COMMUNICATION ELECTRONICS

EEE 224-227

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Course Description including Aims

- To provide the necessary mathematical background for signal and systems analysis, signal processing and its applicability in communication electronics.

- To provide an introduction to the field of communication systems, including nomenclature, methodology and applications.

- To introduce the concept of modulation and examine its influence on system performance.

- To examine typical circuits for implementing both analogue and digital modulation and demodulation.

- To introduce the idea of synthesising circuits to achieve specified transfer functions in the context of active filters.

- To introduce the concepts of oscillators and the circuits that may be employed

Learning outcomes

•manipulate discrete and continuous signals using common techniques such as time shifting, time scaling, amplitude scaling and modulation

•explain the basic principles underlying a communication system.

•choose which type of modulation to use for a specific application.

•display knowledge of representative types of circuitry to implement various modulation and demodulation schemes

•derive and interpret transfer functions for first and second order systems

•use normalised filter polynomials, in conjunction with first and second order circuits to realise basic low pass active filters

•Design linear oscillators for use in communication electronics

SYLLABUS

- Signals and systems
- Fourier analysis and convolution.
- Analogue modulation and demodulation.
- Receivers and multiplexing.
- Digital modulation and demodulation.

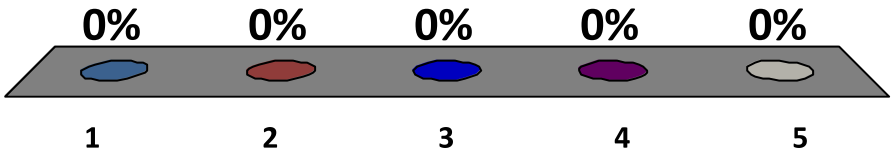
- Introduction to transmission lines.
- 1st and 2nd order systems.
- Linear oscillators.

Lecture style

- Bring your clickers!
- Lectures will include a mixture of me talking and you practicing, where ever possible.

How many electronic communication devices do you own?

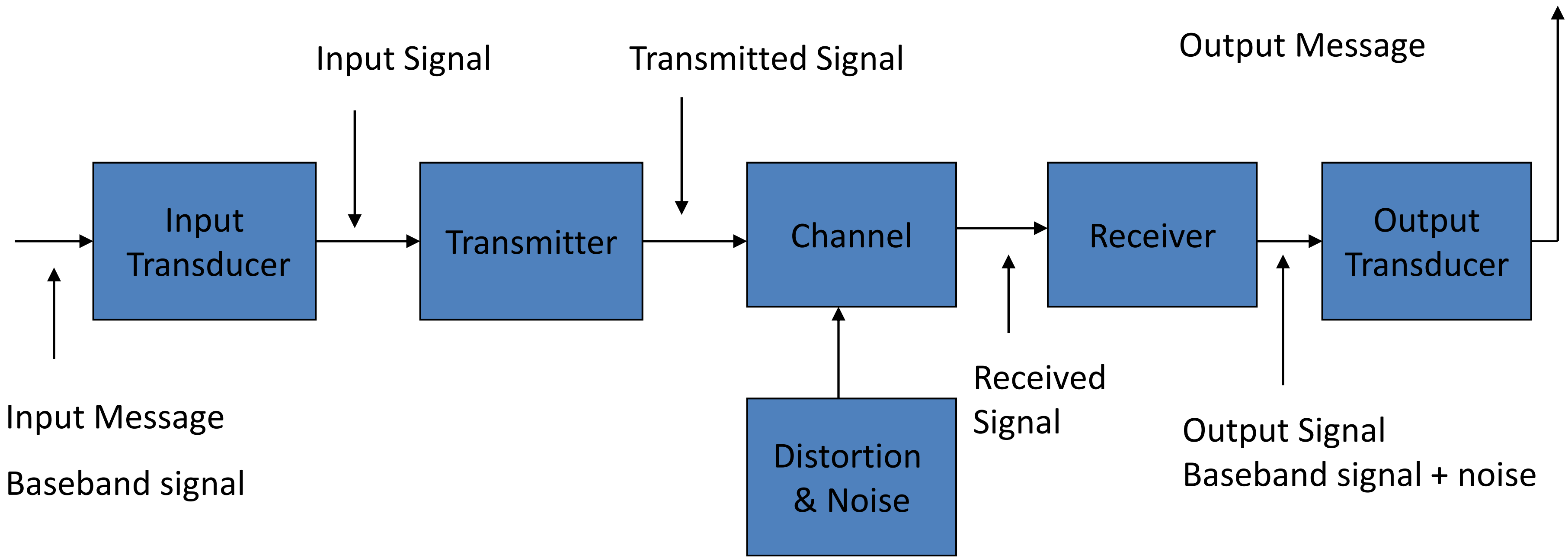
1. Zero
2. <3
3. 3-5
4. 6-9
5. >10



There are many different devices – look at the history!

- **1837 – Samuel Morse invented telegraph.**
- **1858 – First telegraph cable across Atlantic (Canada – Ireland)**
- **1876 – Alexander Graham Bell invented telephone.**
- **1888 – Heinrich Hertz introduce electromagnetic field theory.**
- **1897 – Marconi invented wireless telegraph.**
- **1906 – Radio communication system was invented Marconi.**
- **1923 – Television was invented.**
- **1938 – Radar and microwave system was invented for World War II.**
- **1956 – First telephone cable was installed across Atlantic.**
- **1960 – Laser was invented**
- **1962 – Satellite communication**
- **1969 – Internet DARPA – response to Sputnik launch in DoD**
- **1970 – Corning Glass invented optical fiber.**
- **1975 – Digital telephone was introduced.**
- **1979 - Mobile Telephones introduced NTT Japan 900 MHz**
- **1985 – Facsimile machine.**
- **1988 – Installation of fiber optic cable across Pacific and Atlantic.**
- **1990 – World Wide Web and Digital Communication.**
- **1998 – Digital Television.**

Basic Communication System



Input message examples

- Voice
- Music
- Video
- Digital data
- Heart rate
- etc

Input transducer might be:-
- Microphone
- Camera
- Data logger
- ECG and other medical

Transmitter/Receiver examples

- Mobile phone
- Laptop
- TV or radio antenna
- Satellite
- etc

Channel examples

Channels may be
•Urban environment – many buildings
•Rural
•Optical fibre
•Other wired

Output transceiver examples

Output transducer may be:-

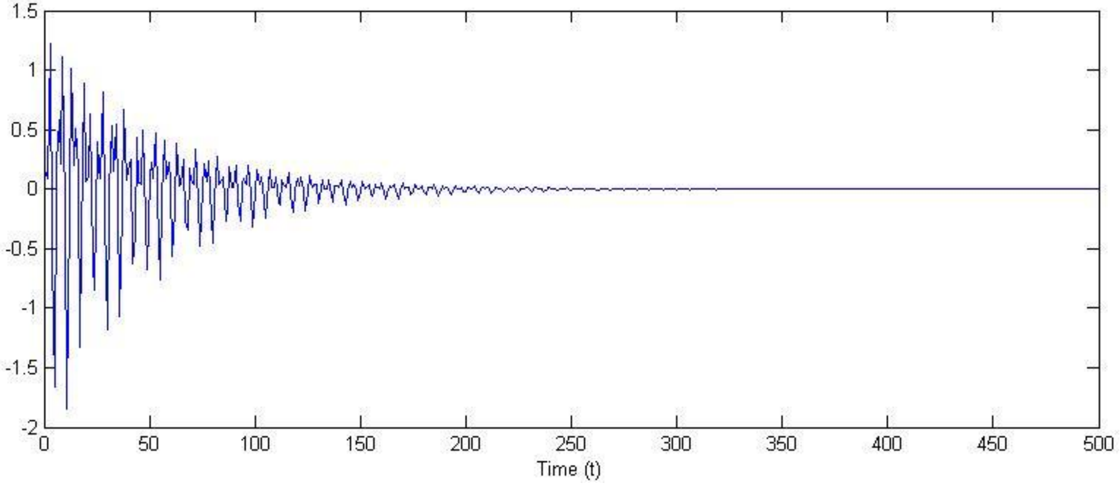
- The human ear!
- A loudspeaker
- TV
- etc

What does the perfect communications engineer need?

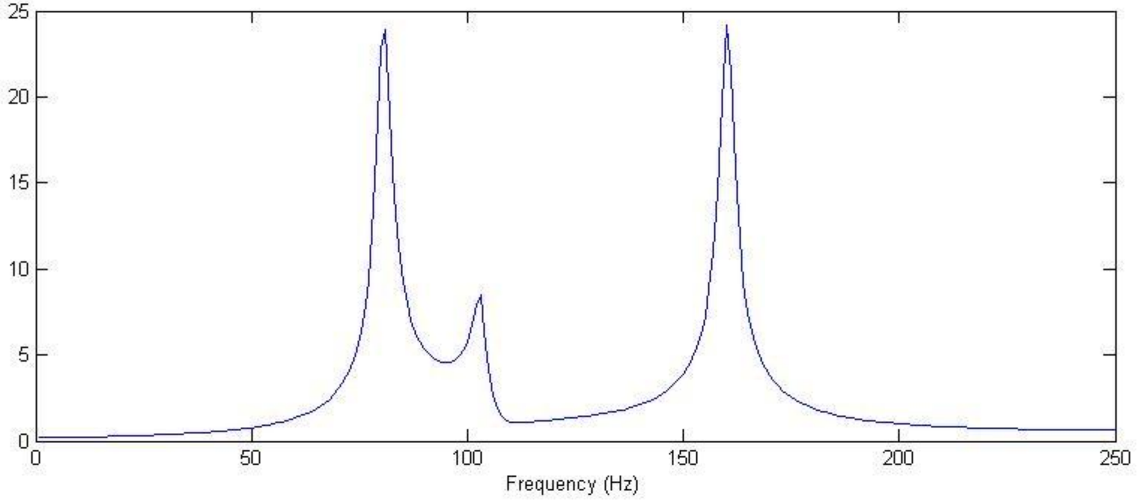
- Ability to
  - manipulate continuous and discrete signals
  - choose the best method of transmission
  - understand how the channel affects the performance of the system
  - Choose the best method of reception of data
  - Design the circuitry needed to carry out the above tasks in a meaningful manner

What is a signal?

A signal can be thought of in 2 domains as illustrated below

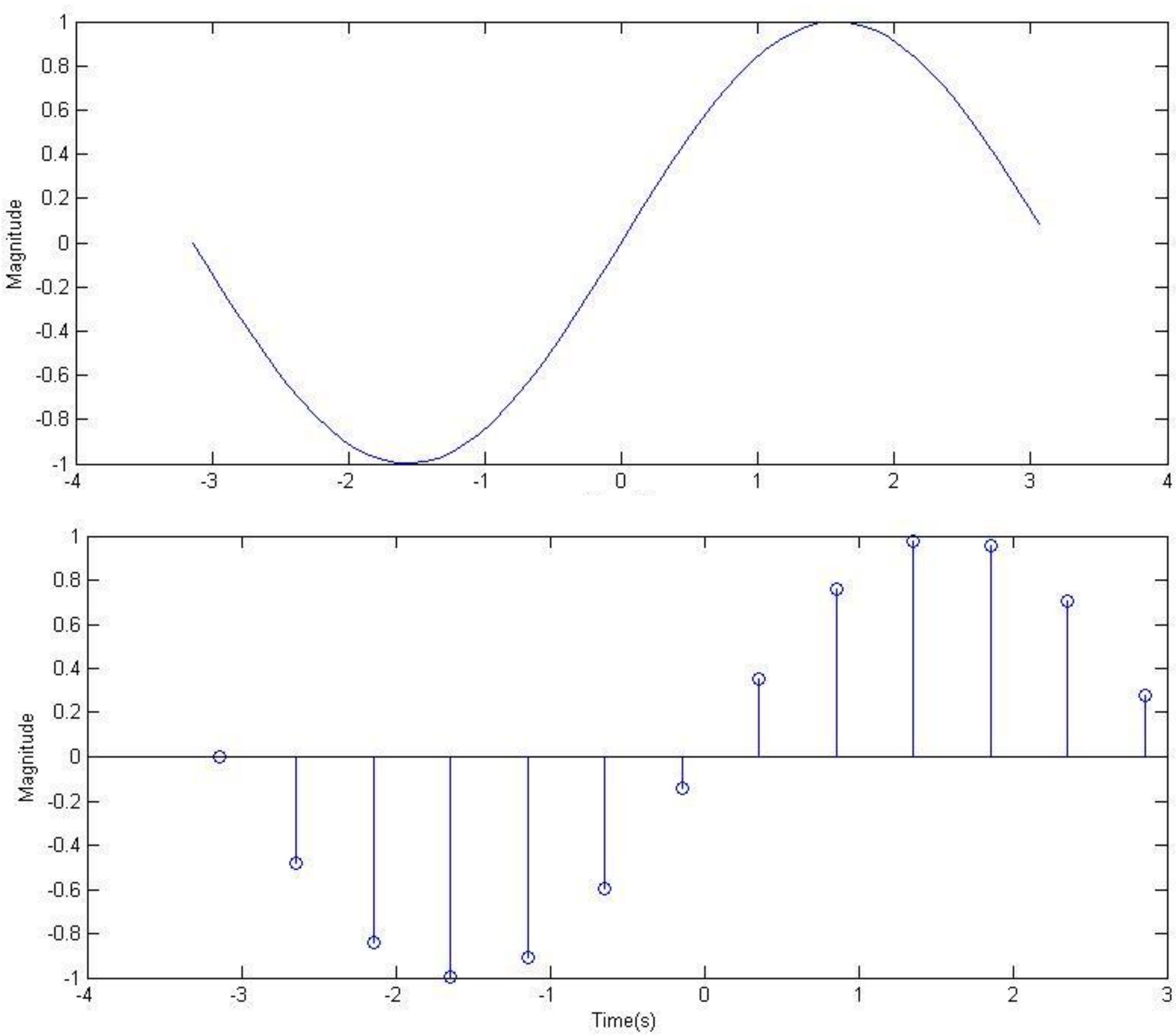


Time domain



Frequency domain
Or frequency spectrum

Continuous or discrete?



Both are just sin(x)

Transmission – Why does everything have a different frequency?

- Why is bbc Radio 1 at 97-99MHz, but bbc Radio 2 transmits at 88-91MHz?
- And why so high in frequency? I can only hear up to 20kHz.

Transmission – An antenna to transmit voice messages

- Antenna theory tells us that to transmit efficiently we need an antenna which is $\lambda/4$
- Typical voice frequencies cover 300Hz-20kHz, so the low frequency requirement would be an antenna of length 1000km!!

Why different frequencies

- Ever tried listening to a conversation in a noisy room?
  - It’s the same with any communications system.
  - If you don’t transmit at different frequencies you get interference.
  - However, there are clever ways to get around this. What if everyone in the room shouted a word on there own and each person got there allotted amount of time to shout?

I need 4 volunteers

- In turn shout what you see

This

Lecture

is

Awful

That is an example of “multiplexing”

- That was time domain multiplexing
- Others include
  - Frequency domain multiplexing
  - Code multiplexing
  - etc

So which frequency do I choose to transmit at?

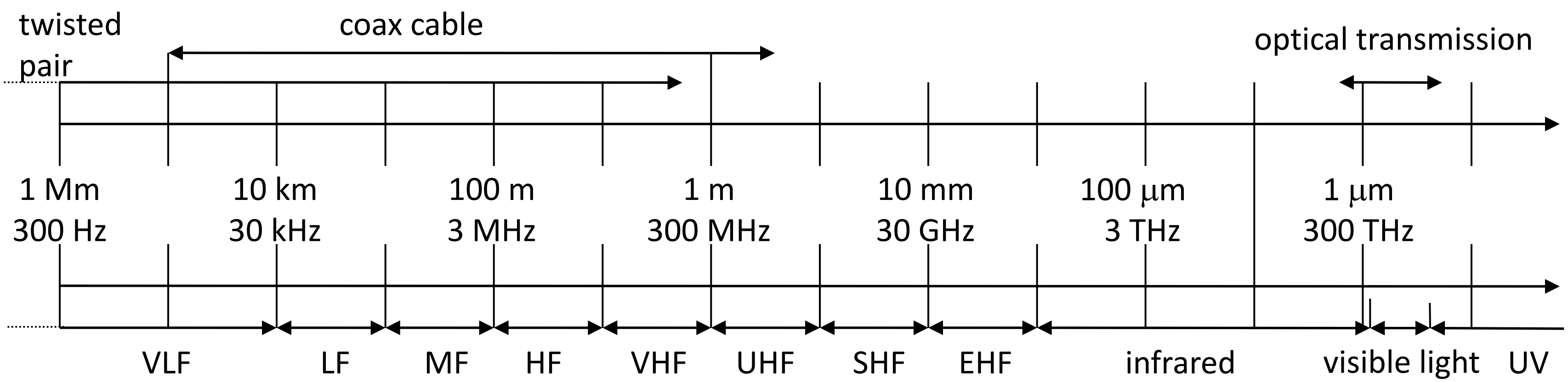
A low frequency wave travels much further than a high frequency

High frequency systems cost more than low frequency systems

A low frequency carrier has less bandwidth -

- Less communication capacity
- Lower quality

Frequencies for communication



- VLF = Very Low Frequency
- LF = Low Frequency
- MF = Medium Frequency
- HF = High Frequency
- VHF = Very High Frequency

UHF = Ultra High Frequency
SHF = Super High Frequency
EHF = Extra High Frequency
UV = Ultraviolet Light

- Frequency and wave length: $\lambda = c/f$
- wave length λ, speed of light $c \cong 3 \times 10^8$ m/s, frequency f

How do I convert my low frequency signal to this high frequency?

- This process is called modulation
- I have to “modulate” my signal, s(t) on to a high frequency “carrier”, for instance sin(2*pi*f*t)
- This can be done using different modulation schemes
  - Amplitude
  - Phase
  - Frequency
  - Digital

Once I’ve transmitted my message how do I get it back?

- This process is called demodulation
- Demodulation effectively takes away the high frequency carrier and leaves you with your message

So that was a brief overview

- We will develop these techniques from a mathematical beginning to a practical completion.
- Please attend the lectures

End lecture 1

Signal analysis - Why bother?

- Communications engineers have to transmit very complex signals. Basic understanding of signals is essential
- The mathematics involved with communications includes functions that you may not be familiar with.

Types of signals

There are 2 types of signal:-

1. Continuous time varying

2. Discrete time varying

$$y(t) = 3x(t)$$

$$y[n] = 3x[n]$$

Specific continuous signals

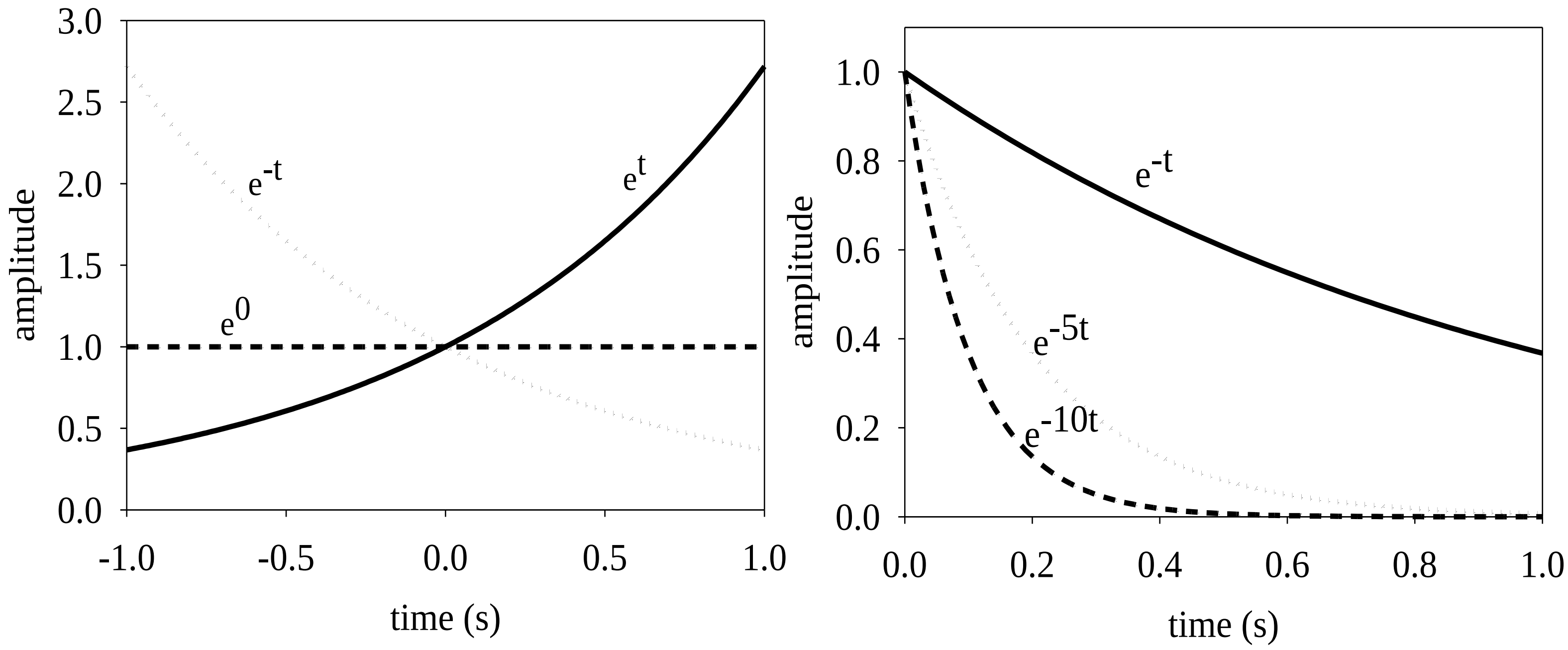
- Two of the most used functions in communications are the exponential and sinusoid.
- This should be a nice revision session on functions you should already be aware of

The exponential

$x(t) = e^{-at}$, $t \geq 0$.

If a is positive $x(t)$ decays exponentially.

If a is negative $x(t)$ grows exponentially.



The exponential

At what time will $x(t) = 0$? Mathematically this happens when $t = \infty$. In practice we often consider $x(t) = 0$ if its magnitude is less than 1% of its peak magnitude.

- At $t = t+\tau$, $\dfrac{e^{-a(t+\tau)}}{e^{-at}} = e^{-a\tau} = e^{-1} = 0.37$ (37% of its original value).

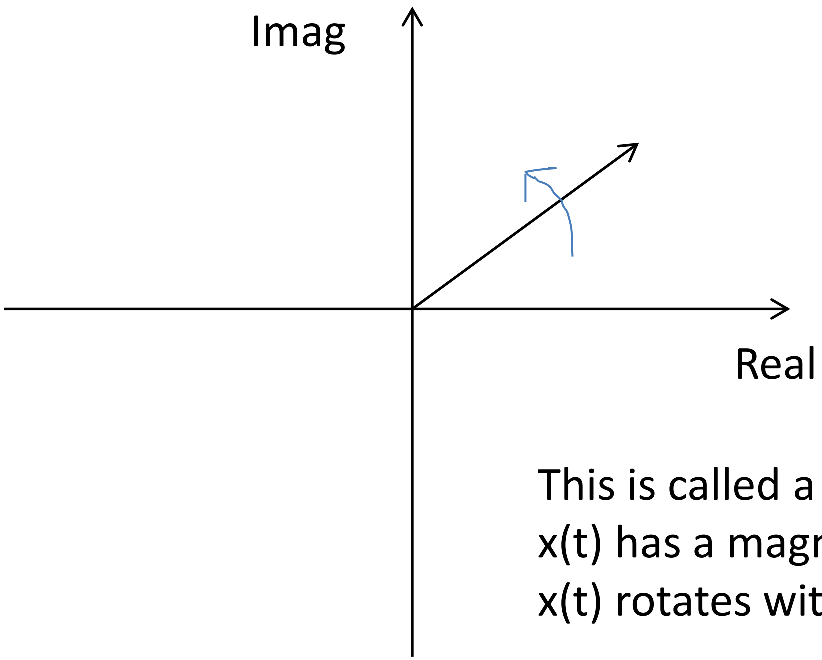
- At $t = t+5\tau$, $\dfrac{e^{-a(t+5\tau)}}{e^{-at}} = e^{-a5\tau} = e^{-5} = 0.007$ (0.7% of its original value).

Thus, we often consider e^{-at} to reach zero after 5τ. Where τ is known as the time constant

The exponential

What happens to $x(t) = e^{-at}$ when "a" is not a real number?

If we set a=-jω_0 then $x(t) = e^{j\omega_o t}$



This is called a phasor diagram

x(t) has a magnitude of 1

x(t) rotates with angular frequency ω_0

A quick exponential test!

If $x(t)=e^{-at}=0.5$ at $t=0.5s$, what is a?

1. 0.6
2. 1.38
3. 1

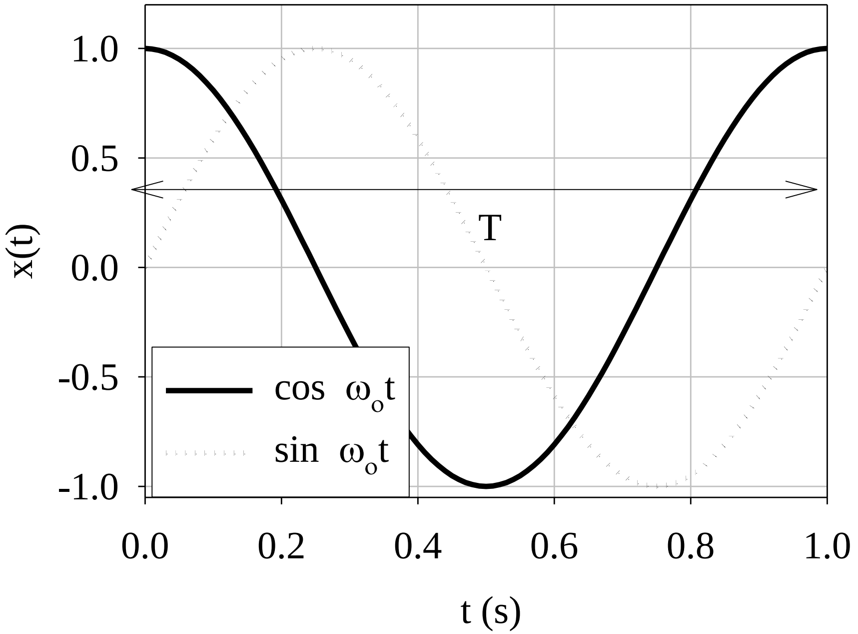
The Sinusoid

A sinusoidal signal is given by

$$v(t) = V\sin(\omega_o t) = V\cos\left(\omega_o t - \pi/2\right) \qquad T = \frac{1}{f_o} = \frac{2\pi}{\omega_o}$$

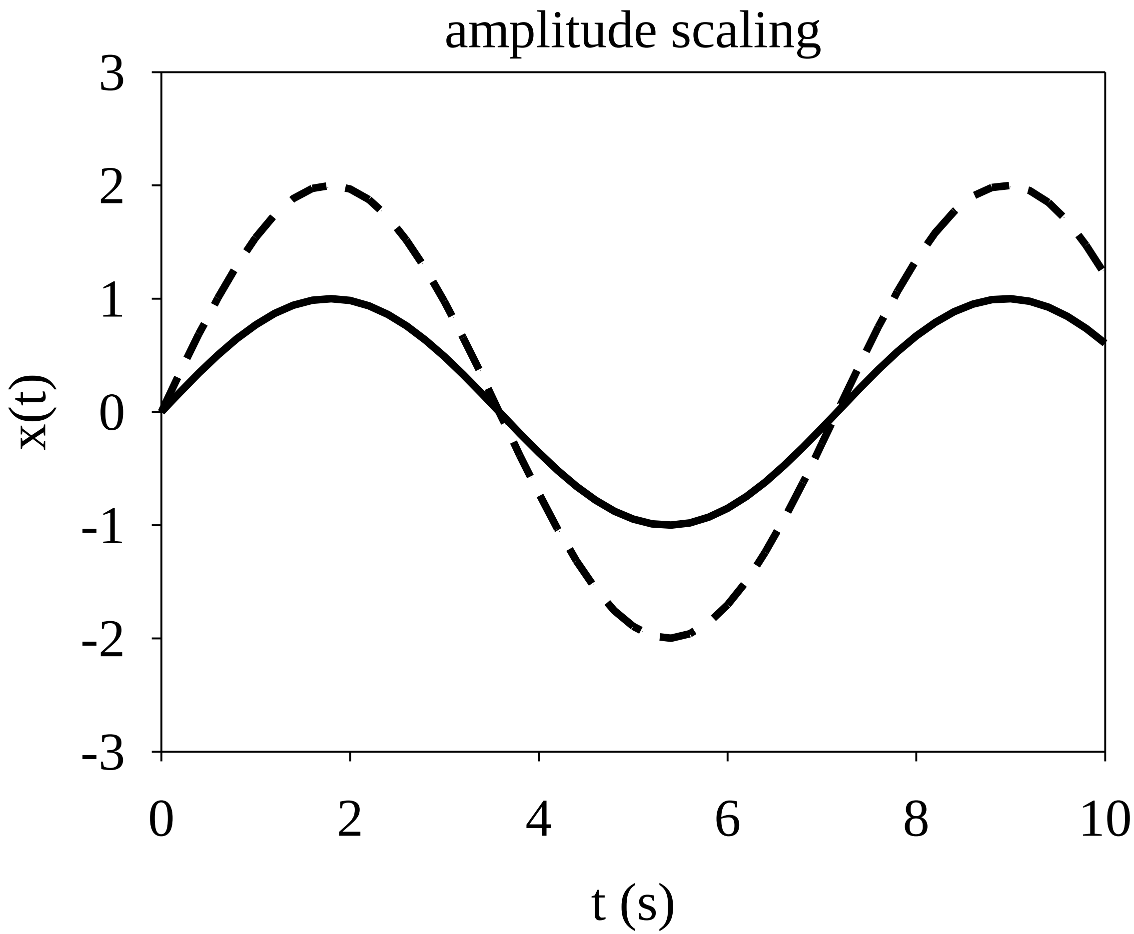
$$\sin(\omega_o t) = \frac{e^{j\omega_o t} - e^{-j\omega_o t}}{2j}$$

$$\cos(\omega_o t) = \frac{e^{j\omega_o t} + e^{-j\omega_o t}}{2}$$



Sinusoid manipulation

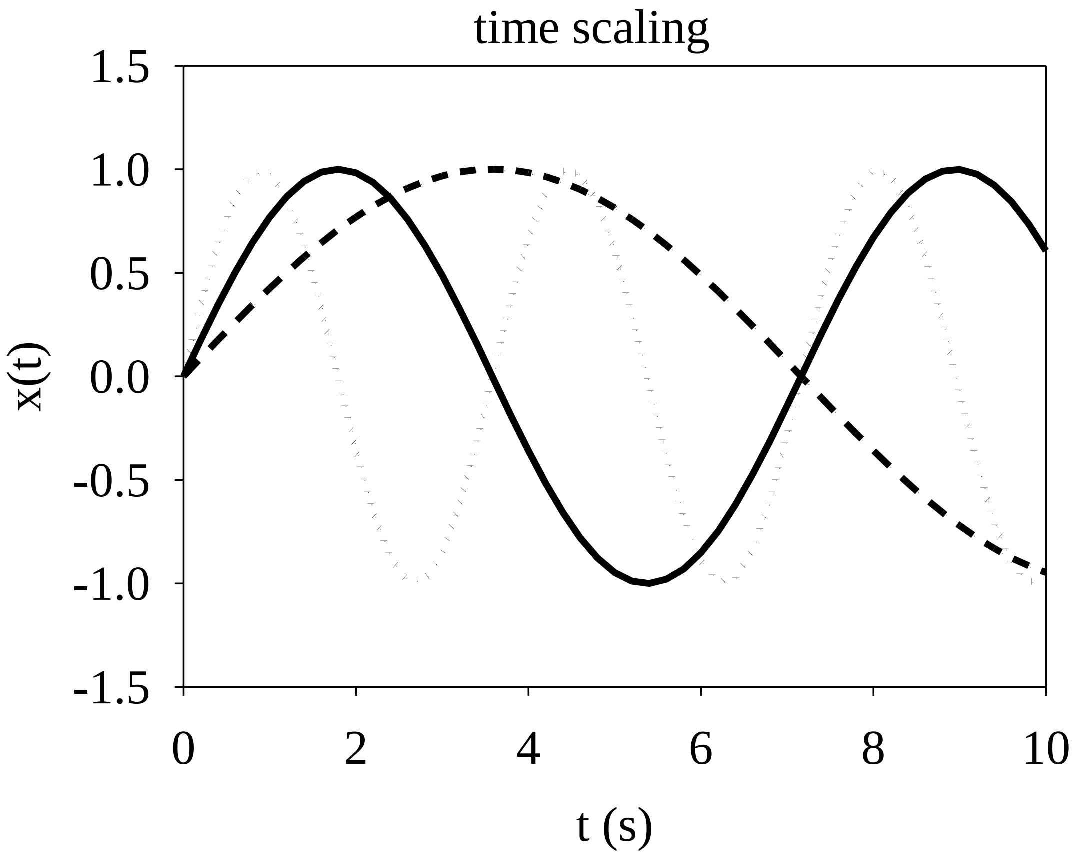
Amplitude scaling: $y(t) = Ax(t)$



Sinusoid manipulation

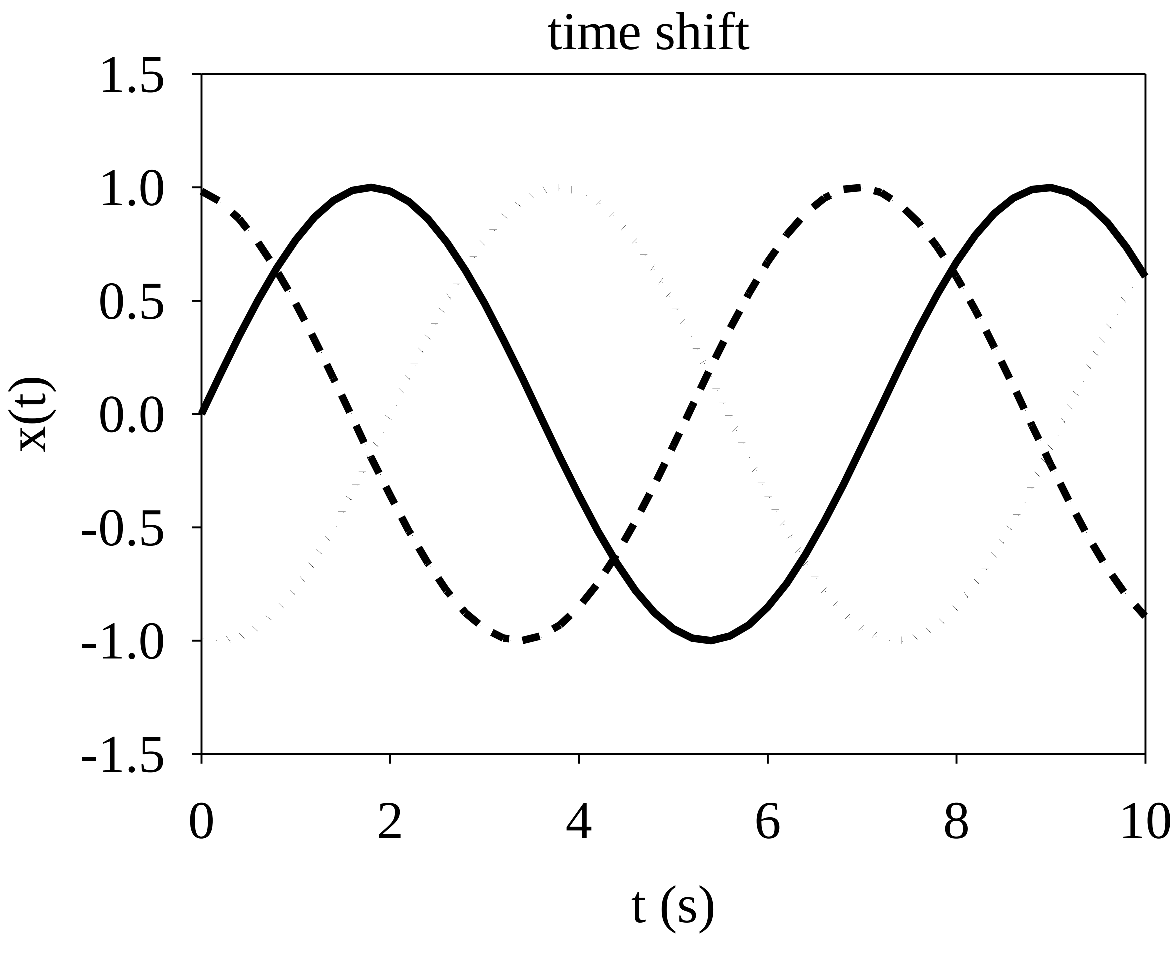
Time scaling: $y(t) = x(At)$

$y(t)$ is a time-compressed (speed up, if $A > 1$) or a time-expanded (slowed down, if $A < 1$) version of $x(t)$.



Sinusoid manipulation

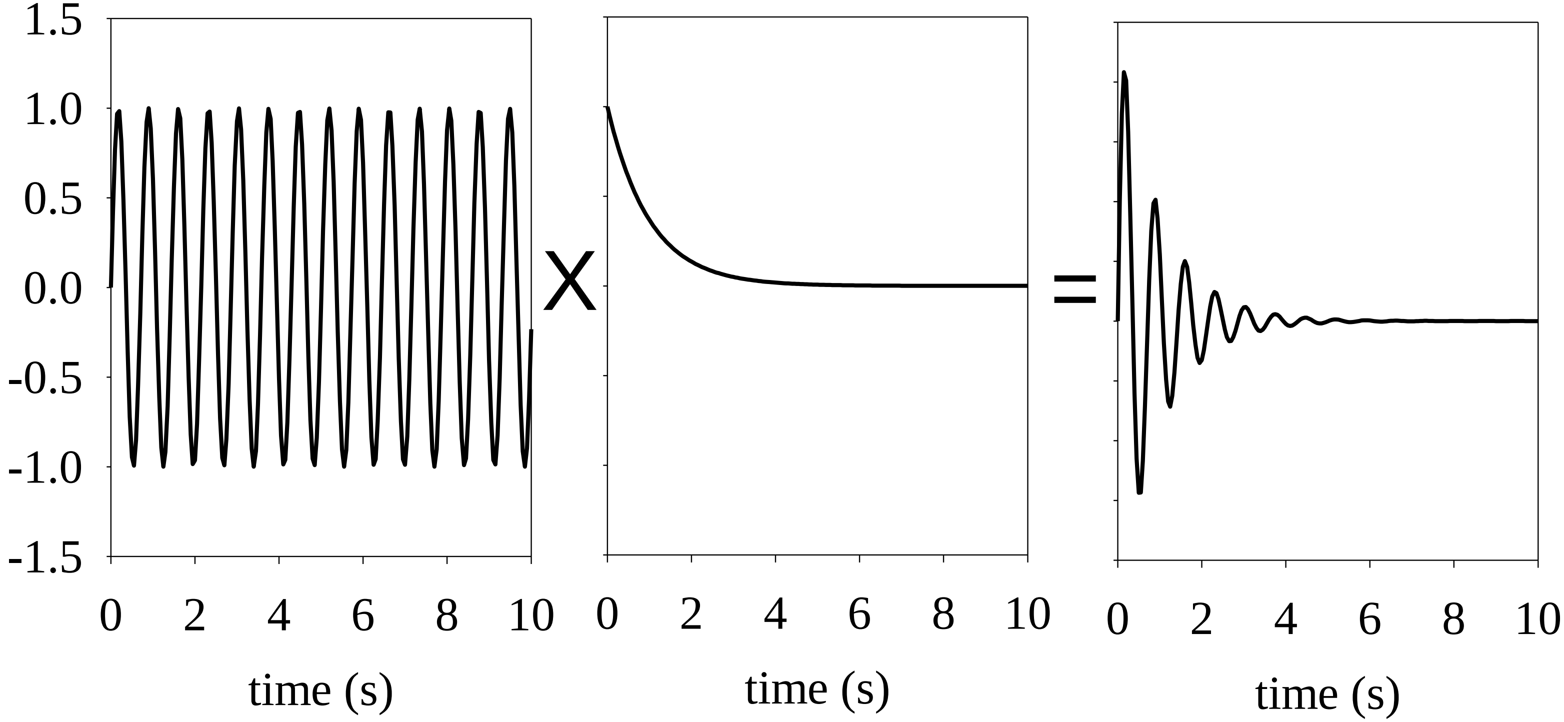
Time shifting: $y(t) = x(t\text{-}t_o)$ or $x(t+t_o)$



Is sin(t-t0) a time delayed version of sin(t)?

1. Yes
2. no

Multiplying signals



Sinusoidal signals multiplied by an exponential are usually known as damped sinusoids

Important trigonometric identities

$$\sin A \sin B = \tfrac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$2\sin^2 A = 1-\cos 2A$$

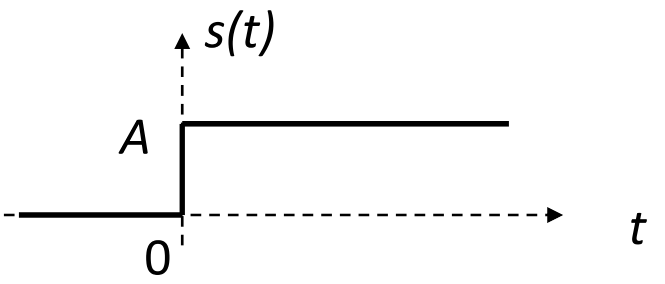
$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

Step functions

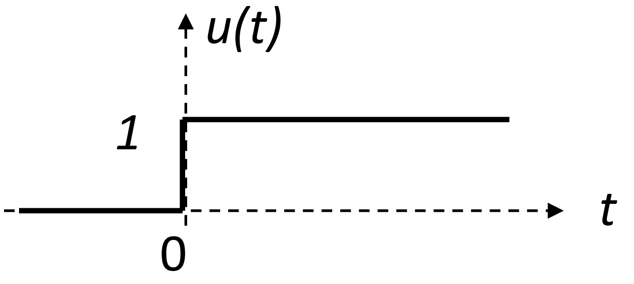
The step function is defined below

$$s(t) = \begin{cases} 0, & t < 0 \\ A, & t \geq 0 \end{cases}$$



When $A = 1$ the step function is known as a **Unit Step Function**

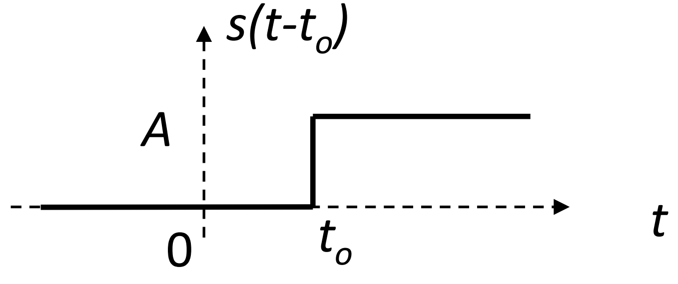
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



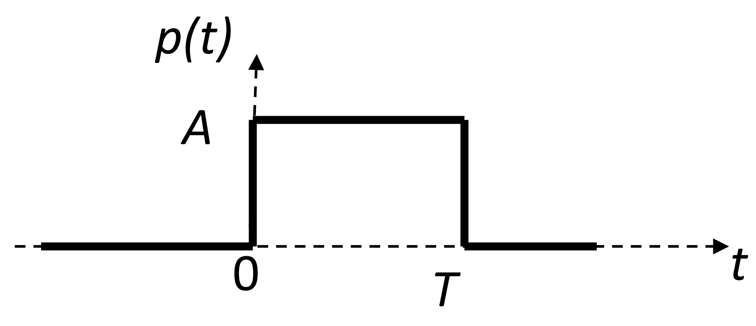
In general $s(t) = Au(t)$ where $A \neq 0$. E.g: current flow through a resistive circuit in which aswitch is closed at time $t = 0$. The current is zero for $t < 0$ and has a constant value for $t \geq 0$.

Step functions

If a switch is closed at $t = t_0$ a delay step signal, $s(t- t_0)$ is obtained.



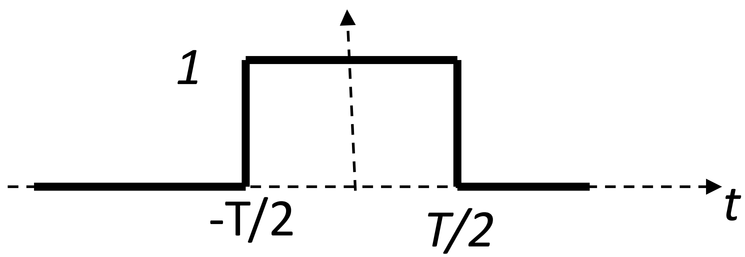
If the switch is closed at $t = 0$ and opened at time $t = T$, a pulse signal, $p(t)$ of width T is obtained.



Rectangular function

A rectangular function, rect(t), is a function that can be described as a manipulation of unit step functions, as below.

$$rect_{-T/2,T/2}(t) = \begin{cases} 0, & |t| > T/2 \\ 1, & |t| \le T/2 \end{cases}$$



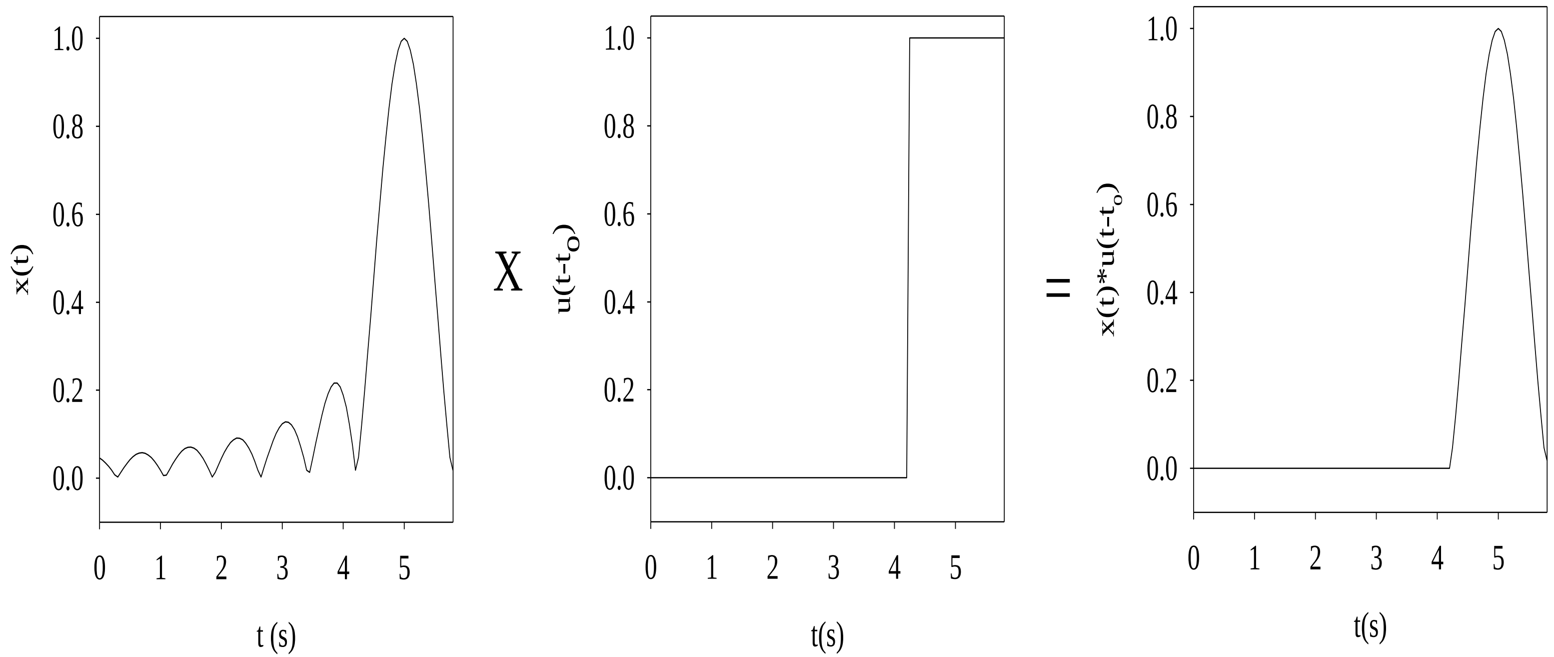
Or

$$rect_{-T/2,T/2}(t) = u(t+T/2) - u(t-T/2)$$

Step and Rect applications

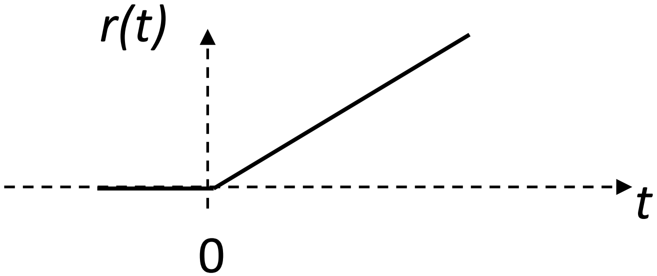
Both the step and rectangular functions are useful as perfect filters, for instance:-

$$x(t)u(t-t_o) = \begin{cases} 0, & t < t_o \\ x(t), & t \geq t_o \end{cases}$$



Ramp and periodic functions

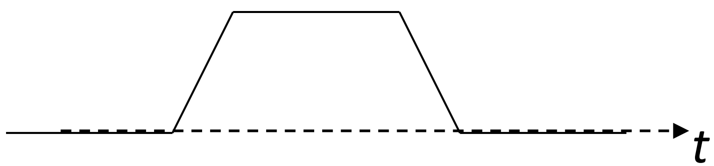
If we integrate *u(t)*, a unit ramp function, *r(t)* is obtained.



$$r(t) = \begin{cases} 0, & t < 0 \\ \int_0^t 1 d\tau = t, & t \geq 0 \end{cases}$$

Ramp functions are often used to describe non-ideal digital transitions. i.e. how does a voltage instantly switch from 0 to 1V??

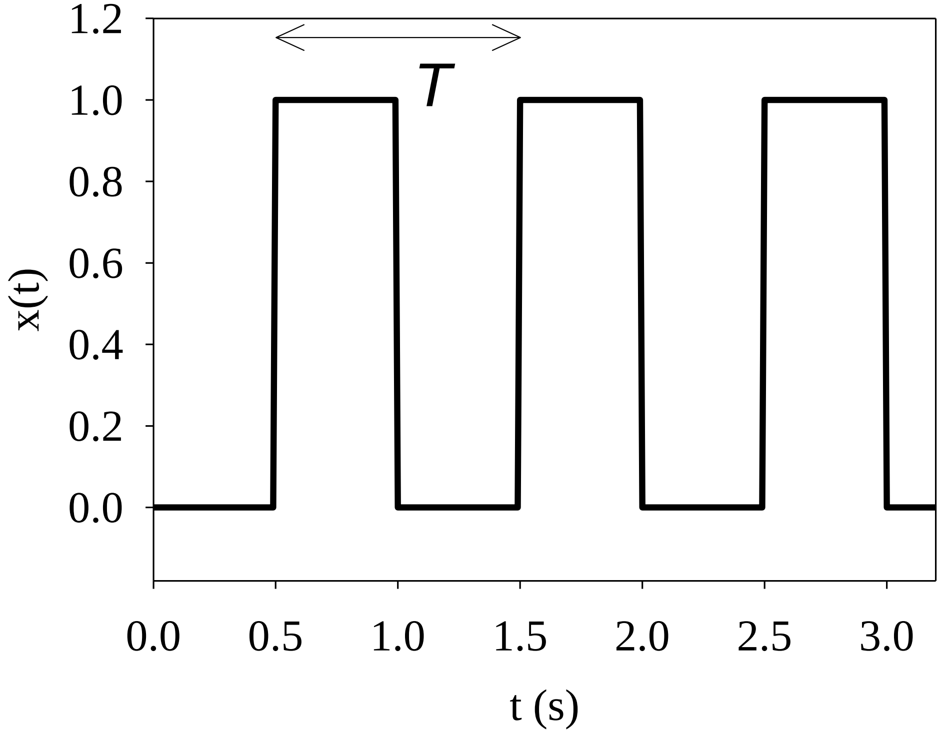
Answer is it has to change gradually as illustrated below.



Periodic functions

A signal *x(t)* is periodic with period *T* if *x(t) = x(t+T)* for all *T*.

Clearly this is just a repeating rectangular function, but is essential in the use of digital systems! Note that x(t) can be any continuous function



Unit impulse

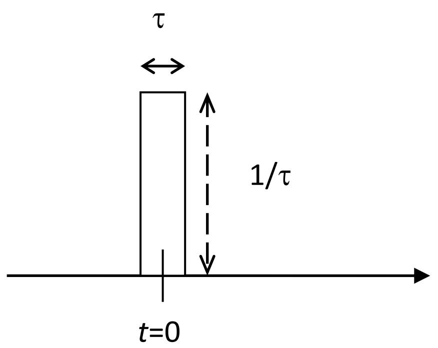
- The unit impulse is an extremely important function and is used throughout communication mathematics
- It’s also a relatively simple concept!

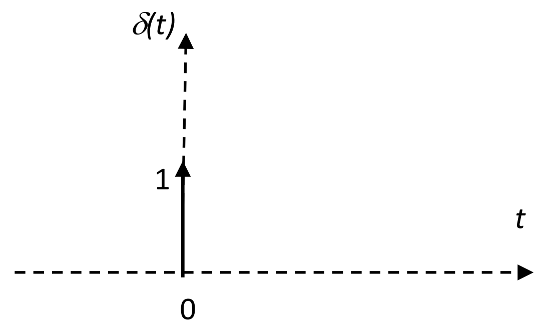
Unit impulse

Unit impulse $\delta(t)$ is an idealisation of a signal that

- is zero for all nonzero values of t: i.e, $\delta(t) = 0$ for $t \neq 0$ and $\delta(t) = 1$ for $t = 0$.
- has an area of unity : $\int_{-a}^{a} \delta(\tau) d\tau = 1$ for any real number $a > 0$.

for example:





Unit impulse approximated by a square pulse when $\tau \to 0$. $\delta(t)$ is often represented by an arrow

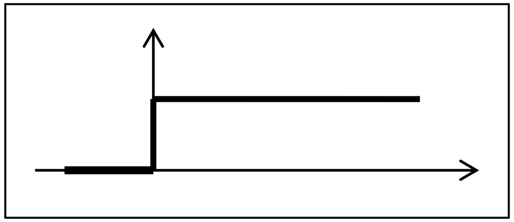
For any real number K, $K\delta(t)$ is the impulse with area K.

Unit impulse properties

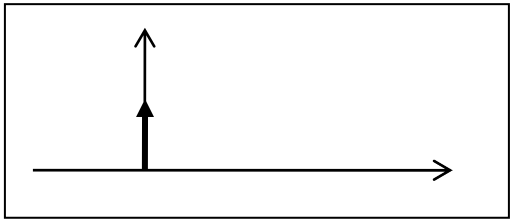
What happens if we integrate $\delta(t)$?

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t) \quad \Leftrightarrow \quad \frac{du(t)}{dt} = \delta(t)$$

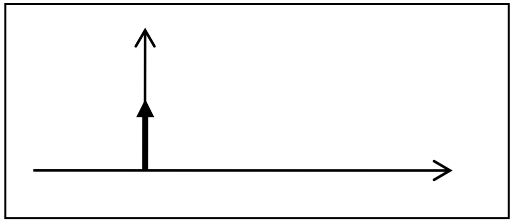
$$\frac{d}{dt}\Bigg($$



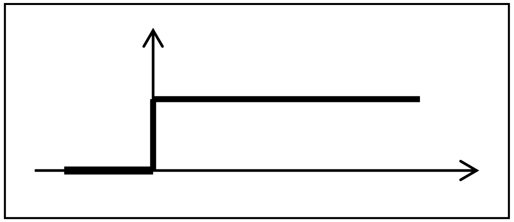
$$\Bigg) =$$



$$\int_{-\infty}^{t}\Bigg($$

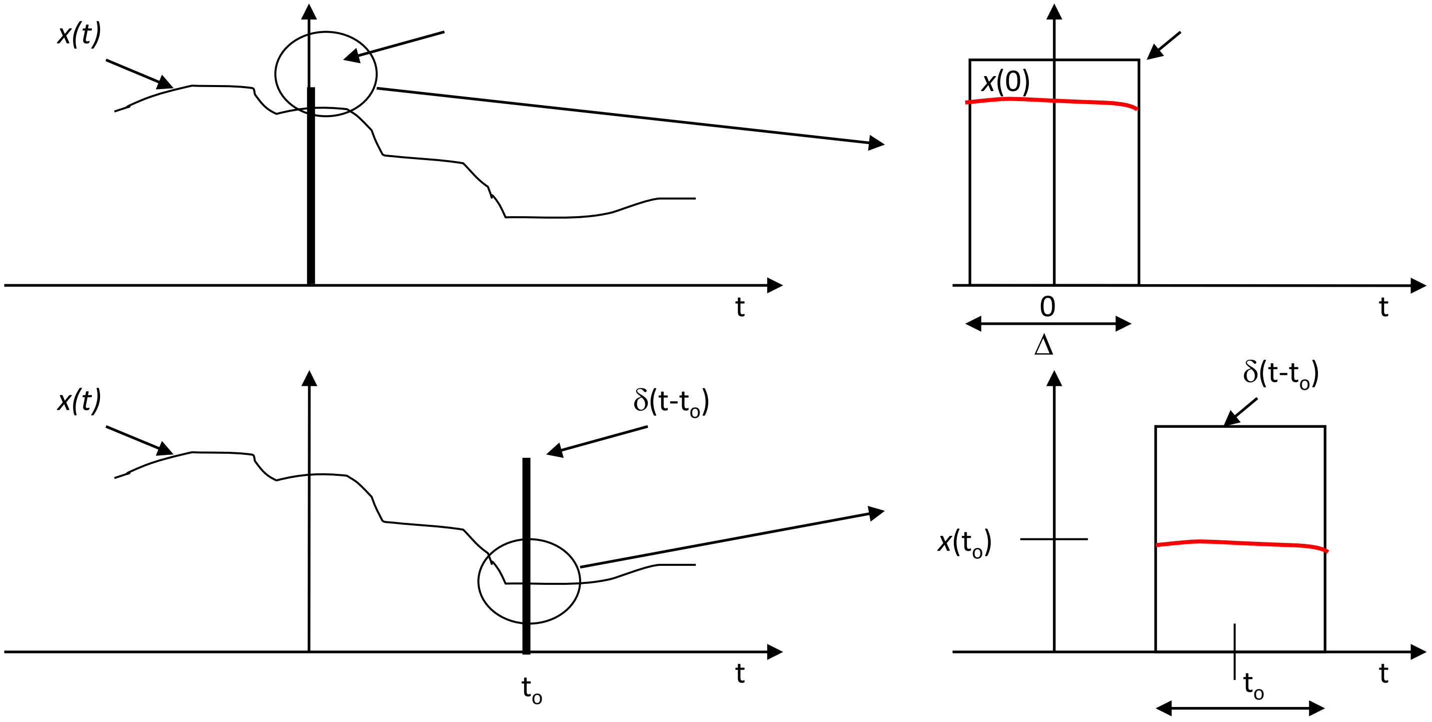


$$\Bigg)\, dt =$$



Unit impulse properties

It can be shown that $x(t)\delta(t-t_o) = x(t_o)\delta(t-t_o)$



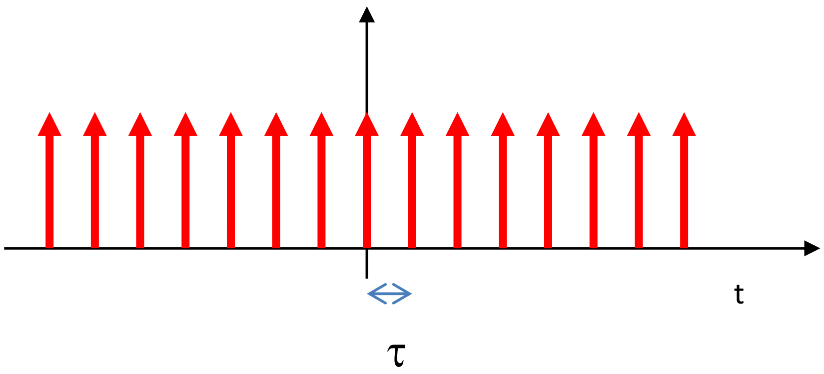
Consider the product $x(t)\delta(t)$ depicted above. If $\Delta \rightarrow 0$, $x(t)\delta(t) \approx x(0)\delta(t)$.
Using similar argument we have $x(t)\delta(t-t_o) = x(t_o)\delta(t-t_o)$.

Unit impulse properties

Consider the case when we have a periodically repeating function of unit impulses, each spaced by a time, τ.

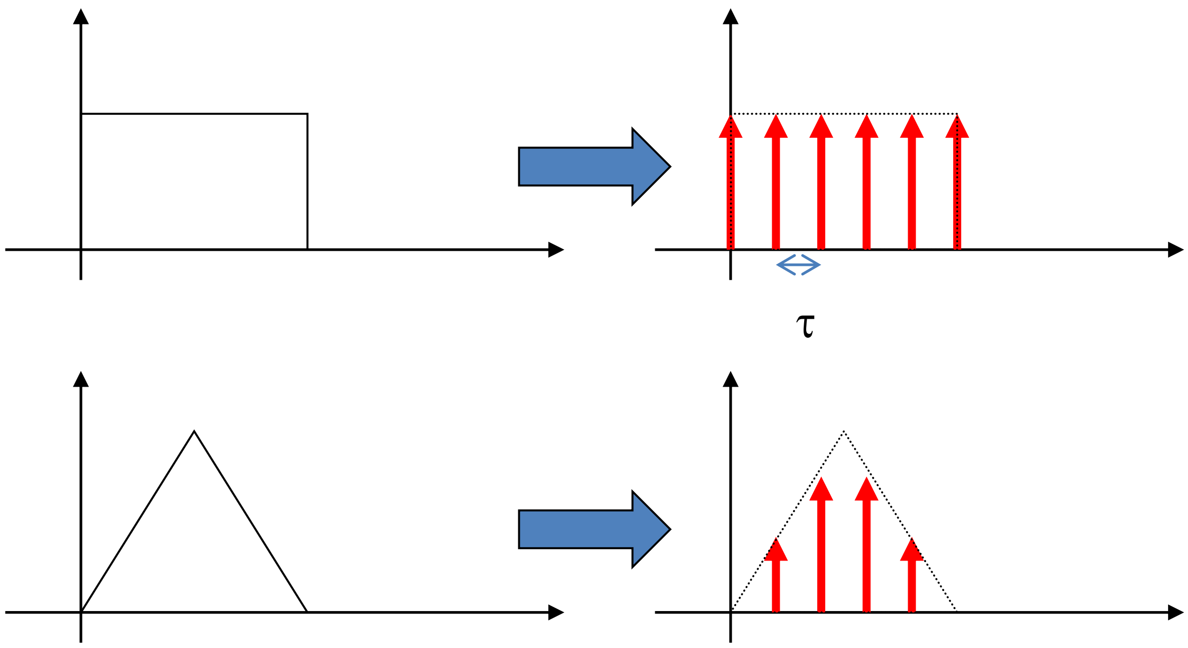
This function would be expressed as, below

$$\sum_{n=-\infty}^{\infty} \delta(t - n\tau) = \ldots\delta(t+2\tau) + \delta(t+\tau) + \delta(t) + \delta(t-\tau) + \delta(t-2\tau)\ldots.$$



Unit impulse properties

If we then multiple this impulse “train” by any continuous function, x(t), then we will obtain a discrete (or sampled) version of x(t). This is the basis of digital communications!



x(t)=…x(0)δ(t) + x(τ)δ(t-τ) + x(2τ)δ(t-2τ)…..

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\tau)\delta(t-n\tau)$$

Discrete signals

All of the previous analysis can be applied to a discrete signal, the notation is given below

$$y(t) = f(x(t))$$

$$y[n] = f(x[n])$$

Brief summary so far

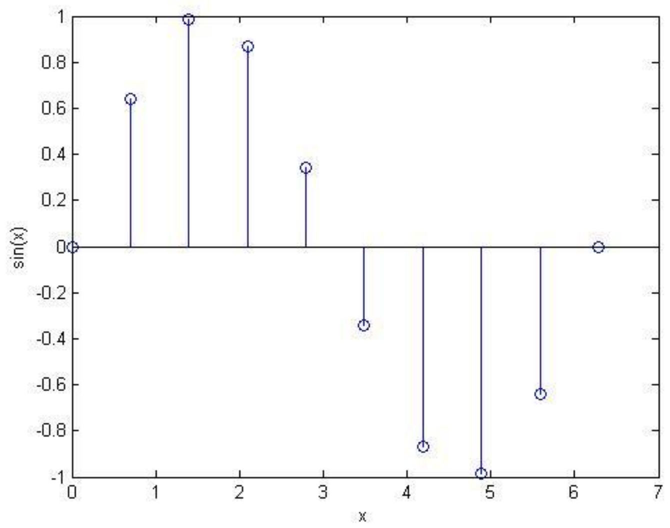
- Step and rect functions useful for mathematical filtering
- Ramp useful for realistic application of digital signals
- Periodic functions useful for describing digital signals
- Impulse response useful for generating a discrete version of a continuous signal

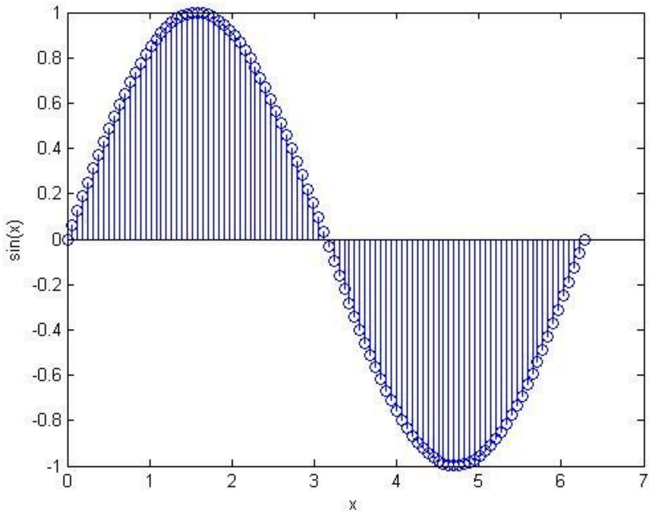
Application of the impulse

Previously we noted that any continuous signal can be "made" discrete by the function below

$$x(t) \approx \sum_{n=-\infty}^{\infty} x(n\tau)\delta(t-n\tau)$$

What happens as we make τ very small?





10 point y[n]=sin[n]

100 point y[n]=sin[n]

Application of the impulse

A very powerful conclusion of this is :-

“Any continuous signal can be made up of an infinite number of weighted impulse responses, separated by an infinitely small distance”

Mathematically we can write this as below

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Discrete version

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

Continuous version

Why is this so important? SYSTEMS analysis!!

Consider the case below

Input signal
(voltage or current)

Output signal
(voltage or current)

This is called a **SYSTEM**

A system can be thought of as a process of transforming an input signal from one form to another as an output signal.

The SYSTEM can be considered as a black box, with at least one input and one output

System properties

- Memory
- Causality
- Stability
- Linearity
- Time invariance

Basic system properties: Memory

A system is said to be ***memoryless*** if its output $y(t_o)$ **depends** only **on the input** $x(t)$, applied **at $t = t_o$**. $y(t_o)$ is independent of the input applied before and after $t = t_o$.

$y[n] = x[n] - 3x[n]$ and $v_o(t) = \frac{R_2}{R_1 + R_2} v_i(t)$ are memoryless.

If the output value depends on past inputs, the system is said to have ***memory***. Examples of system with memory are:

1) Unit time delay $y(t) = u(t-1)$.

2) Voltage across a capacitor $V_c(t) = \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau$

3) An accumulator output $y[n] = \sum_{k=-\infty}^{n} p[k]$

Is $y(t)=x^2(t)$ memoryless?

1. Yes
2. No

Causality

A system is ***causal*** if its output at the current time **depends only on past** and **current inputs** but **is independent of future input**.

For instance the integrator system is causal or ***non-anticipatory*** because $V_c(t)$ does not depend on future input.

The unit-time advance system is non-causal since its output $y(t)$ depends on future input $u(t+1)$. **In practice all memoryless systems are causal.**

Is the function below causal?

1. Yes
2. no

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right)$$

Stability

A ***stable*** system is a system in which the **output does not diverge when the input to the system is bounded** (i.e if its magnitude does not grow indefinitely).

For example a system described by $y_1(t) = tx(t)$ is unstable.

When the input $x(t) = 1$ is bounded, $y_1(t) = t$ in unbounded.

A system $y_2(t) = \cos(x(t))$ is stable since the output is bounded when the input $x(t)$ is bounded.

Is an integrator stable?

1. Yes
2. no

$$y(t) = \int_{-\infty}^{t} x(\tau) d\tau$$

Linearity

A system is linear if

1) The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$ (additivity property).

2) The response to $ax_1(t)$ is $ay_1(t)$ where a is a constant (homogeneity property).

These two properties can be combined into:

$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$, (Continuous signal)

$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$. (Discrete signal)

Linearity example

Consider the equation below, which is a differentiator

$$y(t) = K\frac{dx(t)}{dt}$$

Step 1: consider the input $x_1(t)$

$$y_1(t) = K\frac{dx_1(t)}{dt}$$

Step 2: consider the input $ax_1(t)$

$$y(t) = K\frac{d(ax_1(t))}{dt} = aK\frac{dx_1(t)}{dt} = ay_1(t)$$

This is linear so far and the
Same proof applies for $x_2(t)$

Linearity example

So what happens for the input $ax_1(t)+bx_2(t)$

$$y(t) = K\frac{d\left(ax_1(t)+bx_2(t)\right)}{dt} = aK\frac{dx_1(t)}{dt} + bK\frac{dx_2(t)}{dt} = ay_1(t)+by_2(t)$$

Hence the differentiator is linear

Is $y(t)=3x(t)+4$ linear?

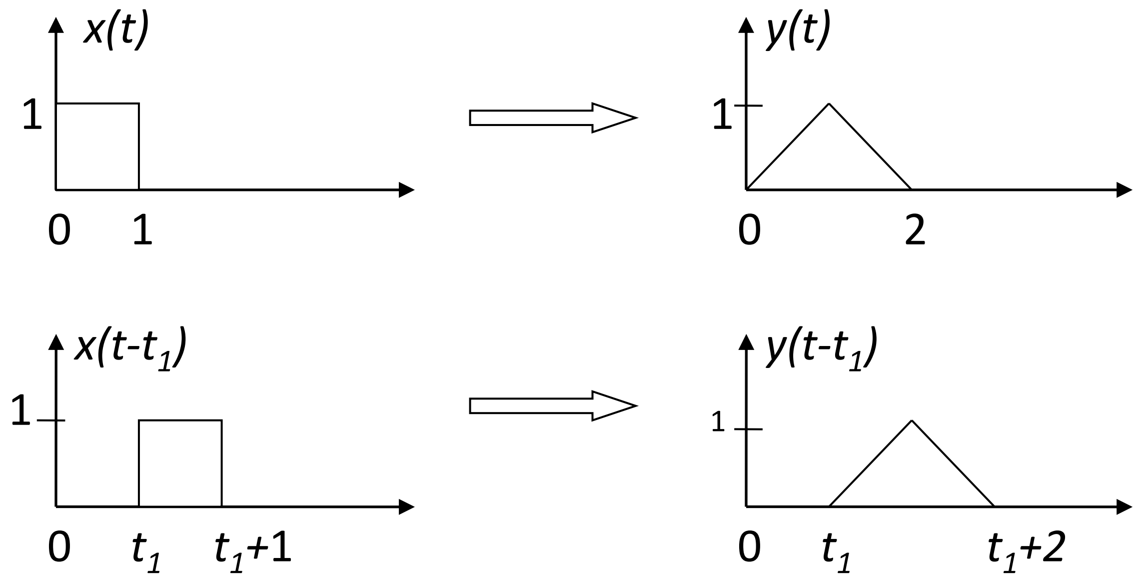
1. Yes
2. No

$y(t)=3x(t)+4$ proof

Time invariant

If the characteristics of a system are independent of time it is said to be ***time invariant.*** The RC low pass circuit is an example of time invariant system since R and C are constant over time.

A time shift in the input signal will result in an identical shift in the output signal of a time invariant system.



Lets go back to our SYSTEM

How do we use what we have learned so far to calculate system performance?

First of all lets make some assumptions

1) The system is linear
2) The system is time invariant

This is often called a Linear Time Invariant (LTI) system

Let's first assess the input signal

We know that any signal can be approximated by a series of impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad\qquad x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

What we are doing here is applying an infinite number of impulses to our system and summing the response for all the time shifted impulses.

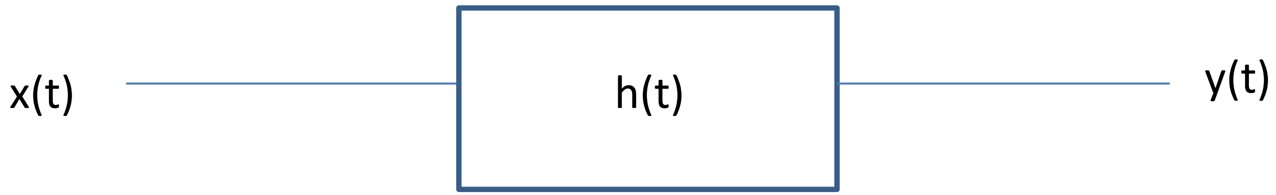
In essence we are using the properties of linearity and time invariance

If we "know" the system response for an impulse, we can calculate the output for any form of input signal.

We denote the **IMPULSE RESPONSE** of the system as h(t) or h[n]

A SYSTEM with any input?

Once we "know" the impulse response we can extend the analysis to the general case



$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

This is known as a the **convolution** integral for Continuous signals

or

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

This is known as a the **convolution** sum for discrete signals

Convolution properties

A **CONVOLUTION** of two signals is often denoted with an asterisk *
This applies to both continuous and discrete signals

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$$

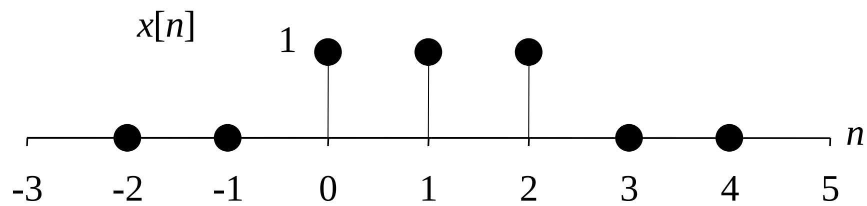
$$y[n] = x[n] * h[n]$$

$$y(t) = x(t) * h(t)$$

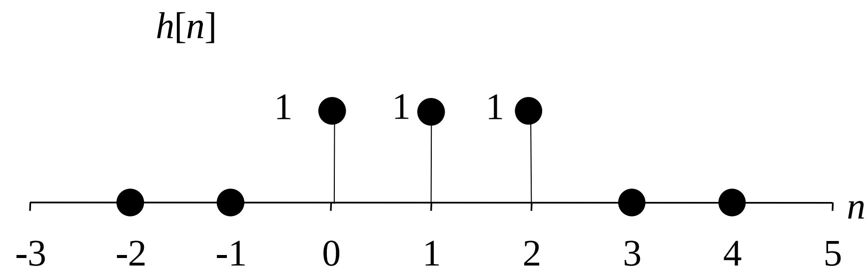
Commutativity	x(t)*h(t)=h(t)*x(t)
Associativity	(x(t)*h(t))*f(t)=x(t)*(h(t)*f(t))
Distributivity	f(t)*(x(t)+h(t))=f(t)*x(t)+f(t)*x(t)

Convolution example

- Consider an LTI system with impulse response *h*[*n*] and input *x*[*n*] shown below



$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$



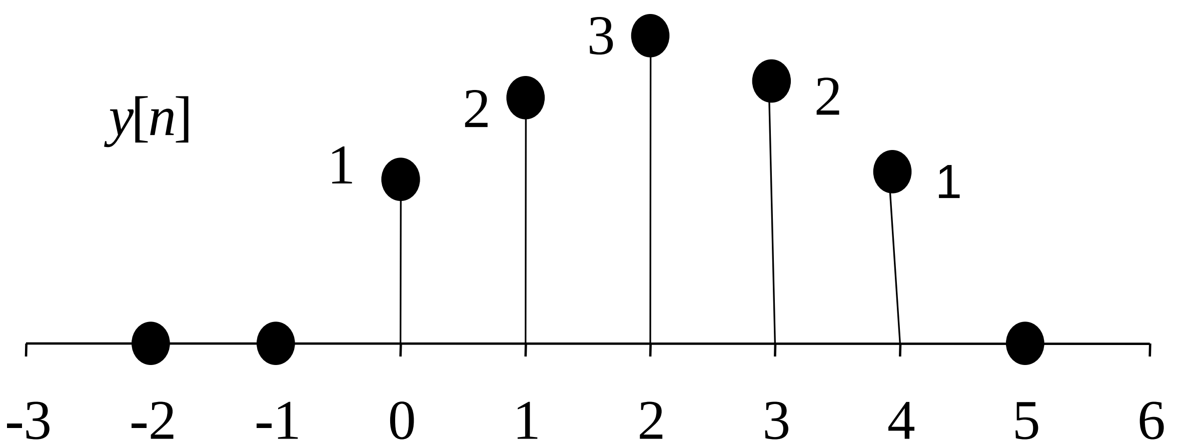
The procedures to compute *y*[*n*] are:

1) Replace the variable *n* with *k*.
2) Flipping *h*[*k*] with respect to *k* = 0 to obtain *h*[-*k*].
3) Shifting *h*[-*k*] to *n* to give *h*[*n*-*k*].
4) Multiply *h*[*n*-*k*] and *x*[*k*] for all *k*.
5) Summing all non-zero products of *h*[*n*-*k*]*x*[*k*] to yield *y*[*n*].

Convolution example

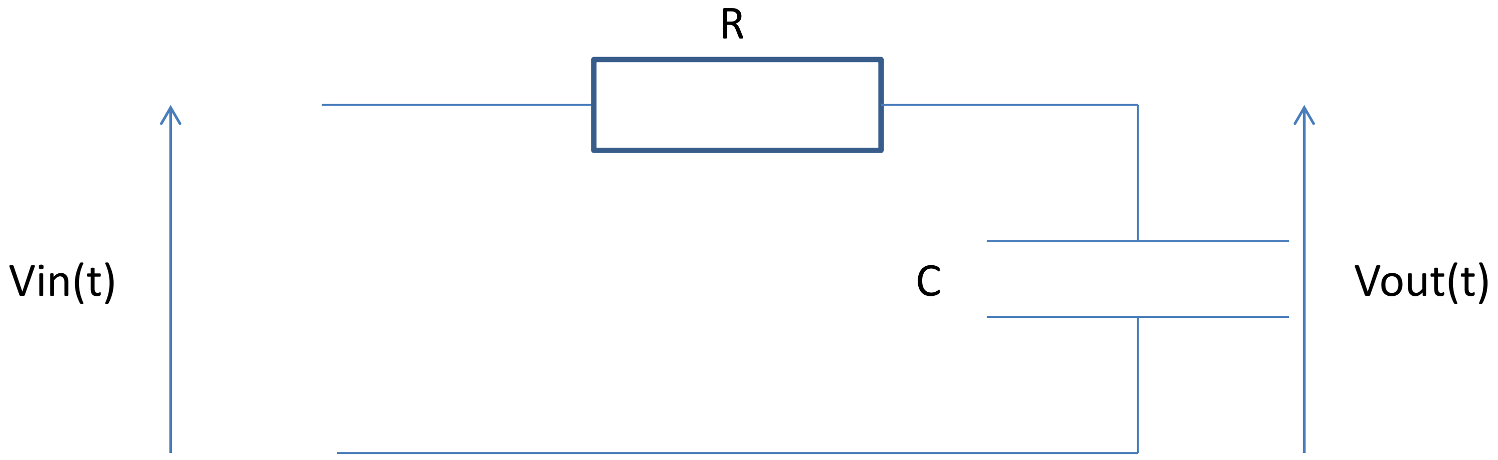
	k	-3	-2	-1	0	1	2	3	4	$y[n]=\Sigma h[n-k]x[k]$
	$x[k]$	**0**	**0**	**0**	**1**	**1**	**1**	**0**	**0**	
	$h[k]$	0	0	0	1	1	1	0	0	
	$h[-k]$									
n = -1	$h[-1-k]$									
n = 0	$h[-k]$									
n = 1	$h[1-k]$									
n = 2	$h[2-k]$									
n = 3	$h[3-k]$									
n = 4	$h[4-k]$									

Convolution example

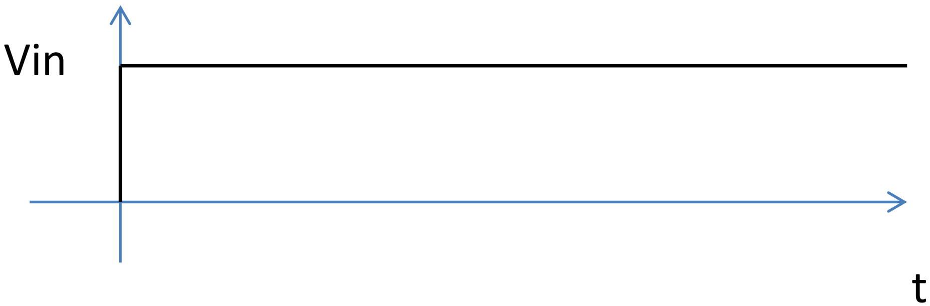


This is a triangular waveform

How might we use convolution in a circuit?



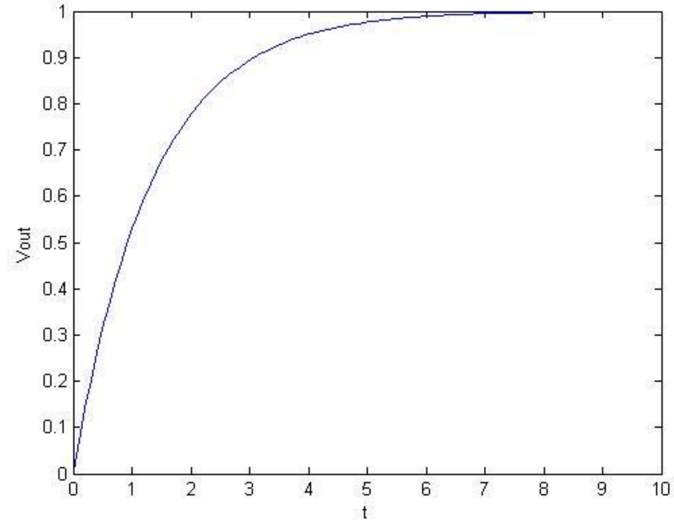
A simple RC circuit? If Vin(t) is given below what would the output be?
Hint, you did this in year 1.



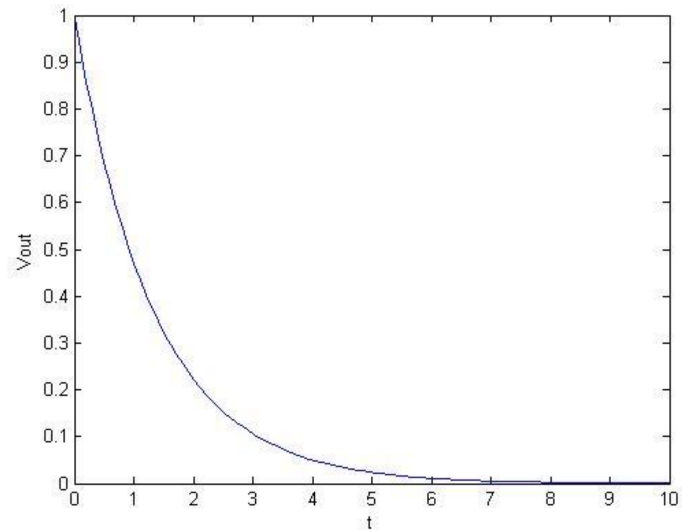
$$V_{in}(t) = V_{in} u(t)$$

Does the voltage look like a or b?

a



b



Does the voltage look like a or b?

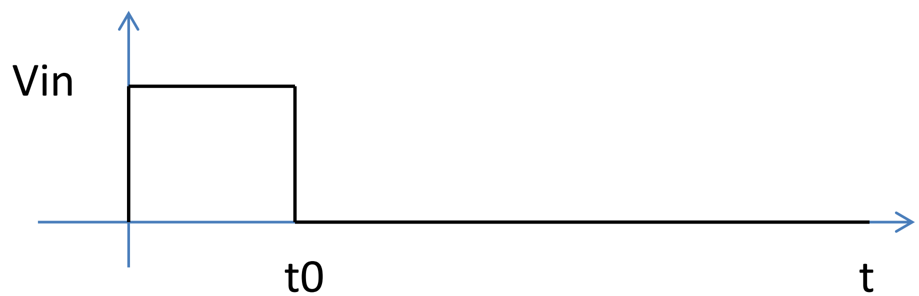
1. A
2. B

RC circuit

The equation below gives the voltage across the capacitor with time

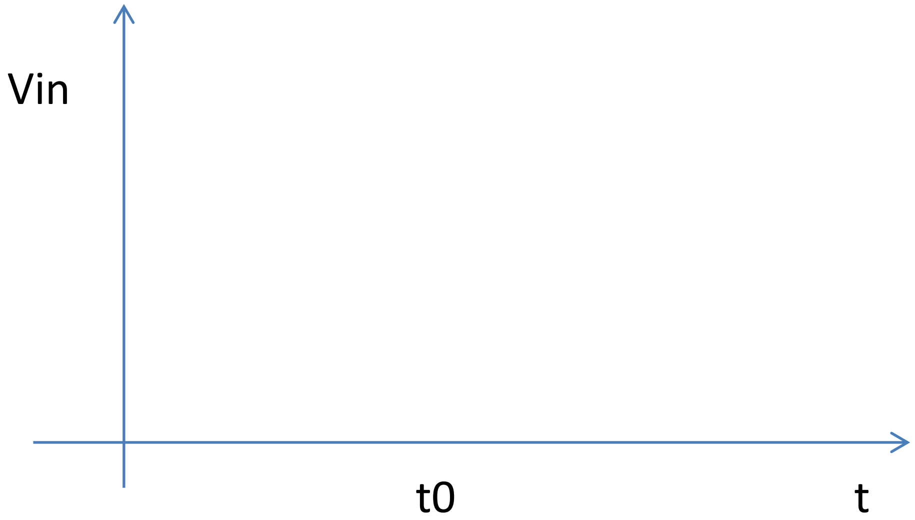
$$V_{out}(t) = V_{in}\left(1 - e^{-t/RC}\right)$$

Now lets assume the input is a single pulse, as below



Assume the time constant, RC<<t0. I.e. the capacitor is fully charged before t0

RC circuit



RC circuit

We could work out the voltage for simple inputs assuming we have the time to work out the charge and discharge curves.

What if the input signal is continually varying, **but not periodic**? e.g. a digital signal

In this case the best solution is to use the convolution

One problem remains, how do we work out h(t) for the RC circuit?

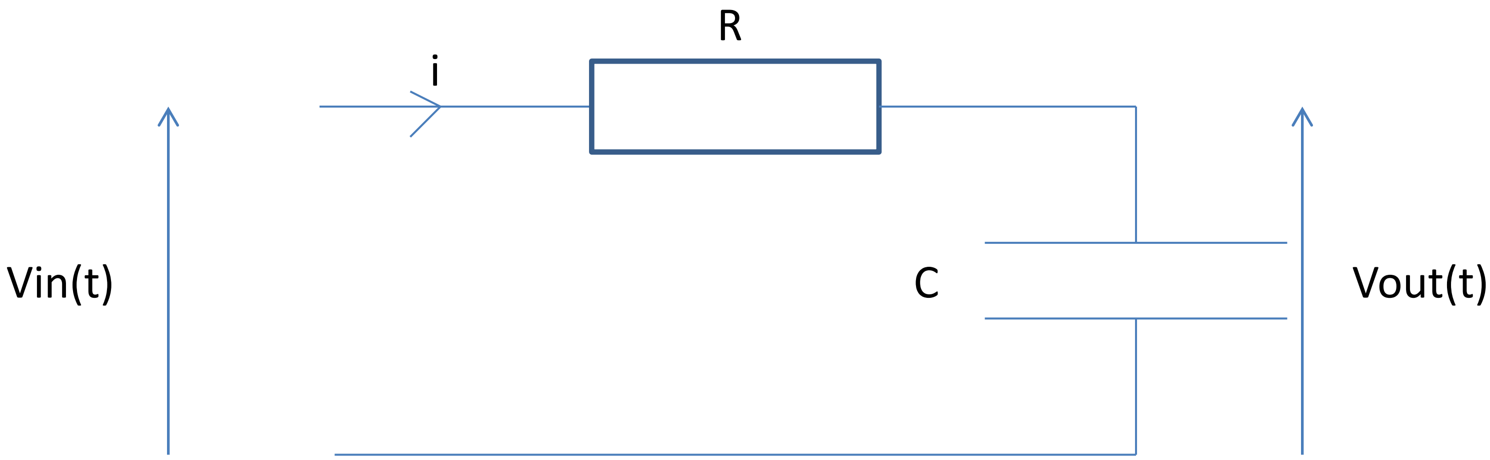
RC circuit

$$V_{in}(t) = i(t)R + V_{out}(t)$$

$$i(t) = C\frac{dV_{out}(t)}{dt}$$

$$V_{in}(t) = RC\frac{dV_{out}(t)}{dt} + V_{out}(t)$$

This is a first order differential equation.
Who remembers Laplace transforms?



RC circuit

The Laplace transform of a time derivative is given below

$$L\left(\frac{df(t)}{dt}\right) = sF(s) - f(0)$$

Where F(s) is the Laplace transform of f(t) and f(0) is the value of f(t) at t=0
Applying this to our circuit gives

$$V_{in}(t) = RC\frac{dV_{out}(t)}{dt} + V_{out}(t)$$

$$\text{Laplace transform all terms}$$

$$V_{in}(s) = sRCV_{out}(s) - RCV_{out}(0) + V_{out}(s)$$

$$V_{in}(s) = V_{out}(s)\left[1 + sRC\right] - RCV_{out}(0)$$

$$V_{out}(s) = \frac{V_{in}(s)}{\left[1 + sRC\right]} + \frac{RCV_{out}(0)}{\left[1 + sRC\right]}$$

RC circuit

To carry out our convolution we need to know the IMPULSE response

So in this case Vin(t)=δ(t)

The Laplace transform of an impulse response =1 so our response is

$$V_{out}(s) = \frac{1}{[1+sRC]} + \frac{RCV_{out}(0)}{[1+sRC]}$$

To obtain our impulse response, h(t), we need to inverse Laplace Transform Vout(s)

$$INV\_L\left[\frac{1}{s+a}\right] = e^{-at}u(t)$$

$$V_{out}(s) = \frac{1}{RC}\frac{1}{\left[\frac{1}{RC}+s\right]} + \frac{V_{out}(0)}{\frac{1}{RC}+s}$$

$$V_{out}(t) = \frac{1}{RC}e^{-t/RC}u(t) + V_{out}(0)e^{-t/RC}u(t)$$

RC circuit

Hence the impulse response of an RC circuit is given below

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) + V_{out}(0) e^{-t/RC} u(t)$$

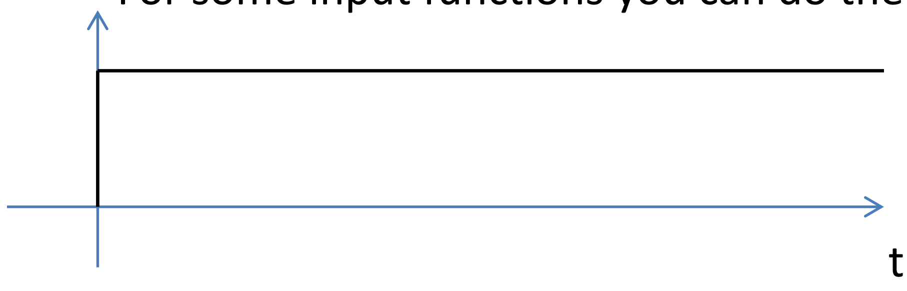
To compute the convolution with an input signal is usually done using software

2 video examples illustrate this for

single pulse
Periodic pulse train

RC circuit

For some input functions you can do the process analytically using Laplace



$$V_{in}(t) = V_{in}u(t)$$

$$V_{out}(s) = \frac{V_{in}(s)}{[1+sRC]} + \frac{RCV_{out}(0)}{[1+sRC]}$$

The Laplace transform of a step function =1/s

$$V_{out}(s) = \frac{V_{in}}{s[1+sRC]} + \frac{RCV_{out}(0)}{[1+sRC]}$$

$$INV\_L\left[\frac{a}{s(s+a)}\right] = (1-e^{-at})u(t)$$

$$V_{out}(t) = V_{in}(1-e^{-t/RC)})u(t) + V_{out}(0)e^{-t/RC}$$

$$V_{out}(t) = V_{in}(1-e^{-t/RC)})u(t)$$

Assume the capacitor is discharged at t=0

RC circuit using convolution

This is not a very pleasant exercise but we'll go through it none the less!

$$x(t) = u(t)$$

The input voltage is a unit step function

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t) + V_{out}(0) e^{-t/RC} u(t)$$

Assume the capacitor is discharged

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

The output voltage is given by the convolution

$$V_{out}(t) = x(t) * h(t)$$

$$V_{out}(t) = \int_{-\infty}^{t} x(\tau) h(t-\tau) d\tau$$

RC circuit using convolution

Step 1: Replace t with τ

$$x(\tau) = u(\tau)$$

$$h(\tau) = \frac{1}{RC} e^{-\tau/RC} u(\tau)$$

Step 2: Flip $h(\tau)$ to give $h(-\tau)$

$$h(-\tau) = \frac{1}{RC} e^{\tau/RC} u(-\tau)$$

Step 3: Shift $h(-\tau)$ to $h(t-\tau)$

$$h(t-\tau) = \frac{1}{RC} e^{(\tau-t)/RC} u(t-\tau)$$

RC circuit using convolution

Step 4: Carry out integral

$$V_{out}(t) = x(t) * h(t)$$

$$V_{out}(t) = \int_{-\infty}^{t} u(\tau) \frac{1}{RC} e^{(\tau - t)/RC} u(t-\tau) d\tau$$

$$V_{out}(t) = \frac{1}{RC} \int_{-\infty}^{t} u(\tau) u(t-\tau) e^{-t/RC} e^{\tau/RC} d\tau$$

Take out constants and split up exponential

$$V_{out}(t) = \frac{e^{-t/RC}}{RC} \int_{-\infty}^{t} u(\tau) u(t-\tau) e^{\tau/RC} d\tau$$

RC circuit using convolution

Integration limits change as $u(\tau)=1$ for $\tau>0$

$$V_{out}(t) = \frac{e^{-t/RC}}{RC}\int_0^t e^{\tau/RC}\,d\tau$$

$$V_{out}(t) = \frac{e^{-t/RC}}{RC} RC\left[e^{\tau/RC}\right]_0^t$$

$$V_{out}(t) = e^{-t/RC}\left[e^{t/RC} - e^{0}\right]$$

$$V_{out}(t) = e^{-t/RC}e^{t/RC} - e^{0}e^{-t/RC}$$

$$V_{out}(t) = 1 - e^{-t/RC}$$

We end up with the exponential charge equation we expect!!!

Convolution summary

- A useful way of analysing LTI systems
- For some input functions it is often simpler to use Laplace transforms rather than convolution
- Convolution has to be used in real applications as you can't determine the input prior to it occurring i.e. Speech

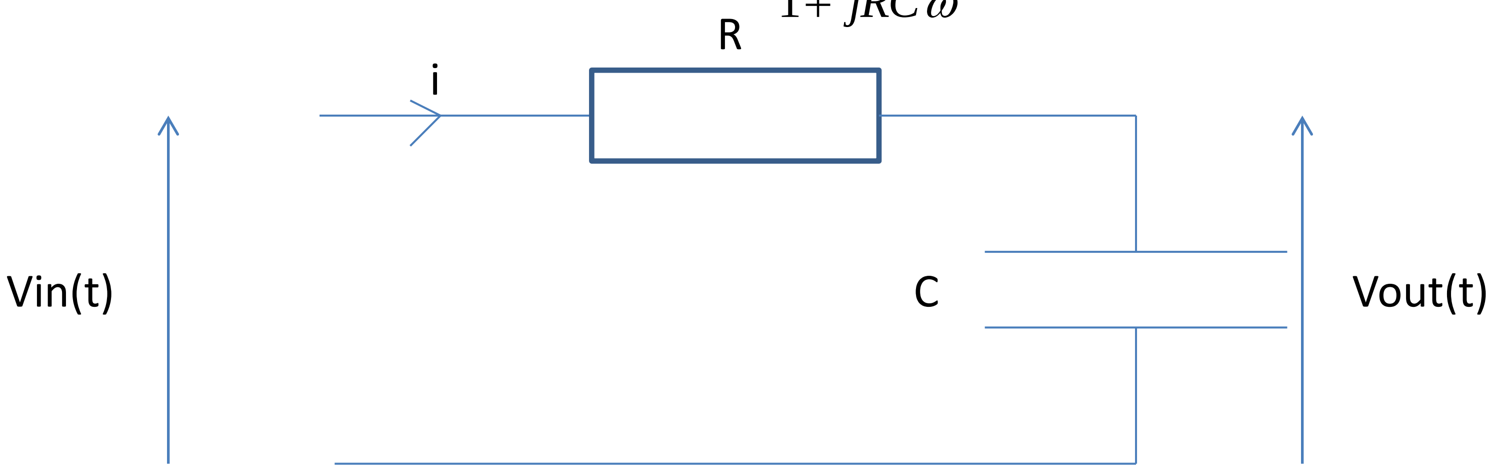
Analysis of periodic functions

What if we know that the input signal to our RC circuit is periodic
For instance a sine wave, what is the output?

$$V_{out}(\omega) = V_{in}(\omega)\frac{1/\,j\omega C}{R + 1/\,j\omega C}$$

$$V_{out}(\omega) = V_{in}(\omega)\frac{1}{1 + jRC\omega}$$

Low pass filter



Nice and easy first year example, but what if the “periodic” waveform isn’t a sine wave?

Fourier Series

Fourier analysis says that any periodic function can be described as an infinite number of sine and cosine waves.

$$f(t) = a_0 + \sum_{n=1}^{N}\left[a_n \cos\left(\frac{2n\pi t}{T}\right) + b_n \sin\left(\frac{2n\pi t}{T}\right)\right]$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt$$

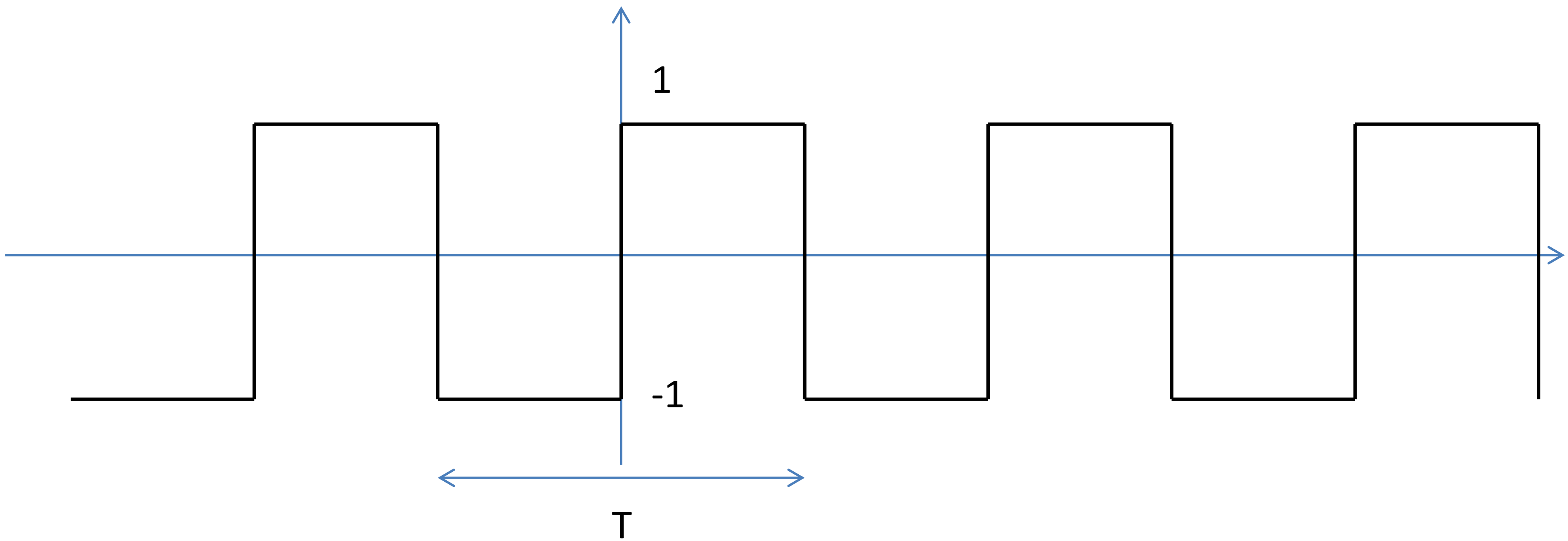
$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

a_0, a_n and b_n are known as Fourier coefficients

If we can decompose any periodic function into sine waves we can use standard Impedance models to calculate voltages and currents, rather than using convolution

Fourier series



Lets analyse a periodic square wave, which has an amplitude of -1 or 1 and repeats every T seconds.

Fourier series

a_0 is just the average value of the function

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} f(t)dt = \frac{1}{T}\int_{-T/2}^{0}(-1)dt + \frac{1}{T}\int_{-T/2}^{0} 1dt = 0$$

Fourier series

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$a_n = \frac{2}{T}\int_{-T/2}^{0} (-1)\cos\left(\frac{2\pi nt}{T}\right)dt + \frac{2}{T}\int_{0}^{T/2} 1\cos\left(\frac{2\pi nt}{T}\right)dt$$

$$a_n = \frac{2}{T}\frac{T}{2\pi n}\left[-\sin\left(\frac{2\pi nt}{T}\right)\right]_{-T/2}^{0} + \frac{2}{T}\frac{T}{2\pi n}\left[\sin\left(\frac{2\pi nt}{T}\right)\right]_{0}^{T/2}$$

$$a_n = \frac{1}{n\pi}\left[-\sin(0) - \left(-\sin\left(-\frac{2\pi nT}{2T}\right)\right)\right] + \frac{1}{n\pi}\left[\sin\left(\frac{2\pi nT}{2T}\right) - \sin(0)\right]$$

$$a_n = \frac{1}{n\pi}\left[\sin(-n\pi)\right] + \frac{1}{n\pi}\left[\sin(n\pi)\right]$$

$$a_n = 0$$

Fourier series

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{0} (-1)\sin\left(\frac{2\pi nt}{T}\right)dt + \frac{2}{T}\int_{0}^{T/2} 1\sin\left(\frac{2\pi nt}{T}\right)dt$$

$$b_n = \frac{2}{T}\frac{T}{2\pi n}\left[\cos\left(\frac{2\pi nt}{T}\right)\right]_{-T/2}^{0} + \frac{2}{T}\frac{T}{2\pi n}\left[-\cos\left(\frac{2\pi nt}{T}\right)\right]_{0}^{T/2}$$

$$b_n = \frac{1}{n\pi}\left[\cos(0) - \cos\left(-\frac{2\pi nT}{2T}\right)\right] + \frac{1}{n\pi}\left[-\cos\left(\frac{2\pi nT}{2T}\right) - (-\cos(0))\right]$$

$$b_n = \frac{1}{n\pi}\left[1 - \cos(n\pi) - \cos(n\pi) + 1\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]$$

Fourier series

$$b_n = \frac{2}{n\pi}\left[1 - \cos(n\pi)\right]$$

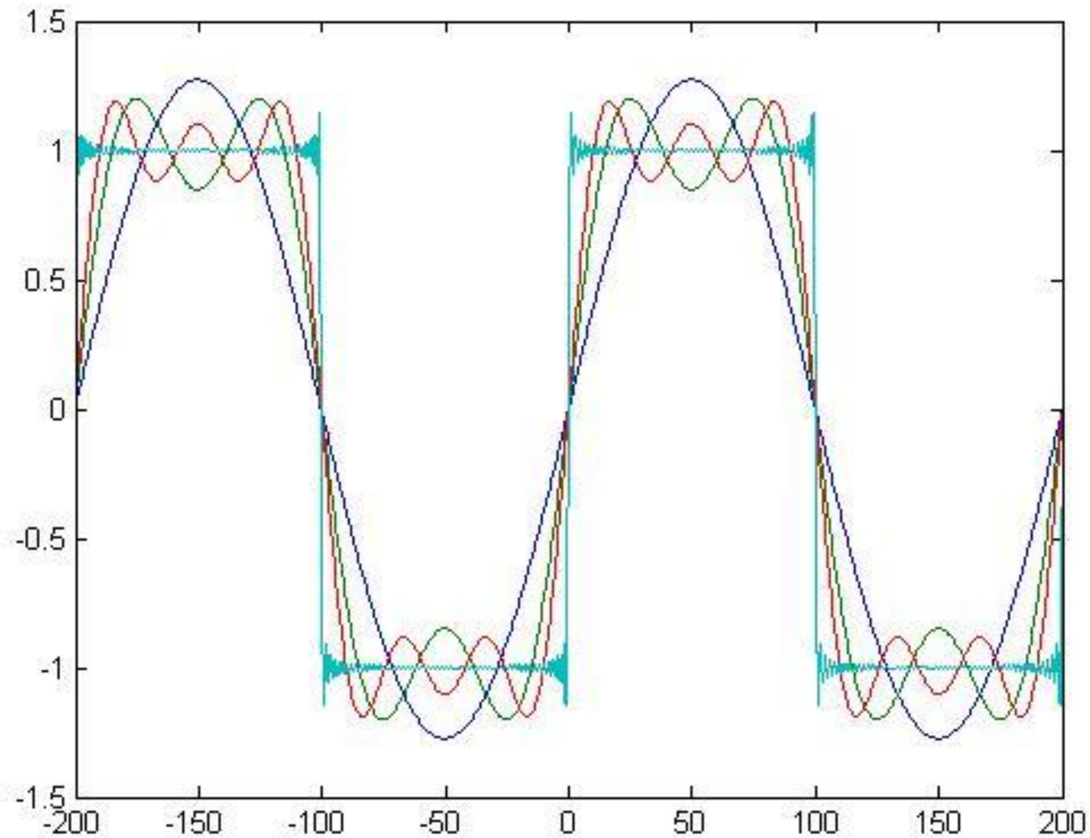
However, b_n can be simplified

$$b_n = \begin{cases} 0 & \text{For even values of n} \\ \dfrac{4}{n\pi} & \text{For odd values of n} \end{cases}$$

$$f(t) = \sum_{n=1,3,5\ldots}^{N} \left[\frac{4}{n\pi}\sin\left(\frac{2n\pi t}{T}\right)\right]$$

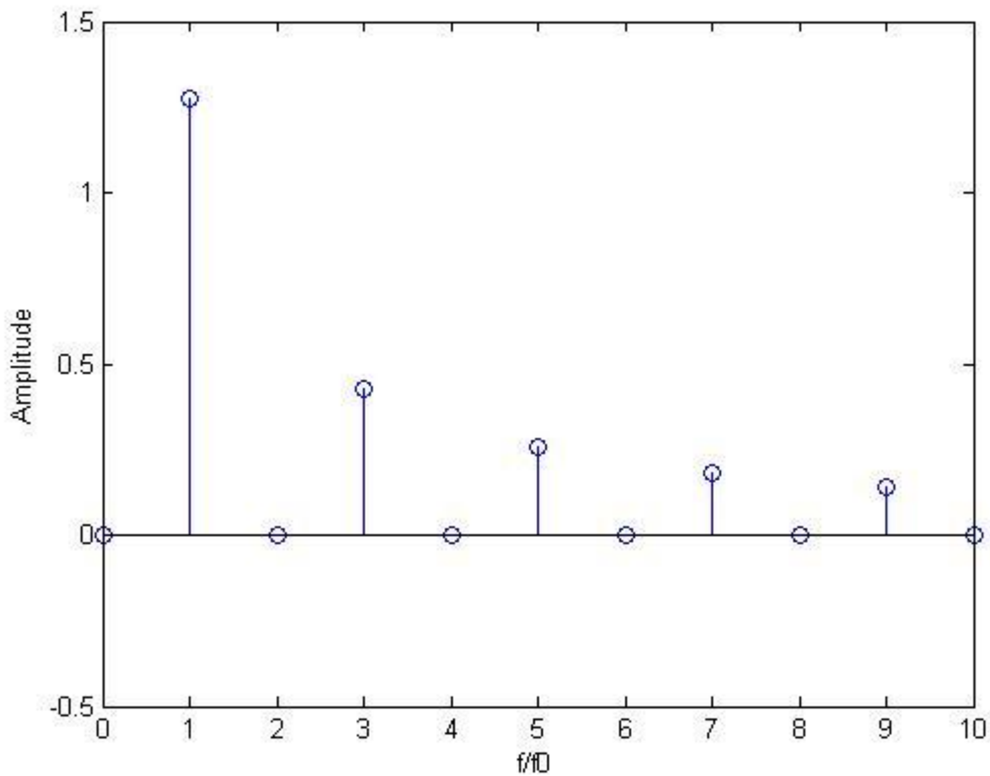
$$f(t) = \frac{4}{\pi}\sin\left(\frac{2\pi t}{T}\right) + \frac{4}{3\pi}\sin\left(\frac{6\pi t}{T}\right) + \frac{4}{5\pi}\sin\left(\frac{10\pi t}{T}\right) + \ldots\ldots.$$

Fourier series



Plot using the 1st, 3rd, 5th and 101st coefficient

Fourier Spectrum



This shows the frequency spectrum of the square wave signal

A square wave signal can occupy a large bandwidth

Back to our RC circuit

Now we know the input voltage in terms of sine waves we can use
Standard potential divider to solve

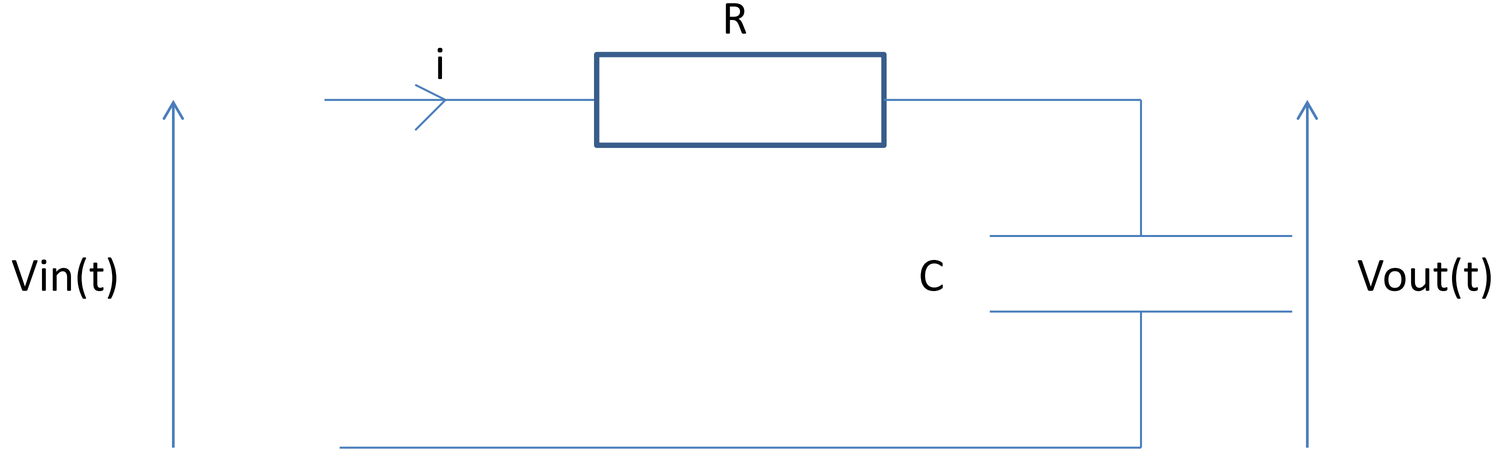
$$\omega = \frac{2\pi}{T} = 2\pi f$$

$$\omega_0 = \frac{1}{RC}$$

$$f_0 = \frac{1}{2\pi RC}$$

$$V_{out}(\omega) = V_{in}(\omega)\frac{1}{1 + jRC\omega} = V_{in}(\omega)\frac{1}{1 + j\dfrac{\omega}{\omega_0}} = V_{in}(f)\frac{1}{1 + j\dfrac{f}{f_0}}$$

$$V_{in}(t) = \frac{4}{\pi}\sin\left(\frac{2\pi t}{T}\right) + \frac{4}{3\pi}\sin\left(\frac{6\pi t}{T}\right) + \frac{4}{5\pi}\sin\left(\frac{10\pi t}{T}\right) + \ldots\ldots$$



Back to our RC circuit

As an example lets assume we use the first 3 terms of the Fourier series only

Also assume that
-f=2kHz
-f0=4kHz

The output for the first Fourier coefficient is

$$V_{out}(f) = V_{in}(f)\frac{1}{1+j\dfrac{f}{f_0}} = \frac{4}{\pi}\sin(2\pi ft)\frac{1}{1+j\dfrac{f}{f_0}}$$

$$V_{out}(f) = \frac{4}{\pi}\sin(2\pi ft)\left(0.894\angle -26.6^\circ\right) = 1.138\sin(2\pi ft - 0.46)$$

The output for the third Fourier coefficient is

$$V_{out}(f) = V_{in}(f)\frac{1}{1+j\dfrac{f}{f_0}} = \frac{4}{n\pi}\sin(2\pi nft)\frac{1}{1+j\dfrac{nf}{f_0}}$$

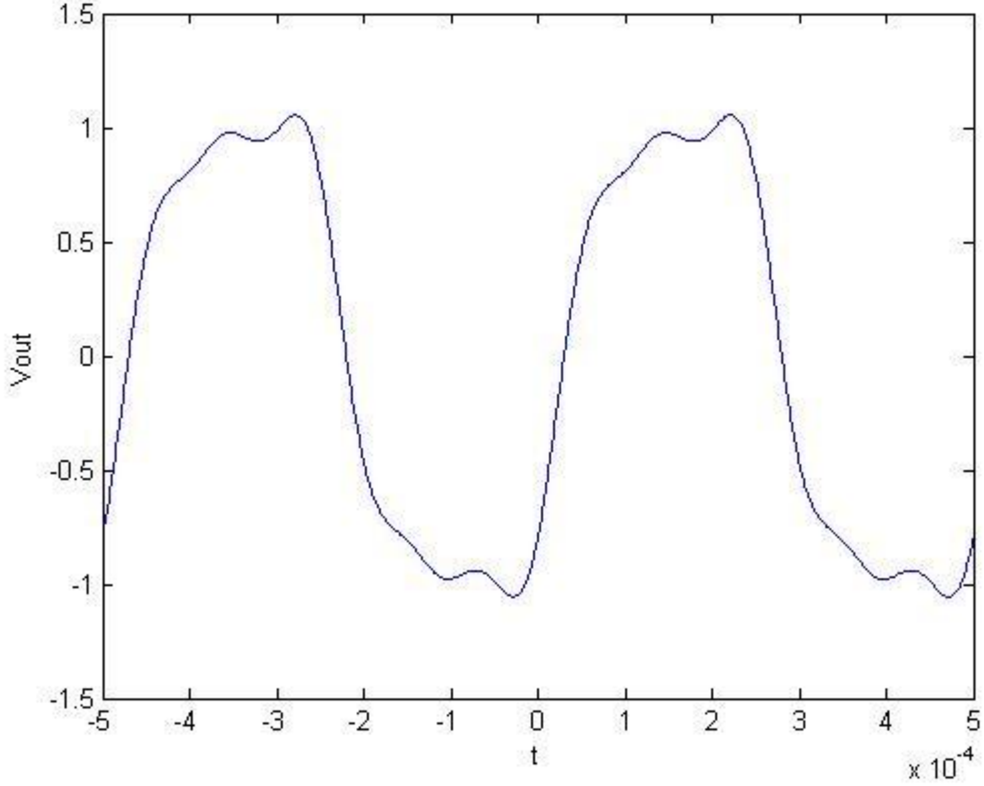
$$V_{out}(f) = \frac{4}{3\pi}\sin(6\pi ft)\left(0.555\angle -56.3^\circ\right) = 0.236\sin(6\pi ft - 0.98)$$

The output for the fifth Fourier coefficient is

$$V_{out}(f) = 0.094\sin(10\pi ft - 1.186)$$

Back to our RC circuit

$$V_{out}(t) = 1.138\sin(2\pi ft - 0.46) + 0.236\sin(6\pi ft - 0.98) + 0.094\sin(10\pi ft - 1.186) + \ldots..$$



Notice the exponential charge and discharge

Fourier series summary

- Very powerful for signal analysis
- Simplifies circuit analysis
- Stops the use of convolution
- Important for spectrum analysis

Fourier transform

The Fourier series can be extended to non-periodic functions, which is similar to the Laplace transform and is given below

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

The Fourier transform converts time domain signals to the frequency domain.
Useful for spectrum analysis

The inverse Fourier transform is given by

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Important Fourier transform properties for communications

The Fourier transform is linear $F(ax_1(t)+bx_2(t)) = aF(x_1(t))+bF(x_1(t))$

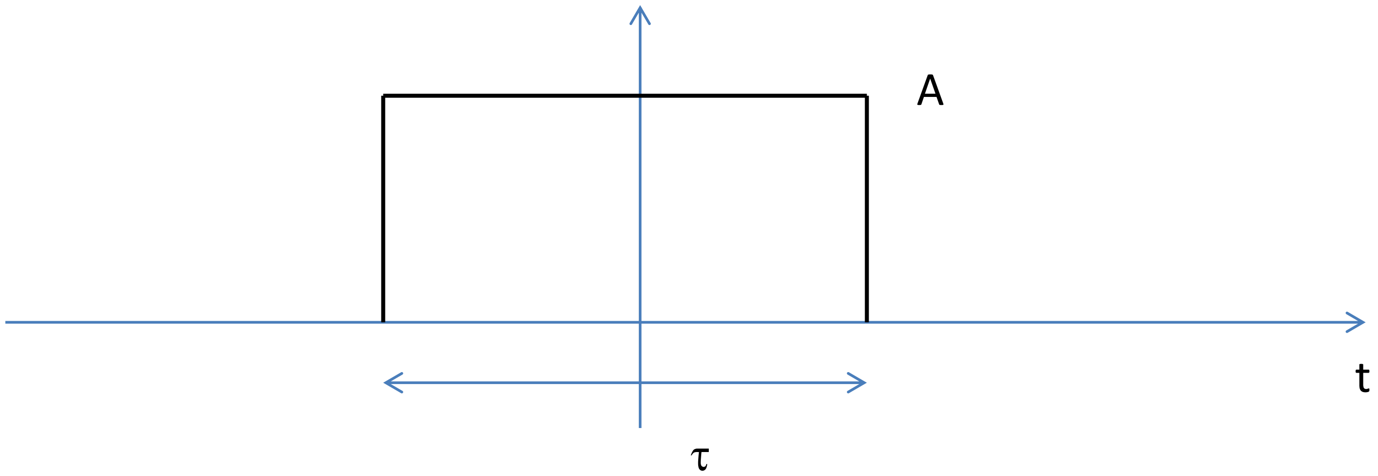
Convolution – It can be shown that the convolution in the time domain is a multiplication in the frequency domain

$$x(t)*h(t) \to X(\omega)H(\omega)$$

$$\frac{1}{2\pi}\left[X(\omega)*H(\omega)\right] \to x(t)h(t)$$

This is important as multiplication is much easier than convolution

Fourier Transform example



What is the Fourier transform of a rect function of amplitude A and width τ?

Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\tau/2}^{\tau/2} A e^{-j\omega t} dt = -\frac{A}{j\omega}\left[e^{-j\omega t}\right]_{-\tau/2}^{\tau/2}$$

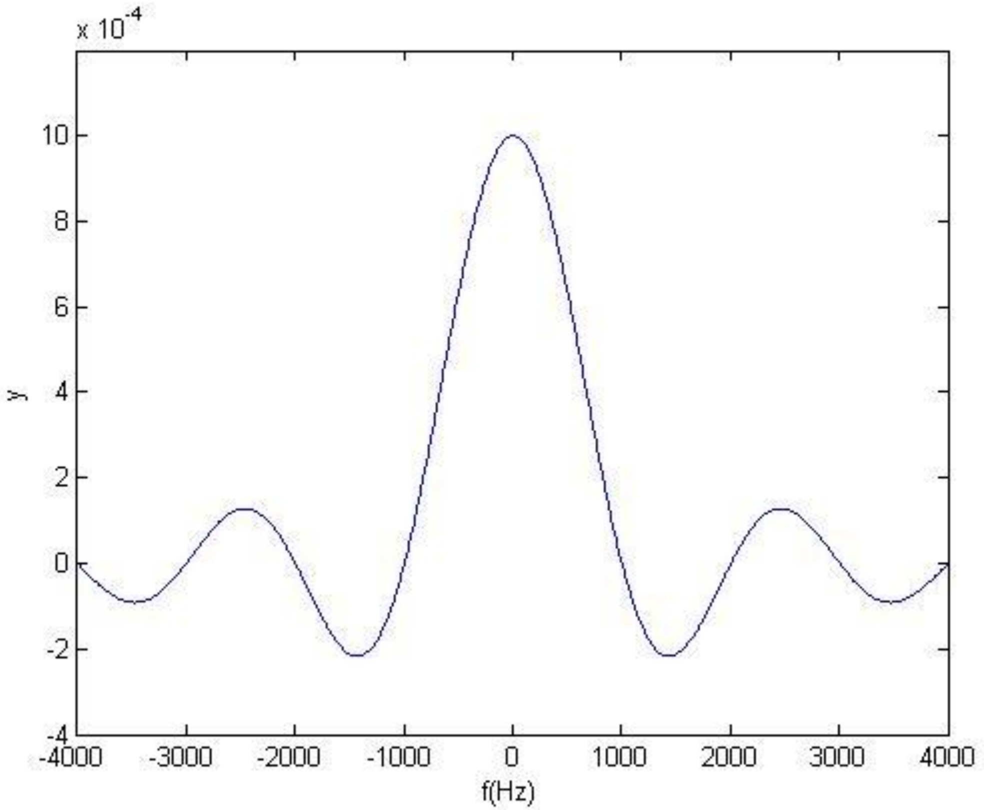
$$F(\omega) = -\frac{A}{j\omega}\left[e^{-j\omega\tau/2} - e^{j\omega\tau/2}\right] = \frac{2A}{\omega}\left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j}\right] = \frac{2A}{\omega}\sin\left[\frac{\omega\tau}{2}\right]$$

$$F(\omega) = A\tau\frac{2}{\omega\tau}\sin\left[\frac{\omega\tau}{2}\right] = A\tau\frac{\sin\left[\frac{\omega\tau}{2}\right]}{\frac{\omega\tau}{2}} = A\tau\,\mathrm{sinc}\left[\frac{\omega\tau}{2}\right]$$

Fourier Transform

Below is an example of the Fourier Transform of a rect function

In this example τ=1ms and A=1



As you can see the amplitude is τ

Important to note the frequency where the function =0 at f=1/ τ.

This zero point is often used to determine bandwidth

At the start of the module I showed you this for the perfect comms engineer

- Ability to
  - **manipulate continuous and discrete signals**
  - choose the best method of transmission
  - understand how the channel affects the performance of the system
  - Choose the best method of reception of data
  - Design the circuitry needed to carry out the above tasks in a meaningful manner

Summary of signals and systems

- You now have the basic tools for
  - Manipulating continuous and discrete functions
  - Understanding the properties of a system
  - Calculating the outputs of a system using convolution and Fourier analysis