an arbitrary number of series connected resistors to arrive at the general rule,

- **The resistance of a number of resistors connected in series equals the sum of their individual resistances.**

Now consider the parallel case of R_3 and R_4. The current I flowing into the parallel combination splits into two components, I_{R3} and I_{R4} so,

$$I = I_{R3} + I_{R4} = \frac{V_P}{R_3} + \frac{V_P}{R_4} = V_P\left(\frac{1}{R_3} + \frac{1}{R_4}\right)$$

The effective resistance, R_P, of the parallel combination R_3 and R_4 is given by

$$\frac{V_P}{I} = R_P = \frac{1}{\dfrac{1}{R_3} + \dfrac{1}{R_4}} \quad \text{or} \quad \frac{1}{R_P} = \frac{1}{R_3} + \frac{1}{R_4} \qquad \textbf{(1.8)}$$

In other words, the reciprocal of the effective resistance of the parallel combination of R_3 and R_4 is the sum of the reciprocals of R_3 and R_4. Again, this argument can be extended to account for an arbitrary number of parallel resistors,

- **The reciprocal of the effective resistance of a number of parallel connected resistors is the sum of the reciprocals of all of the resistors in the parallel combination.**

2.10 Kirchhoff's laws

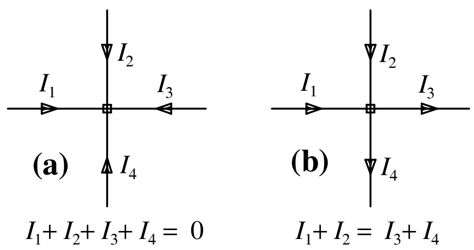

Figure 6

Kirchhoff's laws are the two basic circuit laws that together enable the process of circuit analysis. One of them is called "Kirchhoff's current law" and the other "Kirchhoff's voltage law"

Kirchhoff's current law is

- **The sum of all currents entering a circuit node is zero** - see figure 6a.

Another way of putting this is "the sum of currents entering a particular node must equal the sum of currents leaving that node - see figure 6b. A node might not look like those of figure 6 - see node 1 of figure 4b for a more spread out example of a node.

Kirchhoff's voltage law is

- **The sum of voltages around any closed loop within a circuit must equal zero.**

It is very important here to add up the voltages in the same direction - eg, going clockwise around a loop and taking as (say) positive the first terminal encountered of each element in the loop. There are three loops in figure 7,

$$V_S + V_{R1} - V_{R2} = 0$$

$$V_{R2} + V_{R3} + V_{R4} - V_S = 0$$

$$V_{R1} + V_{R3} + V_{R4} = 0$$

These equations have been obtained from the left hand loop, the right hand loop and the outer loop respectively by going around each loop in a clockwise direction.

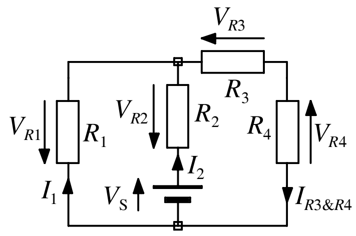

Figure 7

2.11 Examples on the application of circuit laws

The first example uses the circuit of figure 7, redrawn here with modified labelling as figure 8. In this circuit there is only one source and the directions of V and I in various parts of the circuit can be deduced fairly easily by inspection by someone with a little experience in dealing with circuits.

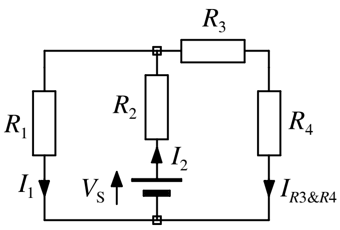

Figure 8

Conventional current is driven into the bottom end of R_2 by V_S. When that current emerges from the top of R_2 it splits between the two paths: through R_1 back to V_S and through R_3 and R_4 back to V_S. The currents through R_1 and R_4 are thus in a downwards direction. R_3 and R_4 are in series and the series combination of R_3 and R_4 is in parallel with R_1 (you can tell this because thr left hand end of R_3 is connected to the top of R_1 and the bottom of R_4 is connected to the bottom of R_1) so the circuit could be redrawn as in figure 9.

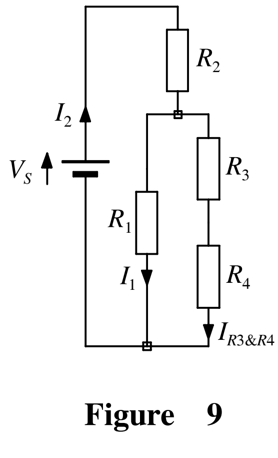

Figure 9

If all the resistors are 1Ω and V_S = 10V,

$R_3 + R_4 = 2\Omega$

$R_1 \,//\, (R_3 + R_4) = 1\Omega \,//\, 2\Omega = 2/3\ \Omega$

Resistance seen by $V_S = R_2 + R_1 \,//\, (R_3 + R_4) = 5/3\ \Omega$

Therefore $I_2 = 10/(5/3)$ A = 30/5 A = **6 A**

I_2 now splits between R_1 and the R_3, R_4 combination. There is a current splitting formula that can be used but its often safer to use basic circuit laws. First work out the voltage that will be generated across $R_1 \,//\, (R_3 + R_4)$ by I_2 . . . .

$V_{R1} = I_2 \text{ x } R_1 \,//\, (R_3 + R_4) = 6 \text{ x } 2/3 =$ **4 V**

so $I_1 = V_{R1}/R_1 = 4/1$ A = **4 A** and $I_{R3\&R4} = V_{R1}/(R_3 + R_4) = 4/2$ A = **2 A**

(Note that once I_2 and I_1 were known, $I_{R3\&R4}$ could have been worked out from Kirchhoff's current law, $I_2 = I_1 + I_{R3\&R4}$.)

This was relatively easy to work out because the directions of the currents involved were fairly easy to deduce by inspection. When it is not clear at the outset which directions circuit currents will be travelling in, one must resort to a formal convention based approach to solving the circuit. We will now use such an approach on the circuit of figure 7, for convenience redrawn here as figure 10.

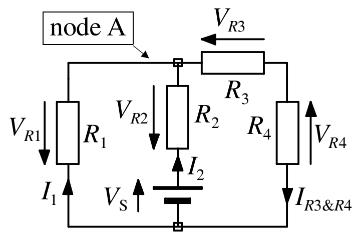

Figure 10

Using Kirchhoff's voltage law we can sum the voltages around the R_1, R_2, V_S loop (loop 1) and around the V_S, R_2, R_3, R_4 loop (loop 2). Using V_S as the reference direction in both cases,

For loop 1, $V_S - V_{R2} + V_{R1} = V_S - I_2R_2 + I_1R_1 = 0$ **(1.9)**

For loop 2, $V_S - V_{R2} - V_{R3} - V_{R4} = V_S - I_2R_2 - I_{R3\&R4}(R_3 + R_4) = 0$ **(1.10)**

The outer loop does not give an independent relationship. If one subtracts equation **(1.10)** from equation **(1.9)**, the result is the third loop equation written for figure 7 on page 5. The third relationship needed is provided by the application of Kirchhoff's current law to the node at the top of R_2, node A,

For node A, $I_2 + I_1 = I_{R3\&R4}$ **(1.11)**

Using equation **(1.11)** with equation **(1.10)** to eliminate $I_{R3\&R4}$ we are left with two simultaneous equations with I_1 and I_2 as unknowns,

$V_S - I_2R_2 + I_1R_1 = 0$ [equation **(1.9)** with no changes] **(1.9)**

$V_S - I_2R_2 - (I_2 + I_1)(R_3 + R_4) = 0$ or $V_S - I_2(R_2 + R_3 + R_4) - I_1(R_3 + R_4) = 0$ **(1.12)**

These equations can be solved algebraically (not difficult but a bit cumbersome) or numbers can be put in to the equations to reduce the size of the coefficients. If, as before, all Rs are 1Ω and $V_S = 10$ V, equations **(1.9) and (1.12)** become,

$10 - I_2 + I_1 = 0$ **(1.13)**

$10 - 3I_2 - 2I_1 = 0$ **(1.14)**

multiplying equation **(1.13)** by 2 and adding the result to equation **(1.14)** eliminates I_1 to give

$30 - 5I_2 = 0$ or $I_2 =$ **6 A**

I_1 can then be found from equation **(1.13)** [or **(1.14)**] as $I_1 =$ **−4 A**

- **Notice that I_1 is negative here. This means that I_1 is actually flowing in the opposite direction to the direction chosen for I_1 in figure 10.** It doesn't matter which direction you choose for a current providing you make sure that the relationship between the current through and the voltage across components adheres to the correct convention.

3 Formal analysis methods

3.1 Nodal Analysis

Nodal analysis begins by defining one node as the reference node and then defining voltages at every other node with respect to the reference node. At each node a sum of currents is performed and those currents are then expressed in terms of node voltage differences and resistance. **The aim of a nodal analysis is to evaluate the node voltages of the circuit.**

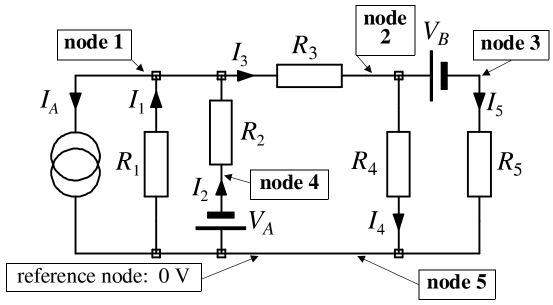

Figure 11

As an example, consider the circuit of figure 11. The significant nodes in figure 11 are 1, 2 and 5; 3 and 4 have only two connections so the current sum is trivial. I_A, V_A and V_B are known driving sources.

Summing currents at node 1 . . .

$$I_1 + I_2 = I_3 + I_A \quad \text{or} \quad \frac{0 - V_1}{R_1} + \frac{-V_A - V_1}{R_2} = \frac{V_1 - V_2}{R_3} + I_A \qquad \textbf{(1.15)}$$

Summing currents at node 2 . . .

$$I_3 = I_4 + I_5 \quad \text{or} \quad \frac{V_1 - V_2}{R_3} = \frac{V_2 - 0}{R_4} + \frac{V_2 - V_B}{R_5} \qquad \textbf{(1.16)}$$

These two equations have V_1 and V_2 as unknowns and the next step is to collect like terms to give . . .

$$-V_1\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right] + \frac{V_2}{R_3} = \frac{V_A}{R_2} + I_A \qquad \textbf{(1.17)}$$

$$\frac{V_1}{R_3} - V_2\left[\frac{1}{R_3} + \frac{1}{R_4} + \frac{1}{R_5}\right] = -\frac{V_B}{R_5} \qquad \textbf{(1.18)}$$

before solving them simultaneously to get values for V_1 and V_2. Once those voltages are known, any of the currents in the circuit can be evaluated with ease.

3.2 Loop Analysis

Loop analysis is very similar in nature to nodal analysis except that instead of defining a current through each branch of the circuit, one defines a current for each independent loop in the circuit. Some components in the loop may have only that loop current flowing through them but components that are common to two (or more) loops will have those two (or more) loop currents flowing through them. The loop currents can be defined as flowing in a clockwise or an anti-clockwise direction and different loops in the same circuit can be chosen to flow in different directions. The only critical factor is that the voltages around the loop must be summed using potential differences across components that are conventionally consistent with the direction of currents chosen. Loop analysis introduces one current variable for each independent circuit loop - this is the minimum number of unknowns needed to solve a problem so in that respect loop analysis is an efficient analytical method.

As an example, consider the circuit of figure 12. Here the choice of loops is fairly obvious and three loops, I_1, I_2 and I_3 are labelled on the diagram. I_1 and I_3 are assumed to flow in a clockwise direction and I_2 in an anti-clockwise direction; these choices are arbitrary. The analysis proceeds as follows,

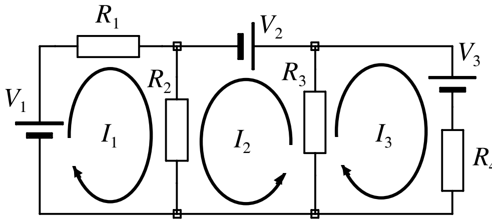

Figure 12

Summing voltages around the three loops using the voltage developed across a resistor by the loop current as the reference direction (one could equally well choose different reference directions),

$$I_1R_1 + (I_1 + I_2)R_2 - V_1 = 0 \quad \text{or} \quad I_1(R_1 + R_2) + I_2R_2 - V_1 = 0 \qquad \textbf{(1.19)}$$

$$(I_2 + I_3)R_3 + (I_2 + I_1)R_2 + V_2 = 0 \quad \text{or} \quad I_1R_2 + I_2(R_2 + R_3) + I_3R_3 + V_2 = 0 \qquad \textbf{(1.20)}$$

$$I_3R_4 + (I_3 + I_2)R_3 + V_3 = 0 \quad \text{or} \quad I_2R_3 + I_3(R_3 + R_4) + V_3 = 0 \qquad \textbf{(1.21)}$$

These three simultaneous equations must then be solved, a task that is easier if the coefficients of the Is in the three equations become numbers. For example if all the Vs are 10 V and all the Rs are 1Ω, Equations **(1.19)**, **(1.20)** and **(1.21)** become,

$$2I_1 + I_2 - 10 = 0 \qquad \textbf{(1.22)}$$

$$I_1 + 2I_2 + I_3 + 10 = 0 \qquad \textbf{(1.23)}$$

$$I_2 + 2I_3 + 10 = 0 \qquad \textbf{(1.24)}$$

Using **(1.22)** to express I_1 in ters of I_2 and **(1.24)** to express I_3 in terms of I_2 and substituting in **(1.23)** gives

$$\frac{10 - I_2}{2} + 2I_2 + \frac{(-10 - I_2)}{2} + 10 = 0 \quad \text{or} \quad I_2 = -10 \text{ A} \qquad \textbf{(1.25)}$$

So, from equation **(1.22)**, $I_1 = 10$ A and from **(1.24)**, $I_3 = 0$ A. Once the currents are known, voltage differences can be easily worked out.

Loop analysis can be used for circuits containing current sources but because the voltage drop across a current source is defined by the interaction between the circuit and the current source and is therefore an unknown, the voltage(s) across the current source(s) must be defined as variables. Superficially this seems unattractive because there is an apparent increase in the number of unknowns but on reflection, the current sources always provide extra information about the relationship between loop currents - ie the loops are usually no longer independent. As an example, look at figure 13. Here the current sources are I_A and I_B ; V_A and V_B, both measured with respect to the bottom node V_C, are the voltage differences across them. Clearly, $I_2 - I_1 = I_A$ and $I_2 - I_3 = I_B$ so both I_2 and I_3 can be written in terms of I_1, I_A and I_B , in other words, in this case there is really only one current variable.

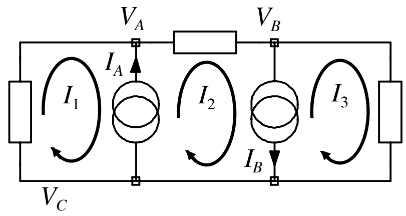

Figure 13

3.3 The Principle of Superposition

Superposition is a very useful tool for solving circuit (and other) problems. It is true for all linear systems and can be used to calculate displacement in air-frames due to the application of a number of forces as effectively as it can be applied to electrical networks.

A statement of the superposition principle applied to circuits is

- **In an electrical network containing several sources (current or voltage or both), the current through or voltage across any element in the circuit is due to the linear sum of the currents and voltage differences across that element due to each source acting alone and with all the other sources replaced in the circuit by their internal resistance - ie 0 Ω for a voltage source and ∞ Ω for a current source.**

This is a bit of a mouthful - the example of figure 14 might help to make it clearer. In figure 14 there are four sources, one of which is a current source. Analysis by superposition requires us to consider the effects of each source in turn as the only source in the circuit. In this example the aim is to work out the voltage across R_3, $V_A - V_B$ or V_{AB}. The sources will be considered in the order V_1, V_2, I_1 and V_3.

V_{AB} due to V_1. Figure 15a shows the partial circuit for V_1. V_1 sees R_1 in series with the parallel combination of R_2 and R_3; V_{AB} is the voltage across this combination.

$$V_{AB(V_1)} = V_1 \frac{R_2 // R_3}{R_1 + R_2 // R_3} \quad \text{where } R_2 // R_3 = \frac{R_2 R_3}{R_2 + R_3}.$$

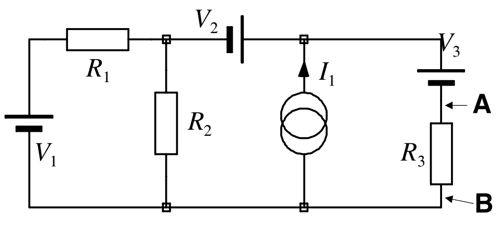

Figure 14

V_{AB} due to V_2. Figure 15b shows this partial circuit.

V_2 sees R_3 in series with the parallel combination R_1 and R_2. V_{AB} is across R_3.

$$V_{AB(V_2)} = V_2 \frac{R_3}{R_3 + R_1 // R_2} \quad \text{where } R_1 // R_2 = \frac{R_1 R_2}{R_1 + R_2} .$$

V_{AB} due to I_1. Figure 15c shows this partial circuit - I_1 sees R_1, R_2 and R_3 as a parallel threesome the voltage across which is V_{AB}.

$$V_{AB(I_1)} = I_1 R_1 // R_2 // R_3 = I_1 \frac{R_1 R_2 R_3}{R_2 R_3 + R_1 R_3 + R_1 R_2} .$$

V_{AB} due to V_3. Figure 15d shows this partial circuit. V_3 sees R_3 in series with the parallel combination R_1 and R_2. V_{AB} is across R_3.

$$V_{AB(V_3)} = -V_3 \frac{R_3}{R_3 + R_1 // R_2} \quad \text{where } R_1 // R_2 = \frac{R_1 R_2}{R_1 + R_2} .$$

The total V_{AB} is obtained by adding the components of V_{AB} due to each source,

$$V_{AB(TOT)} = V_{AB(V_1)} + V_{AB(V_2)} + V_{AB(I_1)} + V_{AB(V_3)} . \quad \textbf{(1.26)}$$

Notice that the V_{AB} due to V_3 is in the opposite polarity to the other contributions.

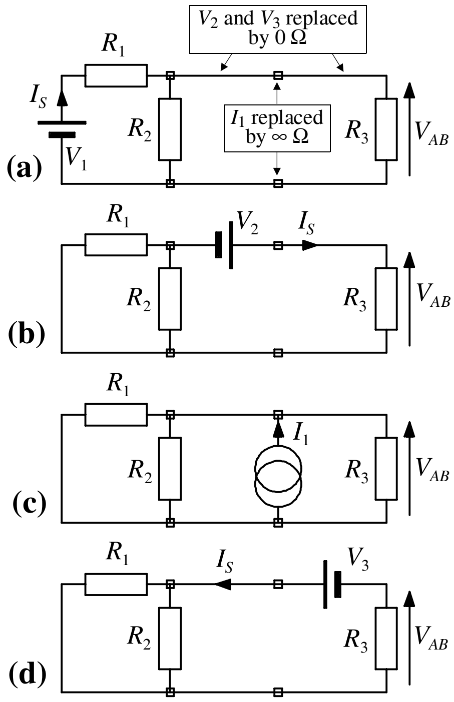

Figure 15

4 Thevenin and Norton Equivalent Circuits

Thevenin and Norton equivalent circuits are ways in which an arbitrary linear, time invariant, two terminal electrical network can be modelled from the point of view of the world outside the network. The Thevenin equivalent is the one used most commonly. It is used to model the internal resistance of a battery and is the model generally used to describe the output resistance of amplifiers and signal sources, both of which tend to be modelled in the same way as batteries. Some of the networks that can be represented by these equivalent circuits give access to the two terminals of interest but no information of what is behind the terminals is given. Networks like this are called "black box" networks. Other networks are described by a circuit diagram that gives details of the circuit behind the terminals of interest.

4.1 Black box models

Figure 16a shows a linear time invariant network. With no load connected between nodes **A** and **B**, the output voltage is V_{OC}. When a short circuit (0 Ω) is connected between nodes **A** and **B**, the current that flows is I_{SC}.

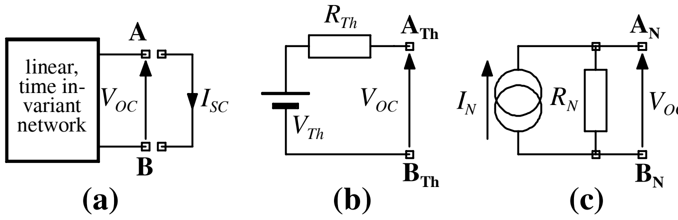

Figure 16

Figure 16b shows a **Thevenin** equivalent circuit. If the circuit of figure 16b is to model the network of figure 16a then V_{Th} and R_{Th} must be found such that figure 16b gives the same open circuit output voltage and short circuit output current as those measured for figure 16a. Thus

$$V_{Th} = V_{OC} \quad \text{and} \quad R_{Th} = \frac{V_{OC}}{I_{SC}} \qquad \textbf{(1.27)}$$

Figure 16c shows a **Norton** equivalent circuit. If the circuit of figure 16c is to model the network of figure 16a then I_N and R_N must be found such that figure 16c gives the same open circuit output voltage and short circuit output current as those measured for figure 16a. Thus

$$I_N = I_{SC} \text{ and } I_N R_N = V_{OC} \text{ or } R_N = \frac{V_{OC}}{I_{SC}} \tag{1.28}$$

Note that R_{Th} and R_N are the same.

In black box systems such as batteries and many electronic circuits, short circuiting the output in order to measure I_{SC} is likely to result in damage so a more subtle measurement approach is used. One of these approaches, illustrated in figure 17, works as follows,

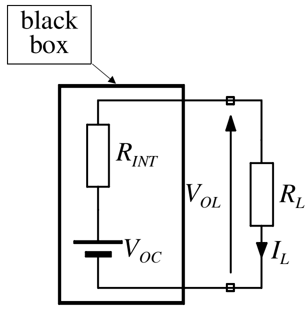

Figure 17

(i) measure the open circuit output voltage, V_{OC} (= V_{Th})

(ii) measure the output voltage with a known resistance, R_L, between the two output terminals. Call this V_{OL}.

(iii) calculate the internal resistance R_{INT} (= R_{Th}) of the black box . . . .

The value of V_{OL} together with the known resistance across which it was measured can be used to find the current I_L flowing in the circuit when loaded with R_L

$$I_L = \frac{V_{OL}}{R_L}$$

This I_L also flows through the internal (Thevenin) resistance of the black box causing a voltage drop across it which is $V_{OC} - V_{OL}$. So R_{Th} is given by

$$R_{INT} = R_{Th} = \frac{V_{OC} - V_{OL}}{I_L}$$

In cases where R_{Th} is very small, other methods may have to be used.

4.2 Thevenin to Norton Transformations

It is possible to convert a Thevenin model into a Norton one very easily ...

$$\text{Thevenin to Norton: } I_N = \frac{V_{Th}}{R_{Th}} \text{ and } R_N = R_{Th} \tag{1.29}$$

$$\text{Norton to Thevenin: } V_{Th} = I_N R_N \text{ and } R_{Th} = R_N \tag{1.30}$$

These conversions, sometimes called Thevenin to Norton (or Norton to Thevenin) transformations, can be very useful tools for simplifying circuit shapes.

4.3 A Couple of Circuit Examples

Consider the circuit of figure 18. This circuit is often called a "potential divider" and occurs very frequently on its own and as part of more sophisticated circuits. To find the Thevenin equivalent of the potential divider circuit of figure 18 one must find V_{OC} and I_{SC} and then use equation **(1. 27)** to find R_{Th} and V_{Th},

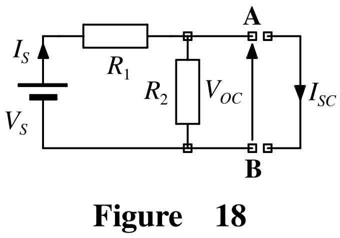

Figure 18

$I_S = \dfrac{V_S}{R_1 + R_2}$ so $V_{OC} = I_S R_2 = V_S \dfrac{R_2}{R_1 + R_2}$ since V_{OC} is the voltage developed across R_2 by I_S.

When nodes **A** and **B** are short circuited, there is zero voltage across R_2 and hence no current through it. Thus $I_{SC} = \dfrac{V_S}{R_1}$.

$$\text{Then } R_{Th} = \frac{V_{OC}}{I_{SC}} = \frac{\dfrac{V_S R_2}{R_1 + R_2}}{\dfrac{V_S}{R_1}} = \frac{R_1 R_2}{R_1 + R_2} = R_1 // R_2 \text{ and } V_{Th} = V_{OC} = \frac{V_S R_2}{R_1 + R_2} . \qquad \textbf{(1.31)}$$

In a general case, V_{OC} can be worked out by using the analysis methods already mentioned earlier in this handout.

NOTE: The Thevenin resistance can often be determined by inspection. The approach is to replace sources by their internal impedances (0 Ω for a voltage source and ∞ Ω for a current source) and look into the network from the two terminals of interest. If one looks at figure 18 in this way it is clear that R_{Th} is the parallel combination of R_1 and R_2.

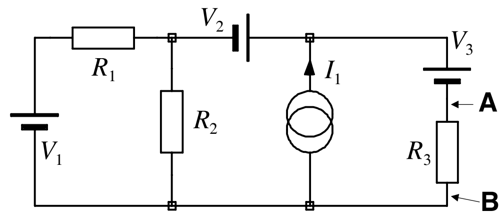

Figure 19

Consider figure 19 - this is in fact the same as figure 14 but re-drawn for convenience. Let the terminals of interest be **A** and **B**. The first task is to calculate $V_{AB\text{-}OC}$ which has been done already, the answer being given by equation **(1.26)**. The only other variable to work out is the I_{SC} that would flow through a short circuit placed between V_A and V_B. We can avoid working out I_{SC} if we can identify the Thevenin equivalent resistance by inspection. Replace all voltage sources by 0 Ω and all current sources by ∞ Ω and look into the circuit from points **A** and **B.** The three resistors are in parallel so $R_{Th} = R_1 // R_2 // R_3$.

The problem could also be solved by successive Thevenin to Norton operations . . .

(i) replace the combination, V_1 R_1 and R_2, by its Thevenin equivalent, as for figure 18 (eq 1.31)

(ii) add V_2 to $\dfrac{V_1 R_2}{R_1 + R_2}$ and convert to a Norton equivalent as shown in figure 20 where $I_A = \dfrac{\dfrac{V_1 R_2}{R_1 + R_2} + V_2}{R_1 // R_2}$

(iii) add the current sources together and convert to a Thevenin equivalent as shown in figure 21

(iv) add V_3 to the thevenin equivalent source and use the process of figure 18 to express the potential divider as a Thevenin equivalent as in figure 22 and get $V_{Th} = \dfrac{((I_A + I_1) R_1 // R_2 - V_3) R_3}{R_1 // R_2 + R_3}$

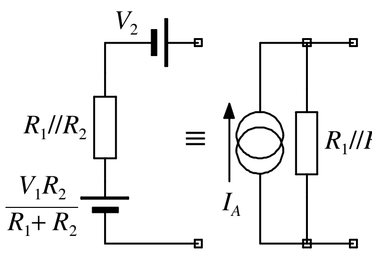

Figure 20

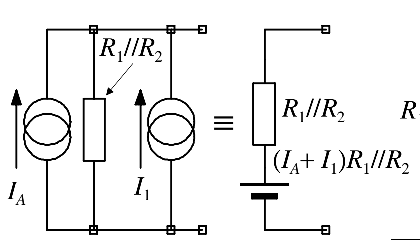

Figure 21

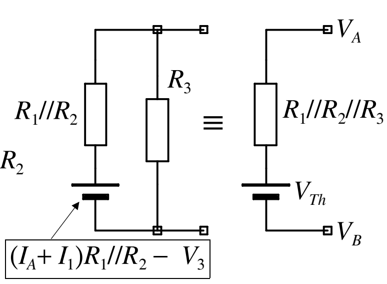

Figure 22

4.4 Star - Delta and Delta - Star Transformations

These are transformations that are related to Thevenin's theorem in terms of equivalent resistance and are useful in simplifying some resistive problems. Figure 23a shows a so-called "star" network and Figure 23b shows a so-called "delta" network; the names arise because of the circuit shapes. Star and delta network topologies occur commonly, but not exclusively, in power systems. If the resistance measured between A and B, R_{AB}, is the same as that between A' and B', $R_{A'B'}$, and also if $R_{BC} = R_{B'C'}$ and $R_{CA} = R_{C'A'}$ the networks are equivalent and cannot be distinguished from a viewpoint outside the networks. To find the condition that will give equivalence . . .

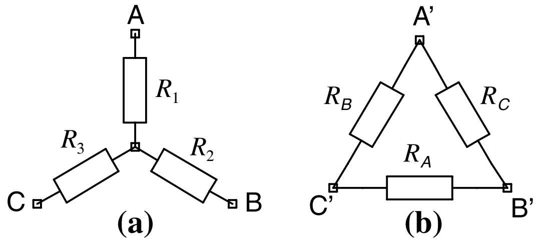

Figure 23

The resistance between A and B must equal that between A' and B', ie

$$R_1 + R_2 = R_C // (R_A + R_B) = \frac{R_C R_A + R_C R_B}{R_A + R_B + R_C} \qquad \textbf{(1.32)}$$

simarly $$R_2 + R_3 = R_A // (R_B + R_C) = \frac{R_A R_B + R_A R_C}{R_A + R_B + R_C} \qquad \textbf{(1.33)}$$

and $$R_3 + R_1 = R_B // (R_A + R_C) = \frac{R_B R_A + R_B R_C}{R_A + R_B + R_C} \qquad \textbf{(1.34)}$$

Subtracting **(1.34)** from **(1.33)**, $$R_2 - R_1 = \frac{R_C R_A - R_C R_B}{R_A + R_B + R_C} \qquad \textbf{(1.35)}$$

and subtracting **(1.35)** from **(1.32)** gives $$R_1 = \frac{R_C R_B}{R_A + R_B + R_C} \qquad \textbf{(1.36)}$$

Similar procedures lead to $$R_2 = \frac{R_C R_A}{R_A + R_B + R_C} \qquad \textbf{(1.37)}$$

and $$R_3 = \frac{R_A R_B}{R_A + R_B + R_C} \qquad \textbf{(1.38)}$$

Equations **(1.36)**, **(1.37)** and **(1.38)** give a "**delta to star**" transformation. It is possible to find R_A, R_B and R_C in terms of R_1, R_2 and R_3 (a "**star to delta**" transformation) by manipulating **(1.36)**, **(1.37)** and **(1.38)** although the manipulation process is not obvious. The result is

$$R_A = R_2 R_3 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \qquad \textbf{(1.39)}$$

$$R_B = R_1 R_3 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \qquad \textbf{(1.40)}$$

$$R_C = R_1 R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \qquad \textbf{(1.41)}$$

The star to delta and delta to star transformations are also useful for transforming "pi" (Π) circuits into "tee" (T) equivalents and vice versa. Figure 24a shows a T circuit and figure 24b a Π circuit. The circuit components have been labelled such that the transformation results **(1.36)** to **(1.41)** are valid. The circuits of figure 24 are often found as sections of filters or attenuators in high frequency circuits. In such applications the circuits would normally have some symmetry - ie, $R_2 = R_3$ and $R_B = R_C$ would normally be true.

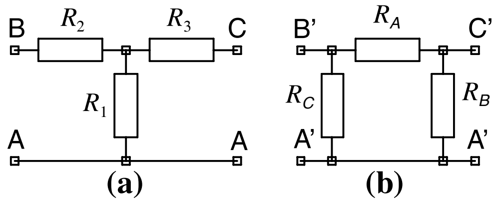

Figure 24

5 Controlled Sources

Controlled sources behave in the same way as fixed sources except that their value is linearly dependent on some other circuit variable. They are used extensively to model the way devices like op-amps and transistors behave towards signals. Most basic amplifying devices (transistors of all types and vacuum tubes (or valves)) behave, from a signal point of view, like voltage controlled current sources; op-amps tend to behave more like voltage controlled voltage sources.

A typical signal equivalent circuit of a single transistor (or vacuum tube) amplifier is shown in figure 25. The parameter g_m is called the transconductance - it tells us the value of the voltage controlled current source in units of A V^{-1}. The objective of this sort of analysis is usually to work out the circuit gain, v_o/v_s. In order to work out v_o, it is necessary first to work out v_1 in terms of v_s.

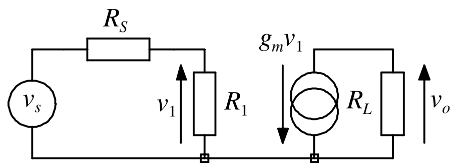

Figure 25

$$v_1 = v_s \frac{R_1}{R_1 + R_S} \quad \text{and} \quad v_o = -g_m v_1 R_L = -g_m v_s \frac{R_1}{R_1 + R_S} R_L$$

$$\text{or} \quad \frac{v_o}{v_s} = -g_m R_L \frac{R_1}{R_1 + R_S}$$