EEE201 Jan 2012 Solutions

Q1(a) $\dfrac{dy(t)}{dt}+5y(t)=5x(t)$

Taking the LT,

$$sY(s)-y(0)+5Y(s)=5X(s)$$
$$Y(s)(s+5)=5X(s)+y(0)$$

To work out the forced response, $Y_{forced}(s)=\dfrac{5X(s)}{(s+5)}=\dfrac{5}{s(s+5)}=\dfrac{1}{s}-\dfrac{1}{(s+5)}$.

Therefore we have time domain forced response given by $y_{forced}(t)=[1-\exp(-5t)]u(t)$.

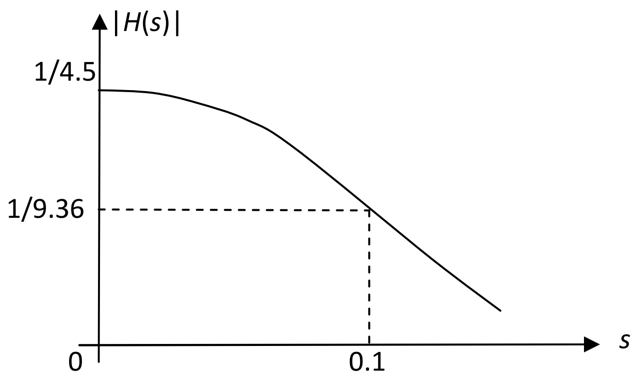
Note that since the input is *u*(*t*), the forced response will approach 1 at large *t*.

To work out the natural response, $Y_{natural}(s)=\dfrac{y(0)}{(s+5)}=\dfrac{1}{(s+5)}$ and therefore the corresponding expression in time domain is $y_{natural}(t)=[\exp(-5t)]u(t)$.

Q1(b) $H(s)=\dfrac{1}{(s+0.1)(s^2+18s+45)}$.

$$H(s)=\frac{1}{(s+0.1)(s^2+18s+45)}=\frac{1}{(s+0.1)(s+15)(s+3)}$$

There is no zeros. Poles are at s= -0.1,-3 and -15. The dominant pole is s=-0.1.



Q1(c) $H(s)=\dfrac{3s-1}{(s^2+s-6)}=\dfrac{2}{(s+3)}+\dfrac{1}{(s-2)}$.

Therefore for a causal system h(t)=(2e^{-3t} +e^{2t})u(t). Since e^{2t}u(t) is not integrable, the system is not stable.

Q2(a) Note that the RC circuit is a linear system. If H(ω) is the transfer function of the RC circuit, W(ω)=Y(ω)/W(ω)

i) Let ω_o = 100π be the fundamental frequency of the signal $w(t)$ so that $\omega=k\omega_o$, where k is an integer. $H(\omega)=\dfrac{1}{1+j\omega RC}=\dfrac{1}{1+jk100\pi RC}$

$$W(\omega)=Y(\omega)/H(\omega)=\frac{\left(\dfrac{1}{1+j100k\pi RC}\right)\left(\dfrac{4}{\pi}\sum_{k=-\infty}^{N}\dfrac{-1^k}{(1-4k^2)}\delta(\omega-100k\pi)\right)}{\left(\dfrac{1}{1+j100k\pi RC}\right)}=\frac{4}{\pi}\sum_{k=-\infty}^{N}\frac{-1^k}{(1-4k^2)}\delta(\omega-100k\pi)$$

ii)Taking the first harmonic only, we have k =-1, 0 and 1.

We have

$$Y(\omega)=\frac{4}{\pi}\left(\left(\frac{-1^{-1}}{(1-4)}\right)\left(\frac{1}{1-j100\pi RC}\right)\delta(\omega+100\pi)+\delta(\omega)+\left(\frac{-1^{1}}{(1-4)}\right)\left(\frac{1}{1+j100\pi RC}\right)\delta(\omega-100\pi)\right)$$

$$Y(\omega)=\frac{4}{\pi}\left(\frac{1}{3}\left(\frac{\delta(\omega+100\pi)}{1-j100\pi RC}\right)+\delta(\omega)+\frac{1}{3}\left(\frac{\delta(\omega-100\pi)}{1+j100\pi RC}\right)\right)$$

$$y(t)=\frac{4}{\pi}\left(\frac{1}{3}\left(\frac{1}{2\pi}\left(\frac{e^{-j100\pi t}}{1-j100\pi RC}\right)\right)+\frac{1}{2\pi}+\frac{1}{3}\left(\frac{1}{2\pi}\left(\frac{e^{j100\pi t}}{1+j100\pi RC}\right)\right)\right)$$

$$y(t)=\frac{2}{\pi^2}+\frac{2}{3\pi^2}\left[\frac{e^{j100\pi t}}{1+j100\pi RC}+\frac{e^{-j100\pi t}}{1-j100\pi RC}\right]$$

iii)The dc (or the average value) is $2/\pi^2$ and the magnitude of the ripple is given by

$$\left|\frac{2}{3\pi^2}\left[\frac{e^{j100\pi t}}{1+j100\pi RC}+\frac{e^{-j100\pi t}}{1-j100\pi RC}\right]\right|.$$

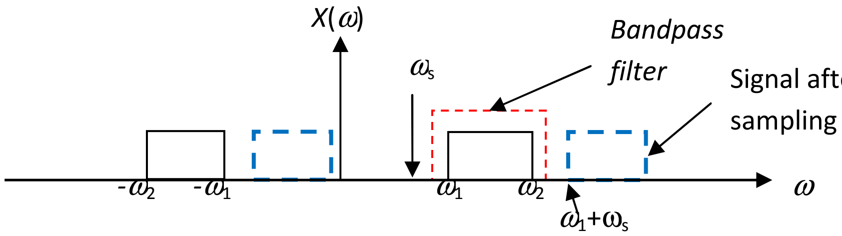
Therefore,

$$\left|\frac{2}{3\pi^2}\left[\frac{e^{j100\pi t}}{1+j100\pi RC}+\frac{e^{-j100\pi t}}{1-j100\pi RC}\right]\right|<0.01\left(\frac{2}{\pi^2}\right)$$

Solving this gives RC > 0.2s.

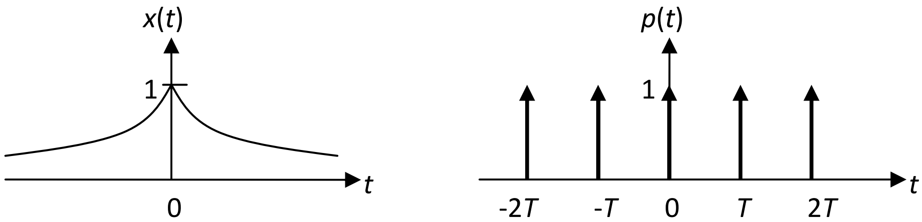
Q3(a) i)The sampling frequency is $\omega_s = 2\pi/T_s$. The largest frequency is ω_2. Hence Nyquist Theorem states that $2\pi/T_s > 2\omega_2$. Therefore $T_s < \pi/\omega_2$.

ii) Here it is best to use a graphical approach to work out the answer. After sampling the signal is illustrated below. There will be a copy of $X(\omega)$ represented by the black lines and a copy of $X(\omega)$ represented by dashed lines.

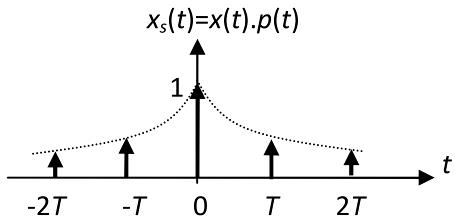


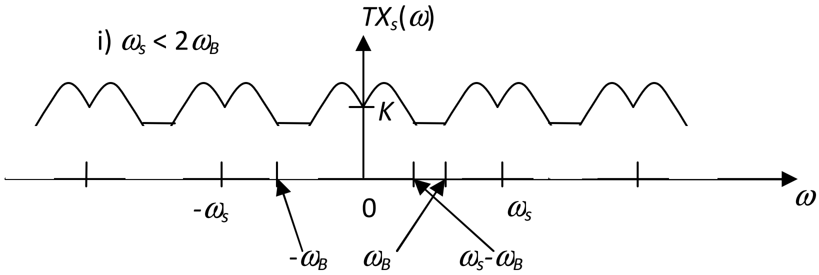
No aliasing if there is no overlap within $\omega_1 \leq \omega \leq \omega_2$ if $\omega_s + \omega_1 > \omega_2$. Therefore we have $\omega_s > \omega_2 - \omega_1$ and $T_s < 2\pi/(\omega_2-\omega_1)$. To recover the signal we need to use a band pass filter as illustrated.

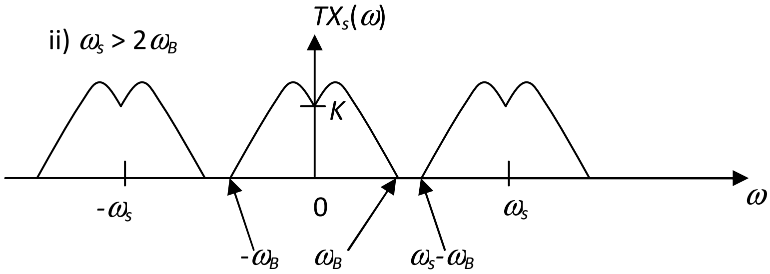
(b)



(c)







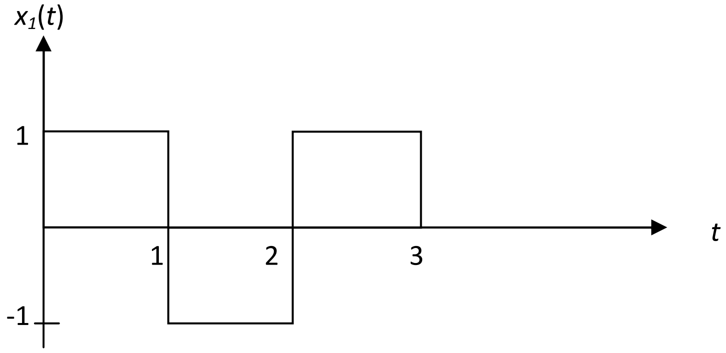
Spectrum of $x(t)$ can be recovered by low pass filtering only when $\omega_s > 2\omega_B$. this is the Nyquist sampling theorem. When $\omega_s < 2\omega_B$ the repetitions of $X(\omega\text{-}n\omega_s)$ will overlap as shown in (ii). This effect is known as aliasing.

(d) $W(\omega) = \int_{-\infty}^{\infty} w(t)e^{-j\omega t}\,dt = \int_{-\tau/2}^{\tau/2} e^{-j\omega t}\,dt$

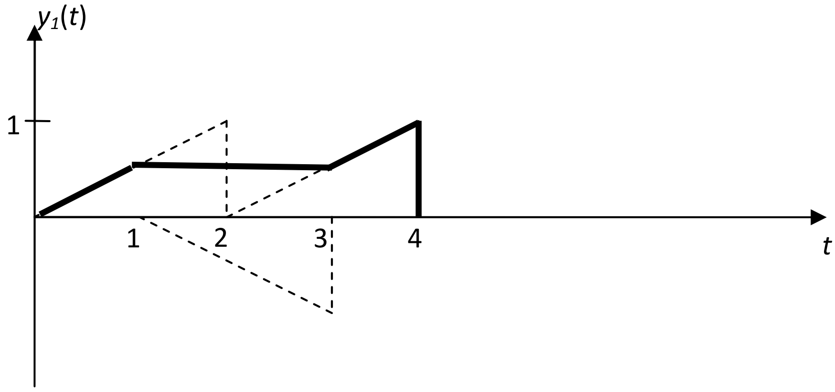
$$W(\omega) = \frac{1}{j\omega}\left[-e^{-j\omega t}\right]_{-\tau/2}^{\tau/2} = \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j\omega} = \frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j2\omega/2} = \frac{\tau}{\omega\tau/2}\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{j2}$$

$$W(\omega) = \tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}.$$

Q4(a)



$x_1(t)=x(t)-x(t-1)+x(t-2)+x(t-3)$. Therefore the output is $y(t)-y(t-1)+y(t-2)+y(t-3)$



Q4(b)

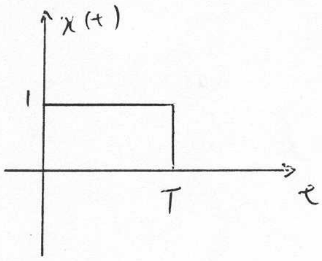
$$x[n]=\begin{cases}1, & 0\le n\le 3\\ 0, & otherwise\end{cases} \text{ and } h[n]=\begin{cases}e^{-n}, & 0\le n\le 4\\ 0, & otherwise\end{cases}$$

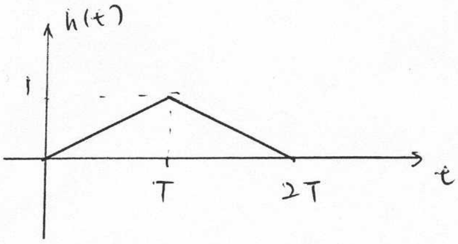
k	-4	-3	-2	-1	0	1	2	3	4	5	$\sum x[k]h[n-k]$
$x[k]$	0	0	0	0	1	1	1	1	0	0	
$h[0\text{-}k]$	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	0	0	0	1
$h[1\text{-}k]$	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	0	0	1.368
$h[2\text{-}k]$	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	0	1.503
$h[3\text{-}k]$	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0	1.553
$h[4\text{-}k]$	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0	0.571
$h[5\text{-}k]$	0	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	1	0.203
$h[6\text{-}k]$	0	0	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	e^{-1}	0.068
$h[7\text{-}k]$	0	0	0	0	0	0	0	e^{-4}	e^{-3}	e^{-2}	0.018
$h[8\text{-}k]$	0	0	0	0	0	0	0	0	e^{-4}	e^{-3}	0

Q4(c)

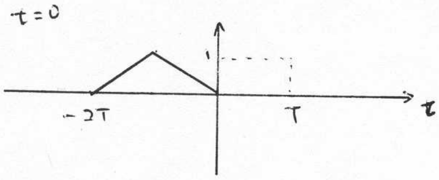
a) $y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & 0 \le t \le T \\ 0 & t > T \end{cases}$$

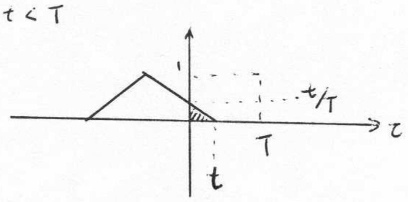




$t = 0$

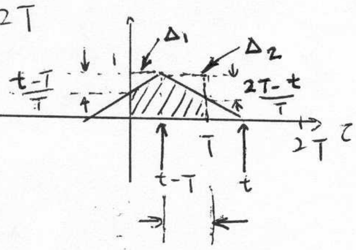


$0 \le t < T$



area $= \frac{1}{2} t\,(t/T) = \underline{\frac{1}{2} t^2/T}$

$\le t < 2T$

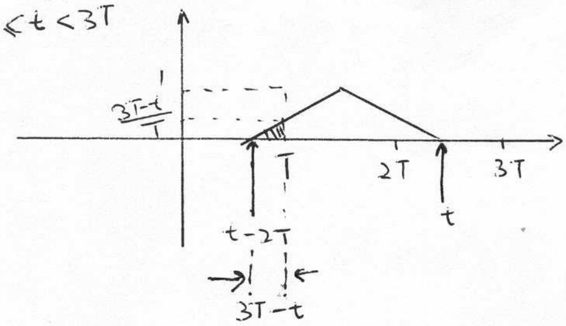


$T - (t-T) = 2T - t$

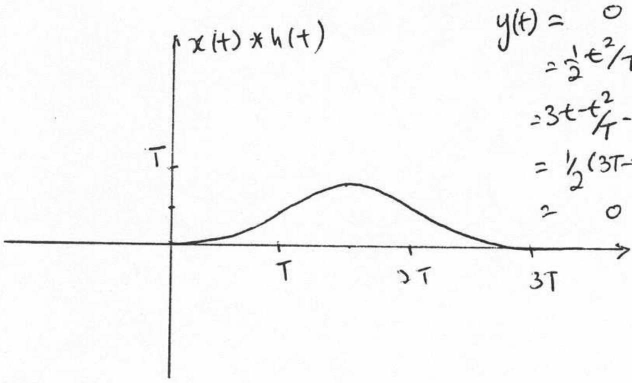
$$\begin{aligned}
\text{area} &= T - \Delta_1 - \Delta_2 \\
&= T - \tfrac{1}{2}(t-T)^2/T - \tfrac{1}{2}(2T-t)^2/T \\
&= T - \tfrac{1}{2T}(t^2 - 2Tt + T^2) - \tfrac{1}{2T}(4T^2 - 4Tt + t^2) \\
&= T - \tfrac{1}{2T}(2t^2 - 6Tt + 5T^2) \\
&= T - \tfrac{5}{2}T - t^2/T + 3t \\
&= \underline{3t - t^2/T - 3/2\,T}
\end{aligned}$$

Q4(c)

$2T \leq t < 3T$



$$\text{area} = \tfrac{1}{2}(3T-t)^2/T$$



$$
\begin{aligned}
y(t) &= 0 && t<0 \\
&= \tfrac{1}{2}t^2/T && 0 \leq t < T \\
&= 3t - t^2/T - 3T/2 && T \leq t < 2T \\
&= \tfrac{1}{2}(3T-t)^2/T && 2T \leq t < 3T \\
&= 0 && t \geq 3T
\end{aligned}
$$