1.

a. Open-loop unstable because it has a pole in the right-hand side of the s-plane, at $s = 1$. The contribution of this to the time response is an increasing exponential, i.e., an unbounded output.

[2 marks]

b. For $C(s) = K$, the closed-loop transfer function is

$$\frac{C(s)G(s)}{1 + C(s)G(s)} = \frac{KG(s)}{1 + KG(s)}$$

The characteristic equation is the denominator:

$$1 + KG(s) = 1 + \frac{K(1 + K_{\mathrm{M}}s)}{s(s-1)}$$

The roots of $1 + KG(s)$ are the poles of the closed-loop system.

$$0 = 1 + \frac{K(1 + K_{\mathrm{M}}s)}{s(s-1)}$$

$$\Rightarrow 0 = s^2 + (KK_{\mathrm{M}} - 1)s + K = q(s)$$

A sufficient condition for stability of a second-order system is for the coefficients of s^2, s^1 and s^0 in the characteristic equation $q(s)$ to be of the same sign. Thus, we require

$$K > 0$$
$$KK_M > 1$$

The same result can be found by using the Routh array, or by determining directly the closed-loop poles.

[5 marks]

c. Comparing $q(s)$ with the standard form of a second-order system

$$s^2 + 2\zeta\omega_{\mathrm{n}}s + \omega_{\mathrm{n}}^2$$

it follows that $\omega_{\mathrm{n}}^2 = K$, thus the natural frequency is

$$\omega_{\mathrm{n}} = \sqrt{K}$$

and $2\zeta\omega_n = KK_{\mathrm{M}} - 1$, thus the damping ratio is

$$\zeta = \frac{KK_{\mathrm{M}} - 1}{2\sqrt{K}}$$

[2 marks]

d. Plot the root locus by following the "textbook" rules

i. Open-loop poles are at $s = 0$ and $s = 1$ (mark with "x"), and the zero is at $s = -1$ (mark with "o").

ii. Since there are $n = 2$ poles and $m = 1$ zero, there are **two** branches of the locus; one ending at the zero, and the other at $\pm\infty$.

iii. On the **real axis**, the locus lies in the intervals $[0,1]$ and $(-\infty, -1]$, BUT NOT in the interval $(-1, 0)$.

The sketch should indicate these three points.

[2 marks]

iv. For the asymptotes as $K \to \infty$, the angles are given by

$$\varphi_{\mathrm{A}} = \frac{(2k+1)\pi}{n-m}, k = 0 \cdots (n - m - 1)$$

$$= \pi \text{ radians (180 degrees)}$$

as measured anti-clockwise from the real axis. Since the asymptote is along the real axis, is not necessary/meaningful to calculate the centre of gravity (the intersection of the asymptotes with the real axis). But if this is calculated, it gives

$$\sigma_{\mathrm{A}} = \frac{\sum \text{poles} - \sum \text{zeros}}{n - m}$$
$$= \frac{(+1+0) - (-1)}{1} = 0$$

[2 marks]

v. The break points on the real axis, σ_{B}, (the points where the locus departs from / arrives at the real axis is **either** found as the solutions of

$$\frac{dK}{ds} = 0$$

or as the solutions of

$$\sum_{i=1}^{n} \frac{1}{\sigma_{\mathrm{B}} - p_i} = \sum_{i=1}^{m} \frac{1}{\sigma_{\mathrm{B}} - z_i}$$

where p_i, z_i are the individual poles and zeros.

The former approach gives

$$K = \frac{-s(s-1)}{1+s}$$

Thus,

$$\frac{dK}{ds} = \frac{(1+s)(1-2s) + s(s-1)}{(1+s)^2} = 0$$

and therefore

$$s^2 + 2s - 1 = 0$$

Finally,

$$\sigma_{\mathrm{B}} = -1 \pm \sqrt{2}$$

[2 marks]

Since the locus is present on the real axis at both of these points (i.e. taking either the plus or the minus),we conclude that one is an exit point and one is an arrival point. In fact, the locus leaves at $-1 + \sqrt{2} \approx 0.41$ and arrives at $-1 - \sqrt{2} \approx -2.41$.

[1 mark]

vi. Two roots leave and arrive at the break points at an angle of $\pm 90°$. This provides the angles of the aforementioned departure and arrival.

[1 mark]

Putting all of this together, the root locus plot is as shown in Figure 1. Note that the centre part is a circle of radius $\sqrt{2}$.

e. The question is deliberately open ended. The correct answer is that this compensator means the closed-loop system is unstable for all values of K.

Some explanation needs to be given as to why. The pole of the compensator $C(s)$ cancels the zero of the system $G(s)$. Hence, the characteristic equation of the closed-loop system is

$$1 + C(s)G(s) = 1 + \frac{K}{s(s-1)}$$

Setting to zero to find the roots, $s^2 - s + K = 0$, which has roots with positive real parts for all K. Hence, the compensated closed-loop system is unstable for all K.

The same conclusion may be reached by other arguments, including sketching the compensated root locus diagram.

[3 marks]

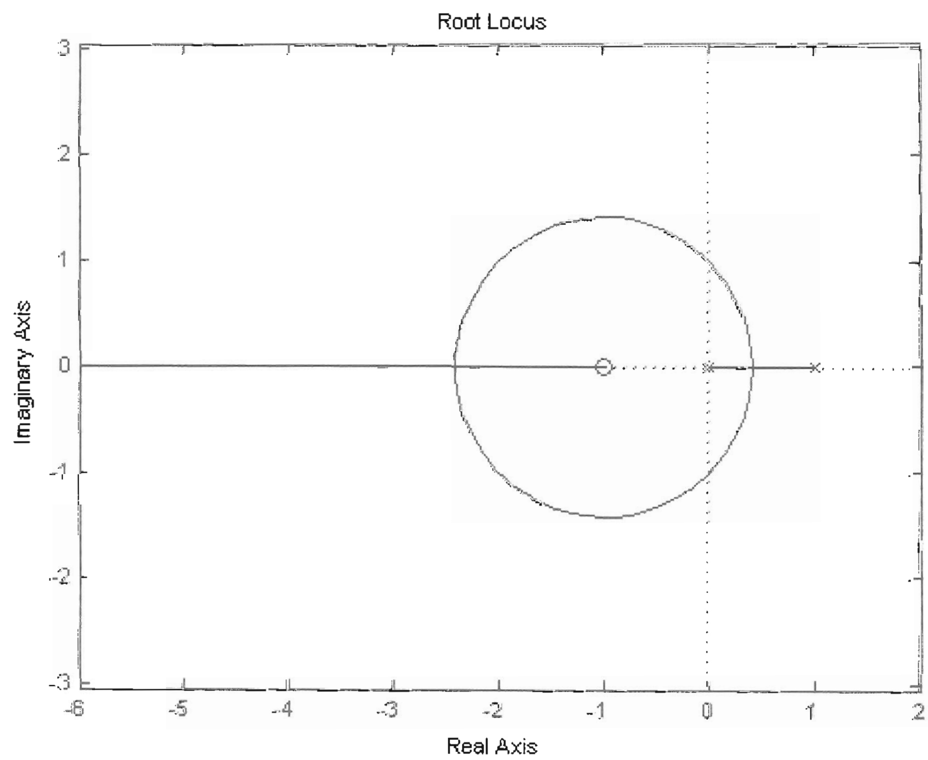

Figure 1: Root locus for Q1d.

2.

a. The following points should be clearly stated or depicted on the Bode sketch.

Magnitude

$$\begin{aligned}
\text{Gain (dB)} &= 20\log_{10}|G(j\omega)| \\
&= 20\left(\log_{10} K + \log_{10} j\omega + \log_{10}\left|\frac{1}{1+j\omega}\right| + \log_{10}\left|\frac{1}{1+0.1j\omega}\right|\right) \\
&= 20\log_{10} K - 20\log_{10}\omega - 10\log_{10}(1+\omega^2) - 10\log_{10}(1+0.01\omega^2)
\end{aligned}$$

i. The breakpoints are at $\omega_1 = 1$ rad/s and $\omega_2 = 1/0.1 = 10$ rad/s.

ii. For $\omega < \omega_1$, the slope is -20 dB/decade.
For $\omega_1 < \omega < \omega_2$, the slope is -40 dB/decade.
For $\omega > \omega_2$, the slope is -60 dB/decade.

iii. To fix the plot verticaly, The gain at $\omega = 0.01$ rad/s is approximately

$$20\log_{10} K - 20\log_{10} 0.01 = 13.98 + 40 = 53.98 \text{ dB}$$

[4 marks]

Phase

i. The gain K contributes nothing to the phase.
ii. The pole at the origin $(j\omega)$ contributes a phase of $-90°$ for all ω.
iii. The pole factor $(1 + j\omega)$ contributes a phase of $-\tan^{-1}\omega$ for all ω.
iv. The pole factor $(1 + 0.1j\omega)$ contributes a phase of $-\tan^{-1} 0.1\omega$ for all ω

Overall, the phase angle as a function of ω is

$$\arg(G(j\omega)) = -\frac{\pi}{2} - \tan^{-1}\omega - \tan^{-1} 0.1\omega$$

Using the linear approximations to the phase contributions, the phase is

- $-90°$ for all $\omega < 0.1$
- The $-\tan^{-1}\omega$ term decreases the phase at 45°/decade between $\omega = 0.1$ and $\omega = 10$.
- The $-\tan^{-1} 0.1\omega$ term decreases the phase at 45°/decade between $\omega = 1$ and $\omega = 100$.
- The net effect of these is, starting from $-90°$, a 45°/decade decrease between $\omega = 0.1$ and $\omega = 1$, to $-135°$ at $\omega = 1$; a 90°/decade decrease between $\omega = 1$ and $\omega = 10$, to to $-235°$ at $\omega = 10$; a 45°/decade decrease between $\omega = 10$ and $\omega = 100$, to $-270°$ at $\omega = 100$.

[4 marks]

Figure 2 shows the Bode plots. A plot of the asymptotes is sufficient.

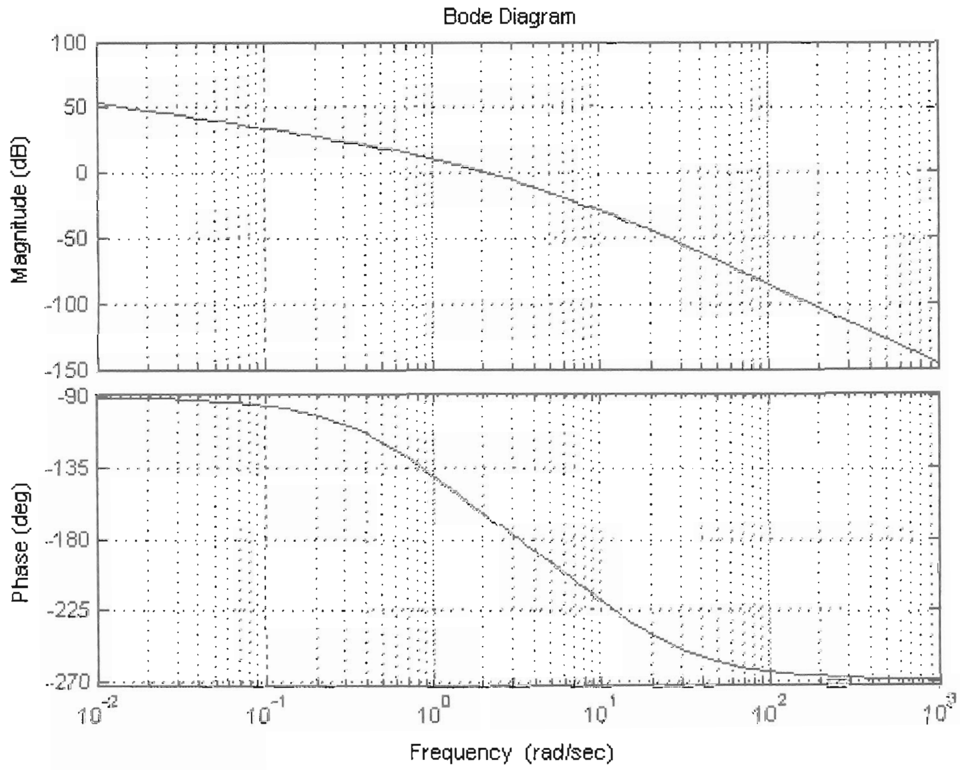

Figure 2: Bode plot for Q2a.

b. The gain margin requires determination of the system gain when the phase is $-180°$. The phase of the system is

$$\arg\big(G(j\omega)\big) = -\frac{\pi}{2} - \tan^{-1}\omega - \tan^{-1}0.1\omega$$

[2 marks]

Setting to $-180°$, and solving for ω,

$$\begin{aligned} -\pi &= -\frac{\pi}{2} - \tan^{-1}\omega - \tan^{-1}0.1\omega \\ \frac{\pi}{2} &= \tan^{-1}\omega + \tan^{-1}0.1\omega \\ \tan\frac{\pi}{2} &= \tan(\tan^{-1}\omega + \tan^{-1}0.1\omega) \end{aligned}$$

Using the relation $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$,

$$\tan\frac{\pi}{2} = \frac{\omega + 0.1\omega}{1 - 0.1\omega^2}$$

It follows that $\omega^2 = 10$, hence $\omega = \omega_c = \sqrt{10} \approx 3.16$ rad/s.

[2 marks]

At this frequency, the gain is

$$\begin{aligned} |G(j\omega_c)|\ (\text{dB}) &= 20\log_{10}K - 20\log_{10}\omega_c - 10\log_{10}(1+\omega_c^2) - 10\log_{10}(1+0.01\omega_c^2) \\ &= 20\log_{10}K - 20.83 \end{aligned}$$

For a gain margin of 10 dB, we need $|G(j\omega_c)| = -10$ dB at ω_c. Therefore,

$$20\log_{10} K = -10 + 20.83$$
$$\therefore K = 10^{\frac{10.83}{20}} = 3.48$$

[2 marks]

c. The gain of the compensator is

$$|C(j\omega)| = \frac{1}{\alpha}\frac{|1+\alpha\tau j\omega|}{|1+\tau j\omega|}$$

In dB,

$$\begin{aligned}|C(j\omega)|\,(dB) &= 20\log_{10}\left(\frac{1}{\alpha}\right) + 20\log_{10}(1+\alpha\tau j\omega) - 20\log_{10}(1+\tau j\omega)\\ &= -20\log_{10}\alpha + 10\log_{10}(1+\alpha^2\tau^2\omega^2) - 10\log_{10}(1+\tau^2\omega^2)\end{aligned}$$

At $\omega_m = \frac{1}{\tau\sqrt{\alpha}}$

$$\begin{aligned}|C(j\omega_m)|\,(dB) &= -20\log_{10}\alpha + 10\log_{10}(1+\alpha^2\tau^2\omega^2) - 10\log_{10}(1+\tau^2\omega^2)\\ &= -20\log_{10}\alpha + 10\log_{10}(1+\alpha) - 10\log_{10}\left(1+\frac{1}{\alpha}\right)\\ &= -20\log_{10}\alpha + 10\log_{10}\alpha\\ &= -10\log_{10}\alpha\end{aligned}$$

[3 marks]

Therefore, for the compensator to reduce the gain by 3 dB at ω_m,

$$-3 = -10\log_{10}\alpha$$
$$\Rightarrow \alpha = 10^{0.3} = 1.995 \approx 2$$

For τ,

$$\tau = \frac{1}{\omega_m\sqrt{\alpha}} = \frac{1}{2\sqrt{1.995}} = 0.354$$

The phase contribution of the compensator is

$$\arg C(j\omega) = \tan^{-1}\alpha\tau\omega - \tan^{-1}\tau\omega$$

At $\omega_m = \frac{1}{\tau\sqrt{\alpha}}$

$$\arg C(j\omega) = \tan^{-1}\sqrt{\alpha} - \tan^{-1}\frac{1}{\sqrt{\alpha}} = 54.7° - 35.3° = 19.4°$$

[3 marks]

3.

a. Denoting the complex current through the resistor R and inductor L as $I_i(s)$, the current through the capacitor C as $I_C(s)$ and the current through the output resistor R_o as $I_o(s)$, we have

$$I_i(s) = \frac{V_i(s) - V_o(s)}{R + sL}$$
$$I_C(s) = sCV_o(s)$$
$$I_o(s) = \frac{V_o(s)}{R_o}$$

and $I_i(s) = I_C(s) + I_o(s)$. Hence,

$$V_i(s) = V_o(s)\left[1 + sC(R + sL) + \frac{R + sL}{R_o}\right]$$

and

$$\frac{V_o(s)}{V_i(s)} = \frac{1}{LCs^2 + \left(RC + \frac{L}{R_o}\right)s + \left(1 + \frac{R}{R_o}\right)}$$
$$= \frac{\left({}^{1}/_{LC}\right)}{s^2 + \left(\frac{RC + L/R_o}{LC}\right)s + \left(\frac{1 + R/R_o}{LC}\right)}$$

The same transfer function may also be obtained by considering the circuit as a potential divider.

[6 marks]

b. The answer may be obtained in a number of ways.

For the first way, recall that the position error constant is $K_p = G(0)$. The steady-state output in response to a step of amplitude A is then simply

$$\lim_{t\to\infty} v_o(t) = AK_p$$

Here $K_p = \frac{1}{1+R/R_o}$ and $A = 0.1$, thus the output is

$$\lim_{t\to\infty} v_o(t) = \frac{0.1}{1 + R/R_o} = 0.099\text{ V}$$

Alternatively, note that capacitors and inductors only affect the transient response, so that in steady state, the circuit is just a simple potential divider, with output

$$v_o = \frac{R_o}{R_o + R}v_i = \frac{0.1}{1 + R/R_o} = 0.099\text{ V}$$

Finally, from first principles. Using the final value theorem,

$$\lim_{t\to\infty} v_o(t) = \lim_{s\to 0} sV_o(s) = \lim_{s\to 0} s\frac{V_o(s)}{V_i(s)}V_i(s)$$

For a step input of 0.1 V, $V_i(s) = \frac{0.1}{s}$. Hence,

$$\lim_{t\to\infty} v_o(t) = \lim_{s\to 0} s\frac{V_o(s)}{V_i(s)}\frac{0.1}{s} = \lim_{s\to 0} G(s) = 0.1G(0)$$

where $G(s) = V_o(s)/V_i(s)$. Therefore,

$$\lim_{t\to\infty} v_o(t) = 0.1G(0) = \frac{0.1}{1 + R/R_o}$$

If $R_o = 100R$, then

$$\lim_{t\to\infty} v_o(t) = 0.099\text{ V}$$

[4 marks]

c. Note that to find the time, t_p, at which the peak occurs, we need to differentiate the provided expression for $v_o(\omega_n t)$ with respect to $\omega_n t$ (or t) and equate to zero.

[1 marks]

Firstly, define $A = \sqrt{1-\zeta^2}$ and $\phi = \tan^{-1}\left(\frac{A}{\zeta}\right)$. Then,

$$v_o(\omega_n t) = 1 - \frac{1}{A}\exp(-\zeta\omega_n t)\sin(A\omega_n t + \phi)$$

Differentiating,

$$\frac{dv_o}{d(\omega_n t)} = \frac{-1}{A}\frac{d}{dx}(\exp(-\zeta\omega_n t)\sin(A\omega_n t + \phi))$$

$$= \frac{-1}{A}[A\exp(-\zeta\omega_n t)\cos(A\omega_n t + \phi) - \zeta\exp(-\zeta\omega_n t)\sin(A\omega_n t + \phi)]$$

Equating to zero, we have

$$A\exp(-\zeta\omega_n t)\cos(A\omega_n t + \phi) = \zeta\exp(-\zeta\omega_n t)\sin(A\omega_n t + \phi)$$

$$A\cos(A\omega_n t + \phi) = \zeta\sin(A\omega_n t + \phi)$$

and so

$$\tan\left(A\omega_n t + \tan^{-1}\left(\frac{A}{\zeta}\right)\right) = \frac{A}{\zeta}$$

Using $\tan(a+b) = \frac{\tan a + \tan b}{1 - \tan a \tan b}$, it follows that

$$\frac{\tan(A\omega_n t) + \frac{A}{\zeta}}{1 - \frac{A}{\zeta}\tan(A\omega_n t)} = \frac{A}{\zeta}$$

which holds when $\tan(A\omega_n t) = 0$, i.e., when $A\omega_n t = k\pi, k = 0,1,\ldots$

The peak occurs at the smallest time after zero, i.e. k = 1. Hence,

$$\omega_n t_p = \frac{\pi}{A} = \frac{\pi}{\sqrt{1-\zeta^2}}$$

[5 marks]

d. By comparing the transfer function obtained in part (a) with that of a standard second order system, i.e.

$$\frac{K\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

we obtain for the undamped natural frequency

$$\omega_n^2 = \frac{1 + \frac{R}{R_o}}{LC}$$

$$\therefore \omega_n = \sqrt{\frac{1 + \frac{R}{R_o}}{LC}}$$

and for the damping ratio

$$2\zeta\omega_n = \frac{RC + \frac{L}{R_o}}{LC}$$

Inserting the values provided, note that $\frac{R}{R_o} = 0.01$ and $LC = 2 \times 10^{-3}$. Hence, $\omega_n = \sqrt{505} =$ 22.47 rad/s.

[1 mark]

For 10% overshoot,

$$10 = 100 \exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Thus,

$$\frac{-\log 0.1}{\pi} = k = \frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$\therefore \zeta^2 = \frac{k^2}{k^2+1}$$

Inserting values, $k = 0.733$, hence $\zeta = 0.59$ is the smallest permissible value.

[2 marks]

Revisiting

$$2\zeta\omega_{\mathrm{n}} = \frac{RC + \frac{L}{R_o}}{LC}$$

rearranging, and substituting $R_{\mathrm{o}} = 100R$, we have

$$2\zeta\omega_{\mathrm{n}} = \frac{RC + \frac{L}{100R}}{LC}$$

$$\mathrm{R}^2 - 2\zeta\omega_{\mathrm{n}}\, LR + \frac{0.01L}{C} = 0$$

$$R^2 - 53.03R + 20 = 0$$

The solution of which gives $R = 52.7\ \Omega$.

[1 mark]

4.

a. Define $E(s) = R(s) - Y(s)$. Then

$$E(s) = \frac{R(s)}{1 + KG(s)}$$

Using the final value theorem,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\frac{R(s)}{1 + KG(s)}$$

For a step input of 1 rad/s, $R(s) = 1/s$. Thus,

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{1}{1 + KG(s)} = \frac{1}{1 + KG(0)}$$

Now, $G(0) = 2$. So for $K = 1$,

$$\lim_{t\to\infty} e(t) = \frac{1}{1+2} = \frac{1}{3}$$

The percentage steady-state error at unity gain is therefore 33%.

[2 marks]

To obtain an error within 1%, require

$$0.01 > \frac{1}{1 + 2K}$$

Therefore, $K > 49.5$.

[1 mark]

b. The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}$$

Inserting the transfer function provided,

$$\frac{Y(s)}{R(s)} = \frac{2K}{0.5s + (2K + 1)}$$

For a step input of 1 rad/s, $R(s) = \frac{1}{s}$. Thus,

$$\begin{aligned} Y(s) &= \frac{\frac{2K}{s}}{0.5s + (2K + 1)} \\ &= \frac{4K}{s\big(s + (4K + 2)\big)} \\ &= \frac{400}{s(s + 402)} \end{aligned}$$

[2 marks]

Using partial fractions,

$$Y(s) = \frac{A}{s + 402} + \frac{B}{s}$$

Solving for A and B, we find that $B = \frac{400}{402} = 0.995$ and $A = -B = -0.995$. Taking the inverse transform,

$$\begin{aligned} y(t) &= 0.995L^{-1}\left\{\frac{1}{s} - \frac{1}{s + 402}\right\} \\ &= 0.995(1 - \mathrm{e}^{-402t}) \end{aligned}$$

[3 marks]

The units of $y(t)$ are rad/s. For the angular acceleration, we differentiate.

$$\frac{dy}{dt} = 400\mathrm{e}^{-402\mathrm{t}}$$

Initially, then, the angular acceleration is 400 rad/s/s.

[2 marks]

Suitability of the controller: to minimize steady-state error, a large value of K must be chosen. However, this makes a very fast, and probably unrealizable, time response. For larger step inputs the problem is exacerbated.

[1 mark]

c. The proportional-plus-integral controller permits a zero steady-state error in response to a step input without the need for a high proportional gain. However, too much integral gain introduces excessive oscillation and overshoot into the closed-loop step response. The most satisfactory time response can be obtained by tuning K_p and K_i, (or introducing a derivative part).

[2 marks]

d. The open-loop continuous-time transfer function is

$$C(s)G(s) = \left(\frac{K_\mathrm{p}s + K_\mathrm{i}}{s}\right)\left(\frac{2}{1+0.5s}\right) = \frac{12s+4}{s(2+s)}$$

using $K_\mathrm{p} = 3$ and $K_\mathrm{i} = 1$. The zero-order hold has the transfer function

$$G_0(s) = \frac{1-e^{-sT}}{s}$$

Thus,

$$F(s) = G_0(s)C(s)G(s) = (1-e^{-sT})\frac{12s+4}{s^2(2+s)}$$

[2 marks]

Taking partial fractions,

$$F(s) = (1-e^{-sT})\frac{12s+4}{s^2(2+s)} = (1-e^{-sT})\left(\frac{2}{s^2}+\frac{5}{s}-\frac{5}{s+2}\right)$$

Taking z-transforms,

$$\begin{aligned} F(z) &= (1-z^{-1}) \times Z\left(\frac{2}{s^2}+\frac{5}{s}-\frac{5}{s+2}\right) \\ &= \left(\frac{z-1}{z}\right)\left(\frac{2Tz}{(z-1)^2}+\frac{5z}{z-1}-\frac{5z}{z-e^{-2T}}\right) \end{aligned}$$

[2 marks]

Inserting $T = 0.5$, and rearranging,

$$\begin{aligned} F(z) &= \frac{(z-e^{-1}) + 5(z-1)(z-e^{-1}) - 5(z-1)^2}{(z-1)(z-e^{-1})} \\ &= \frac{(6-5e^{-1})z + (4e^{-1}-5)}{z^2-(1+e^{-1})z+e^{-1}} \\ &= \frac{4.161z - 3.528}{z^2 - 1.368z + 0.3679} \end{aligned}$$

[3 marks]