Solutions to Tutorial Sheet 2

1.

$\vec{J} = \nabla \times \vec{H}$ and in Cartesian co-ordinates

$$\nabla \times \vec{H} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3 & 7y & 2x \end{vmatrix} = -2\ e_y$$

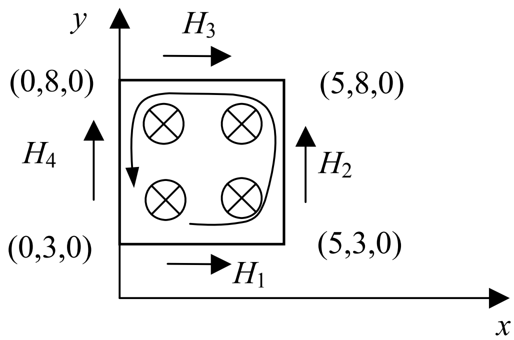
2. Solution 1

From Application of Ampere's law to the path shown below:

$$I = \oint \vec{H} \cdot d\vec{l} = \int H_1 dl_1 + \int H_2 dl_2 + \int H_3 dl_3 + \int H_4 dl_4$$

But ***H*** has only *x* components, thus

$H_2 = H_4 = 0;$
$H_1\ (x, y=3) = 81\ \boldsymbol{e_x}$
$H_3\ (x, y=8) = 1536\ \boldsymbol{e_x}$



$$I = \oint \vec{H} \cdot d\vec{l} = \int H_1 dl_1 + \int H_3 dl_3$$
$$= 81 \times 5 - 1536 \times 5 = -7275 \quad (A)$$

The negative sign indicates that the current flows in the opposite direction of the z axis

Solution 2

Using

$$\vec{J} = \nabla \times \vec{H} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3y^3 & 0 & 0 \end{vmatrix} = -9y^2\ e_x$$

The total current is the integration of the current density over the square region:

$$I = \iint_S \vec{J} \cdot d\vec{s} = \int_0^5 dx \int_3^8 -9y^2 dy = -5(3y^3)\ \Big|_3^8 = -7275(A)$$

3. From the definition

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 4xy & 7 \end{vmatrix} = xy\ e_y + (4y - xz)\ e_z$$
$$= 6\ e_y - 14\ e_z$$

4. Again using

$$\vec{J} = \nabla \times \vec{H} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 7y & 7x & 7xy \end{vmatrix} = 7x\ e_x - 7y\ \ e_x$$

Thus, at (2, 1, 0) $\boldsymbol{J} = 14\ \boldsymbol{e_x} - 7\ \boldsymbol{e_y}$

And at (0, 0, 0) $\boldsymbol{J} = \boldsymbol{0}$

5.

$$\nabla \cdot (\nabla \times \vec{F}) = \frac{\partial}{\partial x}\left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]$$

$$= \frac{\partial^2 F_z}{\partial x \partial y} - \frac{\partial^2 F_y}{\partial x \partial z} + \frac{\partial^2 F_x}{\partial y \partial z} - \frac{\partial^2 F_z}{\partial y \partial x} + \frac{\partial^2 F_y}{\partial z \partial x} - \frac{\partial^2 F_x}{\partial z \partial y} = 0$$

6.

$$\nabla \times (\nabla \varphi) = \nabla \times \left\{ \frac{\partial \varphi}{\partial x}\ e_x + \frac{\partial \varphi}{\partial y}\ e_y + \frac{\partial \varphi}{\partial z}\ e_z \right\} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix}$$

$$= \left(\frac{\partial^2 \varphi}{\partial y \partial z} - \frac{\partial^2 \varphi}{\partial z \partial y} \right) e_x + \left(\frac{\partial^2 \varphi}{\partial z \partial x} - \frac{\partial^2 \varphi}{\partial x \partial z} \right) e_y + \left(\frac{\partial^2 \varphi}{\partial x \partial y} - \frac{\partial^2 \varphi}{\partial y \partial x} \right) e_z = 0$$

7.

$$H_{x2} - H_{x1} = J = (0.32 - 21.6)\cos(2\pi x / 6.3) = -21.28\cos(2\pi x / 6.3)$$

At $x = 3$ $\quad J = -21.25$

Since flux density B is usually limited to a few Tesla in any materials whilst the permeability of iron could be a few thousand times greater than that in air, from $H = B/\mu$, the larger the permeability, the smaller the magnetic field intensity. Thus, it is likely that region 2 with lower H_{x2} contains iron.

8. This is virtually the same problem as *Example* 1 in section 3.6. However, the magnetic field solutions needs to be considered in both regions ($y \geq 0$ and $y < 0$) and in finding the solutions, scalar magnetic potential φ will be used.

$$\vec{H} = -\nabla \varphi$$

and it satisfies the Laplace's equation:

$$\nabla^2 \varphi = 0$$

Also due to the symmetry, φ will be independent of *z,* thus

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

$$H_x = -\frac{\partial \varphi}{\partial x} \quad ; \quad H_y = -\frac{\partial \varphi}{\partial y}$$

Assume solution takes the form:

$$\varphi(x, y) = F(x) \bullet G(y)$$

Where F is a function of x only, and G is a function of y only. Thus:

$$G\frac{d^2 F}{ax^2} + F\frac{d^2 G}{dy^2} = 0$$

Dividing both sides by $\{-F(x)G(y)\}$ gives:

$$-\frac{1}{F}\frac{d^2 F}{dx^2} = \frac{1}{G}\frac{d^2 G}{dy^2} = k^2$$

The standard solutions are:

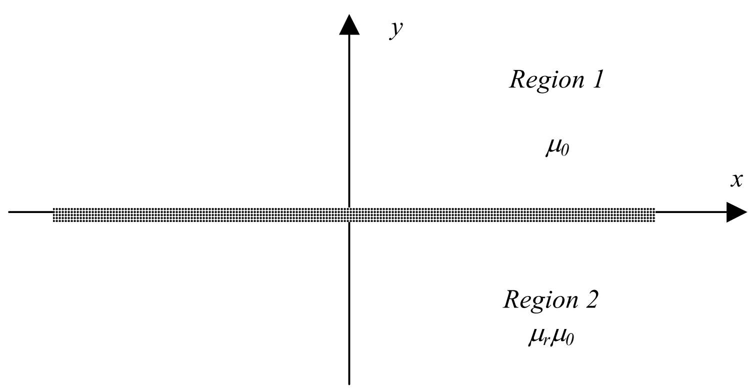
$$F(x) = C_1 \sin(kx) + C_2 \cos(kx)$$
$$G(y) = C_3 e^{ky} + C_4 e^{-ky}$$

where C_1, C_2, C_3 and C_4 are constants to be determined. Hence:

$$\varphi(x, y) = \{C_1 \sin(kx) + C_2 \cos(kx)\}\{C_3 e^{ky} + C_4 e^{-ky}\}$$

Since the excitation (field source) has the form of $J_m \sin(px)$, we would expect H (or $-\partial\varphi/\partial x$) variation with x of the same form as the excitation. Thus:

$k = p$ and $C_1 = 0$



The solution for regions 1 and 2 are given by:
Region 1, $y \geq 0$, $\varphi_1(x,y)$ must be finite as $y \rightarrow \infty$. This implies $C_3 = 0$. Now φ is further simplified to:

$$\varphi_1(x,y) = K_1(\cos px)e^{-py}$$

Region 2, $y < 0$, $\varphi_2(x,y)$ must be finite as $y \rightarrow -\infty$. This implies $C_4 = 0$, and φ is further simplified to:

$$\varphi_2(x,y) = K_2(\cos px)e^{py}$$

Use the boundary condition at $y = 0$:

$$B_{y1}\Big|_{y=0} = B_{y2}\Big|_{y=0}$$
$$H_{x2}\Big|_{y=0} - H_{x1}\Big|_{y=0} = J_m \sin(px)$$

$$H_{x1} = -\frac{\partial \varphi_1}{\partial x} = K_1 p \sin(px)e^{-py}$$
$$H_{y1} = -\frac{\partial \varphi_1}{\partial y} = K_1 p \cos(px)e^{-py}$$
$$H_{x2} = -\frac{\partial \varphi_2}{\partial x} = K_2 p \sin(px)e^{py}$$
$$H_{y2} = -\frac{\partial \varphi_2}{\partial y} = -K_2 p \cos(px)e^{py}$$

At $y = 0$

$$B_{y1} = \mu_0 H_{y1} = \mu_0 K_1 p \cos(px) = B_{y2} = \mu_0 \mu_r H_{y2} = -\mu_0 \mu_r K_2 p \cos(px)$$
or
$$K_1 + \mu_r K_2 = 0$$

$$H_{x2} - H_{x1} = K_2 p \sin(px) - K_1 p \sin(px) = J_m \sin(px)$$
or
$$K_2 - K_1 = J_m / p$$

$$\therefore \qquad K_2 = \frac{J_m}{p(1+\mu_r)} \quad ; \quad K_1 = -\frac{\mu_r J_m}{p(1+\mu_r)}$$

and

$$\varphi_1(x,y) = \frac{-\mu_r J_m}{p(1+\mu_r)}\cos(px)e^{-py}$$
$$\varphi_2(x,y) = \frac{J_m}{p(1+\mu_r)}\cos(px)e^{py}$$

At any point (x, y) in regions 1 and 2, the x and y components of ***B*** are given by:

$$B_{x1} = -\mu_0 \frac{\partial \varphi_1}{\partial x} = \frac{-\mu_0 \mu_r J_m}{(1+\mu_r)} \sin(px) e^{-py}$$

$$B_{y1} = -\mu_0 \frac{\partial \varphi_2}{\partial y} = \frac{-\mu_0 \mu_r J_m}{(1+\mu_r)} \cos(px) e^{-py}$$

$$H_{x2} = -\frac{\partial \varphi_2}{\partial x} = \frac{J_m}{(1+\mu_r)} \sin(px) e^{py}$$

$$H_{y2} = -\frac{\partial \varphi_2}{\partial y} = \frac{-J_m}{(1+\mu_r)} \cos(px) e^{py}$$

$$B_{x2} = \mu_0 \mu_r H_{x2} = \frac{\mu_0 \mu_r J_m}{(1+\mu_r)} \sin(px) e^{py}$$

$$B_{y2} = \mu_0 \mu_r H_{y2} = \frac{-\mu_0 \mu_r J_m}{(1+\mu_r)} \cos(px) e^{py}$$

It should be noted that if the relative permeability μ_r in region 2 is very large, which is often the case with ferromagnetic materials such as silicon steel, $(\mu_r + 1) >> 1$, $H_{y2} << H_{y1}$, and H_{y2} can be approximate to zero, which leads to the same result with the vector magnetic potential approach in the lecture note pp. 3-9 ~ 3-10.

9. This question is very similar to example 2 in section 3.6, except for the boundary conditions

(a) Boundary conditions at $r = R_2$:

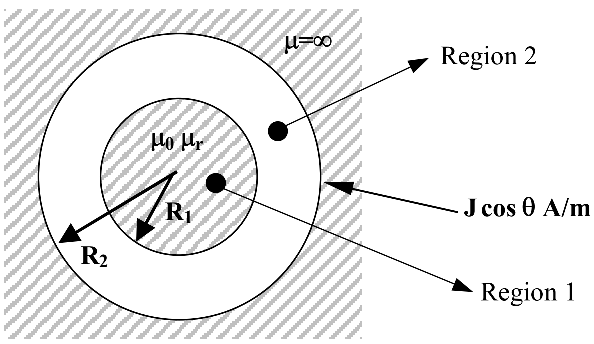
$$H_{\theta 2} = J \cos\theta$$

Interface conditions at $r = R_1$:

$$H_{\theta 1} = H_{\theta 2} \quad ; \quad B_{r1} = B_{r2}$$

(b) Field Solution

Use the scalar magnetic potential, the general solution for Laplace's equation for both regions is
$\varphi = (K_1 \sin k\theta + K_2 \cos k\theta)(K_3 r^k + K_4 r^{-k})$
Since the source of field varies as $J \cos\theta$, and $H_\theta = \frac{-1}{r}\frac{\partial \varphi}{\partial \theta}$
Set $k = 1$ and $K_2 = 0$. The solutions for regions 1 and 2 are given:



In **region** (1): $r < R_1$

$$\varphi_1=(C_1\, r + C_2 r^{-1}) \sin\, \theta$$

In **region** (2): $r > R_1$

$$\varphi_2 = (C_3\, r + C_4\, r^{-1}) \sin\, \theta$$

To find the 4 unknown constants, 4 boundary/interface conditions are applied

Boundary Conditions

(i) At r = 0, $\varphi_1 = 0$, $\therefore C_2 = 0$

$$\therefore \varphi_1 = C_1\, r \sin\, \theta$$

(ii) At $r = R_2$,

$H_{\theta 2} = J \cos\, \theta$ **Note:** $H_\theta = -\nabla\varphi\big|_{r=R_2} = -\dfrac{1}{r}\dfrac{\partial\varphi}{\partial\theta}$

$$\therefore -(C_3 + C_4 \frac{1}{R_2^2}) \cos\theta = J\, \cos\theta$$

$$\therefore -(C_3 + C_4 \frac{1}{R_2^2}) = J \qquad \textbf{(1)}$$

(iii) At $r = R_1$ $B_{r1} = B_{r2}$ $B_r = -\mu \dfrac{\partial\varphi}{\partial r}$

$$\therefore -\mu_0\mu_r \frac{\partial\varphi_1}{\partial r} = -\mu_0 \frac{\partial\varphi_2}{\partial r}$$

$$\therefore -\mu_0\mu_r\, C_1 \sin\, \theta = (-\mu_0\, C_3 + \mu_0\, C_4\, R_1^{-2}) \sin\, \theta$$

$$\therefore\ -\mu_r\, C_1 = -C_3 + C_4\, R_1^{-2} \qquad \textbf{(2)}$$

(iv) At $r = R_1$, $H_{\theta 1} = H_{\theta 2}$ $H_\theta = \dfrac{-1}{r}\dfrac{\partial\varphi}{\partial\theta}$

$$\therefore \frac{C_1}{R_1} R_1\, \cos\theta = C_3 \frac{R_1}{R_1}\cos\theta + \frac{C_4}{R_1{}^2}\cos\theta$$

$$\therefore C_1 = C_3 + \frac{C_4}{R_1{}^2} \qquad \textbf{(3)}$$

Solving for C_1, C_3 and C_4

$$C_1 = \frac{-2(1+\mu_r)J}{(1-\mu_r)R_1^2\left[\frac{1+\mu_r}{R_1^2}+\frac{1-\mu_r}{R_2^2}\right]}$$

$$C_3 = \frac{-(1+\mu_r)J}{R_1^2\left[\frac{1+\mu_r}{R_1^2}+\frac{1-\mu_r}{R_2^2}\right]}$$

$$C_4 = \frac{-(1-\mu_r)J}{\left[\frac{1+\mu_r}{R_1^2}+\frac{1-\mu_r}{R_2^2}\right]}$$

Finally,

$$\varphi_2 = (C_3\, r + C_4\, r^{-1}) \sin\theta$$

$$\varphi_2 = \frac{-J}{\left[\frac{1+\mu_r}{R_1^2}+\frac{1-\mu_r}{R_2^2}\right]}\left(\left[\frac{(1+\mu_r)r}{R_1^2}\right]+\left[\frac{(1-\mu_r)}{r}\right]\right)\sin\theta$$

Note: A useful method to verify your solution is to check if it satisfies the boundary/interface conditions