Lecture content

- Laplace Transform
  - Region of Convergence (ROC)
  - Pole-zero plot
  - Laplace Transform pairs and properties

Definition

A generalised or extended Fourier Transform which is known as the bilateral or two-sided Laplace Transform of a signal $x(t)$ is defined as

$$X_B(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

where $s = \sigma + j\omega$. If $\sigma = 0$, $X(s)$ becomes $X(j\omega)$, sometimes written as $X(\omega)$, the Fourier Transform. In practice most systems are causal, that is $x(t) = 0$ for $t < 0$, resulting in the single-sided (uni-lateral) form of the Laplace Transform

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}\,dt$$

Definition

The inverse Laplace Transform is defined as

$$x(t) = \frac{1}{j2\pi} \int_{c-j\infty}^{c+j\infty} X(s) e^{st} ds$$

where c is a constant chosen to be within the ***region of convergence*** (ROC).

Definition

Examples:

1. Consider a signal $x(t) = e^{2t}$, defined for $t \geq 0$. Its Laplace Transform is

$$X(s) = \int_0^\infty e^{2t} e^{-st}\, dt = \int_0^\infty e^{-(s-2)t}\, dt$$

Substituting $s = \sigma + j\omega$ into $e^{-(s-2)t}$ we have

$$X(s) = \int_0^\infty e^{2t} e^{-st}\, dt = \int_0^\infty e^{2t} e^{-\sigma t} e^{-j\omega t}\, dt = \int_0^\infty e^{-(\sigma-2)t} e^{-j\omega t}\, dt$$

Examples of Laplace Transform

We see that the Laplace Transform can be interpreted as the Fourier Transform of the signal $e^{-(\sigma-2)t}$. If $\sigma < 2$, $e^{-(\sigma-2)t}$ is a growing exponential and $X(s)$ does not converge.
However for $\sigma > 2$,

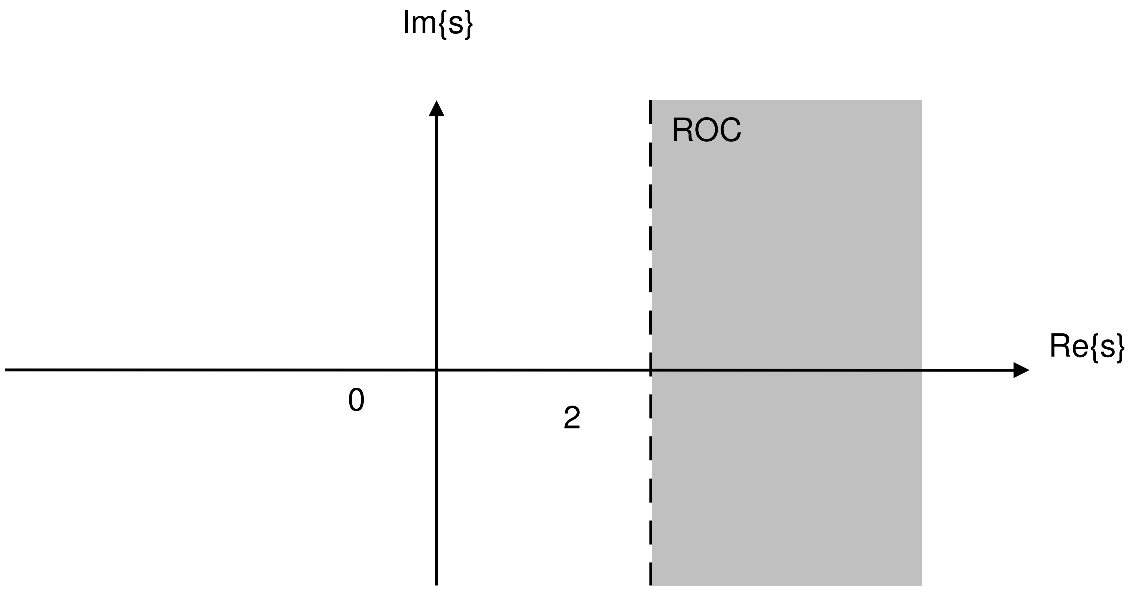
$$X(s) = \int_0^\infty e^{-(s-2)t}\,dt = \frac{-1}{s-2} e^{-(s-2)t}\Big|_{t=0}^{\infty} = \frac{1}{s-2}\left[1 - e^{-(s-2)t}\Big|_{t=\infty}\right] = \frac{1}{s-2}$$

$X(s)$ is not defined if $\sigma = \mathrm{Re}\{s\} < 2$. If $\mathrm{Re}\{s\} > 2$, the Laplace Transform of $x(t)$ becomes

$$X(s) = \frac{1}{s-2}, \ \mathrm{Re}\{s\} > 2.$$

ROCs

The region Re{s} > 2 is called ***region of convergence*** (ROC) and is displayed as



ROCs

2. Let $x(t) = e^{-at}u(t)$. The Laplace Transform is

$$X(s) = \int_0^{\infty} e^{-at} e^{-st} dt = \int_0^{\infty} e^{-(s+a)t} dt = \frac{-1}{s+a} e^{-(s+a)t} \Big|_{t=0}^{\infty} = \frac{1}{s+a}, \ \mathrm{Re}\{s\} > -a.$$

3. Let $x(t) = -e^{-at}u(-t)$. The Laplace Transform is

$$X(s) = -\int_{-\infty}^{0} e^{-at} e^{-st} dt = -\int_{-\infty}^{0} e^{-(s+a)t} dt = \frac{1}{s+a} e^{-(s+a)t} \Big|_{t=-\infty}^{0} = \frac{1}{s+a}, \ \mathrm{Re}\{s\} < -a.$$

ROCs

The Laplace Transforms are **identical** but the ROCs are **different** in examples 2 and 3.

This demonstrates that the ROC is needed to compute the inverse Laplace Transform.

Without specifying the ROC the inverse Laplace transform may not produce the original signal $x(t)$.

However, we will not be computing the inverse Laplace Transform but instead we will use a lookup table containing Laplace Transform pairs.

Examples

4. Consider a signal that is the sum of two real exponentials:

$$x(t) = 2e^{-t}u(t) + 5e^{-3t}u(t).$$

The Laplace Transform is

$$X(s) = \int_0^\infty (2e^{-t} + 5e^{-3t})e^{-st}\,dt = 2\int_0^\infty e^{-(s+1)t}\,dt + 5\int_0^\infty e^{-(s+3)t}\,dt$$

$$= \frac{2}{s+1} + \frac{5}{s+3} = \frac{7s+11}{(s+1)(s+3)}$$, Re{s} >-1

$2e^{-t}u(t) \leftrightarrow \frac{2}{s+1}$, Re{s} >-1 $\quad$ $5e^{-3t}u(t) \leftrightarrow \frac{5}{s+3}$, Re{s} >-3

Pole-zero

The Laplace Transform in each of the examples 1 to 4 is rational, i.e it can be written as a ratio of polynomial

$$X(s) = \frac{N(s)}{D(s)}$$

where $N(s)$ and $D(s)$ are the numerator polynomial and denominator polynomial, respectively.
The roots of $N(s)$ are called zeros and usually indicated with "O" while the roots of $D(s)$ are called poles and are usually indicated with "X" as illustrated in figure 2. The s-plane representation of $X(s)$ via the poles and zeros is also known as the pole-zero plot of $X(s)$.

Pole-zero plot