FOURIER SERIES REPRESENTATION OF PERIODIC SIGNALS

Consider a CT signal given by

$$x(t) = A_1 \cos t + A_2 \cos(3t + \pi/4) + A_3 \cos(5t + \pi/2),\ -\infty < t < \infty.$$

This signal consists of three components with amplitudes of A_1, A_2 and A_3, frequencies of 1, 3 and 5 rad/s and phases of 0, π/4 and π/2 rad respectively.

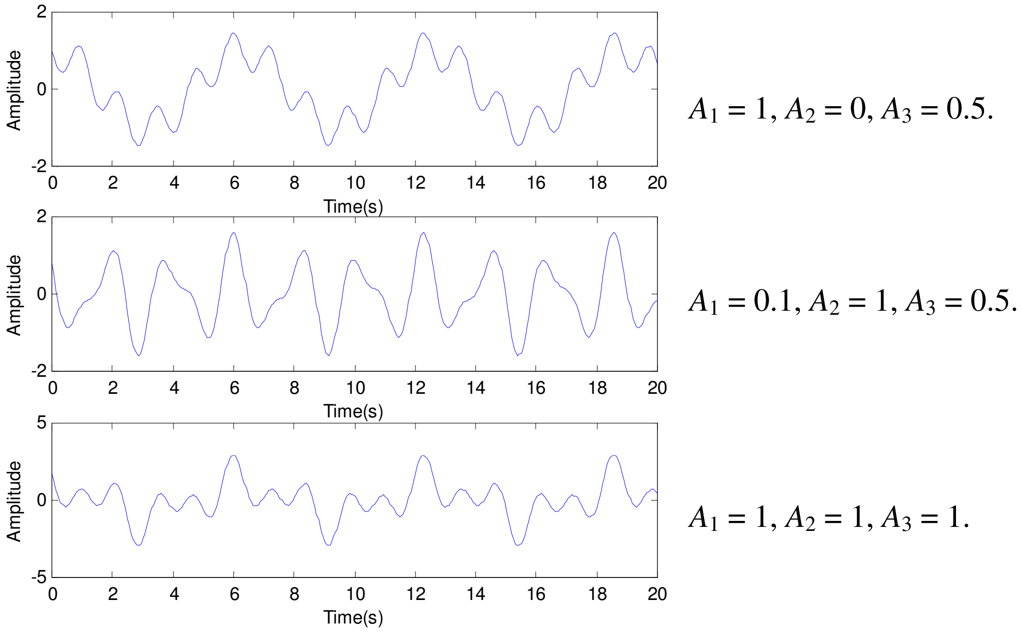


Figure 1: Plot of x(t) with different combinations of A_1, A_2 and A_3.

Figure 1 shows that the frequency of $x(t)$ is 1 rad/s, independent of the amplitudes of each component. This frequency is the same as that of the fundamental sinusoid, the component with the lowest frequency. The sinusoids with frequencies of 3 and 5 rad/s are known as the harmonics, defined as sinusoids with frequencies of integer multiple of the fundamental. The phases of the sinusoids can also alter the shape of $x(t)$ as shown in figure 2 when the amplitudes are $A_1 = 1$, $A_2 = A_3 = 0.5$, the frequencies are 1, 3 and 5 rad/s and the phases are p_1, p_2, and p_3.

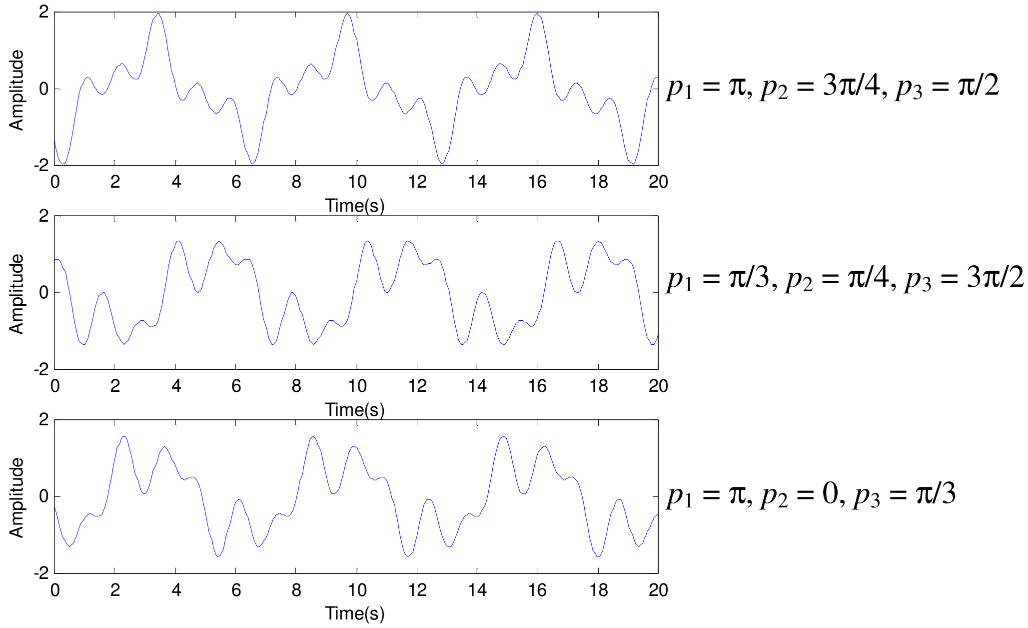


Figure 2: Plot of x(t) with different combinations of p_1, p_2 and p_3.

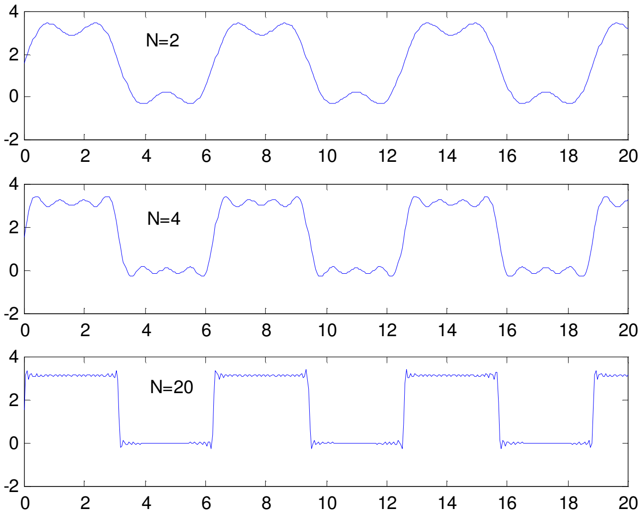
In fact we can express a CT periodic signal, $x(t)$ as a sum of sinusoids

$$x(t) = \sum_{k=1}^{N} A_k \sin(\omega_k t + \theta_k),$$

where N is a positive integer, A_k is the amplitude, ω_k is the frequency in rad/s and θ_k is the phase angle. This is the ***Fourier Series*** representation of the periodic signal $x(t)$. We can approximate any periodic signal by using the Fourier Series and the converse is true, any periodic signal may be broken down into a series of sinusoidal components that are harmonically related.

Examples:

1. The square waveform shown below can be represented as $x(t) = \dfrac{\pi}{2} + \sum_{n=0}^{N} \dfrac{2}{2n+1} \sin\big((2n+1)t\big)$.



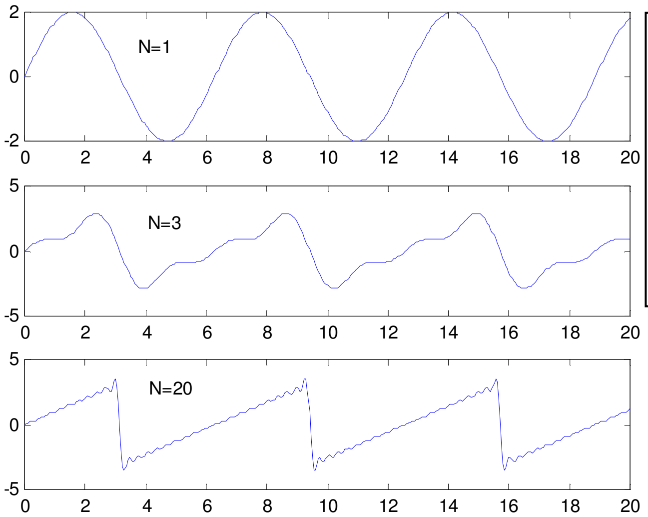

```
%generate squarewaves with diff no. Of
%harmonics
t=0:20/400:20;
H=1;    w0=1;  x=0;
for N=[0:1:H]
   x1=sin((2*N+1)*w0*t)/(2*N+1);
   x=x+x1;
end;
y=pi/2+2*x;

H2=3;   x=0;
for N=[0:1:H2]
    x1=sin((2*N+1)*w0*t)/(2*N+1);
   x=x+x1;
end;
y2=pi/2+2*x;

H3=19;   x=0;
for N=[0:1:H3]
    x1=sin((2*N+1)*w0*t)/(2*N+1);
   x=x+x1;
end;
y3=pi/2+2*x;

subplot(3,1,1),plot(t,y);
subplot(3,1,2),plot(t,y2);
subplot(3,1,3),plot(t,y3);
```

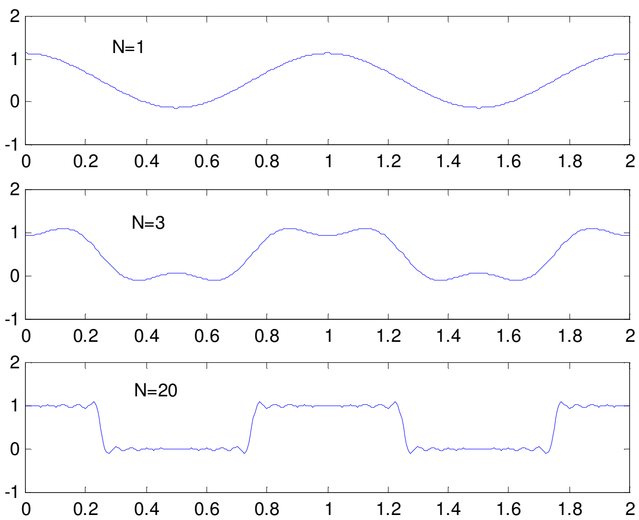
2. The sawtooth waveform shown below can be represented as $x(t) = \sum_{n=1}^{N} \dfrac{2}{n}(-1)^{n+1} \sin(nt)$.




```
function generate_sawtooth(H)
%generate sawtooth
%number of harmonics = H
t=-9:20/400:10;
w0=1;
x=0;
for N=[1:1:H]
   x1=(2*power(-1,N+1)/N)*sin(N*w0*t);
   x=x+x1;
end;
y=x;

plot(t,y);
xlabel ('t(s)'), ylabel('y(t)');
```

3. The square waveform shown below can also be approximated by $x(t)=\frac{1}{2}+\sum_{n=1}^{N}\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\cos(2\pi t)=\frac{1}{2}+\sum_{n=1}^{N}\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\left(e^{j2n\pi t}+e^{-j2n\pi t}\right)$.




```
%generate squarewaves
f=125;
fs=16000;
t=-199/fs:1/fs:200/fs;
H=1;    %number of harmonics
w0=2*pi*f;
x=0;

for N=[1:1:H]
   an=sin(N*w0*0.25)/(N*pi);
   x=x+an*(exp(j*N*w0*t)+exp(-j*N*w0*t));
end;
y=1/2+x;

H2=3;
x=0;
for N=[1:1:H2]
   an=sin(N*w0*0.25)/(N*pi);
   x=x+an*(exp(j*N*w0*t)+exp(-j*N*w0*t));
   end;
y2=1/2+x;

H3=20;
x=0;
for N=[1:1:H3]
   an=sin(N*w0*0.25)/(N*pi);
   x=x+an*(exp(j*N*w0*t)+exp(-j*N*w0*t));
end;
y3=1/2+x;

subplot(3,1,1),plot(t,y);
subplot(3,1,2),plot(t,y2);
subplot(3,1,3),plot(t,y3);
```

Note that the periodic square waveform can also be represented as sum of complex exponentials since $\sin(\omega_o t)=\frac{e^{j\omega_o t}-e^{-j\omega_o t}}{2j}$ and $\cos(\omega_o t)=\frac{e^{j\omega_o t}+e^{-j\omega_o t}}{2}$.

Determination of the Fourier Series representation of a CT periodic signal

Consider a periodic signal that can be represented by

$$x(t)=\sum_{k=-\infty}^{\infty}|c_k|e^{j(k\omega_o t+\angle c_k)} \quad ,$$

where ω_o is the fundamental frequency of a periodic exponential and c_k is the Complex Fourier Series Coefficients. Assuming that $|c_k| = c_k$ and $\angle c_k = 0$.
Multiplying both sides by $e^{-jn\omega ot}$ gives

$$x(t)e^{-jn\omega_o t} = \sum_{k=-\infty}^{\infty}c_k e^{j(k-n)\omega_o t} .$$

Integrating both sides from 0 to $T = \omega_o/2\pi$, we have

$$\int_0^T x(t)e^{-jn\omega_o t}dt = \int_0^T \sum_{k=-\infty}^{\infty}c_k e^{j(k-n)\omega_o t}dt .$$

Interchanging the order of integration and summation, we have

$$\int_0^T x(t)e^{-jn\omega_o t}dt = \sum_{k=-\infty}^{\infty} c_k \int_0^T e^{j(k-n)\omega_o t}dt \ .$$

We know that $e^{j(k-n)\omega_o t} = \cos(k-n)\omega_o t + \mathrm{j}\sin(k-n)\omega_o t$.

For $k \neq n$, $\int_0^T e^{j(k-n)\omega_o t}dt = 0$ since $\int_0^T \cos(k-n)\omega_o t dt = 0$ and $\int_0^T \sin(k-n)\omega_o t dt = 0$ and therefore $c_k \int_0^T e^{j(k-n)\omega_o t}dt = 0$.

For $k = n$, we have $c_n \int_0^T e^{j(n-n)\omega_o t}dt = c_n T$ since $\int_0^T e^0 dt = T$.

We can therefore write $\int_0^T x(t)e^{-jn\omega_o t}dt = c_n T$ and consequently the Complex Fourier Series Coefficients can be determined as

$$c_n = \frac{1}{T}\int_{<T>} x(t)e^{-jn\omega_o t}dt \ ,$$

where $\int_{<T>}$ denotes integration over an interval of length T. The coefficients c_n is a amplitude of the n^{th} harmonic of the periodic signal $x(t)$. Sometimes, there is a constant or d.c component in the signal $x(t)$ given by

$$c_0 = \frac{1}{T}\int_{<T>} x(t)dt \ .$$

As shown in example 3, the square waveform can be expressed as a sum of sinusoids or complex exponentials. We can replace $x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$ with

$$x(t) = a_0 + \sum_{n=1}^{\infty}[a_n \cos\omega_n t + b_n \sin\omega_n t] \ ,$$

where $a_0 = c_0 = \frac{1}{T}\int_{<T>} x(t)dt$ is the d.c term,

$$a_n = 2\,\mathrm{Re}[c_n] = \frac{2}{T}\int_{<T>} x(t)\cos n\omega_0 t dt \ ,$$

$$b_n = -2\,\mathrm{Im}[c_n] = \frac{2}{T}\int_{<T>} x(t)\sin n\omega_0 t dt \ .$$

If $x(t)$ is an even function $b_n = 0$. If $x(t)$ is an odd function $a_o = 0$ and $a_n = 0$.

Examples:

Consider the periodic square wave $x(t)$ shown in figure 3. Find the Fourier Series coefficients for $x(t)$.

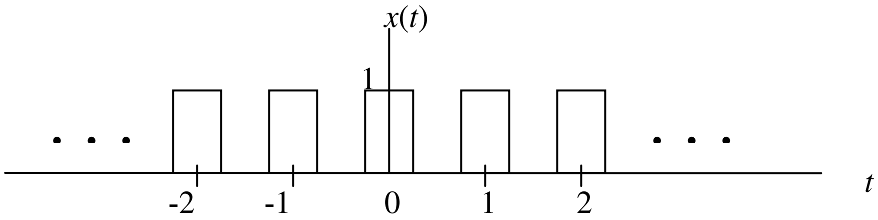


Figure 3: Periodic square wave.

Conditions for the existence of Fourier Series

A CT periodic signal $x(t)$ can be decomposed into Fourier Series if it satisfies the ***Dirichlet conditions*** (p.197, Oppenheim):

1. $x(t)$ is absolutely integrable over any period.

$$\int_{<T>} |x(t)| dt < \infty ,$$

which ensures that the Fourier Series coefficients will be finite since

$$|c_n| = \frac{1}{T} \int_{<T>} |x(t) e^{-jn\omega_o t}| dt = \frac{1}{T} \int_{<T>} |x(t)| dt .$$

So, if $\int_{<T>} |x(t)| dt < \infty$, $|c_n| < \infty$.

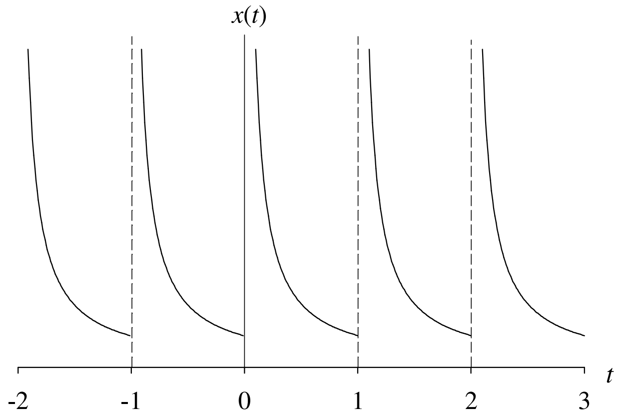


Figure 4: x(t) = 1/t, 0 < t ≤ 1 is not absolutely integrable.

2. $x(t)$ has a finite number of maxima and minima over any period.

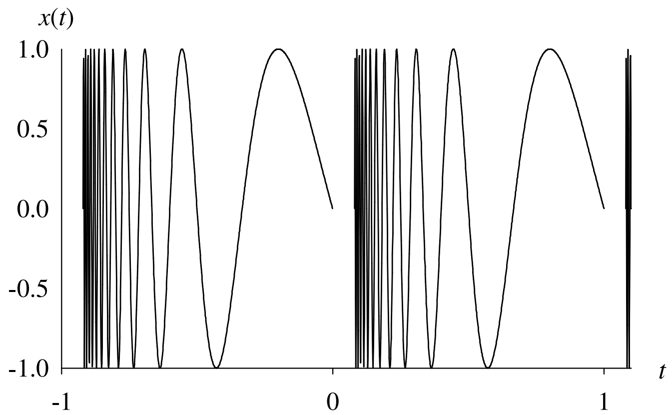


Figure 5: x(t) = sin (2π/t), 0 < t ≤ 1, is absolutely integrable but has an infinite number of maxima and minima.

3. $x(t)$ has a finite number of discontinuities over any period.

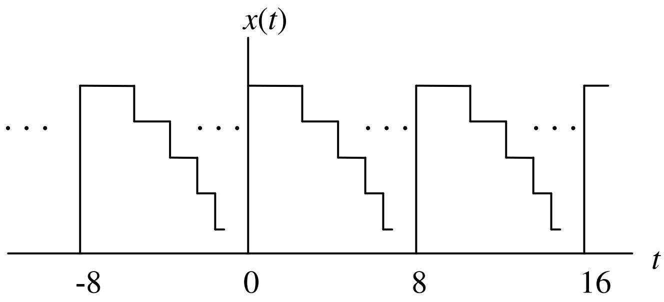


Figure 6: A periodic signal with period of 8 and an infinite number of discontinuities.

<u>Gibbs phenomenon</u>

We have learned that a periodic signal that meets the Dirichlet conditions can be decomposed into Fourier Series. We have also shown that the Fourier Series representation improves with higher number of harmonically related components.

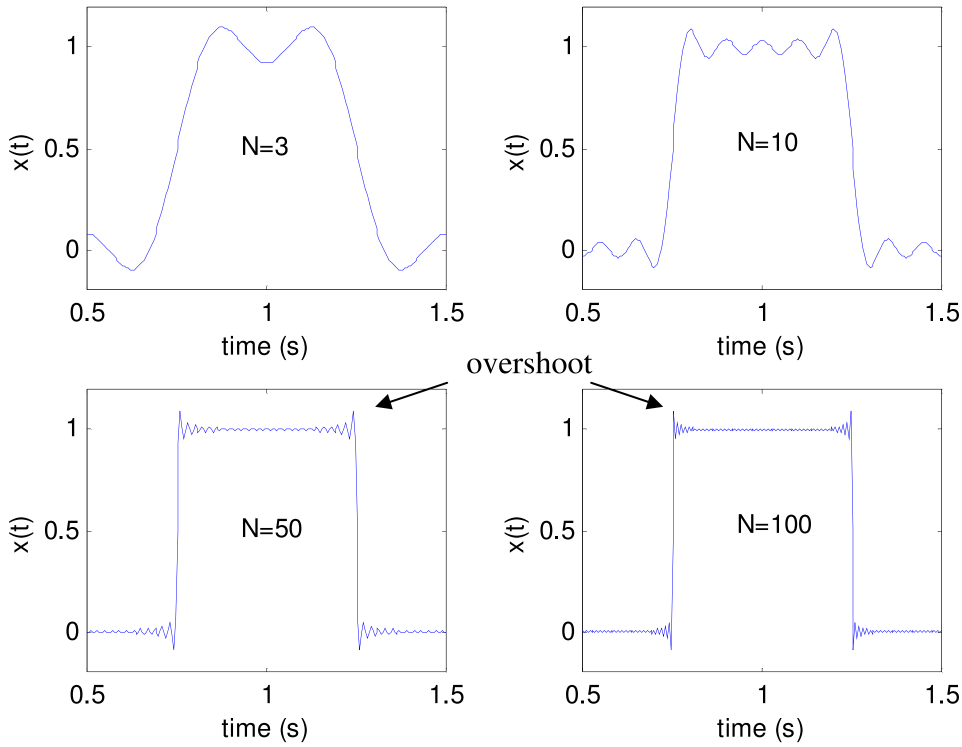


Figure 7: Convergence of the Fourier Series representation of a square wave.

Figure 7 shows that there are ripples in the Fourier Series representation of a square wave shown in figure 3. These ripples become smaller as the number of components N in the Fourier Series representation increases. However there is an overshoot of 9% at the discontinuity independent of N. This behaviour is known as Gibbs phenomenon. The implication is that the Fourier Series representation of a discontinuous signal, such as the square wave, will in general exhibits high-frequency ripples and overshoot near the discontinuity. It is therefore necessary to use sufficiently large value of N if such approximation is used in practice, so that the total energy in the ripples is insignificant.

Example: low pass filter

Consider a RC low pass filter shown in figure 8.

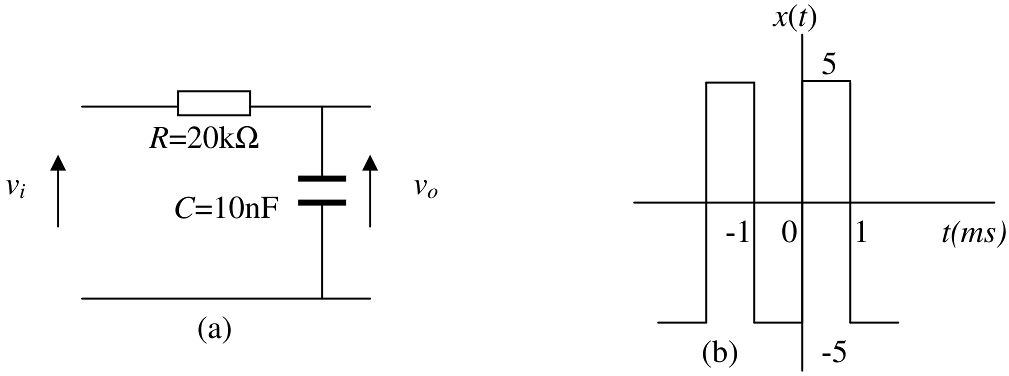


Figure 8: Low pass filter.

We can show that this RC circuit is a low pass filter by analysing the response of the circuit to the harmonics of a periodic signal. Consider an input signal shown in figure 8(b).

No d.c component since $x(t)$ is an odd function, $a_o = 0$ and $a_n = 0$.

Let $D_n = \dfrac{amplitude \quad of \quad nth \quad harmonic}{amplitude \quad of \quad the \quad fundamental}$.

Before filtering $D_1 = 1$, $D_3 = 1/3$, $D_5 = 1/5$,……$D_n = 1/n$.

$\omega_c = 1/RC = 5000$ rad/s and $\omega_o = 2\pi/(2\text{ms}) = 1000\pi$.

After filtering:

At $\omega = \omega_o, H(\omega) = \dfrac{1}{1 + j\left(\dfrac{1000\pi}{5000}\right)}$ $= 0.85\angle\text{-}32^o$.

The amplitude of the fundamental is reduced to $0.85 \times 20/\pi = 17/\pi$ (15% reduction).

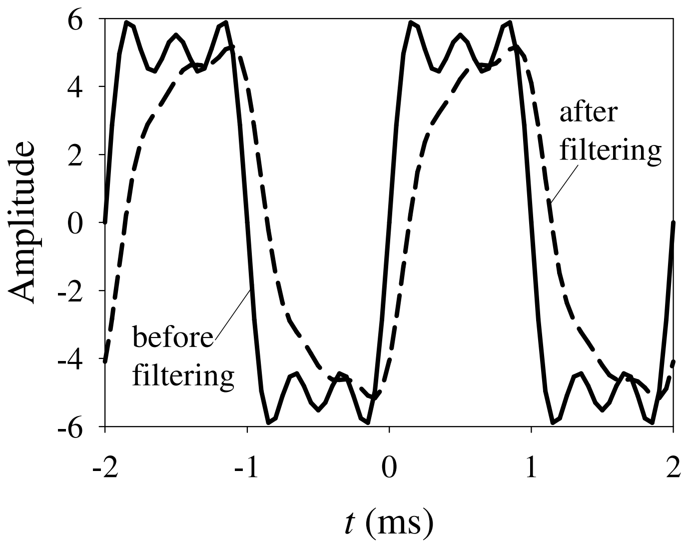
At $\omega = 3\omega_o, H(\omega) = \dfrac{1}{1 + j\left(\dfrac{3000\pi}{5000}\right)}$ $= 0.47\angle\text{-}62^o$.

The amplitude of the 3rd harmonic is reduced to $0.47 \times 20/3\pi = 9.4/3\pi$ (53% reduction). $D_3 = (9.4/3\pi)/(17/\pi) = 0.184$.

At $\omega = 5\omega_o, H(\omega) = \dfrac{1}{1 + j\left(\dfrac{5000\pi}{5000}\right)}$ $= 0.3\angle\text{-}72.3^o$.

The amplitude of the 5^{th} harmonic is reduced to $0.3 \times 20/5\pi = 6/5\pi$ (70% reduction). $D_5 = (6/5\pi)/(17/\pi) = 0.07$.

The signal $x(t)$, approximated by including up to the 5^{th} harmonics, before and after filtering are shown below.



Clearly, the higher frequency harmonics are attenuated more significantly than the fundamental showing that the RC circuit in figure 8 behaves like a low pass filter. We find that the harmonic components can be considered separately and the overall response can then be obtained by adding the individual response of each harmonic. For instance the rms value can be obtained from

$(\text{rms}_{\text{tot}})^2 = (\text{rms}_1)^2 + (\text{rms}_2)^2 + (\text{rms}_3)^2$………..

exercise: Obtain the rms voltage of the signal after filtering in the above example.

The response will contain the same frequency components as the input but the amplitude and the phase of each harmonic will be modified differently according to the frequency response of the system.

Parseval's theorem

Consider a voltage waveform $x(t)$ applied to a 1Ω resistor. The current flowing through the resistor is $x(t)$ and the power is $x(t)^2$. The total energy supplied by x(t) from -∞ to ∞ is

$$E = \int_{-\infty}^{\infty} x(t)^2 \, dt ,$$

where $x(t)$ is assumed to be real. If $x(t)$ is complex the total energy is given by

$$E = \int_{-\infty}^{\infty} x(t) x^*(t) dt = \int_{-\infty}^{\infty} |x(t)|^2 \, dt .$$

This equation shows that every periodic signal has $E = \infty$. It is therefore more meaningful to compute the average power over one period as

$$P_{av} = \frac{1}{T} \int_0^T |x(t)|^2 \, dt = \sum_{n=-\infty}^{\infty} |c_n|^2 .$$

The average power can be calculated if the complex Fourier Series coefficients are known.

Example:
Obtain the average power in the signal shown in figure 3, within the frequency range [-7πrad/s, 7πrad/s].
From figure 3, $\omega_o = 2\pi$. Therefore only c_m with $m = \pm 1$, 0 and ±3 exist within the frequency range [-7πrad/s, 7πrad/s].

$c_o = 1/2$

$|c_1| = |c_{-1}| = 1/\pi$

$|c_3| = |c_{-3}| = 1/3\pi$

The average power within this frequency range is

$$\sum_{-3}^{3} |c_n|^2 =$$

Notes: