EEE349 Power Engineering Electromagnetics Solutions – 15/16

1.

a) $\vec{D} = 3x^2y\,\overrightarrow{u_x} + 4x^2y^2\,\overrightarrow{u_y} + 3z\,\overrightarrow{u_z} \quad C/m^2$

The electric field is given by:

$$\vec{E} = \frac{\vec{D}}{\varepsilon_0} = \frac{1}{\varepsilon_0}\left(3x^2y\,\overrightarrow{u_x} + 4x^2y^2\,\overrightarrow{u_y} + 3z\,\overrightarrow{u_z}\right) V/m$$

Substituting in the coordinates (1,3,4)yields:

$$\vec{E} = \frac{1}{\varepsilon_0}\left(9\,\overrightarrow{u_x} + 36\,\overrightarrow{u_y} + 12\,\overrightarrow{u_z}\right) V/m$$

(OK to leave as is or divide through by ε_0)

The divergence is given by:

$$\nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = 6xy + 8x^2y + 3\ C/m^3$$

Substituting in the coordinates (1,3,4) yields:

$$\nabla \cdot \vec{D} = 18 + 24 + 3 = 45\ C/m^3$$

(1 for correct E calculation, 2 for divergence calculation, 1 for units)

b) Consider general point in the region within conductor and located at a radius r relative to the centre of the conductor. Applying Gauss's Law over a cylindrical surface at a radius r and noting that the electric field is purely radial yields:

$$\oiint \vec{D}\,.\overrightarrow{ds} = \oiiint q\,dv$$

$$2\pi r l \overrightarrow{D_r} = \pi r^2 l\,q$$

Hence,

$$\overrightarrow{D_r} = \frac{r^2 q}{2r} = \frac{rq}{2}$$

But $\vec{D} = \varepsilon\vec{E}$ and so the variation in the radial component of electric field outside the conductor is given by:

$$\overrightarrow{E_r} = \frac{rq}{2\varepsilon}$$

In a similar manner, consider a general point P which is the region outside by the conductor and located at a radius r relative to the centre of the conductor. Applying Gauss's Law over a cylindrical surface at a radius r and noting that the electric field is purely radial yields:

$$\oiint \vec{D}\,.\overrightarrow{ds} = \oiiint q\,dv$$

$$2\pi r l \overrightarrow{D_r} = \pi R_c^2 l\,q$$

Hence,

$$\overrightarrow{D_r} = \frac{R_c^2 q}{2r}$$

But $\vec{D} = \varepsilon\vec{E}$ and so the variation in the radial component of electric field outside the conductor is given by:

$$\overrightarrow{E_r} = \frac{R_c^2 q}{2\varepsilon r}$$

(6 – 3 for each case)

c) The contribution to the overall potential difference between conductors 1 and 2 due to conductor 1 is given by:

$$V_{1-2,1} = \int_{R_c}^{D} \overrightarrow{E_r} \cdot \overrightarrow{dr} = \left[\frac{R_c^2 q_1}{2\varepsilon} log_e r\right]_{R_c}^{D} = \frac{R_c^2 q_1}{2\varepsilon} log_e \left(\frac{D}{R_c}\right)$$

Similarly the contribution to the overall potential difference between conductors 2 and 1 due to conductor 2 is given by:

$$V_{2-1,2} = \int_{R_c}^{D} \overrightarrow{E_r} \cdot \overrightarrow{dr} = \left[\frac{R_c^2 q_2}{2\varepsilon} log_e r\right]_{R_c}^{D} = \frac{R_c^2 q_2}{2\varepsilon} log_e \left(\frac{D}{R_c}\right)$$

Hence, the net potential difference between the conductors is:

$$V_{1-2} = V_{1-2,1} - V_{2-1,2} = \frac{R_c^2 q_1}{2\varepsilon} log_e \left(\frac{D}{R_c}\right) - \frac{R_c^2 q_2}{2\varepsilon} log_e \left(\frac{D}{R_c}\right)$$

But for the simple arrangement, $q_1=-q_2=q$

$$V_{1-2} = V_{1-2,1} - V_{2-1,2} = \frac{R_c^2 q}{2\varepsilon}\left(log_e \left(\frac{D}{R_c}\right) + log_e \left(\frac{D}{R_c}\right)\right) = \frac{R_c^2 q}{\varepsilon} log_e \left(\frac{D}{R_c}\right)$$

(7 for derivation)

d) The voltage expression derived in part (c) can be used as a starting point to derive straightforwardly an expression for the capacitance per unit, although as the question is posed, a simple quoting of the equation with appropriate substitution will accrue full marks providing the units are quoted.

$$c = \frac{C}{l} = \frac{\pi\varepsilon_0}{log_e\left(\frac{D}{R_c}\right)} = \frac{\pi \times 8.85 \times 10^{-12}}{log_e(50)} = 7.11 \times 10^{-12}\ Fm^{-1}$$

2.

a) Given $V = (3yx^3 + 5zx + 7xyz + 12)\ V$ then:

i) The electric field strength is given by:

$$\vec{E} = -\left(\overrightarrow{u_x}\frac{\partial V}{\partial x} + \overrightarrow{u_y}\frac{\partial V}{\partial y}\overrightarrow{u_z}\frac{\partial V}{\partial z}\right) = -\left(\overrightarrow{u_x}(9yx^2 + 5z + 7yz) + \overrightarrow{u_y}(3x^3 + 7xz) + \overrightarrow{u_z}(5x + 7xy)\right)\ V/m$$

Substituting in for the point (3,2,1)m gives:

$\vec{E} = -\overrightarrow{u_x}181 - \overrightarrow{u_y}102 - \overrightarrow{u_z}57\ V/m$

(3 marks – 2 for establishing correct expression and indicating that it is a vector and 1 for substitution. Half mark deduction for not including units)

ii) The charge density is given by:

$\rho = \nabla \cdot \vec{D} = \varepsilon_0 \nabla \cdot \vec{E} = \varepsilon_0 \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} \right) = \varepsilon_0 (18yx)\ \ C/m^3$

Substituting in for the point defined and ε_0 gives:

$\rho = \varepsilon_0 (18 \times 2 \times 3) = 9.56\ \times 10^{-10} C/m^3$

(3 marks - 2 marks for correct expression and a further 1 mark for correct substitution –OK to leave in terms of ε_0-full marks awarded. ½ mark deduction for no or incorrect units)

b) Any integration path can be chosen, but the most straightforward path involves integration from (2,2) in the x-direction to (5,2) and then in the y-direction to (5,5):

$$\vec{E} = 3x^2 y\, \overrightarrow{u_x} + 4x^2 y^2 \overrightarrow{u_y}\ \ V/m$$

For the first section of the path and substituting for y=2

$V_1 = \int_2^5 6x^2\, dx = [2x^3]_2^5 = 234\ V$

For the first section of the path and substituting for x=5

$V_2 = \int_2^5 100y^2\, dy = \left[\frac{100}{3} y^3 \right]_2^5 = 3900\ V$

Total voltage change:

$V = V_1 + V_2 = 4134V$

c)

i) Gauss's Law states:

$\oiint_S \vec{D} \bullet d\vec{S} = Q$

Consider a spherical surface at a distance r which is centred on the charged sphere with a charge density q and radius R_c

$4\pi r^2 \vec{D}_r = \frac{4}{3} \pi R_c^3 q$

Rearranging yields:

$\vec{D}_r = \frac{R_c^3 q}{3r^2}$

(3 marks)

ii) Noting that $\vec{D} = \varepsilon_0 \vec{E}$:

$\vec{E}_r = \frac{R_c^3 q}{3\varepsilon_0 r^2}$

Rearranging yields:

$$r = \sqrt{\frac{R_c^3 q}{3\varepsilon_0 |\vec{E}_r|}}$$

For the values quoted in the question

$$r = \sqrt{\frac{0.01^3 \times 1 \times 10^{-3}}{3 \times 8.85 \times 10^{-12} \times 10{,}000}} = 61.3mm$$

(2 marks)

d) The maximum electric occurs at the surface of the conductor (r=R_c) and is given by:

$$\overrightarrow{E_r}max = \frac{R_c^2 q}{2\varepsilon r} = \frac{R_c q}{2\varepsilon}$$

[The key to this question is recognising that the capacitance provides the link between q and V]

The total charge on the 50m length of cable is:

$$Q = CV = 0.334 \times 10^{-6} \times 30{,}000 = 0.0100C$$

The charge density in the cable is:

$$q = \frac{Q}{\pi R_c^2 l}$$

Hence,

$$\overrightarrow{E_r}max = \frac{R_c q}{2\varepsilon} = \frac{R_c Q}{2\varepsilon\, \pi R_c^2 l} = \frac{Q}{2\varepsilon\, \pi R_c l}$$

Rearranging yields:

$$R_c = \frac{Q}{2\varepsilon\pi l \overrightarrow{E_r}} = \frac{0.010}{2 \times 6 \times 8.85 \times 10^{-12} \times \pi \times 50 \times 30 \times 10^6} = 0.02m = 20mm$$

(5)

3.

a) Starting from the equations provided:

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

The second equation can be re-written:

$$\nabla \times \vec{H} = \sigma\vec{E} + \varepsilon_0\varepsilon_r \frac{\partial \vec{E}}{\partial t}$$

At power frequencies, the displacement current term is negligible and hence this equation reduces to:

$$\nabla \times \vec{H} = \sigma \vec{E}$$

Taking the curl of the above and substituting for $\vec{E}$ yields:

$$\nabla \times (\nabla \times \vec{H}) = \sigma \nabla \times \vec{E} = -\mu\sigma \frac{\partial \vec{H}}{\partial t}$$

Using the trigonometric identity provided:

$$\nabla \times (\nabla \times \vec{H}) = \nabla(\nabla . \vec{H}) - \nabla^2 \vec{H}$$

But

$$\nabla . \vec{H} = \frac{1}{\mu}(\nabla . \vec{B})$$

But $\nabla . \vec{B} = 0$

Hence,

$$\nabla \times (\nabla \times \vec{H}) = -\nabla^2 \vec{H} = -\mu\sigma \frac{\partial \vec{H}}{\partial t}$$

Which gives the diffusion equation:

$$\nabla^2 \vec{H} = \mu\sigma \frac{\partial \vec{H}}{\partial t}$$

(7)

b) The diffusion equation is

$$\nabla^2 H = \sigma\mu \frac{\partial H}{\partial t}$$

For the geometry shown, this reduced to:

$$\frac{\partial^2 H_z}{\partial y^2} = \sigma\mu \frac{\partial H_z}{\partial t}$$

But $H_z = H_z e^{j\omega t}$

$$\therefore \frac{\partial^2 H_z}{\partial y^2} = j\omega\sigma\mu H = \alpha^2 H_z$$

The general solution to this equation takes the form:

$H_z = K_1 e^{\alpha y} + K_2 e^{-\alpha y}$

Boundary conditions

(i) At $y = b$, $H_z = H_s = K_1 e^{\alpha b} + K_2 e^{-\alpha b}$

(ii) At $y = -b$, $H_z = H_s = K_1 e^{-\alpha b} + K_2 e^{\alpha b}$

$\therefore K_1 = K_2$

$\therefore H_s = K_1 (e^{\alpha b} + e^{-\alpha b})$

$$\therefore K_1 = \frac{H_s}{e^{\alpha b} + e^{-\alpha b}} = K_2$$

$H_z = K_1 e^{\alpha y} + K_2 e^{-\alpha y}$

$$\therefore H_z = H_s \frac{\left(e^{\alpha y} + e^{-\alpha y}\right)}{\left(e^{\alpha b} + e^{-\alpha b}\right)} e^{j\omega t} = H_s \frac{\cosh \alpha y}{\cosh \alpha b} e^{j\omega t}$$

From Curl ***H*** = ***J***

$$J_x = \frac{\partial H_z}{\partial y} = \alpha H_s \frac{\sinh \alpha y}{\cosh \alpha b} e^{j\omega t}$$

(9)

c) The skin effect is given by:

$$\delta = \sqrt{\frac{2}{\omega \mu \sigma}}$$

Substituting in for the parameters provided, yields:

$$\delta = \sqrt{\frac{2}{2\pi \times 100 \times 500 \times 4\pi \times 10^{-7} \times 3 \times 10^{6}}} = 1.3mm$$

It is good design practice to make the lamination thickness < skin depth, so any answer <1.3mm is OK if the workings for the skin depth are correct

(4)

4.

a) $\vec{A} = 8y^3x\,\overrightarrow{u_x} + 14xz\,\overrightarrow{u_y} + 3xy^3z^2\,\overrightarrow{u_z}\ \ Wb/m$

$\nabla \times \vec{A} = \vec{B}$

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\overrightarrow{u_x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\overrightarrow{u_y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\overrightarrow{u_z}$$

$$\nabla \times \vec{A} = (9xy^2z^2 - 14x)\overrightarrow{u_x} + (0 - 3y^3z^2)\overrightarrow{u_y} + (14z - 24y^2x)\overrightarrow{u_z}$$

Substitute in coordinates of point (0.1,0.7,0.15):

$$\vec{B} = -1.39\overrightarrow{u_x} - 0.02\overrightarrow{u_y} + 0.924\overrightarrow{u_z}\ T$$

$\vec{H} = \frac{1}{\mu_0}(-1.39\overrightarrow{u_x} - 0.02\overrightarrow{u_y} + 0.924\overrightarrow{u_z}\ ) = \left(-1.10 \times 10^6\overrightarrow{u_x} - 18.4 \times 10^3\overrightarrow{u_y} + 732 \times 10^3\overrightarrow{u_z}\right) A/m$

(OK to leave as $1/\mu_0$)

(4- 1 mark reduction for no units)

b)

$\vec{H} = 6y^2z\,\overrightarrow{u_x} + 4yx^3\,\overrightarrow{u_y} + 2x^2z^3\,\overrightarrow{u_z}\ \ A/m$

$\nabla \times \vec{H} = \vec{J}$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\overrightarrow{u_x} + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\overrightarrow{u_y} + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\overrightarrow{u_z}$$

$= (0 - 0)\overrightarrow{u_x} + (6y^2 - 4xz^3)\overrightarrow{u_y} + (12yx^2 - 12yz)\overrightarrow{u_z}$

Substitute in coordinates of point (2,5,2):

$\vec{J} = 86\,\overrightarrow{u_y} - 120\,\overrightarrow{u_z}\ \ A/m^2$

(4- 1 mark reduction for no units)

c) For the geometry shown:

Boundary condition on normal component of flux density

$B_{n1} = B_{n2}$

Boundary condition on tangential component of flux density:

$H_{t1} - H_{t2} = 0$

Hence,

$H_{t1} = H_{t2}$

(also accept boundary condition specified in terms of B_y or H_x and if labelled as air and iron)

The only boundary condition that can be applied to the magnetic vector potential

$A_z = 0\ at\ y = \infty$

(3- 1 mark for each – ½ mark deduction if z subscript missing off A)

d) The difference in tangential H at the interface is:

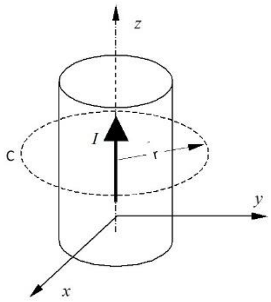
$H_{t1} - H_{t2} = 100\cos(1.3) = 28.4\ A/m$

But since the permeability of the iron is infinite, then $H_{t2} = 0\ A/m$

Hence,

$H_{t1} = 28.4\ A/m$

e) Consider an infinitely long circular wire carrying a current I



In this case, Ampere's Law takes the form

$$\oint_{C2} \vec{H}_1 \cdot d\vec{l}_1 = I_1$$

In the region outside the conductor (r > a), and selecting the integration path C, which encloses all the current I:

$$\oint_C \vec{H}.\vec{dl} = \int_0^{2\pi} H(\overrightarrow{u_\theta}.\overrightarrow{u_\theta})r\,d\theta = 2\pi r H = I$$

$$\vec{H} = H\overrightarrow{u_\theta} = \frac{I}{2\pi r}\overrightarrow{u_\theta}$$

$$\vec{B} = \mu_0 H\overrightarrow{u_\theta} = \frac{\mu_0 I}{2\pi r}\overrightarrow{u_\theta}$$

Rearranging and taking the magnitude of B yields:

$$r = \frac{\mu_0 I}{2\pi B} = \frac{4\pi \times 10^{-7} \times 4000}{2\pi \times 450 \times 10^{-6}} = 1.78m$$

f) Two practical solutions

Introduce some soft magnetic material between the cable and the instrument to act as 'magnetic shielding'.

Locate the return cable close to in the inlet cable to partially cancel out the field.