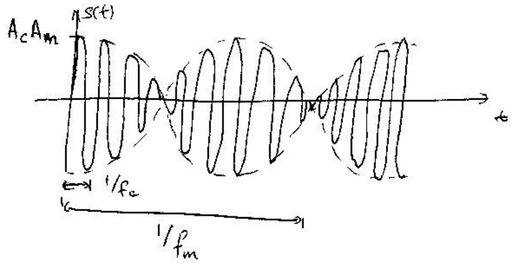
Q1. a)

i) $s(t) = m(t)c(t) = A_c A_m \cos(\omega_c t)\cos(\omega_m t)$.



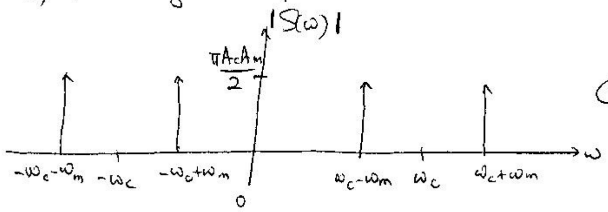
(2)

$$s(t) = \frac{A_c A_m}{2}\left[\cos(\omega_c - \omega_m)t + \cos(\omega_c + \omega_m)t\right]$$ (1)

use FT pairs to give

ii) $$S(\omega) = \pi\frac{A_cA_m}{2}\left[\delta(\omega+\omega_c-\omega_m) + \delta(\omega-\omega_c+\omega_m) + \delta(\omega+\omega_c+\omega_m) + \delta(\omega-\omega_c-\omega_m)\right]$$ (1)

iii) The magnitude spectrum looks like



(2)

iv) Double sideband-suppressed carrier modulation does not require transmission of carrier signal, hence lower power consumption. (2)
However it requires synchronisation between the transmitter and the receiver.

Q1. b) i) let $A_c = 1$.

$m(t) = A_m \cos(\omega_m t)$ $\quad c(t) = \cos(\omega_c t)$

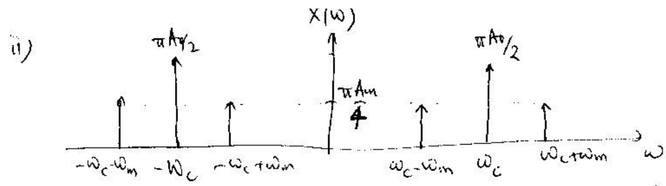
$$x(t) = \left(A_0 + m(t)\right) c(t)$$

$$= \left(A_0 + A_m \cos(\omega_m t)\right)\cos(\omega_c t)$$

$$= A_0 \cos(\omega_c t) + A_m \cos(\omega_m t)\cos(\omega_c t)$$

$$= A_0 \cos(\omega_c t) + \frac{A_m}{2}\left[\cos(\omega_m - \omega_c)t + \cos(\omega_m + \omega_c)t\right]$$

ii)



This modulation scheme requires transmission of carrier signal, and have higher power consumption.

c) For this envelope detector, C must charge rapidly when the diode is conducting. This is achieved if $R_S C \ll 2\pi/\omega_c$ — ①

C must also discharge slowly through R_ℓ when the diode is not conducting, but not too slow so that it can discharge at a maximum rate determined by the modulating signal.

So we need to satisfy $\quad \frac{2\pi}{\omega_c} \ll R_\ell C \ll \frac{2\pi}{\omega_m}$ — ②.

From ①

$$C \ll \frac{2\pi}{\omega_c R_S}$$

$$C \ll \frac{2\pi}{2\pi \times 10^5 \times 75}$$

$$C \ll 0.13\,\mu F$$

From ②

$$C \ll \frac{2\pi}{\omega_m R_\ell}$$

$$C \ll \frac{2\pi}{0.01 \times 2\pi \times 10^5 \times 10^4}$$

$$C \ll 0.1\,\mu F$$

A suitable value for C is $\sim 0.01\,\mu F$.

$$X(\omega) = \pi A_0[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)] + \frac{\pi A_m}{2}[\delta(\omega+\omega_m-\omega_c) + \delta(\omega-\omega_m-\omega_c) + \delta(\omega+\omega_m+\omega_c) + \delta(\omega-\omega_m+\omega_c)]$$

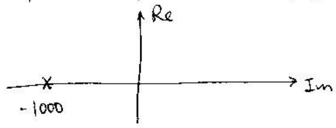
Q2. a) i) If $y_0(t)$ is the output,

$$H_0(s) = \frac{Y_0(s)}{X(s)} = \frac{1/sC}{1/sC + R} = \frac{1}{1+sRC} \quad \text{or} \quad \frac{1}{RC}\cdot\frac{1}{s+1/RC}$$

ii) if $y_1(t)$ is the output,

$$H_1(s) = \frac{Y_1(s)}{X(s)} = \frac{R}{1/sC + R} = \frac{sRC}{1+sRC} \quad \text{or} \quad \frac{s}{s+1/RC}$$

No zeros for part (i), pole at $s = -\frac{1}{RC} = -\frac{1}{1\times10^3 \times 1\times10^{-6}} = -1000$

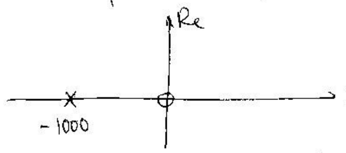


$s \to 0 \quad \frac{Y_0(s)}{X(s)} \to 1$

$s \to \infty \quad \frac{Y_0(s)}{X(s)} \to 0.$

The system behaves like a low pass filter.

Zero at $s=0$, pole at $s = -\frac{1}{RC}$ for part (ii).



$s \to 0 \quad \frac{Y_1(s)}{X(s)} \to 0$

$s \to \infty \quad \frac{Y_1(s)}{X(s)} \to 1$

The system behaves like a high pass filter.

b) $H_0(s) = \frac{Y_0(s)}{X(s)} = \frac{1}{RC}\cdot\frac{1}{s+1/RC}$

Therefore the impulse response $h_0(t) = \frac{1}{RC}\exp(-t/RC)\cdot u(t) = \frac{1}{RC}\exp(-t/RC),\ t \geq 0.$

$H_1(s) = \frac{Y_1(s)}{X(s)} = s\left(\frac{1}{s+1/RC}\right)$ [redacted] $= \left(s + 1/RC - 1/RC\right)/(s+1/RC) = 1 - \frac{1/RC}{s+1/RC}$

$h_1(t) = \delta(t) - \frac{1}{RC}\exp(-t/RC).$ [redacted]

Q2. c) $\frac{d^2}{dt^2} y(t) + 6\frac{d}{dt} y(t) + 8y(t) = x(t)$

Taking the Laplace transforms

$$s^2 Y(s) + 6s Y(s) + 8 Y(s) = X(s)$$

$$Y(s)(s^2 + 6s + 8) = X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2+6s+8} = \frac{1}{(s+2)(s+4)}$$

$$H(s) = \frac{k_1}{(s+2)} + \frac{k_2}{(s+4)}$$

$$k_1 = \frac{1}{\cancel{(s+2)}(s+4)} \cancel{(s+2)} \Big|_{k=-2} = \frac{1}{2}$$

$$k_2 = \frac{1}{(s+2)(s+4)} (s+4) \Big|_{k=-4} = -\frac{1}{2}$$

$$\therefore H(s) = \frac{1}{2(s+2)} - \frac{1}{2(s+4)}$$

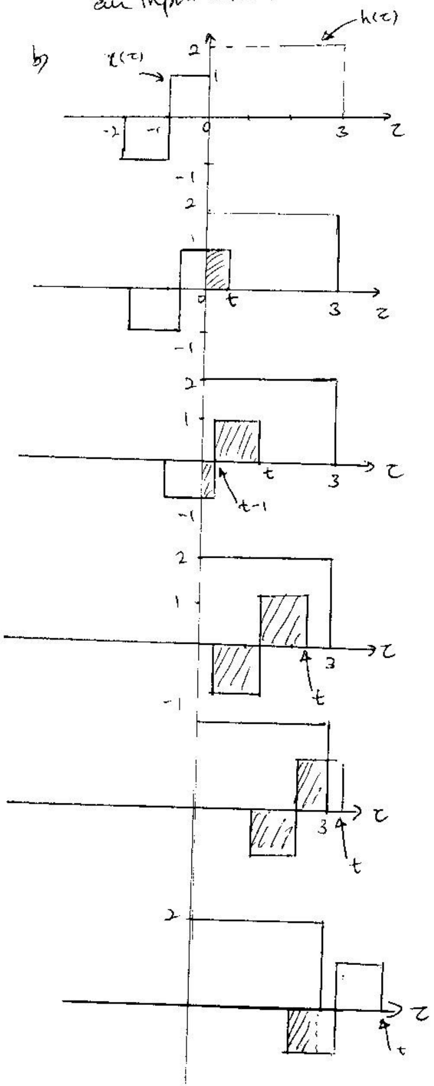
$$h(t) = \frac{1}{2}\left[\exp(-2t) - \exp(-4t) \right] u(t).$$

Q3 a) For an LTI system with an impulse response $h(t)$ we have

input	output
$\delta(t)$	$h(t)$
$\delta(t-\tau)$	$h(t-\tau)$
$x(\tau)\,\delta(t-\tau)$	$x(\tau)h(t-\tau)$
$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$	$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$.

Therefore $y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ is the response of the LTI system to an input $x(t)$.

b)



$t \leq 0 \quad y(t) = 0$.

$0 < t \leq 1$

$y(t) = 2t$.

$1 < t \leq 2$

$y(t) = 2\left[1-(t-1)\right] = 4-2t$

$2 < t \leq 3$

$y(t) = 0$

$3 < t \leq 4$

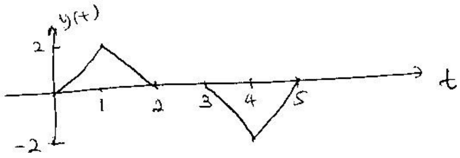
$y(t) = 2\left[3-(t-1)-1\right] = 6-2t$

$4 < t \leq 5$

$y(t) = 2\left[-(3-(t-2))\right] = 2t-10$

$t > 5 \quad y(t) = 0$.

Q3 b).



Q3. c) $x[n] = \begin{cases} 1 & 0 \le n \le 3 \\ 0 & \text{otherwise} \end{cases}$ $\quad h[n] = \begin{cases} \frac{1}{n} & 1 \le n \le 4 \\ 0, & \text{otherwise}. \end{cases}$

k	-4	-3	-2	-1	0	1	2	3	4	5	$\sum x[k]h[n-k]$
x[k]	0	0	0	0	1	1	1	1	0	0	
h[0-k]	1/4	1/3	1/2	1	0	0	0	0	0	0	0
h[1-k]	0	1/4	1/3	1/2	1	0	0	0	0	0	1
h[2-k]	0	0	1/4	1/3	1/2	1	0	0	0	0	1 1/2 (1.5)
h[3-k]	0	0	0	1/4	1/3	1/2	1	0	0	0	11/6 (1.833)
h[4-k]	0	0	0	0	1/4	1/3	1/2	1	0	0	25/12 (2.083)
h[5-k]	0	0	0	0	0	1/4	1/3	1/2	1	0	13/12 (1.083
h[6-k]	0	0	0	0	0	0		1/3	1/2	1	7/12 (0.583)
h[7-k]	0	0	0	0	0	0	0	1/4	1/3	1/2	1/4 (0.25)
h[8-k]	0	0	0	0	0	0	0	0	1/4	1/3	0

Q4. a) The d.c term is given by

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,dt = \int_{-\tau}^{\tau} 1\,dt = 2\tau \quad \text{since } T=1.$$

~~The~~ is an even signal, $b_n = 0$.

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos n\omega_0 t\,dt \quad \text{where } \omega_0 = 2\pi/T = 2\pi.$$

$$= 2\int_{-\tau}^{\tau} x(t)\cos n\omega_0 t\,dt$$

$$= 2\int_{-\tau}^{\tau} \cos n\omega_0 t\,dt$$

$$= \frac{2}{n\omega_0}\Big[\sin n\omega_0 t\Big]_{-\tau}^{\tau}$$

$$= \frac{2}{2n\pi}\left[\sin 2n\pi\tau - \sin(-2n\pi\tau)\right]$$

$$= \frac{2}{n\pi}\sin(2n\pi\tau).$$

Therefore the Trigonometric Fourier Series is

$$x(t) = 2\tau + \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(2n\pi\tau)\cos(2n\pi t).$$

if $\tau = 1/4$

$$x(t) = 2\left(\tfrac{1}{4}\right) + \frac{2}{\pi}\left[\sin\left(2\pi\left(\tfrac{1}{4}\right)\right)\cos(2\pi t) + \frac{1}{3}\sin\left(6\pi\left(\tfrac{1}{4}\right)\right)\cos(6\pi t)\right.$$
$$\left.\cancel{+\frac{1}{5}\sin\left(10\pi\left(\tfrac{1}{4}\right)\right)\cos(10\pi t)}\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\cos(2\pi t) - \frac{1}{3}\cos(6\pi t) + \frac{1}{5}\cos(10\pi t)\right]$$

Q4. b) i) $Y(\omega) = H(\omega) \cdot W(\omega)$ where $H(\omega)$ is the transfer function of the RC circuit.

$$H(\omega) = \frac{1}{1+j\omega RC} = \frac{1}{1+jk\omega_0 RC} = \frac{1}{1+j100k\pi RC}$$

Therefore $$Y(\omega) = \left(\frac{1}{1+j100k\pi RC}\right)\left(\frac{4}{\pi}\sum_{k=-\infty}^{N}\frac{(-1)^k}{(1-4k^2)}\delta(\omega-k\omega_0)\right)$$

$$= \left(\frac{1}{1+j100k\pi RC}\right)\left(\frac{4}{\pi}\sum_{k=-\infty}^{N}\frac{(-1)^k}{(1-4k^2)}\delta(\omega-100k\pi)\right)$$

ii) Assuming that harmonics ≥ 2 can be ignored,

$$Y(\omega) \simeq \frac{4}{\pi}\left[\frac{1}{3}\underbrace{\frac{\delta(\omega+100\pi)}{(1-j100\pi RC)}}_{k=-1} + \underbrace{\delta(\omega)}_{k=0} + \frac{1}{3}\underbrace{\frac{\delta(\omega-100\pi)}{(1+j100\pi RC)}}_{k=1}\right]$$

Since $e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0)$ we have

$$\frac{1}{2\pi}e^{-j100\pi t} \longleftrightarrow \delta(\omega+100\pi)$$

$$\frac{1}{2\pi}e^{j100\pi t} \longleftrightarrow \delta(\omega-100\pi)$$

Therefore $$y(t) \simeq \frac{4}{\pi}\left\{\frac{1}{2\pi} + \frac{1}{3}\left[\frac{1}{2\pi}\frac{e^{-j100\pi t}}{1-j100\pi RC}\right] + \frac{1}{3}\left[\frac{1}{2\pi}\frac{e^{j100\pi t}}{1+j100\pi RC}\right]\right\}$$

$$= \frac{4}{\pi}\left[\frac{1}{2\pi} + \frac{1}{6\pi}\left(\frac{e^{-j100\pi t}}{1-j100\pi RC}\right) + \frac{1}{6\pi}\left(\frac{e^{j100\pi t}}{1+j100\pi RC}\right)\right]$$

$$= \frac{2}{\pi^2} + \frac{2}{3\pi^2}\left[\frac{e^{-j100\pi t}}{1-j100\pi RC} + \frac{e^{j100\pi t}}{1+j100\pi RC}\right].$$

Q4. b iii)

The ripple voltage is given by

$$\frac{2}{3\pi^2}\left[\frac{e^{j100\pi t}}{1+j100\pi RC}+\frac{e^{-j100\pi t}}{1-j100\pi RC}\right]$$

To achieve

$$\left|\frac{2}{3\pi^2}\left[\frac{e^{j100\pi t}}{1+j100\pi RC}+\frac{e^{-j100\pi t}}{1-j100\pi RC}\right]\right| < 2\times10^{-3}$$

$$\frac{2}{3\pi^2}\left|\frac{\cancel{2}}{\sqrt{1+(100\pi RC)^2}}\right| < \cancel{2}\times10^{-3}$$

$$\frac{1}{\sqrt{1+(100\pi RC)^2}} < \frac{3\pi^2\times10^{-3}}{2}$$

$$RC > 0.2\ s.$$