The University Of Sheffield.

Data Provided: List of useful formulae at the end of paper

DEPARTMENT OF ELECTRONIC AND ELECTRICAL ENGINEERING

Autumn Semester 2011-12 (2.0 hours)

EEE201 Signals and Systems

Answer **THREE** questions. **No marks will be awarded for solutions to a fourth question.** Solutions will be considered in the order that they are presented in the answer book. Trial answers will be ignored if they are clearly crossed out. **The numbers given after each section of a question indicate the relative weighting of that section.**

1. **a.** The input signal $x(t)$ and the output signal $y(t)$ of a Linear Time Invariant (LTI) system are described by the differential equation

$$\frac{dy(t)}{dt} + 5y(t) = 5x(t)$$

The initial conditions are $y(0) = 1$ and $x(t) = u(t)$. Determine the time domain forced and natural responses for this LTI system. **(8)**

b. Consider a system described by the transfer function

$$H(s) = \frac{1}{(s+0.1)(s^2+18s+45)}$$

i) Determine the poles and zeros.

ii) Identify the dominant pole and sketch the magnitude spectrum for $s > 0$. **(7)**

c. The transfer function of a causal system is given by

$$H(s) = \frac{3s-1}{(s^2+s-6)}.$$

Verify whether the system is stable. **(5)**

2.

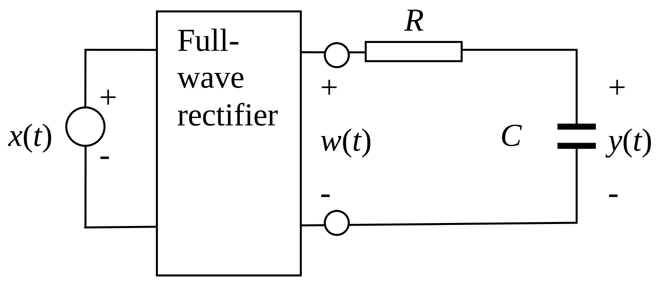

Figure 2.1

A simple dc power supply is shown in figure 2.1. The output of the fullwave rectifier is given by $w(t) = |x(t)|$ while the output signal of the circuit, in the frequency domain, is described by

$Y(\omega) = \left(\frac{4}{\pi} \sum_{k=-\infty}^{N} \left(\frac{-1^k}{(1-4k^2)} \right) \left(\frac{1}{1+j100k\pi RC} \right) \delta(\omega - 100k\pi) \right)$, where k is an integer, R is the resistance and C is the capacitance.

i) For an input signal $x(t) = \cos(100\pi t)$, derive $W(\omega)$, the frequency domain expression for the output of the fullwave rectifier. **(6)**

ii) If the 2nd (i.e k = 2) and higher harmonics are negligible, show that the time domain output is given by $y(t) = \frac{2}{\pi^2} + \frac{2}{3\pi^2}\left[\frac{e^{j100\pi t}}{1+j100\pi RC} + \frac{e^{-j100\pi t}}{1-j100\pi RC} \right]$. **(7)**

iii) Find a suitable value for the time constant RC such that the ripple in $y(t)$ is less than 1% of its average value. **(7)**

3. **a.** Consider a continuous time signal $x(t)$ with the magnitude spectrum shown below.

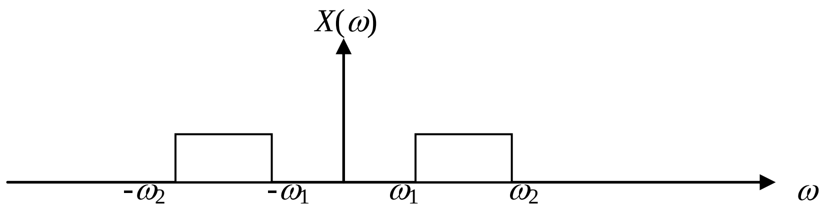

Figure 3.1

i) Based on the Nyquist Theorem, state the sampling interval, T_s, required to avoid aliasing. **(2)**

ii) Assuming that $\omega_1 > \omega_2 - \omega_1$, work out the maximum sampling interval such that it is still possible to reconstruct $x(t)$ perfectly. **(6)**

b.

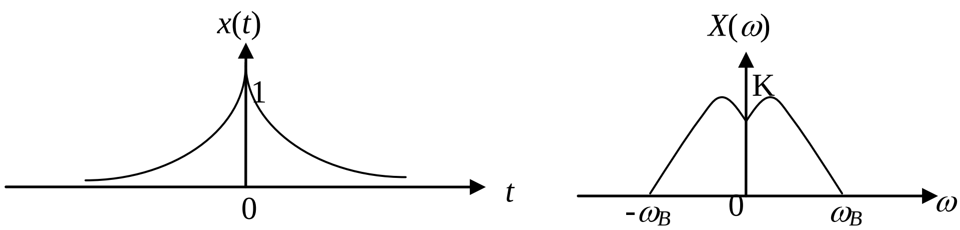

Figure 3.2

i) The signal $x(t)$, shown figure 3.2, is multiplied by a sampling function $p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ to obtain $x_s(t)$, the sampled version of $x(t)$. Sketch and label $x_s(t)$. **(2)**

ii) The Fourier Transform of the sampling function is given by $P(\omega) = \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$, where ω_s is the sampling frequency in rad/s. Sketch and label $TX_s(\omega)$ if

(a) $\omega_s < 2\omega_B$

and

(b) $\omega_s > 2\omega_B$.

State whether the spectrum of $x(t)$ can be recovered using a low pass filter in each case and describe the aliasing effect. **(6)**

3. c.

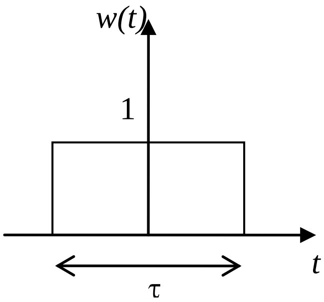

Figure 3.3

Show that the Fourier Transform $W(\omega)$ of the rectangular pulse $w(t)$ shown in figure 3.3 is given by $W(\omega) = \tau \dfrac{\sin(\omega\tau/2)}{(\omega\tau/2)}$. **(4)**

4. **a.** The response of a Linear Time Invariant (LTI) system, $y(t)$, is shown in figure 4.1 when subjected to the input signal $x(t)$. Derive $y_1(t)$, the response of this LTI system when the input signal $x_1(t)$ is shown in figure 4.2.

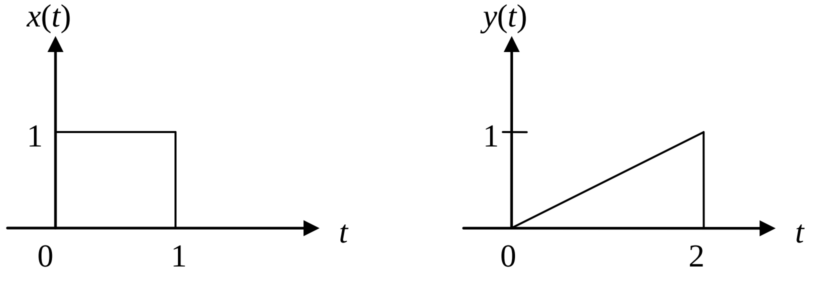

Figure 4.1

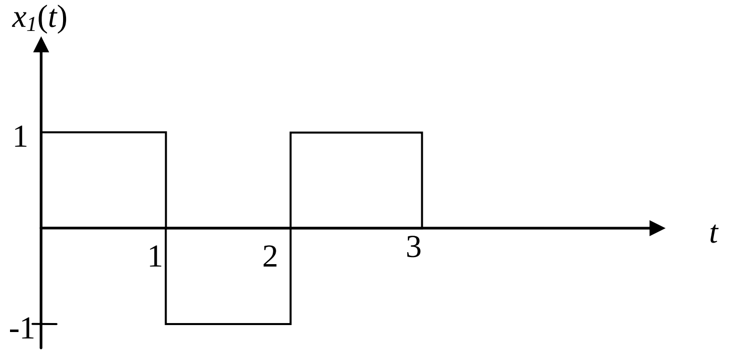

Figure 4.2 **(5)**

b. Compute the response of an LTI discrete system if the input and impulse response are described by $x[n]=\begin{cases}1, & 0\le n\le 3\\ 0, & otherwise\end{cases}$ and $h[n]=\begin{cases}e^{-n}, & 0\le n\le 4\\ 0, & otherwise\end{cases}$, respectively. **(6)**

c.

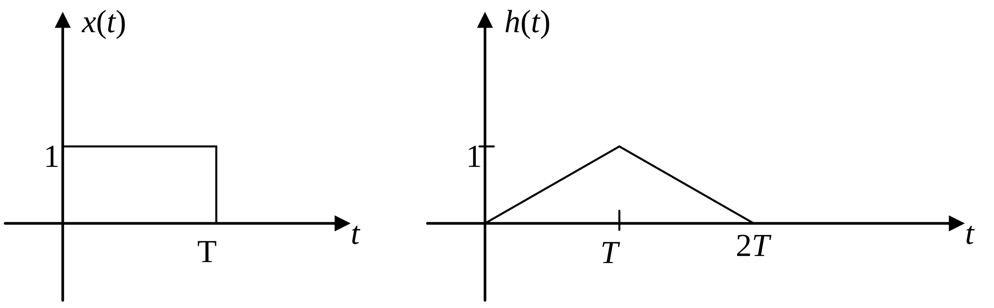

Figure 4.3

The response of an LTI system is given by the convolution of the continuous-time signals $x(t)$ and $h(t)$ shown in figure 4.3. Derive the analytical expression of $y(t)$. **(9)**

CHT

List of useful formulae

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda$$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \qquad x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

$$c_0 = \frac{1}{T}\int_{<T>} x(t)dt \qquad c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_o t}dt$$

$$a_n = 2\,\mathrm{Re}[c_n] = \frac{2}{T}\int_{<T>} x(t)\cos n\omega_0 t dt \qquad b_n = -2\,\mathrm{Im}[c_n] = \frac{2}{T}\int_{<T>} x(t)\sin n\omega_0 t dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \qquad X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$X(\omega) = 2\int_{0}^{\infty} x(t)\cos \omega t dt \qquad X(\omega) = -j2\int_{0}^{\infty} x(t)\sin \omega t dt$$

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt \qquad x(t) = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} X(s)e^{st}dt$$

$$\cos(x)\cos(y) = \frac{1}{2}\left[\cos(x-y) + \cos(x+y)\right] \qquad \sin(x)\sin(y) = \frac{1}{2}\left[\cos(x-y) - \cos(x+y)\right]$$

$$\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x-y) + \sin(x+y)\right] \quad \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y)$$

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

$$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j} \qquad \cos(x) = \frac{e^{jx} + e^{-jx}}{2j}$$

Fourier Transform Pairs

Signal	Fourier Transfrom
$\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$	$2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_o)$
$e^{j\omega_o t}$	$2\pi\delta(\omega - \omega_o)$
$\cos\omega_o t$	$\pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$
$\sin\omega_o t$	$j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_o)$	$e^{-j\omega t_o}$
$e^{-at}u(t), a > 0$	$\frac{1}{a + j\omega}$
$x(t) = \begin{cases} 1, & \lvert t \rvert < \tau \\ 0, & \lvert t \rvert > \tau \end{cases}$	$\frac{2\sin\omega\tau}{\omega} = 2\tau \sin c(\omega\tau)$
$\frac{\sin\omega_c t}{\pi t} = \frac{\omega_c}{\pi} \sin c(\omega_c t)$	$X(\omega) = \begin{cases} 1, & \lvert\omega\rvert < \omega_c \\ 0, & \lvert\omega\rvert > \omega_c \end{cases}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$

Properties of Fourier Transform

Property	Aperiodic signal, $x(t)$	Fourier Transfrom, $X(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time Shifting	$x(t - t_o)$	$e^{-j\omega t_o} X(\omega)$
Frequency Shifting	$e^{j\omega_o t} x(t)$	$X(\omega - \omega_o)$
Time Scaling	$x(at)$	$\frac{1}{a} X\left(\frac{\omega}{a}\right)$
Differentiation in Time	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{dX(\omega)}{d\omega}$
Integration in time	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Convolution	$x(t)*h(t)$	$X(\omega).H(\omega)$
Multiplication in time	$x(t).h(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)H(\omega-\lambda)d\lambda$
Parseval's Theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	
Duality	$x(t) \leftrightarrow X(\omega)$	
	$X(t) \leftrightarrow 2\pi x(-\omega)$	

<u>Laplace Transform pairs</u>

Signal	Transform
Unit step: $u(t)$	$\dfrac{1}{s}$
Unit impulse: $\delta(t)$	1
Unit ramp: $tu(t)$	$\dfrac{1}{s^2}$
$e^{-at}u(t)$	$\dfrac{1}{s+a}$
$t^n e^{-at}u(t)$	$\dfrac{n!}{(s+a)^{n+1}}$
$(\cos\omega_o t)u(t)$	$\dfrac{s}{\left(s^2+\omega_o{}^2\right)}$
$(\sin\omega_o t)u(t)$	$\dfrac{\omega_o}{\left(s^2+\omega_o{}^2\right)}$
$(e^{-at}\cos\omega_o t)u(t)$	$\dfrac{s+a}{\left((s+a)^2+\omega_o{}^2\right)}$
$(e^{-at}\sin\omega_o t)u(t)$	$\dfrac{\omega_o}{\left((s+a)^2+\omega_o{}^2\right)}$
$(t\cos\omega_o t)u(t)$	$\dfrac{s^2-\omega_o{}^2}{\left(s^2+\omega_o{}^2\right)^2}$
$(t\sin\omega_o t)u(t)$	$\dfrac{2\omega_o s}{\left(s^2+\omega_o{}^2\right)^2}$

Properties of Laplace Transform

Property	Transform Property
Linearity	$ax_1(t) + bx_2(t) \leftrightarrow aX_1(s) + bX_2(s)$.
Time shift	$x(t-t_o)\,u(t-t_o) \leftrightarrow X(s)e^{-st_o}$, $t_o > 0$
Multiplication by a complex exponential	$x(t)e^{s_o t} \leftrightarrow X(s-s_o)$
Time scaling	$x(at) \leftrightarrow X(s/a)/\lvert a\rvert$
Differentiation in time domain	$\dfrac{dx(t)}{dt} \leftrightarrow sX(s) - x(0)$
	$\dfrac{d^2x(t)}{dt^2} \leftrightarrow s^2X(s) - sx(0) - \left.\dfrac{dx(t)}{dt}\right\rvert_{t=0}$
Differentiation in *s* domain	$t^n x(t) \leftrightarrow \dfrac{d^n X(s)}{ds^n}(-1)^n$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \dfrac{1}{s}X(s)$
Convolution in time domain	$x(t)*h(t) \leftrightarrow X(s).H(s)$
Initial value theorem	$x(0) = \lim_{s\to\infty} sX(s)$
Final value theorem	$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)$

(if $x(t)$ has a finite value as $t \to \infty$)

Unit step response for 2nd order systems

Damping factor, ζ	Unit step response
>1	$y(t) = \dfrac{k}{p_1 p_2} + k_2 e^{p_1 t}.u(t) + k_3 e^{p_2 t}.u(t)$
1	$y(t) = \dfrac{k}{\omega_n^{\ 2}}\left(1 - (1 + \omega_n t)e^{-\omega_n t}.u(t)\right)$
$0 < \zeta < 1$	$y(t) = \dfrac{k}{\omega_n^{\ 2}}\left(1 - \dfrac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \theta).u(t)\right)$
0	$y(t) = \dfrac{k}{\omega_n^{\ 2}}\left(1 - \cos(\omega_n t).u(t)\right)$