EEE345 exam 2016: exam questions and model solutions

1. Maxwell's equations and waves

5 points

a. State all four Maxwell's equations for electric and magnetic field vectors $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{H}}$ in differential form and state what the parameters on the right hand sides mean.

Solution:

(i) div $\underline{\boldsymbol{E}}=\rho/\varepsilon_0$

(ii) div $\underline{\boldsymbol{H}}=0$

(iii) rot $\underline{\boldsymbol{E}} = -\mu\, \partial\underline{\boldsymbol{H}}/\partial t$

(iv) rot $\underline{\boldsymbol{H}} = \underline{\boldsymbol{j}}_{\text{free}} + \varepsilon\, \partial\underline{\boldsymbol{E}}/\partial t$

ρ = charge density, ε_0 = permittivity of vacuum, t = time, $\underline{\boldsymbol{j}}_{\text{free}}=\underline{\boldsymbol{j}}/\varepsilon_r$ =free current density, $\mu=\mu_0\mu_r$ = permeability (product of permeability constant and relative permeability), $\varepsilon = \varepsilon_0\varepsilon_r$. Each correct equation and each definition gives a half point. Note that corresponding equations for the magnetic induction $\underline{\boldsymbol{B}}=\mu\underline{\boldsymbol{H}}$ would only be acceptable if this relationship was clearly stated as well.

5 points

b. Consider a damped electromagnetic wave propagating along z-direction of form

$$\underline{\boldsymbol{E}}(z,t)= E_0 \cos(\omega t - kz) \exp(-\alpha z)\, \underline{\boldsymbol{a}}_x \qquad \text{(equation 1)}$$

and

$$\underline{\boldsymbol{H}}(z,t)= H_0 \cos(\omega t - kz-\varphi) \exp(-\alpha z)\, \underline{\boldsymbol{a}}_y \qquad \text{(equation 2)}$$

where $\underline{\boldsymbol{a}}_x$ and $\underline{\boldsymbol{a}}_y$ are unit vectors along x- and y-directions, respectively.

(i) Define the Poynting vector.

(ii) Calculate its time averaged value.

(iii) What physical meaning does the phase angle φ have?

Solution:

$\underline{\boldsymbol{P}}=\underline{\boldsymbol{E}}\times\underline{\boldsymbol{H}}$ points along the direction of propagation. (1 point)

$\underline{\boldsymbol{P}}(z,t)= E_0H_0 \cos(\omega t - kz) \cos(\omega t - kz-\varphi) \exp(-2\alpha z)\, \underline{\boldsymbol{a}}_z$.

Use $\xi=E_0/H_0=E/H= [(\mu_0\mu_r)/(\varepsilon_0\varepsilon_r)]^{1/2}$ and cosA cosB= ½ [cos(A–B)+cos (A+B)] to get

$\underline{\boldsymbol{P}}(z,t)= E_0^2/(2\xi)\ [\cos\varphi + \cos(2\omega t - 2kz-\varphi)]\, \exp(-2\alpha z)\, \underline{\boldsymbol{a}}_z$ (2 points)

The term with $\cos(2\omega t - 2kz-\varphi)$ oscillates with time and so averages out over time:

$\langle \underline{\boldsymbol{P}}(z,t)\rangle = E_0^2/(2\xi) \cos\varphi \exp(-2\alpha z)\, \underline{\boldsymbol{a}}_z$ (1point)

The cosine of angle φ thus describes a power factor (=1 for φ=0, =0 for φ=90°). (1point)

10 points

c. Using Gauss' Law and the following relationship for any vectors $\underline{\boldsymbol{A}}$, $\underline{\boldsymbol{B}}$

$$\text{div}\,(\underline{\boldsymbol{A}}\times\underline{\boldsymbol{B}})= \underline{\boldsymbol{B}}\ \text{rot}\underline{\boldsymbol{A}} - \underline{\boldsymbol{A}}\ \text{rot}\underline{\boldsymbol{B}} \qquad \text{(equation 3)}$$

integrate the Poynting vector $\underline{\boldsymbol{P}}$ over a closed surface S and interpret the physical meaning of all three terms obtained. This is called Poyntings' theorem.

Solution:

$\underline{\boldsymbol{P}}=\underline{\boldsymbol{E}}\times\underline{\boldsymbol{H}}$ points along the direction of propagation and has the magnitude of power density. With this the total dissipated power is:

$\oint_S \boldsymbol{P}\, d\boldsymbol{S} = \oint_S (\boldsymbol{E}\times\boldsymbol{H})\, d\boldsymbol{S}$ (definition with surface normal $\underline{\boldsymbol{S}}$)

$= \int \text{div}\,(\boldsymbol{E}\times\boldsymbol{H})\, dV$ (Gauss' law)

$= \int \underline{\boldsymbol{H}}\ \text{rot}\underline{\boldsymbol{E}} - \underline{\boldsymbol{E}}\ \text{rot}\ \underline{\boldsymbol{H}}\ dV$ (application of given identity)

$= \int \underline{\boldsymbol{H}} \, (-\mu \, \partial\underline{\boldsymbol{H}}/\partial t) - \underline{\boldsymbol{E}} \, (\boldsymbol{j}_{\text{free}} + \varepsilon \, \partial\underline{\boldsymbol{E}}/\partial t) \, dV$

(3rd and 4th of Maxwell's equations)

$= \int [-\tfrac{1}{2} \mu \, \partial/\partial t(\underline{\boldsymbol{H}} \cdot \underline{\boldsymbol{H}}) - \tfrac{1}{2} \varepsilon \, \partial/\partial t(\underline{\boldsymbol{E}} \cdot \underline{\boldsymbol{E}}) - \sigma \, \underline{\boldsymbol{E}}^2 \,] \, dV$

(using diff. rule and Ohm's law $\boldsymbol{j}_{\text{free}} = \sigma \underline{\boldsymbol{E}}$)

$= -\tfrac{1}{2} \, \partial/\partial t \, [\int(\mu \underline{\boldsymbol{H}}^2 + \varepsilon \underline{\boldsymbol{E}}^2) \, dV \,] - \int \sigma \, \underline{\boldsymbol{E}}^2 \, dV$

(or $= -\partial/\partial t \, [\int(\tfrac{1}{2}\underline{\boldsymbol{HB}} + \tfrac{1}{2}\underline{\boldsymbol{ED}}) \, dV \,] - \int \sigma \, \underline{\boldsymbol{E}}^2 \, dV$)

The first term is the decrease in magnetic field strength, the second the decrease in electric field strength and the third Ohmic power dissipation.

2. Transmission Lines

9 points

a. The voltage as a function of position, x, and time, t, along a transmission line can generally be written as a superposition of forward and backward travelling waves in the form:

$$V(x,t)=V_0^+ \exp[j(\omega t - k'x)] + V_0^- \exp[j(\omega t + k'x)] \qquad \text{(equation 4)}$$

where ω is the angular frequency and k' is a complex propagation constant. For a lossy transmission line with impedance Z^* per unit length and admittance Y^* per unit length it can be shown that

$$k'^2 = -Z^*Y^* \qquad \text{(equation 5).}$$

(i) Use the definitions of Z^* and Y^* in terms of the standard parameters R^*, G^*, L^* and C^* to derive an exact expression for k' in terms of these parameters.

(ii) Interpret the physical meaning of all three terms you get.

(iii) For the lossless case, calculate the characteristic impedance $Z_0=Z^*/(jk')$.

(iv) For the lossless case, calculate the phase velocity of the signal on the line.

Solution:

(i) $Z^*=R^*+j\omega L^*$ and $Y^*=G^*+j\omega C^*$ (1 point)
Inserting into equation (5) gives
$k'^2 = -(G^*+j\omega C^*)(R^*+j\omega L^*)$
$= \omega^2 L^*C^* - G^*R^* - j\omega(R^*C^*+L^*G^*)$ (1 point)

(ii) The first term describes the (frequency-dependent and hence dispersive) phase velocity for the lossless case, the second a redshift and third an exponential damping term due to losses. (3 points)

(iii) $Z_0 = Z^*/(jk_0')$
with $Z^*=j\omega L^*$ (as $R^*=0$ for the lossless case) (1 point)
and $k_0' = \omega\sqrt{(L^*C^*)}$ (as $R^*=G^*=0$ for the lossless case) (1 point)
gives
$Z_0 = j\omega L^*/[j\omega\sqrt{(L^*C^*)}] = \sqrt{(L^*/C^*)}$ (1 point)

(iv) $v = \omega/k_0' = 1/\sqrt{(L^*C^*)} = (L^*C^*)^{-½}$ (1 point)

6 points

b. A 50MHz signal is fed into a computer's printed circuit board that can be described as a lossy transmission line with the characteristics of L^*=1mH/m, C^*=1nF/m, R^*=1Ω/m, G^*=0.001/(Ω m). Use your above solution for k'^2 to calculate over what length the signal can be transferred so that at least 90% of the voltage of the input signal arrives. Compare this to what you would get for 50kHz.

Solution:

Use $\omega=2\pi f$ where f=50MHz and get $\omega=3.1416\times10^8\ \text{s}^{-1}$ (1 point)
and insert numbers in above equation for k'^2 to get
$k'^2 = \omega^2 L^*C^* - G^*R^* - j\omega(R^*C^*+L^*G^*)$
$= (98696-0.001-314.5j)\ \text{m}^{-2}$
If you neglect the second term due to G^*R^* and consider that the real part, $\Re=98696\ \text{m}^{-2}$, is much bigger than the imaginary part, $\Im=314.5\ \text{m}^{-2}$, and that $\sqrt{(1-x)} \approx 1-0.5x$ for small x, then
$k' = \sqrt{k'^2} = \sqrt{(\Re - j\Im)} = \sqrt{\Re}\,\sqrt{(1-j\Im/\Re)} \approx \sqrt{\Re}\,(1-j\,0.5\Im/\Re) = \sqrt{\Re} - j\,0.5\Im/\sqrt{\Re}$
$= (314.16-0.50j)\ \text{m}^{-1}$ (2 points)
If $k' = a - jb$ with real component $a=314.16\text{m}^{-1}$ and imaginary part $b=0.50\text{m}^{-1}$, then the damping is described by $0.90=|V/V_0|=\exp(-bx)$. (1point)

This yields x=−(ln 0.90)/b=2.1 ×10^{-4}m=0.21m. (1 point)
The previous exam used a much lower frequency of 50kHz but since the imaginary part of k' scales with the ratio $\Im/\sqrt{\Re}$ ratio of k'^{2}, which scale with ω and ω^2 respectively, the result is independent of ω as long as the approximation $\Im<\Re$ is valid! (1 point)

5 points

c. For a lossless coaxial cable of outer diameter R and inner diameter r one has a capacitance per unit length of

$$C^{*}=2\pi\varepsilon_0\varepsilon_r/\ln(R/r) \qquad \text{(equation 6)}$$

and an inductance per unit length of

$$L^{*}=\mu_0\mu_r\ln(R/r)/(2\pi) \qquad \text{(equation 7).}$$

(i) Determine the phase velocity.
(ii) Derive the refractive index n of a dielectric medium wherein the velocity of light is the same as the above and compare to the definition of optics.

Solution:

i) $v=(L^{*}C^{*})^{-1/2}$ (1 point)

$= \{[(2\pi)\ln(R/r)]/[\mu_0\mu_r\ln(R/r)(2\pi\varepsilon_0\varepsilon_r)]\}^{1/2} = (\mu_0\mu_r\varepsilon_0\varepsilon_r)^{-1/2}$

(1 point)

ii) With $c=(\mu_0\varepsilon_0)^{-1/2}$ as speed of light in vacuum (1 point)
and the definition of the refractive index as the ratio of this to the speed of light in the dielectric

$n=c/v=(\mu_0\varepsilon_0)^{-1/2}/(\mu_0\mu_r\varepsilon_0\varepsilon_r)^{-1/2}=(\mu_r\varepsilon_r)^{1/2}$ (1 point)

This is exactly the definition of the refractive index in optics if the medium is a dielectric ($\varepsilon_r\geq1$) and magnetic ($\mu_r\geq1$). (1 point)

3. Electric potential and electronic devices: capacitors and pn-junctions

12 points

a. Consider a plate capacitor of width w, length l, distance d between the plates that is filled with a dielectric of relative permittivity ε_r.

(i) Using Coulomb's Law and Gauss' Law, calculate the magnitude D of the dielectric flux between the plates as a function of the charge Q_{free} on the plates. Give a physical interpretation of your result.

(ii) Perform the same calculation for the magnitude E of the electric field.

(iii) From the above, calculate the magnitude of polarisation in the dielectric as

$$P=D-\varepsilon_0 E \qquad \text{(equation 8)}$$

and give a physical interpretation of your result.

Solution:

(i) $Q_{free} = \iiint \rho_{free}\, dV$ (definition of density of free charge Q on plates)

$= \iiint \operatorname{div} \underline{\boldsymbol{D}}\, dV$ (Coulomb's Law: $\operatorname{div} \underline{\boldsymbol{D}} = \rho_{free}$)

$= \oint \boldsymbol{D}\, d\boldsymbol{s}$ (Gauss' Law, where the surface integral is over the closed surface)

$= D \int_S \underline{\boldsymbol{n}}\, d\underline{\boldsymbol{s}}$ (where $\underline{\boldsymbol{n}}$ is the unity normal vector pointing from one to the other plate, so $\int_S \underline{\boldsymbol{n}}\, d\underline{\boldsymbol{s}} = wl = A$ is the area of each plate)

$= DA$,

hence $D = Q_{free}/A$ is the free surface charge density (6 points)

(ii) $Q = \iiint \rho\, dV$ (definition of density of total charge Q)

$= \varepsilon_0 \iiint \operatorname{div} \underline{\boldsymbol{E}}\, dV$ (Coulomb's Law: $\operatorname{div} \underline{\boldsymbol{E}} = \rho/\varepsilon_0$)

$= \varepsilon_0 \oint \boldsymbol{E}\, d\boldsymbol{s}$ (Gauss' Law)

$= \varepsilon_0 E \int_S \underline{\boldsymbol{n}}\, d\underline{\boldsymbol{s}}$ (where $\underline{\boldsymbol{n}}$ is the unity normal vector)

$= \varepsilon_0 E A$,

hence $E = Q/(A\varepsilon_0)$ (3 points)

(iii) $P = D - \varepsilon_0 E = Q_{free}/A - Q/A = (Q_{free} - Q)/A = -Q_{bound}/A$ (1 point)

describes the bound surface charge density of the dielectric, and the negative sign means it opposes the definition of $\underline{\boldsymbol{E}}$ and $\underline{\boldsymbol{D}}$, pointing from negative to positive charges. Note this is similar to the defintion of dipoles. (2 points)

8 points

b. Consider a p-i-n diode where p- and n-doped regions are 100nm wide and the intrinsic region in-between is 500nm wide.

(i) State Poisson's equations for the one-dimensional case along x-direction.

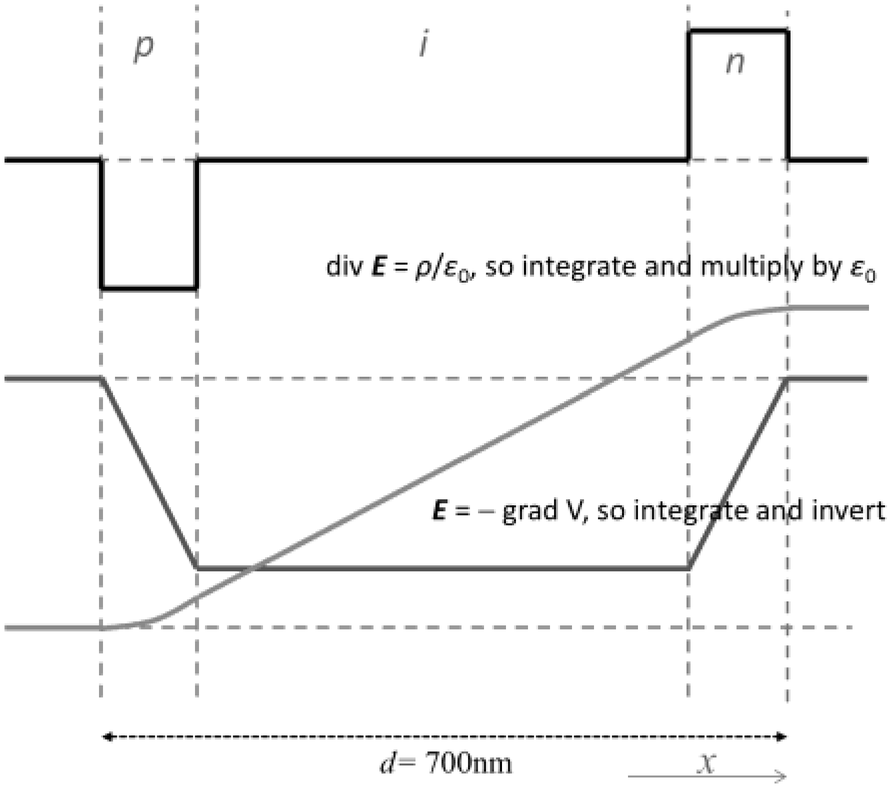
(ii) Provide a sketch of the x-dependence of the free charge $\rho(x)$, the electric field strength $E(x)$ and the potential $V(x)$ for the above p-i-n diode. Plot the graphs under each other, with the p-region to the left, so the interfaces between the differently doped regions are vertically aligned to each other.

(iii) Compare the built-in voltage qualitatively to that of a p-n diode without intrinsic layer and use the standard model of a plate capacitor to explain your result.

Solution:

(i) Poisson's equation: $\partial^2 V/\partial x^2 = -\rho(x)/\varepsilon_0$ (1 point)

(ii) The three plots should roughly look like the figure below. Each plot gets one point if the alignment at the interfaces is clear. Getting the sign of the ionic charges in p- and n-regions right, and the sign reversal from E to V gives an additional point each.



(5 points)

(iii) The built-in voltage of the p-i-n diode is much higher than for the p-n diode because it increases linearly in the intrinsic region. If p- and n-regions are considered as plates of a plate capacitor, with area A and average distance d, then its capacity is given by $Q/V_{bi}=C=\varepsilon_0\varepsilon_r A/d$ where Q is the total charge on the plates (remains constant) and V_{bi} the built-in voltage (between the plates), hence V_{bi} increases with d as the capacitance shrinks. (2 points)

4. Wave Optics

6 points

a. Consider light transversing from a medium with refractive index n_1 to another, denser one with refractive index n_2, as sketched in figure 1. Assume the speed of light in medium i is given by $v_i = c/n_i$ where c is the speed of light in vacuum. Derive Snell's Law of refraction under the assumption that the light travels from point A to point C along the fastest possible route.

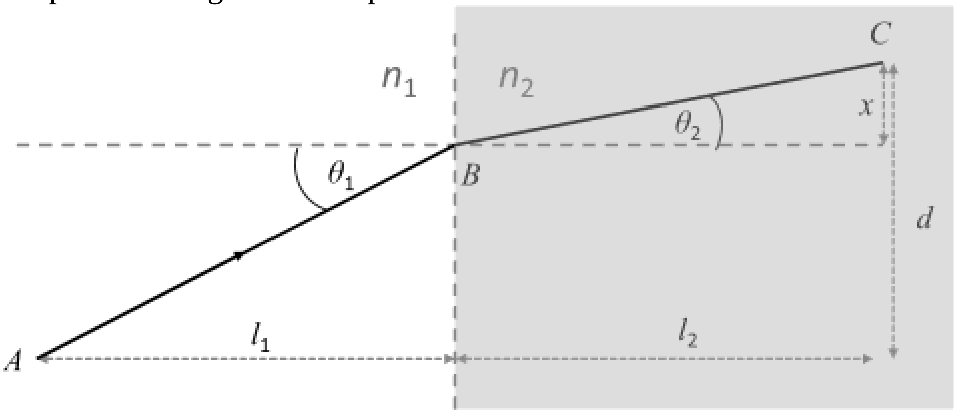


Figure 1

Solution:
The time taken by the light to travel from A to B and from B to C must be summed. With the definition of velocity as distance travelled per time:

$t = AB/v_1 + BC/v_2$ (1 point)

Inserting the given relationships $v_1 = c/n_1$ and $v_2 = c/n_2$ and using Pythagoras theorem:

$t = 1/c\ [n_1\ \{l_1^2 + (d-x)^2\}^{½} + n_2\ (l_2^2 + x^2)^{½}]$ (1 point)

Differentiating with respect to x and setting this to zero to get extremum:

$0 = \partial t/\partial x = 1/c\ [-2n_1(d-x)\ /[2\{l_1^2 + (d-x)^2\}]^{½} + 2n_2x/[2(l_2^2 + x^2)^{½}]]$ (1 point)

Equating the terms on the right gives

$n_1(d-x)\ /[\{l_1^2 + (d-x)^2\}]^{½} = n_2x/[(l_2^2 + x^2)^{½}]$ (1 point)

Now realise that $(d-x)\ /[\{l_1^2 + (d-x)^2\}]^{½} = \sin\theta_1$ and $x/[(l_2^2 + x^2)^{½}] = \sin\theta_2$ (1point)
and one directly gets Snell's Law:

$n_1 \sin\theta_1 = n_2 \sin\theta_2$ (1 point)

8 points

b. Consider the surface of a dielectric material with refractive index $n_1 > 1$, above which is air ($n_2 = 1$), as sketched in figure 2 below. Light shines at the interface under an angle θ_1 within the (x,z) plane where the x-direction points to the right and z upwards. Neglect the reflected wave. Assume another wave is generated in point A of the general form

$$\underline{\boldsymbol{E}}_A = \underline{\boldsymbol{E}}_0 \exp j(\omega t - \underline{\boldsymbol{K}} \cdot \underline{\boldsymbol{r}}) \qquad \text{(equation 9)}$$

where $\underline{\boldsymbol{K}} = (K_x, K_y K_z) = k_T(\sin\theta_T, 0, -\cos\theta_T)$, $\underline{\boldsymbol{r}} = (x,y,z)$ and ω and t have their usual meaning where Snell's law determines the relationship between θ_1 and θ_T.

(i) Calculate $\underline{\boldsymbol{E}}_A$ for the case that $\sin\theta_1 = n_2/n_1$ and explain your result physically.

(ii) Calculate $\underline{\boldsymbol{E}}_A$ for the case that $\sin\theta_1 > n_2/n_1$ and explain your result physically.

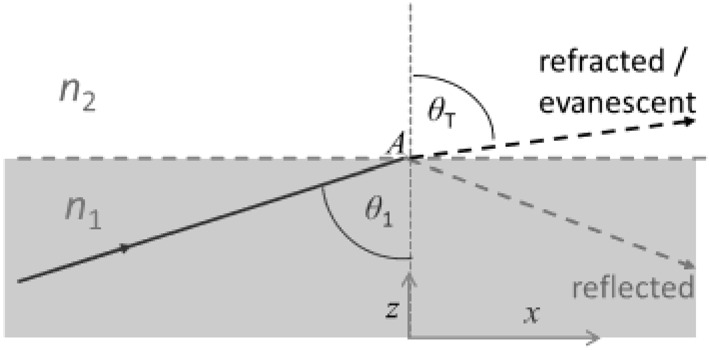


Figure 2

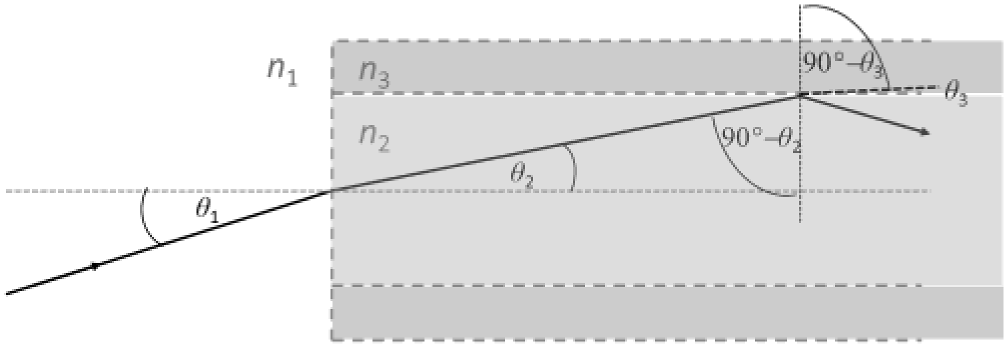
Solution:
If $\sin\theta_1 = n_2/n_1$, then $\sin\theta_T = \sin\theta_1\, n_1/n_2 = 1$, hence $\theta_T = 90°$ and $\cos\theta_T = 0$ (1 point)
Thus $\underline{\boldsymbol{E}}_A = \underline{\boldsymbol{E}}_0 \exp j(\omega t - \underline{\boldsymbol{K}}\cdot\underline{\boldsymbol{r}}) = \underline{\boldsymbol{E}}_0 \exp j(\omega t - k_T x)$.
This describes a simple plane wave travelling along x-direction, <u>in the interface plane defined by the two media</u>. (1 point)

If $\sin\theta_1 > n_2/n_1$, then $\sin\theta_T = \sin\theta_1\, n_1/n_2 > 1$, hence there is no obvious geometrical solution to θ_T, however, $\cos\theta_T = \sqrt{(1-\sin^2\theta_T)} = j\sqrt{(\sin^2\theta_T - 1)}$ now becomes complex. (1 point)
With this $\underline{\boldsymbol{E}}_A = \underline{\boldsymbol{E}}_0 \exp j(\omega t - k_T \sin\theta_T\, x + k_T \cos\theta_T\, z)$
$= \underline{\boldsymbol{E}}_0 \exp j(\omega t - k_T \sin\theta_T\, x) \exp -(k_T \sqrt{(\sin^2\theta_T - 1)}\, z)$
Using for the above medium $k_T = \omega/v_2 = \omega n_2/c$ (1 point)
and re-inserting $\sin\theta_T = \sin\theta_1\, n_1/n_2$
yields for the products of wave numbers and angles:
$K_x = k_T \sin\theta_T = \omega/c\; n_1 \sin\theta_1$, (1 point)
$\kappa = k_T \sqrt{(\sin^2\theta_T - 1)} = \omega/c \sqrt{((n_1 \sin\theta_1)^2 - n_2^2)}$, (1 point)
so $\underline{\boldsymbol{E}}_A = \underline{\boldsymbol{E}}_0 \exp j(\omega t - K_x x) \exp -(\kappa z)$ (1 point)
where the first (imaginary) exponential describes a wave travelling along $+x$ direction with wavevector K_x and the second (real) exponential describes damping along z-direction of this evanescent surface wave. (1 point)

6 points

c. Consider a coated optical fibre where the inner core material has a larger refractive index (n_2=1.5) than the outer cladding (n_3=1.4). Assume light falls onto the fibre cross-section from outside (air, n_1=1) under an angle θ_1 to the long axis of the fibre. Sketch what happens for sufficiently large θ_1 at the interface between core and cladding and calculate the so-called numerical aperture of the fibre which is defined as the maximal $\sin\theta_1$ value for which light that enters the fibre stays confined within it.

Solution:



The sketch should show the light refraction at the entrance towards the fibre centre as $n_2>n_1$ (where Snell's law states $n_1 \sin\theta_1 = n_2 \sin\theta_2$, with n_1=1 provided), the second refraction at the core/cladding interface away from the normal so again towards the fibre centre as $n_3<n_2$, and an internal reflection). The point to note is that if the ray falls in under angle θ_1 and continues to travel in the core under angle θ_2, then it hits the vertical to the core/cladding interface under angle $90°-\theta_2$. Full sketch with all aspects and labelling gives 2 points.

At the core/cladding interface: $n_2 \cos\theta_2 = n_2 \sin(90°-\theta_2) = n_3 \sin(90°-\theta_3) < n_3 = 1.4$ must alsways be smaller than 1.4, and if $\sin(90°-\theta_3)=1$ (or $\theta_3 \to 0°$) then the refracted ray travels parallel to the core/cladding interface and can no longer escape: total internal reflection occurs (1 point)

This limits the numerical aperture to

$$NA = \sin\theta_1^{max} = n_2/n_1 \sin\theta_2^{max} = n_2/n_1 (1-\cos^2\theta_2^{max})^{½} \quad \text{(as } \sin^2\theta+\cos^2\theta=1)$$
$$= n_2/n_1 [1-(n_3/n_2)^2)^{½} \quad \text{(inserting above)}$$
$$= (n_2^2-n_3^2)^{½} \quad (n_1=1)$$
$$= \sqrt{}(1.5^2-1.4^2)=0.5385\approx 0.54 \quad \text{(3 point for complete solution)}$$