3. Magnetostatic Fields

While stationary charges produce static electric fields, steady (no time varying) currents produce magnetic fields. For $\partial / \partial t = 0$, the magnetic fields in a medium with magnetic permeability μ are governed by the second pair of Maxwell's equations:

$$\left.\begin{aligned} \nabla \bullet \vec{B} &= 0 \\ \nabla \times \vec{H} &= \vec{J} \end{aligned}\right\} \quad \textit{magnetostatics} \qquad \begin{aligned} &(3.1a) \\ &(3.1b) \end{aligned}$$

where $\boldsymbol{J}$ is the current density. The magnetic flux density $\boldsymbol{B}$ and the magnetic field intensity $\boldsymbol{H}$ are related by:

$$\vec{B} = \mu \vec{H} \tag{3.2}$$

With the exception of ferromagnetic materials, such as pure iron, mild steel, silicon steel, etc., for which the relationship between $\boldsymbol{B}$ and $\boldsymbol{H}$ is non-linear, most material are characterised by constant magnetic permeability. Furthermore, $\mu = \mu_0$ for most dielectrics and metals (excluding ferromagnetic materials). However, ferromagnetic materials are often used in most electrical machineries and actuators, and the magnetic fields we have to deal with in these devices are fundamentally non-linear. Our objective in this chapter is to develop an understanding of the relationships between steady currents and the magnetic fields $\boldsymbol{B}$ and $\boldsymbol{H}$ for various types of current distributions and in various types of media and to introduce a number of related quantities, such as the magnetic vector potential $\boldsymbol{A}$, etc, as well as techniques for magnetic field calculations.

3.1 Ampere's Law

Ampere's Law states that the line integral of magnetic field strength H around a closed path is equal to the total current enclosed, i.e.,

$$\oint_C \vec{H} \cdot d\vec{l} = I \tag{3.3}$$

where c is the closed contour and I is the total current enclosed by c. By way of illustration, for both configuration shown in Fig. 3.1 (a) and (b), the line integral of $\boldsymbol{H}$ is equal to the current I, even though the paths have very different shapes and the magnitude of $\boldsymbol{H}$ is not uniform along the path of configuration (b). By the same token, because path (c) in Fig. 3.1 does not enclose the current I, its line integral of $\boldsymbol{H}$ is identically zero, even though $\boldsymbol{H}$ is not zero along the path.

Clearly Ampere's Law is very useful in calculating the field around a wire as we shall see its application in the following example: *magnetic field of a long wire*.

A long (practically infinite) straight wire of radius a carries a steady current I that is uniformly distributed over the cross section of the wire. Determine the magnetic field $\boldsymbol{H}$ a distance r from the axis of the wire both (a) inside the wire ($r \leq a$) and (b) outside the wire ($r > a$).

Solution:

(a) We choose I to be along the z direction as shown in Fig. 3.2. To determine $\boldsymbol{H}_1$ at a distance ($r \le a$), we choose the Ampere's contour C_1 to be a circular path of radius r_1 as shown in Fig 3.2.

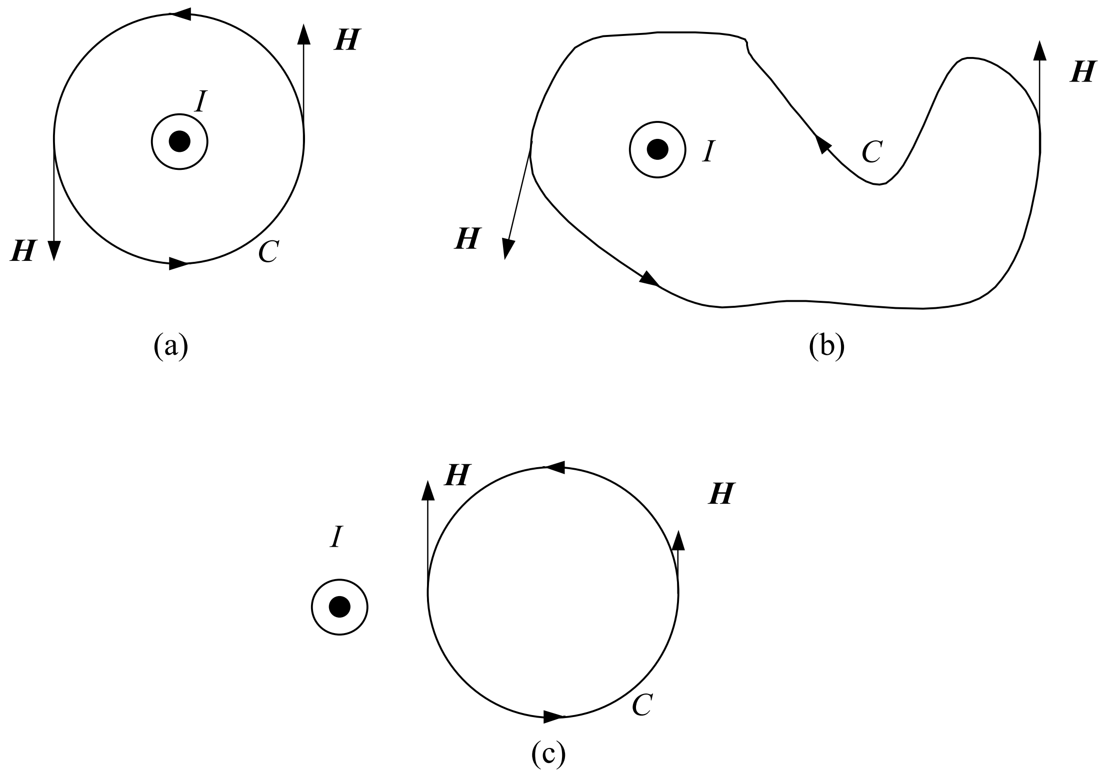


Fig. 3.1 Illustration of Ampere's Law

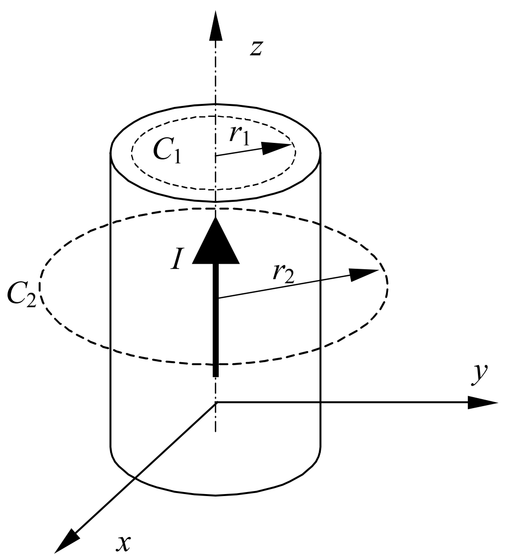


Fig. 3.2 Application of Ampere's law to a straight, infinitely long wire

In this case, Ampere's Law takes the form

$$\oint_{C_1} \vec{H}_1 \cdot d\vec{l}_1 = I_1$$

where I_1 is the fraction of the total current I flowing through area enclosed by C_1. From symmetry, $\boldsymbol{H}_1$ must be constant in magnitude and parallel to the contour at any point along the path. Furthermore, to satisfy the right-hand rule and given that I is alone the z-direction, $\boldsymbol{H}_1$ must be along the θ direction in a cylindrical co-ordinates. Hence $\boldsymbol{H}_1 = H_1\mathbf{e}_\theta$. With $d\boldsymbol{l}_1 = r_1 d\theta\, \mathbf{e}_\theta$, the left-hand side of Ampere's law gives

$$\oint_{C_1} \vec{H}_1 \cdot d\vec{l}_1 = \int_0^{2\pi} H_1 (e_\theta \cdot e_\theta) r_1 d\theta = 2\pi r_1 H_1$$

The current I_1 flowing through the area enclosed by C_1 is equal to the total current I multiplied by the ratio of the area enclosed by C_1 to the total cross-sectional area of the wire:

$$I_1 = \left(\frac{\pi r_1^2}{\pi a^2}\right) I = \left(\frac{r_1}{a}\right)^2 I$$

Equating both sides of Ampere's law and then solving for H_1 leads to:

$$\vec{H}_1 = H_1 e_\theta = \frac{r_1}{2\pi a^2} I\, e_\theta \qquad (\text{ for } r_1 \le a)$$

(b) For ($r > a$), we choose path C_2, which encloses all the current I. Hence,

$$\oint_{C2} \vec{H}_2 \cdot d\vec{l}_2 = \int_0^{2\pi} H_2 (e_\theta \cdot e_\theta) r_2 d\theta = 2\pi r_2 H_2 = I$$

and

$$\vec{H}_2 = H_2 e_\theta = \frac{I}{2\pi r_2} e_\theta \qquad (\text{ for } r_2 > a)$$

3.2 The Curl Operator

This is in essence the application of Ampere's law at a point rather than around a closed loop. If we consider Ampere's law for an elemental plane area ΔS, and divide by the area enclosed by the path, ΔS

$$\oint_C \frac{\vec{H} \cdot d\vec{l}}{\Delta S} = \frac{\Delta I}{\Delta S} \tag{3.4}$$

In the limit $\dfrac{\Delta I}{\Delta S}$ is the current density at the point of interest. The left-hand side of the above expression is defined as the curl of $\boldsymbol{H}$, i.e.,

$$\lim_{\Delta S \to 0} \oint_C \frac{\vec{H} \cdot d\vec{l}}{\Delta S} = curl\, \vec{H} \tag{3.5}$$

Note curl ***H*** is a vector, and its direction is normal to ΔS in the direction of the current density vector ***J***. Given this general definition of curl, we can deduce its operation for vectors in Cartesian co-ordinates.

With reference to Fig. 3.3, if a uniform current density J_x flows in the x direction, then from Ampere's law with due account of the sense of the integration path:

$$\oint \vec{H} \cdot d\vec{l} = H_y \Delta y + H_z \Delta z + \frac{\partial H_z}{\partial y}\Delta y \Delta z - H_y \Delta y - \frac{\partial H_y}{\partial z}\Delta z \Delta y - H_z \Delta z = J_x \Delta y \Delta z$$

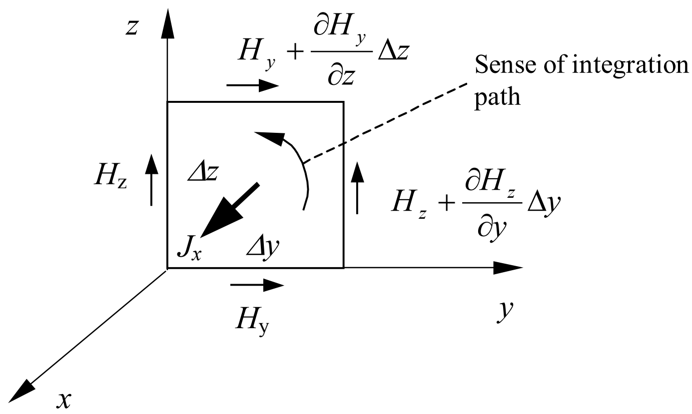


Fig. 3.3 Application of Ampere's law to an elemental plane

Dividing by the area of the elemental plane:

$$\oint \frac{\vec{H} \cdot d\vec{l}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x$$

In the limit as ΔyΔz → 0, this equals curl ***H*** in the x direction, i.e.,

$$(Curl\, \vec{H})_x = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \tag{3.6}$$

If there are components of currents in the other directions, there will also be a curl in these directions. The expression for curl_y and curl_z can be derived in an identical manner. The complete expression for curl ***H*** is:

$$\begin{aligned} Curl\, \vec{H} &= \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)e_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)e_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)e_z \\ &= J_x e_x + J_y e_y + J_z e_z = \vec{J} \end{aligned} \tag{3.7}$$

which can also be written as:

$$Curl\, \vec{H} = \vec{J} \tag{3.8}$$

Curl ***H*** has a value whenever current is present. Curl ***H*** is often expressed in vector notation as the cross product of the del ∇ operator and ***H***, i.e.,

$$\nabla = \frac{\partial}{\partial x}e_x + \frac{\partial}{\partial y}e_y + \frac{\partial}{\partial z}e_z$$

$$\vec{H} = H_x e_x + H_y e_y + H_z e_z$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)e_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)e_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)e_z$$

Thus

$$\nabla \times \vec{H} = \vec{J} \tag{3.9}$$

The curl operation can be expressed in determinant form of:

$$\nabla \times \vec{H} = \begin{vmatrix} e_x & e_y & e_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ H_x & H_y & H_z \end{vmatrix} \tag{3.10}$$

3.3 Scalar and Vector Magnetic Potentials

<u>Scalar magnetic potential</u>

For the region in which there is no current present, the governing equation for magnetic field distribution reduces to:

$$\left.\begin{aligned} \nabla \bullet \vec{B} &= 0 \\ \nabla \times \vec{H} &= 0 \end{aligned}\right\} \tag{3.11}$$

Since for any scalar function φ, the curl of its gradient is always zero, or $\nabla \times (\nabla \varphi) \equiv 0$, and from the second equation of (3.11) $\nabla \times \vec{H} = 0$, it is evident that ***H*** can be described in terms of a scalar magnetic potential φ, i.e.,

$$\vec{H} = -\nabla \varphi \tag{3.12}$$

Since $\nabla \cdot \vec{B} = 0$ and $\vec{B} = \mu \vec{H}$, the scalar magnetic potential φ will be governed by:

$$\nabla \cdot (\mu \vec{H}) = \nabla \cdot (-\mu \nabla \varphi) = 0 \tag{3.13}$$

or

$$\frac{\partial}{\partial x}\left(\mu \frac{\partial \varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\mu \frac{\partial \varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\mu \frac{\partial \varphi}{\partial z}\right) = 0 \tag{3.14}$$

For linear problems, μ = constant, and equation (3.13) can be simplified to:

$$\nabla^2 \varphi = 0 \tag{3.15}$$

That is, the scalar magnetic potential φ satisfies Laplace's equation. By introducing a scalar magnetic potential, we can solve the magnetic field problem associated with vector quantities (***B*** and ***H***) using equations (3.14) or (3.15). This greatly reduces the complexity of solutions.

Vector magnetic potential

However, the concept of scalar magnetic potential is not applicable to current carrying regions, in which $\nabla \times \vec{H} = \vec{J}$.

On the other hand, it can be shown that for any vector function ***F***:

$$\nabla \cdot (\nabla \times \vec{F}) = 0$$

i.e., the divergence of the curl is always zero. There are no isolated magnetic charges, i.e., for every N pole there is an S pole. Flux does not "disappear" nor "emerge" from a point. Hence the divergence of the magnetic flux density is always zero, or $\nabla \cdot \vec{B} = 0$. Consequently, ***B*** can be expressed as the curl of another vector function ***A*** since its divergence is zero, i.e.,

$$\vec{B} = \nabla \times \vec{A} \tag{3.16}$$

A is known as vector magnetic potential. Since $\vec{H} = \vec{B} / \mu$ and

$$\nabla \times \vec{H} = \nabla \times (\vec{B} / \mu) = \nabla \times \left(\frac{1}{\mu} \nabla \times \vec{A} \right) = \vec{J} \tag{3.17}$$

For constant μ, equation (3.17) becomes:

$$\nabla \times \left(\nabla \times \vec{A} \right) = \mu \vec{J} \tag{3.18}$$

but from the vector differential identity:

$$\nabla \times \left(\nabla \times \vec{A} \right) = \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

By specifying $\nabla \cdot \vec{A} = 0$ known as Coulomb Gauge, we have:

$$\nabla^2 \vec{A} = -\mu \vec{J} \tag{3.19}$$

In Cartesian co-ordinates, the above equation represents the vector sum of three scalar equations of the following form:

$$
\begin{aligned}
\nabla^2 A_x &= -\mu J_x \\
\nabla^2 A_y &= -\mu J_y \\
\nabla^2 A_z &= -\mu J_z
\end{aligned}
\tag{3.20}
$$

each of which has the same form as Poisson's equation. Since A is a vector, the solution to magnetic field at a given point from specific boundary conditions is more complex, and can involve the solution of up to three equations, depending on the existence of current in all three dimensions.

3.4 Vector differential operations in Cylindrical system

When solving an electromagnetic field problem, the geometry of field regions is of great importance. In many cases, it is advantageous to use the cylindrical co-ordinate system, in the case of electrical motors for example. Hence we also need the mathematical treatment of various vector differential operators in this co-ordinate systems. They are summaries below along with the operators in Cartesian co-ordinates for comparison. The derivation of these operators can be found in most engineering mathematics books.

<u>Cartesian Co-ordinates (x, y, z)</u>

$$\nabla\Phi = \frac{\partial\Phi}{\partial x}e_x + \frac{\partial\Phi}{\partial y}e_y + \frac{\partial\Phi}{\partial z}e_z$$

$$\nabla \bullet \vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)e_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)e_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)e_z$$

$$\nabla^2\Phi = \frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\vec{A} = \nabla^2 A_x e_x + \nabla^2 A_Y e_y + \nabla^2 A_z e_z$$

<u>Cylindrical Co-ordinates (r, θ, z)</u>

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}e_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}e_\theta + \frac{\partial\Phi}{\partial z}e_z$$

$$\nabla \bullet \vec{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial\theta} + \frac{\partial D_z}{\partial z}$$

$$\nabla \times \vec{H} = \left[\frac{1}{r}\frac{\partial H_z}{\partial\theta} - \frac{\partial H_\theta}{\partial z}\right]e_r + \left[\frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r}\right]e_\theta + \left[\frac{1}{r}\frac{\partial(rH_\theta)}{\partial r} - \frac{1}{r}\frac{\partial H_r}{\partial\theta}\right]e_z$$

$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{\partial^2\Phi}{\partial r^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}$$

$$\nabla^2\vec{A} = \left(\nabla^2 A_r - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{A_r}{r^2}\right)e_r + \left(\nabla^2 A_\theta + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{A_\theta}{r^2}\right)e_\theta + \left(\nabla^2 A_z\right)e_z$$

3.5 Boundary Conditions:

In order to obtain a unique solution to the partial differential equation (pde) which governs the field distribution, boundary conditions must be imposed by specifying:

(i) the value of the function (φ, **A, H** etc) whose distribution is to be determined, and/or

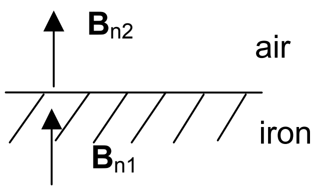
(ii) the normal derivative of the function $\left(\partial\varphi/\partial n \text{ etc}\right)$ whose distribution is to be determined.

These conditions enable a unique solution to the pde to be obtained.

Interface Conditions:

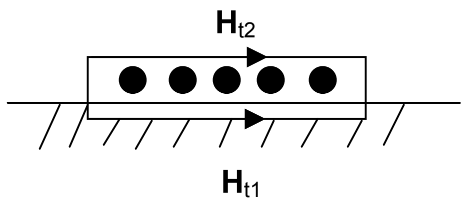
At the interface between different regions (e.g. iron, air) in the bounded domain, the following conditions must be satisfied.

(i) Continuity of the normal component of **B** (follows from div **B** = 0)



$$\mathbf{B}_{n1} = \mathbf{B}_{n2}$$

(ii) Discontinuity in tangential **H**, if a current sheet **J** (A/m) exists at the interface (follows from Amperes Law)



$$\oint \vec{H}\cdot d\vec{l} = I$$

$$\mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{J}$$

3.6 Field Calculation

Example 1

Sinusoidal current sheet on an infinitely permeable iron surface (e.g. a "snapshot" of the field produced by the stator of a linear induction motor, such as a MAGLEV train), find the magnetic field distribution above the surface.

Solution: Assume that the current sheet flows in the z direction, and the infinitely permeable surface lies on the plane where $y = 0$, as shown in Fig. 3.4.

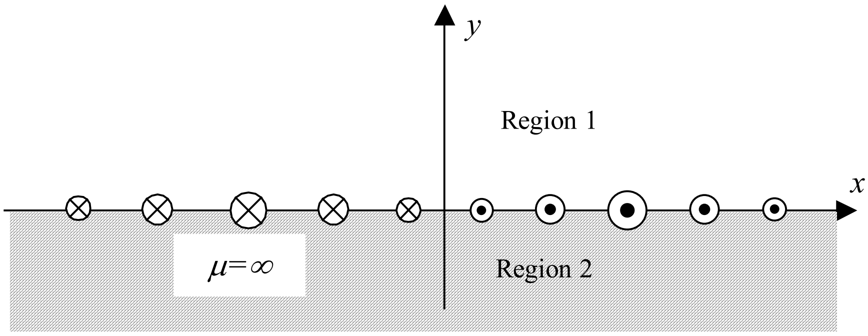


Fig. 3.4 Field region of sinusoidal current sheet on an infinitely permeable iron surface

The current sheet distribution can be represented by:

$J_{zs} = J_m \sin(px)$ (A/m)

Since only J_z is present, only A_z exists, i.e., the field distribution is two-dimensional. If we can find the solution for A_z, the flux density component can be obtained by:

$$B_x = \frac{\partial A_z}{\partial y} \quad ; \quad B_y = -\frac{\partial A_z}{\partial x}$$

In the region where y > 0, there is no current present and therefore, A_z satisfies the Laplace's equation:

$$\frac{\partial^2 A_z}{\partial x^2} + \frac{\partial^2 A_z}{\partial y^2} = 0$$

Assume solution takes the form:

$$A_z(x, y) = F(x) \bullet G(y)$$

Where F is a function of x only, and G is a function of y only. Thus:

$$G\frac{d^2 F}{ax^2} + F\frac{d^2 G}{dy^2} = 0$$

Dividing both sides by $\{-F(x)G(y)\}$ gives:

$$-\frac{1}{F}\frac{d^2 F}{dx^2} = \frac{1}{G}\frac{d^2 G}{dy^2} = k^2$$

where k^2 is known as separation constant, and standard solutions are:

$$F(x) = C_1 \sin(kx) + C_2 \cos(kx)$$
$$G(y) = C_3 e^{ky} + C_4 e^{-ky}$$

where C_1, C_2, C_3 and C_4 are constants to be determined. Hence:

$$A_z(x, y) = \{C_1 \sin(kx) + C_2 \cos(kx)\}\{C_3 e^{ky} + C_4 e^{-ky}\}$$

Since the excitation (field source) has the form of $J_m \sin(px)$, we would expect an A_z variation with x of the same form as the excitation. Thus:

$k = p$ and $C_2 = 0$

Also A_z must be finite as y goes infinite. This implies $C_3 = 0$. Now A_z is further simplified to:

$$A_z(x, y) = C \sin(px) e^{-py}$$

where C_1 and C_4 are merged into one constant C. To determine the value of C, we need to use the boundary condition at $y = 0$:

$$H_{x1} - H_{x2} = J_m \sin(px)$$

but since $\mu = \infty$ in the region 2, there can be no x component of H, hence $H_{x2} = 0$. Thus the boundary condition at $y = 0$ is given by:

$$H_{x1}\big|_{y=0} = J_m \sin(px)$$

$$H_{x1} = B_{x1} / \mu_0 = \frac{1}{\mu_0}\frac{\partial A_z}{\partial y} = -\frac{C}{\mu_0} p e^{-py} \sin(px)$$

Applying the boundary condition at $y = 0$ yields:

$$-\frac{C}{\mu_0} p \sin(px) = J_m \sin(px) \quad or \quad C = -\mu_0 J_m / p$$

and

$$A_z(x, y) = -\frac{\mu_0 J_m}{p} \sin(px) e^{-py}$$

At any point (x, y) in region 1, the x and y components of $\boldsymbol{B}$ are given by:

$$B_x = \frac{\partial A_z}{\partial y} = \mu_0 J_m \sin(px) e^{-py}$$

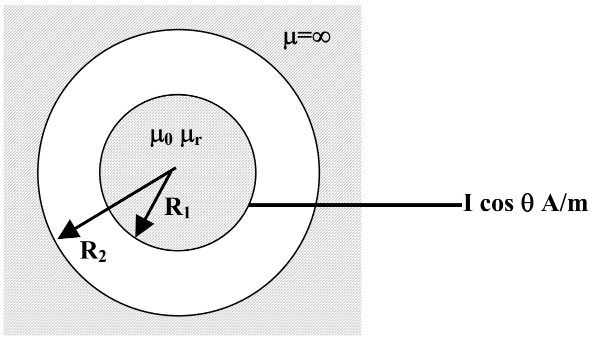
$$B_y = -\frac{\partial A_z}{\partial x} = \mu_0 J_m \cos(px) e^{-py}$$

The above field distribution can also be solved using scalar magnetic potential φ. See the tutorial question.

Example 2

The figure below shows an idealised representation of a 2-pole synchronous motor in which the stator iron is assumed to be infinitely permeable (i.e. $H_\theta = 0$ at $r = R_2$), and the rotor has a permeability $\mu_0\mu_r$ and carries a surface current distribution $I\cos\theta$ (A/m). The stator winding is unexcited.

Derive, by the method of separation of variables, an expression for the scalar magnetic potential distribution in the air-gap, and magnetic field distribution in the air-gap.



Solution

Let's denote the rotor and air-gap as regions 1 and 2. Since in both regions, there is no volume current density present, the ***H*** satisfies : $\nabla \times \vec{H} = 0$. Hence a scalar magnetic potential φ may be introduced:

$$\vec{H} = -\nabla \varphi$$

and it satisfies the Laplace's equation:

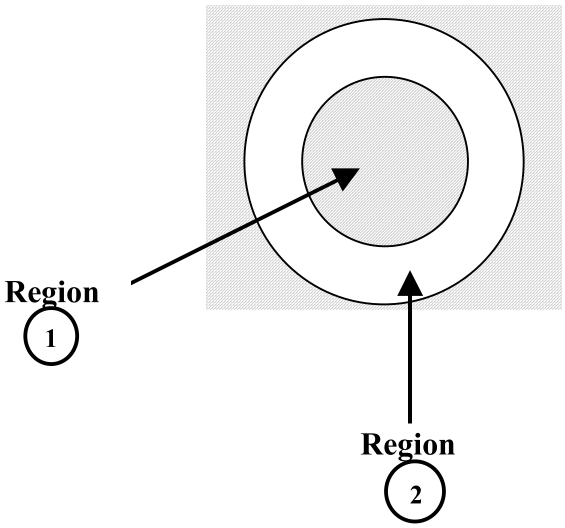
$$\nabla^2 \varphi = 0$$

$$\text{i.e. } \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0$$

Assume solution takes the form:

$$\varphi = F(r) \bullet G(\theta)$$

where F is a function of r only, and G is a function of θ only.



$$\therefore G\frac{d^2F}{dr^2}+\frac{G}{r}\frac{dF}{dr}+\frac{F}{r^2}\frac{d^2G}{d\theta^2}=0$$

Divide by $\dfrac{Gf}{r^2}$

$$\underbrace{\frac{r^2}{F}\frac{d^2F}{dr^2}+\frac{r}{F}\frac{dF}{dr}}_{\text{Function of } r \text{ only}}=\underbrace{-\frac{1}{G}\frac{d^2G}{d\theta^2}}_{\text{Function of } \theta \text{ only}}=k^2$$

Separation constant

Standard solutions are:

$F = K_1\, r^k + K_2\, r^{-k}$

$G = K_3 \sin k\theta + K_4 \cos k\theta$

General Solution:

$\varphi = (K_1 \sin k\theta + K_2 \cos k\theta)\,(K_3\, r^k + K_4\, r^{-k})$

Since source of field varies as $I\cos\theta$, and $H_\theta = \dfrac{-1}{r}\dfrac{\partial\varphi}{\partial\theta}$

Set $k = 1$ and $K_2 = 0$. The solutions for regions 1 and 2 are given:

Region 1 $\varphi_1 = (C_1\, r + C_2\, r^{-1}) \sin\theta$
Region 2 $\varphi_2 = (C_3\, r + C_4\, r^{-1}) \sin\theta$

Note: the constants are merged

Boundary/Interface Conditions

(i) At r = 0, $\varphi_1 = 0$

$\varphi_1 = (C_1\, r + C_2\, r^{-1}) \sin\,\theta$

$\therefore\ C_2 = 0$

$\therefore\ \varphi_1 = C_1\, r \sin\,\theta$

$\varphi_2 = (C_3\, r + C_4\, r^{-1}) \sin\,\theta$

(ii) At $r = R_1$, $B_{r1} = B_{r2}$ where $B_r = -\mu \dfrac{\partial\varphi}{\partial r}$

$$\therefore -\mu_0\mu_r \frac{\partial\varphi_1}{\partial r} = -\mu_0 \frac{\partial\varphi_2}{\partial r}$$

$\therefore -\mu_0\mu_r\, C_1 \sin\,\theta = (-\mu_0\, C_3 + \mu_0\, C_4\, R_1^{-2}) \sin\,\theta$

$\therefore\ -\mu_r\, C_1 = -C_3 + C_4\, R_1^{-2}$ **(1)**

(iii) At $r = R_1$, $H_{\theta 1} - H_{\theta 2} = I \cos\,\theta$, where $H_\theta = \dfrac{-1}{r}\dfrac{\partial\varphi}{\partial\theta}$

$$\therefore \frac{-C_1}{R_1} R_1 \cos\theta + C_3 \frac{R_1}{R_1}\cos\theta + \frac{C_4}{R_1{}^2}\cos\theta = I\, \cos\theta$$

$$\therefore -C_1 + C_3 + \frac{C_4}{R_1{}^2} = I \qquad \textbf{(2)}$$

(iv) At $r = R_2$, $H_{\theta 2} = 0 = \dfrac{-1}{r}\dfrac{\partial\varphi}{\partial\theta}$

$$\therefore -C_3 R_2 \cos\theta - \frac{C_4}{R_2{}^2}\cos\theta = 0$$

$\therefore\ C_3 = -C_4\, R_2^{-2}$ **(3)**

The 3 equations enable the 3 unknown constants to be determined.

$$C_1 = \frac{-I\left[\dfrac{1}{R_1^2} + \dfrac{1}{R_2^2}\right]}{\left[(1+\mu_r)\left(\dfrac{1}{R_2^2}\right) + (1-\mu_r)\left(\dfrac{1}{R_1^2}\right)\right]}$$

$$C_3 = \frac{-\mu_r I\left[\dfrac{1}{R_2^2}\right]}{\left[(1+\mu_r)\left(\dfrac{1}{R_2^2}\right)+(1-\mu_r)\left(\dfrac{1}{R_1^2}\right)\right]}$$

$$C_4 = \frac{\mu_r I}{\left[(1+\mu_r)\left(\dfrac{1}{R_2^2}\right)+(1-\mu_r)\left(\dfrac{1}{R_1^2}\right)\right]}$$

In the rotor region

$$\varphi_1 = \frac{-I}{\mu_r}\frac{\left[\dfrac{1}{R_2{}^2}+\dfrac{1}{R_1{}^2}\right]r\sin\theta}{\left[\dfrac{1}{R_2{}^2}\left(\dfrac{1}{\mu_r}-1\right)+\dfrac{1}{R_1{}^2}\left(\dfrac{1}{\mu_r}+1\right)\right]}$$

In the annular air-gap region

$$\varphi_2 = (C_3\, r + C_4\, r^{-1})\sin\theta = \frac{\mu_r I}{\left[(1+\mu_r)\left(\dfrac{1}{R_2^2}\right)+(1-\mu_r)\left(\dfrac{1}{R_1^2}\right)\right]}\left[\left(\frac{-r}{R_2^2}\right)+\frac{1}{r}\right]\sin\theta$$

$$B_{r2} = -\mu_0\frac{\partial\varphi_2}{\partial r} = \frac{\mu_0\mu_r I}{\left[(1+\mu_r)\left(\dfrac{1}{R_2^2}\right)+(1-\mu_r)\left(\dfrac{1}{R_1^2}\right)\right]}\left[\frac{1}{R_2^2}+\frac{1}{r^2}\right]\sin\theta$$

$$B_{\theta 2} = \frac{-\mu_0 1}{r}\frac{\partial\varphi_2}{\partial\theta} = \frac{\mu_0\mu_r I}{\left[(1+\mu_r)\left(\dfrac{1}{R_2^2}\right)+(1-\mu_r)\left(\dfrac{1}{R_1^2}\right)\right]}\left[\frac{1}{R_2^2}-\frac{1}{r^2}\right]\cos\theta$$

As can be seen, the flux density in the air-gap is sinusoidally distributed.

Tutorial Sheet 2

1. Calculate $\boldsymbol{J}$ if $\boldsymbol{H} = (3\,\boldsymbol{e_x} + 7y\,\boldsymbol{e_y} + 2x\,\boldsymbol{e_z})$ A/m.

   (ans $\boldsymbol{J} = -2\boldsymbol{e_y}$)

2. If $\boldsymbol{H} = 3y^3\,\boldsymbol{e_x}$ (A/m) find the current through a square area 5mm on each side, with corners at (0,3,0); (0,8,0); (5,8,0); (5,3,0).

3. The magnetic vector potential **A** in a region of space is given by $xyz\,\boldsymbol{e_x} + 4yx\,\boldsymbol{e_y} + 7\boldsymbol{e_z}$. Calculate $\boldsymbol{B}$ at the point (6, 1, 3)m

   (ans $\boldsymbol{B} = 6\,\boldsymbol{e_y} - 14\,\boldsymbol{e_z}$)

4. The magnetic field strength $\boldsymbol{H}$ in a region of space is given by $(7y\,\boldsymbol{e_x} + 7x\,\boldsymbol{e_y} + 7xy\,\boldsymbol{e_z})$ A/m. Are there current densities present at the points (2,1,0) and (0,0,0)?

5. Show that $\nabla \cdot (\nabla \times \vec{F}) = 0$, where $\vec{F}$ is any vector function.

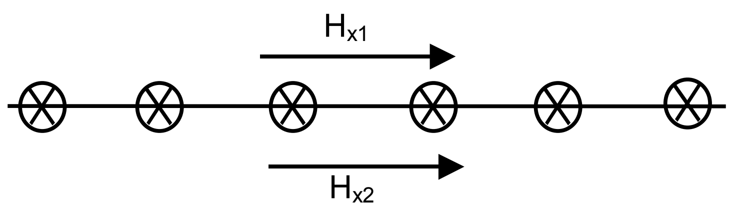
6. Shaw that $\nabla \times (\nabla \varphi) = 0$, where φ is any scalar function..

7. The figure below shows the interface between two regions 1 and 2 which has a surface current density. Calculate the magnitude of the surface current density at the point $x = 3$ if the magnetic field strength in the two materials is given by:

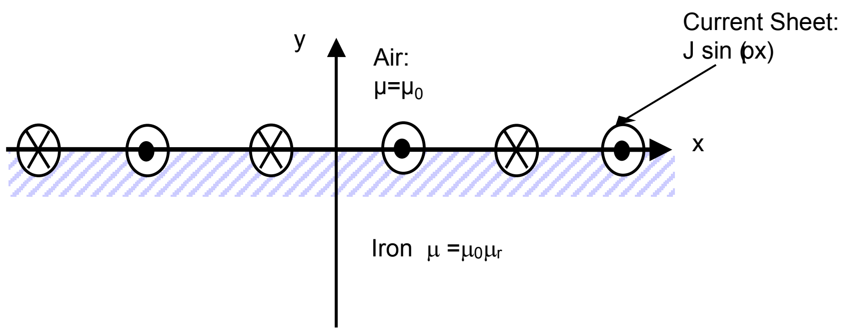
$$H_{x1} = 21.6\cos\frac{2\pi x}{6.3}$$

$$H_{x2} = 0.32\cos\frac{2\pi x}{6.3}$$

   Which of the regions is likely to be iron?



8. The figure below shows a schematic representation of the track section of a levitated rapid transport system. It consists of a sheet of finitely permeable iron ($\mu = \mu_0\mu_r$) on which there is a surface current sheet $J\sin(px)$ A/m. Starting from the appropriate governing magneto-static field equation, and listing any assumptions that you make, derive expressions for the x and y components of flux density in the air region above the track.



9. The figure below shows an idealised representation of a 2-pole synchronous motor in which the stator iron is assumed to be infinitely permeable and the rotor has a permeability $\mu_0\mu_r$. The inner bore of the stator carries a surface current distribution $J\cos\theta$ (A/m).
   (a) Determine the boundary condition at $r = R_2$ and the interface condition at $r = R_1$
   (b) Derive, by the method of separation of variables, an expression for the scalar magnetic potential distribution in the air-gap.