Solutions EEE309 (2015-2016)

Q1 a.

i)
Causality
For a causal system, its output at time index *n* depends only on the values of the input at *n* and earlier time instants.
(1 mark)
Stability
For every bounded input sequence, the system produces a bounded output sequence.
A bounded input x[n]: $|x[n]| \leq B_x < \infty$; a bounded output y[n]: $|y[n]| \leq B_y < \infty$, where B_x and B_y are fixed with a finite positive value.
(1 mark)
Linearity
It is defined by the principle of superposition which has two requirements for the system to meet. Suppose the sequence $y_1[n]$ is the response of the system to the input sequence $x_1[n]$, and $y_2[n]$ the response to $x_2[n]$, then we have
additivity property:
$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$
scaling property:
$$T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n]$$
(2 marks)
Time invariance
A time shift or delay of the input sequence causes a corresponding shift in the output sequence. Given the input and output sequence x[n] and y[n], for all n_0, the input sequence with values $x_1[n]=x[n-n_0]$ produces the output sequence with values $y_1[n]=y[n-n_0]$.
(1 mark)

ii)
Non-causal, stable, linear time-invariant. (3 marks, one mark for each)

Q1 b.

i) The equivalent system impulse response h[n] can be obtained by
$h[n]= (h_2[n]+h_3[n])*h_1[n]$ and the result is
(3 marks)

h[0]=1; h[1]=h[2]=3; h[3]=1. It is zero for other values of *n*.
(3 marks)

ii) For $\Omega = 0$
the gain is h[0]+h[1]+h[2]+h[3]=8;

For $\Omega=\pi$
the gain is h[0]-h[1]+h[2]-h[3]=0.
(2 marks, one mark for each case)

Q1 c.

Nyquist Sampling Theorem:
Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\omega) = 0 \qquad for \quad |\omega| \geq \omega_N$$

Then $x_c(t)$ is uniquely determined by its samples $x[n]=x_c(nT)$, where T is the sampling period, if

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_N$$

The frequency ω_N is commonly referred to as the Nyquist frequency, and the frequency $2\omega_N$ that must be exceeded by the sampling frequency is called the Nyquist rate.
(3 marks)

The sampling frequency must be at least twice the highest frequency of interest, i.e. 2X100π/(2π)=100Hz.
(1 marks)

Q2a.
i)
The unit sample sequence is defined as

$$\delta[n] = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$$

(1 mark)
The unit step sequence is given by

$$u[n] = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0. \end{cases}$$

(1 mark)
ii)
Two expressions:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k]$$

(1 mark)

$$u[n] = \delta[n] + \delta[n-1] + \delta[n-2] + \cdots = \sum_{k=0}^{\infty} \delta[n-k]$$

(1 mark)

Q2b.

i)
Suppose the input is given by x[n]. Then the LCCD equation describing the input-output relationship is given by

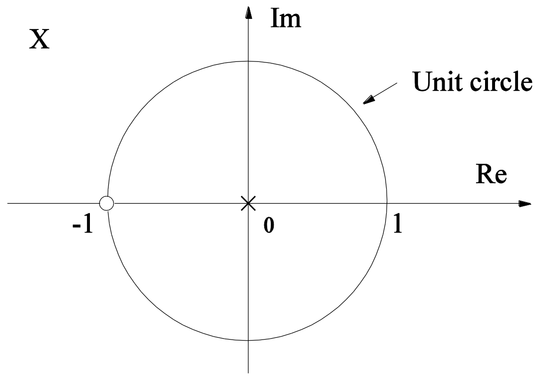
$$y[n] = \frac{1}{2}x[n] + \frac{1}{2}x[n-1]$$

(2 marks)

ii)

z-transform: $H(z) = \frac{1}{2} + \frac{1}{2} z^{-1} = \frac{z+1}{2z}$

one zero at z=-1
one pole at z=0



(4 marks)

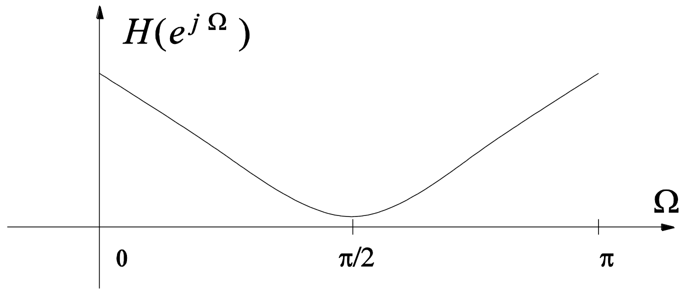
iii)
It is not a minimum-phase system. A minimum-phase system must have all its poles and zeros inside the unit circle. The above system has a zero on the unit circle.
(3 marks)

Q2 c.

Its transfer function H(z) can be obtained by placing the zeros in the numerator and the poles in the denominator:

$$H(z) = C\frac{(z-0.9j)(z+0.9j)}{(z-0.2)(z+0.2)} = C\frac{z^2+0.81}{z^2-0.04} = C\frac{1+0.81z^{-2}}{1-0.04z^{-2}}$$

where C is an unknown constant and can not be determined by the pole-zero plot.
(3 marks)



(3 marks)
It has a bandstop characteristic.
(1 mark)

Q3(a)

The z-transform of h[n] is given by
$H(z)=1+3z^{-1}-z^{-2}$
The z-transform of x[n] is given by
$X(z)=1+3z^{-1}-z^{-2}+3z^{-3}$
(2 marks)

The z-transform Y(z) of the output y[n] is therefore
$Y(z)=X(z)H(z)=1+6z^{-1}+7z^{-2}-3z^{-3}+10z^{-4}-3z^{-5}$
(2 marks)

Then y[n] is given by
x[n]=δ[*n*]+6([n-1]+7([n-2]-3([n-3]+10([n-4]-3([n-5].
(1 mark)

Q3 b.

The DFT of a sequence x[n] with length N is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

(2 marks)
The IDFT is given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j(2\pi/N)kn}$$

(2 marks)

For x[n]={.5, 1, 1, .5}, N=4, then

$$X[k] = \sum_{n=0}^{3} x[n]e^{-jkn\frac{\pi}{2}} = 0.5 + e^{-jk\frac{\pi}{2}} + e^{-jk\pi} + 0.5e^{-jk\frac{3\pi}{2}}$$

(1 mark)
X[0]=3, X[1]=-.5-.5j, X[2]=0, X[3]=-.5+.5j
(1 mark)

Q3 c.

Using linear convolution, the third sequence $x_3[n]$ is given by

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

The product $x_1[m]x_2[n-m]$ is zero for all *m* whenever n<0 and n>L+P-2. Therefore, (L+P-1) is the maximum length of the sequence $x_3[n]$.
(1 mark)

To calculate $x_3[n]$ using DFT, we first form the N-point sequence $\hat{x}_1[n]$ by adding N-L zeros to the L-points sequence $x_1[n]$ and the N-point sequence $\hat{x}_2[n]$ by adding N-P zeros to the P-points sequence $x_2[n]$ (N=L+P-1).
(1 mark)

Then we calculate the DFT $X_1[k]$ and $X_2[k]$ of $\hat{x}_1[n]$ and $\hat{x}_2[n]$ for k=0, 1, …, N-1. The product of the two DFTs is given by $X_3[k] = X_1[k]X_2[k]$ with a length of N. Applying the inverse DFT to $X_3[k]$, we then obtain the desired sequence $x_3[n]$.
(3 marks)

Q3 d.

Impulse invariance method:
Inverse Laplace transform

$$h_a(t) = 5e^{-5t}$$

At sampling instants nT (T=1/8 sec), we have

$$h_a(nT) = 5e^{-\frac{5}{8}n} = 5 \times 0.535^n$$

(2 marks)
From z-transform table, we have

$$H_d(z) = \frac{5z}{z - 0.535}$$

(2 marks)

Q4 a.

Suppose the impulse response of the system is h[n]. Then given the input $x[n]=\alpha^n$, its output y[n] is given by

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\alpha^{(n-k)}$$

$$= \alpha^n \sum_{k=-\infty}^{+\infty} h[k]\alpha^{-k} = \beta x[n]$$

where $\beta = \sum_{k=-\infty}^{+\infty} h[k]4^{-k}$ is a scalar.
(3 marks)

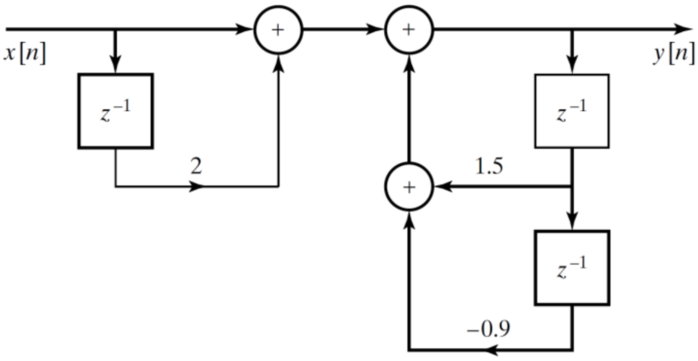
So α^n is the eigenfunction of the system.
(1 mark)

Q4 b.

For the system function

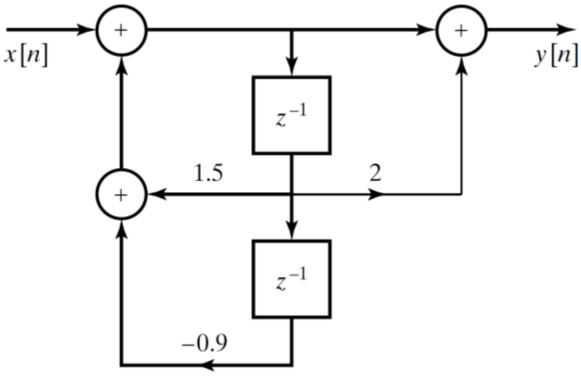
$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

Its direct form I implementation is



(2 marks)

Its direct form II implementation is

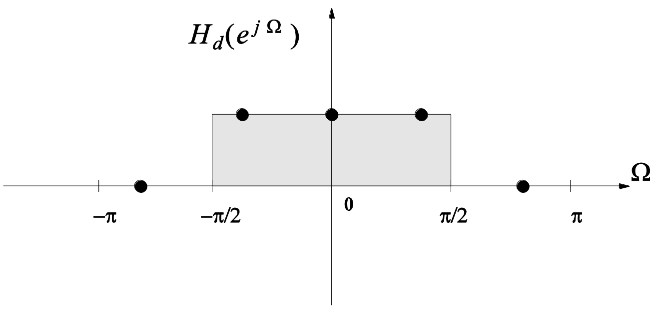


(2 marks)

Q4c.

The passband range between 0.5kHz and 1kHz corresponds to the normalised frequency Ω_l=0.5*2π/2=π/2 to Ω_h=1*2π/2=π

As the spectrum is symmetric with respect to the origin, we can then design a lowpass filter between –π/2 to π/2 first. The ideal frequency response of the corresponding lowpass filter is given by



We approximate the ideal one by 5 equally spaced samples, each 2π/5=1.26 rad apart.

(2 mark)

Using the provided equation, we have

h[0]=0.6, h[1]=0.3236, h[2]=-0.1236, h[3]=-0.1236, h[4]=0.3236.

(2 marks)

Note that the relationship between the impulse response $h_{hp}[n]$ of the highpass filter and the impulse response $h_{lp}[n]$ of the lowpass filter is given by $h_{hp}[n]=(-1)^n h_{lp}[n]$, then we have the final design result for the desired highpass filter:

h[0]=0.6, h[1]=-0.3236, h[2]=-0.1236, h[3]=0.1236, h[4]=0.3236.

(2 marks)

Q4 d.

To derive this property, we consider the new sequence

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

(2 marks)

Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}$$

(2 marks)

Interchange the order of summation, we have

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$$

Changing the index of summation in the second sum from n to m=n-k, we have

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k}$$

Then for values of z inside the ROCs of both $X_1(z)$ and $X_2(z)$, we have

$$Y(z) = X_1(z)X_2(z)$$

(2 marks)