Circuits and Signals

6 Introduction to a.c. Circuits

The term "a.c. circuit analysis" usually means the analysis of circuits that are driven by sinusoidal sources. All periodic signals can be represented as the sum of harmonics of the fundamental frequency of the signal (ie, the reciprocal of the periodic time) so if the response of a circuit to an input signal consisting of a sinusoid of arbitrary frequency can be worked out, the response to a general periodic signal can be worked out.

In circuits containing only resistors the analytical process used for solving a.c. problems is exactly the same as that for d.c problems. In a.c. circuit analysis problems, however, there are almost always one or both of **inductance** and **capacitance** to deal with and it is the behaviour of these elements and how to deal with it that is the subject of the next section.

7 Reactive Components

7.1 Capacitors

A capacitor is a circuit element that exhibits the physical property of capacitance. It stores energy in the form of charge and electric field. Its basic form is two parallel conducting plates separated by a non-coducting medium called the "dielectric" as shown in figure 23. Capacitors made of flexible materials are usually rolled up into a cylindrical shape in order to achieve a high surface area in as small as possible a volume. The capacitance of the structure of figure 23 is

$$C = \frac{A\,\varepsilon}{d} \text{ where } \varepsilon = \text{permittivity} = \varepsilon_r\varepsilon_0 \qquad \textbf{(2.1)}$$

Permittivity is a property of a dielectric that describes its ability to concentrate electric flux. It is often written as the product of ε_r, the "**relative permitivity**" or "**dielectric constant**", and ε_0, the permittivity of free space. The dielectric can be air, inorganic materials (such as mica, ceramic or, as is common in digital and analogue integrated circuits, silicon dioxide) or organic polymers (such as polyester, polycarbonate or polystyrene); all these dielectrics give rise to non-polarised capacitors that can be put in the circuit either way round. The symbol for a non-polarised capacitor is shown in figure 24 (i).

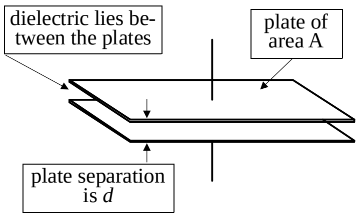


Figure 23

A class of capacitors known as "electrolytic" capacitors are polarised and must be inserted into the circuit in the correct polarity - ie the capacitor terminal designed to be the positive one must be positively biased with respect to the other (negative) terminal. Failure to observe this rule usually leads to a short capacitor life and a violent failure. Electrolytic capacitors, usually made of aluminium or tantalum, achieve very large values of capacitance by using as a dielectric an electrolytically grown layer of oxide which is extremely thin. Incorrect biasing reverses the layer forming process until the layer becomes too thin to withstand the voltage drop across it and hence fails. The symbol for an electrolytic capacitor is shown in figure 24 (ii) - the hollow electrode is the one that must be positive.

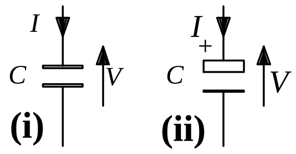


Figure 24

In fact all capacitors have a maximum working voltage set by the ability of the dielectric to withstand electric fields and specified by the manufacturer.

The relationship between current through and voltage across a capacitor is given by,

$$I = C\frac{dV}{dt} \text{ or } V = \frac{1}{C}\int I dt + \text{ constant} \qquad \textbf{(2.2)}$$

and the relationship between charge stored and capacitor voltage is

$$Q = CV \qquad \textbf{(2.3)}$$

The unit of capacitance is the **Farad**, **F**. The Farad is a large unit and until recently most capacitors that could be bought in shops lay in the range 1pF (10^{-12} F) to tens of mF ($\approx 10^{-2}$ F). Recently a new breed of capacitor, known as a super-capacitor, has emerged for special applications and these devices have capacitances of the order of kF (10^{3} F).

Stored energy in a capacitor

Consider a capacitor such as the one in figure 24 **(i)**. Imagine that at some instant of time, the voltage across the capacitor is v and the current through it is i. The incremental energy stored in a time δt is then $\delta E = vi\,\delta t$ and since $i = C\frac{dv}{dt}$, $\delta E = v\,C\frac{dv}{dt}\delta t$. In the limit $\delta t \to 0$, $\delta t \to dt$ and $\delta E \to dE$ so the energy increment becomes $dE = v\,C\frac{dv}{dt}dt = Cv\,dv$.

The stored energy in the capacitor at a voltage V V is the sum of all the increments of energy between 0 V and V V, or

$$E = C\int_0^V v\,dv = \frac{C\,V^2}{2} \text{ J} \qquad \textbf{(2.4)}$$

where J is the energy unit, Joules. The total energy stored in the capacitor at any time is related only to the voltage across the capacitor at that time and not to the history of how the voltage reached that value.

7.2 Inductors

Inductors, like capacitors, are energy storage elements. In an inductor energy is stored in the magnetic field associated with a wire or a coil of wire. The magnetic flux might be contained within a magnetic circuit as shown in figure 25 or alternatively the coil might be air cored in which case the magnetic flux paths will not be so tightly confined. Magnetic circuits are usually made from special iron alloys or ceramic materials based on magnetic iron oxides and other trace elements; these materials are usually called "ferrite materials". Iron tends to be used at power frequences (50 or 60 Hz) whereas ferrite tends to be used from a few kHz to a few MHz.

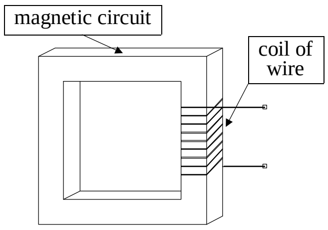


Figure 25

In power and power management circuits inductors are common because

- Electromechanical energy converters are based on electro-magnets, magnetic circuits and the interaction between current carrying conductors and magnetic fields.
- Magnetically coupled inductors - i.e., transformers - are used extensively in power systems.
- High values of current mean that modest values of inductance can store relatively large amounts of energy.

In electronic circuitry designed to process or condition signals inductors are less commonly used. (One notable exception to this is the area of radio circuitry at frequencies of MHz and above where inductors are often used as part of resonant circuits.) There are many reasons why designers of non radio circuitry avoid the use of inductors in signal conditioning circuits:

- Inductors are relatively bulky in comparison to resistors and capacitors .
- The presence of a magnetic circuit introduces non-linear behaviour into a circuit but without a magnetic circuit, excessive amounts of wire are needed for quite modest inductance values.
- The magnetic field from a air cored inductor can easily interact with nearby circuitry.

The circuit symbols used for some of the commonest types of inductor are shown in figure 26. Figure 26 **(i)** shows an air cored inductor, figure 26 **(ii)** a ferrite cored inductor and figure 26 **(iii)** an iron cored inductor. In each case, L is the algebraic variable used to represent inductance and this usage is universal. The convention for V and I is as shown in figure 26.

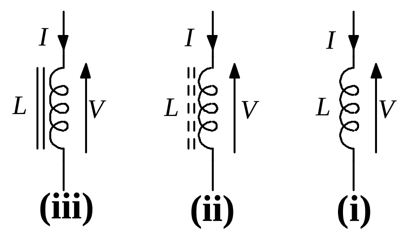


Figure 26

The relationship between current through and voltage across an inductor is given by,

$$V = L\frac{dI}{dt} \text{ or } I = \frac{1}{L}\int V dt + \text{ constant} \qquad \textbf{(2.5)}$$

The unit of inductance is the **Henry**, **H**. Inductance values from µH (10^{-6} H) to around 10^{2} H are common in various applications. Inductances of nH (10^{-9} H) tend to be parasitic and troublesome - especially in high frequency or high current circuitry.

Stored energy in an inductor

Consider an inductor such as the one of those in figure 24. Imagine that at some instant of time, the voltage across the inductor is v and the current through it is i. The incremental energy stored in a time δt is then $\delta E = vi\,\delta t$ and since $v = L\frac{di}{dt}$, $\delta E = i\,L\frac{di}{dt}\delta t$. In the limit $\delta t \to 0$, $\delta t \to dt$ and $\delta E \to dE$ so the energy increment becomes $dE = i\,L\frac{di}{dt}dt = Li\,di$.

The stored energy in the inductor at a current I A is the sum of all the increments of energy between 0 A and I A, or

$$E = L\int_0^I i\,di = \frac{L\,I^2}{2} \text{ J} \qquad \textbf{(2.6)}$$

where, as for the capacitor case, J is the energy unit, Joules. The total energy stored in the inductor at any time is related only to the current through the inductor at that time and not to the history of how the current reached that value.

Note the similarity between the capacitor relationships of equations **(2.2)** and **(2.4)** and the inductor relationships of equations **(2.5)** and **(2.6)** respectively. The shapes of the realtionships are the same but where there is L in equations **(2.5)** and **(2.6)** there is C in equations **(2.2)** and **(2.4)**, where there is V in equations **(2.5)** and **(2.6)** there is I in equations **(2.2)** and (**2.4)** and where there is I in equations **(2.5)** and **(2.6)** there is V in equations **(2.2)** and **(2.4).**

7.3 Inductors and capacitors at d.c.

Dealing with reactive components under d.c. conditions is easy;

- Inductors behave like short circuits - i.e., their impedance = 0 Ω

This is because for the inductor, V must be zero since dI/dt must be zero - nothing changes with time for d.c. conditions.

- Capacitors behave like open circuits - i.e., their impedance is infinite

This is because for the capacitor, I must be zero since dV/dt must be zero - nothing changes with time for d.c. conditions.

For example, consider the circuit of figure 27. Under d.c. conditions the circuit can be drawn as figure 28 where all Ls have been replaced by short circuits and all Cs by open circuits.

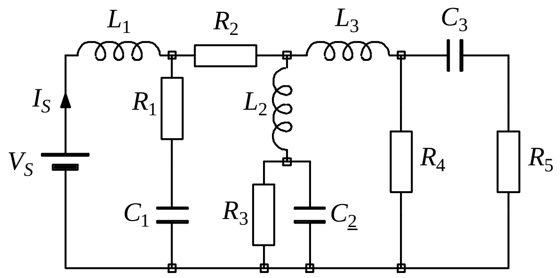


Figure 27

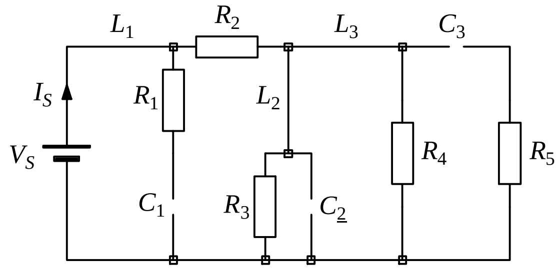


Figure 28

Inspection of the circuit then reveals that

$$\frac{V_S}{I_S} = R_2 + R_3 \,//\, R_4$$

Note that although the capacitors and inductors are open circuits and short circuits respectively, they will all store energy because in this circuit, all the Ls will be carrying a current and there will be a voltage drop across all the Cs.

8 Representing a.c. Quantities

The sinusoid is the basic a.c. signal. Sinusoidal sources can be of the voltage source or the current source variety but voltage sources are more common. This is probably because power systems use sinusoidal waveforms and most people have encountered power systems. From a signal point of view, sinusoids have properties that make them particularly attractive as test signals and as the signals of interest - but those special properties are beyond the scope of this module.

The basic sinusoid comes in two forms, a sine wave and a cosine wave. These two are really the same thing - the only difference between them is the position of the time origin, as can be seen in figure 29.

The sine wave is given by $V(t) = V_P \sin \theta$ and it is easy to see that the only difference between the sine and cosine waves is an angular shift of $\pi/2$ radians. The cosine wave can be expressed in terms of the sine wave by adding $\pi/2$ to θ. For the cosine

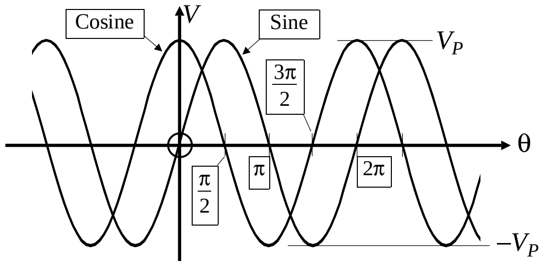


Figure 29

$$V(t) = V_P \cos \theta = V_P \sin (\theta + \frac{\pi}{2}) \qquad \textbf{(2.7)}$$

In the electronic engineering area, the angle on which the sinusoid operates is usually linearly increasing with time at a rate ω radians per second so equation **(2.7)** becomes

$$V(t) = V_P \cos \omega t = V_P \sin (\omega t + \frac{\pi}{2}) \qquad \textbf{(2.8)}$$

8.1 Sinusoids and reactive components.

(i) Capacitors

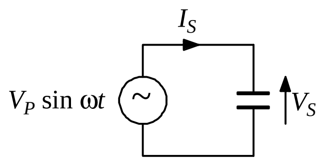


Figure 30

The current through a capacitor that has a sinusoidal voltage imposed across it, as shown in figure 30, can be derived with the help of equation **(2.2).**

$$I_S = C\,\frac{dV_S}{dt} = C\,\frac{d\,(V_P \sin \omega t)}{dt}$$

$$= C\,\omega V_P \cos \omega t = C\,\omega V_P \sin (\omega t + \frac{\pi}{2}) \qquad \textbf{(2.9)}$$

The capacitive reactance is $X_C = \left|\frac{V_S}{I_S}\right| = \frac{V_P}{V_P\,\omega C} = \frac{1}{\omega C}$ **(2.10)**

and V_S lags I_S by $\frac{\pi}{2}$ radians or 90^o. **(2.11)**

(ii) Inductors

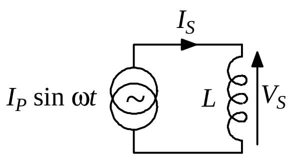


Figure 31

The voltage across an inductor that has a sinusoidal current driven through it, as shown in figure 31, can be derived with the help of equation **(2.5).**

$$V_S = L\,\frac{dI_S}{dt} = L\,\frac{d\,(I_P \sin \omega t)}{dt}$$

$$= \omega L\, I_P\, Cos\, \omega t = \omega L\, I_P \sin (\omega t + \frac{\pi}{2}) \qquad \textbf{(2.12)}$$

The inductive reactance is $X_L = \left|\frac{V_S}{I_S}\right| = \frac{I_P\,\omega L}{I_P} = \omega L$ **(2.13)**

and V_S leads I_S by $\frac{\pi}{2}$ radians or 90^o. **(2.14)**

8.2 Adding voltages of different phase

The main problem in dealing with a.c. circuit analysis lies in combining quantities that have a relative phase other than zero. The problem could be solved graphically by adding the two quantities together at each instant of time. This approach would be very tedious. Fortunately the quantities can be added together vectorially in what is known as a phasor diagram.

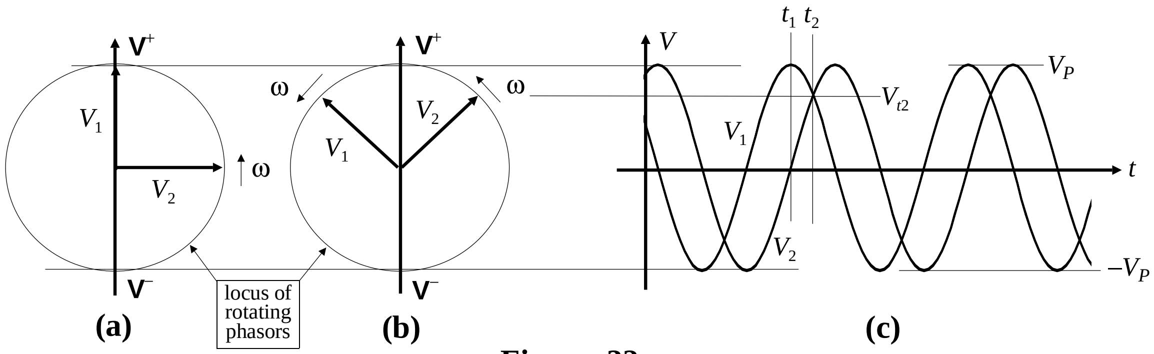


Figure 32

Consider figure 32. The sinusoids in figure 32 (c) are the projections of the rotating vectors (phasors) shown in figures 32 (a) and (b) onto the $\mathbf{V}^+\mathbf{V}^-$ axis. The phasors rotate at a rate of ω radians per second and conventionally rotate in an anticlockwise direction. Figure 32 (a) shows the phasor positions at time t_1 and figure 32 (b) shows their positions at time t_2. At time t_1 the sum of V_1 and V_2 is V_P but at t_2 the sum is $(V_P\sin 45^o + V_P\cos 45^o) = V_P\sqrt{2}$. This is the same as the vector sum of the phasors representing V_1 and V_2. In fact, the vector sum is a third rotating vector with a magnitude of $V_P\sqrt{2}$ and an angle that is half way between that of V_1 and V_2. The angle of the resultant with respect to V_1 is the phase of the resultant with respect to V_1.

The phasor diagram behaves like a rotating vector diagram and by taking a snapshot of this rotating system at some convenient time, well tried vector diagram methods can be used to identify the relationship between various circuit parameters. The rotation is an indication of the frequency of the sinusoid and is not explicitly included in the phasor diagram.

The length of the phasors can be their peak value (as in figure 32) or their rms value - the latter measure is commonly used in power systems because in that subject area the primary objective is to transmit power and rms has meaning only in terms of pwer. Assume in this module that the phasor is defined by its peak value unless you are told otherwise.

8.3 Some a.c. circuit examples

(i) An *L-R* series circuit

A phasor diagram can be used to investigate the relationship between the various quantities in figure 33.

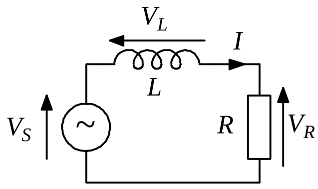


Figure 33

- The first thing to do is choose a reference direction. For a series circuit this would normally be the current because it is common to all the circuit elements.
- Next draw the phasors representing the circuit quantities - in this case V_R and V_L.
- Summing votages around the circuit gives $V_L + V_R = V_S$ so a construction that vectorially adds V_L and V_R will have V_S as the resultant.
- Next sketch the resultant.

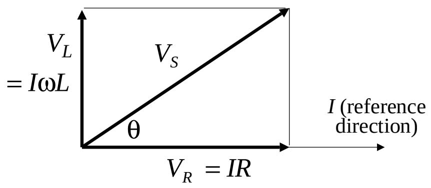


Figure 34

The result is shown in figure 34. Since V_R and V_L are 90^o out of phase, with V_L leading V_R, the triangle is a right angled triangle and so

$$V_S^2 = (IR)^2 + (I\,\omega L)^2.$$

The phase of V_S with respect to I is given by the angle θ, i.e.,

$$\theta = \tan^{-1}\frac{\omega L}{R}$$

(ii) A *C-R* parallel circuit

In the case of a parallel circuit, the voltage across each element is the same so it makes sense to choose the voltage direction as the reference and sum the currents.

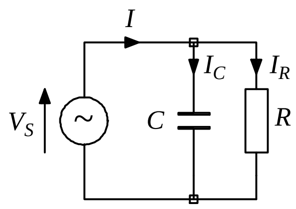


Figure 35

- Draw the reference phasor
- Draw I_R and I_C
- Sum these to find I as the resultant

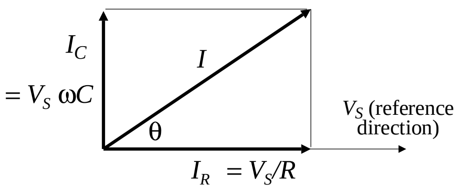


Figure 36

The completed diagram is shown in figure 36. Again the result is a right angled triangle so

$$I^2 = \frac{V_S^2}{R^2} + V_S^2\,\omega^2 C^2$$

and the phase of I with respect to V_S is

$$\theta = \tan^{-1}\omega CR$$

(iii) An *L-C-R* circuit with a series and parallel mix

Phasor diagrams can get quite complicated. Combinations of components that are purely in series or in parallel always reduce to a relatively simple vector summation with an obvious reference direction choice. In circuits that are mixtures of series and parallel elements, such as that of figure 37, the choice of reference is not so obvious . . .

The diagram for this circuit can be constructed as follows,

- Draw a partial phasor diagram to perform the series arm summation, $V_L + V_R = V_S$ - it makes sense to use the direction of IR as the reference direction here. This produces a diagram similar to figure 34 for the series *L*-*R* case which shows the phase relationship between V_S and I_R.
- Armed with knowledge of the phase relationship between V_S and I_R, it is possible to draw a diagram to do the summation $I = I_C + I_R$. The most convenient reference direction here is that of V_S. A diagram similar to figure 36 is the result, the main difference being that I_R is not in the same direction as V_S, as it is in figure 36.

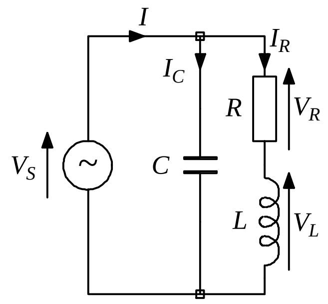


Figure 37

The complete phasor diagram is shown in figure 38. Notice how the left hand diagram defines the direction needed for the right hand one. Notice also that if *C* is reduced, the resultant, *I*, will rotate in an anti-clockwise direction, getting smaller as it does so. At some particular value of *C*, *I* and V_S are in phase with each other - a condition known as "resonance". The phenomenon of resonance will be discussed in more detail in a later section.

The ability to get V_S and *I* in phase is also inportant for power factor correction, a technique for ensuring that the load presented to the power source, V_S, is purely resistive.

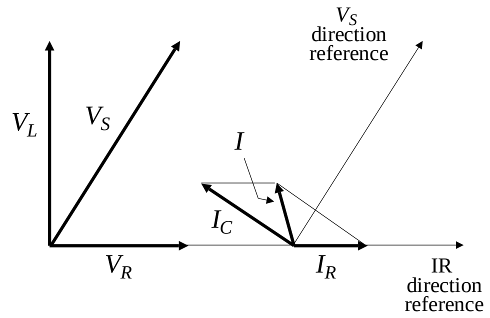


Figure 38

8.4 Complex number representation of a.c. circuits

Sometimes called the "*j*" notation, the complex number representation of reactive elements is a powerful tool that makes a.c. analysis much easier than it would otherwise be. "*j*" is the square root of -1 and mathematicians call it "*i* " (electronic engineers call it *j* to avoid the confusion that would arise if *i* were used both for current and for the imaginary number operator). It is tempting to ask what $\sqrt{-1}$ has to do with electronics; the answer is that it offers a way of coding the phase behaviour of a network in a way which can be dealt with using the algebra of complex numbers.

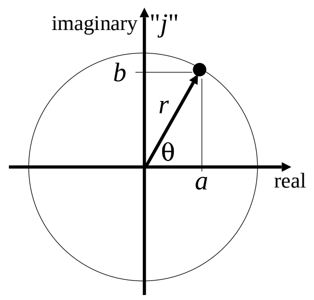
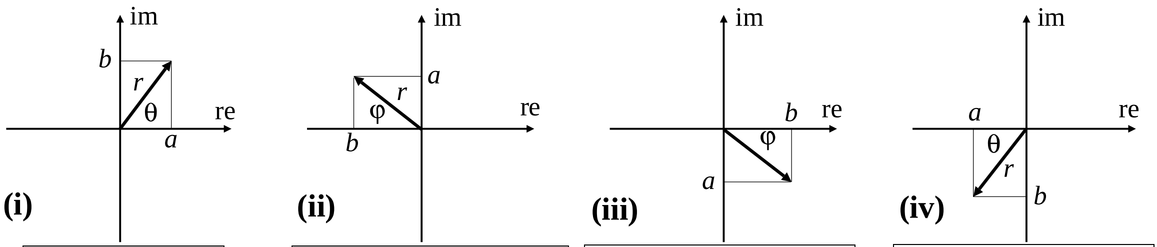


Figure 39

The use of complex numbers is really an extension of the phasor idea. Figure 39 shows an Argand diagram - a two dimensional representation of a complex number. A point $(a + jb)$ can be plotted on the Argand diagram as shown. The same point can also be expressed as a complex exponential $r\,e^{j\,\theta}$ where r and θ are as shown. Euler's theorem, $r\,e^{j\,\theta} = r\cos\theta + j\,r\sin\theta$, gives a relationship between the polar (r,θ) and Cartesian $(a + jb)$ forms of the same number and thus $a = r\cos\theta$ and $b = r\sin\theta$. So as θ increases, a cosine function is projected on to the real axis and a sine function onto the imaginary axis - behaviour very similar to that associated with the phasor.

If θ increases linearly with time at a rate ω radians per second, $r\,e^{j\,\theta}$ becomes $r\,e^{j\,\omega t}$. This source can be advanced or retarded in time simply by adding a phase term $r\,e^{j(\omega t\,+\,\theta)}$ which in turn can be expressed as $r\,e^{j\,\omega t}e^{j\,\theta}$.

From a practical point of view, multiplying a complex number by j adds a phase shift of 90^{o} to that quantity whilst dividing by j adds a phase shift of $-\,90^{\mathrm{o}}$. In other words j acts as a 90^{o} phase shifting operator. This behaviour is illustrated in figure 40.



(i)	(ii)	(iii)	(iv)
$(a+jb)$ or in polar form $r\,e^{j\,\theta}$ $\theta = \tan^{-1}\dfrac{b}{a}$	$j(a+jb) = (-b+ja)$ or in polar form $r\,e^{j\,(\theta+\frac{\pi}{2})}$ $\varphi = \tan^{-1}\dfrac{a}{-b} = -\tan^{-1}\dfrac{a}{b}$	$(-j)(a+jb) = (b-ja)$ or in polar form $r\,e^{j\,(\theta-\frac{\pi}{2})}$ $\varphi = \tan^{-1}\dfrac{-a}{b} = -\tan^{-1}\dfrac{a}{b}$	$(j^2)(a+jb) = (-a-jb)$ or in polar form $r\,e^{j\,(\theta\pm\pi)}$ $\theta = \tan^{-1}\dfrac{-b}{-a} = \tan^{-1}\dfrac{b}{a}$

Notes: The angle θ is positive in case **(i)** and **(iv)** because it is measured in an anticlockwise direction with respect to its reference in both cases.

The angle φ is negative in cases **(ii)** and **(iii)** because it is measured in a clockwise direction with respect to its reference in both cases.

Figure 40

8.5 Resistance, capacitance and inductance with complex number model

(i) Resistance

Resistance is straightforward to deal with because there are no reactive effects. V is proportional to I whatever the form of V or I. For example, if a voltage $V_P\,e^{j(\omega t\,+\,\theta)}$ is imposed across a resistor, the resulting current will be

$$I = \frac{V_P\,e^{j(\omega t\,+\,\theta)}}{R}$$

(ii) Capacitance

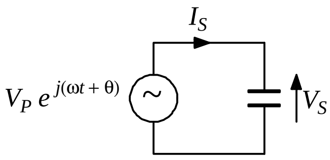


Figure 41

If a voltage V_S is imposed across a capacitor as shown in figure 41, the capacitor current is

$$I_S = C\,\frac{dV_S}{dt} = C\,j\,\omega V_P\,e^{j(\omega t\,+\,\theta)}$$

and so the reactance of the capacitor, X_C , is given by

$$X_C = \frac{V_S}{I_S} = \frac{V_P\, e^{j(\omega t + \theta)}}{C\, j\, \omega V_P\, e^{j(\omega t + \theta)}} = \frac{1}{j\, \omega C} \qquad \textbf{(2.15)}$$

Notice that the driving function, $e^{j\omega t}$, and its arbitrary phase shift, θ, have cancelled out leaving a modulus of $X_C = 1/\omega C$ and a phase relationship described by $1/j$ or $-j$. Since j is a 90^o phase shift operator this indicates a phase of V_S with respect to I_S of -90^o which is consistent with equation **(2.11).**

(iii) Inductance

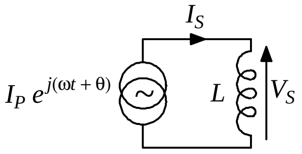


Figure 42

If a current I_S is driven through an inductor, as shown in figure 42, the inductor voltage is

$$V_S = L\, \frac{dI_S}{dt} = L\, j\, \omega I_P\, e^{j(\omega t + \theta)}$$

and so the reactance of the inductor, X_L , is given by

$$X_L = \frac{V_S}{I_S} = \frac{L\, j\, \omega I_P\, e^{j(\omega t + \theta)}}{I_P\, e^{j(\omega t + \theta)}} = j\, \omega L \qquad \textbf{(2.16)}$$

As in the case of the capacitor, the driving function, $e^{j\omega t}$, and its arbitrary phase shift, θ, have cancelled out leaving a modulus of $X_L = \omega L$ and a phase relationship described by j . Since j is a 90^o phase shift operator this indicates a phase of V_S with respect to I_S of 90^o which is consistent with equation **(2.14).**

(iv) Application examples

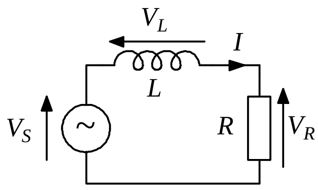


Figure 43

These examples are the same three examples as those used in section **8.3**. Using the same examples allows a direct comparison of the phasor diagram and j notation approaches. In figure 43, summing voltages gives $V_S = V_L + V_R$ or,

$$V_S = I\, j\, \omega L + IR = I\, (j\, \omega L + R) \text{ and thus,}$$

$$Z = \frac{V_S}{I} = j\, \omega L + R$$

the modulus or magnitude of this function is

$$|Z| = (R^2 + \omega^2 L^2)^{0.5} \text{ and the phase of } V_S \text{ with respect to } I \text{ is } \theta = \tan^{-1} \frac{\omega L}{R}$$

Figure 44 shows a capacitor in parallel with a resistor. The capacitive reactance combines with the resistance in the same way as a parallel resistor pair to form an impedance given by,

$$Z = \frac{RX_C}{R + X_C} = \frac{\dfrac{R}{j\, \omega C}}{R + \dfrac{1}{j\, \omega C}} = \frac{R}{1 + j\, \omega CR}$$

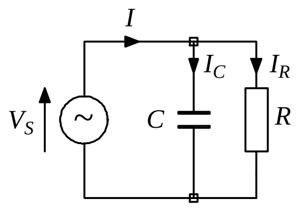


Figure 44

The result would normally be processed as it is . . .

$$|Z| = \frac{R}{(1+\omega^2C^2R^2)^{0.5}} \text{ and phase is } \theta = -\tan^{-1}\omega CR$$

Figure 45 shows the series - parallel combination of figure 37. Using the j approach to find Z,

$$Z = X_C\,//(R + X_L) = \frac{\dfrac{R + j\,\omega L}{j\,\omega C}}{R + j\,\omega L + \dfrac{1}{j\,\omega C}}$$

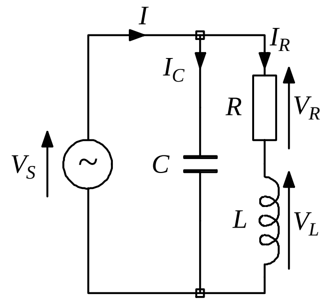


Figure 45

By multiplying top and bottom by $j\omega C$ this simplifies to

$$Z = \frac{R + j\,\omega L}{1 - \omega^2LC + j\,\omega CR}$$

In the phasor diagram approach to this circuit it was demonstrated that it was possible to choose a value of C for a given frequency that would make I in phase with V_S. This is equivalent to identifying the C that makes the j terms disappear from Z for that frequency. Alternatively one could look for the value of frequency that, with a given C, will make the j terms vanish. When the j terms vanish, the circuit is said to be "resonant" and the phenomenon of resonance will be discussed in more detail later in the module.

In this example, to find the condition that will make the j terms vanish, Z must be expressed in the form $(a + jb)$ and then b must be equated to zero,

$$Z = \frac{R + j\,\omega L}{1 - \omega^2LC + j\,\omega CR} = \frac{(R + j\,\omega L)\,((1 - \omega^2LC) - j\,\omega CR)}{(1 - \omega^2LC)^2 + (\omega CR)^2}$$

$$Z = \frac{R\,(1 - \omega^2LC) + \omega^2LCR \;+\; j\,\omega(L\,(1 - \omega^2LC) - CR^2)}{(1 - \omega^2LC)^2 + (\omega CR)^2}$$

equating j terms to zero

$$L\,(1 - \omega^2LC) - CR^2 = 0$$

which for the resonant frequency at a given C gives

$$\omega = \left(\frac{1}{LC} - \frac{R^2}{L^2}\right)^{0.5}$$

and for the C necessary for resonance at a given ω gives

$$C = \frac{L}{\omega^2L^2 + R^2}$$

The purpose of this section was to provide an example of manipulations using complex number representations of currents, voltages and impedances. Up to now the complex quantities have always been in the $a + jb$ form but the polar form is sometimes more useful - especially when multiplying and dividing complex numbers.

8.6 Polar and Cartesian representations

The relationship between $(a + jb)$ and $r\,e^{j\theta}$ is

polar to Cartesian: $a = r \cos\theta$ and $b = r \sin\theta$ **(2.17)**

Cartesian to polar: $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\dfrac{b}{a}$ **(2.18)**

Most calculators have a dedicated Cartesian to polar (and vice - versa) transformation facility - you can use this to perform conversions if you wish but practice the conversion process thoroughly so that you know exactly what keys you need to press and in what order they should be pressed. If you would rather not use your calculator to perform these transformations, you can use equations **(2.17)** and **(2.18)** to achieve the same function.

(i) Addition and subtraction

In general, the linear operations of addition and subtraction are most effectively performed using the Cartesian form because it is easier to add complex numbers in this form. For example,

$$(a + jb) + (c + jd) - (f + jg) = (a + c - f) + j\,(b + d - g) \qquad \textbf{(2.19)}$$

ie, the real parts are added together and the imaginary parts are added together.

(ii) Multiplication and division

These are both multiplicative operations which, although possible in Cartesian form, are most conveniently done in polar form. For example, multiply $r_1\angle\theta_1$ by $r_2\angle\theta_2$. Remembering that $r\angle\theta$ is shorthand for $r\,e^{j\theta}$,

$$r_1\angle\theta_1 \times r_2\angle\theta_2 = r_1\,e^{j\theta_1} \times r_2\,e^{j\theta_2} = r_1\,r_2\,e^{j(\theta_1+\theta_2)} = (r_1\,r_2)\angle(\theta_1 + \theta_2) \qquad \textbf{(2.20)}$$

so to multiply in polar co-ordinates, **multiply magnitudes and add phases**.

[To do the same thing in Cartesian form,

$(a + jb)(c + jd) = (ac - bd) + j(bc + ad)$ - more complicated.]

Division is very similar to multiplication. As an example, divide $r_1\angle\theta_1$ by $r_2\angle\theta_2$.

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = \frac{r_1\,e^{j\theta_1}}{r_2\,e^{j\theta_2}} = \frac{r_1}{r_2}\,e^{j(\theta_1-\theta_2)} = \frac{r_1}{r_2}\angle(\theta_1 - \theta_2) \qquad \textbf{(2.21)}$$

ie, for division, **divide the magnitudes and subtract the phases.**

[To do the same thing in Cartesian form,

$\dfrac{a + jb}{c + jd} = \dfrac{a + jb}{c + jd} \times \dfrac{c - jd}{c - jd} = \dfrac{(ac + bd) + j(bc - ad)}{c^2 + d^2}$ - much more complicated!]

9 Power in ac circuits

9.1 Introduction

Working out power dissipation in dc circuits is relatively straightforward because the voltage and current variables in the circuit have values that do not vary with time. When ac sources are involved there are two issues that have to be faced:

- How is the time varying nature of V (or I) accommodated?
- How are the effects of phase accommodated?

9.2 Time variation

For ac circuits the power is governed by the power integral

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt\,, \text{ where } T \text{ is the periodic time of the signal.} \qquad \textbf{(2.22)}$$

just as it was for dc circuits in section **2.9**. If a resistor has a voltage $V_P \sin \omega t$ across it, the current that flows through it is $I(t) = I_P \sin \omega t = \frac{V_P}{R} \sin \omega t$. Thus the power integral becomes,

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt = \frac{1}{R}\frac{V_P^2}{T}\int_0^T \sin^2\omega t\,dt = \frac{\text{mean squared voltage}}{\text{R}} \qquad \textbf{(2.23)}$$

[Note that one could have supposed that the resistor had a current $I(t) = I_P \sin \omega t$ flowing through it and in that case the voltage across the resistor would be $V(t) = V_P \sin \omega t = R\, I_P \sin \omega t$. Putting these expressions for $V(t)$ and $I(t)$ into equation **(2.22)** would give

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt = R\,\frac{I_P^2}{T}\int_0^T \sin^2\omega t\,dt = \text{ mean squared current} \times R \qquad \textbf{(2.23a)}$$]

Equation **(2.23)** is important because it makes it clear that the power dissipated in a resistive circuit element is proportional to the mean squared voltage across the resistor (or, alternatively, is proportional to the mean squared current through the resistor). The relationship between the peak voltage across the resistor and the mean squared voltage is a function of the voltage waveshape and is found by proceeding with the integral in equation **(2.23)** . . .

$$P = \frac{1}{R}\frac{V_P^2}{T}\int_0^T \sin^2\omega t\,dt = \frac{1}{R}\frac{V_P^2}{T}\int_0^T \frac{(1-\cos 2\omega t)}{2}\,dt = \frac{1}{R}\frac{V_P^2}{T}\left[\frac{t}{2} - \frac{1}{4\omega}\sin 2\omega t\right]_0^T$$

where T is the periodic time and $T = \frac{1}{f} = \frac{2\pi}{\omega}$. Putting in the limits leads to

$$P = \frac{1}{R}\frac{V_P^2}{2} = \frac{V_{rms}^2}{R}\ . \qquad \textbf{(2.24)}$$

The mean squared value of the sinusoidal voltage used here is thus $\frac{V_P^2}{2}$. **(2.25)**

The power could also be expressed as $P = \frac{I_P^2 R}{2} = I_{rms}^2 R$. **(2.26)**

It is very common for the mean squared voltage to be expressed in terms of V rather that V^2 and when expressed in terms of volts it is referred to as the "root mean square" voltage, a name often abbreviated to "rms". It is important to remember that although rms is commonly used as a voltage measure, it means nothing except in the context of the power delivering capability of the waveshape concerned. The rms voltage associated with equation **(2.25)** is

$$V_{rms} = \frac{V_P}{\sqrt{2}}\ . \qquad \textbf{(2.27)}$$

Many text books deal with rms as a separate entity rather than as part of the power calculation. This is unfortunate because rms has no meaning other than in the context of average power. The rms value of a quantity $V(t)$ is found by evaluating

$$V_{rms} = \left[\frac{1}{T}\int_0^T [V(t)]^2\,dt\right]^{\frac{1}{2}} \tag{2.28}$$

so at least the name - root-mean-square - describes what this measure is.

9.3 Phase effects

The effects of phase are also worked out by using the power integral of **(2.22)**. Suppose a voltage source V_S is driving a current I through an impedance Z such that

$$V_S = V_P \sin \omega t \quad \text{and}$$

$$I = I_P \sin(\omega t + \varphi)$$

Notice that there is a phase difference between V_S and I and the purpose of this section is to work out the effect of this phase difference on power dissipation.

The power dissipation in Z (which equals the power delivered by V_S) is

$$P = \frac{1}{T}\int_0^T V(t)I(t)\,dt = \frac{V_P I_P}{T}\int_0^T \sin\omega t\ \sin(\omega t + \varphi)\,dt$$

$$= \frac{V_P I_P}{2T}\int_0^T (\cos(-\varphi) - \cos(2\omega t + \varphi))\,dt = \frac{V_P I_P}{2T}\left[\left(t\cos\varphi - \frac{\sin(2\omega t + \varphi)}{2\omega}\right)\right]_0^T$$

where,
- $T = \dfrac{2\pi}{\omega}$,
- the first term in line 2 is the sine product in line 1 expressed as a cosine sum, and
- the second term in line 2 recognises that $\cos(-\varphi) = \cos\varphi$.

Putting in the limits gives

$$P = \frac{V_P I_P}{2}\cos\varphi = V_{rms}I_{rms}\cos\varphi \tag{2.29}$$

and since $V_P = I_P|Z|$, power can also be written as

$$P = \frac{V_P^2}{2|Z|}\cos\varphi = \frac{V_{rms}^2}{|Z|}\cos\varphi = \frac{I_P^2}{2}|Z|\cos\varphi = I_{rms}^2|Z|\cos\varphi \tag{2.30}$$

This means that power is the product of the **in phase** components of V and I. **cos** φ is called the "**power factor**" of the circuit - a very important parameter of power systems.

Remember that if the current through or the voltage across a purely resistive circuit element is known, the power dissipated within that element can be worked out simply from equations **(2.24)** or **(2.26)** respectively.

Note that these results are for a sinusoid. For non sinusoidal waveforms the concept of phase doesn't really mean much and an apparent phase shift would normally be expressed in terms of time delay. The power delivered by such a waveform can be calculated from equation **(2.22) .**

9.4 Power dissipation due to more than one source

The way in which multiple sources combine to give rise to power dissipation within a resistive circuit element depends upon the nature of the sources. Four example combinations of different sources are described below to illustrate the key points. Assume that in each case we are dealing with the voltage across a resistor and remember that $P \propto V^2$.

(i) $\boldsymbol{V = V_{DC1} + V_{DC2}}$

$$\begin{aligned} P &\propto \overline{V}^2 \propto \overline{(V_{DC1} + V_{DC2})^2} \\ &\propto \overline{V_{DC1}^2} + \overline{2V_{DC1}V_{DC2}} + \overline{V_{DC2}^2} \end{aligned}$$

There are three terms in this mean square expansion. The first and the last are the mean squared values of the components due to each source - these are always positive numbers for non-zero source magnitudes. The middle term is a cross product term. Cross product terms always appear when the sum of two or more terms is squared. If V_{DC1} and V_{DC2} are non zero then the mean value of their product will also be non zero and will therefore contribute to power dissipation.

For two (or more) dc sources contributing to a voltage across a resistor,

the contributions must be added (as voltages) before mean squaring.

(ii) $\boldsymbol{V = V_{DC} + V_P \sin \omega t}$

$$\begin{aligned} P &\propto \overline{V}^2 \propto \overline{(V_{DC} + V_P \sin \omega t)^2} \\ &\propto \overline{V_{DC}^2} + \overline{2V_{DC}V_P \sin \omega t} + \overline{V_P^2 \sin^2 \omega t} \\ &\propto \overline{V_{DC}^2} + \overline{V_P^2 \sin^2 \omega t} \quad \text{since} \quad \overline{2V_{DC}V_P \sin \omega t} = 0 \end{aligned}$$

In this case the first and last terms are non zero, as for case **(i)**. The cross product term multiplies V_{DC} and V_P with the mean value of sin ωt. The mean, or average, value of a sinusoid is zero so the cross product is zero and power is proportional to the sum of the first and third terms.

For a dc and an ac source contributing to a voltage across a resistor

the contributions due to each source must be mean squared before adding.

(iii) $\boldsymbol{V = V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_2 t}$

$$\begin{aligned} P &\propto \overline{V}^2 \propto \overline{(V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_2 t)^2} \\ &\propto \overline{V_{P1}^2 \sin^2 \omega_1 t} + \overline{2V_{P1}V_{P2} \sin \omega_1 t \sin \omega_2 t} + \overline{V_{P2}^2 \sin^2 \omega_2 t} \\ &\propto \overline{V_{P1}^2 \sin^2 \omega_1 t} + \overline{V_{P2}^2 \sin^2 \omega_2 t} \quad \text{since} \quad \overline{2V_{P1}V_{P2} \sin \omega_1 t \sin \omega_2 t} = 0 \end{aligned}$$

This case is very similar to case **(ii)**. The cross product here is equal to zero because the mean value of the product of two sinusoids of different frequency is zero. The easiest way to see why this is so is to use the identity, $2 \sin\omega_1 t \sin \omega_2 t = \cos(\omega_1 - \omega_2)t - \cos(\omega_1 + \omega_2)t$. The mean value of the two cosine functions is zero.

If two or more ac sources of different frequencies contribute to the voltage across a resistor

the contributions due to each source must be mean squared before adding.

(iv) $\boldsymbol{V = V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_1 t}$

$$P \propto \overline{V}^2 \propto \overline{(V_{P1} \sin \omega_1 t + V_{P2} \sin \omega_1 t)^2}$$

$$\propto \overline{V_{P1}^2 \sin^2\omega_1 t} + \overline{2V_{P1}V_{P2} \sin \omega_1 t \sin \omega_1 t} + \overline{V_{P2}^2 \sin^2\omega_1 t}$$

$$\propto (\overline{V_{P1}^2} + \overline{2V_{P1}V_{P2}} + \overline{V_{P2}^2})\, \overline{\sin^2\omega_1 t}$$

This case is similar to case **(i)**. When two contributing ac sources are at the same frequency, a $\sin^2$ term appears in the cross product and this gives a non zero cross product.

So for two or more sources at the same frequency and with the same phase,

the magnitudes must be added before mean squaring.

If the sources are not in the same phase, the magnitudes must be added vectorially using phasor diagrams or complex numbers to find the resultant magnitude.

9.5 Maximum power transfer

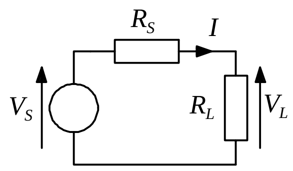


Figure 46

Consider the circuit of figure 46 where a source with a series resistance R_S supplies a load R_L.

- If $R_L \Rightarrow \infty$ and R_S is finite, $V_L \Rightarrow V_S$ and $I \Rightarrow 0$. Thus P_{RL} (which $= V_S I$) $\Rightarrow 0$ as $R_L \Rightarrow \infty$.
- If $R_L \Rightarrow 0$, $I \Rightarrow V_S/R_S$ and $V_{RL} \Rightarrow 0$. Once again, $P_{RL} \Rightarrow 0$ but this time it happens as $R_L \Rightarrow 0$.

Between these two extremes in value of R_L the product of V_L and I will not be zero - in fact there must be a particular value of R_L that maximises the power transferred to R_L.

To identify the value of R_L that will cause maximum power transfer to R_L, we need to derive an expression that describes the power dissipated in R_L in terms of the other circuit elements and then differentiate to find the maximum value. First write down the current I . . .

$I = \dfrac{V_S}{R_S + R_L}$ and then use this expression to write down the power dissipated in R_L,

$$P = I^2 R_L = \frac{V_S^2 R_L}{(R_S + R_L)^2} \qquad \textbf{(2.31)}$$

There are several ways of finding the maximum value of P. One way would be to differentiate directly using the quotient rule. Another is to recognise that a maximum in P is the same as a minimum in $1/P$ and this has the advantage that it is easier to differentiate $1/P$ than it is to differentiate P directly. There are several situations in electronics where this trick is useful so it will be pursued here. From equation **(2.31)** . . .

$$\frac{1}{P} = \frac{(R_S + R_L)^2}{V_S^2 R_L} = \frac{1}{V_S^2} \frac{R_S^2 + 2R_S R_L + R_L^2}{R_L} = \frac{1}{V_S^2}\left(\frac{R_S^2}{R_L} + 2R_S + R_L\right)$$

Differentiating, $\dfrac{d\,(1/P)}{d\,R_L} = \dfrac{1}{V_S^2}\left(-\dfrac{R_S^2}{R_L^2} + 0 + 1\right)$ and equating this to zero gives $1/P$ is a minimum

(i.e., P is a maximum) when $\boldsymbol{R_S = R_L}$.

Notes

(i) This rule is especially important at high frequencies. In general the elements R_S and R_L will be impedances Z_S and Z_L both with real and imaginary parts. In this case it can be shown that maximum power is transferred when Z_L is the complex conjugate of Z_S, i.e., if $Z_S = a + jb$, $Z_L = Z_S^* = a - jb$ for maximum power transfer. You don't need to know the $Z_S^* = Z_L$ rule for this module but it is easy to remember and you'll recognise it when you come across it in later modules.

(ii) For power supply applications the maximum power transfer theorem is irrelevant. For example, figure 46 could be interpreted as a battery supplying a load through its own internal resistance or a generator supplying the power grid through its own internal resistance. In these instances if $R_L = R_S$ half the available power will be lost in the internal resistance of the source - the last thing one wants to happen in a power drill, mobile phone or generator. Power sources are usually designed to approximate to ideal voltage sources - their internal resistances are designed to be very low and the current that would flow if they were loaded by their own internal resistance would be sufficient to cause severe damage to them.

10 Resonance

10.1 Introduction

Most physical systems exhibit resonance. In order to be capable of resonance, a system must be able to store energy in potential and kinetic forms and to allow transfer of energy from one to the other. Resonant systems abound in the field of music; guitars, flutes, violins, pianos (real ones!) and horns are all examples that either use the mass and elasticity of stretched string or the mass and elasticity of columns of air. Pendulums were once commonly used in clocks but are now largely confined to grandfather clocks. In electric circuits it is inductor - capacitor (L-C) circuits that form the basis of resonant systems. They were originally used in radio sets to selectively "magnify" one particlar frequency but are now used extensively over a whole range of specialisms from communication systems at one end to power management systems at the other.

The purpose of this section is to describe electrical resonance and identify the parameters used to quantify resonant behaviour. There are two basic forms of electrical resonant behaviour, series resonance and parallel resonance.

10.2 Series resonance

The circuit of figure 47 is a classical series resonant circuit. The impedance of the circuit is

$$Z = \frac{V_S}{I} = j\,\omega L + \frac{1}{j\,\omega C} + R = j\left(\omega L - \frac{1}{\omega C}\right) + R \qquad \textbf{(2.32)}$$

The condition for resonance is

- ***V_S* and *I* must be in phase with each other**

or in different words

- **the "*j*" terms must vanish from *Z* at the resonant frequency**

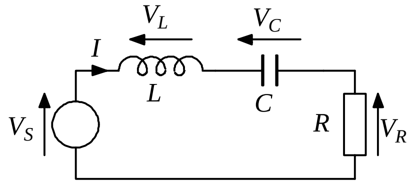


Figure 47

so at the resonant frequency, Z is purely real and the circuit behaves like a resistance.

The resonant frequency, ω_r, of the circuit is found by equating the "j" terms in equation **(2.32)** to zero, i.e., $\left(\omega L - \dfrac{1}{\omega C}\right) = 0$, to give

$$\omega_r^2 = \frac{1}{LC} \quad \text{or} \quad \omega_r = \frac{1}{\sqrt{LC}} \quad \text{or} \quad f_r = \frac{1}{2\pi\sqrt{LC}} \tag{2.33}$$

Note that at resonance the series L - C combination has an impedance of 0Ω.

Phasor diagram view of series resonance

It is sometimes conceptually useful to consider resonance from a phasor diagram point of view. For the circuit of figure 47 the current is common to all circuit elements so it makes sense to choose current as the reference direction as shown in figure 48. In figure 48, the voltage across the inductance, which leads the current by 90^o, is larger than the voltage across the capacitor, which lags the current by 90^o. This leads to a resultant V_S as shown. As frequency reduces, the voltage across the inductance will decrease and that across the capacitor will increase. Resonance occurs when $|I\,\omega L| = |I/(\omega C)|$ - i.e., the 90^o terms sum to zero and V_S lies in the same direction as I.

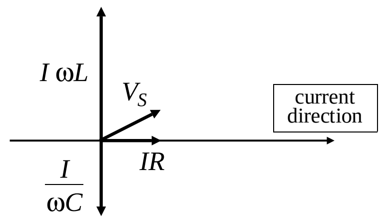


Figure 48

Frequency dependence of impedance and current

The way in which the circuit impedance changes with frequency is shown in figure 49. Also shown on figure 49 is the magnitude and phase of the voltage across the resistor which is proportional to the circuit current.

For frequencies below f_r the circuit behaves capacitively because $|V_C| > |V_L|$ so the sum $V_C + V_L$ is in a V_C direction. For frequencies above f_r the circuit behaves inductively - this is the situation in figure 48. On a logarithmic frequency scale these graphs are symmetrical about f_r.

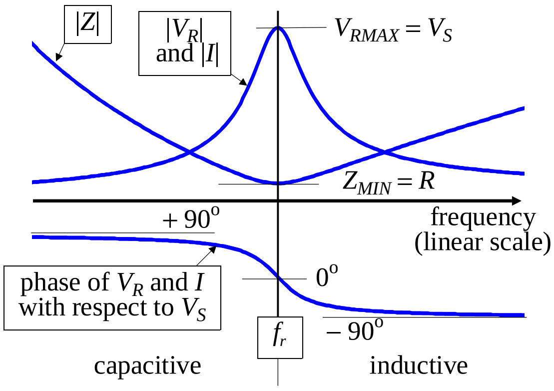


Figure 49

"Q" or "magnification" factor

The sharpness of the resonant peak in V_R or I is dependent upon the particular values of L, C and R used. In the early radio applications of resonant circuits, circuits that produced a very narrow peak and as a consequence a large magnification were regarded as high quality resonators. It wasn't long before someone defined a technically sensible figure of merit for resonant circuits . . . and called it Q factor. Q is an abbreviation for quality and essentially describes the shape of the circuit response.

There are several ways of defining Q factor; they all measure the same thing. The simplest of these is often used experimentally and relates to the shape of the resonant peak as shown in figure 50. Firstly the resonant frequency is identified by finding the frequency at which the maximum value of V_R occurs. Secondly the frequencies f_1 and f_2 at which the value of V_R has fallen by a factor of 0.707 (or -3dB) from its maximum value are determined. The Q factor is then given by:

$$Q = \frac{f_r}{f_2 - f_1} = \frac{f_r}{\Delta f} \qquad \textbf{(2.34)}$$

f_r is not half way between f_1 and f_2; it is the geometric mean of f_1 and f_2,

$$f_r = \sqrt{f_1 f_2} \qquad \textbf{(2.35)}$$

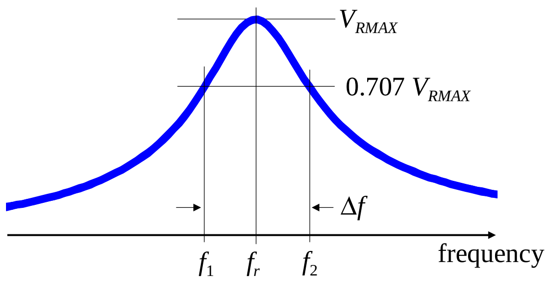


Figure 50

The Q factor can be expressed in terms of the circuit components by using the magnification definition of Q factor. For the ideal circuit of figure 47

$$Q = \frac{|V_L|}{|V_S|} = \frac{|V_L|}{|V_R|} = \frac{I\,\omega_r L}{IR} = \frac{\omega_r L}{R} \qquad \textbf{(2.36)}$$

Remember that at resonance the magnitude of V_L is equal to that of V_C so V_C could equally well have been used in equation **(2.36)**. Using $\omega_r = 1/(LC)^{0.5}$,

$$Q = \frac{\omega_r L}{R} = \frac{1}{\omega_r CR} = \frac{1}{R}\left(\frac{L}{C}\right)^{0.5} \qquad \textbf{(2.37)}$$

The $1/(\omega_r CR)$ term in equation **(2.37)** is the result of using V_C rather than V_L to calculate the magnification in equation **(2.36)**. The justification of equation **(2.35)** and of the fact that equation **(2.34)** is consistent with equations **(2.36)** and **(2.37)** is given in the third handout.

Non-ideal components

In reality all three components in the circuit of figure 47 will deviate from their ideal behaviour. Resistors have a series inductance and a parallel capacitance, capacitors have series inductance and series and parallel resistance and inductors have a series resistance and a parallel capacitance. It is usually possible to find a technology of resistor or capacitor that makes their defects insignificant for the application of interest but the deviations from ideality associated with inductors are less easy to reduce to insignificant proportions. In this module we will look at the effects of inductor series resistance which is important in communication systems because it limits Q factor (and alters f_r for parallel circuits) and important in power management systems because of the energy that is dissipated within it.

Figure 51 is the circuit of figure 47 with the inductor series resistance included. The circuit behaves in the same basic way as that of figure 47. At resonance the impedance of the series combination of C and the ideal bit of L is zero so at resonance the circuit consists of two resistors in series. V_R at resonance is thus

$$V_R = V_S \frac{R}{R + R_L}$$

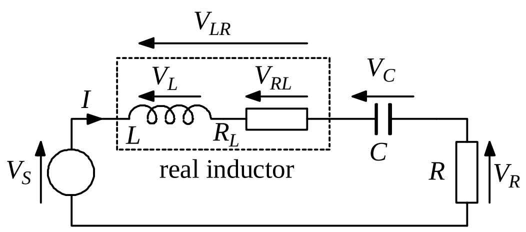


Figure 51

and if V_S and V_R are measured and R is known, this provides an easy way of measuring R_L at the resonant frequency.

It is important to know the resistance at the resonant frequency because the inductor resistance will tend to increase as frequency increases. This means that the dc resistance will not be the same as the inductor resistance at the resonant frequency.

> The increase happens because of something called the "skin effect" - as frequency increases, current flow is confined to a "skin" on the outside surface of the conductor. The thickness of this skin, called the "skin depth", is inversely proportional to the square root of frequency and is of the order of 10^{-2} m at 100 Hz - power frequencies - and 10^{-6} m at 10 GHz - satellite television frequencies. To find out more, look at any good book on electromagnetism.

The Q of the circuit is affected by R_L because the total series resistance is now $(R + R_L)$. Consequently one would expect the measured Q (measured perhaps by $f_r/\Delta f$) to be

$$Q_E = \frac{1}{R + R_L}\left(\frac{L}{C}\right)^{0.5} \tag{2.38}$$

Dividing equation **(2.38)** by the expression for the Q factor with ideal components, equation **(2.37)**, leaves $\frac{Q_E}{Q} = \frac{R}{R + R_L}$. If Q_E is measured and Q is calculated from equation **(2.37)** assuming ideal components, this offers another route to finding R_L experimentally. If the magnification approach is used to measure Q, it must be $\frac{|V_C|}{|V_S|}$ that is used since V_R is no longer interchangeable with V_S at resonance (because some voltage is lost across R_L at resonance) and the terminals of the ideal part of L are not accessible (so the voltage measured across L will contain a component due to R_L).

10.3 Parallel resonance

Parallel resonant circuits are used much more commonly than series resonant circuits in communications systems but they are not as common as series resonant circuits in power management systems. The basic parallel resonant circuit is shown in figure 52. Sometimes the circuit is drawn as in figure 53 but a moment's thought will reveal that Thevenin's theorem identifies the two circuits as functionally the same if $V_S = I_S R$. The approach to identifying the resonance condition is the same as for the series circuit, i.e., write down the impedance and force j terms to zero. For the figure 52 form of the circuit,

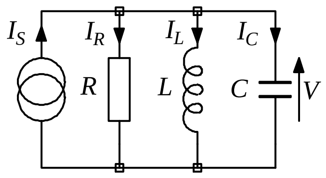


Figure 52

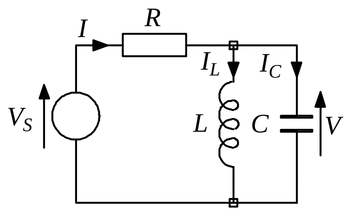


Figure 53

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{j\,\omega L} + j\,\omega C = \frac{1}{R} + j\,(\omega C - \frac{1}{\omega L})$$

The objective is to make the j terms vanish and if they can be made to vanish from $1/Z$, they will also vanish from Z. Thus, as for the series case

$$\omega_r^2 = \frac{1}{LC} \text{ or } \omega_r = \frac{1}{\sqrt{LC}} \text{ or } f_r = \frac{1}{2\pi\sqrt{LC}} \tag{2.39}$$

Phasor diagram view of parallel resonance

For the parallel circuit the voltage is the same across all the elements so it makes sense to use voltage as the reference direction. The currents through L and C are in antiphase, I_C leading the voltage by 90^o and I_L lagging the voltage by 90^o. The current through R is in phase with V. Since in this diagram I_L has a slightly bigger magnitude than I_C, the resultant I has a slightly inductive behaviour. At resonance, the magnitudes of I_C and I_L are equal so they sum to zero.

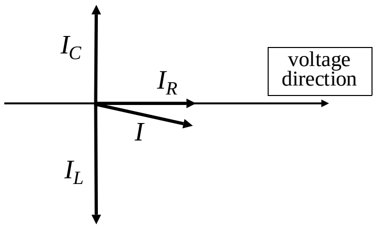


Figure 54

For the parallel circuit the total current flowing through the L-C combination at resonance is zero - i.e., **the impedance of the parallel L-C circuit at resonance is infinite.**

Frequency dependence of impedance and voltage

Figure 55 shows, for the circuit of figure 52, the relationship between V, $|Z|$ and the phase of V with respect to I_S for an I_S of varying frequency but constant amplitude. Since $V = I_S Z$, the shape of $|V|$ and $|Z|$ as a function of frequency is the same.

At resonance, since the L-C combination has infinite impedance, $I_S = I_R$. At frequencies below resonance $I_L > I_C$ so the circuit behaves inductively, at frequencies above resonance, $I_C > I_L$ so the circuit behaves capacitively. Since the maximum V is proportional to R, the Q factor of the parallel circuit will increase with increasing R.

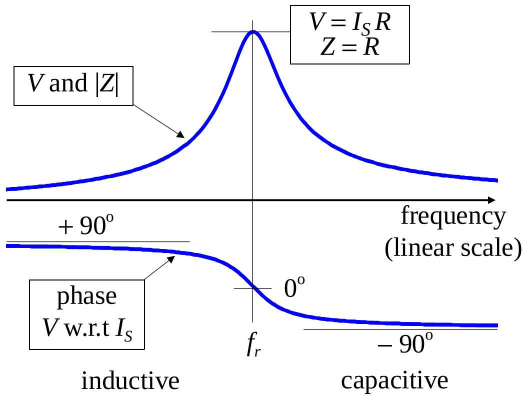


Figure 55

"Q" or "magnification" factor

For a parallel circuit the Q factor is either measured using a similar approach to the one used for a series circuit or calculated (or measured) using magnification behaviour. Figure 55 indicates that the voltage across the parallel network has a similar frequency dependent shape to that of the current through a series resonant circuit such as that shown in figure 50. The $f_r/\Delta f$ measurement approach follows the same process as was described in the series case. The magnification approach must be based on current magnification since all elements have the same voltage across them. So

$$Q = \frac{|I_L|}{|I_S|} = \frac{|I_C|}{|I_S|} \quad \text{at resonance}$$

$$= \frac{V\omega_r C}{\frac{V}{R}} = \omega_r CR = R\left(\frac{C}{L}\right)^{0.5} \qquad \textbf{(2.40)}$$

Again, the same result would be obtained from $|I_L|/|I_S|$.

Notice that the parallel expression for Q is the inverse of the series expression. In terms of remembering which is which, it might help to remember that in a series circuit, a large Q arises when only a small fraction of the supply voltage is lost across R. For a parallel circuit, a large Q occurs when the parallel R draws only a small current compared with the magnitudes of I_L and I_C at resonance.

Inductor with series resistance

A parallel resonant circuit including the effects of inductor series resistance is shown in figure 56. The R in parallel with the real L and C has been omitted - it does not affect the resonant frequency*(see below) but it does affect the Q factor. The circuit impedance is the impedance of the real L in parallel with the reactance of C,

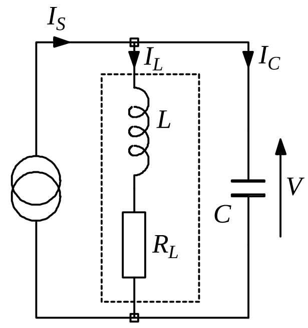


Figure 56

$$Z = \frac{V}{I_S} = \frac{\dfrac{R_L + j\,\omega L}{j\,\omega C}}{R_L + j\,\omega L + \dfrac{1}{j\,\omega C}} = \frac{R_L + j\,\omega L}{j\,\omega C R_L - \omega^2 LC + 1}$$

Since this is complex in both the numerator and denominator it must be rationalised to get the complex part in the numerator alone,

$$Z = \frac{(R_L + j\,\omega L)\,((1-\omega^2 LC) - j\,\omega C R_L)}{(1-\omega^2 LC)^2 + \omega^2 C^2 R_L^2} = \frac{R_L + j\,\omega(L - \omega^2 L^2 C - C R_L^2)}{(1-\omega^2 LC)^2 + \omega^2 C^2 R_L^2} \qquad \textbf{(2.41)}$$

To find the resonant frequency, the j terms must be forced to zero, i.e.,

$$(L - \omega^2 L^2 C - C R_L^2) = 0 \text{ which gives}$$

$$\omega_r^2 = \frac{L - C R_L^2}{L^2 C} \text{ or } \omega_r^2 = \frac{1}{LC} - \frac{R_L^2}{L^2} \qquad \textbf{(2.42)}$$

Equation **(2.42)** shows the modification to the resonant frequency referred to in the "non-ideal components" section of section **10.2**. For high Q circuits - i.e., ones containing carefully designed low loss inductors, the modification is small because R_L is small.

The real part of Z at resonance is $Z_r = \dfrac{R_L}{(1-\omega^2 LC)^2 + \omega^2 C^2 R_L^2}$ and this is the effective value of R_L transformed into a parallel equivalent resistance across the current source. This effective parallel R could be used in the Q factor relationship of equation **(2.40)** to find Q - but only if a similar effective parallel value of L was also used. Series to parallel transformations are dealt with in handout 3. Alternatively one could work out $|I_C| / |I_S|$ at resonance - possible but awkward. Q factor can be measured relatively easily by either the magnification or the $f_r/\Delta f$ approach.

When components are not ideal, defining resonant parameters can become very awkward. You will be pleased to hear that you don't need to worry about that for EEE101 - except of course the stuff covered in this handout.

* This is so because if a parallel R existed, as in figure 52, the current through it would always be in phase with V. Since the resonant condition is that I_S be in phase with V, the presence of an extra current that is always in phase with V will not alter the condition for resonance. It will, however alter the rate of energy loss in the circuit and hence will alter Q.