Solutions EEE309

Q1 a.

For the two signals $x[n]=e^{j\omega n}$ and $x(t)= e^{j\omega t}$, since *n* is always an integer, there are two important differences between them:

1). Consider the frequency $(\omega+2\pi)$, we have
$\underline{x}[n]=e^{j(\omega+2\pi)n}= e^{j\omega n}\, e^{j2\pi n} =e^{j\omega n}=x[n]$
More generally, complex exponential sequences with frequencies $(\omega+2\pi r)$, where r is an integer, are indistinguishable from one another, and we can conclude that, when discussing complex exponential signals, we need only consider frequencies in an interval of length 2π.

2). Periodicity. In the continuous-time case, the complex exponential signal $x(t)= e^{j\omega t}$ is always periodic, with the period equal to $2\pi/\omega$. In the discrete-time case, a sequence is periodic when

$$x[n] = x[n + N] \qquad \text{for all } n$$

where N is an integer. Then for the discrete-time signal, for

$\underline{x}[n+N]=e^{j(\omega n+\omega N)}= e^{j\omega n}\, e^{j\omega N} =e^{j\omega n}=x[n]$

to hold, we require

$\omega N=2\pi k$

where k is an integer. As a result, the complex exponential sequence $x[n]=e^{j\omega n}$ is not necessarily periodic.

However, there are indeed N distinguishable frequencies for which $x[n]=e^{j\omega n}$ is periodic with period N. One such set of frequencies is

$$\omega_k = \frac{2\pi k}{N}$$

with k =0, 1, …, N-1.

Q1 b.

Suppose the impulse response of the system is h[n]. Then given the input $x[n]=\alpha^n$, its output y[n] is given by

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]\alpha^{(n-k)}$$

$$= \alpha^n \sum_{k=-\infty}^{+\infty} h[k]\alpha^{-k} = \beta x[n]$$

where $\beta = \sum_{k=-\infty}^{+\infty} h[k]\alpha^{-k}$ is a scalar.

So α^n is the eigenfunction of the system.

Q1 c.

To find the equivalent impulse response of the cascaded system, consider the response of the system to the impulse $\delta[n]$. Then the output of the first system with an impulse response $h_1[n]$ will be exactly $h_1[n]$.
(2 marks)
Now the output $h_1[n]$ of the first system is fed into the second system with an impulse response $h_2[n]$. Then the output of the second system will be the convolution of its input $h_1[n]$ and its impulse response $h_2[n]$.
(1 mark)
So for the whole system, given input $\delta[n]$, the output is $h_1[n]*h_2[n]$, i.e. the impulse response of the cascaded system is the by the convolution of $h_1[n]$ and $h_2[n]$.
(1 mark)

Q1 d.

i) Complex Fourier series

For a continuous signal x(t) with a period T, its complex Fourier series coefficients are given by

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt \,, \quad \omega_0 = \frac{2\pi}{T}$$

Here C_k, k=-∞, …, +∞ is a discrete series and not periodic with respect to k.

ii) Fourier transform

For a non-periodic continuous signal x(t), its Fourier transform is given by

$$X(e^{j\omega}) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$$

The function $X(e^{j\omega})$ is a non-periodic continuous function.

iii) Discrete-time Fourier transform

For a non-periodic discrete-time sequence x[n], its discrete-time Fourier transform (DTFT) is given by

$$X(e^{j\Omega}) = \sum_{-\infty}^{+\infty} x[n] e^{-jn\Omega}$$

$X(e^{j\Omega})$ is a periodic continuous function of Ω.

iv) Discrete Fourier transform

For the finite-duration sequence x[n] of length N (not periodic), its DFT for 0<=k<=N-1 is defined as (analysis equation)

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)kn}$$

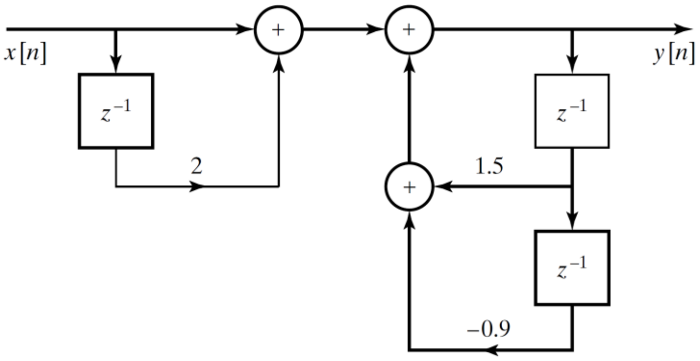
Although the DFT is defined on a finite-duration sequence for the range 0<=n<=N-1, it implicitly assumes the sequence itself is periodic with a period of N and its values for one period are identical to the original finite-duration sequence x[n] for n=0, 1, …, N-1. The DFT sequence X[k] is also periodic with a period N.

Q2 a.

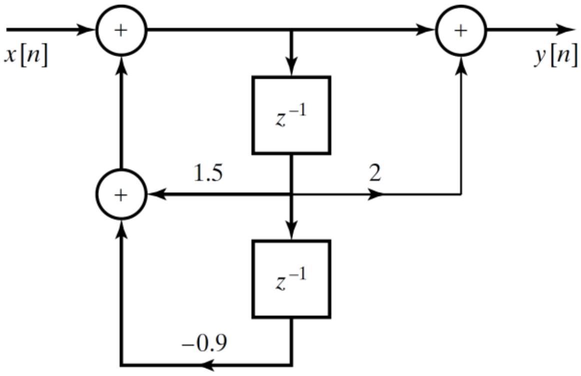
For the system function

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

Its direct form I implementation is



Its direct form II implementation is



Q2 b.

i) For the discrete-time signal

$$x[n] = (\frac{1}{3})^n u[n] + (-\frac{1}{4})^n u[n]$$

its z-transform is

$$X(z) = \sum_{n=-\infty}^{+\infty} [(\frac{1}{3})^n u[n] + (-\frac{1}{4})^n u[n]] z^{-n}$$

$$= \sum_{n=0}^{+\infty} [(\frac{1}{3} z^{-1})^n + (-\frac{1}{4} z^{-1})^n]$$

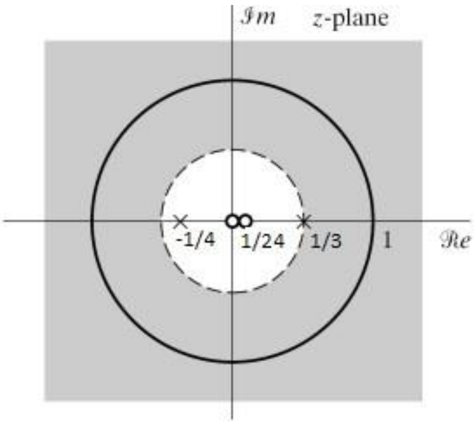
$$= \frac{1}{1-\frac{1}{3}z^{-1}} + \frac{1}{1+\frac{1}{4}z^{-1}} = \frac{3z}{3z-1} + \frac{4z}{4z+1}$$

$$= \frac{3z}{3z-1} + \frac{4z}{4z+1} = \frac{z(24z-1)}{(3z-1)(4z+1)}$$

(2 marks)

For the convergence of X(z), both sums must converge. Then we have |z|>1/3 and |z|>1/4. As a result, the ROC is the region of overlap |z|>1/3.
(2 marks)

Pole-zero plot and its ROC:



(2 marks)

Q2 c.

The z-transform of the system is $H(z)=2-z^{-1}+z^{-2}-0.4z^{-3}$
The z-transform of the input sequence is $X(z)=1+z^{-1}+z^{-2}+z^{-3}$
(1 mark)

Then the z-transform of the output is

$Y(z)=X(z)H(z)= 2+z^{-1}+2z^{-2}+1.6z^{-3}-0.4z^{-4}+0.6z^{-5}-0.4z^{-6}$

(2 marks)

So the output sequence is

$y[n]=2\delta [n]+ \delta [n-1]+2\delta [n-2]+1.6\delta [n-3]-0.4\delta [n-4]+0.6\delta [n-5]-0.4\delta [n-6]$.

(2 marks)

Q2 d.

Using linear convolution, the third sequence $x_3[n]$ is given by

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

The product $x_1[m]x_2[n-m]$ is zero for all *m* whenever $n<0$ and $n>L+P-2$. Therefore, $(L+P-1)$ is the maximum length of the sequence $x_3[n]$.
(1 mark)

To calculate $x_3[n]$ using DFT, we first form the N-point sequence $\hat{x}_1[n]$ by adding N-L zeros to the L-points sequence $x_1[n]$ and the N-point sequence $\hat{x}_2[n]$ by adding N-P zeros to the P-points sequence $x_2[n]$ $(N=L+P-1)$.
(1 mark)

Then we calculate the DFT $X_1[k]$ and $X_2[k]$ of $\hat{x}_1[n]$ and $\hat{x}_2[n]$ for k=0, 1, …, N-1. The product of the two DFTs is given by $X_3[k] = X_1[k]X_2[k]$ with a length of N. Applying the inverse DFT to $X_3[k]$, we then obtain the desired sequence $x_3[n]$.
(3 marks)

Q3 a.

The DFT of a sequence x[n] with length N is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

(2 marks)

For x[n]={1, 2, 2, 1}, N=4, then

$$X[k] = \sum_{n=0}^{3} x[n]e^{-jkn\frac{\pi}{2}} = 1 + 2e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + e^{-jk\frac{3\pi}{2}}$$

X[0]=6, X[1]=-1-j, X[2]=0, X[3]=-1+j

(2 marks)

Q3 b.

i)
Nyquist Sampling Theorem:
Let $x_c(t)$ be a bandlimited signal with

$$X_c(j\omega) = 0 \qquad for \quad |\omega| \geq \omega_N$$

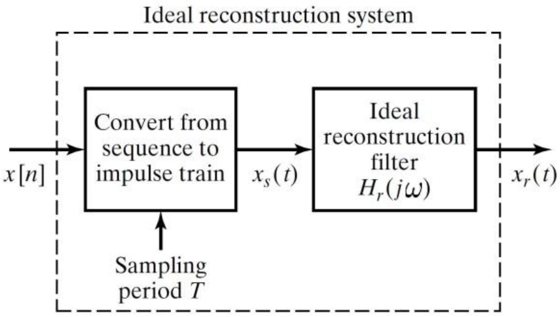
Then $x_c(t)$ is uniquely determined by its samples $x[n]=x_c(nT)$, where T is the sampling period, if

$$\omega_s = \frac{2\pi}{T} \geq 2\omega_N$$

The frequency ω_N is commonly referred to as the Nyquist frequency, and the frequency $2\omega_N$ that must be exceeded by the sampling frequency is called the Nyquist rate.
(3 marks)

The sampling frequency must be at least twice the highest frequency of interest, i.e. $f_s=2X50\pi/(2\pi)=50Hz$.
(1 marks)

ii)
A block diagram representation of the process is given below



(1 mark)

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)$$

(1 mark)

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n]h_r(t - nT)$$

(1 mark)

The ideal lowpass filter $H_r(j\omega)$ ($h_r(t)$) has a gain of *T* and a cutoff frequency of $f_s/2$=25Hz.

(1 mark)

Q3 c.

i)

Impulse invariance method:

Inverse Laplace transform

$h_a(t) = 40e^{-40t}$

(1 mark)

At sampling instants nT (T=1/40 sec), we have

$h_a(nT) = 40e^{-n} = 40 \times 0.368^n$

(1 mark)

From the z-transform table, we have

$$H_d(z) = \frac{40z}{z - 0.368}$$

(2 marks)

ii)

Bilinear transform method

$$\frac{Y(s)}{X(s)} = \frac{40}{s + 40}$$

With

$$s = \frac{2(z-1)}{T(z+1)}$$

(2 marks)

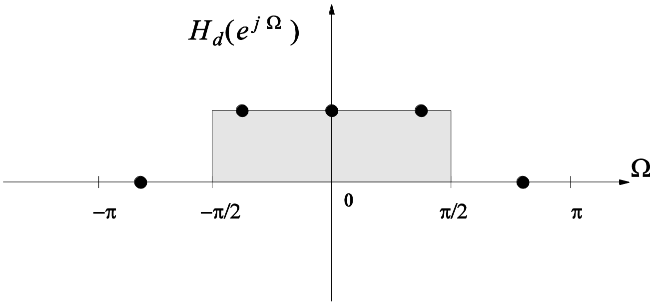
we have

$$H_d(z) = \frac{40}{\frac{80(z-1)}{(z+1)} + 40} = \frac{z+1}{3z-1}$$

(2 marks)

Q4 a.

The passband range between 0.5kHz and 1kHz corresponds to the normalised frequency Ω_l=0.5*2π/2=π/2 to Ω_h=1*2π/2=π
As the spectrum is symmetric with respect to the origin, we can then design a lowpass filter between –π/2 to π/2 first. The ideal frequency response of the corresponding lowpass filter is given by



We approximate the ideal one by 5 equally spaced samples, each 2π/5=1.26 rad apart.

(2 mark)

Using the provided equation, we have

h[0]=0.6, h[1]=0.3236, h[2]=-0.1236, h[3]=-0.1236, h[4]=0.3236.

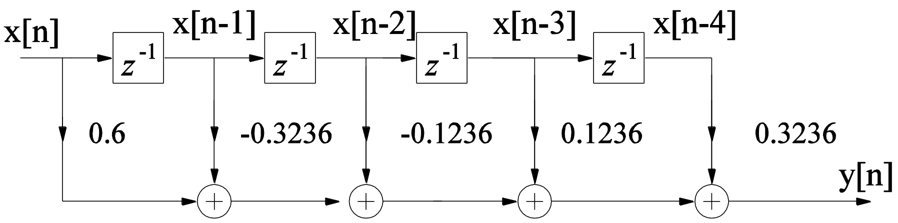
(2 marks)

Note that the relationship between the impulse response $h_{hp}[n]$ of the highpass filter and the impulse response $h_{lp}[n]$ of the lowpass filter is given by $h_{hp}[n]=(-1)^n h_{lp}[n]$, then we have the final design result for the desired highpass filter:

h[0]=0.6, h[1]=-0.3236, h[2]=-0.1236, h[3]=0.1236, h[4]=0.3236.

(2 marks)

ii)



(1 mark)

iii)

y[n]=0.6x[n]-0.3236x[n-1]-0.1236x[n-2]]+0.1236x[n-3]+0.3236x[n-4]

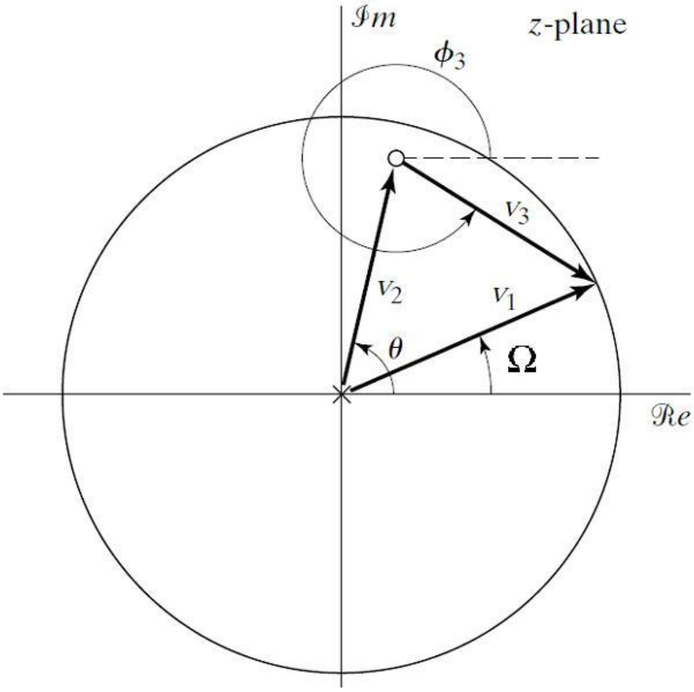
(1 mark)

Q4 b.

i)
For the first-order system function, we have

$$H(z) = (1 - re^{j\theta}z^{-1}) = \frac{z - re^{j\theta}}{z}$$

Such a factor has a pole at z=0 and a zero at $z = re^{j\theta}$.
(1 mark)



The vector $\mathbf{v}_3$ (from the zero to the unit circle) is the zero vector and the vector $\mathbf{v}_1$ (from the pole to the unit circle) is the pole vector.

(3 marks)

ii)

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$=$\mathbf{v}_1$-$\mathbf{v}_2$ represent respectively the complex numbers $e^{j\Omega}$, $re^{j\theta}$ and $e^{j\Omega} - re^{j\theta}$. Then we can express the magnitude response in terms of the three vectors:

$$|H(j\Omega)| = |1 - re^{j\theta}e^{-j\Omega}| = |\frac{e^{j\Omega} - re^{j\theta}}{e^{j\Omega}}|$$

$$= |\frac{v_3}{v_1}| = |v_3|$$

(2 marks)

The corresponding phase is

$$\angle H(j\Omega) = \angle(1 - re^{j\theta}e^{-j\Omega}) = \angle v_3 - \angle v_1$$
$$= \phi_3 - \phi_1 = \phi_3 - \Omega$$

(2 marks)

Q4 c.

To derive this property, we consider the new sequence

$$y[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k]$$

(1 mark)

Its z-transform is given by

$$Y(z) = \sum_{n=-\infty}^{\infty} y[n]z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \right\} z^{-n}$$

(2 marks)

Interchange the order of summation, we have

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \sum_{n=-\infty}^{\infty} x_2[n-k]z^{-n}$$

Changing the index of summation in the second sum from n to m=n-k, we have

$$Y(z) = \sum_{k=-\infty}^{\infty} x_1[k] \left\{ \sum_{m=-\infty}^{\infty} x_2[m]z^{-m} \right\} z^{-k}$$

Then for values of z inside the ROCs of both $X_1(z)$ and $X_2(z)$, we have

$$Y(z) = X_1(z)X_2(z)$$

(2 marks)