Tutorial 3: Solutions

1. Find the Fourier Transforms of the following signals:
(i) $x(t) = 1$ (use duality property)

Since we can't evaluate $\int_{-\infty}^{\infty} 1e^{-j\omega t}dt$, we will need to find an alternative.

Here we can use the duality property which states that if we have a Fourier Transform pair, $x(t) \longleftrightarrow X(\omega)$, we can derive a second Fourier Transform pair by interchanging the frequency and time parameters, that is changing ω to t and any constant in the time domain such as τ to the frequency domain constant such as W. Therefore we have $X(t) \longleftrightarrow 2\pi x(-\omega)$,

We know that $\delta(t) \leftrightarrow 1$. Using the duality property of Fourier Transform we have,
$\delta(t) \leftrightarrow 1$
$1 \leftrightarrow 2\pi\delta(-\omega) = 2\pi\delta(\omega)$.

(ii) $x(t) = e^{j\omega_o t}$ (use frequency shift property)
The frequency shift property states that if $x(t) \leftrightarrow X(\omega)$ then $x(t)e^{j\omega_o t} \leftrightarrow X(\omega - \omega_o)$.
We know that $1 \leftrightarrow 2\pi\delta(\omega)$.
Using the frequency shift property of Fourier Transform we have,
$1 \times e^{j\omega_o t} \leftrightarrow 2\pi\delta(\omega - \omega_o)$ and hence $e^{j\omega_o t} \leftrightarrow 2\pi\delta(\omega - \omega_o)$

(iii) $x(t) = \delta(t-t_o)$ (use time shift property)
We know that $\delta(t) \leftrightarrow 1$. Using the time shift property of Fourier Transform we have,
$\delta(t - t_o) \leftrightarrow e^{-j\omega t_o}$.

2. Verify that the Fourier Transform of a train of impulse $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$, is given by $P(\omega) = \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$, where T_s is the sampling time and $\omega_s = 2\pi/T_s$.

The complex Fourier Series coefficient is given by

$$c_n = \frac{1}{T_s}\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jn\omega_s t}dt = \frac{1}{T_s}.$$

Note that $\int_{-T_s/2}^{T_s/2} \delta(t)e^{-jn\omega_s t}dt = e^{-jn\omega_s(0)} = 1$. Therefore we can write $p(t)$ as

$$p(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t}.$$

We know that $e^{j\omega_s t} \leftrightarrow 2\pi\delta(\omega - \omega_s)$, therefore

$$p(t) = \frac{1}{T_s}\sum_{n=-\infty}^{\infty} e^{jn\omega_s t} \leftrightarrow \frac{1}{T_s}\sum_{n=-\infty}^{\infty} 2\pi\delta(\omega - n\omega_s) = \frac{2\pi}{T_s}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s).$$

3. Prove the convolution property of Fourier Transform, $\mathcal{F}[x(t)*h(t)] = X(\omega).H(\omega)$.

We know that $x(t)*h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$.

$$\mathcal{F}[x(t)*h(t)] = \int_{-\infty}^{\infty} [x(t)*h(t)]e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\right]e^{-j\omega t}dt \qquad \text{eqn(3)}$$

Note that $X(\omega) = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau$, we will need to rewrite eqn (3) so that we can obtain $X(\omega)$. To do this we can write $e^{-j\omega t} = e^{-j\omega t}.e^{j\omega\tau}.e^{-j\omega\tau} = e^{-j\omega(t-\tau)}.e^{-j\omega\tau}$ and eqn (3) becomes

$$\mathcal{F}[x(t)*h(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\tau)h(t-\tau)e^{-j\omega(t-\tau)}\right]e^{-j\omega\tau}d\tau dt$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau\int_{-\infty}^{\infty} h(t-\tau)e^{-j\omega(t-\tau)}dt.$$

Let $\lambda = t-\tau$, $dt = d\lambda$. Therefore we have,

$$\mathcal{F}[x(t)*h(t)] = X(\omega)\int_{-\infty}^{\infty} h(\lambda)e^{-j\omega\lambda}d\lambda = X(\omega)H(\omega).$$

4. Show that $\mathcal{F}[x(t).h(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega')H(\omega-\omega')d\omega'$.

$$\mathcal{F}[x(t).h(t)] = \int_{-\infty}^{\infty} x(t)h(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega')e^{j\omega' t}d\omega'\right]e^{-j\omega t}dt$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{j\omega' t}e^{-j\omega t}dtH(\omega')d\omega' = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega')t}dtH(\omega')d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega-\omega')H(\omega')d\omega' = \frac{1}{2\pi}H(\omega)*X(\omega)$$

$$= \frac{1}{2\pi}X(\omega)*H(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega')H(\omega-\omega')d\omega'.$$

5. The Fourier Transform of a signal $x(t) = \begin{cases}1, & |t| < \tau \\ 0, & |t| > \tau\end{cases}$, is $X(\omega) = \frac{2\sin\omega\tau}{\omega}$. Use this Fourier Transform pair and the duality property to find the Fourier Transform of a signal described by $y(t) = \frac{\sin t}{\sqrt{\pi t}}$. Calculate the total energy contained in $y(t)$ using Parseval's theorem.

It will be difficult to evaluate $Y(\omega) = \int_{-\infty}^{\infty} \frac{\sin t}{\sqrt{\pi t}}e^{-j\omega t}dt$. We will use the duality property to find $Y(\omega)$. First we will derive the Fourier Transform pair $X(t) \leftrightarrow 2\pi x(-\omega)$.

replacing t with ω and τ with W gives,

$$X(t)=\frac{2\sin Wt}{t}\leftrightarrow 2\pi x(-\omega)=\begin{cases}2\pi & |\omega|<W\\ 0 & |\omega|>W\end{cases}$$

let $W = 1$, we have

$$\frac{2\sin t}{t}\leftrightarrow 2\pi x(-\omega)=\begin{cases}2\pi & |\omega|<1\\ 0 & |\omega|>1\end{cases}$$

However we are interested in finding the Fourier Transform of

$$y(t)=\frac{1}{\sqrt{\pi}}\frac{\sin t}{t}=\frac{1}{2\sqrt{\pi}}\times X(t)$$

Therefore

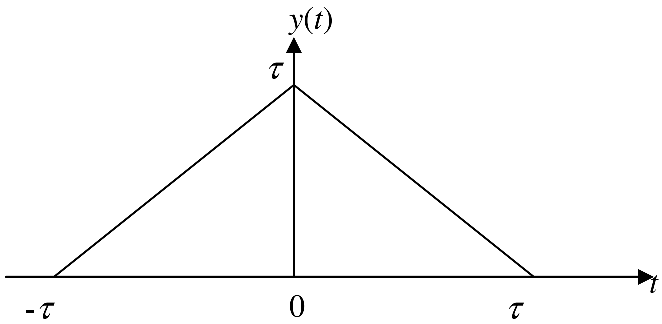
$$y(t)=\frac{1}{2\sqrt{\pi}}\times\frac{2\sin t}{t}\leftrightarrow Y(\omega)=\begin{cases}\frac{1}{2\sqrt{\pi}}\times 2\pi & |\omega|<1\\ 0 & |\omega|>1\end{cases}$$

$$y(t)=\frac{\sin t}{\sqrt{\pi}t}\leftrightarrow Y(\omega)=\begin{cases}\sqrt{\pi} & |\omega|<1\\ 0 & |\omega|>1\end{cases}$$

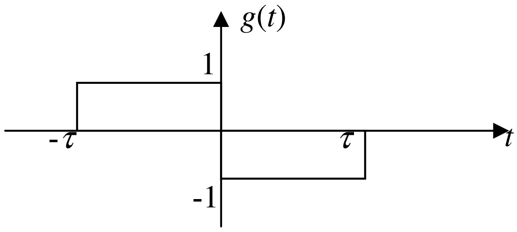
Using Parseval's theorem, the total energy is

$E=\int_{-\infty}^{\infty}|y(t)|^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(\omega)|^2\,d\omega=\frac{1}{2\pi}\int_{-1}^{1}\pi d\omega=\frac{2\pi}{2\pi}=1$. Note that the integration limit is -1 and +1 since Y(ω)=0 for |ω|>1.

6. Using the integration property and the Fourier Transform of the rectangular pulse, derive the Fourier Transform of the triangular signal shown below.



$Y(\omega)=\int_{-\tau}^{0}(t+\tau)e^{-j\omega t}dt+\int_{0}^{\tau}(-t+\tau)e^{-j\omega t}dt$. However this is difficult to evaluate. In general it is easier to work with a rectangular time domain function since a rectangular function in the time domain corresponds to a sinc function in the frequency domain. To convert y(t) to a rectangular signal let $g(t)=\frac{dy(t)}{dt}$,



Let $g(t) = x(t+\tau/2) - x(t-\tau/2)$ where $x(t)$ is a rectangular signal with a duration τ.

$$G(\omega) = X(\omega)e^{j\omega\tau/2} - X(\omega)e^{-j\omega\tau/2} = X(\omega)\left(e^{j\omega\tau/2} - e^{-j\omega\tau/2}\right) = \frac{\tau\sin(\omega\tau/2)}{(\omega\tau/2)}\left(2j\sin(\omega\tau/2)\right)$$

We know that $y(t) = \int_{-\infty}^{t} g(\tau)d\tau$. Using the integration property we have,

$$Y(\omega) = \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega).$$

To find $G(0)$,

$$\lim_{\omega\to 0}\frac{j2\tau\sin^2(\omega\tau/2)}{(\omega\tau/2)} = \lim_{\omega\to 0}\frac{j4\tau\sin(\omega\tau/2)\cos(\omega\tau/2)(\tau/2)}{\tau/2} = 0.$$ [use l-Hopital rule]

Therefore $G(0) = 0$ and

$$Y(\omega) = \frac{G(\omega)}{j\omega} = \frac{2\tau\sin^2(\omega\tau/2)}{\omega(\omega\tau/2)} = \left(\frac{\tau\sin(\omega\tau/2)}{(\omega\tau/2)}\right)^2.$$

7. The carrier frequency used in an AM wave is typically in the range of 0.535-1.605 MHz. A superheterodyne receiver, consisting of a product modulator and a local oscillator followed by a bandpass filter, is usually used as the receiver. Obtain the tuning frequency range of the oscillator that is required to translate an input AM wave, with a bandwidth of 8 kHz, to a frequency band with an intermediate frequency (IF) of 0.455 MHz.

Let f_{local} and f_s be the frequencies of the local oscillator and the signal respectively. Note that the frequency shift property states that if we multiply a signal $x(t)$ with a sinusoid we have

$x(t)\cos\omega_o t \leftrightarrow \frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$ so if $\omega_o = 2\pi f_{local}$ we have

$x(t)\cos 2\pi f_{local}\, t \leftrightarrow \frac{1}{2}[X(f+f_{local})+X(f-f_{local})]$, i.e the signal will be shifted by - f_{local} and + f_{local}.

To shift a signal with f_s = 0.535 MHz to the IF of 0.455 MHz, we can use the frequency shift property of FT. Therefore we have $f_s - f_{local}$ = 0.455 MHz, i.e f_{local} = (0.535 – 0.455) MHz = 0.08 MHz.

To shift a signal with f_s = 1.605 MHz to the IF of 0.455 MHz, we can use the frequency shift property of FT. Therefore we have $f_s - f_{local}$ = 0.455 MHz, i.e f_{local} = (1.605 – 0.455) MHz = 1.15 MHz. Therefore the tuning range of the local oscillator is 0.08 – 1.15 MHz independent of the signal bandwidth.

8. In a pulse amplitude modulation system, an analogue signal $x(t)$ is multiplied by a periodic train of rectangular pulses, $p(t)$. The Complex Fourier Series representation of $p(t)$ is given by $p(t) = \sum_{n=-\infty}^{\infty} \left(\frac{\tau \sin(n\omega_s \tau/2)}{T(n\omega_s \tau/2)} \right) e^{jn\omega_s t}$, where τ is the pulse width and $\omega_s = \frac{2\pi}{T}$ is the repetition frequency of $p(t)$. Find the spectrum of the modulated signal, $m(t)$.

We have $m(t) = x(t).p(t)$.

$$m(t) = \sum_{n=-\infty}^{\infty} x(t) \left(\frac{\tau \sin(n\omega_s \tau/2)}{T(n\omega_s \tau/2)} \right) e^{jn\omega_s t} = \sum_{n=-\infty}^{\infty} \left(\frac{\tau \sin(n\omega_s \tau/2)}{T(n\omega_s \tau/2)} \right) x(t) e^{jn\omega_s t}.$$

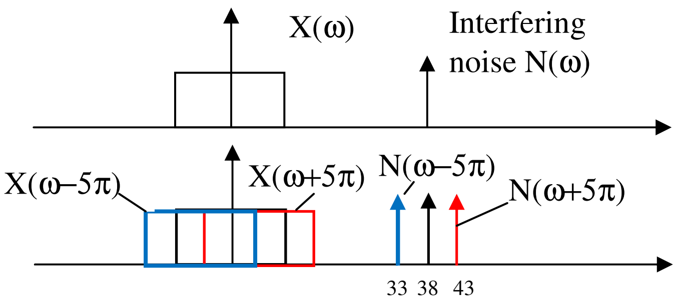
Therefore using the frequency shift property of FT gives

$$M(\omega) = \sum_{n=-\infty}^{\infty} \left(\frac{\tau \sin(n\omega_s \tau/2)}{T(n\omega_s \tau/2)} \right) X(\omega - n\omega_s),$$

where $X(\omega)$ is amplitude the spectrum of $x(t)$. Thus, the original spectrum $X(\omega)$ has been replicated at $n\omega_s$ with the nth replica scaled by the factor $\frac{\tau \sin(n\omega_s \tau/2)}{T(n\omega_s \tau/2)}$.

9. Consider a continuous time signal, $x(t)$, that lies in the frequency band $|\omega| < 10\pi$ rad/s. Due to inadequate shielding the signal is contaminated by a large sinusoid with a frequency of 38π rad/s. This contaminated signal is now sampled at a frequency of 5π rad/s.

i) At what frequencies does the interfering sinusoid appear after sampling?



N(ω) reproduced at 38π ±5nπ

We can solve this problem using a graphical method. Since the sampling frequency is 5π, we will have copy of $X(\omega)$ at frequencies of $5n\pi$, where n = ..-2, -1, 0, 1, 2... For example the signal and noise is replicated at -5π (blue) and $+5\pi$ (red) in the diagram above. Therefore the interfering signal will be reproduced at frequencies of $38\pm5n\pi$.

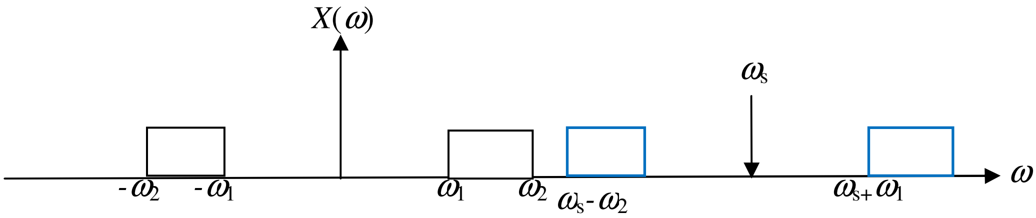
ii) A low pass filter is used to reduce aliasing. A sufficient condition is to attenuate the interfering sinusoid by a factor of 100. Work out the RC time constant required to achieve this.

The transfer function of the RC low pass filter is $1/(1+j\omega\tau)$ where τ=RC. Before sampling the signal is passed through the low pass filter. To attenuate the interfering signal by 100, $|1/(1+j\omega\tau)| = 0.01$ when $\omega = 38\pi$.

$$\frac{1}{\sqrt{1+(38\pi\tau)^2}} = 0.01$$

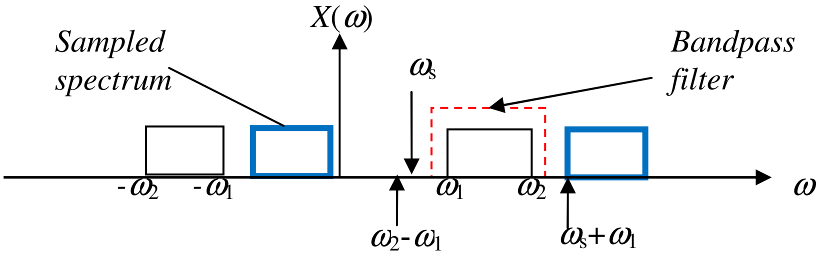
Hence τ=RC = 0.84s.

12. Consider a continuous time signal $x(t)$ with a magnitude spectrum shown below.



i) Based on the Nyquist Theorem, state the sampling interval, T_s, required to avoid aliasing.
The sampling frequency is $\omega_s = 2\pi/T_s$. Nyquist Theorem states that $2\pi/T_s > 2\omega_2$. Therefore $T_s < \pi/\omega_2$.

ii) Assuming that $\omega_1 > \omega_2 - \omega_1$. Work out the maximum sampling interval such that it is still possible to reconstruct $x(t)$ perfectly. (Note that in this case T_s can be smaller than in part (i)).



No aliasing occurs if there is no overlap of spectrum within $\omega_1 \leq \omega \leq \omega_2$. This is the case if $\omega_s + \omega_1 > \omega_2$. Therefore we have $\omega_s > \omega_2 - \omega_1$ and $T_s < 2\pi/(\omega_2 - \omega_1)$. The maximum sampling interval is therefore $T_s = 2\pi/(\omega_2 - \omega_1)$. To recover the signal we need to use a band pass filter as illustrated.