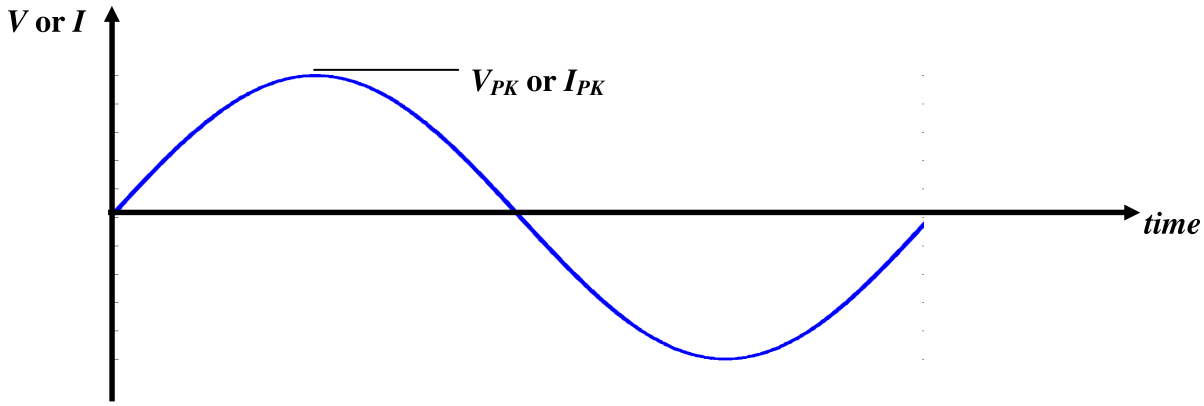
Power Dissipation in an AC Circuit

For a d.c. circuit (e.g. a battery supply) the power dissipated, P, in a resistance, R, is:

$$P = I^2 R = \frac{V^2}{R} = VI$$

For an a.c. circuit, however, how can power be defined since the voltage and current are continuously varying with time?

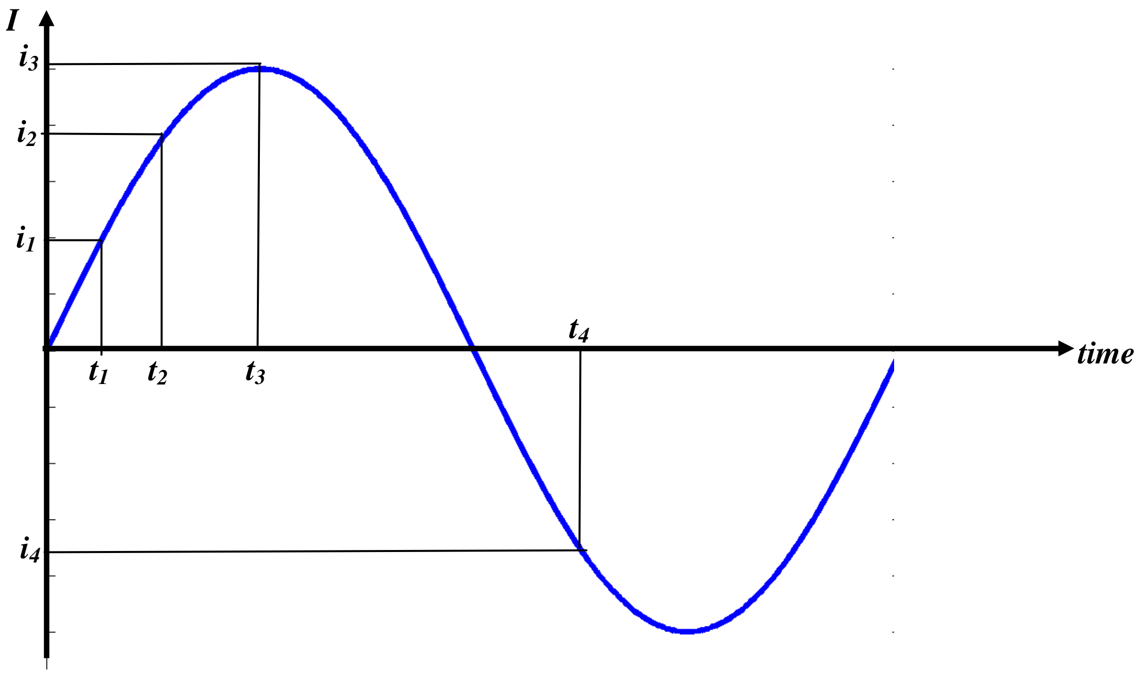


For a sine wave current:

$$I(t) = I_{PK} \sin \omega t = I_{PK} \sin 2\pi f t$$

i.e. the MEAN current is zero. Clearly, however, an a.c. current passing through a resistance does dissipate power (e.g. an electric fire). A value known as the "Root-Mean-Square" value or RMS value is used to specify a.c. voltages and currents. This defines a level of alternating current that has the same heating effect in a load as a d.c. current with the same numerical value. I.e. if an a.c. current has a value of $3A_{rms}$ this has the same heating effect as a 3A d.c. current flowing through the same resistive load.

Consider an a.c. current waveform:



At any instant in time the instantaneous power dissipated is equal to the square of the current times the resistance, R, e.g.:

At t_1 $\quad P_1 = i_1^2 R \qquad$ At t_2 $\quad P_2 = i_2^2 R \qquad$ At t_3 $\quad P_3 = i_3^2 R \qquad$ At t_4 $\quad P_4 = i_4^2 R$

So the average power can be written as:

$$P = \frac{i_1^2 R + i_2^2 R + i_3^2 R + i_4^2 R + \cdots + i_n^2 R}{n}$$

Now suppose I is the value of the d.c. current to produce the same heating effect then:

$$I^2 R = \frac{R\left(i_1^2 + i_2^2 + i_3^2 + i_4^2 + \cdots + i_n^2\right)}{n}$$

therefore:

$$I = \sqrt{\frac{\left(i_1^2 + i_2^2 + i_3^2 + i_4^2 + \cdots + i_n^2\right)}{n}}$$

which is the square **Root** of the **Mean** of the **Squares**.

Formally:

$$I_{rms} = \sqrt{\frac{\int_0^t i(t)^2\, dt}{\int_0^t dt}}$$

For the usual case of a sinusoidal waveform then if:

$$i(t) = I_{PK} \sin \omega t$$

then:

$$I_{rms} = \frac{I_{PK}}{\sqrt{2}}$$

<u>Example</u>
What is the peak value of a sinusoidal current waveform which has the same heating effect as a 3A d.c. current?

$$I_{DC} = 3A = I_{rms}$$
$$\therefore I_{PK} = \sqrt{2} \times I_{rms} = \sqrt{2} \times 3 = 4.24A$$

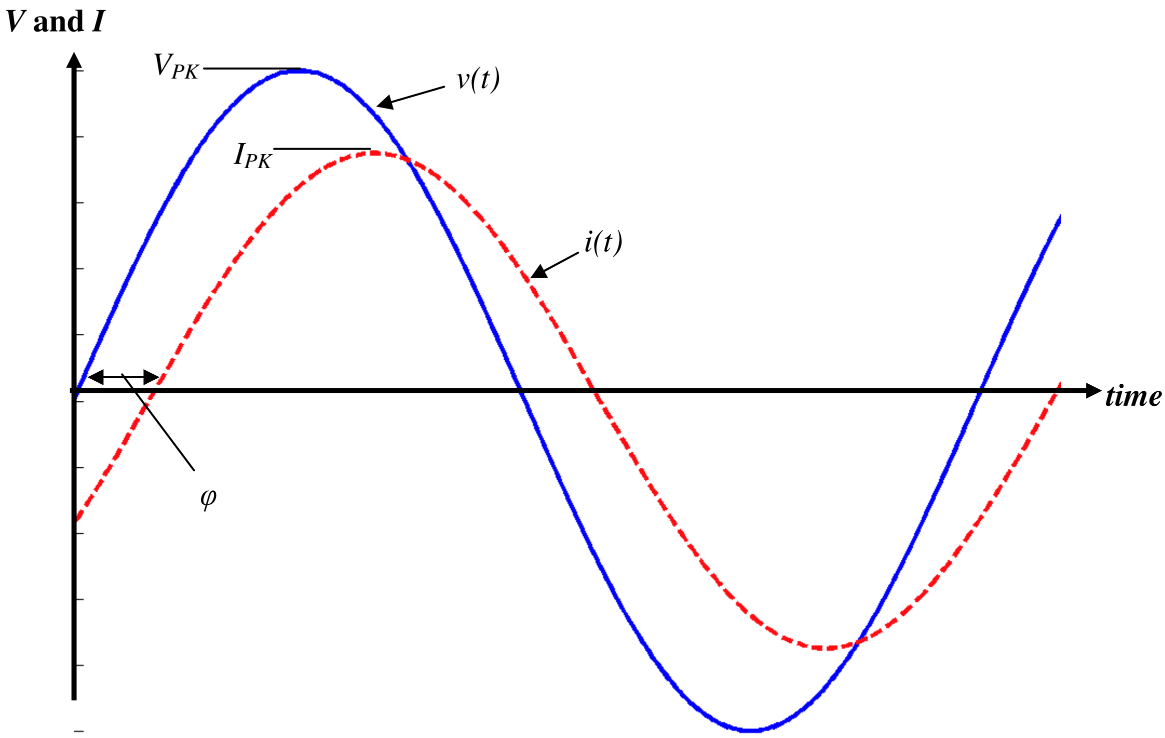
The same method can be applied to voltage. The mains electricity supply in the UK is $230V_{rms}$ and hence the peak voltage is:

$$V_{PK} = \sqrt{2} \times V_{rms} = \sqrt{2} \times 230 = 325V$$

Previously we have seen that there are three aspects to an a.c. waveform:

- Amplitude
- Frequency
- Phase

Let us now look at a general case of power dissipation in an a.c. circuit. Consider the case where the current lags the voltage by a phase angle, φ.



We have also seen that the power is the product of the voltage and the current. So for sinusoidal voltage and current waveforms shown in the figure above we can write:

$$v(t) = V_{PK}\sin(\omega t)$$
$$i(t) = I_{PK}\sin(\omega t - \varphi)$$

$$P = v(t)\times i(t) = V_{PK}\sin(\omega t)\times I_{PK}\sin(\omega t - \varphi)$$

$$P = \frac{1}{2}V_{PK}I_{PK}\cos(\varphi) \quad + \quad \frac{1}{2}V_{PK}I_{PK}\cos(2\omega t - \varphi)$$

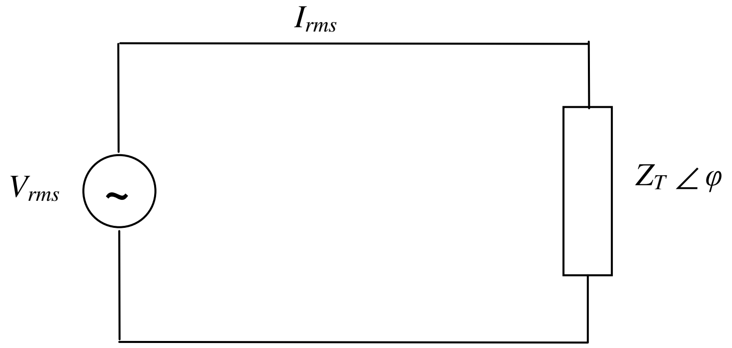
Clearly the first term in the above expression is independent of ω and t whereas the second term has an average value of zero. Thus the average power dissipated in the circuit will be:

$$P_{AVE} = \frac{1}{2}V_{pk}I_{PK}\cos(\varphi) = \frac{V_{pk}}{\sqrt{2}}\frac{I_{PK}}{\sqrt{2}}\cos(\varphi) = V_{rms}I_{rms}\cos(\varphi)$$

$\cos(\varphi)$ is called the **Power-Factor**.

For a purely resistive circuit, the current is in-phase with the voltage and $\varphi = 0$ so $\cos(\varphi) = 1$.
For a purely inductive or capacitive circuit $\varphi = \pm 90°$ so $\cos(\varphi) = 0$. i.e. there is no power dissipation in a purely inductive or purely capacitive circuit.

For a general case:



$$Z_T = R + jX$$

$$Z_T = |Z_T| \times \cos(\varphi) + j|Z_T| \times \sin(\varphi)$$

$$|Z_T| = \sqrt{R^2 + X^2} \qquad \text{and} \qquad \varphi = \tan^{-1}\frac{X}{R}$$

The impedance forces a phase difference between v and i and determines the magnitude of I_{rms}.

$$I_{rms} = \frac{V_{rms}\angle 0^\circ}{Z_T \angle \phi}$$

therefore:

$$|I_{rms}| = \frac{|V_{rms}|}{|Z_T|}$$

Power dissipated:

$$P = V_{rms} I_{rms} \cos(\varphi)$$

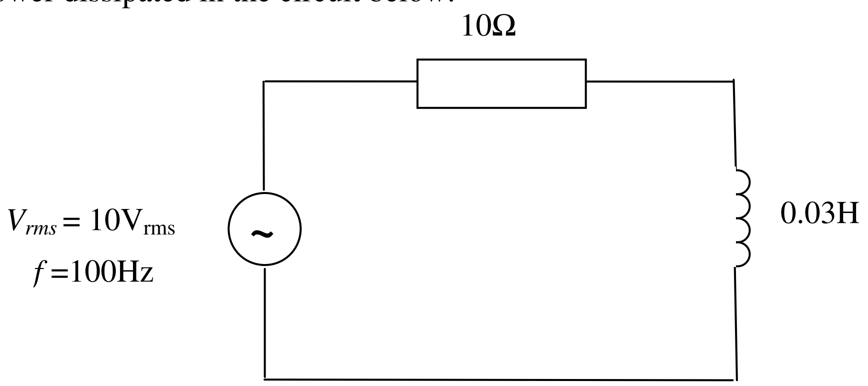
This is also equal to the power dissipated in the real (resistive) part of the impedance.

$$P = I_{rms}^2 R \qquad \text{or} \qquad P = I_{Rrms}^2 \operatorname{Re}(Z_T) \qquad \textbf{\underline{BUT NOT}} \qquad I_{rms}^2 Z_T$$

Use Z_T to find I_{rms} then power is equal to $P = I_{rms}^2 R$

<u>Example</u>

Find the power dissipated in the circuit below:



$$Z_T = R + j\omega L = R + j2\pi f L = 10 + j2\pi 100 \times 0.03 = 10 + \text{j } 18.85\ \Omega$$

$$|Z_T| = \sqrt{R^2 + (\omega L)^2} = \sqrt{10^2 + 18.85^2} = 21.34\ \Omega$$

$$\varphi = \tan^{-1}\left(\frac{\omega L}{R}\right) = \tan^{-1}\left(\frac{18.85}{10}\right) = 62^\circ$$

So the total impedance can be written as:

$$Z_T = 10 + \text{j } 18.85 = 21.34 \angle 62^\circ\ \Omega$$

Hence:

$$\left|I_{rms}\right| = \frac{\left|V_{rms}\right|}{\left|Z_T\right|} = \frac{10}{21.34} = 0.469\ \mathrm{A_{rms}}$$

And the power may be calculated from:

$$P = V_{rms} I_{rms} \cos(\varphi) = 10 \times 0.469 \times \cos 62^\circ = \mathbf{2.2\ W}$$

Alternatively:

$$P = I_{rms}^2\ \mathrm{Re}(Z_T) = 0.469^2 \times 10 = \mathbf{2.2\ W}$$

So we have shown that in this *RL* circuit example a current of 0.469 A_{rms} flows from the 10 V_{rms} supply to produce 2.2 W of power in our load. If the circuit had been purely resistive (i.e. cos $\varphi = 1$) then to produce 2.2 W of power the current would only have needed to be 0.22 A_{rms}. Thus to transfer the same amount of useful power (2.2 W) the cables (and switches etc.) need only to be rated to carry 0.22 A_{rms} and not 0.469 A_{rms}.

This makes little difference in this example where we are dealing with quite small currents, but for much larger loads and currents (e.g. industrial loads) this is very important. Electricity suppliers implement tariff structures to encourage users to operate at or near unity power-factor (cos $\varphi = 1$).

Power-factor correction, as it is known, is achieved by using a capacitor to cancel the effect of an inductor (or vice-versa in the case of capacitive loads). Usually the capacitors (inductors) are added in parallel so as not to alter the voltage across the load. When we have an inductive circuit (current lagging behind the voltage) we say that we have a lagging power-factor. For a capacitive circuit where the current leads the voltage we say we have a leading power-factor.

VA Rating

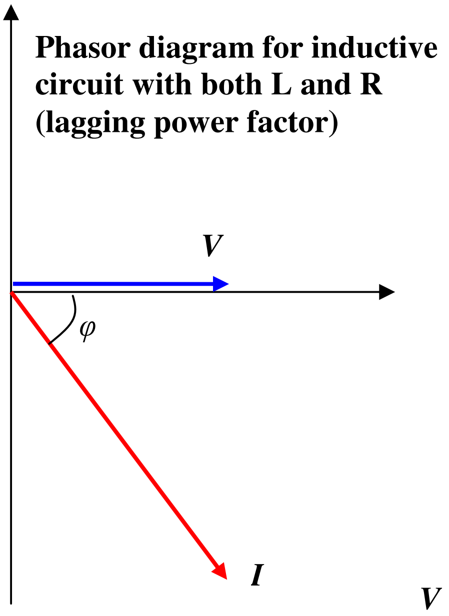
Often large equipment for industrial use is rated in VA (Volt-Amps) or kVA (kilo Volt-Amps).

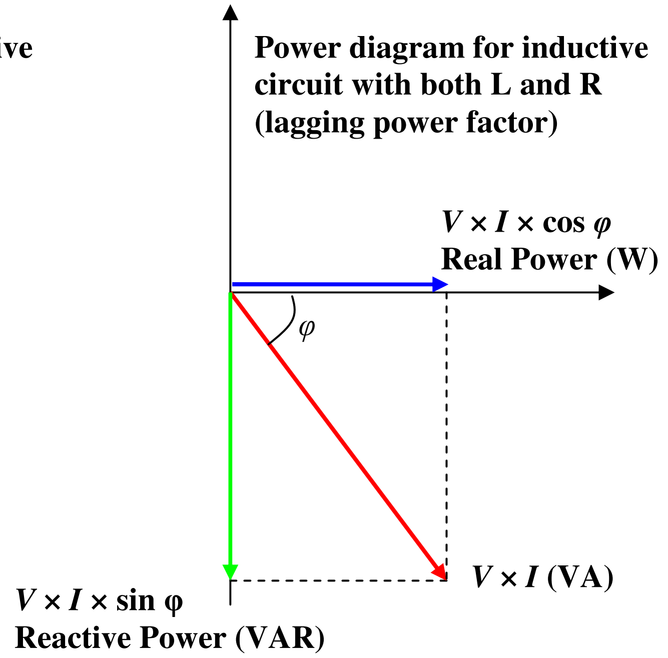
For example, a generator may be rated to produce (say) 2000A_{rms} at a voltage of 400V_{rms}. These are maximum limits on the current and voltage the machine can produce without overheating of the insulation breaking down. Such a generator would hence be rated as a 2000 × 400 = 800 000 VA or 800 kVA generator.

The phase angle between the voltage and the current will depend on the load the generator is supplying and not on the generator itself.

At a p.f. of 1 the generator supplies 800 kW of power to the load
At a p.f. of 0.5 the generator supplies 400 kW of power to the load even though it is operating at 800 kVA.

I.e. it will only transmit half the power although the magnitude of the current, and hence temperature rise of the generator is the same. In a similar way as we have drawn phasor diagrams for voltages and currents, we can draw a phasor diagram for power and introduce the concept of imaginary, or reactive power.





Reactive power is a measure of how much energy is flowing between our source (generator) and load which is not being dissipated. It has the same units as real power but we use the terminology Volt-Amps-Reactive (or VARs) to distinguish it from the real power.

The above example is for an inductive circuit where the current lags behind the voltage. Adding some capacitance would have the effect of reducing the length of the reactive power phasor and hence reduce the phase angle. When performing power calculations it is vitally important that we consider real and imaginary parts separately – we cannot simply add VAs together – as we will see in the next example.

<u>Example</u>

A large industrial company has the following loads connected to an a.c. supply:

(a) Process heaters rated at 15kW
(b) A motor load of 40kVA at 0.6 power-factor lagging
(c) A load of 20kW at 0.8 power factor-lagging

Calculate the load from the supply in kW, kVA and the overall power factor. Find the kVAr rating of a parallel capacitor required to correct the overall power factor to unity.

The first thing to do is look at each load in turn and calculate the real power in kW, the reactive power in kVAR (if applicable) and the total kVA.

(a) The heaters are purely resistive and will operate at a unity power factor, therefore the real power is 15kW, they draw no reactive power and the kVA rating is simply:

$$Power = V_{rms} I_{rms} \times \cos\varphi \quad \text{therefore} \quad V_{rms} I_{rms} = \frac{Power}{\cos\phi} = \frac{15}{1} = \mathbf{15kVA}$$

(b) For the motor operating at p.f. = 0.6, since it draws 40kVA, the real power is given by:

$$Power = V_{rms} I_{rms} \times \cos\varphi = 40 \times 0.6 = \mathbf{24kW}$$

and the reactive (or imaginary power) is:

$$Reactive\ power = V_{rms} I_{rms} \times \sin\varphi = 40 \times \sin(\cos^{-1}(0.6)) = 40 \times 0.8 = \mathbf{32kVAR}$$

(Real power = $V_{rms} I_{rms} \cos\varphi$ and Imaginary (reactive) power is $V_{rms} I_{rms} \sin\varphi$)

(c) For the 20kW load at 0.8pf lagging, obviously the real power is 20kW as this is given in the question. The total VA can be found from:

$$V_{rms} I_{rms} = \frac{Power}{\cos\varphi} = \frac{20}{0.8} = \mathbf{25\ kVA}$$

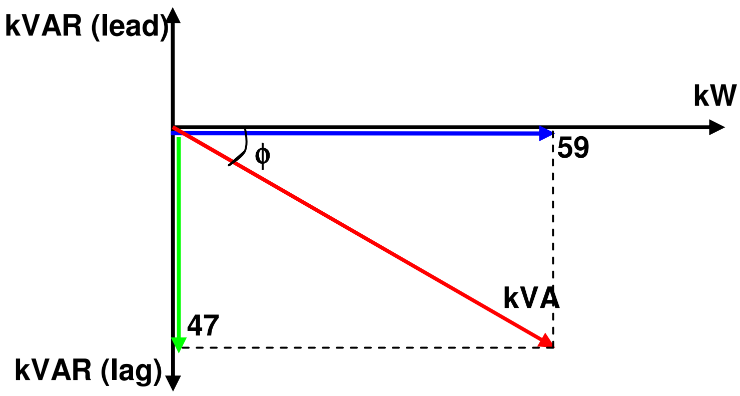
and the reactive power is found from:

$$Reactive\ \ power = V_{rms} I_{rms} \times \sin\varphi = 25 \times \sin(\cos^{-1}(0.8)) = 40 \times 0.6 = \mathbf{15kVAR}$$

These can be best displayed in the form of a table:

	kVA	**Real Power kW**	**React. Power kVAR**
Heater	15	15	0
Motor (0.6 lag)	40	24	32
Load (0.8 lag)	25	20	15
Total	**Do Not Sum!**	**59**	**47**

Note that because we are dealing with phasors we cannot simply add up the kVAs, instead we need to draw a phasor diagram:



From the table the total real power is **59kW** and the total reactive power is **47kVAR.**

The total kVA is thus:

$$kVA\ rating = \sqrt{real\ \ power^2 + reactive\ power^2} = \sqrt{59^2 + 47^2} = \mathbf{75.5kVA}$$

The phase angle is given by:

$$\phi = \tan^{-1}\frac{Imag}{Real} = \frac{47}{59} = \mathbf{38.5}°$$

and the power factor is:

$$power - factor = \cos\varphi = \cos 38.5 = \mathbf{0.78}$$

Alternatively we could have obtained the power factor from the total kVA and the total real power since:

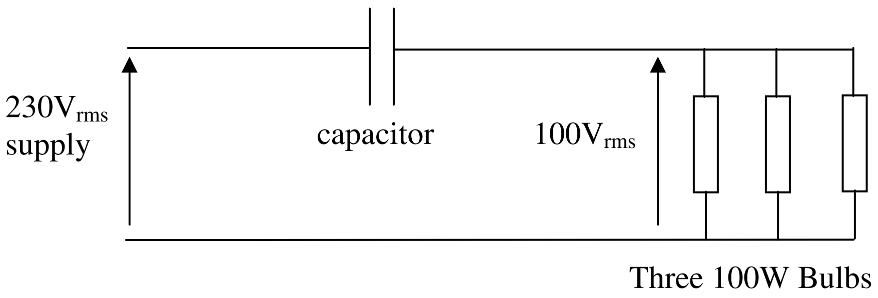
$$Power = V_{rms} I_{rms} \times \cos\varphi \quad \text{so} \quad \cos\varphi = \frac{Power}{V_{rms} I_{rms}} = \frac{59}{75.5} = \mathbf{0.78}$$

To correct the overall system to have a unity power factor we need to effectively cancel out 47kVAR lagging by adding 47kVAR leading. This can be done by adding a parallel **capacitor of rating 47 kVAR leading**. If we had knowledge of the voltage and frequency we could calculate the actual value of the capacitor. Power-factor correction capacitors or inductors are normally added in parallel with the load so as not to affect the voltage across the load itself as can be illustrated in the following example.

This next example will show how a series capacitor may be used to drop the supply voltage to power three lower voltage light bulbs. Later in the question a parallel inductor is used to correct the power-factor to unity. (Alternatively we could use a series inductor to obtain the required bulb voltage, but then we would need to use a parallel capacitor to provide power-factor correction).

Example

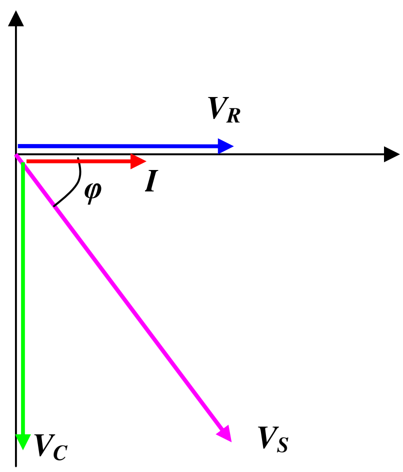
Three 100V$_{rms}$, 100W light bulbs, which may be assumed to be pure resistances, are connected in parallel and are to be used with a 230V$_{rms}$, 50Hz supply. A series capacitor is included in the circuit to reduce the voltage across the bulbs from 230V$_{rms}$ to the 100V$_{rms}$ required as shown in the figure below. Calculate the current required to operate the bulbs at their rated voltage and power, and the voltage across the series capacitor. Hence find the value of capacitor required, the power-factor of the circuit, and the real and reactive power drawn from the supply.



Since there are three 100W bulbs the total power is 300W. Since the bulbs are pure resistances then the current through the bulbs is in phase with the voltage, V_R, across them, hence:

$$I = \frac{P}{V_R} = \frac{300}{100} = \mathbf{3A_{rms}}$$

Now the same current must flow through the series capacitor, but the voltage across the capacitor, V_C, will lag the current by 90° (CIVIL), so we can now sketch the phasor diagram:



The supply voltage (V_S = 230V$_{rms}$) will also lag the current by the phase angle, φ, hence:

$$V_S^2 = V_R^2 + V_C^2$$

So:

$$V_C = \sqrt{V_S^2 - V_R^2} = \sqrt{230^2 - 100^2} = 207\ \mathrm{V_{rms}}$$

The power-factor is given by:

$$p.f. = \cos\varphi = \frac{V_R}{V_S} = \frac{100}{230} = \mathbf{0.435}$$

The actual value of the series capacitor can now be found, since:

$$V_C = I \times X_C = I \times \frac{1}{2\pi f C}$$

then:

$$C = \frac{I}{2\pi f V_C} = \frac{3}{2\pi \times 50 \times 207} = \mathbf{46\mu F}$$

The real power can be calculated from:

$$P = VI\cos\varphi = 230 \times 3 \times 0.435 = \mathbf{300\ W} \text{ (as expected!)}$$

and the reactive power is:

$$Q = VI\sin\varphi = 230 \times 3 \times \sqrt{\left(1 - 0.435^2\right)} = \mathbf{621\ VAR}$$

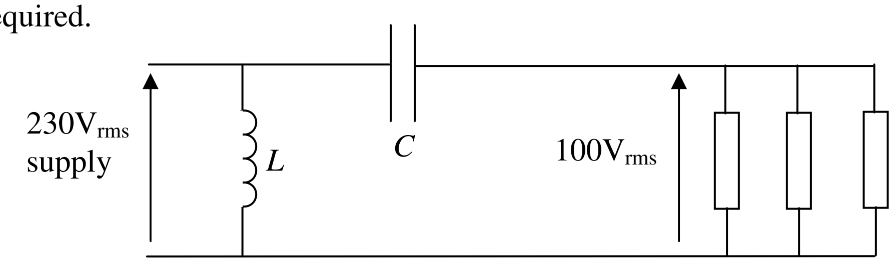
Alternatively find the total resistance of the three bulbs and reactance of the capacitor:

$$R_{BULBS} = \frac{V_R}{I} = \frac{100}{3} = 33.3\Omega \quad \text{and} \quad X_C = \frac{V_C}{I} = \frac{207}{3} = 69\Omega$$

Then:

$$P = I^2 R_{BULBS} = 3^2 \times 33.3 = 300\ \text{W} \quad \text{and} \quad Q = I^2 X_C = 3^2 \times 69 = 621\ \text{VAR}$$

In order to correct the overall power-factor of the circuit to unity an inductor, L, is connected in parallel across the input to the circuit as shown in the figure below. Calculate the value of the inductor required.



Three 100W Bulbs

No real power is taken by the inductor so the overall power taken from the supply remains at 300W. However, to correct the overall power-factor of the circuit to unity the inductor must cancel out the 621VAR from the capacitor. Since the voltage across the inductor is 230V$_{rms}$ then:

$$Q_L = I_L^2 X_L = V_L I_L = \frac{V_L^2}{X_L} \quad \text{or} \quad X_L = \frac{V_L^2}{Q_L} = \frac{230^2}{621} = 85.2\Omega$$

Hence:

$$L = \frac{X_L}{2\pi f} = \frac{85.2}{2\pi \times 50} = 0.271\text{H}$$

Check by calculating the currents in each branch. For the branch containing the capacitor and bulbs:

$$I_{BULBS} = 3\angle\cos^{-1}(0.435) = 3\angle 64.2^\circ = 1.306 + j2.7 \ \ \text{A}_{rms}$$

and for the branch containing the inductor:

$$I_L = \frac{V_L}{X_L} = \frac{230\angle 0^\circ}{85.2\angle 90^\circ} = 2.7\angle -90^\circ = -j2.7 \ \ \text{A}_{rms}$$

Total current is therefore:

$$1.306 + j2.7 - j2.7 = 1.306\ \text{A}_{rms} \qquad (230 \times 1.306 \times 1 = 300\text{W})$$

If we had used a series inductor instead of a capacitor its inductance would have been 0.22H. We would then have had to use a parallel capacitor of value 37μF to counteract the effect of the series inductor. (Verify this for yourselves).