SIGNALS

A signal *x(t)* is formed from any physical quantity made to vary with time or other variables such as space and altitude. The signal incorporates and relays information. Therefore signal processing is a very important component of technology as it allows the extraction of information in a signal.

Examples of signals are:
1) Electrocardiogram (ECG) signals are used to monitor heart activity. A number of electrodes monitor different parts of the heart activity. An experienced doctor can determine whether a patient's heart is normal by analysing the electrical signals from these electrodes

2) Audio signals such as speech waveform or music. Signal processing can be developed to characterise the speech signals in terms of their frequency spectrum.

3) Current fluctuation in electronic equipments due to random motion of electrons.

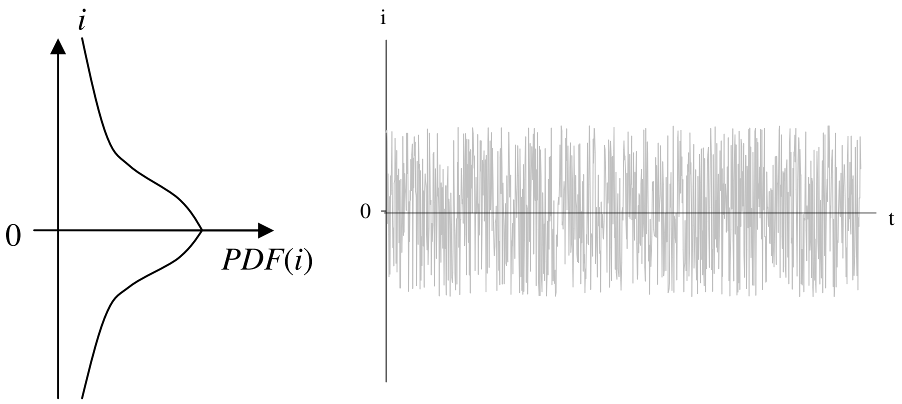
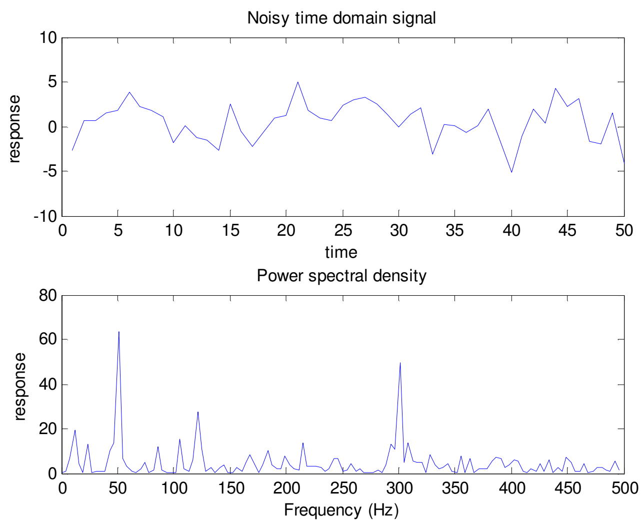


Fig. 1: Gaussian PDF of current.

TRY THIS:
In this example we show that it is possible to extract the dominant frequency components from a rather noisy time domain signal.



```
t = 0:.001:.25;
x = sin(2*pi*50*t) + sin(2*pi*120*t) + sin(2*pi*300*t);
y = x + 2*randn(size(t));
subplot (2,1,1),plot(y(1:50))
title('Noisy time domain signal')
xlabel('time')
ylabel ('response')

Y = fft(y,256);
Pyy = Y.*conj(Y)/256;
f = 1000/256*(0:127);
subplot(2,1,2),plot(f,Pyy(1:128))
title('Power spectral density')
xlabel('Frequency (Hz)')
ylabel ('response')
```

A signal *x(t)* is real-valued or scalar valued if the value of *x(t)* is a real number at time *t*. *x(t)* is a vector signal if it is a vector of some dimension for example the voltage at 3 points of an antenna. So far we have only mentioned signal that varies with time. In image processing the variation of brightness across the image is important.

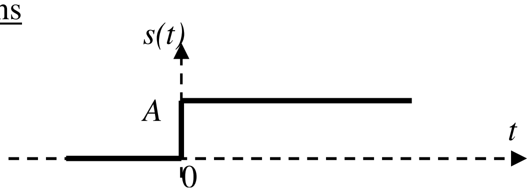
Fig. 2: Edge detection of the image.

Variation of air pressure, wind speed, water content and temperature as functions of altitude are extremely important in meteorological studies. However in this course EEE201 we will only study two basic types of signal: continuous time (CT) signals and discrete time (DT) signals. Suppose we take the temperature using a thermometer with an analogue readout. The temperature is a CT signal since it is defined for a continuum of values of time. On the other hand, if we record the temperature every hour the signal is DT signal since it is defined only at discrete time instants.

Continuous Time (CT) Signals

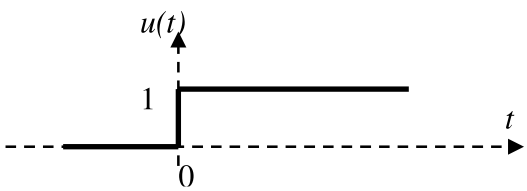
The Step and Ramp Functions

$$s(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}.$$



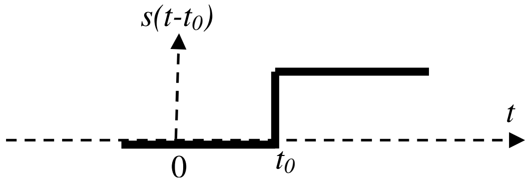
When $A = 1$ the step function is known as a **Unit Step function** defined as:

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}.$$

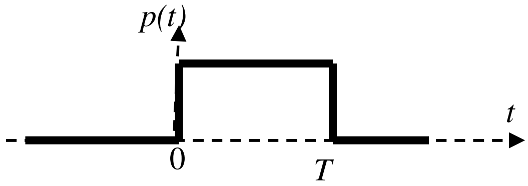


In general $s(t) = Au(t)$ where $A \neq 0$. E.g: current flow through a resistive circuit in which the switch is closed at time $t = 0$. The current is zero for $t < 0$ and has a constant value for $t \geq 0$.

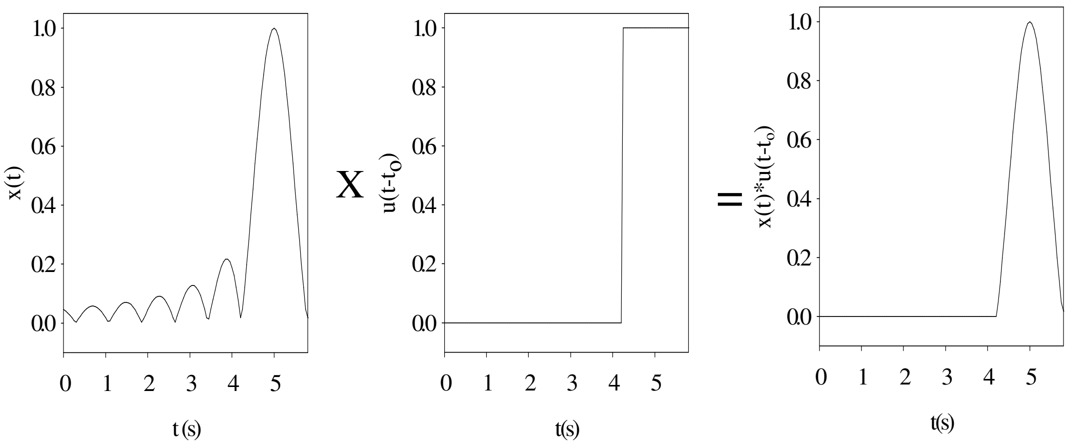
If the switch is closed at $t = t_0$ a delay step signal, $s(t-t_0)$ is obtained.



If the switch is closed at $t = 0$ and opened at time $t = T$ a pulse signal, $p(t)$ of width T is obtained.



$u(t)$ is useful for deleting parts of signals in time

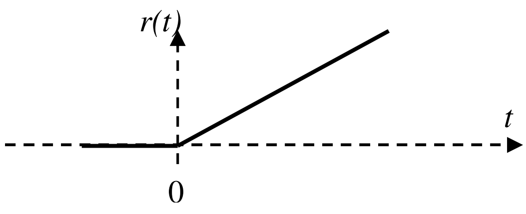


$$x(t)u(t-t_o) = \begin{cases} 0, & t < t_o \\ x(t), & t \geq t_o \end{cases}.$$

If we integrate $u(t)$, a unit ramp function, $r(t)$ is obtained.

$$r(t) = \int_{-\infty}^{t} u(\tau)d\tau$$

$$r(t) = \begin{cases} 0, & t < 0 \\ \int_0^t 1d\tau = t, & t \geq 0 \end{cases}.$$



For any slope other than unity we can use $kr(t)$ where $k \neq 0$.

Periodic CT Signals

A CT signal $x(t)$ is periodic with period T if $x(t) = x(t+T)$ for all T.

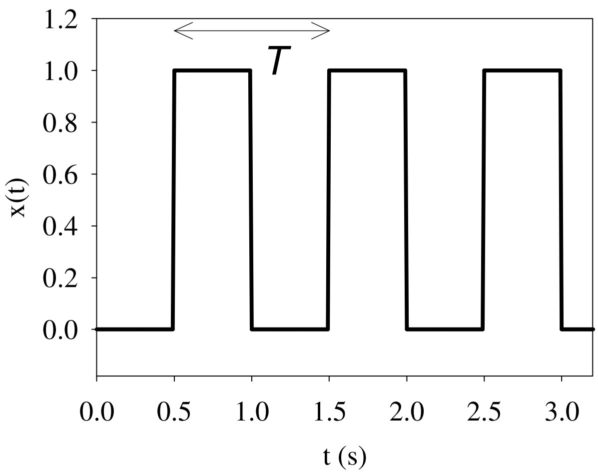


Fig. 3: A square wave with period T.

Unit impulse

Unit impulse $\delta(t)$ is an idealisation of a signal that

- has an area of unity : $\int_{-a}^{a} \delta(\tau)d\tau = 1$ for any real number $a > 0$.
- is zero for all nonzero values of t: $\delta(t) = 0$ for $t \neq 0$.

for example:

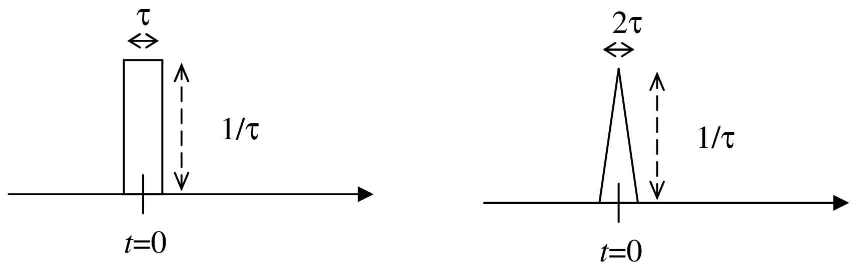
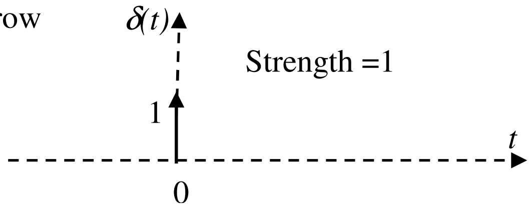


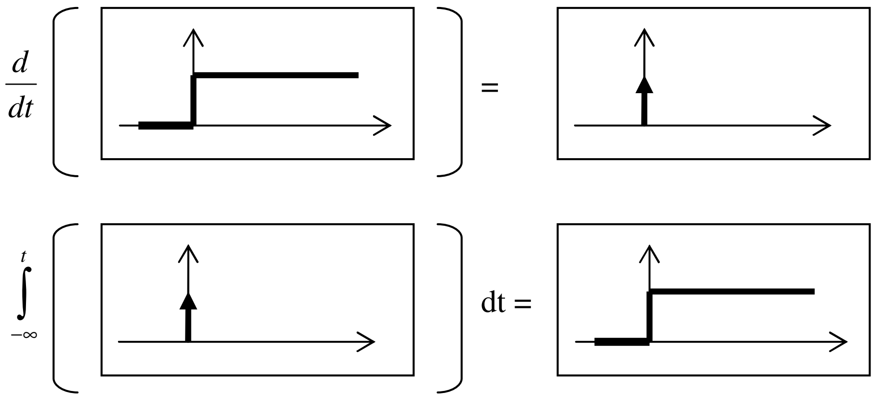
Fig. 4: Unit impulse approximated by a square pulse and a triangle pulse when $\tau \to 0$.

$\delta(t)$ is often represented by an arrow



For any real number K, $K\delta(t)$ is the impulse with area K. What happens if we integrate $\delta(t)$?

$$\int_{-\infty}^{t}\delta(\tau)d\tau = u(t) \Leftrightarrow \frac{du(t)}{dt} = \delta(t)$$



$\delta(t\text{-}t_o)$ is the impulse $\delta(t)$ shifted from $t = 0$ to $t = t_o$.

It can be shown that $x(t)\delta(t - t_o) = x(t_o)\delta(t - t_o)$.

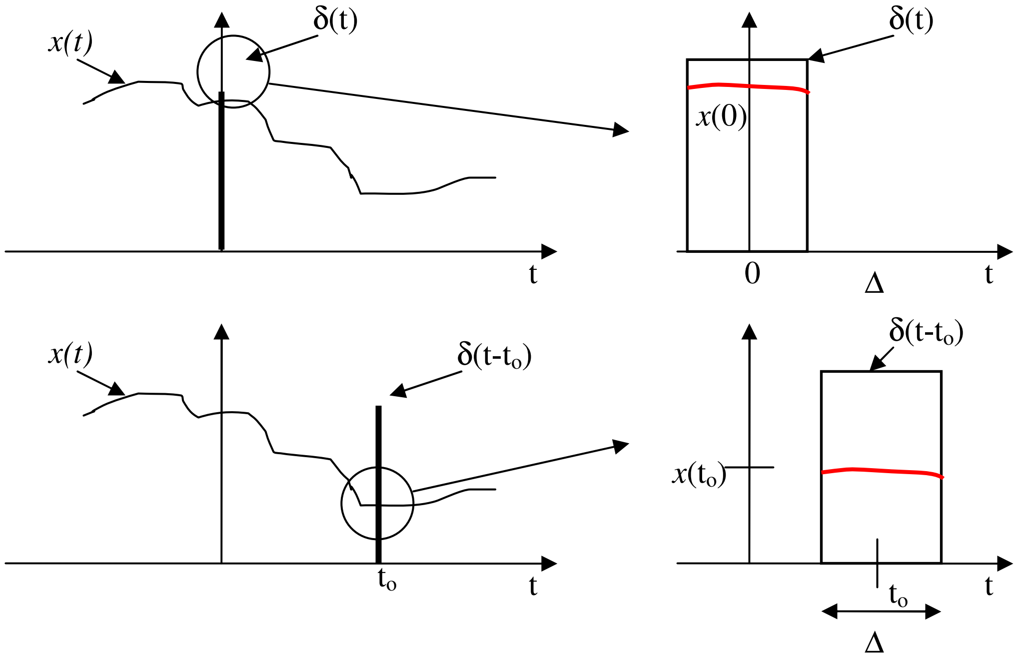


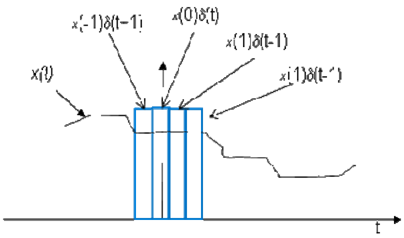
Fig. 5: Product of $x(t)\delta(t)$ and $x(t)\delta(t\text{-}t_o)$.

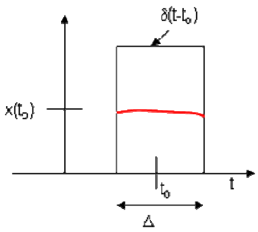
Consider the product $x(t)\delta(t)$ depicted above. If $\Delta \rightarrow 0$, $x(t)\delta(t) \approx x(0)\delta(t)$. Using similar arguments for an impulse shifted to time t_o we have,

$x(t)\delta(t-t_o) = x(t_o)\delta(t-t_o)$. This product of a signal $x(t)$ with an impulse is very important in sampling of CT signals because any CT signals can be considered as a summation of an infinite number of adjacent scaled impulses, i.e $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$.

$$x(t) = \ldots + x(-1)\delta(t+1)\Delta + x(0)\delta(t)\Delta + x(1)\delta(t-1)\Delta + x(2)\delta(t-2)\Delta + \ldots\ldots$$

$$x(t) = \sum_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\Delta$$





When $\Delta \to 0$, $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$.

Consider,

$\int_{-a}^{a} f(t)\delta(t-t_o)dt = \int_{-a}^{a} f(t_o)\delta(t-t_o)dt = f(t_o)\int_{-a}^{a} \delta(t-t_o)dt = f(t_o)$, provided that $-a < t_o < a$. Therefore we can move the function $f(t)$ out of the integration and substitute $t = t_o$.

exercise:

$$\int_{-1}^{3} f(t)(3-\delta(t-1)+2\delta(t+3))dt.$$

exercise: Show that $\delta(kt) = \frac{1}{k}\delta(t)$.

$\int_{-\infty}^{\infty} f(t)\delta(t)dt = f(0)$. Let $u =$ kt.

The exponential

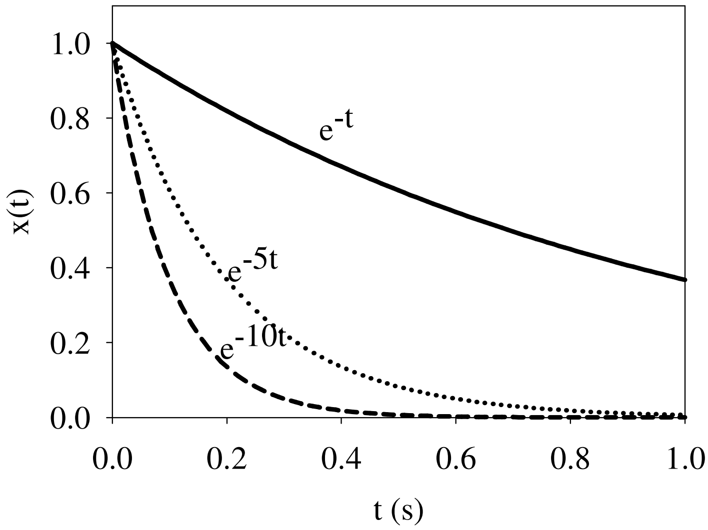


Fig. 6: Exponential signals.

$x(t) = e^{-at}$, $t \geq 0$. If a is negative $x(t)$ grows exponentially. If $a = 0$, $x(t) = u(t)$.

At what time will $x(t) = 0$? Mathematically this happens when $t = \infty$. In practice we often consider $x(t) = 0$ if its magnitude is less than 1% of its peak magnitude. Let us define $1/\tau = a$ where τ is called the time constant.

At $t = t+\tau$, $\dfrac{e^{-a(t+\tau)}}{e^{-at}} = e^{-a\tau} = e^{-1} = 0.37$ (37% of its original value).

At $t = t+5\tau$, $\dfrac{e^{-a(t+5\tau)}}{e^{-at}} = e^{-a5\tau} = e^{-5} = 0.007$ (0.7% of its original value). Thus, we often consider e^{-at} to reach zero after 5τ.

If a =- $j\omega_o$ (purely imaginary) then $x(t) = e^{j\omega_o t}$ is periodic, where ω_o is the fundamental frequency. We will use the exponential signals in much of our signals and systems analysis. Exponential signals with frequencies that are integer multiple of ω_o are known as the harmonics.

The sinusoidal

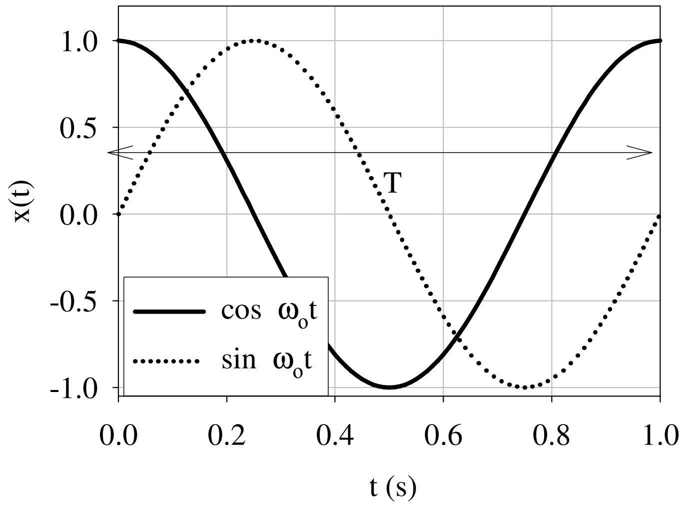


Fig. 7: Sinusoidal signals with period T.

A sinusoidal signal is given by $v(t) = V\sin(\omega_o t) = V\cos\left(\omega_o t - \pi/2\right)$. T is the periodic time in s. $T = \dfrac{1}{f_o} = \dfrac{2\pi}{\omega_o}$ where ω_o (f_o = $\omega_o/2\pi$) is the fundamental frequency. The sinusoidal signal is closely related to exponential signal via Euler's relation which can be written as

$e^{j\omega_o t} = \cos\omega_o t + j\sin\omega_o t$. Alternatively,

$\sin(\omega_o t) = \dfrac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}$ and $\cos(\omega_o t) = \dfrac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}$.

exercise: Sketch the magnitude of $y(t) = e^{j2t} + e^{jt}$.

<u>Manipulations of CT signals</u>

Amplitude scaling: $y(t) = Ax(t)$

The signal $x(t)$ is scaled by a factor A to give another signal $y(t)$.

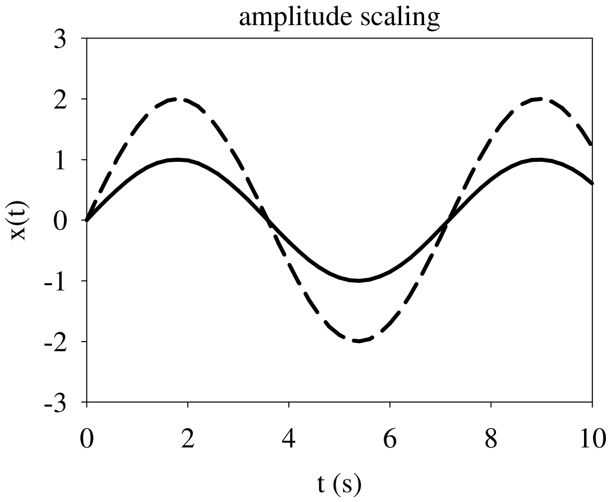


Fig. 8: Amplitude scaling.

Time scaling: $y(t) = x(At)$

$y(t)$ is a time-compressed (speed up, if $A > 1$) or a time-expanded (slowed down, if $A < 1$) version of $x(t)$.

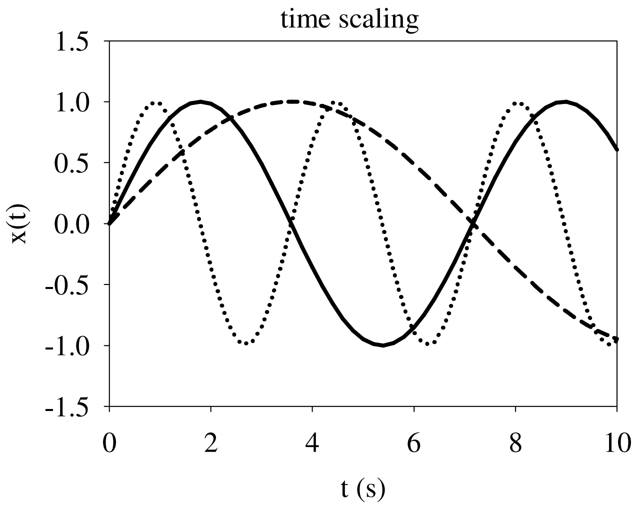


Fig. 9: Time-compressed (dotted-line) and time-expanded (dashed-line).

Time shifting: $y(t) = x(t - t_0)$ or $x(t+t_0)$

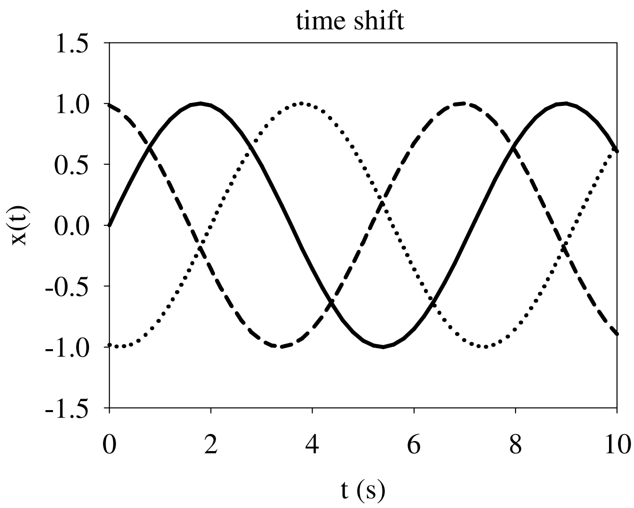


Fig. 10: Time shift.

To add/subtract two continuous signals, their individual values at every instant are added/subtracted together.

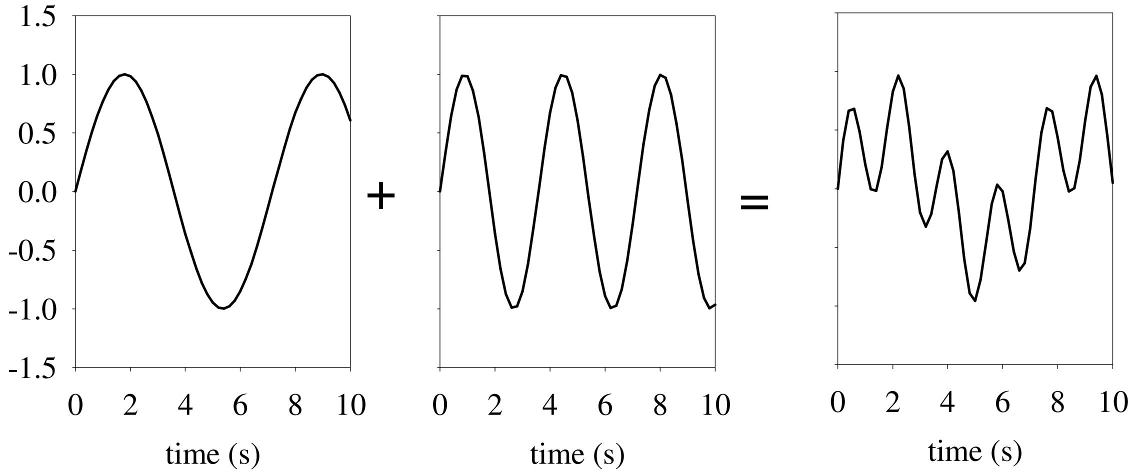
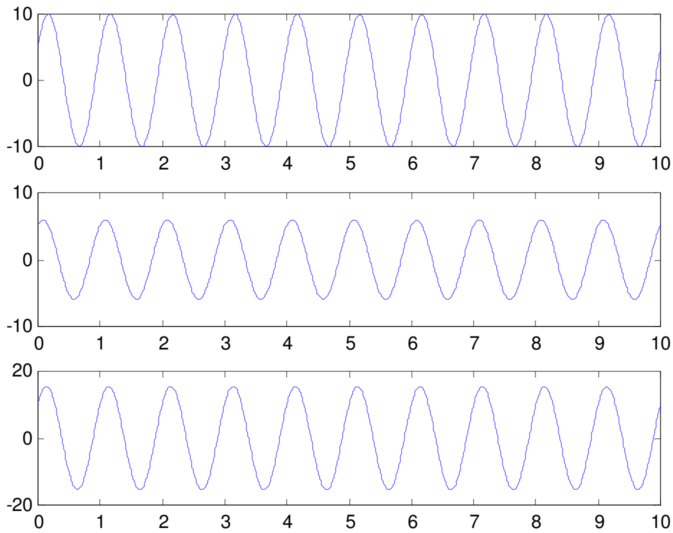


Fig.11: Addition of two sinusoidal signals.

If two sinusoids of the same frequency are added, the result is another sinusoid of the same frequency, regardless of the amplitude and phase of the original components. $A\sin(\omega t+\alpha)+B\sin(\omega t+\beta)=C\sin(\omega t+\gamma)$ where

$C=\sqrt{X^2+Y^2}$, $\tan\gamma = Y/X$, $X = A\cos\alpha + B\cos\beta, Y = A\sin\alpha + B\sin\beta$.

exercise: Evaluate $10\sin(2\pi t+\pi/6)+6\sin(2\pi t+\pi/3)$.



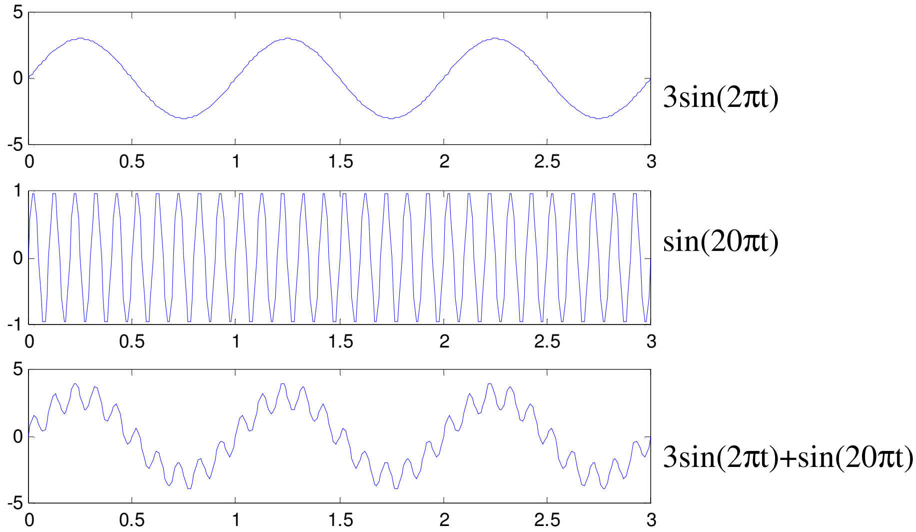
10sin(2πt+π/6)

6sin(2πt +π/3)

15.5sin(2πt+0.72)

If two sinusoids of different frequencies are added and if the frequency of one is an exact multiple of the other a non-sinusoidal repetitive signal is obtained. This non-sinusoidal has a fundamental frequency, which equals the lower frequency of the two sinusoids. The higher frequency refers to as harmonic will determine the waveshape.

In fact any periodic signal can be decomposed into a sum of sinusoids. We will analyse this in more detail in Fourier Series.



To multiply two continuous signals, their individual values at every instant are multiply together.

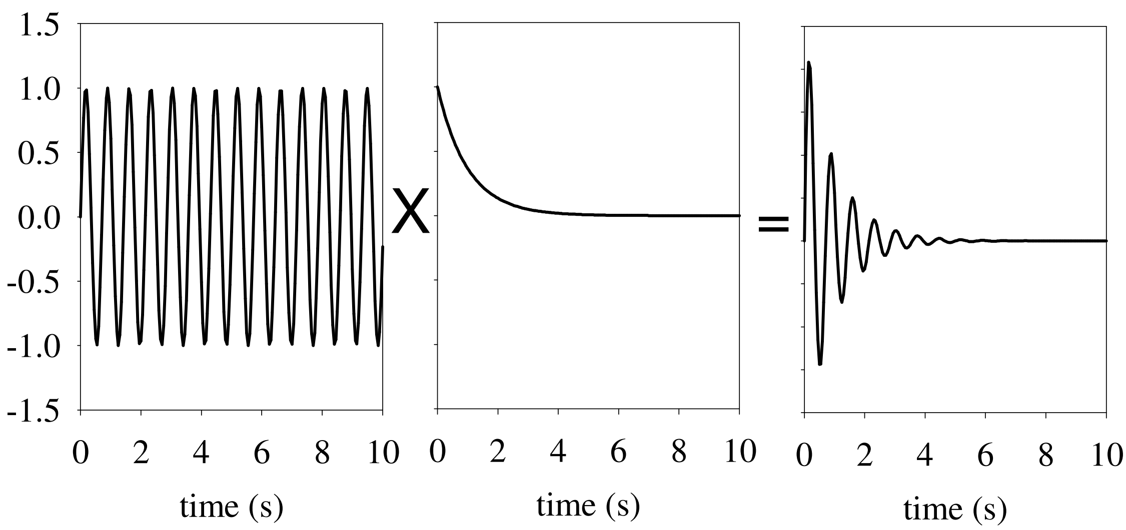


Fig. 12: Multiplication of a sinusoidal signal and an exponential signal.

Sinusoidal signals multiplied by an exponential are usually known as damped sinusoids.

Even and Odd Signals

A signal $x(t)$ has an even symmetry if $x(t) = x(-t)$. On the other hand if $x(t) = -x(-t)$ the signal $x(t)$ is said to have an odd symmetry.

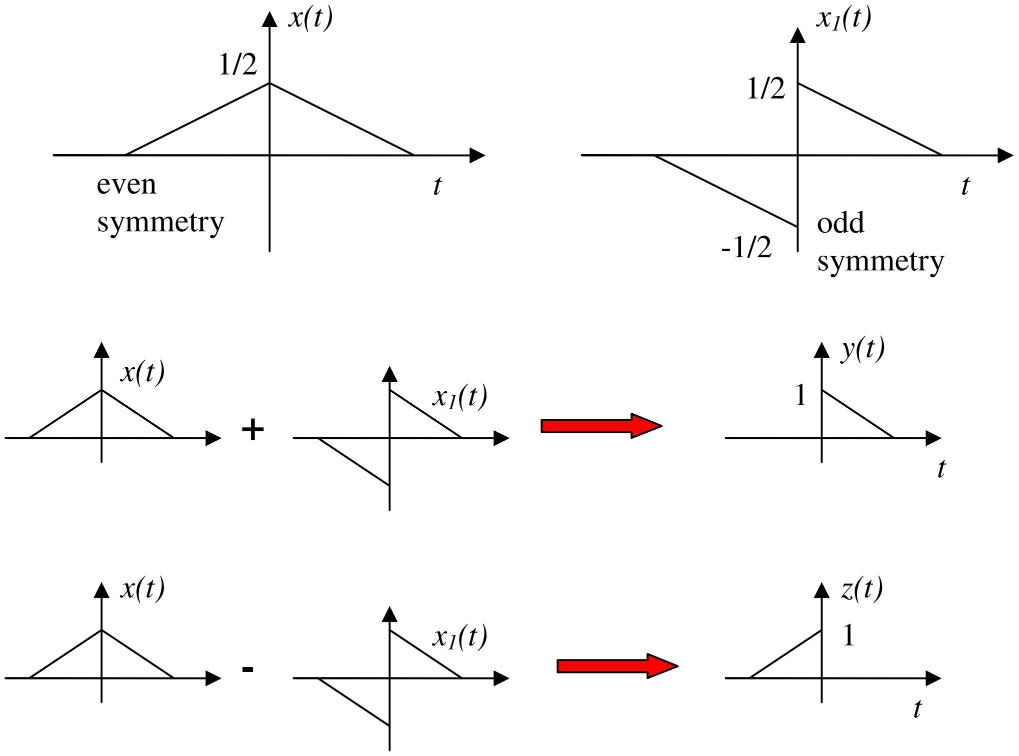


Fig.13: Even and odd symmetry.

Any signal can be broken into a sum of an even and an odd signal. e.g: $y(t) = x(t) + x_1(t)$ and $z(t) = y(-t) = x(t) - x_1(t)$.

The even component of a signal $y(t)$ is given by $y_{even}(t) = \frac{1}{2}\left[y(t) + y(-t)\right]$.

The odd component of a signal $y(t)$ is given by $y_{odd}(t) = \frac{1}{2}\left[y(t) - y(-t)\right]$.

If $y_1(t)$ and $y_2(t)$ are even signals, $y_1(t) \pm y_2(t)$, $y_1(t) \times y_2(t)$ and $y_1(t) \div y_2(t)$ are also even signals.

If $y_1(t)$ and $y_2(t)$ are odd signals, $y_1(t) \pm y_2(t)$ is odd but $y_1(t) \times y_2(t)$ and $y_1(t) \div y_2(t)$ are even signals.

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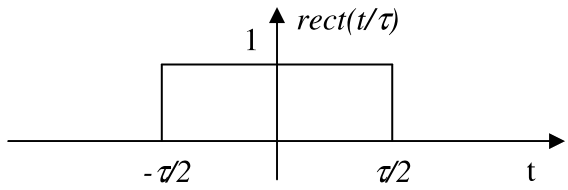
Piecewise continuous signals

Consider: A rectangular pulse

$$rect(t/\tau) = \begin{cases} 1 & \frac{-\tau}{2} \le t < \frac{\tau}{2} \\ 0 & t < \frac{-\tau}{2}, t \ge \frac{\tau}{2} \end{cases},$$



where τ is a fixed positive number (the pulse duration). *rect(t/τ)* is discontinuous at $t = \pm\frac{\tau}{2}$.

A **piecewise continuous function** is **continuous at all *t* except at a finite collection of t_1**, e.g:train of pulses.

A CT signal can be approximated by piecewise-constant or stair-step function.

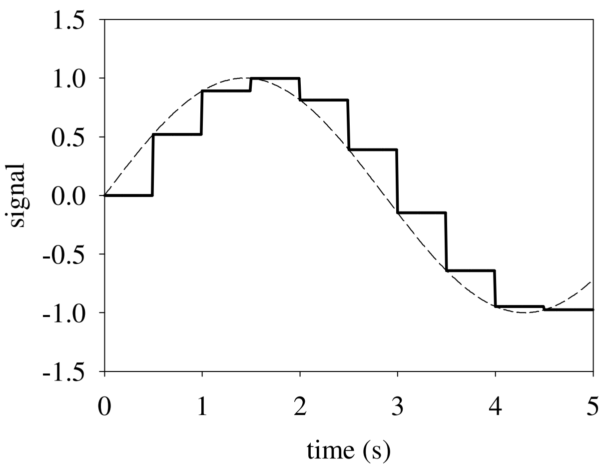


Fig.14: Piece-wise approximation of a sinusoidal signal.

Discrete signals (or Discrete-time signals)

We can make time take on discrete values by only allowing *t* values to be given by $t = kT$ where *k* is an integer, *T* is fixed, positive and real. A discrete signal $x[kT]$ is defined only at *kT* and has values only at *0*, *±T*, *±2T* etc. (use square brackets to remind ourselves we are using discrete-time signals).

Sampling

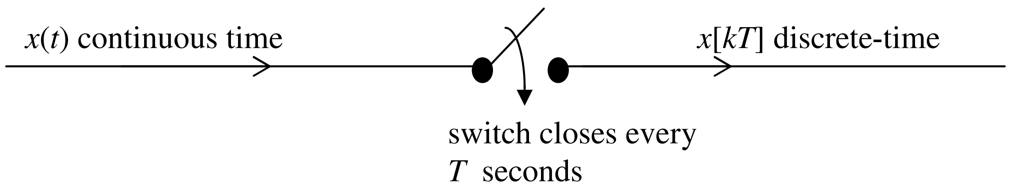


Fig. 15: Sampling of a CT signal x(t) to produce a DT signal x[kT].

Discrete-time signals often arise from sampling a continuous-time signal (in which case we call it a **sampled** version of the original). In the example below the sinusoidal signal is sampled every 0.2s.

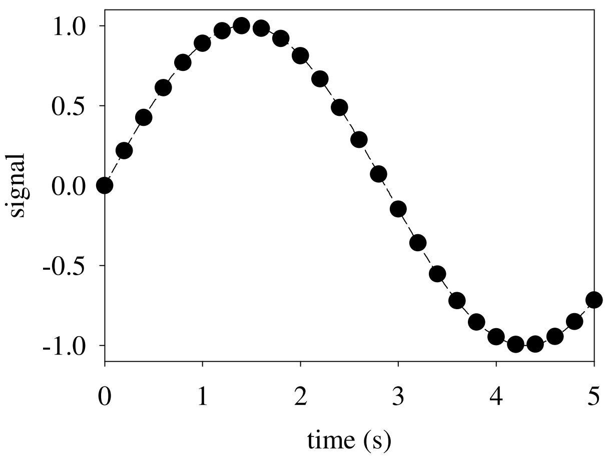


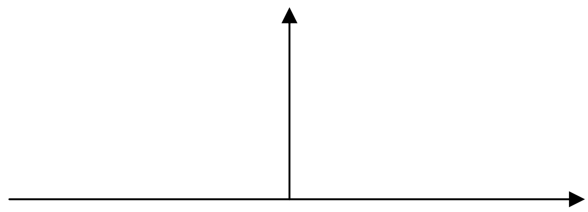
Fig.16: Sampled sinusoidal signal.

Let's look at some common signals:

<u>The unit-step function</u>

$$u[n] = \begin{matrix} 0 & n < 0 \\ 1 & n \geq 0 \end{matrix}$$

$u[n\text{-}N]$ is the delayed unit step.



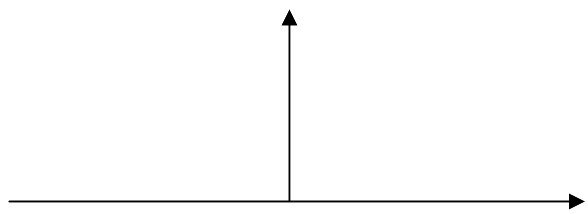
<u>The rectangular pulse signal</u>

$rect[N] = u[n] - u[n\text{-}N]$

$= 0,\ n < 0$

$= 1,\ 0 \leq n \leq N\text{-}1$

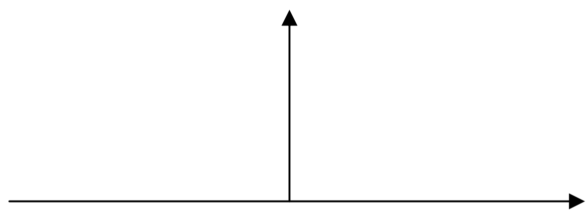
$= 0,\ n \geq N$



<u>The unit impulse function</u>

$$\delta[n] = \begin{matrix} 0 & n \neq 0 \\ 1 & n = 0 \end{matrix}$$

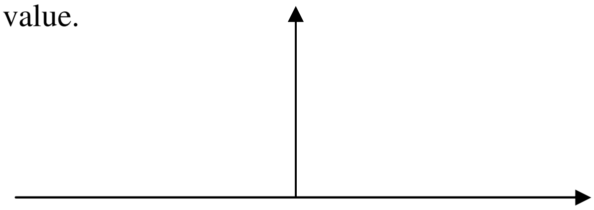
There is no sampled version of



$\delta[n]$ (as $\delta(0)$ is not defined) so we make up our own discrete-time counterpart like this: $\boxed{\delta[n] = u[n] - u[n\text{-}1]}$

The unit ramp function

$r[n] = nu[n]$ i.e n times unit step value.



Periodic DT Signals

A DT signal $x[n]$ is periodic if $x[n] = x[n+r]$ for all integers n and r is a positive integer.

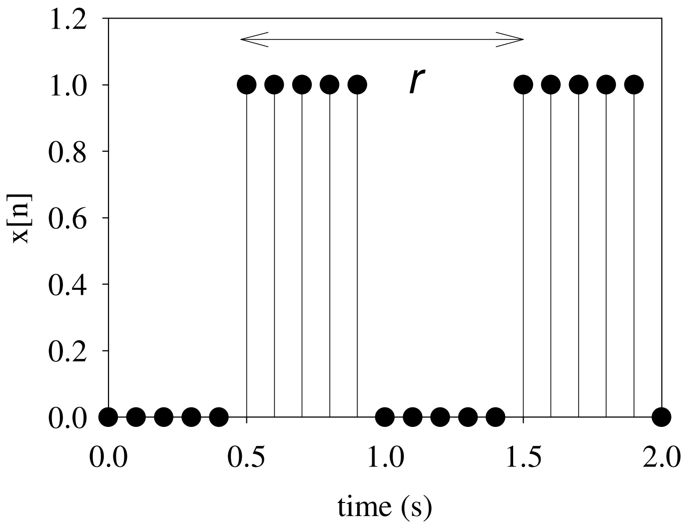


Fig.17: Periodic discrete-time signal.

DT signals and CT signals have many similarities such as the amplitude and time scaling, time shift and addition and multiplication operations of signals. DT exponentials and sinusoidal signals can be obtained from their CT equivalent by sampling.

Notes: