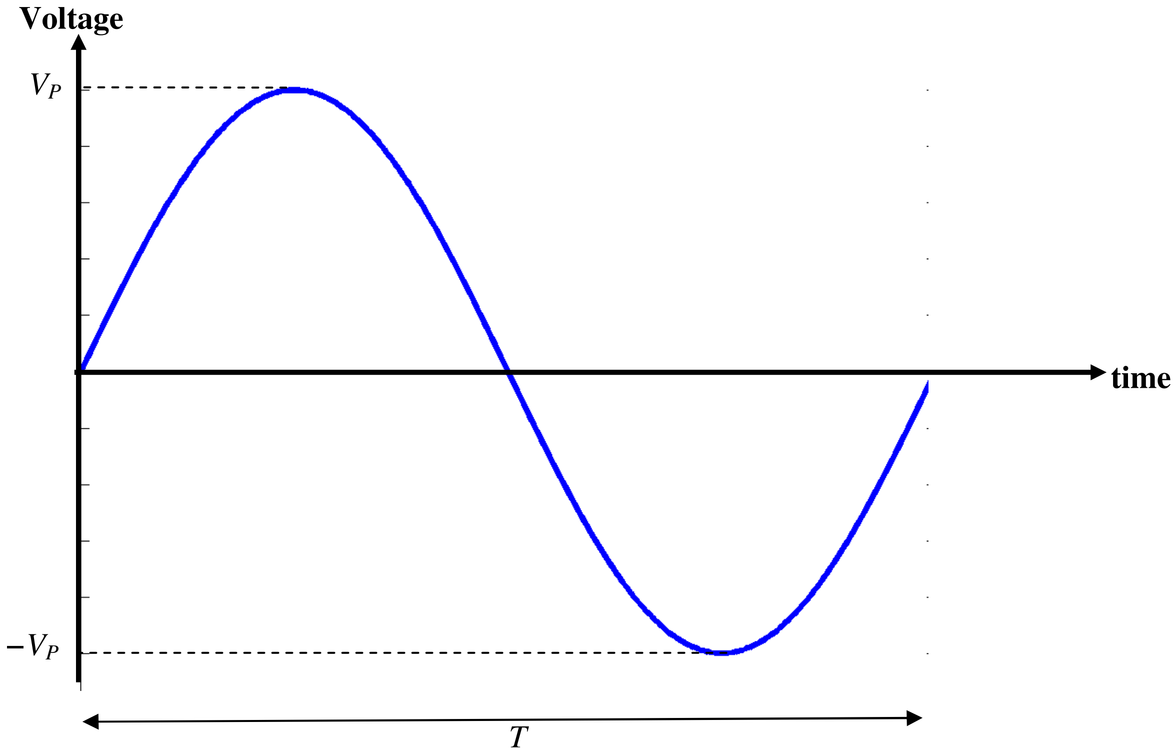
Steady State AC Circuits

So far on the course we have limited our study to circuits supplied by a constant value d.c. source either containing purely resistive elements, in which case the current is also constant and unidirectional, or containing $R+L$ or $R+C$ in which case we performed a transient analysis. In the latter cases the current was also unidirectional, but its value varied with time.

We will now proceed to study circuits supplied with time varying voltage sources, in particular sources in which the voltage varies sinusoidally and the currents flowing around the circuit are also sinusoidal. These types of Alternating Current (a.c.) sources are of great importance since the majority of generation, transmission, distribution and consumption of electrical energy occur under essentially sinusoidal steady-state conditions. The key reasons for the use of a.c. is that it is more easily generated than d.c. (generators in power stations) and it is readily changed from one voltage level to another (via a device called a transformer).

Terminology

Consider a sinusoidal voltage waveform:



The voltage at any instant, v, at time, t, may be obtained from:

$$v(t) = V_P \sin(\omega t) = V_P \sin(2\pi f t)$$

where:

t	=	Time (s)
V_P	=	Amplitude or peak value (V)
ω	=	Angular frequency (rad/s)
ω	=	$2\pi f$
f	=	Frequency (Hz)
f	=	1 / T
T	=	Period of 1 cycle (s)

Note: It is normal to use small letters (e.g. v) to represent time varying quantities of voltage and current.

Normally we do not define the voltage or current of an a.c. waveform in terms of the peak value, V_P, instead we use a quantity known as the Root Mean Square (rms) value which will be discussed in more detail later. For a sinusoidal waveform the rms value is related to the peak value by:

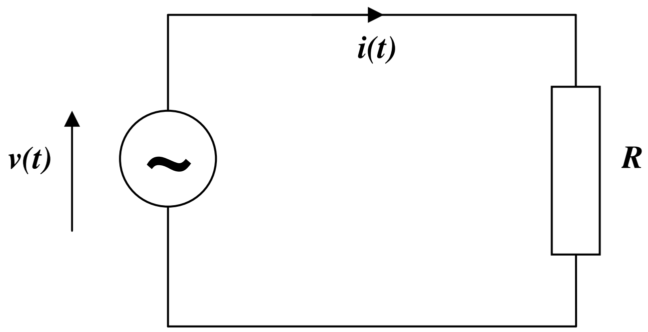
$$V_{rms} = \frac{V_P}{\sqrt{2}}$$

Note: We need rms quantities to ensure correct calculation of power which we shall consider later in the course. However, if we perform calculations using the peak value of the voltage then we shall simply get the peak value of the current.

The mains electricity supply in the UK is 230 V_{rms} and has a frequency of 50Hz. This means it will vary between ±325 Volts (230×√2) fifty times per second. The period is 1 / 50 or 20ms and the angular frequency will be 314 rad/s.

AC circuit containing a pure resistance

The circuit below shows a circuit consisting of a sinewave voltage source connected across a resistor.



We have already used Ohm's law to solve d.c. circuits – it is equally applicable here:

$$I = \frac{V}{R} \qquad \Rightarrow \qquad i(t) = \frac{v(t)}{R}$$

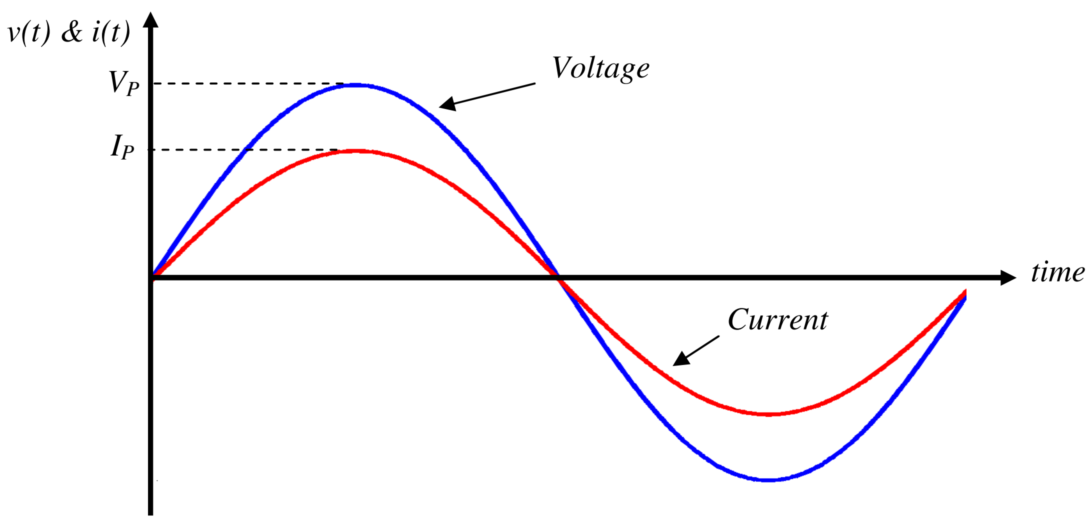
Substituting our sinewave voltage:

$$v(t) = V_P \sin(\omega t)$$

in Ohm's law gives an expression for the current flowing in the circuit:

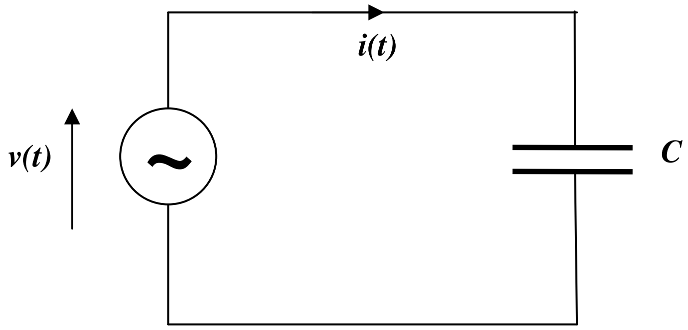
$$i(t) = \frac{V_P}{R}\sin(\omega t) = I_P \sin(\omega t)$$

where I_P is the peak value of the current. This is a waveform of the same shape and frequency as the voltage waveform and there is no phase shift, i.e. the current waveform is 'in-phase' with the voltage waveform. The ratio of the amplitudes depends on the value of resistance



AC circuit containing a pure capacitance

The circuit below shows a circuit consisting of a sinewave voltage source connected across a capacitor.



The expression relating the current through, to voltage across the terminals of a capacitor has been given previously. This expression is equally applicable to a.c. circuits:

$$I = C\frac{dV}{dt} \qquad \Rightarrow \qquad i(t) = C\frac{dv(t)}{dt}$$

Substituting our sinewave voltage:

$$v(t) = V_P \sin(\omega t)$$

in the above gives:

$$i(t) = C\frac{dv(t)}{dt} = C\frac{d(V_P \sin(\omega t))}{dt} = CV_P\,\omega\cos(\omega t) = I_P\cos(\omega t)$$

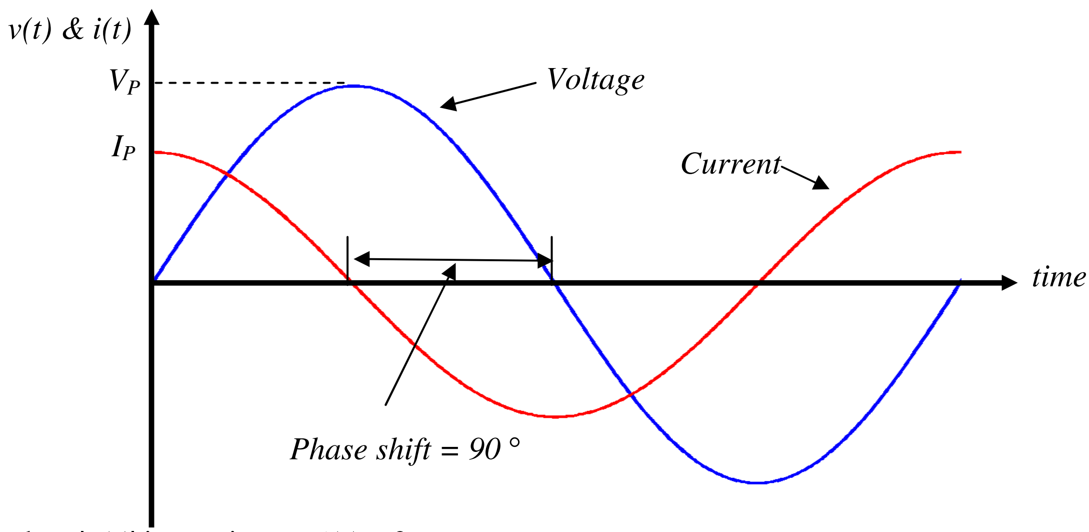
Alternatively this may be written as:

$$i(t) = CV_P\,\omega\sin\left(\omega t + \frac{\pi}{2}\right) = I_P\sin\left(\omega t + \frac{\pi}{2}\right)$$

Hence, $1/\omega C$ can be thought of as a constant of proportionality relating the magnitude of the voltage to the magnitude of the current (c.f. Ohm's law):

$$\frac{|V_P|}{|I_P|} = \frac{1}{\omega C} = \frac{1}{2\pi f C}$$

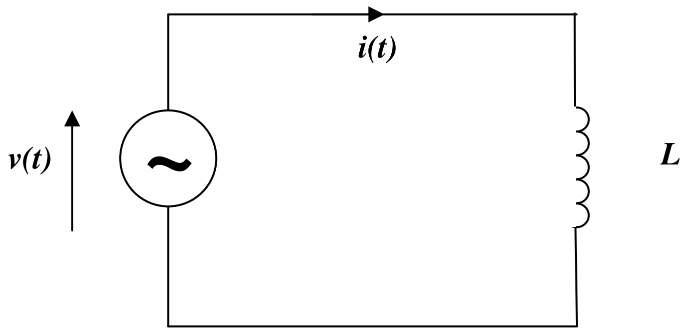
This is known as the (capacitive) reactance and has the units of Ohms. However, the current and voltage waveforms are now out of phase with one another with the voltage lagging the current by $\pi/2$ or 90°. (or current leads the voltage by 90°).



I.e. when $|v(t)|$ is maximum, $i(t) = 0$
when $|i(t)|$ is maximum, $v(t) = 0$

AC circuit containing a pure inductance

The circuit below shows a circuit consisting of a sinewave voltage source connected across a inductor.



The expression relating the current through, to voltage across the terminals of an inductor has been given previously. This expression is equally applicable to a.c. circuits:

$$V = L\frac{dI}{dt} \qquad \Rightarrow \qquad v(t) = L\frac{di(t)}{dt}$$

Substituting our sinewave voltage in the above gives:

$$V_P \sin(\omega t) = L\frac{di(t)}{dt}$$

Rearranging:

$$di(t) = \frac{V_P}{L}\sin(\omega t)dt$$

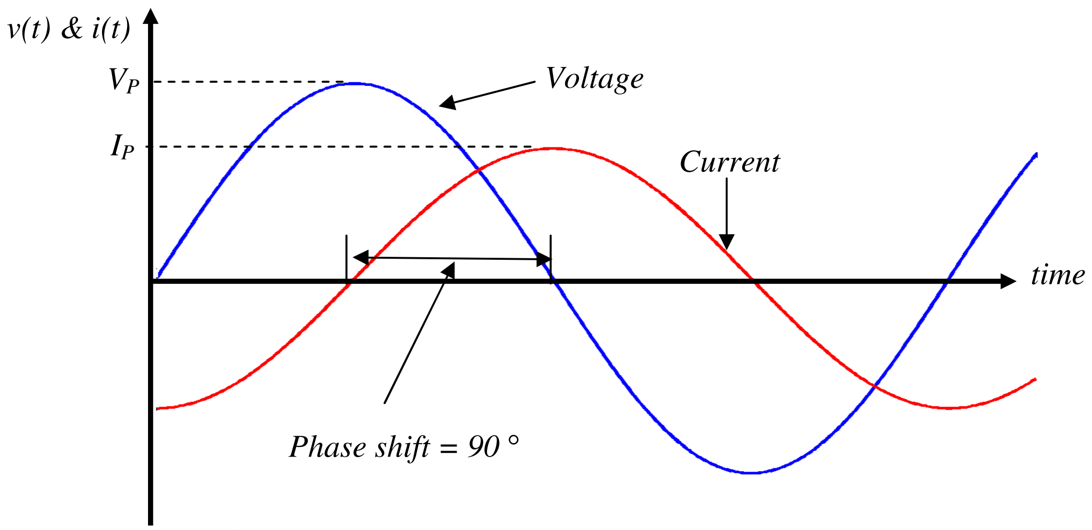
Integrating both sides of the equation:

$$i(t) = \frac{V_P}{L}\int \sin(\omega t) = -\frac{V_P}{\omega L}\cos(\omega t) = \frac{V_P}{\omega L}\sin\left(\omega t - \frac{\pi}{2}\right) = I_P \sin\left(\omega t - \frac{\pi}{2}\right)$$

Hence, ωL can be thought of as a constant of proportionality relating the magnitude of the voltage to the magnitude of the current (c.f. Ohm's law):

$$\frac{|V_P|}{|I_P|} = \omega L = 2\pi f L$$

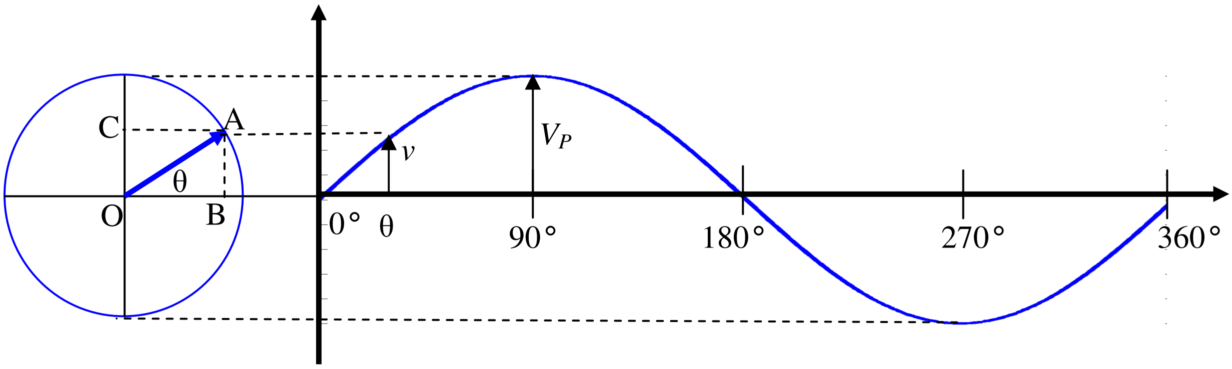
This is known as the (inductive) reactance and has the units of Ohms. However, the current and voltage waveforms are now out of phase with one another with the current lagging the voltage by $\pi/2$ or 90° (or voltage leads the current by 90°).



I.e. when $|v(t)|$ is maximum, $i(t) = 0$
when $|i(t)|$ is maximum, $v(t) = 0$

Phasor representation

So far we have indicated the relationship between a.c. currents and voltages for individual components by drawing the sinusoidal waveforms, however for this becomes cumbersome and time consuming if more voltages and currents are involved. Is there an easier way of conveying the relative magnitudes and phase of different waveforms?



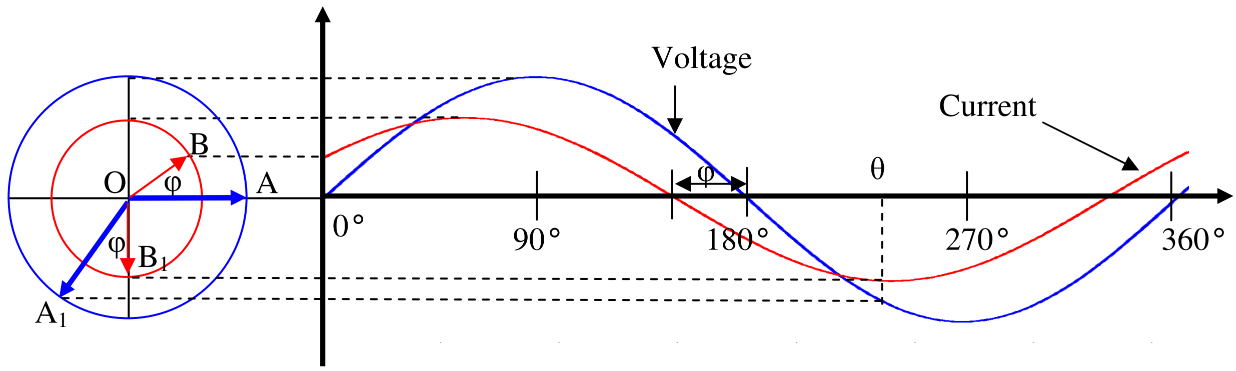
Suppose OA in the diagram to represent to scale the maximum value of an alternating quantity, (e.g. voltage, V_P). Now suppose OA rotates in an anticlockwise direction about O at a uniform angular velocity, so that it completes 1 revolution in the period of the waveform. The figure shows OA when it has rotated through an angle, θ, from the position occupied when the voltage waveform was passing through its zero value. If AB and AC are drawn perpendicular to the horizontal and vertical axes respectively then:

$$OC = AB = OA\sin\theta = V_P \sin\theta = v \text{ (}i.e. \text{the value of the voltage at that instant)}$$

Hence the projection of OA onto the vertical axis represents to scale the instantaneous value of voltage. Thus when θ = 90° the projection is OA (V_P) itself; when θ = 180° the projection is zero and corresponds to the voltage passing through zero from a positive to negative value. When θ = 270° the projection is OA on the negative axis (which corresponds to $-V_P$) and when θ = 360° the projection is again zero and corresponds to the voltage passing through zero from a negative to positive value.

Thus we have shown that we can represent our sinusoidal voltage waveform by a rotating vector, or phasor, having a length equal to the peak value of the waveform. The magnitude of the voltage at any instant in time will be equal to the projection of the phasor onto the vertical axis.

Now let us consider two waveforms displaced from one another by a certain angle. For example consider a voltage and current waveform where the current leads the voltage by an angle, φ as shown in the figure below. OA represents a phasor, the length of which corresponds to the maximum voltage, and OB another phasor representing the maximum value of the current. The angle between these two phasors must be φ. Consequently when OA is along the horizontal axis, the voltage at that instant is zero and the value of the current is represented by the projection of OB onto the vertical axis. This would correspond to the 0° point on our waveforms. After the phasors



have rotated through an angle θ they will occupy positions OA_1 and OB_1, with OB_1 still leading OA_1 by the angle φ. The instantaneous values of voltage and current are again given by the projections of OA_1 and OB_1 on the vertical axis as shown by the horizontal dotted lines. If the instantaneous value of the voltage is represented by:

$$v = V_P \sin\theta$$

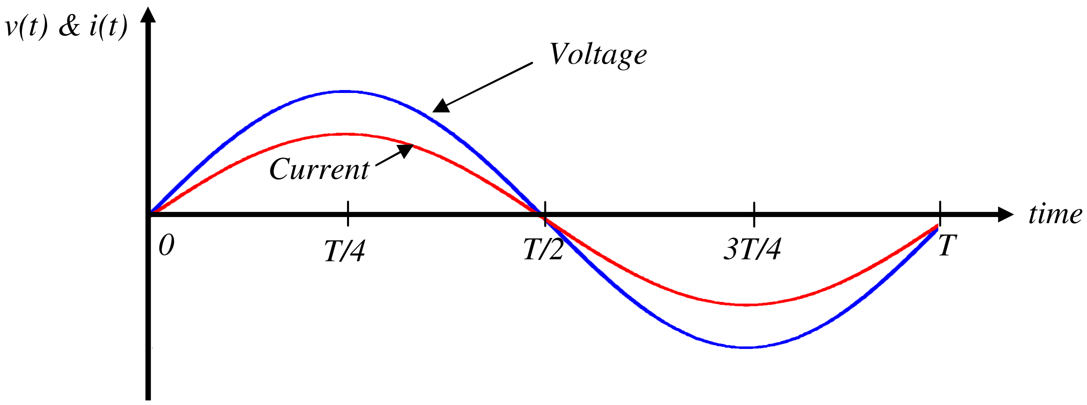
then the instantaneous value of the current is represented by:

$$i = I_P \sin(\theta + \varphi)$$

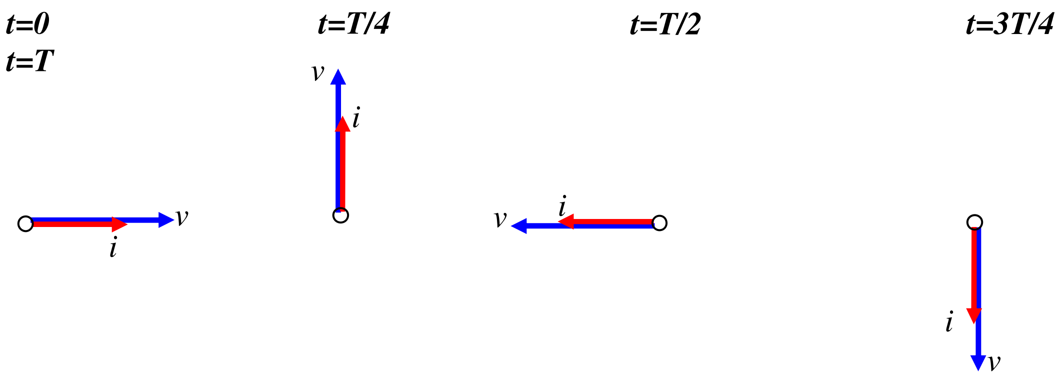
The current is said to lead the voltage by an angle φ, or the voltage is said to lag the current by an angle φ.

When we draw a phasor diagram to represent several waveforms we could choose any point on the waveform, however it is usual to choose the 0° point for one of our phasors.

Consider again the simple circuit with a resistance connected to a sinusoidal voltage supply. The voltage and current waveforms are in-phase.

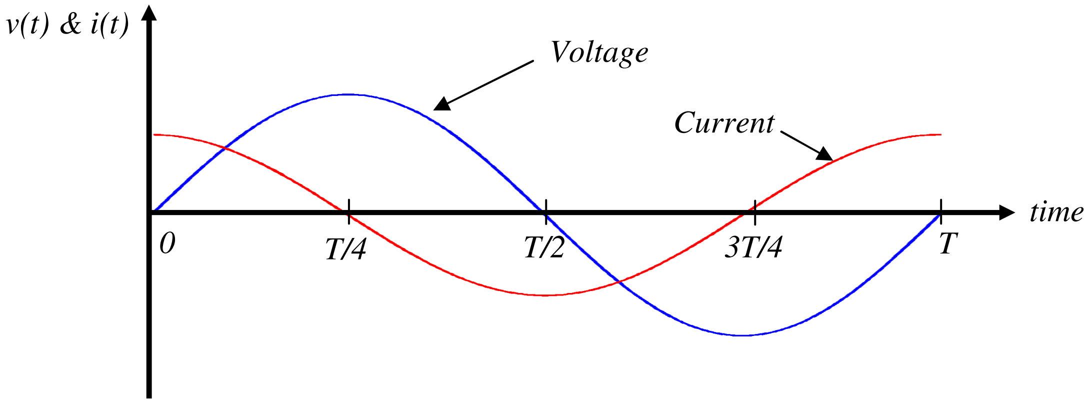


We can now represent the above waveforms by two rotating phasors, one representing the voltage, the other the current. Let us choose five different instances in time and draw the phasor diagram:

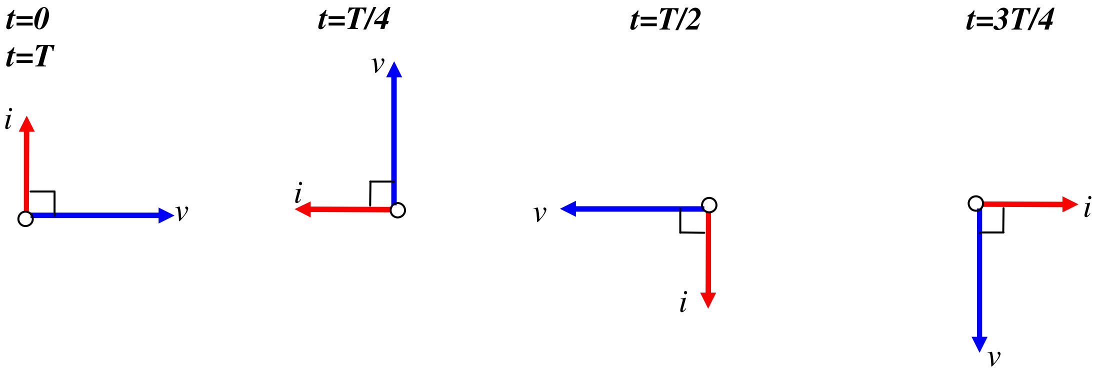


In this case the phasors are co-incident (although they have been drawn slightly displaced for clarity). We could have chosen any instant in time, however it is usual to choose a time when one of our phasors lies along the positive horizontal axis (t = 0 or t = T in this case).

Now consider the circuit containing a capacitor connected to a sinusoidal voltage source. Previously it was shown that the current waveform leads the voltage waveform by 90° as shown in the figure below.

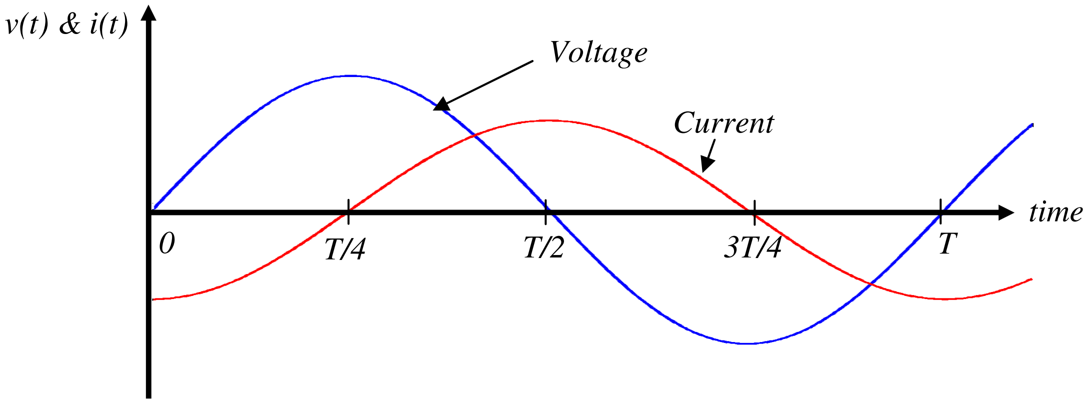


Drawing the phasor diagrams for this circuit gives:

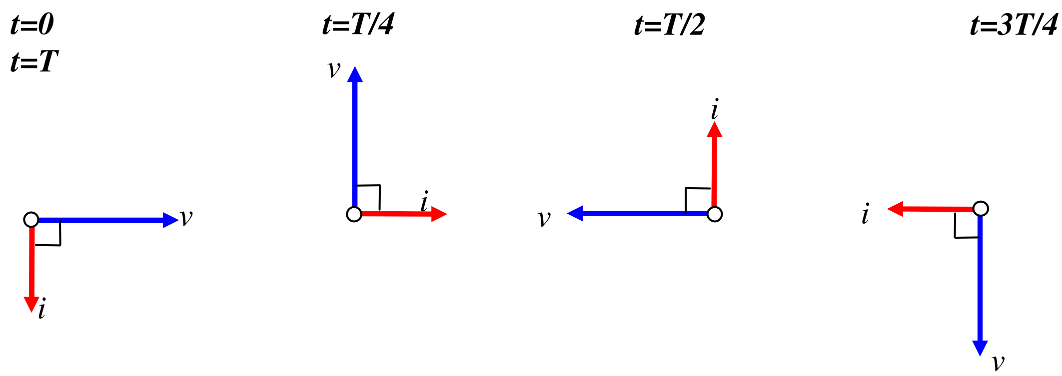


Normally we would choose to draw the phasor diagram at t = 0 (voltage phasor as reference along the positive horizontal axis) or at t = 3T/4 (current phasor as reference along the positive horizontal axis).

Finally let us reconsider the circuit containing an inductance connected to a sinusoidal supply where the current waveform lags the voltage waveform by 90°



Drawing the phasor diagrams for this circuit gives:



In this case we would either choose to draw the phasor diagram at t = 0 (voltage phasor as reference) or at t = T/4 (current phasor as reference).

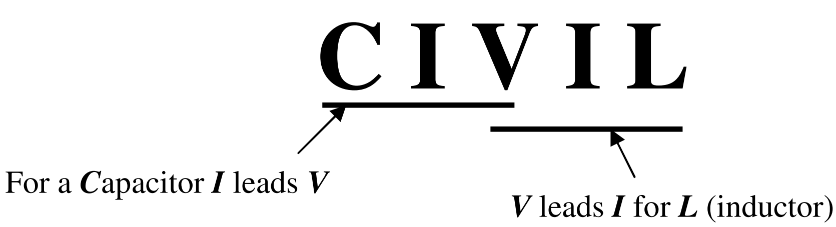
So for an inductive, ***L***, circuit:

The voltage, ***V***, **leads** the current, ***I***
Or the current, ***I,*** **lags** the voltage, ***V***

And for the capacitive, ***C***, circuit:

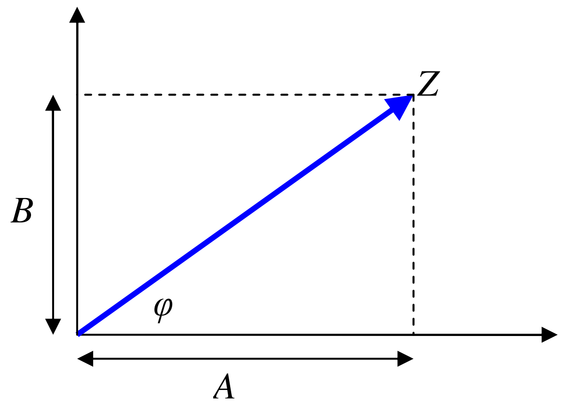
The current, ***I*** **leads** the voltage, ***V***
Or the voltage, ***V*** **lags** the current, ***I***

The following is a useful memory aid.



Phasor notation

So far we have looked at the graphical representation of a phasor – how can we represent this mathematically to allow us to add, subtract, multiply and divide phasors together? Consider a phasor of amplitude *Z*. This can be represented on our phasor diagram as:



$$A = Z\cos\varphi$$
$$B = Z\sin\varphi$$

We can express the phasor is a number of different ways, but the two common ones in electrical engineering are:

$Z\angle\varphi$ Z is the magnitude and φ is the phase angle (e.g. $30\angle 10°$ Volts or $5\angle -90°$ Amps).

$Z = A + \mathrm{j}B$ (where *j* is the complex operator $\sqrt{-1}$) and $A = Z\cos\varphi$ and $B = Z\sin\varphi$ – more on the use of complex numbers in solving electrical circuits will be discussed shortly.

Multiplication of two phasors

Multiply the amplitudes and add the angles:

$$C = X\angle\alpha \qquad \text{and} \qquad D = Y\angle\beta$$

then:

$$C \times D = XY\angle(\alpha + \beta)$$

Division of two phasors

Divide the amplitudes and subtract the angles:

$$C = X\angle\alpha \qquad \text{and} \qquad D = Y\angle\beta$$

then:

$$\frac{C}{D} = \frac{X}{Y}\angle(\alpha - \beta)$$

Addition and subtraction of two phasors

Before the two phasors can be added or subtracted it is necessary to resolve them into quadrature components, sum these components and recombine into the resultant phasor, $R\angle\delta$:

$$C = X\angle\alpha \quad \Rightarrow \quad C_x = X\cos\alpha \ \text{ and } \ C_y = X\sin\alpha$$
$$D = Y\angle\beta \quad \Rightarrow \quad D_x = Y\cos\beta \ \text{ and } \ D_y = Y\sin\beta$$
$$R = \sqrt{(X\cos\alpha + Y\cos\beta)^2 + (X\sin\alpha + Y\sin\beta)^2}$$
$$\delta = \tan^{-1}\frac{(X\sin\alpha + Y\sin\beta)}{(X\cos\alpha + Y\cos\beta)}$$

We shall see later that for addition and subtraction it is better to use the complex form to represent the phasors and use complex algebra.

Complex numbers – Revision

So far we have looked at solving a.c. circuits by sketching waveforms or using phasor diagrams. However with more involved circuits calculation can be simplified by the use of complex algebra. This system enables equations representing alternating voltages and currents and their phase relationships to be expressed in simple algebraic form. It is based on the idea that a phasor can be resolved into two components at right angles to each other.

The complex operator *j* is defined as:

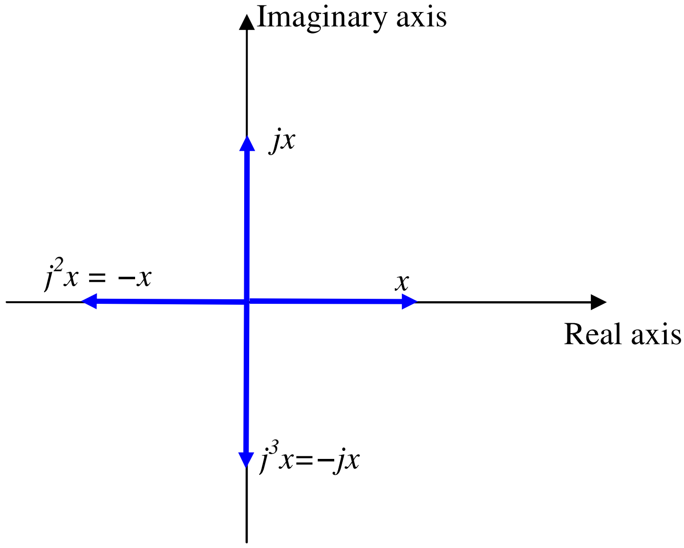
$$j = \sqrt{-1}$$

(Note: often *i* is used to represent the complex operator in maths text books, however electrical engineers tend to use *j* to avoid confusion when *i* is used for current).

Hence:

$$j^2 = -1 \qquad j^3 = -j \qquad j^4 = 1 \qquad \frac{1}{j} = \frac{j}{j^2} = -j$$

In many ways *j* can be thought of as a $90°$ operator. Consider *j* acting on a phasor *x*:



When manipulating complex numbers it is important to consider the real (Re) and imaginary (Im) parts separately.

Addition and subtraction:

$$(a + jb) + (c + jd) = (a + c) + j(b + d)$$
$$(a + jb) - (c + jd) = (a - c) + j(b - d)$$

Multiplication:

$$(a + jb)(c + jd) = ac + jad + jbc + j^2bd = (ac - bd) + j(ad + bc)$$

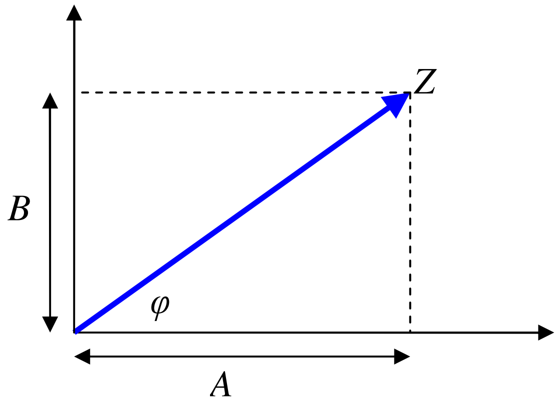
Division:

$$\frac{(a + jb)}{(c + jd)} = \frac{(a + jb)(c - jd)}{(c + jd)(c - jd)} = \frac{ac - jad + jbc - j^2bd}{c^2 - jcd + jcd - j^2d^2} = \frac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$$

In general use the polar form, $Z\angle\varphi$, for multiplication and division and the complex form, a+jb for addition and subtraction.

Conversion between polar and complex notation

Consider the phasor $Z\angle\varphi = A + jB$ plotted in the figure below:



Clearly to obtain complex form from polar form use:

$$A = Z\cos\varphi \quad \text{and} \quad B = Z\sin\varphi$$

and to obtain polar form from complex form use:

$$|Z| = \sqrt{A^2 + B^2} \quad \text{and} \quad \varphi = \tan^{-1}\frac{B}{A}$$

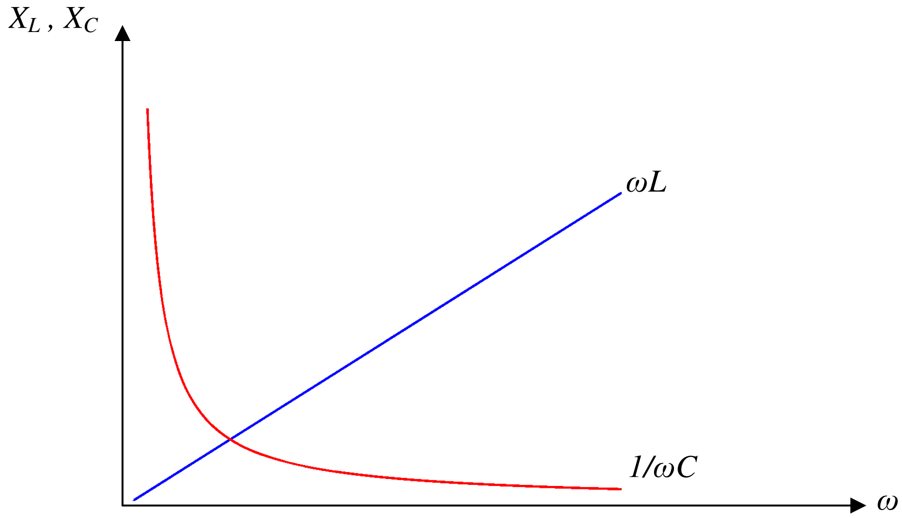
Use of complex numbers in solving a.c. circuits

Previously we have defined the magnitude of the reactance as:

$$Capacitive\ Reactance = \frac{1}{\omega C} = \frac{1}{2\pi fC} = X_C \qquad \text{(units } \Omega\text{)}$$

$$Inductive\ Reactance = \omega L = 2\pi fL = X_L \qquad \text{(units } \Omega\text{)}$$

Clearly the value of reactance will vary with frequency:



The reactances, $X_C = \frac{1}{\omega C}$ and $X_L = \omega L$ define the relationship between the magnitudes of the voltage and current (Ohm's law) but do not convey any information about the phase difference.

Complex numbers provide a way of conveying information about the phase angle.

Capacitor

$$X_C = \frac{V\angle 0^\circ}{I\angle 90^\circ}$$

$$X_C = \left|\frac{V}{I}\right|\angle -90^\circ$$

$$\text{Let } X_C = 0 - \frac{j}{\omega C}$$

$$|X_C| = \sqrt{(\mathrm{Re})^2 + (\mathrm{Im})^2}$$

$$= \sqrt{0 + \left(\frac{1}{\omega C}\right)^2}$$

$$= \frac{1}{\omega C}$$

$$\varphi_C = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

$$= \tan^{-1}\left(\frac{-1}{0 \times \omega C}\right)$$

$$= \tan^{-1}(-\infty)$$

$$\varphi_C = -90^\circ$$

Inductor

$$X_L = \frac{V\angle 0}{I\angle -90}$$

$$X_L = \left|\frac{V}{I}\right|\angle 90^\circ$$

$$\text{Let } X_L = 0 + j\omega L$$

$$|X_L| = \sqrt{(\mathrm{Re})^2 + (\mathrm{Im})^2}$$

$$= \sqrt{0 + (\omega L)^2}$$

$$= \omega L$$

$$\varphi_L = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right)$$

$$= \tan^{-1}\left(\frac{\omega L}{0}\right)$$

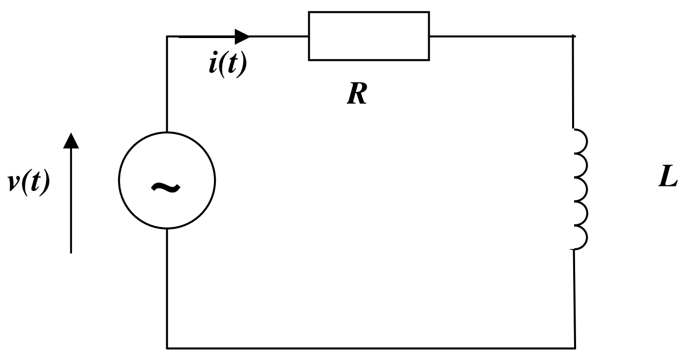
$$= \tan^{-1}(\infty)$$

$$\varphi_L = 90^\circ$$

So if we use $jX_C = -\frac{j}{\omega C} = \frac{1}{j\omega C}$ and $jX_L = j\omega L$ to represent our capacitive and inductive reactances respectively we can correctly calculate both the magnitude and phase angle.

<u>Combining components in a.c. circuits – Series Connection</u>

Consider a circuit containing both resistance, R, and inductance, L, connected in series across an a.c. sinusoidal voltage source. The characteristics of each component must remain even when they are connected together.



Define the **Impedance**, Z, which is the opposition to current flow of both components combined. Since this circuit consists of two components in series (i.e. the same current flows through each) we simply add the resistance and reactance:

$$Z = R + j\omega L$$

The overall magnitude of the impedance is given by:

$$|Z| = \sqrt{R^2 + (\omega L)^2}$$

and the phase angle of the impedance is:

$$\varphi = \tan^{-1}\frac{\omega L}{R}$$

So if ωL is dominant then $\varphi \Rightarrow 90°$, whereas if R is dominant $\varphi \Rightarrow 0°$.
If we assume the supply voltage as our reference, i.e. $v\angle 0°$ then:

$$current = \frac{v\angle 0^\circ}{Z\angle\varphi} = i\angle -\varphi$$

i.e. the current will lag behind the voltage and if the reactance is very much greater than the resistance then the phase angle will approach $-90°$ as we have shown previously.

Example

A 10Ω resistor and a 10mH inductor are connected in series across a 50Hz 5V sinusoidal supply. Calculate the magnitude and phase of the impedance, the magnitude and phase of the current and the voltage across each component.

The magnitude and phase of the impedance are given by:

$$|Z| = \sqrt{R^2 + (2\pi fL)^2} = \sqrt{10^2 + (2\pi\times 50\times 0.01)^2} = 10.5\ \Omega$$

$$\varphi = \tan^{-1}\left(\frac{2\pi f L}{R}\right) = \tan^{-1}\left(\frac{2\pi\times 50\times 0.01}{10}\right) = \tan^{-1} 0.314 = 17.4°$$

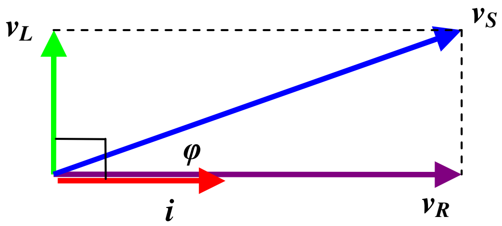
$$Z = 10.5\angle 17.4^\circ\ \Omega$$

The current may be found from:

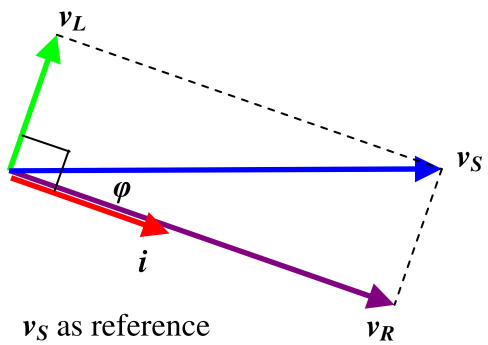
$$i = \frac{v\angle 0^\circ}{Z\angle\varphi} = \frac{5\angle 0^\circ}{10.5\angle 17.4^\circ} = 0.476\angle -17.4°\ \text{A}$$

i.e. the current is lagging behind the voltage by 17.4°.

Now from our previous studies we know that the voltage across the resistor, v_R, will be in-phase with the current and the voltage across the inductor, v_L, will lead the current by 90°. Draw the phasor diagram for the system. Since the current is common to both it is usual to draw this as the reference phasor.



i as reference



v_S as reference

The magnitudes of the voltages across the resistor and inductor are:

$$|v_R| = I\times R = 0.476\times 10 = 4.76\ \text{V}$$

$$|v_L| = I\times X_L = I\times 2\times\pi\times f\times L = 0.476\times 2\times\pi\times 50\times 0.01 = 1.495\ \text{V}$$

Check:

$$v_S = \sqrt{v_R^2 + v_L^2} = \sqrt{4.76^2 + 1.495^2} = 5\ \text{V}$$

Now lets see what would happen if the frequency increased to 10kHz.

$$|Z| = \sqrt{R^2 + (2\pi fL)^2} = \sqrt{10^2 + (2\pi \times 10000 \times 0.01)^2} = 628.4\ \Omega$$

$$\varphi = \tan^{-1}\left(\frac{2\pi f L}{R}\right) = \tan^{-1}\left(\frac{2\pi \times 10000 \times 0.01}{10}\right) = \tan^{-1} 62.8 = 89°$$

$$Z = 628.4\angle 89^\circ\ \Omega$$

The current may be found from:

$$i = \frac{v\angle 0^\circ}{Z\angle\varphi} = \frac{5\angle 0^\circ}{628\angle 89^\circ} = 0.00795\ \angle -89°\ \text{A} = 7.95\angle -89°\ \text{mA}$$

i.e. the current is lagging behind the voltage by 89°.

The magnitudes of the voltages across the resistor and inductor are:

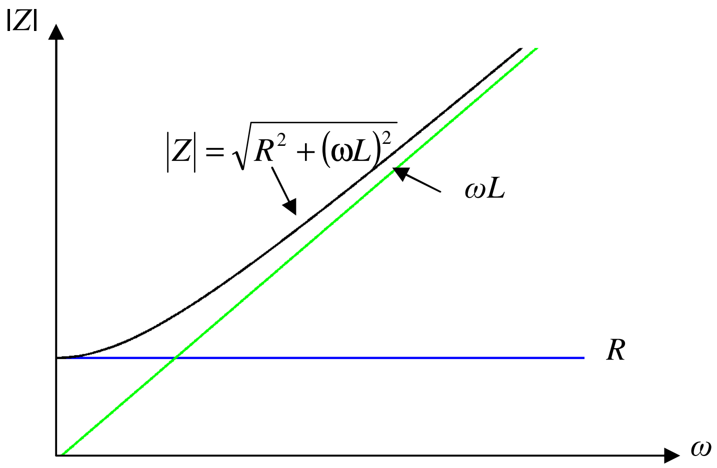
$$|v_R| = I \times R = 0.00795 \times 10 = 0.0795\ \text{V} = 79.5\ \text{mV}$$

$$|v_L| = I \times X_L = I \times 2 \times \pi \times f \times L = 0.00795 \times 2 \times \pi \times 10000 \times 0.01 = 4.995\ \text{V}$$

Check:

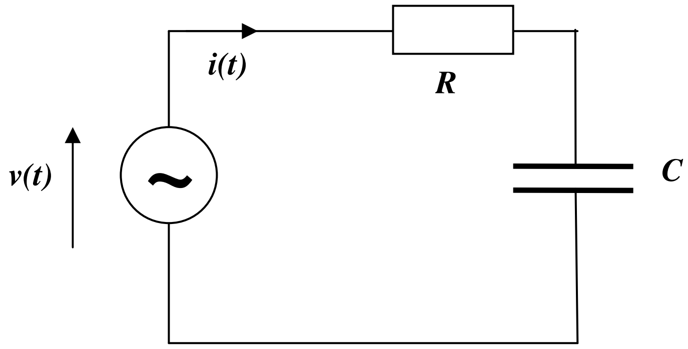
$$v_S = \sqrt{v_R^2 + v_L^2} = \sqrt{0.0795^2 + 4.995^2} = 5\ \text{V}$$

Clearly at high frequencies the inductor is the dominant component in the circuit.



At low frequencies $\omega \Rightarrow 0$ and $|Z| \Rightarrow R$ and $\varphi \Rightarrow 0°$
At high frequencies $\omega \Rightarrow \infty$ and $|Z| \Rightarrow \omega L \Rightarrow \infty$ and $\varphi \Rightarrow 90°$

Now lets look at a circuit containing a resistor and a capacitor in series.



Since this circuit consists of two components in series (i.e. the same current flows through each) we simply add the resistance and reactance:

$$Z = R - \frac{j}{\omega C}$$

The overall magnitude of the impedance is given by:

$$|Z| = \sqrt{R^2 + \left(\frac{1}{\omega C}\right)^2}$$

and the phase angle of the impedance is:

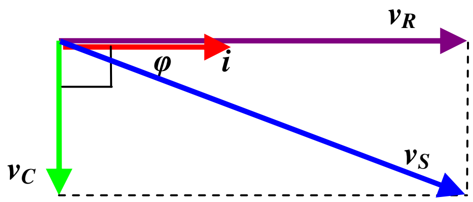
$$\varphi = \tan^{-1}\frac{-1}{\omega CR}$$

So if $1/\omega C$ is dominant then $\varphi \Rightarrow -90°$, whereas if R is dominant $\varphi \Rightarrow 0°$.
The voltage across the resistor is in phase with the current and has a magnitude:

$$|v_R| = i \times R$$

The voltage across the capacitor lags behind the current by 90° and has a magnitude:

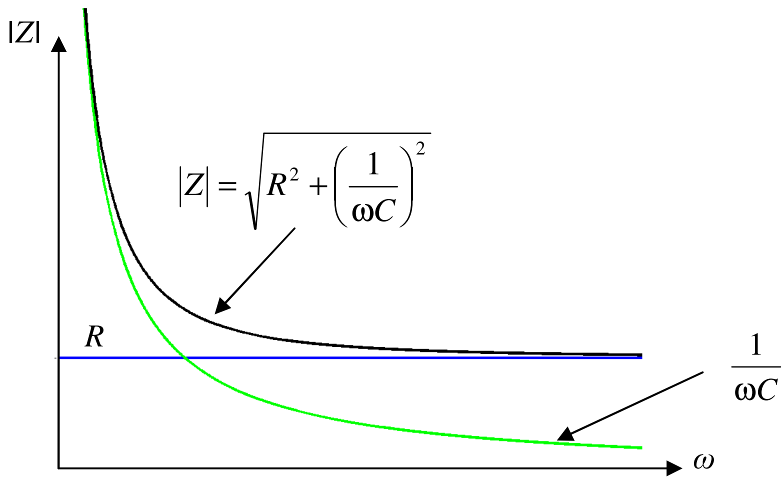
$$|v_C| = i \times X_C = \frac{i}{2\pi f C}$$



i as reference

Once again the supply voltage is equal to the vector sum of these two components:

$$v_S = \sqrt{v_R^2 + v_C^2}$$



At low frequencies $\omega \Rightarrow 0$ and $|Z| \Rightarrow 1/\omega C \Rightarrow \infty$ and $\varphi \Rightarrow -90°$
At high frequencies $\omega \Rightarrow \infty$ and $|Z| \Rightarrow R$ and $\varphi \Rightarrow 0°$

Example

An 8Ω resistor and a 10μF capacitor are connected in series across a 2kHz 10V sinusoidal supply. Calculate the magnitude and phase of the impedance, the magnitude and phase of the current and the voltage across each component.

The impedance of the circuit is:

$$Z = R - \frac{j}{2\pi f C}$$

The magnitude and phase of the impedance are given by:

$$|Z| = \sqrt{R^2 + \left(\frac{1}{2\pi fC}\right)^2} = \sqrt{8^2 + \left(\frac{1}{2\pi \times 2000 \times 10 \times 10^{-6}}\right)^2} = 11.3\ \Omega$$

$$\varphi = \tan^{-1}\left(\frac{-1}{2\pi f RC}\right) = \tan^{-1}\left(\frac{-1}{2\pi \times 2000 \times 8 \times 10 \times 10^{-6}}\right) = \tan^{-1} - 0.995 = -45°$$

$$Z = 11.3\angle -45^\circ\ \Omega$$

The current may be found from:

$$i = \frac{v\angle 0^\circ}{Z\angle\varphi} = \frac{10\angle 0^\circ}{11.3\angle -45^\circ} = 0.885\ \angle\ 45°\ \text{A}$$

i.e. the current is leading the voltage by 45°.

Now from our previous studies we know that the voltage across the resistor, v_R, will be in-phase with the current and the voltage across the capacitor, v_C, will lag the current by 90°.

The magnitudes of the voltages across the resistor and capacitor are:

$$|v_R| = I \times R = 0.885 \times 8 = 7.08\ \text{V}$$

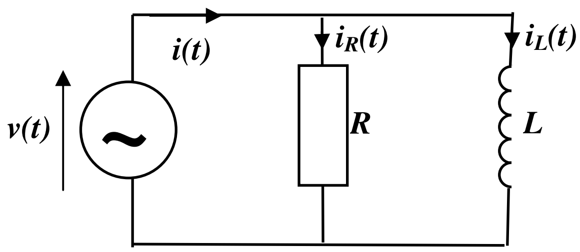
$$|v_C| = I \times X_C = I \times \frac{1}{2 \times \pi \times f \times C} = 0.885 \times \frac{1}{2 \times \pi \times 2000 \times 10 \times 10^{-6}} = 7.04\ \text{V}$$

Check:

$$v_S = \sqrt{v_R^2 + v_C^2} = \sqrt{7.08^2 + 7.04^2} = 10\ \text{V}$$

Combining components in a.c. circuits – Parallel Connection

Consider a circuit containing both resistance, R, and inductance, L, connected in parallel across an a.c. sinusoidal voltage source.



The impedance, Z, may be calculated using the inverse rule, as the same voltage appears across each component:

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{j\omega L} = \frac{j\omega L + R}{j\omega LR}$$

Hence:

$$Z = \frac{j\omega LR}{R + j\omega L}$$

To obtain an expression in conventional form we multiply by the complex conjugate:

$$Z = \frac{j\omega LR(R - j\omega L)}{(R + j\omega L)(R - j\omega L)} = \frac{\omega^2 L^2 R + j\omega LR^2}{R^2 + (\omega L)^2}$$

Clearly this is quite a complicated expression and often, if we are only interested in finding the current, it is not necessary to calculate the impedance. This will be illustrated in the following example.

Example

A 10Ω resistor and a 100mH inductor are connected in parallel across a 50Hz 5V sinusoidal supply. Calculate the magnitude and phase of the impedance, and the magnitude and phase of the total current drawn from the supply.

Using the above expression, the magnitude and phase of the impedance are given by:

$$Z = \frac{\omega^2 L^2 R}{R^2 + (\omega L)^2} + j\frac{\omega LR^2}{R^2 + (\omega L)^2} = \frac{(2\pi\times 50\times 0.1)^2 \times 10}{10^2 + (2\pi\times 50\times 0.1)^2} + j\frac{2\pi\times 50\times 0.1\times 10^2}{10^2 + (2\pi\times 50\times 0.1)^2}$$

$$= \frac{9869.6}{1087} + j\frac{3141.6}{1087} = 9.08 + j2.89 = 9.53\angle 17.6^\circ\,\Omega$$

The current may be found from:

$$i = \frac{v\angle 0^\circ}{Z\angle\varphi} = \frac{5\angle 0^\circ}{9.53\angle 17.6^\circ} = 0.525\angle -17.6^\circ\text{ A}$$

i.e. the current is lagging behind the voltage by 17.6°.

If we only require the current then we can find the current through each component and sum these to obtain the total:

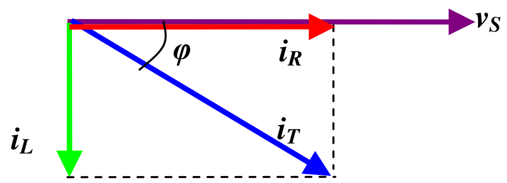
$$i_R = \frac{v\angle 0^\circ}{R\angle 0^\circ} = \frac{5\angle 0^\circ}{10\angle 0^\circ} = 0.5\angle 0^\circ\text{ A or } 0.5\text{ A}$$

$$i_L = \frac{v\angle 0^\circ}{X_L\angle 90^\circ} = \frac{5\angle 0^\circ}{2\times\pi\times 50\times 0.1\angle 90^\circ} = \frac{5\angle 0^\circ}{31.4\angle 90^\circ} = 0.159\angle -90^\circ\text{ A or } -j\,0.159\text{ A}$$

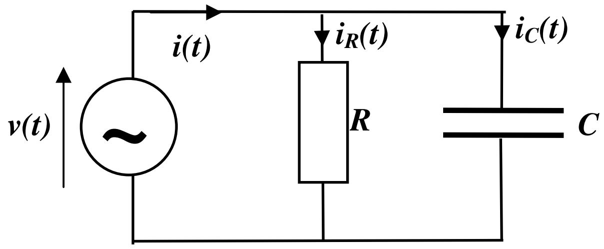
The total current may be found by summing these two components:

$$i_T = 0.5 - j0.159 = 0.525\angle -17.6^\circ\text{ A}$$

which the same answer as before.



Now consider the case of a circuit containing both resistance, *R*, and capacitance, *C*, connected in parallel across an a.c. sinusoidal voltage source.



The impedance, *Z*, may be calculated using the inverse rule, as the same voltage appears across each component:

$$\frac{1}{Z}=\frac{1}{R}+\frac{1}{\frac{1}{j\omega C}}=\frac{1}{R}+j\omega C=\frac{1+j\omega CR}{R}$$

Hence:

$$Z=\frac{R}{1+j\omega CR}$$

To obtain an expression in conventional form we multiply by the complex conjugate:

$$Z=\frac{R(1-j\omega CR)}{(1+j\omega CR)(1-j\omega CR)}=\frac{R-j\omega CR^2}{1+(\omega CR)^2}$$

As with the previous case of the resistor in parallel with the inductor it is often easier to obtain the total current by summing the two components.

<u>Example</u>

An 8Ω resistor and a 10μF capacitor are connected in parallel across a 2kHz 10V sinusoidal supply. Calculate the magnitude and phase of the impedance, and the magnitude and phase of the total current drawn from the supply.

Using the above expression, the magnitude and phase of the impedance are given by:

$$Z=\frac{R}{1+(\omega CR)^2}-j\frac{\omega CR^2}{1+(\omega CR)^2}=\frac{8}{1+\left(2\pi\times2000\times10^{-5}\times8\right)^2}-j\frac{2\pi\times2000\times10^{-5}\times8^2}{1+\left(2\pi\times2000\times10^{-5}\times8\right)^2}$$

$$=\frac{8}{2.011}-j\frac{8.042}{2.011}=3.98-j4=5.64\angle-45.1^\circ\,\Omega$$

The current may be found from:

$$i=\frac{v\angle0^\circ}{Z\angle\varphi}=\frac{10\angle0^\circ}{5.64\angle-45.1^\circ}=1.77\angle45.1^\circ\text{ A}$$

i.e. the current is leading the voltage by 45.1°.

The total current can be more easily found in a similar manner to the previous example:

$$i_R=\frac{v\angle0^\circ}{R\angle0^\circ}=\frac{10\angle0^\circ}{8\angle0^\circ}=1.25\angle0^\circ\text{ A or }1.25\text{ A}$$

$$i_C=\frac{v\angle0^\circ}{X_C\angle-90^\circ}=\frac{10\angle0^\circ}{\frac{1}{2\pi\times2000\times10^{-5}}\angle-90^\circ}=\frac{10\angle0^\circ}{7.96\angle-90^\circ}=1.256\angle90^\circ\text{ A or }j\,1.256\text{ A}$$

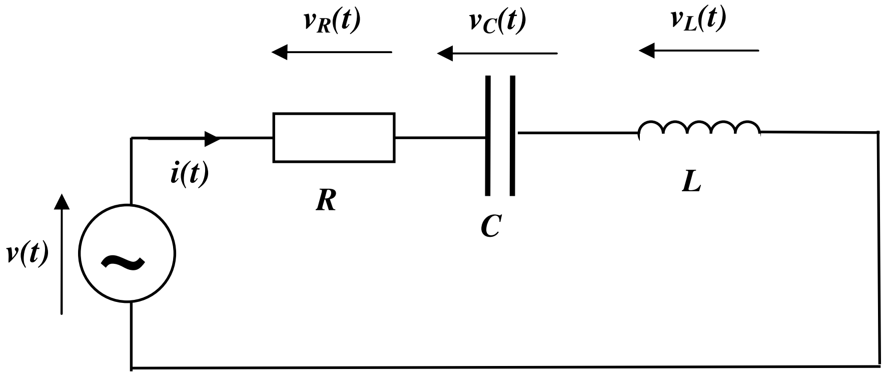
The total current may be found by summing these two components:

$$i_T=1.25+j1.256=1.77\angle45.1^\circ\text{ A}$$

Resonant circuits

Series resonant circuit

Let us now consider a circuit with all three basic elements, R, C, and L, connected in series across a sinusoidal voltage supply.



The total impedance of the circuit is:

$$Z = R + j\omega L + \frac{1}{j\omega C} = R + j\omega L - \frac{j}{\omega C}$$

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

The minus sign in the imaginary term means that at a certain angular frequency, ω_R, the imaginary term will become zero $\left(\omega_R L = \frac{1}{\omega_R C}\right)$ and the impedance will be purely resistive. This frequency is called the resonant frequency and the circuit is said to be at resonance. Hence at resonance:

$$Z = R + j0 = R$$

i.e. it appears that the circuit is purely resistive.

The resonant angular frequency may be derived from:

$$\omega_R L = \frac{1}{\omega_R C}$$

Therefore:

$$\omega_R^2 = \frac{1}{LC}$$

or:

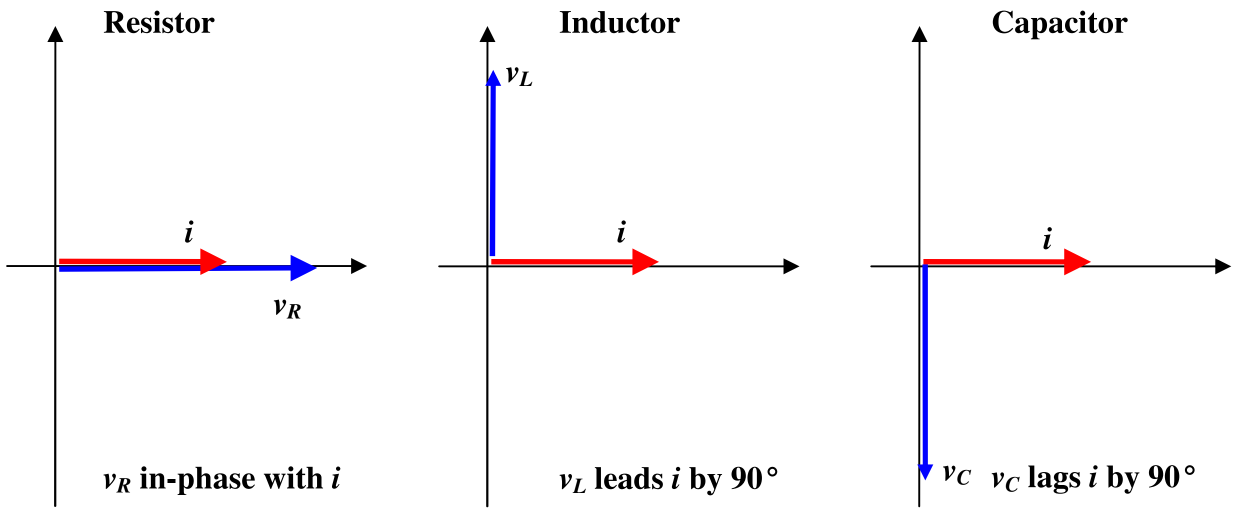
$$\omega_R = \frac{1}{\sqrt{LC}}$$

Alternatively, since $\omega = 2\,\pi\,f$ then:

$$f_R = \frac{1}{2\pi\sqrt{LC}}$$

The phenomenon of resonance can be more easily explained by considering the phasor diagrams. We will take the current as the reference phasor since this is the same in each component. The voltage across the resistor, v_R, will be in-phase with the current, the voltage across the capacitor, v_C, will lag the current by 90°, and the voltage across the inductor, v_L, will lead the current by 90°.

Phasor diagrams for a series resonant circuit, taking the current phasor as reference:



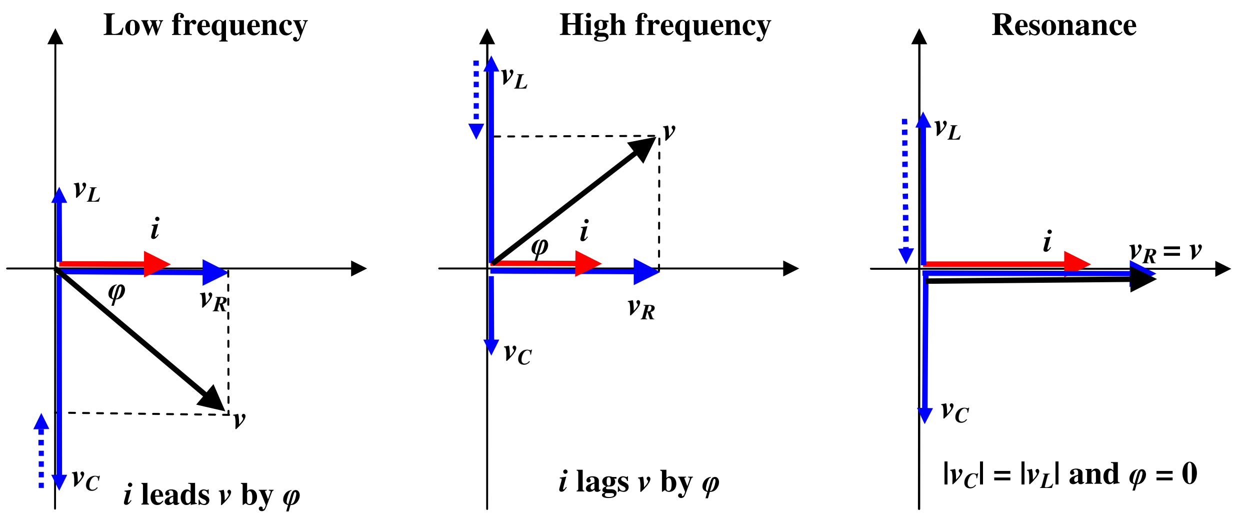
From previous analysis it was shown that:

$$|v_R| = i \times R$$

$$|v_L| = i \times X_L = i \times \omega L = i \times 2\pi f L$$

$$|v_C| = i \times X_C = i \times \frac{1}{\omega C} = i \times \frac{1}{2\pi f C}$$

Clearly v_L and v_C are frequency dependant, at low frequencies v_L will be small and v_C large whereas the opposite is true at high frequencies. At the resonant frequency v_L and v_C will be equal in magnitude. Now combine the above phasor diagrams onto a single diagram.

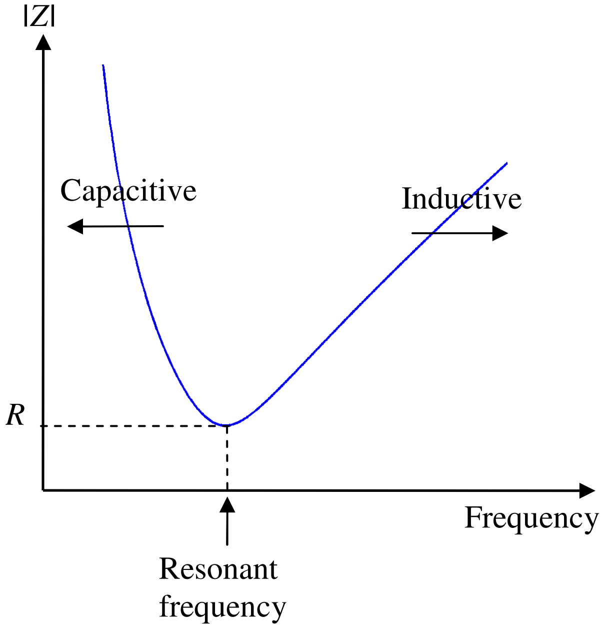


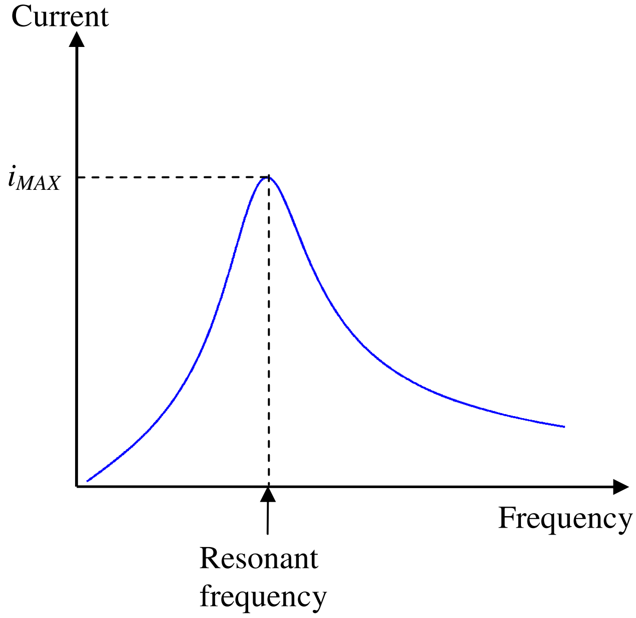
At resonance the voltage across the inductor, v_L, and the voltage across the capacitor, v_C, are equal in magnitude but 180° out of phase (i.e. opposite in direction) so they cancel out. Hence the voltage across the resistor is equal to the supply voltage, v, the phase angle, φ, is zero, and the circuit appears resistive.

At resonance:

$$|v_L| = |v_C| \qquad v = v_R \qquad |Z| = \sqrt{R^2 + 0} = R \qquad \varphi = \tan^{-1}\frac{0}{R} = 0^\circ$$

The magnitude of the impedance will vary with frequency, being dominated by the capacitance at low frequencies and by the inductance at high frequencies. At the resonant frequency the impedance will be a minimum and equal to the resistance. Since the impedance is minimum then the current will be a maximum at the resonant frequency and only dependant on the value of the resistor.





Example

A circuit consists of a 10Ω resistor, a 100mH inductor and a 0.253μF capacitor connected in series across a 10V sinusoidal voltage supply. Calculate the frequency at which the circuit becomes resonant, the current flowing and the voltages across the inductor and capacitor at that frequency.

The resonant frequency is calculated from:

$$f_R = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{100\times10^{-3}\times0.253\times10^{-6}}} = 1000 \text{ Hz}$$

At resonance the impedance is purely resistive:

$$|Z| = R = 10\Omega$$

Using Ohm's law to find the current:

$$|i| = \frac{v}{R} = \frac{10}{10} = 1 \text{ A}$$

The voltage across the inductor is:

$$|v_L| = iX_L = i\times2\pi fL = 1\times2\pi\times1000\times100\times10^{-3} = 628 \text{ V}$$

The voltage across the capacitor is:

$$|v_C| = iX_C = i\times\frac{1}{2\pi fC} = 1\times\frac{1}{2\pi\times1000\times0.253\times10^{-6}} = 628 \text{ V}$$

The voltages are equal in magnitude as expected and cancel each other out. Despite having only a 10V supply the voltages across the capacitor and inductor are many times greater (628 V).

We can define the 'Q' factor (or 'Quality factor' or 'Magnification factor') as the voltage magnification at resonance.

$$Q = \frac{voltage\ magnification\ at\ resonance}{supply\ voltage} = \left|\frac{v_L}{v_R}\right| = \frac{2\pi f_R L\times i}{R\times i} = \frac{2\pi f_R L}{R}$$

since the supply voltage is equal to the voltage across the resistor at resonance.

Substituting for the resonant frequency:

$$f_R = \frac{1}{2\pi\sqrt{LC}}$$

so:

$$Q = \frac{2\pi L}{R} \times \frac{1}{2\pi\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

Alternatively we could have achieved the same result if we had used the capacitor voltage:

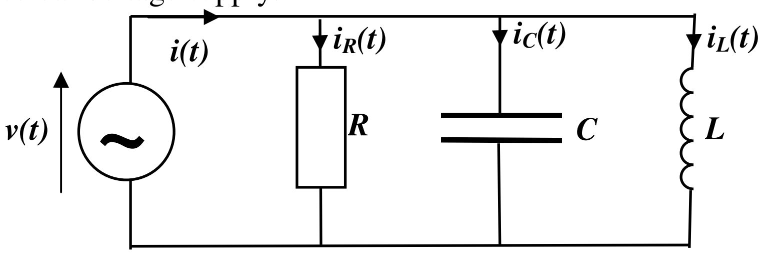
$$Q = \left|\frac{v_C}{v_R}\right| = \frac{i}{2\pi f_R CR \times i} = \frac{1}{2\pi f_R CR} = \frac{1}{2\pi CR} \times 2\pi\sqrt{LC} = \frac{1}{R}\sqrt{\frac{L}{C}}$$

For the example above:

$$Q = \frac{1}{R}\sqrt{\frac{L}{C}} = \frac{1}{10}\sqrt{\frac{0.1}{0.253 \times 10^{-6}}} = 62.8$$

Parallel resonant circuit

Let us now consider a circuit with all three basic elements, *R*, *C*, and *L*, connected in parallel across a sinusoidal voltage supply.



The total impedance of the circuit is:

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{j\omega L} + \frac{1}{\frac{1}{j\omega C}} = \frac{1}{R} - \frac{j}{\omega L} + j\omega C = \frac{\omega L - jR + j\omega^2 RCL}{\omega RL}$$

Therefore:

$$Z = \frac{\omega RL}{\omega L - j\left(R - \omega^2 RCL\right)}$$

Multiplying numerator and denominator by the complex conjugate:

$$Z = \frac{\omega RL\left[\omega L + j\left(R - \omega^2 RCL\right)\right]}{\left[\omega L - j\left(R - \omega^2 RCL\right)\right]\left[\omega L + j\left(R - \omega^2 RCL\right)\right]} = \frac{\omega RL\left[\omega L + j\left(R - \omega^2 RCL\right)\right]}{(\omega L)^2 + \left(R - \omega^2 RCL\right)^2}$$

Once again it is possible for the imaginary term to be zero. This occurs at the resonant frequency when:

$$R - \omega_R^2 RCL = 0$$

Rearranging this equation and re-writing in terms of f_R instead of ω_R gives:

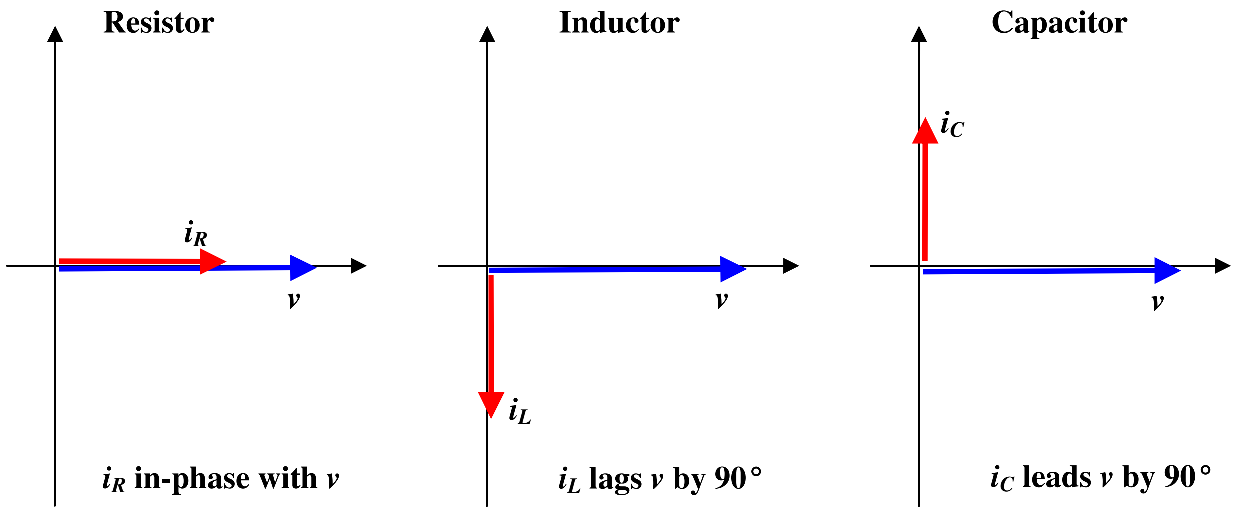
$$f_R = \frac{1}{2\pi\sqrt{LC}}$$

Back substituting in the equation for the impedance shows that once again at resonance the circuit behaves as if it were purely resistive, i.e.:

$$Z = R$$

These results are the same as for the series resonant circuit, however this time it is the capacitor current, i_C, and inductor current, i_L, which are equal in magnitude but 180° out of phase at resonance. Since each component has the same voltage (supply voltage) across it we will use this as the reference when drawing the phasor diagrams.

Phasor diagrams for a Parallel resonant circuit, taking the voltage as reference:



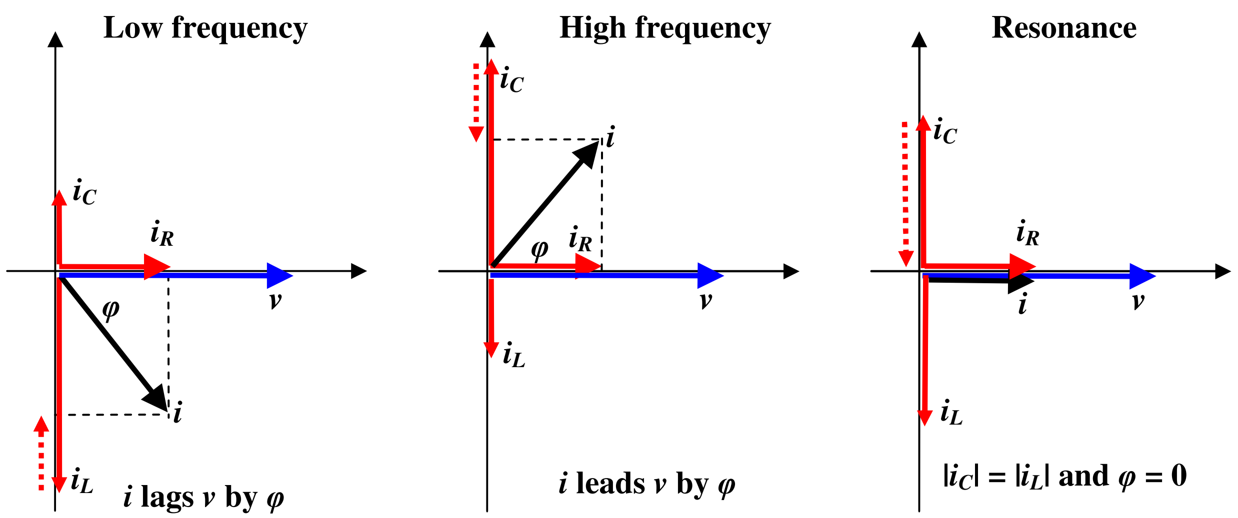
Applying Ohm's law the current in each branch may be found:

$$|i_R| = \frac{v}{R}$$

$$|i_L| = \frac{v}{X_L} = \frac{v}{\omega L} = \frac{v}{2\pi fL}$$

$$|i_C| = \frac{v}{X_C} = v\omega C = v2\pi f C$$

Clearly i_L and i_C are frequency dependant, at low frequencies i_L will be large and i_C small whereas the opposite is true at high frequencies. At the resonant frequency i_L and i_C will be equal in magnitude. Now combine the above phasor diagrams onto a single diagram.

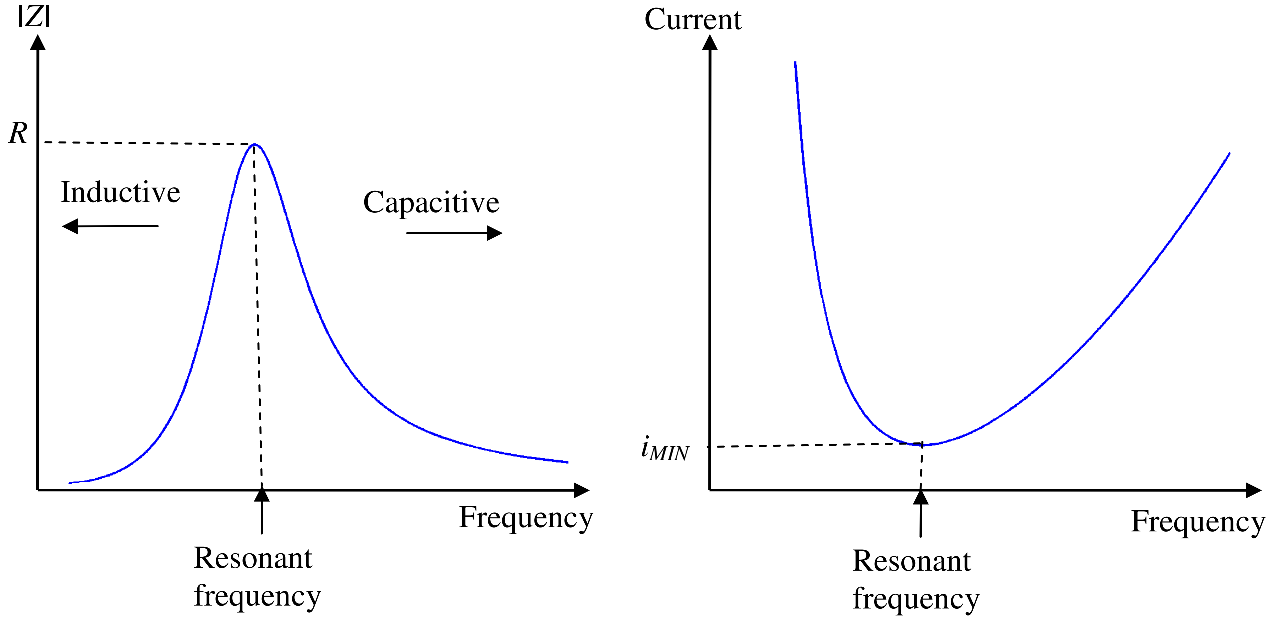


At resonance the current through the inductor, i_L, and the current through the capacitor, i_C, are equal in magnitude but 180° out of phase (i.e. opposite in direction) so they cancel out. Hence the current through the resistor is equal to the current drawn from the supply, i, the phase angle, φ, is zero, and the circuit appears resistive.

At resonance:

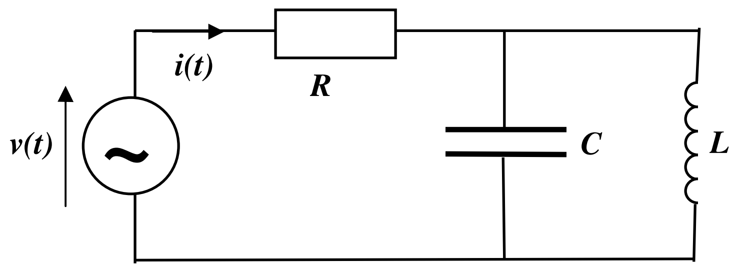
$$|i_L| = |i_C| \qquad i = i_R \qquad |Z| = R \qquad \varphi = 0^\circ$$

The magnitude of the impedance will vary with frequency, and since this is a parallel circuit the inductive reactance will be very low at low frequencies and hence dominate the overall impedance. At high frequencies the reactance of the capacitance branch will be low and this will dominate the impedance. At the resonant frequency the impedance will be a maximum and equal to the resistance. Since the impedance is maximum then the current will be a minimum at the resonant frequency and only dependant on the value of the resistor.



At low frequencies ($\omega \Rightarrow 0$) $X_L \Rightarrow 0$ and $i_L \Rightarrow$ large value
At high frequencies ($\omega \Rightarrow \infty$) $X_C \Rightarrow 0$ and $i_C \Rightarrow$ large value

With all three components connected in parallel very large currents can be drawn from the supply. To limit these it is usual to employ a slightly different arrangement where the current is limited by the resistor.



Example

A circuit consists of a 10Ω resistor, a 100μH inductor and a 253μF capacitor connected in parallel across a 10V sinusoidal voltage supply. Calculate the frequency at which the circuit becomes resonant, the total current, and the current through the inductor and capacitor at that frequency.

The resonant frequency is calculated from:

$$f_R = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{100\times10^{-6}\times253\times10^{-6}}} = 1000 \text{ Hz}$$

At resonance the impedance is purely resistive:

$$|Z| = R = 10\Omega$$

Using Ohm's law to find the current:

$$|i| = \frac{v}{R} = \frac{10}{10} = 1 \text{ A}$$

The current through the inductor is:

$$|i_L| = \frac{v}{X_L} = \frac{v}{2\pi f_R L} = \frac{10}{2\pi \times 1000 \times 100 \times 10^{-6}} = 15.9 \text{ A}$$

The current through the capacitor is:

$$|i_C| = \frac{v}{X_C} = v \times 2\pi f C = 10 \times 2\pi \times 1000 \times 253 \times 10^{-6} = 15.9 \text{ A}$$

At resonance energy circulates between L and C.

The currents are equal in magnitude as expected and cancel each other out. Despite only drawing a current of 1A from the supply, the currents through the capacitor and inductor are many times greater (15.9 A).

We can define the 'Q' factor (or 'Quality factor' or 'Magnification factor') as the current magnification at resonance.

$$Q = \frac{current\ magnification\ at\ resonance}{supply\ current} = \left|\frac{i_L}{i_R}\right| = \frac{R}{v} \times \frac{v}{2\pi f_R L} = \frac{R}{2\pi f_R L}$$

since the supply current is equal to the current through the resistor at resonance. Substituting for the resonant frequency:

$$f_R = \frac{1}{2\pi\sqrt{LC}}$$

so:

$$Q = \frac{R}{2\pi L} \times 2\pi\sqrt{LC} = R\sqrt{\frac{C}{L}}$$

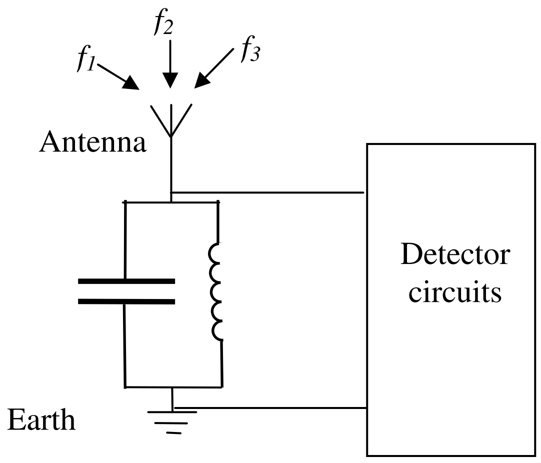
Alternatively we could have achieved the same result if we had used the capacitor current:

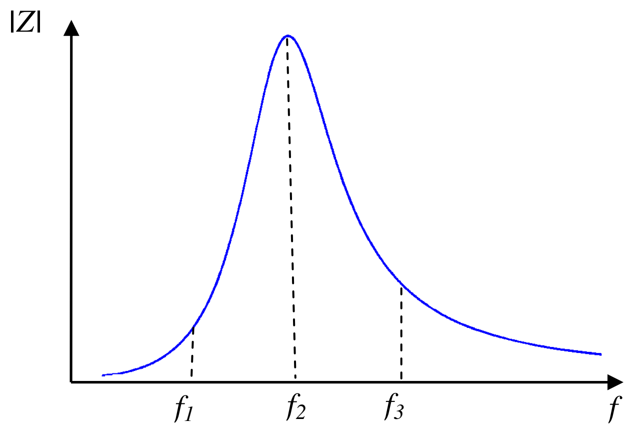
$$Q = \left|\frac{i_C}{i_R}\right| = \frac{R}{v} \times v \times 2\pi f_R C = 2\pi C R \times \frac{1}{2\pi\sqrt{LC}} = R\sqrt{\frac{C}{L}}$$

For the example above:

$$Q = R\sqrt{\frac{C}{L}} = 10\sqrt{\frac{253 \times 10^{-6}}{100 \times 10^{-6}}} = 15.9$$

<u>Application of a parallel tuned circuit - TV or radio tuner</u>





A number of frequencies (e.g. TV channels) are received by the antenna. For our diagram at f_2 the circuit presents a high impedance, whereas at other frequencies (f_1, f_3) it presents a low impedance. Hence the signal at f_2 is passed to the detector circuits for amplification, whereas f_1 and f_3 are shorted to ground. To change the selected frequency the capacitance or inductance is changed.