2. Electrostatic Fields

Electrostatics deals with electric fields resulting from electric sources, charges for example, which do not vary with time. Under these circumstances, the time derivatives of $\vec{B}$ and $\vec{D}$ in the Maxwell equations are zero and separated into two uncoupled pairs, with the first pair involving only the electric field quantities $\vec{E}$ and $\vec{D}$, and the second pair involving only the magnetic quantities $\vec{B}$ and $\vec{H}$.

$$\left.\begin{array}{r}\nabla \bullet \vec{D} = \rho_V \\ \nabla \times \vec{E} = 0\end{array}\right\} \quad \textit{electrostatics} \qquad \begin{array}{r}(2.1a)\\(2.1b)\end{array}$$

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$$\left.\begin{array}{r}\nabla \bullet \vec{B} = 0 \\ \nabla \times \vec{H} = \vec{J}\end{array}\right\} \quad \textit{magnetostatics} \qquad \begin{array}{r}(2.1c)\\(2.1d)\end{array}$$

The electric and magnetic fields are no longer interconnected in the static case. This allows us to study electricity and magnetism as two distinct and separate phenomena, as long as the spatial distributions of charges and current flow remain constant in time. We refer to study of electric and magnetic phenomena under static conditions as electrostatics and magnetostatics, respectively. In this chapter, we discuss electrostatics.

We study electrostatics not only as a prelude to the study of time-varying fields, but also because it is an important field of study in its own right. Many electronic devices and systems are based on the principles of electrostatics. They include x-ray machines, oscilloscopes, ink-jet printers, liquid crystal displays, copying machines, capacitance keyboards, and many solid-state control devices.

2.1 Gauss's Law

Consider the electric field around a simple isolated point charge, Q, in free space, the electric field strength $\vec{E}$ is given by:

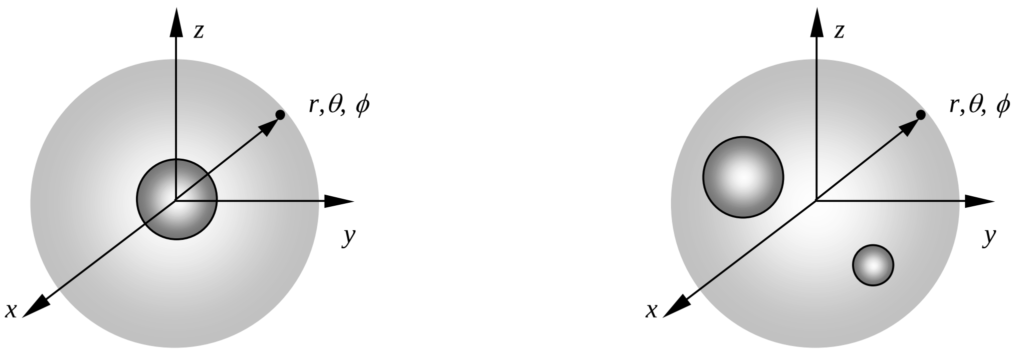


Fig. 2.1 Point charge

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{r} \qquad (2.2)$$

where $\vec{r}$ is the outward radial unit vector. If we surround the charge by an imaginary sphere with radius r_o and centre at the position of the charge, the electric field at any point on its surface is:

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r_o^2}\vec{r} \tag{2.3}$$

The electric flux density is given by $\vec{D} = \varepsilon_0 \vec{E}$, or

$$\vec{D} = \frac{\varepsilon_0 Q}{4\pi\varepsilon_0 r_o^2}\vec{r} = \frac{Q}{4\pi r_o^2}\vec{r} \tag{2.4}$$

Therefore, the surface integration of the normal component of $\vec{D}$ (in this case the radial component) over the surface of this sphere is:

$$\oiint_S \vec{D} \bullet d\vec{S} = \oiint_S \frac{Q}{4\pi r_o^2}(\vec{r} \bullet \vec{r})dS = \frac{Q}{4\pi r_0^2} 4\pi r_0^2 = Q \tag{2.5}$$

i.e. the total charge enclosed. The same relationship holds if the charge is distributed at any position within the sphere. The more general proof involves surface integrals in the spherical coordinate system. See Kraus ---- Electromagnetics page 89-90 for details.

Since equation (2.5) holds for a charge at any position within the sphere, it follows that it also applies to a number of point charges or charge distribution, as shown in Fig. 2.1. In general term, the Gauss's Law states:

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \rho_v dV = Q \tag{2.6}$$

where $\iiint_V \rho_v dV$ is the volume integration of charge density, and Q is the total charge enclosed within the surface of S.

In summary, Gauss's Law states THE ELECTRIC FLUX THROUGH ANY CLOSED SURFACE EQUALS THE TOTAL CHARGE ENCLOSED.

Application of Gauss's Law: *Capacitance of a co-axial cable*

A long length of co-axial cable can be represented by two infinitely long concentric cylinder separated by a material of permittivity ε, as shown in Fig. 2.2.

Due to symmetry, the electric field will only have a radial component, E_r. At any value of r, Gauss's Law state that:

$$\oiint_S \vec{D} \bullet d\vec{S} = \iiint_V \rho_v dV = Q \text{ (charge enclosed)}$$

Applying Gauss's Law at any radius *r*, (*a* < *r* <*b*) gives:

$$\oiint_S \vec{D} \bullet d\vec{S} == Q$$

where *Q* is the charge on the inner surface of the conductor. But ***D*** = ε***E***, thus

$$\oiint_S \vec{D} \bullet d\vec{S} = \oiint_S \varepsilon\vec{E} \bullet d\vec{S} = l\varepsilon\int_0^{2\pi} E_r r d\theta = l\varepsilon E_r 2\pi r = Q$$

where l is the length of the cable.

$$E_r = Q/l\varepsilon 2\pi r$$

$$V = \int_a^b E_r dr = \frac{Q}{2\pi l\varepsilon}\int_a^b \frac{1}{r}dr = \frac{Q}{2\pi l\varepsilon}\ln(b/a)$$

Finally,

$$C = \frac{Q}{V} = \frac{2\pi l\varepsilon}{\ln(b/a)}$$

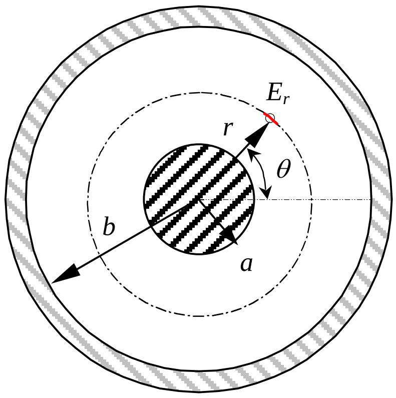


Fig. 2.2 Capacitance of co-axial cable

2.2 The gradient operator

The line integral of the electric field $\vec{E}$ between two points gives the electric potential, V. The inverse is also true, viz:

$$\lim_{\Delta l \to o}\left(\frac{\Delta V}{\Delta l}\right) = \frac{dV}{dl} = \vec{E} \text{ (V/m)} \tag{2.7}$$

where *V* is a scalar, but $d\vec{l}$ and $\vec{E}$ are vectors. When $d\vec{l}$ is in the same direction as $\vec{E}$, this derivative is a maximum. This maximum value is referred to as the gradient of *V*(*x, y, z*), or

$$Gradient\ V = \left(\frac{dV}{dl}\right)_{\max} \tag{2.8}$$

Note since a potential rise occurs when moving against the electric field, the direction of the gradient is opposite to that of the field:

$$Gradient\ V = -\vec{E} \tag{2.9}$$

This is often written as Grad V or ∇V . From this general definition of the gradient operator, we can derive the operator specific to Cartesian coordinates:

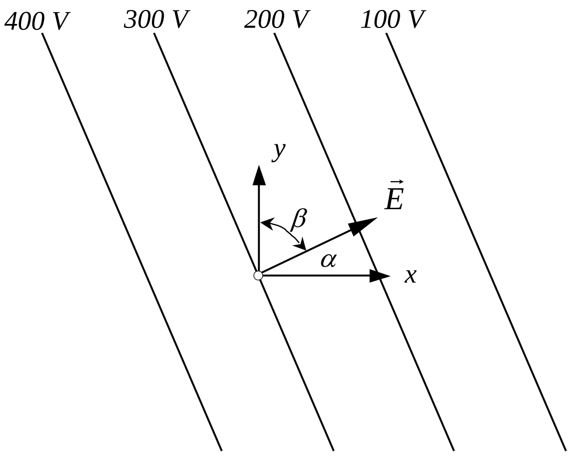


Fig. 2.3 Equal-potential lines (constant increments)

With reference to Fig. 2.3, the potential V is a function of x, and y. For an infinitesimal path $d\vec{l}$ at an arbitrary position P, we have

$$\begin{aligned} -\frac{\partial V}{\partial x} &= \vec{E}\cos\alpha = E_x \\ -\frac{\partial V}{\partial y} &= \vec{E}\sin\alpha = E_y \end{aligned} \tag{2.10}$$

Adding these components vectorally gives:

$$\vec{E} = E_x e_x + E_y e_y = -\left(\frac{\partial V}{\partial x}\right)e_x - \left(\frac{\partial V}{\partial y}\right)e_y = -\nabla V \tag{2.11}$$

In general, for a three dimensional case,

$$\nabla V = \left(\frac{\partial V}{\partial x}\right)e_x + \left(\frac{\partial V}{\partial y}\right)e_y + \left(\frac{\partial V}{\partial z}\right)e_z = -\vec{E} \tag{2.12}$$

In summary, GRADIENT OF A SCALAR YIELDS A VECTOR WHOSE DIRECTION IS THAT OF THE MAXIMUM RATE OF CHANGE OF THE SCALAR.

2.3 The divergence operator

An important concept in fields is that of flux. Flux is a convenient method of visualising and occasionally calculating fields, and it is usually represented schematically by flux lines which are imaginary lines along which forces occur.

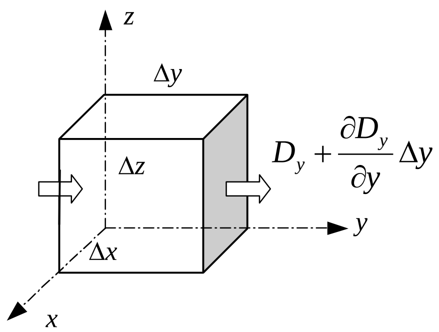
The total flux passing through a given area is given simply by the integral of the flux density over the area, e.g.

$$\text{Electric flux} = \Phi_e = \oint_S \vec{D} \cdot d\vec{S}$$

$$\text{Magnetic flux} = \Phi_m = \oint_S \vec{B} \cdot d\vec{S}$$

Note that flux is usually represented by the symbol Φ or ϕ in both magnetic and electric fields.

To define the divergence operator we need to consider an elemental cube (for Cartesian) with flux flowing through it:



At y = 0, the outward flux is $\approx -D_y \Delta x \Delta z$

At $y = \Delta y$, the outward flux is $\approx D_y \Delta x \Delta z + \frac{\partial D_y}{\partial y}\Delta x \Delta y \Delta z$

Therefore the net flux leaving the two areas perpendicular to the y axis is:

$$\vec{D} \cdot d\vec{S}\Big|_{y=\Delta y} - \vec{D} \cdot d\vec{S}\Big|_{y=0} = \left(D_y + \frac{\partial D_y}{\partial y}\Delta y\right)\Delta x \Delta z - D_y \Delta x \Delta z = \left(\frac{\partial D_y}{\partial y}\Delta y \Delta x \Delta z\right)$$

and similarly for the x and z directions. Summing all the fluxes leaving the cube

$$\Phi_e = \oint_S \vec{D} \cdot d\vec{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right)\Delta x \Delta y \Delta z$$

i.e. this is the additional flux produced by sources (or sinks) within the element. If there are no sources (for electric fields these would be charges) within the cube, then no net flux would flow out of the volume (As much would flow in as out). Dividing this net outward flux by the volume and taking the limit of $\Delta v = \Delta x \Delta y \Delta z \rightarrow 0$ gives:

$$\lim_{\Delta v \to 0} \frac{\oint_S \vec{D} \cdot d\vec{S}}{\Delta v} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

which is the divergence operator, often called *div*:

$$div\,\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Note: the divergence of a vector yields a scalar. The divergence of a vector allows the sources and sinks responsible for the flux to be found. For electric fields, therefore, $div\vec{D}$ only has a value when there is charge present.

The divergence of $\vec{D}$ can also be written as the scalar product of the del operator ∇ and $\vec{D}$, i.e.,

$$div\,\vec{D} = \nabla \cdot \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

The quantity $\nabla \cdot$ can be represented as the divergence operator. In summary, at any point in space:

$$\nabla \cdot \vec{D} = \begin{cases} \rho & \text{if a charge density } \rho \text{ is present at the point} \\ 0 & \text{if no charge density is present at the point} \end{cases}$$

The above relationship is very important in electromagnetics and is one of Maxwell's equation.

<u>Divergence Theorem</u>

From Gauss's Law, we have

$$\underbrace{\oint_S \vec{D} \cdot d\vec{S}}_{\text{Surface integral}} = \underbrace{\oint_V \rho\, dv}_{\text{Volume integral}}$$

However, we can substitute $\rho = \nabla \cdot \vec{D}$, yielding so called Divergence Theorem:

$$\underbrace{\oint_S \vec{D} \cdot d\vec{S}}_{\text{Surface integral}} = \underbrace{\oint_V \nabla \cdot \vec{D}\, dv}_{\text{Volume integral}}$$

which states that ***the integral of the normal component of a vector function over a closed surface equals the integral of the divergence of the vector throughout the volume enclosed by the surface.***

2.4 The Laplacian operator

From the divergence operator:

$$\nabla \cdot \vec{D} = \rho$$

but

$$\vec{D} = \varepsilon\vec{E} \quad \text{and} \quad \vec{E} = -\nabla V$$

Thus

$$\vec{D} = -\varepsilon\nabla V$$

which leads to:

$$\nabla \cdot \vec{D} = \nabla \cdot (-\varepsilon\nabla V) = \rho \qquad \text{or} \quad \nabla \cdot (\nabla V) = -\frac{\rho}{\varepsilon}$$

This is Poisson's equation and in general it is written as:

$$\nabla^2 V = -\frac{\rho}{\varepsilon}$$

where ∇^2 (del ^{2}) is called the Laplacian operator. It is in fact the divergence of the gradient, but is given a specific name because of its importance. If there is no charge density Poisson's equation reduces to:

$$\nabla^2 V = 0$$

which is known as Laplace's equation.

2.5 Electrostatic field calculation

Both Laplace's and Poisson's equations form the cornerstone of electrostatic field analysis. As with most equations which govern field distributions, they can be solved by one of two methods:

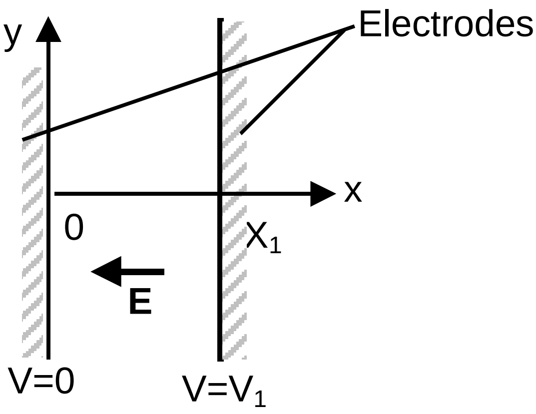
(i) An approximate numerical solution
(ii) An exact analytical solution

Whenever they can be used, analytical techniques are the preferred method of solution. They are exact and can be calculated at an arbitrarily isolated point. However they tend to be limited to fairly simple geometries and linear problems.

Numerical techniques by contrast can accommodate very complex geometries and non-linear material properties. They are however normally iterative (i.e. take a number of passes to obtain a solution) and produce approximate solution. The accuracy of these solutions often depending critically on the spatial discretisation of calculation points used.

We shall use two examples to illustrate how electric field solutions can be found analytically from Laplace's or Poisson's equations with appropriate boundary conditions.

Example 1: Electric field of parallel plate capacitor

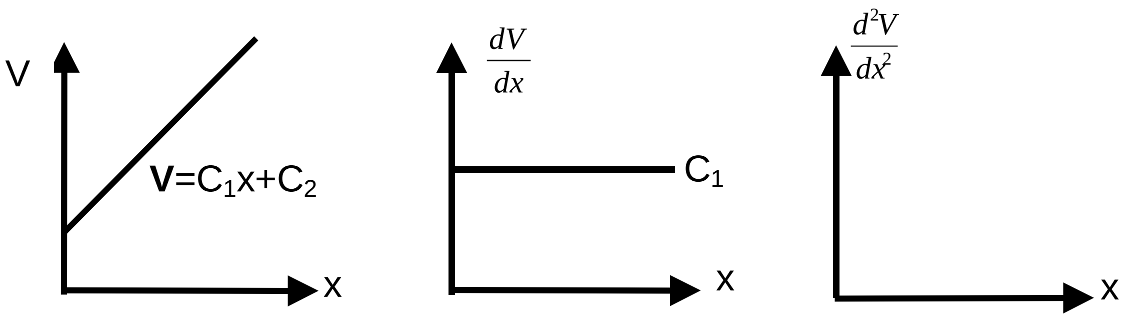


Neglect end-effects (i.e. assume no variation in potential in y and z directions). Laplace's equation reduces to:

$$\frac{d^2V}{dx^2} = 0$$

(Note: For a function which depends on x only $\frac{\partial}{\partial x} = \frac{d}{dx}$)

For the 2nd derivative to be zero, the first derivative must be a constant (i.e. there is no rate of change of slope).



$$\therefore \frac{dV}{dx} = C_1 \qquad \text{where } C_1 \text{ is a constant}$$

$$dV = C_1\,dx$$

$$\int dV = C_1 \int dx$$

$$\therefore V = C_1 x + C_2$$

where C_2 is a constant of integration. To obtain a unique solution, boundary conditions must be applied to determine C_1 and C_2

Boundary Conditions

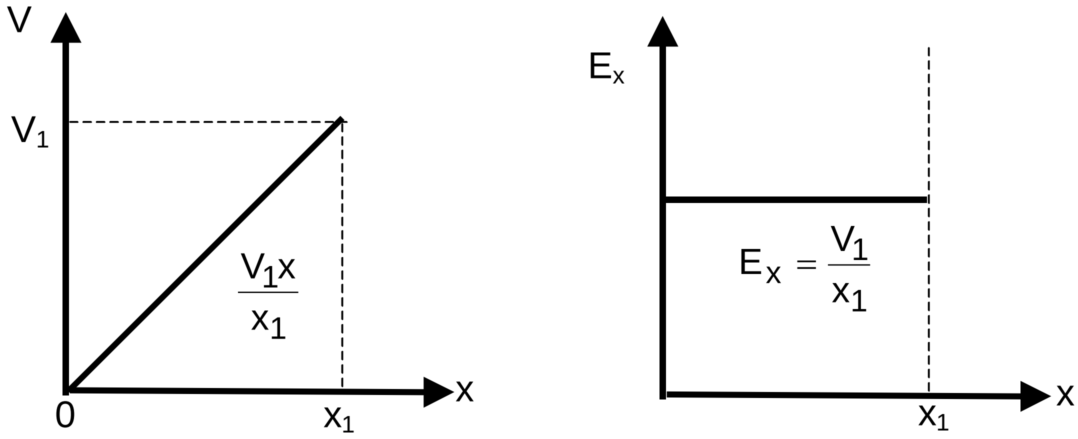
(i) At $x = 0$, $V = 0$

(ii) At $x = x_1$, $V = V_1$

From (i), $0 = 0C_1 + C_2$ $\quad \therefore C_2 = 0$

From (ii), $V_1 = C_1\, x_1$ $\quad \therefore C_1 = \dfrac{V_1}{x_1}$

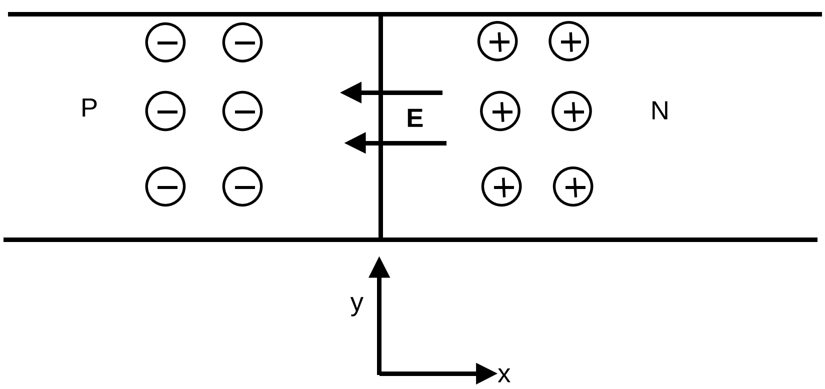
$$\therefore\ V = V_1\,\frac{x}{x_1}$$



NOTE: Application of boundary conditions yields a unique solution – which satisfies Laplace's equation and the boundary conditions.

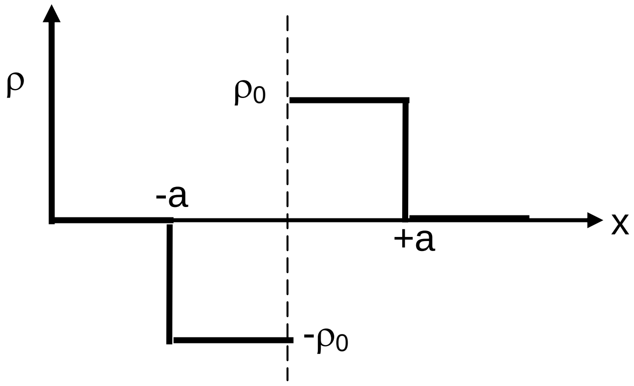
Example 2 Electric field at p-n junction

Consider simplified model of semiconductor junction as follows:



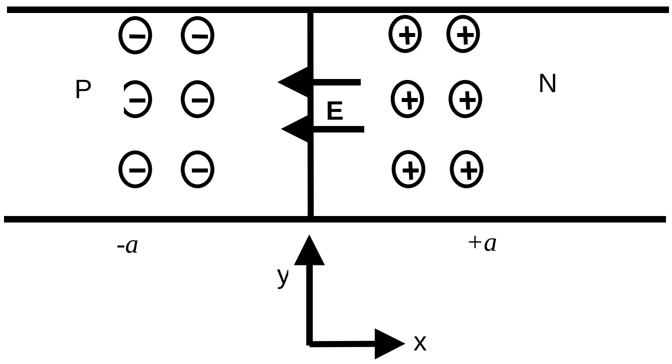
To a reasonable approximation, the 2 charge layers are confined to a region known as the depletion layer, which has a width 2*a*.

The charge distribution approximates to a step function.



For $0 < x < a$, $\rho = \rho_0$

For $-a < x < 0$, $\rho = -\rho_0$



For the n-layer (x > 0), Poisson's equation becomes

$$\frac{d^2V}{dx^2} = -\frac{\rho_0}{\varepsilon} \qquad (1)$$

Boundary Conditions

(i) At x = a, **E** = 0

(ii) At x = 0, V = 0

Integrating (1)

$$\frac{dV}{dx} = -\frac{\rho_0}{\varepsilon}x + C_1$$

From (i), $0 = -\frac{\rho_0 a}{\varepsilon} + C_1$

$$\therefore C_1 = \frac{\rho_0 a}{\varepsilon}$$

$$\therefore \frac{dV}{dx} = \frac{\rho_0}{\varepsilon}(a - x)$$

$$\therefore \mathrm{E}_x = \frac{\rho_0}{\varepsilon}(x - a) \leftarrow \text{ Electric field in n-layer}$$

Integrating again

$$V = -\frac{p_0}{\varepsilon}\left(\frac{x^2}{2} - ax\right) + C_2$$

From (ii) $$0 = -\frac{p_0}{\varepsilon}\left(\frac{0}{2} - a0\right) + C_2$$

$\therefore$ $C_2 = 0$

$$\therefore V = \frac{p_0}{2\varepsilon}\left(2ax - x^2\right) \leftarrow \text{potential at any point in n} - \text{layer}$$

Similar expressions can be derived for ***E*** and *V* in the p-layer.

When $x = a$, the value of V is the total potential across the n-layer, i.e.

$$V_n = \frac{p_0}{2\varepsilon}a^2$$

Since the junction is symmetrical, the total potential across the junction is

$$V_j = V_n + V_p = 2V_n = \frac{p_0}{\varepsilon}a^2$$

If the cross-sectional area of the junction is A_j, the charge contained in one layer is

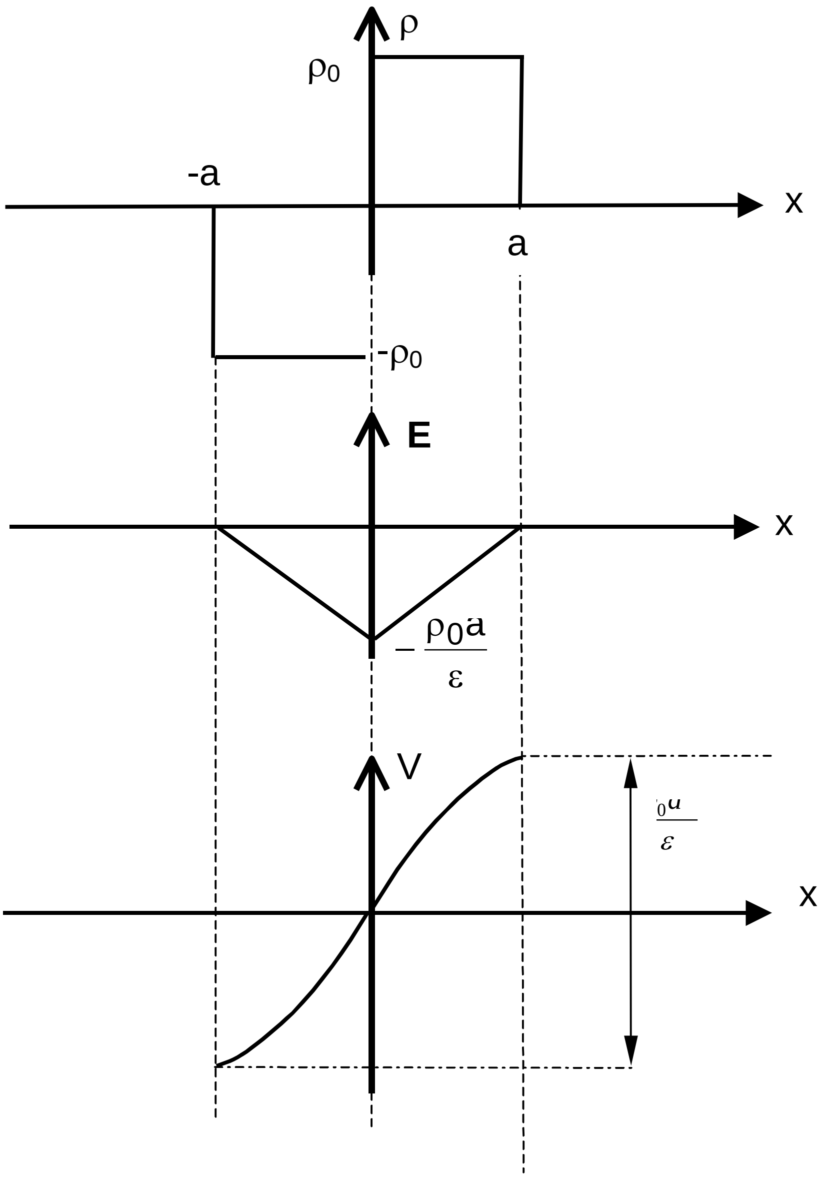
$$Q_n = \rho_0 \, a \, A_j$$

The capacitance of the layer is, therefore,

$$C_n = \frac{Q_n}{V_n} = \frac{\rho_0 a A_j}{\rho_0 \, a^2 / 2\varepsilon} = \frac{2\varepsilon A_j}{a}$$

The capacitance of the junction (series-connected layers) is

$$C_j = \frac{C_n}{2} = \frac{\varepsilon A_j}{a} \quad Farads$$



Tutorial Sheet 1

1. Vector **A** extends from the origin to the point (x, y, z) = (4, 5, 2) in Cartesian co-ordinates. Vector B extents from the origin to the point (r, ϕ, z) = (4, 75^0, 5) in cylindrical co-ordinates. Calculate, in rectangular co-ordinates where appropriate:

   a. $\mathbf{A} + \mathbf{B}$
   b. $\mathbf{A} \cdot \mathbf{B}$
   c. The angle between **A** and **B**
   d. $\mathbf{A} \times \mathbf{B}$

2. If the electric potential V decreases by 2V/m in x-direction and by 1V/m in y-direction, calculate electric field $\boldsymbol{E}$

3. The electric scalar potential within a region of space is given by $V = x^2 + yz^3$ (V). Find the electric field $\boldsymbol{E}$.

4. If the electric scalar potential within a region varies as $V = 10xz + 15\ yz^2$ (V), find V and E at the point (x, y, z) = (5, 4, 3) m.

5. Calculate the electric flux from a point charge of 1.0 (C) which passes through a spherical surface at a radius of 1.3m, extending from θ = 0^0 to 180^0 and from ϕ = 0^0 to 45^0.

6. The electric flux density, $\boldsymbol{D}$, over an imaginary sphere is $1.25\mathbf{e_r}$ C/m^2 (in spherical co-ordinates). Assuming that this is due to a single charge of 0.3C, calculate the radius of the imaginary sphere relative to the position of the charge.

7. The electric flux density $\boldsymbol{D}$, at a point in space is $9yx^3\ \mathbf{e_x} + 6y\ \mathbf{e_y} + 7zxy^2\ \mathbf{e_z}$. Calculate

   a. the electric field $\boldsymbol{E}$ and
   b. the divergence of $\boldsymbol{D}$

   at the point (x, y, z) = (2, 1, 3) m.

8. The electric flux density $\boldsymbol{D}$, at a point in space is $r\boldsymbol{e}_r + r\sin\theta\,\boldsymbol{e}_\theta$, calculate the divergence of $\boldsymbol{D}$

9. The electric flux density $\boldsymbol{D}$, at a point in space is $9x^3\ \mathbf{e_x} + 5y^2\ \mathbf{e_y} + 2z\ \mathbf{e_z}$. Calculate the charge density at the point (x, y, z) = (1, 5, 9) m.

10. The electric field, $\boldsymbol{E}$, at a point is given by $3y\ \mathbf{e_x} + 5xz^4\ \mathbf{e_y} + 2xy^3\ \mathbf{e_z}$. Is there any charge present at the point?

11. Starting from Laplace's equation, derive an expression for the electric field strength mid way between the plates of a parallel plate capacitor which is storing an energy of 0.354 μJ. The capacitor plates are separated by a material with a thickness of 50 μm, a total surface area of 400mm^2 and a relative permittivity of 1.0. Assume that the ratio of capacitor area to plate separation is sufficiently high that the plates can be approximated as being of infinite extent.

12. A p-n semiconductor junction has a depletion layer width of 200nm, a volume charge density of $5\text{x}10^3$ C/m^3, a relative permittivity of 10 and a cross-sectional area of 10^{-9} m^2. Starting from Poisson's equation, and listing any assumptions you make, calculate the maximum value of electric field strength, the voltage across the junction and its capacitance.

13. Derive interface conditions at the boundary between a conductor and dielectric material (*hint conductor is an equi-potential body*).

14. Derive expressions for potential V and electric field strength E in cylindrical coordinate system for the coaxial cable as shown below: