Number Systems and Codes

- Decimal Numbers
- Binary, Hexadecimal
- Number Base Conversions
- Binary Codes

Decimal Numbers

The decimal number system has ten digits (0,1,2,3,4,5,6,7,8,9).

The position of each digit is assigned a weight, which is a power of ten. Ten is the base or **RADIX** of the number system. The weights increase from right to left by a power of the radix for each digit.

$$263 = (2 \times 10^{2}) + (6 \times 10^{1}) + (3 \times 10^{0})$$

$$= (2 \times 100) + (6 \times 10) + (3 \times 1) = 200 + 60 + 3 = 263$$

For fractional numbers, the powers are negative powers of ten.

$$75.42 = (7 \times 10^{1}) + (5 \times 10^{0}) + (4 \times 10^{-1}) + 2 \times 10^{-2})$$

$$= (7 \times 10) + (5 \times 1) + (4 \times 0.1) + (2 \times .01)$$

$$= 70 + 5 + 0.4 + .02 = 75.42$$

The same principle is true for a number in any base.

Binary Numbers

The binary system has two digits: **0** or **1**. These are called 'bits'.

The position of the bit in a binary number indicates its weight which is a power of two. A binary number can be converted to decimal by adding the weights of all bits that are **1**.

$$1101_2 = (1 \times 2^3) + (1 \times 2^2) + (0 \times 2^1) + (1 \times 2^0)$$

$$= (1 \times 8) + (1 \times 4) + (0 \times 2) + (1 \times 1) = 8 + 4 + 1 = 13_{10}$$

The left-most bit is known as the most significant bit or MSB.
The right-most bit is known as the least significant bit or LSB.

For fractional numbers, the powers are negative powers of two.

$$10.01_2 = (1 \times 2^1) + (0 \times 2^0) + (0 \times 2^{-1}) + (1 \times 2^{-2})$$

$$= (1 \times 2) + (0 \times 1) + (0 \times .5) + (1 \times .25) = 2.25_{10}$$

Number Base Conversions

The conversion of a number in base **r** to decimal can be achieved by expanding the number in a power series and adding all of the terms.

To convert from decimal to base **r**, it is necessary to first consider the whole part and then the fractional part.

Repeated division by the base will give the whole part. Division continues until the quotient is 0.

The remainders give the coefficients of the number in the new base.

Consider converting decimal 35 to binary.

2 | 35		
2 | 17	$1 = b_0$	
2 | 8	$1 = b_1$	
2 | 4	$0 = b_2$	
2 | 2	$0 = b_3$	$35_{10} = 100011_2$
2 | 1	$0 = b_4$	
0	$1 = b_5$	

The fractional part can be obtained by repeated multiplication by the base. The coefficients are obtained from the integer part of the product.

Consider converting decimal .53 to binary.

$(0.53)(2) = 1.06 \quad 1 = b_{-1}$

$(0.06)(2) = 0.12 \quad 0 = b_{-2}$

$(0.12)(2) = 0.24 \quad 0 = b_{-3}$

$(0.24)(2) = 0.48 \quad 0 = b_{-4}$

$(0.48)(2) = 0.96 \quad 0 = b_{-5}$

$(0.96)(2) = 1.92 \quad 1 = b_{-6}$

$\vdots$

$.53_{10} = 0.100001_2 \ldots\ldots$

Notice, 0.53_{10} does not have an exact binary representation.

Example: Convert 29.75 to binary

2	29	
2	14	$1 = b_0$
2	7	$0 = b_1$
2	3	$1 = b_2$
2	1	$1 = b_3$
	0	$1 = b_4$

$(0.75)(2) = 1.5 \quad 1 = b_{-1}$

$(0.5)(2) \ = 1.0 \quad 1 = b_{-2}$

$$29.75_{10} \quad = 11101.11_2$$

Hexadecimal (radix 16)

Hex : there are 16 digits (0,1,2,3,4,5,6,7,8,9,A,B,C,D,E,F)

Each digit corresponds to four binary digits as $2^4 = 16$.

To convert from binary to hex, group the binary number in fours from the LSB and assign the corresponding hex digit to each group.

$$(110\ \ 1010\ \ 0101\ \ 1111)_2 = (6A5F)_{16}$$

6 A 5 F

Count Sequences

decimal	binary	hex
0	0 0 0 0 0	0
1	0 0 0 0 1	1
2	0 0 0 1 0	2
3	0 0 0 1 1	3
4	0 0 1 0 0	4
5	0 0 1 0 1	5
6	0 0 1 1 0	6
7	0 0 1 1 1	7
8	0 1 0 0 0	8
9	0 1 0 0 1	9
10	0 1 0 1 0	A
11	0 1 0 1 1	B
12	0 1 1 0 0	C
13	0 1 1 0 1	D
14	0 1 1 1 0	E
15	0 1 1 1 1	F
16	1 0 0 0 0	10
17	1 0 0 0 1	11
18	1 0 0 1 0	12

Conversion between decimal and hex

In the case of hex to decimal, each hex digit is multiplied by its weight and added.

$A2C_{16} = (10 \times 16^2) + (2 \times 16^1) + (12 \times 16^0)$

$= (10 \times 256) + (2 \times 16) + (12 \times 1) = 2560 + 32 + 12 = 2604_{10}$

Repeated division by 16 will give the hex equivalent of a decimal number. Consider the decimal number 37420.

16	37420	
16	2338	$C = h_0$
16	146	$2 = h_1$
16	9	$2 = h_2$
	0	$9 = h_3$

$37420_{10} = 922C_{16}$

Binary Codes

The representation of numbers in binary is called straight binary coding. Other codes have been developed for different applications.

Binary Coded Decimal (BCD) : each decimal digit is encoded with its binary equivalent. Consider the decimal number 916.

binary code for 9 is 1001

binary code for 1 is 0001

binary code for 6 is 0110

$916_{10} = 100100010110_{BCD}$

With four bits, it is possible to represent $2^4 = 16$ codes. As only ten of these are used, there are six invalid codes.

This code is often used for tasks that involve a lot of input/output.

Gray Code

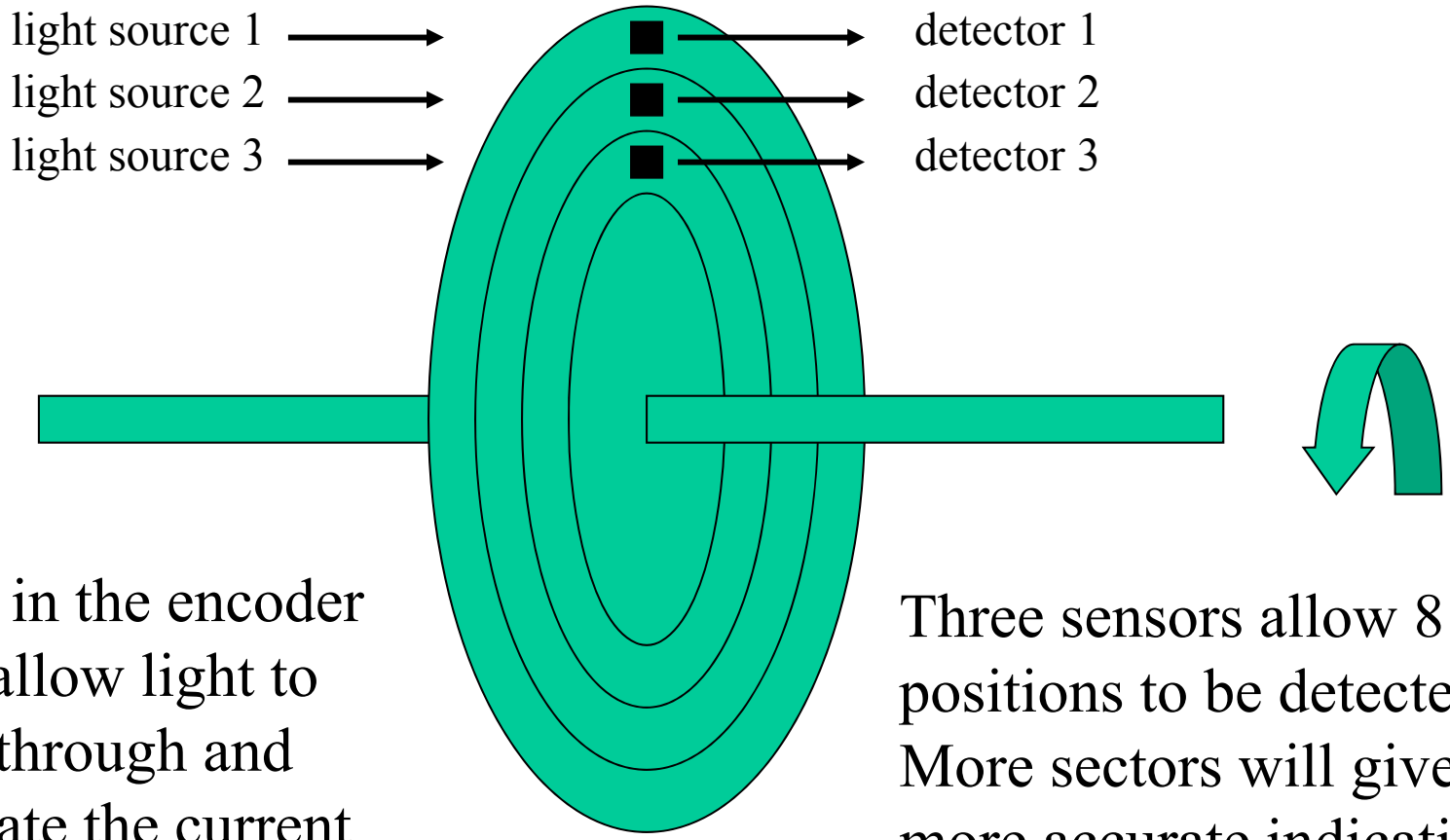
Gray code is an unweighted code in which only one bit position changes value between any two consecutive Gray codes.

Binary	Gray Code
0 0 0	0 0 0
0 0 1	0 0 1
0 1 0	0 1 1
0 1 1	0 1 0
1 0 0	1 1 0
1 0 1	1 1 1
1 1 0	1 0 1
1 1 1	1 0 0

This property is important in many applications where errors can occur if several bits change at the same time, causing an intermediate state.

It also helps to reduce crosstalk in physically adjacent wires as only changing one bit at a time reduces simultaneous electrical switching.

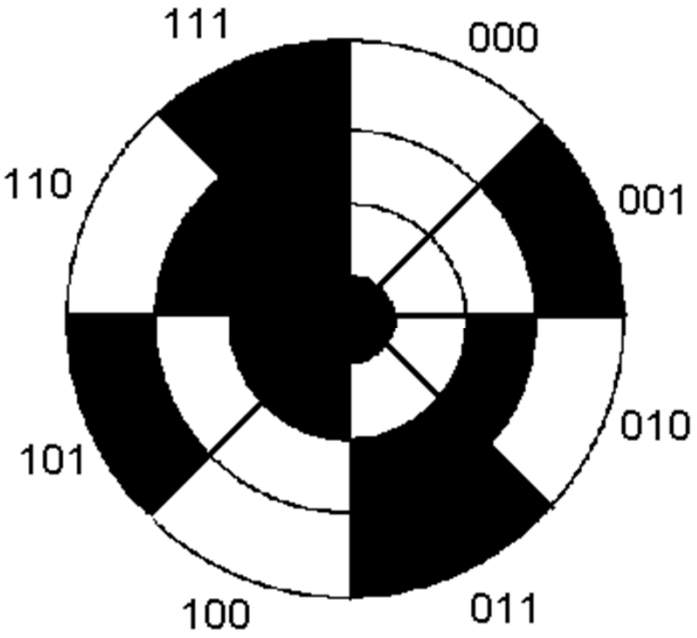
Rotary Position Encoder (Shaft Encoder)



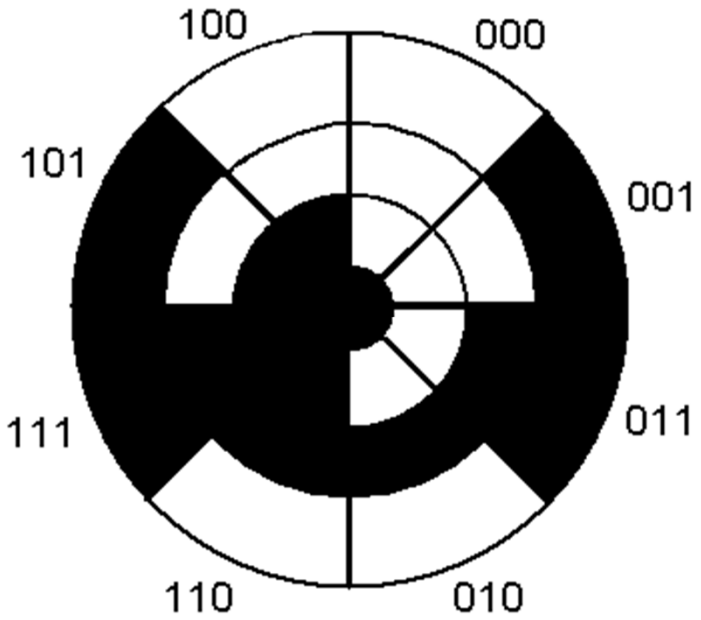
Slots in the encoder disc allow light to pass through and indicate the current position of the shaft.

Three sensors allow 8 positions to be detected. More sectors will give a more accurate indication of the position.

Binary

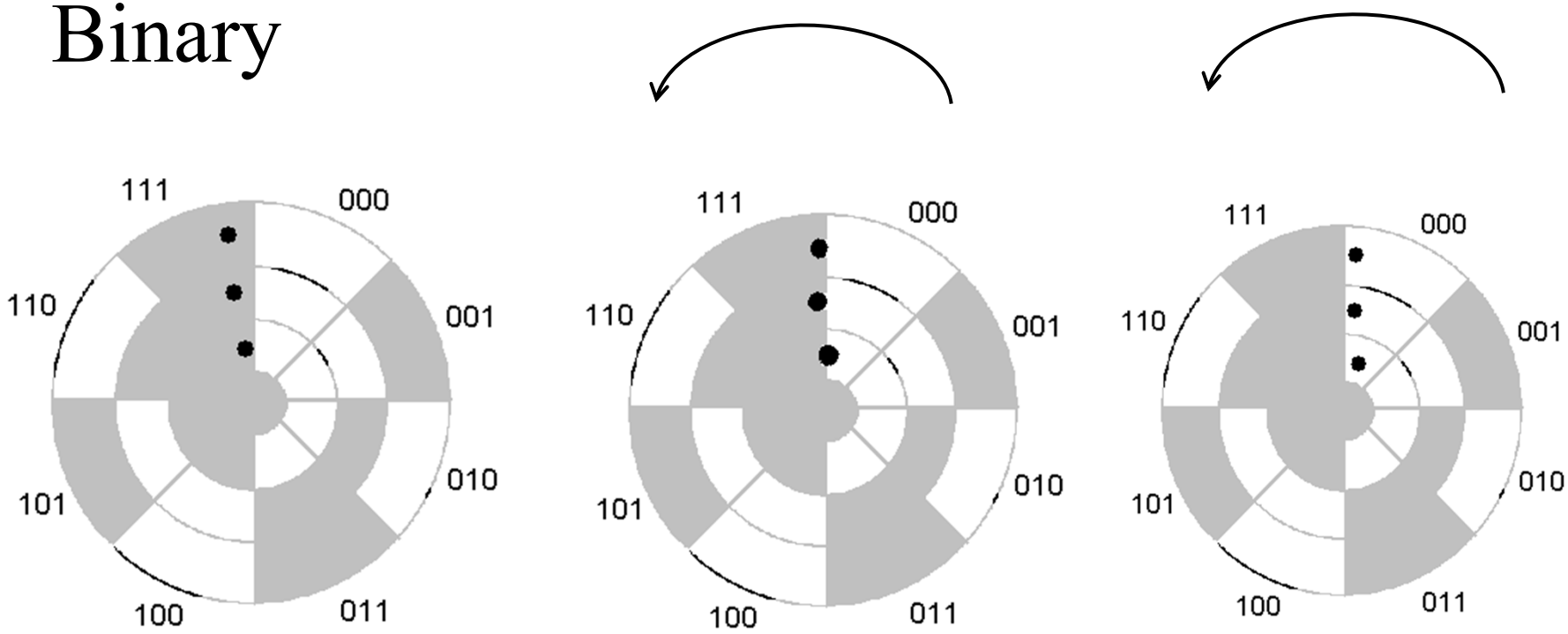


Gray code



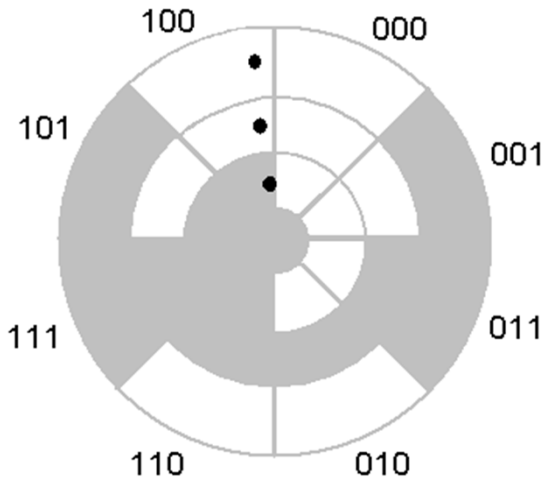
Eight positions of the shaft can be detected by three sensors, with the MSB at the centre, black indicating ‘1’ white indicating ‘0’.

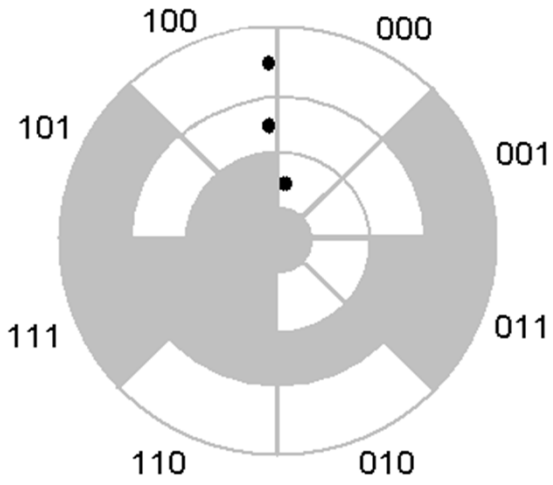
Binary

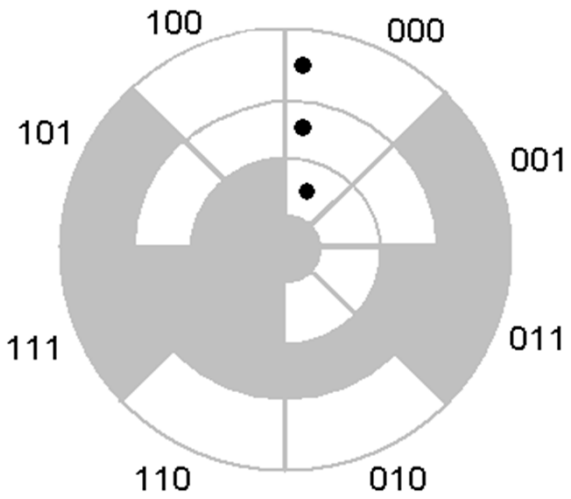


If the MSB sensor is slightly ahead, when moving from 111 to 000 the incorrect position 011 may recorded. Gray code eliminates this problem as only one bit changes between consecutive values.

Gray code





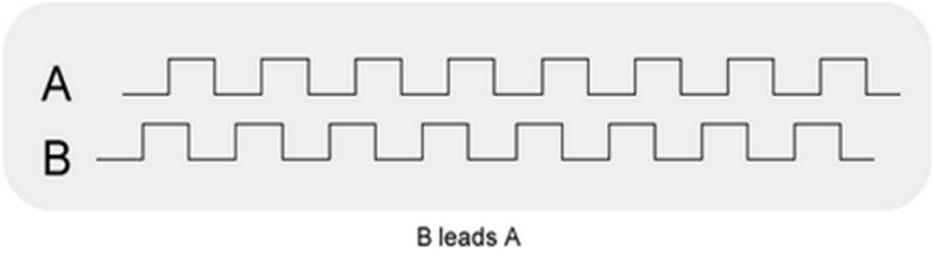


100 000 000

Absolute and Incremental Encoders

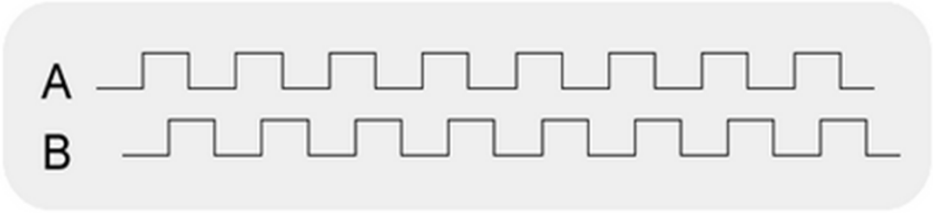
The slotted disc is an example of an absolute encoder which gives the current angular position of the rotor.

An incremental encoder records changes in position by providing two square waves (A,B) in quadrature (90 degrees out of phase). The pulses are decoded to give a count up or a count down.



B leads A

Clockwise coding (A,B):
00 > 01 > 11 > 10



A leads B

Anti-clockwise coding (A,B):
10 > 11 > 01 >00

Conversion between binary and Gray code

$$g_k = b_k \oplus b_{k+1}$$

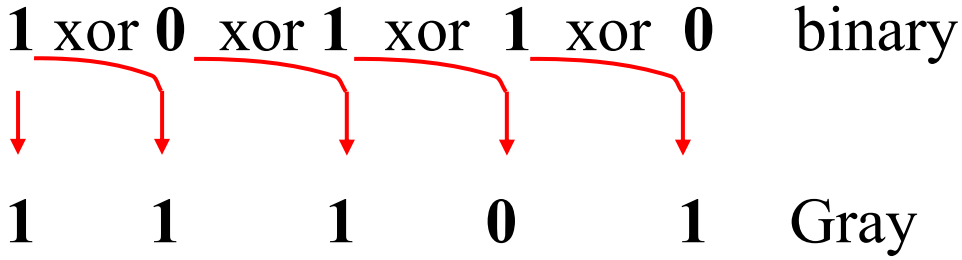
where g is the Gray code bit, b is the binary bit and k is the bit position.

Binary-to-Gray:

1. Write down the MSB. (The MSB is the same in binary and Gray).

2. Going from left to right, XOR adjacent pairs of the straight binary to get the next Gray code bit.

Convert 10110 to Gray code.



Gray-to-Binary:

$$g_k = b_k \oplus b_{k+1}$$

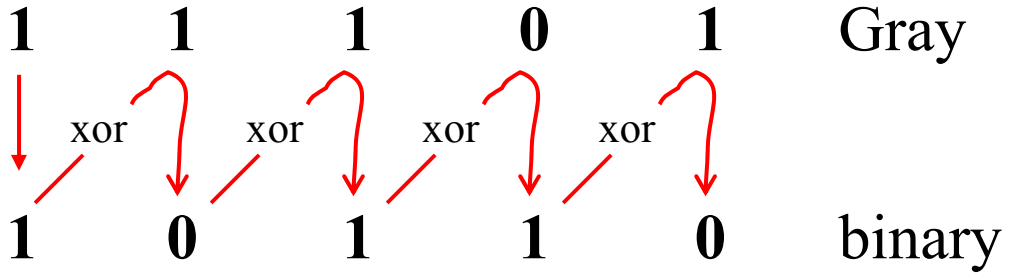
$$g_k \oplus b_{k+1} = b_k \oplus b_{k+1} \oplus b_{k+1}$$

$$g_k \oplus b_{k+1} = b_k \qquad (\text{as } x \oplus x = 0,\ 0 \oplus x = x)$$

1. Write down the MSB. (The MSB is the same in binary and Gray).

2. Going from left to right, XOR each binary digit generated with the Gray code bit in the next adjacent position.

$$b_k = g_k \oplus b_{k+1}$$



Alphanumeric Codes

Characters can be represented by binary codes. The ASCII code uses seven bits to represent the alphabet in upper and lower case, together with numbers, punctuation symbols and control characters.

ASCII (American Standard Code for Information Interchange)

A = 1000001, 6 = 0110110

Symbol	d	a	t	e	CR
Binary	1100100	1100001	1110100	1100101	0001101
Hex	64	61	74	65	0D

The extended ASCII character set uses 8 bits to provide more symbols.

Summary

- Humans use the decimal system but digital systems use a binary system.
- It often more convenient to express these binary numbers using the hexadecimal system.
- Codes exist to represent different types of data.
- The ASCII code represents alphanumeric characters