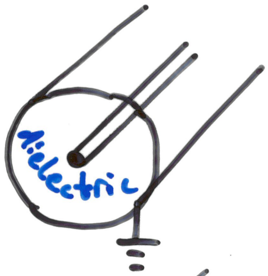
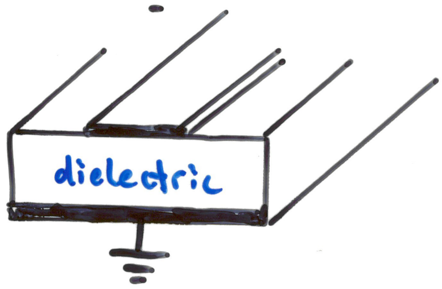
transmission lines (1D)

forms: twin wires

coaxial cable



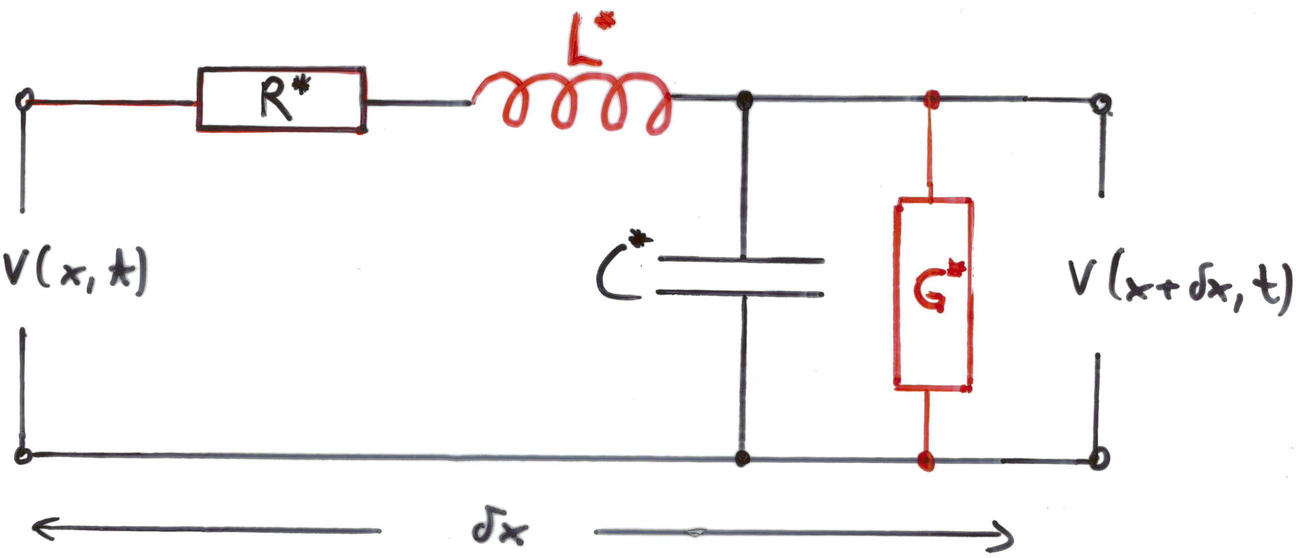
microstrip



black = metal

consider infinitesimal length δx of transmission line to get a

model of transmission line element ideal / with losses



R^*, G^*, C^*, L^* are specific resistance per length (Ω/m)
conductance (S/m)
capacitance (F/m)
inductance (H/m)

now consider for lossy transmission line

a) <u>voltage drop along the lines:</u>

$$-\Delta V = R^* \delta x \cdot I + L^* \delta x \frac{\partial I}{\partial t} \Rightarrow \boxed{-\frac{\partial V}{\partial x} = R^* I + L^* \frac{\partial I}{\partial t}}$$

b) <u>leakage current between the lines:</u>

$$-\Delta I = G^* \delta x \cdot V + C^* \delta x \frac{\partial V}{\partial t} \Rightarrow \boxed{-\frac{\partial I}{\partial x} = G^* V + C^* \frac{\partial V}{\partial t}}$$

<u>1. case: ideal case $R^* = G^* = 0$</u>

a) $-\frac{\partial V}{\partial x} = L^* \frac{\partial I}{\partial t} \xrightarrow{\partial_x} -\frac{\partial^2 V}{\partial x^2} = L^* \frac{\partial^2 I}{\partial t \partial x}$ (i)

b) $-\frac{\partial I}{\partial x} = C^* \frac{\partial V}{\partial t} \xrightarrow{\partial_t} -\frac{\partial^2 I}{\partial x \partial t} = C^* \frac{\partial^2 V}{\partial t^2}$ (ii)

insert (ii) into (i)

$$\Rightarrow -\frac{\partial^2 V}{\partial x^2} = L^* (-C^*) \frac{\partial^2 V}{\partial t^2}$$

$$\Rightarrow \frac{\partial^2 V}{\partial t^2} = \frac{1}{LC} \frac{\partial^2 V}{\partial x^2}$$ is propagating $V(x,t)$

a) analogue: $\xrightarrow{\partial_t} -\frac{\partial^2 V}{\partial t \partial x} = L^* \frac{\partial^2 I}{\partial t^2}$ (iii)

b) $\xrightarrow{\partial_x} -\frac{\partial^2 I}{\partial x^2} = C^* \frac{\partial^2 V}{\partial t \partial x}$ (iv)

insert (iii) into (iv)

$$\Rightarrow -\frac{\partial^2 I}{\partial x^2} = C^* \left(-L^* \frac{\partial^2 I}{\partial t^2}\right)$$

$$\Rightarrow \frac{\partial^2 I}{\partial t^2} = \frac{1}{L^* C^*} \frac{\partial^2 I}{\partial x^2}$$ is propagating $I(x,t)$

if $R^* = 0 = G^*$ (ideal transmission line), then both V and I obey wave equations $\frac{\partial^2 f}{\partial t^2} = v^2 \frac{\partial^2 f}{\partial x^2}$ with wave velocity $v = 1/\sqrt{L^* C^*}$

As $v = \lambda \cdot f = \frac{\omega}{k}$ we could also write $\boxed{k = \omega \sqrt{L^* C^*}}$

2. case : zero resistance but small, finite conductance
$(R^* = 0)$ $(G^* \text{ small})$

a) $-\frac{\partial V}{\partial x} = L^* \frac{\partial I}{\partial t}$

b) $-\frac{\partial I}{\partial x} = G^* V + C^* \frac{\partial V}{\partial t}$

from a) $\xrightarrow{\partial t}$ $\frac{\partial^2 V}{\partial t\, \partial x} = -L^* \frac{\partial^2 I}{\partial t^2}$ (i)

from b) $\xrightarrow{\partial x}$ $\frac{\partial^2 I}{\partial x^2} = -G^* \frac{\partial V}{\partial x} - C^* \frac{\partial^2 V}{\partial x \partial t}$ (ii)

eliminate V by inserting a) and (i) into (ii):

$$\frac{\partial^2 I}{\partial x^2} = G^* L^* \frac{\partial I}{\partial t} + C^* L^* \frac{\partial^2 I}{\partial t^2}$$
$$= \underbrace{L^* C^* \frac{\partial^2 I}{\partial t^2}}_{\substack{\text{as before} \\ (2^{nd} \text{ derivative})}} + \underbrace{L^* G^* \frac{\partial I}{\partial t}}_{\substack{\text{new} \\ (1^{st} \text{ derivative})}}$$

assume $V = V_0 e^{j(\omega t - \tilde{k} x)}$

and $I = I_0 e^{j(\omega t - \tilde{k} x)}$ for a modified $\tilde{k}$ (iii)

insert (iii) into above and get

with $\frac{\partial I}{\partial x} = -j\tilde{k} I$, $\frac{\partial^2 I}{\partial x^2} = -\tilde{k}^2 I$

$\frac{\partial I}{\partial t} = j\omega I$, $\frac{\partial^2 I}{\partial t^2} = -\omega^2 I$

$$\Rightarrow \quad -\tilde{k}^2 \cancel{I} = L^* C^* (-\omega^2) \cancel{I} + L^* G^* j\omega \cancel{I}$$

$$\Rightarrow \quad \tilde{k}^2 = \omega^2 L^* C^* - j\omega L^* G^*$$

$$\Rightarrow \quad \tilde{k} = \sqrt{\omega^2 L^* C^* - j\omega L^* G^*}$$
$$= \omega \sqrt{L^* C^*} \underbrace{\sqrt{1 - \frac{jG^*}{\omega C^*}}}_{\approx 1 - \frac{jG^*}{2\omega C^*}} \quad \text{for small } G^*$$

$$\tilde{k} \approx \underbrace{\omega \sqrt{L^* C^*}}_{\text{as before}} \underbrace{\left(1 - j\frac{G^*}{2\omega C^*}\right)}_{\text{new}}$$

meaning of the new term can be appreciated by inserting result into (iii):

$$I = I_0 \, e^{j(\omega t - \tilde{k} x)} \quad \text{with } \tilde{k} \approx \omega\sqrt{L^*C^*}\left(1 - \frac{jG^*}{2\omega C^*}\right)$$

$$= I_0 \, e^{j\left[\omega t - \omega\sqrt{L^*C^*}\,x + j\sqrt{L^*C^*}\frac{G^*}{2C^*}x\right]}$$

$$= \underbrace{I_0 \, e^{j(\omega t - \omega\sqrt{L^*C^*}\,x)}}_{\text{propagating wave as before, with } k = \omega\sqrt{L^*C^*}\text{, as before}} \; \underbrace{e^{-\sqrt{L^*C^*}\frac{G^*}{2C^*}x}}_{\text{attenuation with exponential where } \beta = \sqrt{L^*C^*}\frac{G^*}{2C^*} \text{ is inverse of } 1/e \text{ decay length}}$$

propagating wave as before, with $k = \omega\sqrt{L^*C^*}$, as before

attenuation with exponential where $\beta = -\sqrt{L^*C^*}\,\frac{G^*}{2C^*}$ is inverse of $1/e$ decay length

3. case: general solution

of
$$-\frac{\partial V}{\partial x} = R^* I + L^* \frac{\partial I}{\partial t} \qquad \text{from page 14} \qquad \text{(i)}$$
$$-\frac{\partial I}{\partial x} = G^* V + C^* \frac{\partial V}{\partial t} \qquad \text{(ii)}$$

$$\Rightarrow \quad -\frac{\partial^2 I}{\partial x^2} = G^* \frac{\partial V}{\partial x} + C^* \frac{\partial^2 V}{\partial x \partial t}$$

$\downarrow$ from (i): $-R^* I - L^* \frac{\partial I}{\partial t}$ $\qquad$ $\downarrow$ from $\frac{\partial}{\partial t}$ of (i): $-R^* \frac{\partial I}{\partial t} - L^* \frac{\partial^2 I}{\partial t^2}$

$$\Rightarrow \quad \frac{\partial^2 I}{\partial x^2} = L^*C^* \frac{\partial^2 I}{\partial t^2} + R^*G^* I + (R^*C^* + L^*G^*)\frac{\partial I}{\partial t}$$

Assume again $I = I_0 \, e^{j(\omega t - \tilde{k} x)}$

$$\Rightarrow \quad -\tilde{k}^2 \cancel{I} = -L^*C^*\omega^2 \cancel{I} + R^*G^* \cancel{I} + (R^*C^* + L^*G^*)\, j\omega \cancel{I}$$

$$\Rightarrow \quad \boxed{\tilde{k}^2 = \underbrace{\omega^2 L^*C^*}_{\text{green}} - \underbrace{R^*G^*}_{\text{blue}} - \underbrace{j\omega (R^*C^* + L^*G^*)}_{\text{red}}}$$

k_0^2 for wave without any attenuation, depends on $\omega \rightarrow$ dispersion

reduction in $\tilde{k}$, i.e. redshift of wave length

describes exponential attenuation