Tutorial 4: Solutions

1. Evaluate the Laplace transform of the signal $x(t) = e^{-t}u(t) + e^{-4t}u(t)$.

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt = \int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt + \int_{-\infty}^{\infty} e^{-4t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+1)t}dt + \int_{0}^{\infty} e^{-(s+4)t}dt$$

$$X(s) = \frac{1}{s+1}\left[-e^{-(s+1)t}\right]_0^\infty + \frac{1}{s+4}\left[-e^{-(s+4)t}\right]_0^\infty = \frac{1}{s+1} + \frac{1}{s+4}, \ \mathrm{Re}\{s\} > -1.$$

2. Verify the following Laplace transform pairs

(i) $\dfrac{dx(t)}{dt} \leftrightarrow sX(s)$

We know that $x(t) = \dfrac{1}{j2\pi}\displaystyle\int_{c-j\infty}^{c+j\infty} X(s)e^{st}ds$. Differentiate both sides w.r.t. to t gives

$$\frac{dx(t)}{dt} = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} X(s)se^{st}ds = \frac{1}{j2\pi}\int_{c-j\infty}^{c+j\infty} \left(sX(s)\right)e^{st}ds.$$

Therefore $\dfrac{dx(t)}{dt}$ is the inverse Laplace Transform of $sX(s)$, i.e, $\dfrac{dx(t)}{dt} \leftrightarrow sX(s)$.

(ii) $-tx(t) \leftrightarrow \dfrac{dX(s)}{ds}$

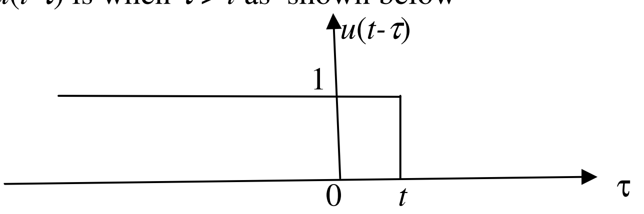
Start with $X(s) = \displaystyle\int_{-\infty}^{\infty} x(t)e^{-st}dt$. Differentiate both sides w.r.t. s gives,

$$\frac{dX(s)}{ds} = \int_{-\infty}^{\infty} -tx(t)e^{-st}dt = \int_{-\infty}^{\infty} \left(-tx(t)\right)e^{-st}dt.$$

Therefore the Laplace Transform of $-tx(t)$ is $\dfrac{dX(s)}{ds}$, i.e, $-tx(t) \leftrightarrow \dfrac{dX(s)}{ds}$.

(iii) $\displaystyle\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{X(s)}{s}$.

Consider, the time domain convolution defined by $x(t)*u(t) = \displaystyle\int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau$. The signal $u(t-\tau)$ is when $\tau > t$ as shown below



$$x(t)*u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

Using the convolution property we know that convolution in time domain is equivalent to multiplication in s-domain.

$$\int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t) \leftrightarrow X(s)U(s) = \frac{X(s)}{s}.$$

3. Find the values of $y(t) = 2e^{-2t}u(t) - e^{-t}u(t)$ for $t = 0$ and $t \to \infty$. Verify your answer using the initial and the final value theorems.

At $t = 0$, $y(0) = 2 - 1 = 1$.

As $t \to \infty$, $\lim_{t\to\infty} y(t) = 0$.

Taking the Laplace Transform of $y(t)$ gives,

$$Y(s) = \frac{2}{s+2} - \frac{1}{s+1} = \frac{2(s+1)-(s+2)}{(s+2)(s+1)} = \frac{s}{s^2+3s+2}.$$

$$sY(s) = \frac{s^2}{s^2+3s+2} = \frac{1}{1+3/s+2/s^2}.$$

Using the initial value theorem,

$$y(0) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty} \frac{1}{1+3/s+2/s^2} = 1.$$

Using the final value theorem,

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{s^2}{s^2+3s+2} = 0.$$

These values are the same as those calculated above.

4. Compute the impulse response and the unit step response of a system with transfer function described by $H(s) = \dfrac{3s}{2s^2+10s+12}$.

$$H(s) = \frac{3s}{2s^2+10s+12} = \frac{3s}{(2s+4)(s+3)} = \frac{3s/2}{(s+2)(s+3)} = k_0 + \frac{k_1}{s+2} + \frac{k_2}{s+3}.$$

Using partial fraction expansion,

$$k_o = \left.\frac{3s}{s^2+10s+12}\right|_{s=\infty} = \left.\frac{3/s}{1+10/s+12/s^2}\right|_{s=\infty} = 0,$$

$$k_1 = \left.\frac{3s/2}{(s+2)(s+3)}(s+2)\right|_{s=-2} = \left.\frac{3s/2}{(s+3)}\right|_{s=-2} = \frac{-3}{1} = -3,$$

$$k_2 = \left.\frac{3s/2}{(s+2)(s+3)}(s+3)\right|_{s=-3} = \left.\frac{3s/2}{(s+2)}\right|_{s=-3} = \frac{-9/2}{-1} = \frac{9}{2}.$$

Alternatively,

$$\frac{3s/2}{(s+2)(s+3)} = \frac{k_0(s+2)(s+3)+k_1(s+3)+k_2(s+2)}{(s+2)(s+3)}$$

$$3s/2 = k_0(s^2+5s+6)+(k_1+k_2)s+3k_1+2k_2 = k_0 s^2 + (5k_0+k_1+k_2)s + (6k_0+3k_1+2k_2)$$

Comparing the coefficients for s^2 gives $k_0 = 0$.

Comparing the coefficients for s gives $k_1 + k_2 = 3/2$.

We also have $3k_1 + 2k_2 = 0$ and hence $k_2 = -3k_1/2$.

Substituting k_2 gives $k_1 - 3k_1/2 = 3/2$ and hence $k_1 = -3$ and $k_2 = 9/2$

The transfer function is

$$H(s) = -\frac{3}{s+2} + \frac{9/2}{s+3} = \frac{9}{2}\frac{1}{s+3} - \frac{3}{s+2}$$

Therefore the impulse response is described by $h(t) = \frac{9}{2}e^{-3t}u(t) - 3e^{-2t}u(t)$ in the time domain.

If the input is a unit step, The output is given by

$$Y(s) = H(s)U(s) = \frac{3s/2}{(s+2)(s+3)}\frac{1}{s} = \frac{3/2}{(s+2)(s+3)} = \frac{k_1}{(s+2)} + \frac{k_2}{(s+3)}.$$

Using partial fraction expansion,

$$k_1 = \frac{3/2}{(s+2)(s+3)}(s+2)\bigg|_{s=-2} = \frac{3/2}{(s+3)}\bigg|_{s=-2} = \frac{3/2}{1} = \frac{3}{2},$$

$$k_2 = \frac{3/2}{(s+2)(s+3)}(s+3)\bigg|_{s=-3} = \frac{3/2}{(s+2)}\bigg|_{s=-3} = \frac{3/2}{-1} = -\frac{3}{2}.$$

Alternatively,

$$\frac{3/2}{(s+2)(s+3)} = \frac{k_1(s+3)+k_2(s+2)}{(s+2)(s+3)}$$

$$3/2 = (k_1+k_2)s + (3k_1+2k_2)$$

Comparing the coefficients for s, $k_1 = -k_2$.

$k_1=3/2$ *and* $k_2= -3/2$.

$$Y(s) = \frac{3}{2(s+2)} - \frac{3}{2(s+3)}.$$

Therefore the unit step response in time domain is

$$y(t) = \frac{3}{2}e^{-2t}u(t) - \frac{3}{2}e^{-3t}u(t) = \frac{3}{2}u(t)(e^{-2t} - e^{-3t}).$$

5. Determine the poles, the natural frequency and the damping factor of systems with the following transfer functions and state the nature of the system response:

(i) $G(s) = \frac{0.3}{s^2+7s+10} = \frac{0.3}{(s+5)(s+2)}$

Compare $G(s) = \frac{0.3}{s^2+7s+10}$ with $\frac{k}{s^2+2\zeta\omega_n+\omega_n{}^2}$.

Natural oscillating frequency is $\omega_n = \sqrt{10}$ rad/s.

Damping factor is $\zeta = \frac{7}{2\omega_n} = \frac{7}{2\sqrt{10}} = 1.107$.

The system is lightly overdamped. Poles are p_1 = -5 and p_2 = -2.

The unit step response is in the form, $y(t) = K_o + K_1e^{-5t}u(t) + K_2e^{-2t}u(t)$, where K_o, K_1 and K_2 are constants.

(ii) $G(s)=\dfrac{1}{s^2+4s+13}=\dfrac{1}{(s^2+4s+4)+9}=\dfrac{1}{(s+2)^2+(3)^2}$.

Compare $G(s)=\dfrac{1}{s^2+4s+13}$ with $\dfrac{k}{s^2+2\zeta\omega_n+\omega_n^{\,2}}$ and

$G(s)=\dfrac{1}{(s+2)^2+(3)^2}$ with $\dfrac{k}{(s+\zeta\omega_n)^2+\omega_d^{\,2}}$.

Natural oscillating frequency is $\omega_n=\sqrt{13}$ rad/s.

Damping factor is $\zeta=\dfrac{4}{2\omega_n}=\dfrac{2}{\sqrt{13}}=0.555.$

The system is underdamped. The poles are complex and are given by $p_{1,2}=-\zeta\omega_n\pm j\omega_d$, where $\omega_d = 3$.

$p_{1,2}=-\dfrac{2}{\sqrt{13}}\sqrt{13}\pm j3=-2\pm j3$. The unit step response is

$y(t)=K_0+K_1e^{-2t}\sin(3t+\theta)u(t)$, where K_0 and K_1 are constants. $y(t)$ is a sinusoid with frequency 3 rad/s that decays exponentially with at a time constant 1/2 s.

(iii) $G(s)=\dfrac{0.1}{s^2+16}$

Compare $G(s)=\dfrac{0.1}{s^2+16}$ with $\dfrac{k}{(s^2+\omega_n^{\,2})}$.

Natural oscillating frequency is $\omega_n=\sqrt{16}=4$ rad/s.

Damping factor is $\zeta=0$.

The system is undamped. The poles are $p_{1,2}=\pm j\omega_n=\pm j4$.

The unit step response is $y(t)=K_0+K_1\cos(4t)u(t)$, where K_0 and K_1 are constants.

(iv) $G(s)=\dfrac{15}{s^2+6s+9}$.

Compare $G(s)=\dfrac{15}{s^2+6s+9}$ with $\dfrac{k}{s^2+2\zeta\omega_n+\omega_n^{\,2}}$

Natural oscillating frequency is $\omega_n=\sqrt{9}=3$ rad/s.

Damping factor is $\zeta=\dfrac{6}{2\omega_n}=\dfrac{3}{3}=1$.

The system is critically damped. The poles are $p_{1,2}=-\omega_n=-3$.

The unit step response is $y(t)=K_0\left(1-(1+3t)e^{-3t}u(t)\right)$ where K_0 is a constant.

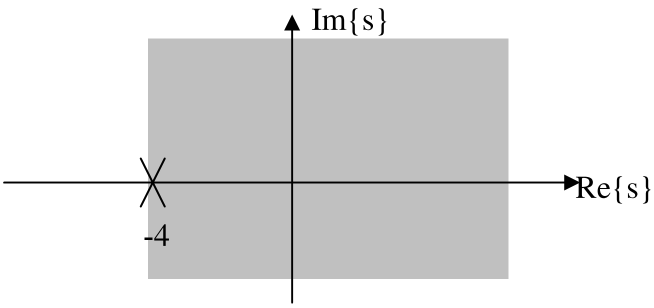
6. Determine the Laplace transforms of the following signals. Sketch the pole-zero plot and region of convergence (if it exists).

(i) $x(t) = e^{-4t}u(t)$

$X(s) = \int_{-\infty}^{\infty} e^{-4t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-4t}e^{-(\sigma+j\omega)t}dt = \int_{0}^{\infty} e^{-(4+\sigma)t}e^{-j\omega t}dt$. $X(s)$ exists if $4 + \sigma > 0$, i.e

$\sigma = \mathrm{Re}\{s\} > -4$. Therefore we have $X(s) = \int_{0}^{\infty} e^{-(s+4)t}dt = \frac{1}{s+4}\left[-e^{-(s+4)t}\right]_0^\infty = \frac{1}{s+4}$.

$X(s) = \frac{1}{s+4}, \mathrm{Re}\{s\} > -4$. Pole = -4.



(ii) $x(t) = e^{-t}u(t) + e^{-3t}u(t)$

The Laplace Transform of $e^{-t}u(t)$ is

$$\int_{-\infty}^{\infty} e^{-t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-t}e^{-(\sigma+j\omega)t}dt = \int_{0}^{\infty} e^{-(1+\sigma)t}e^{-j\omega t}dt$$

with an region of convergence described by $\sigma=\mathrm{Re}\{s\}>-1$.

The Laplace Transform of $e^{-3t}u(t)$ is

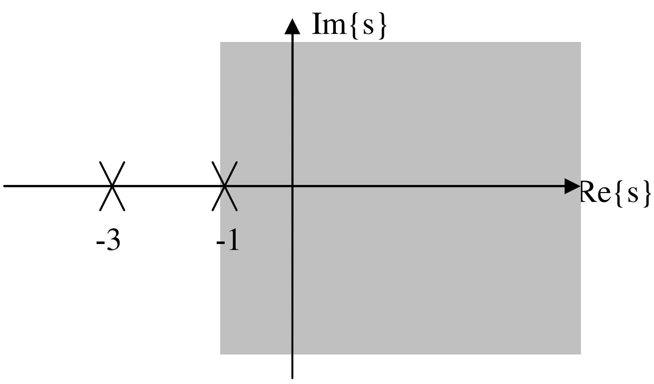
$$\int_{-\infty}^{\infty} e^{-3t}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-3t}e^{-(\sigma+j\omega)t}dt = \int_{0}^{\infty} e^{-(3+\sigma)t}e^{-j\omega t}dt$$

with an region of convergence described by $\sigma=\mathrm{Re}\{s\}>-3$.

Therefore for

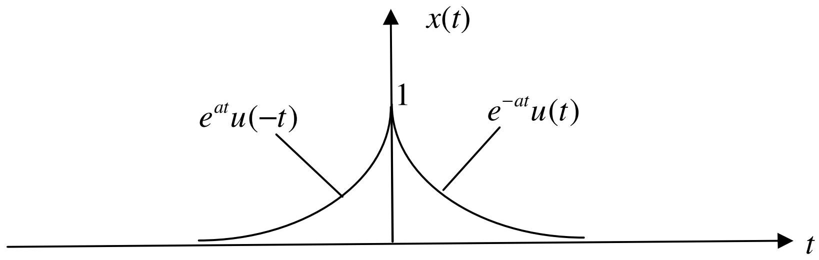
$$X(s) = \frac{1}{s+1} + \frac{1}{s+3}.$$

we have $\sigma=\mathrm{Re}\{s\}>-1$.



(iii) $x(t) = e^{-a|t|}$, $a > 0$

The signal is shown below



We have $x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$ since $a > 0$.

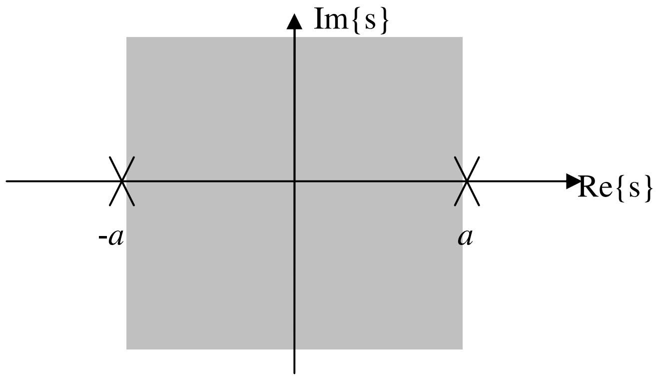
The Laplace transform for $e^{at}u(-t)$ is $\int_{-\infty}^{\infty} e^{at}u(-t)e^{-st}dt = \int_{-\infty}^{0} e^{at}e^{-(\sigma+j\omega)t}dt = \int_{-\infty}^{0} e^{(a-\sigma)t}e^{-j\omega t}dt$.

The Laplace transform for $e^{at}u(-t)$ exists if a - $\sigma > 0$, i.e $\sigma = \mathrm{Re}\{s\} < a$.

$$\int_{-\infty}^{0} e^{(a-s)t}dt = \frac{1}{a-s}\left[e^{(a-s)t}\right]_{-\infty}^{0} = -\frac{1}{s-a}$$

The region of convergence for $\frac{1}{s+a}, \mathrm{Re}\{s\} > -a$ and for $-\frac{1}{s-a}, \mathrm{Re}\{s\} < a$

We have, $X(s) = \frac{1}{s+a} - \frac{1}{s-a}$ with region of convergence defined by $-a < \mathrm{Re}\{s\} < a$.



Therefore $x(t) = e^{-a|t|} \leftrightarrow X(s) = \frac{1}{s+a} - \frac{1}{s-a}, -a < \mathrm{Re}\{s\} < a$.

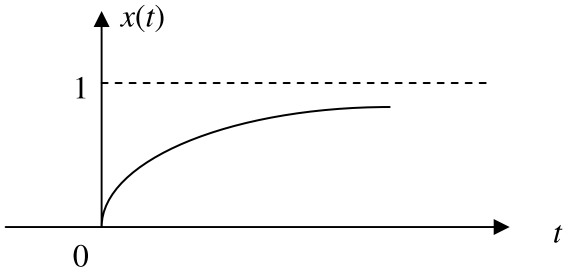
7. Find the Laplace transforms of the signal $x(t) = e^{-t}u(t) * u(t)$ and sketch $x(t)$.

Using the convolution property, $X(s) = \frac{1}{s+1} \times \frac{1}{s} = \frac{k_1}{s} + \frac{k_2}{s+1}$.

Using partial fraction expansion,

$$k_1 = \left.\frac{1}{s+1}\right|_{s=0} = 1 \text{ and } k_2 = \left.\frac{1}{s}\right|_{s=-1} = -1 .$$

Therefore $X(s) = \frac{1}{s} - \frac{1}{s+1}, \mathrm{Re}\{s\} > -1$ and $x(t) = 1 - e^{-t}u(t)$.



8. Determine the initial and the final values of the signal with Laplace transform

$$X(s) = \frac{10s}{s^2 + 10s + 300}.$$

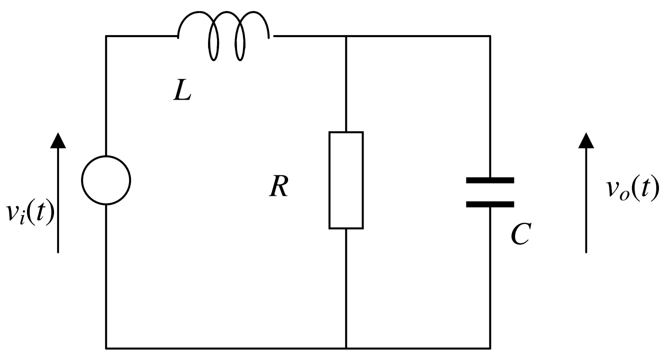
Using the initial value theorem,

$$x(0) = \lim_{s\to\infty}(sX(s)) = \lim_{s\to\infty}\left(s\frac{10s}{s^2+10s+300}\right) = \lim_{s\to\infty}\left(\frac{10}{1+10/s+300/s^2}\right) = 10.$$

Using the final value theorem,

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0}(sX(s)) = \lim_{s\to 0}\left(s\frac{10s}{s^2+10s+300}\right) = \lim_{s\to 0}\left(\frac{10s^2}{s^2+10s+300}\right) = 0$$

9. Determine the transfer function of the circuit shown below.



(i) If $R = 1\ \Omega$ and $C = 1$ pF calculate the value of L required so that the circuit is critically damped. Sketch $v_o(t)$ if $v_i(t)$ is a unit step function.

Let $Z_1 = \dfrac{R/sC}{R+1/sC} = \dfrac{R}{1+sRC}$.

The transfer function, $H(s) = \dfrac{V_o(s)}{V_i(s)} = \dfrac{Z_1}{Z_1 + sL} = \dfrac{\frac{R}{1+sRC}}{\frac{R}{1+sRC} + sL}$.

$$H(s) = \frac{R}{R+sL(1+sRC)} = \frac{R}{R+sL+s^2RLC} = \frac{1/LC}{s^2+(1/RC)s+(1/LC)}.$$

The natural oscillating frequency is, $\omega_n = \dfrac{1}{\sqrt{LC}}$ and the damping factor is

$\zeta = \dfrac{1/RC}{2\omega_n}$. To achieve critical damping, $\zeta = \dfrac{1/RC}{2\omega_n} = 1$.

If R = 1Ω and C = 1pF, $\omega_n = \dfrac{1}{2RC} = \dfrac{1}{2\times1\times1\times10^{-12}} = \dfrac{1}{2\times10^{-12}}$.

$\dfrac{1}{L} = C\omega_n^{\ 2}$. Therefore $L = \dfrac{1}{C\omega_n^{\ 2}} = \dfrac{\left(2\times10^{-12}\right)^2}{1\times10^{-12}} = 4\times10^{-12}H$.

If $v_i(t) = u(t)$, $V_i(s) = 1/s$.

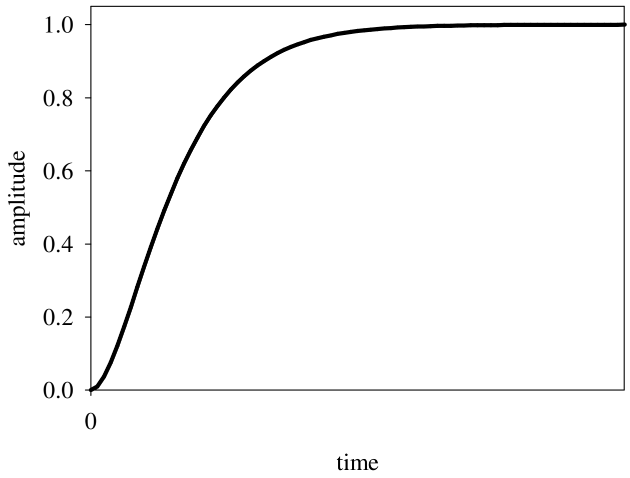
$$H(s) = \frac{1/LC}{s^2 + (1/RC)s + 1/LC} = \frac{(1/4\times10^{-24})}{s^2 + (1/1\times10^{-12})s + (1/4\times10^{-12})} = \frac{(1/4\times10^{-24})}{(s + 1/2\times10^{-12})^2} = \frac{k}{(s+\omega_n)^2}$$

The Laplace Transform of the unit step response is

$$V_o(s) = H(s)V_i(s) = \frac{k}{s(s+\omega_n)^2}.$$

From lecture notes, the unit step response in time domain is

$$v(t) = \frac{k}{\omega_n^{\ 2}} - \frac{k}{\omega_n^{\ 2}}[1+\omega_n t]e^{-\omega_n t}u(t) = 1 - [1 + 5\times10^{11}t]e^{-t/2\times10^{-12}}u(t).$$



(ii) If $R = 50\ \Omega$, $C = 1$ nF and $L = 2.5\mu$H calculate the damping factor and natural oscillating frequency. Sketch and describe $v_o(t)$ if $v_i(t)$ is a unit step function.

If R = 50Ω, C = 1nF and L = 2.5μH , $\omega_n = \sqrt{\dfrac{1}{LC}} = \dfrac{1}{\sqrt{2.5\times10^{-6}\times1\times10^{-9}}} = 20\times10^6$ rad/s.

$$\zeta = \frac{1/RC}{2\omega_n} = \frac{1/50\times10^{-9}}{2\times20\times10^6} = 0.5.$$

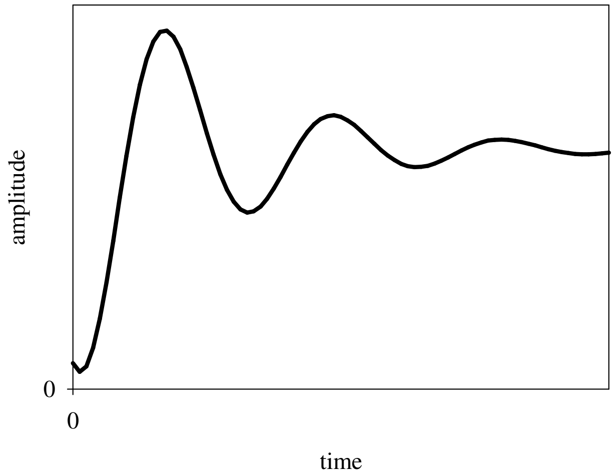
The system is underdamped. The unit step response is

$$y(t) = \frac{k}{\omega_n^{\ 2}}\left(1 - \frac{k}{\omega_n\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)u(t)\right).$$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 20\times10^6\sqrt{1-0.5^2} = 17.32\times10^6$ rad/s.

$\phi = \tan^{-1}(\omega_d/\zeta\omega_n) = \tan^{-1}(17.32\times10^6/0.5\times20\times10^6) = 1.565$ rad.

$1/\zeta\omega_n = 1/0.5\times20\times10^6 = 1/10\times10^6 = 0.1\times10^{-6}$ s.



The unit step response is a sinusoid with a frequency of $\omega_d = 17.32\times10^6$ rad/s that decays exponentially with a time constant of $1/\zeta\omega_n = 0.1\times10^{-6}$ s.

10. (i) The response is

$$Y(s) = \frac{sL.X(s)}{sL + R + \frac{1}{sC}} = \frac{s^2LC.X(s)}{s^2LC + sRC + 1}$$

The response to a unit step function is

$$Y(s) = \frac{s^2(2)(0.25)1/s}{s^2(2)(0.25) + (6)(0.25)s + 1} = \frac{s/2}{(\frac{1}{2})s^2 + (3/2)s + 1}$$

$$Y(s) = \frac{s}{s^2 + 3s + 2} = \frac{s}{(s+1)(s+2)} = \frac{2}{(s+2)} - \frac{1}{(s+1)}$$

Taking the inverse Laplace Transform gives

$y(t) = (2e^{-t} - e^{-t})u(t)$.

We have $X(s) = Y(s) + I(s)R + I(s)/sC$. When $x(t) = 0$, we have

$0 = Y(s) + I(s)R + I_c(s)/sC$

$I(s)$ and $I_c(s)$ are currents flowing through the resistor and capacitor associated with the initial conditions.

$I(s) = I_L(s) = \frac{Y(s)}{sL} + \frac{i(0)}{s}$ [note that $Y(s) = sLI_L(s) - Li(0)$] and

$$v_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau + v_c(0)$$

$$V_c(s) = \frac{I(s)}{sC} + \frac{v_c(0)}{s}$$

$$I_c(s) = sCV_c(s) = I(s) + Cv_c(0) = \frac{Y(s)}{sL} + \frac{i(0)}{s} + Cv_c(0)$$

Substituting $I(s)$ and $I_c(s)$ we have

$$0 = Y(s) + \left[\frac{Y(s)}{sL} + \frac{i(0)}{s}\right]R + \frac{1}{sC}\left[\frac{Y(s)}{sL} + \frac{i(0)}{s} + Cv_c(0)\right]$$

$$Y(s)\left[\frac{s^2LC + sRC + 1}{s^2LC}\right] = -\left[\frac{i(0)sRC + i(0) + v_c(0)sC}{s^2C}\right]$$

$$Y(s) = -\left[\frac{i(0)sLRC + i(0)L + v_c(0)sLC}{s^2LC + sRC + 1}\right]$$

$$Y(s) = -\left[\frac{i(0)L(sRC + 1) + v_c(0)sLC}{s^2LC + sRC + 1}\right]$$

$$Y(s) = -\left[\frac{1(2)(3s/2 + 1) + s/2}{(\frac{1}{2})s^2 + 3s/2 + 1}\right] = -\frac{7s + 4}{(s + 2)(s + 1)} = \frac{3}{(s + 1)} - \frac{10}{(s + 2)}$$

Therefore $y(t) = (3e^{-t} - 10e^{-2t})u(t)$

11. Consider a system with a transfer function $H(s) = \frac{1}{s+3}$. Find the forced and natural responses of this system if the input signal is given by $x(t) = \exp(-3t)u(t)$ and an initial condition of $y_o(0) = 1$, where $y_o(0)$ is the output signal at $t = 0$.

To work out the forced response, $Y_{forced}(s) = \frac{X(s)}{(s+3)} = \frac{1}{(s+3)(s+3)} = \frac{1}{(s+3)^2}$.

Therefore we have time domain forced response given by $y_{forced}(t) = t\exp(-3t)]u(t)$.

To work out the natural response, $Y_{natural}(s) = \frac{y(0)}{(s+3)} = \frac{1}{(s+3)}$ and therefore the corresponding expression in time domain is $y_{natural}(t) = [\exp(-3t)]u(t)$.