Data Provided: $\varepsilon_0 = 8.854\times 10^{-12}$ F/m, $\mu_0 = 4\pi\times10^{-7}$ H/m

Formulae for Vector Differential Operations

The University Of Sheffield.

DEPARTMENT OF ELECTRONIC AND ELECTRICAL ENGINEERING

Spring Semester 2009-2010 (2 hours)

Applied Electromagnetics 6

Answer **THREE** questions. **No marks will be awarded for solutions to a fourth question.** Solutions will be considered in the order that they are presented in the answer book. Trial answers will be ignored if they are clearly crossed out. **The numbers given after each section of a question indicate the relative weighting of that section.**

1. Figure 1 shows a model of a short section Δx of a transmission line.

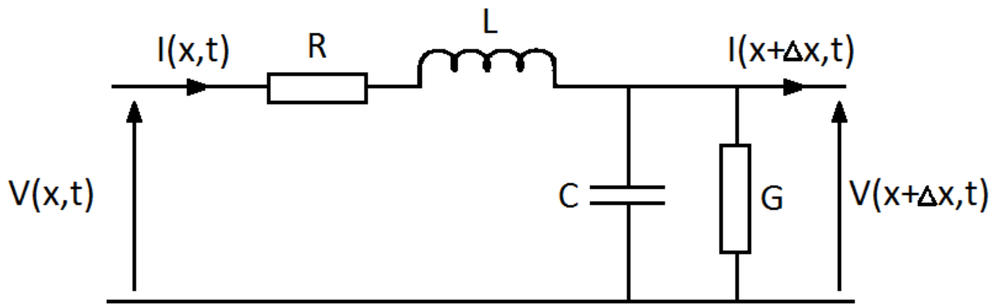

Figure 1 Model of a short section Δx of a transmission line

a. What do R, L, C and G represent and what are their units? **(3)**

b. The line is driven by a purely sinusoidal source. The voltage as a function of position along the line (x) and time (t) can be written as:-

$$V(x,t) = V_0^+ e^{j(\omega t - \beta x)} + V_0^- e^{j(\omega t + \beta x)} \qquad \text{equation 1}$$

Briefly explain what is physically represented by the two terms on the right-hand side of equation 1. If $V_0^- = 0$, what can you say about the load to which the transmission line is attached? **(3)**

c. Assume $V_0^- = 0$. By writing the parameter β as a-jb (b>0), show that the imaginary part leads to attenuation of the signal as it travels down the line.

If b = $2m^{-1}$, at what distance from the source will the amplitude of the signal fall to 10% of its original level? **(4)**

d. For a lossy transmission line, it can be shown that β can be expressed as:-

$$-\beta^2 = \gamma^2 = (G + j\omega C)(R + j\omega L), \qquad \text{equation 2}$$

where ω is the angular frequency of the source and $\gamma = j\beta$ is the complex propagation constant.

If $G \ll \omega C$ and $R \ll \omega L$, show that this expression can be approximated by:-

$$\beta \approx \omega\sqrt{LC}\left(1 - \frac{j}{2}\left(\frac{G}{\omega C} + \frac{R}{\omega L}\right)\right), \qquad \text{equation 3}$$

(3)

e. In general, the velocity of a signal on a transmission line varies with its frequency. Explain why this dependence of velocity on frequency can cause a square-wave signal fed into a transmission line to become distorted.

The first transatlantic telegraph cable suffered from distortion by this mechanism to such an extent that morse-code operatives were restricted to keying only one letter every 15 seconds. Oliver Heaviside realised that this distortion could be countered, and thus higher signalling rates achieved, provided the following Heaviside condition was met by the transmission line:-

$$\frac{G}{C} = \frac{R}{L} \qquad \text{equation 4}$$

Show that the Heaviside condition implies that equation 2 can be written as:-

$$\gamma^2 = (A + j\omega B)^2 \qquad \text{equation 5}$$

with *A* and *B* constants, and explain why this will eliminate frequency-dependent distortion on the transmission line. **(7)**

2. Figure 2 represents a cross-section through a coaxial cable. At a given instant in time, a current I flows into the paper, through the inner core of diameter $2a$, and returns out from the paper via the outer sheath, whose diameter is $2b$. The dielectric between the core and sheath has relative permeability $\mu_r=1$ and relative permittivity ε_r.

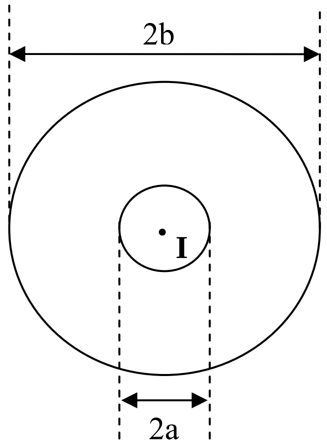

Figure 2 Cross-section through a coaxial cable

a. On a copy of this sketch indicate lines of constant electric and magnetic field. Show the direction of each field and the relative density of the field lines. **(3)**

b. **i)** Assuming that the current flows only on the surface of the inner core, show that the flux-density $B(r)$ a distance r (with $a<r<b$) from the centre of the cable is given by:

$$B(r) = \frac{\mu_0 I}{2\pi r} \qquad \text{equation 1}$$

ii) Using this expression, show that the inductance of the cable per unit length is:

$$L = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \qquad \text{equation 2}$$

iii) If the inner core diameter is 2a=1mm and the diameter of the outer sheath is 2b=3mm, what must the capacitance per unit length of the cable be to achieve a characteristic impedance of 70Ω (assuming the cable is lossless)? **(9)**

c. The coaxial cable from parts a and b (lossless and with characteristic impedance 70Ω) is terminated by a purely resistive 100Ω load. At its opposite end the line is driven by a purely sinusoidal source of frequency f=50MHz and amplitude 5V. What is:

i) The voltage reflection coefficient **(3)**

ii) The voltage standing wave ratio

iii) If the phase velocity on the line is 2.5×10^8 m/s, what is the wavelength λ of the voltage waves on the line?

d. The impedance mismatch of the cable and the load from part b will result in waves travelling both forwards and backwards along the line. These waves will interfere to produce a standing wave. Sketch the envelope of this voltage standing wave along a length of the line from the load to a distance λ towards the source. Label the *x*-axis in terms of λ and indicate the values of any maxima and minima on the *y*-axis. Remember that the cable is lossless. **(5)**

3.

3. a. The electric flux density $\boldsymbol{D}$ in a region of free space ($\varepsilon_0 = 8.85\times10^{-12}$ F/m) is given by:

$$(2x^3\,\boldsymbol{e_x} + 3y^2\,\boldsymbol{e_y} + 4z\,\boldsymbol{e_z}) \times 10^{-9}\ (\text{C/m}^2)$$

(i) Calculate the electric field strength and charge density at the point $(x, y, z) = (1, 1, 0)$m.

(ii) Show that the electrical potential $V(x, y, z)$ referred to the origin (0,0,0) is given by:

$$V(x, y, z) = (0.5x^4 + y^3 + 2z^2)\times 10^{-9} / \varepsilon_0$$

(iii) Determine the potential difference between the two points given by (1,0,0) and (1,1,0). **(6)**

b. Using Gauss's law

$$\oiint_S \boldsymbol{D}\cdot \boldsymbol{dS} = Q$$

show that the capacitance of a co-axial cable with inner radius a, outer radius b and length l is given by:

$$C = \frac{2\pi l\varepsilon}{\ln(b/a)}$$

where ε is the permittivity of the dielectric material of the cable. **(4)**

c. A parallel plate capacitor has a total area of 400 mm^2. The space between the two plates is filled by two homogeneous perfect dielectric materials of equal volume with a thickness of 50μm and permittivity of ε_1 and ε_2. The interface boundary between the two dielectric regions is perpendicular to the plates, as shown in Figure 3. Assume that the ratio of the capacitor area to plate separation is sufficiently large that the plate can be approximated as being of infinite extent.

(i) Specify the interface condition between the two regions.
(ii) Starting from Laplace's equation, derive an expression for electric field strength in the region between the two plates, and find the voltage across the capacitor if the energy stored in it is 0.177 μJ, and $\varepsilon_1 = 10\varepsilon_0$, $\varepsilon_2 = 15\varepsilon_0$.
(iii) Determine the surface charge density on the positive plate associated with regions 1 and 2, respectively.

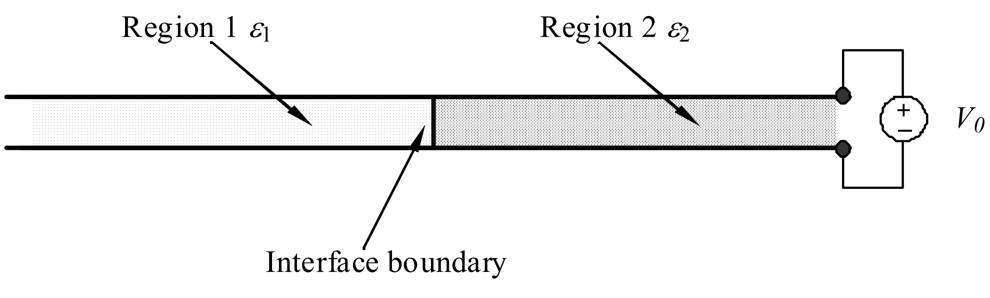

Figure 3 Schematic of parallel plate capacitor with two dielectric materials **(10)**

4. **a.** Explain why magnetic field problems are often formulated in terms of either a scalar or a vector magnetic potential function, and show how the field vectors $\boldsymbol{B}$ and $\boldsymbol{H}$ are derived from such functions. **(6)**

b. Starting from the field vectors $\boldsymbol{B}$ and $\boldsymbol{H}$, show that the scalar magnetic potential φ satisfies Laplace's equation:

$$\nabla^2 \varphi = 0$$

if the permeability in the regions of concern is constant. **(4)**

c. Figure 4 shows an idealised representation of a 2-pole synchronous motor in which the stator iron is assumed to be infinitely permeable and the rotor has a permeability $\mu_0\mu_r$. The inner bore of the stator carries a surface current distribution given by $J\cos\theta$ (A/m).

(i) Determine the boundary condition at $r = R_2$ and the interface condition at $r = R_1$.

(ii) Derive, by the method of separation of variables, an expression for the scalar magnetic potential distribution in the air-gap which satisfies Laplace equation:

$$\nabla^2 \varphi = 0$$

(10)

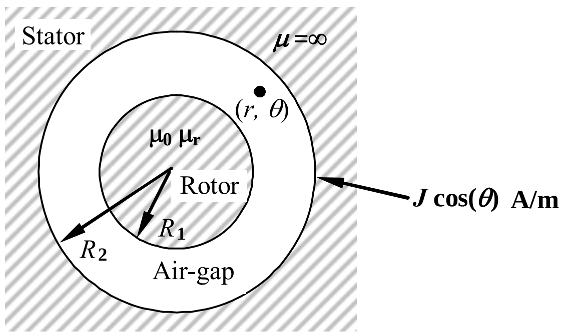

Figure 4 Magnetic field model of 2-pole synchronous motor

JBW / SL

Vector differential operations

Let Φ be a scalar function and $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{A}$ be vector functions.

Cartesian Co-ordinates (x, y, z)

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}\boldsymbol{e}_x + \frac{\partial\Phi}{\partial y}\boldsymbol{e}_y + \frac{\partial\Phi}{\partial z}\boldsymbol{e}_z$$

$$\nabla \bullet \boldsymbol{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \boldsymbol{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\boldsymbol{e}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\boldsymbol{e}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\boldsymbol{e}_z$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\boldsymbol{A} = \nabla^2 A_x\boldsymbol{e}_x + \nabla^2 A_y\boldsymbol{e}_y + \nabla^2 A_z\boldsymbol{e}_z$$

Cylindrical Co-ordinates (r, θ, z)

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial\Phi}{\partial z}\boldsymbol{e}_z$$

$$\nabla \bullet \boldsymbol{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial\theta} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \boldsymbol{H} = \left[\frac{1}{r}\frac{\partial H_z}{\partial\theta} - \frac{\partial H_\theta}{\partial z}\right]\boldsymbol{e}_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]\boldsymbol{e}_\theta + \left[\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial\theta}\right]\boldsymbol{e}_z$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\boldsymbol{A} = \left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{A_r}{r^2}\right)\boldsymbol{e}_r + \left(\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2}\right)\boldsymbol{e}_\theta + \left(\nabla^2 A_z\right)\boldsymbol{e}_z$$