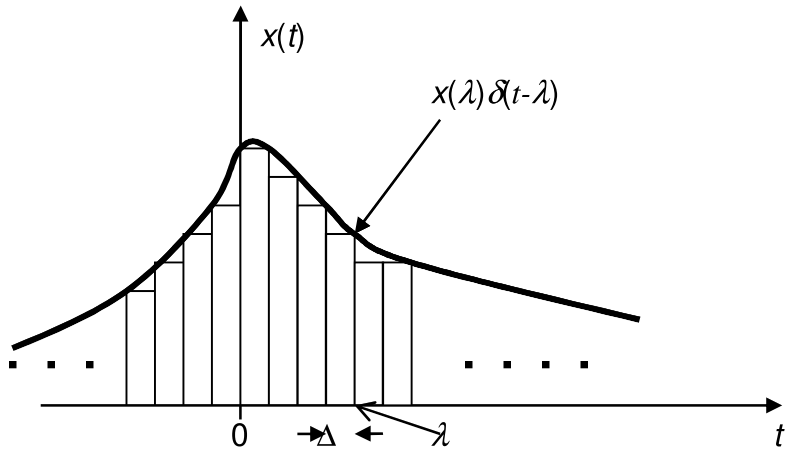
Lecture content

- Continuous Time Convolution
- Derivation of convolution integral for CT signals
- Convolution procedures for CT signals

Convolution of CT signals



Staircase approximation to a CT signal x(t).

Any CT signal can be approximated by a combination of delayed impulses if the impulse is defined as

$$\delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & otherwise \end{cases}$$

where $\Delta \to 0$. Using the sifting property of impulse the signal $x(t)$ can be represented as

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)\,d\lambda$$

If the impulse response of an LTI system is $h(t)$ we have

Convolution of CT signals

input		**response**	
$\delta(t)$	$\rightarrow$	$h(t)$	(definition),
$\delta(t\text{-}\lambda)$	$\rightarrow$	$h(t\text{-}\lambda)$	(time shifting),
$x(\lambda)\delta(t\text{-}\lambda)$	$\rightarrow$	$x(\lambda)h(t\text{-}\lambda)$(homogeneity),	

$$x(t) = \int_{-\infty}^{\infty} x(\lambda)\delta(t-\lambda)d\lambda \rightarrow \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda \quad \text{(additivity).}$$

Thus, the response of the LTI system to an input $x(t)$ is

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t-\lambda)d\lambda$$

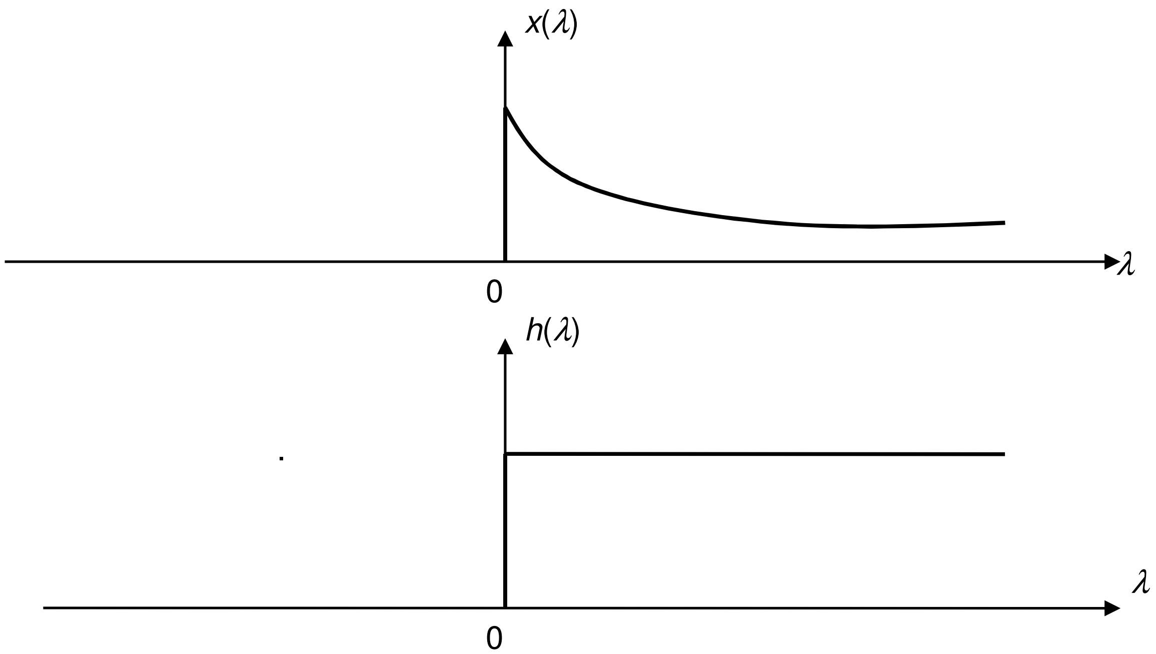
This equation is known as the ***convolution integral*** and the convolution of two signals will be represented symbolically as

$$y(t) = x(t) * h(t).$$

CT convolution procedures

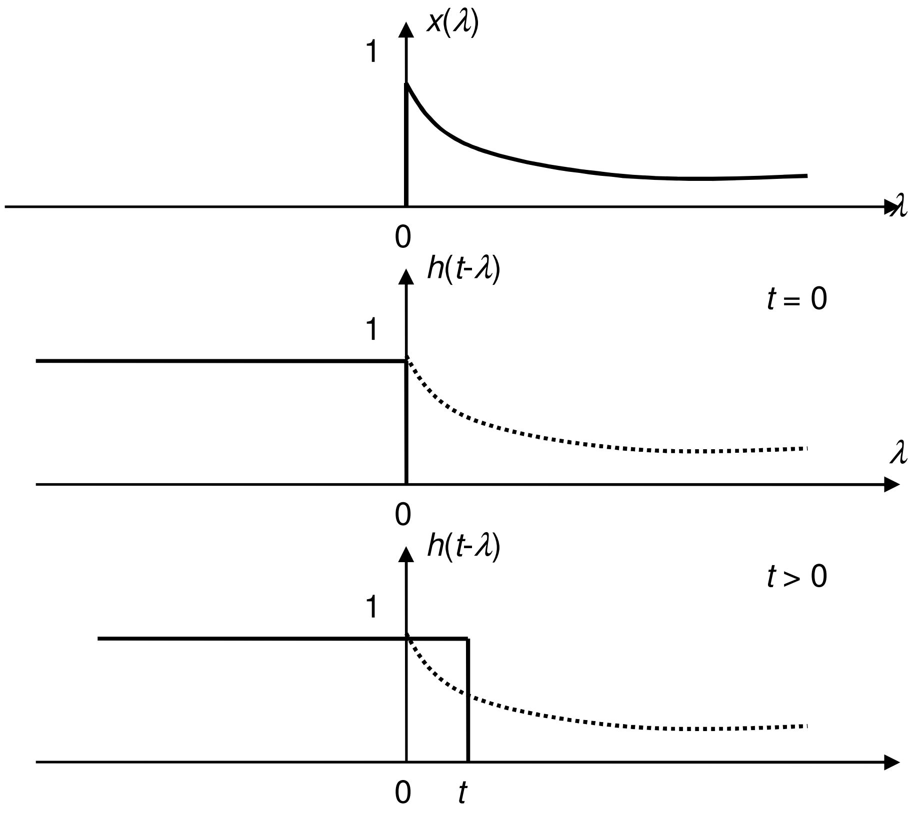
exercise: Let $h(t) = u(t)$ and $x(t) = e^{-at}u(t)$, $a > 0$. Evaluate $y(t) = h(t)*x(t)$.

1.Replacing the variable t with λ.to yield $h(\lambda)$ and $x(\lambda) = e^{-a\lambda}u(\lambda)$.



CT convolution procedures

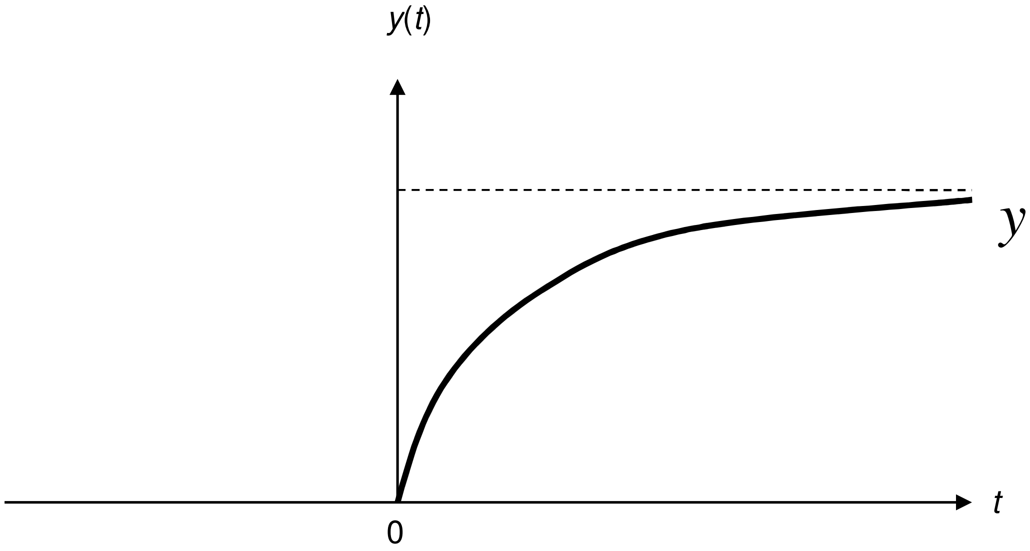
2. Flipping $h(\lambda)$ with respect to $\lambda = 0$ to obtain $h(-\lambda)$.



CT convolution procedures

3. Shift $h(\lambda)$ along the λ-axis by t to give $h(t-\lambda)$.
4. Multiply $x(\lambda)$ and $h(t-\lambda)$ for all λ. For t > 0,
5. Integrate $x(\lambda)h(t-\lambda)$ to yield

$$\boxed{y(t) = \int_0^t x(\lambda)h(t-\lambda)d\lambda = \int_0^t e^{-a\lambda}d\lambda = -\frac{1}{a}(e^{-at} - e^{-0}) = \frac{1}{a}(1-e^{-at})}$$

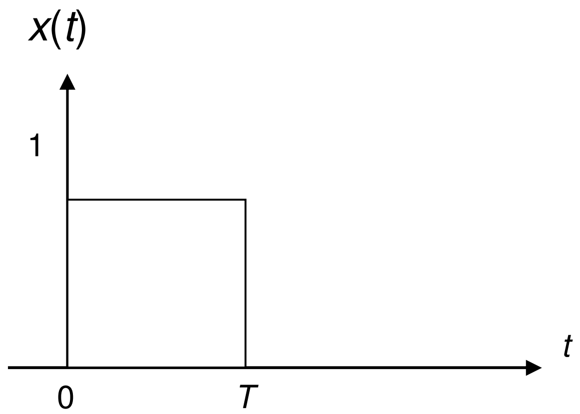


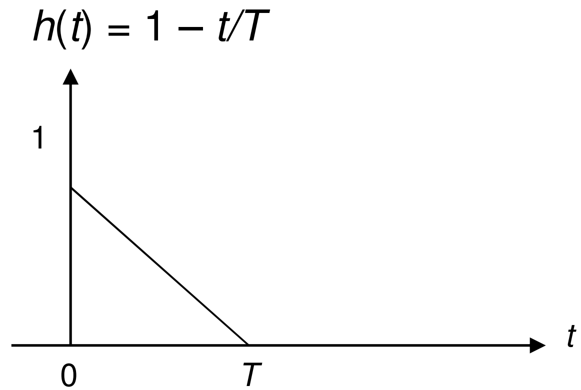
$$y(t) = \frac{1}{a}(1-e^{-at})u(t)$$

CT convolution

More examples:

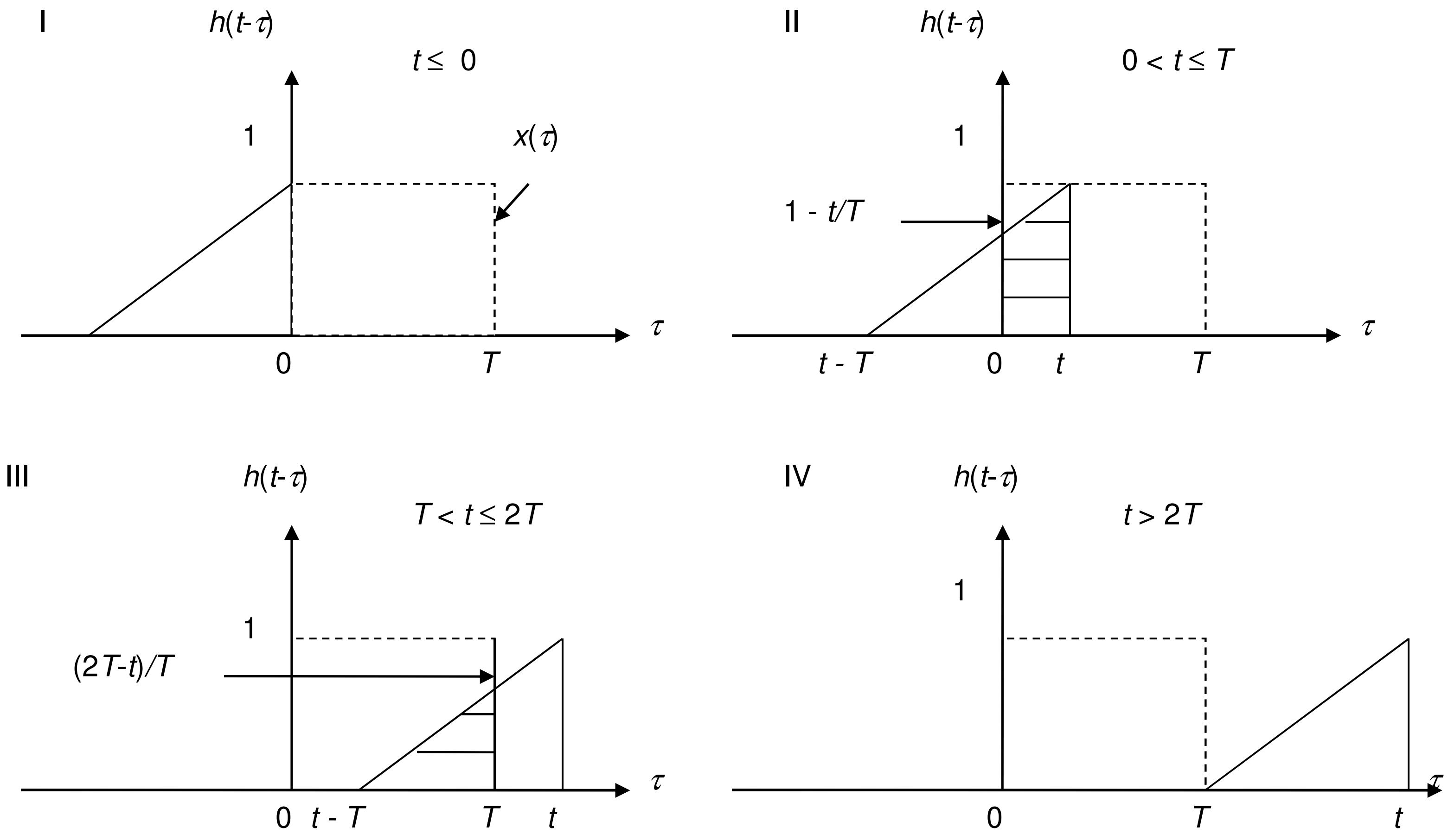
Consider the input signal $x(t)$ and impulse $h(t)$ illustrated in fig 3.13.





CT convolution

Consider the following intervals:



CT convolution

The signals are $x(t)=\begin{cases}1, & 0<t<T\\ 0, & otherwise\end{cases}$ and $h(t)=\begin{cases}1-\dfrac{t}{T}, & 0<t<T\\ 0, & otherwise\end{cases}$

Interval I: For $t \le 0$, $x(\tau)h(t-\tau) = 0$, hence $y(t) = 0$.

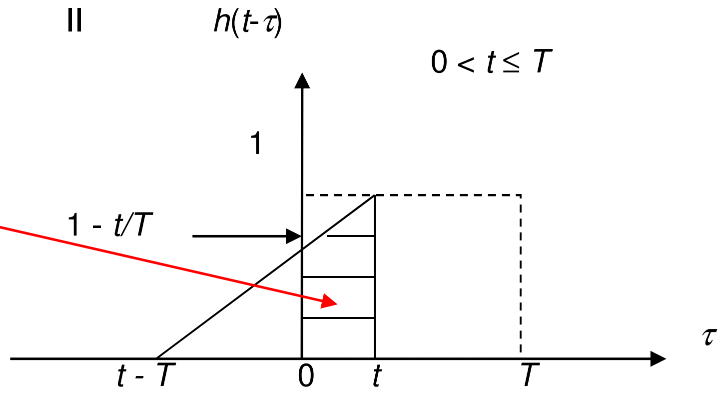
Interval II: *For $0 < t \le T$,*

$$x(\tau)h(t-\tau)=\begin{cases}1-\dfrac{(t-\tau)}{T}, & 0<\tau\le t\\ 0, & otherwise\end{cases}$$

.

Hence

$$y(t)=\int_0^t\left(1-\frac{(t-\tau)}{T}\right)d\tau$$



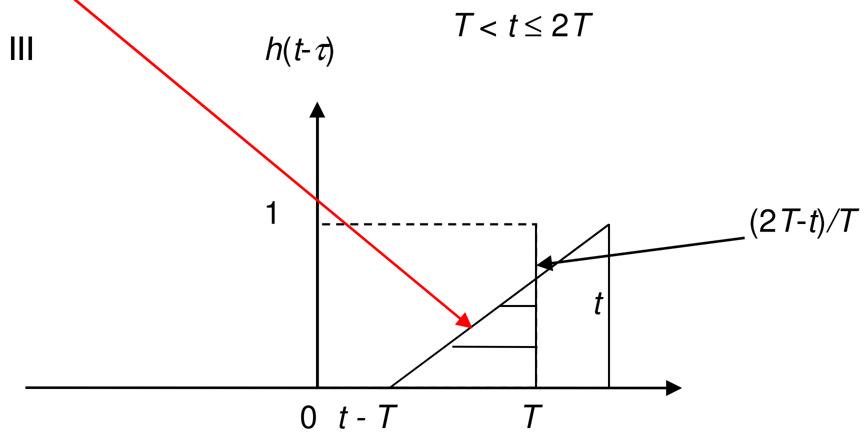
.

$$y(t)=\frac{1}{2}t\left(1+1-\frac{t}{T}\right)=t-\frac{t^2}{2T}$$

CT convolution

Interval III: For $T < t \le 2T$, $\quad x(\tau)h(t-\tau) = \begin{cases} 1 - \dfrac{(t-\tau)}{T}, & t - T < \tau \le T \\ 0, & otherwise \end{cases}$

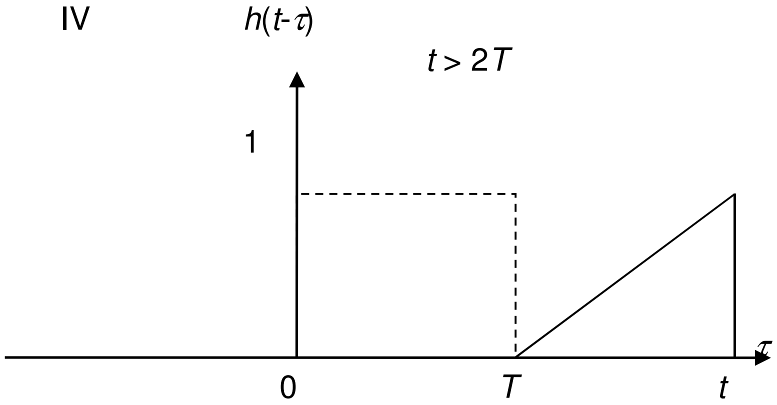
$$y(t) = \int_{t-T}^{T} \left(1 - \frac{(t-\tau)}{T}\right) d\tau$$ = overlapping area in the fig below.



$$y(t) = \frac{1}{2}\left(T - (t - T)\right)\left((2T - t)/T\right) = \frac{1}{2}(2T - t)(2T - t)/T = \frac{1}{2T}(2T - t)^2$$

CT convolution

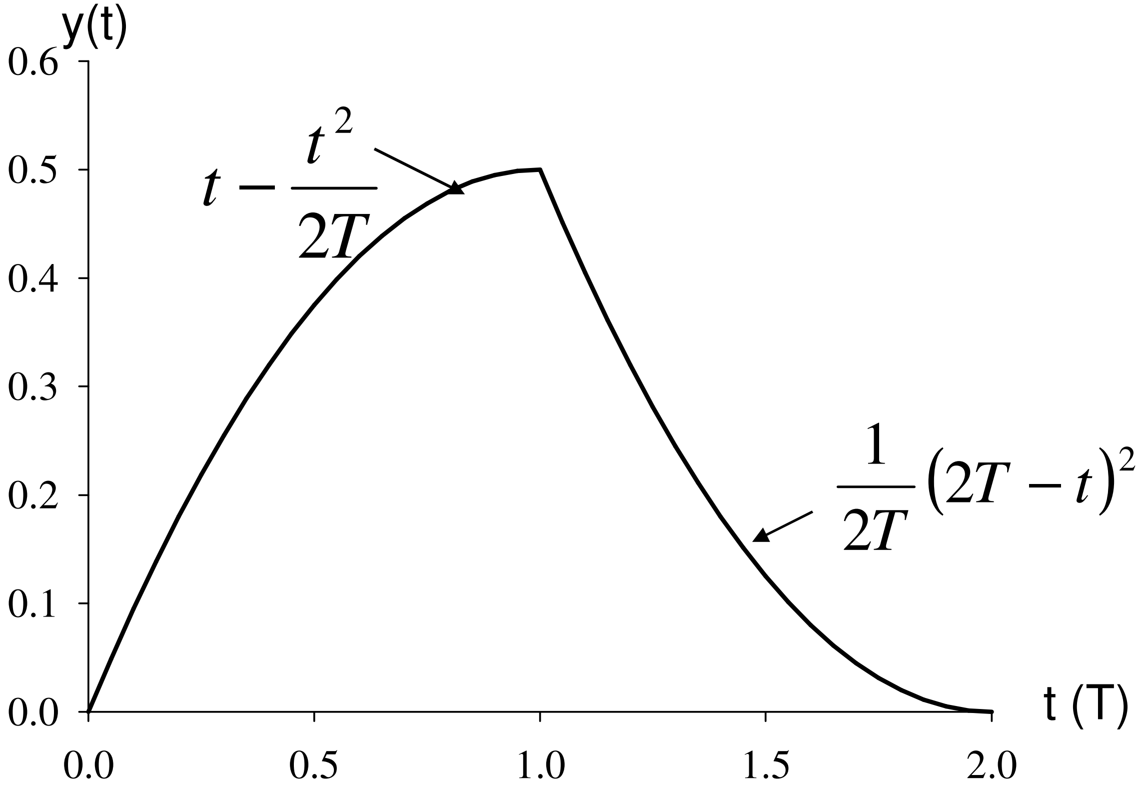
Interval IV: For $t > 2T$, $x(\tau)h(t-\tau) = 0$, hence $y(t) = 0$.



In summary we have

$$y(t)=\begin{cases} 0, & t \le 0 \\ t-\dfrac{t^2}{2T} & 0 < t \le T \\ \dfrac{1}{2T}(2T-t)^2 & T < t \le 2T \\ 0, & t > 2T \end{cases}$$

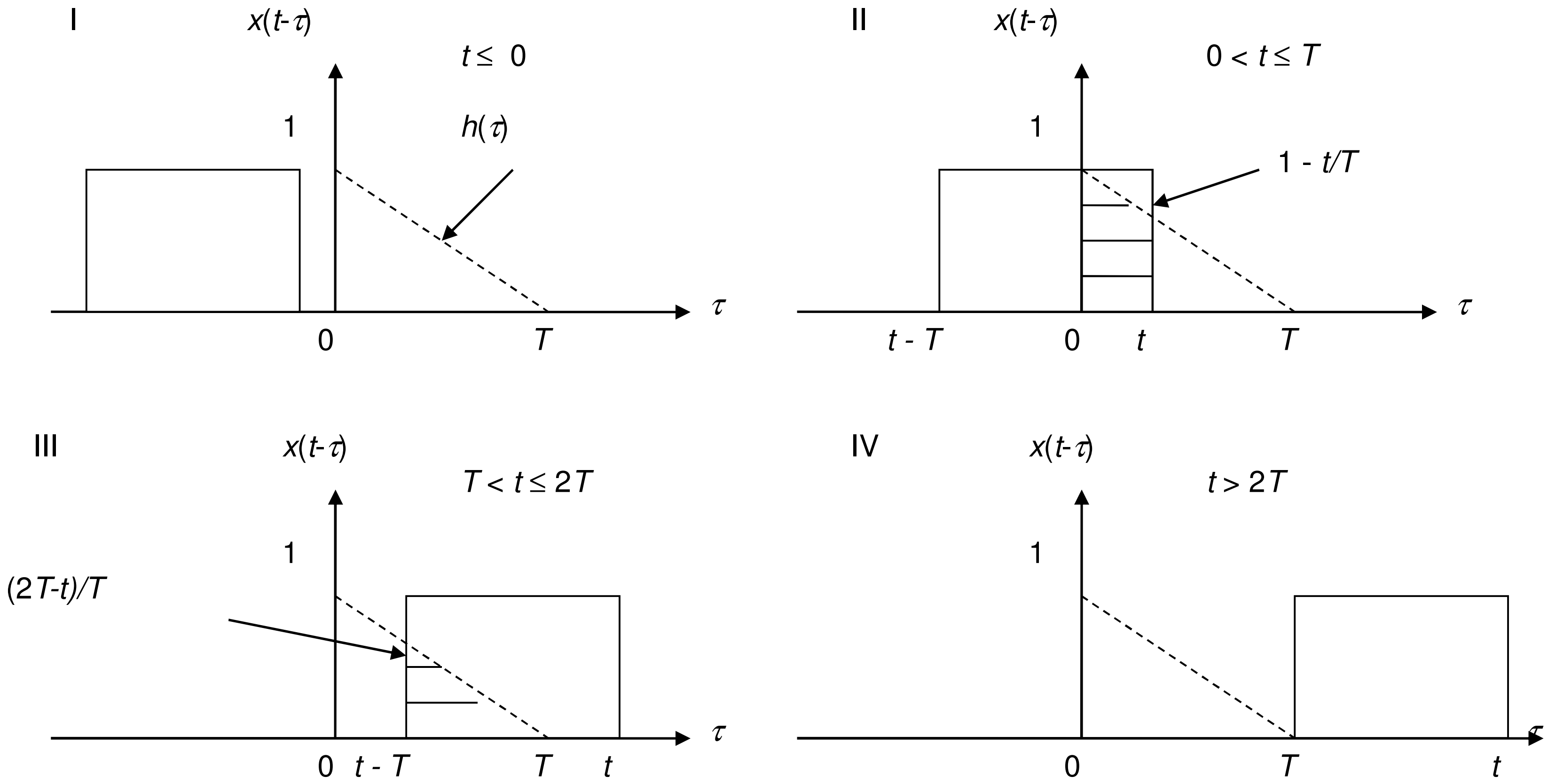
CT convolution



t	$T/4$	$T/2$	$3T/4$	T	$5T/4$	$3T/2$	$7T/4$	$2T$
$y(t)$	$7T/32$	$3T/8$	$15T/32$	$T/2$	$9T/32$	$T/8$	$T/32$	0

CT convolution

$$x(t)*h(t) = h(t)*x(t)$$

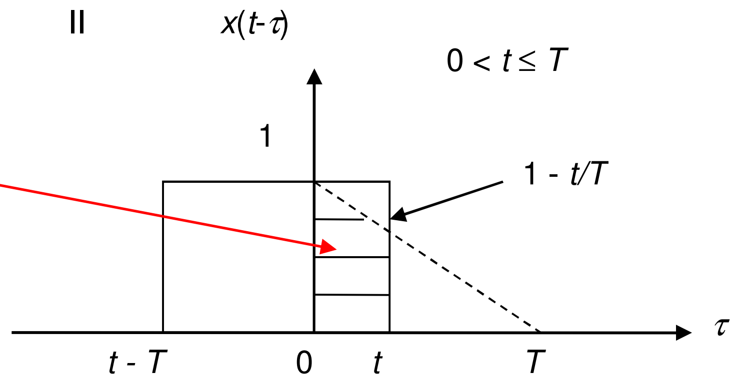


CT convolution

***Interval I**: For $t \le 0$, $h(\tau)x(t-\tau) = 0$, hence $y(t) = 0$.*

***Interval II**: For $0 < t \le T$,* $h(\tau)x(t-\tau) = \begin{cases} 1-\dfrac{\tau}{T}, & 0 < \tau \le t \\ 0, & otherwise \end{cases}$

$$y(t) = \int_0^t \left(1-\frac{\tau}{T}\right) d\tau$$

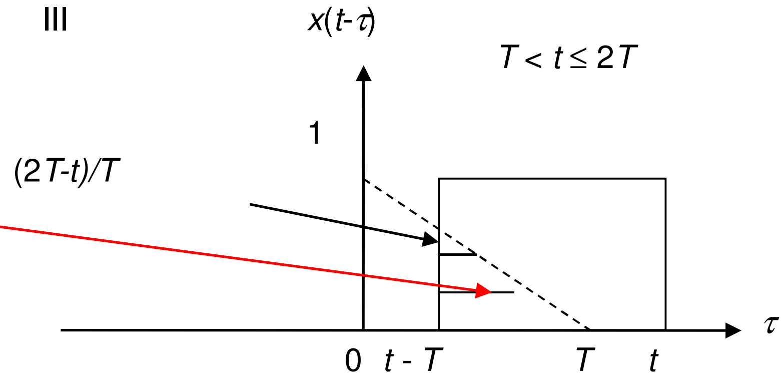


$$y(t) = \frac{1}{2} t\left(1+1-\frac{t}{T}\right) = t - \frac{t^2}{2T}$$

CT convolution

Interval III: For $T < t \le 2T$, $h(\tau)x(t-\tau) = \begin{cases} 1-\dfrac{\tau}{T}, & t-T < \tau \le T \\ 0, & otherwise \end{cases}$

$$y(t) = \int_{t-T}^{T} \left(1-\frac{\tau}{T}\right) d\tau$$



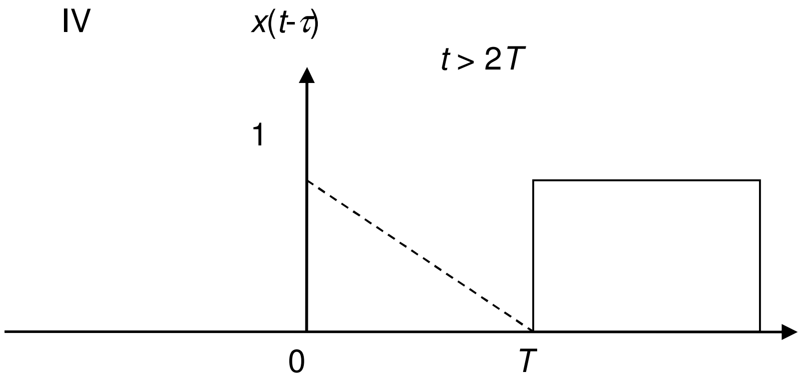
$$y(t) = \frac{1}{2}(T-(t-T))((2T-t)/T) = \frac{1}{2}(2T-t)(2T-t)/T = \frac{1}{2T}(2T-t)^2$$

CT convolution

Interval IV: For $t > 2T$, $h(\tau)x(t-\tau) = 0$, hence $y(t) = 0$.

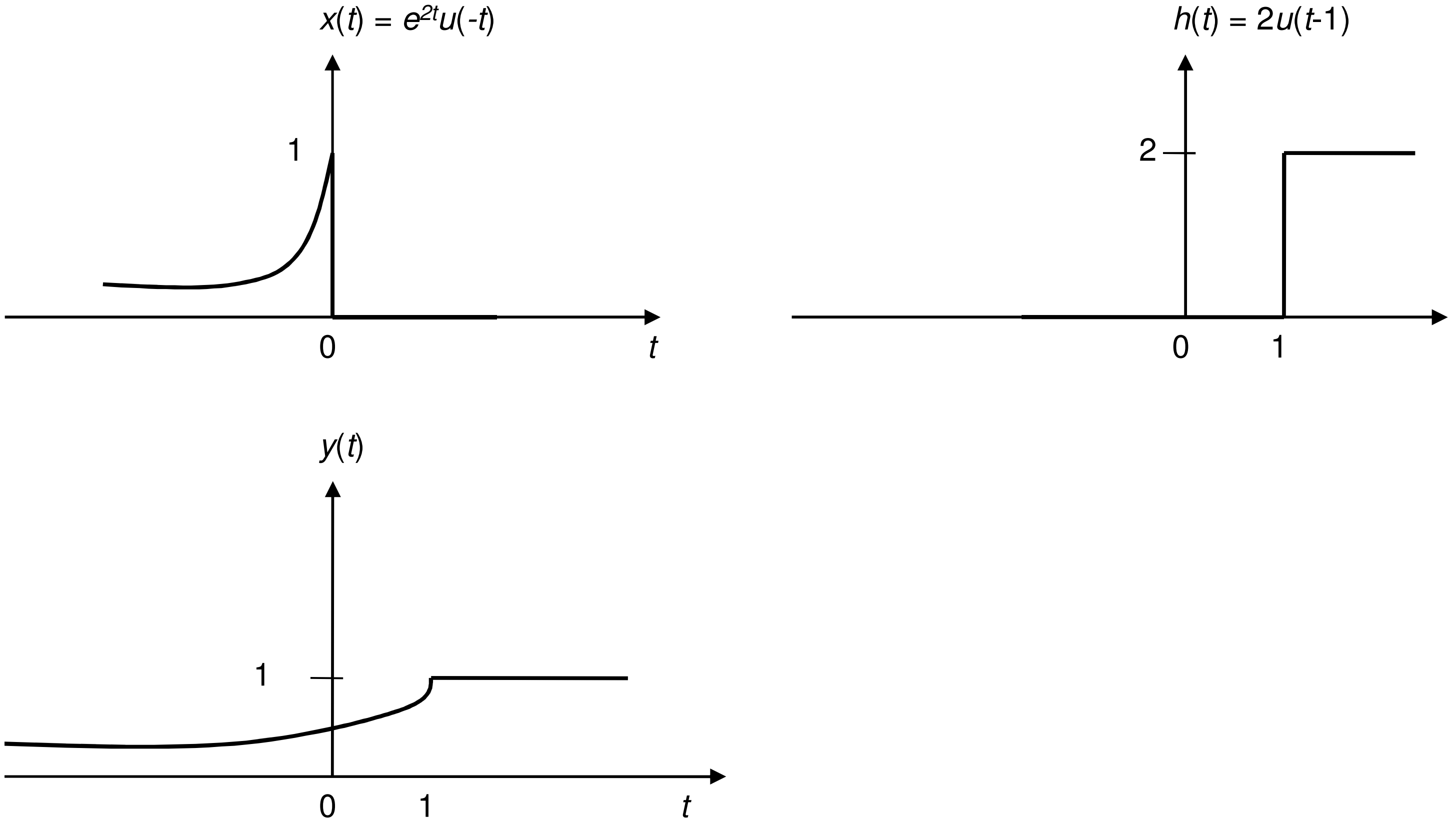
In summary we have,

$$y(t) = \begin{cases} 0, & t \le 0 \\ t - \dfrac{t^2}{T} & 0 < t \le T \\ \dfrac{1}{2T}(2T - t)^2 & T < t \le 2T \\ 0, & t > 2T \end{cases}$$



CT convolution example

$x(t) = e2tu(-t)$ and $h(t) = 2u(t-1)$

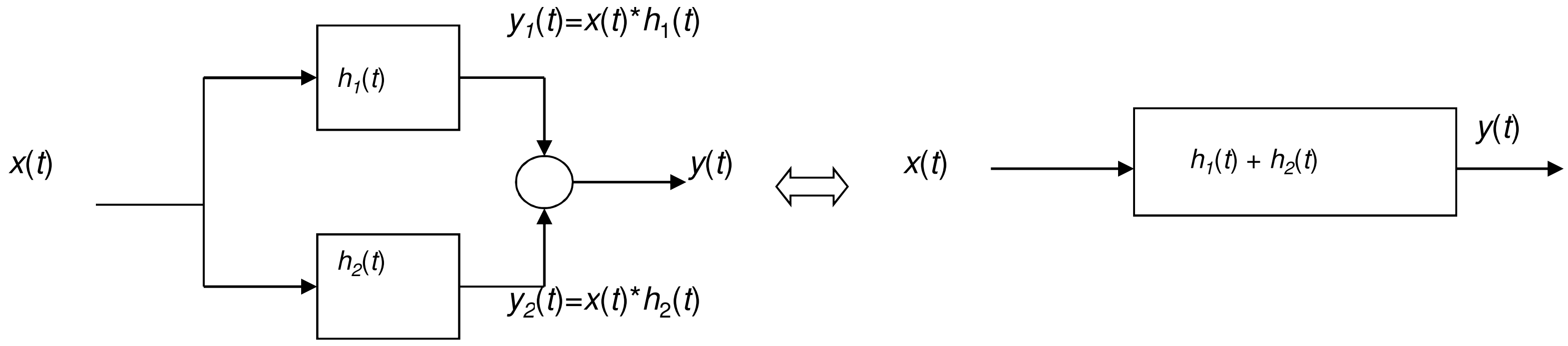


Convolution properties

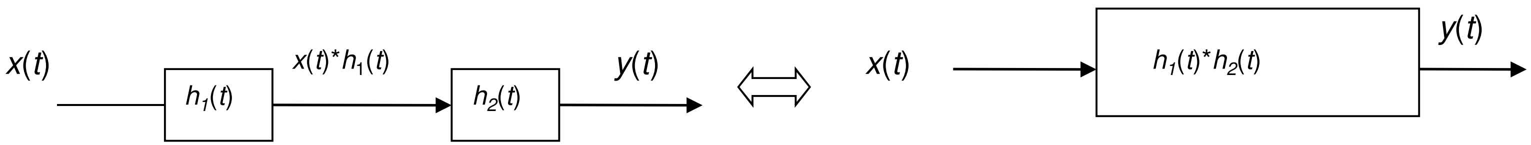
Another basic property of convolution is the ***distributive*** property. In DT

$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n] + x[n]*h_2[n]$ and

in CT $x(t)*(h_1(t) + h_2(t)) = x(t)*h_1(t) + x(t)*h_2(t)$



(a)



(b)

Another useful property of convolution is that it is ***associative***. In DT

$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n]$ and

in CT $x(t)*(h_1(t)*h_2(t)) = (x(t)*h_1(t))*h_2(t)$