The University of Sheffield, Department EEE

Electronic Devices in Circuits Tutorial Solutions: RC Circuit Transfer Functions

The first order transfer functions fall into the three basic types:

low pass $$\frac{v_o}{v_i} = k\,\frac{1}{1+j\dfrac{\omega}{\omega_0}} \tag{1}$$

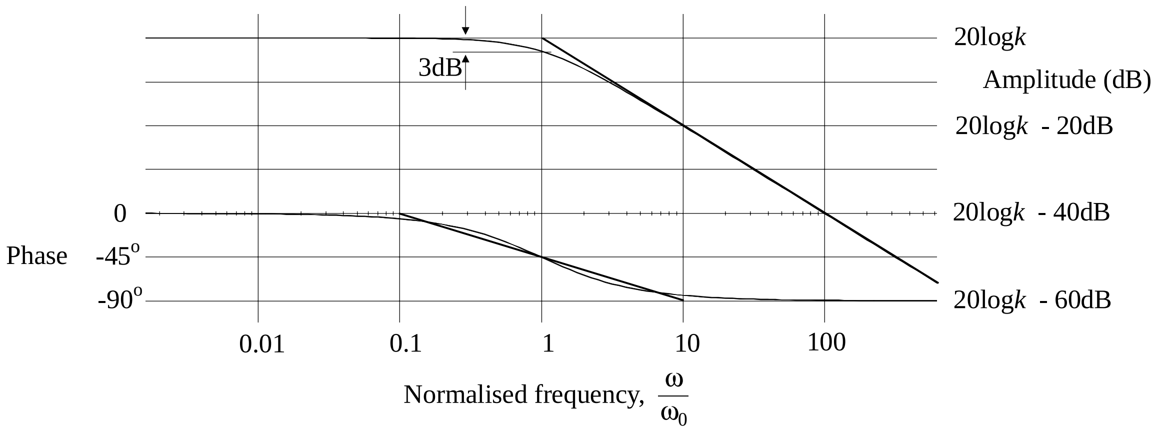
high pass $$\frac{v_o}{v_i} = k\,\frac{j\dfrac{\omega}{\omega_0}}{1+j\dfrac{\omega}{\omega_0}} \tag{2}$$

pole-zero $$\frac{v_o}{v_i} = k\,\frac{1+j\dfrac{\omega}{\omega_1}}{1+j\dfrac{\omega}{\omega_0}} \tag{3}$$

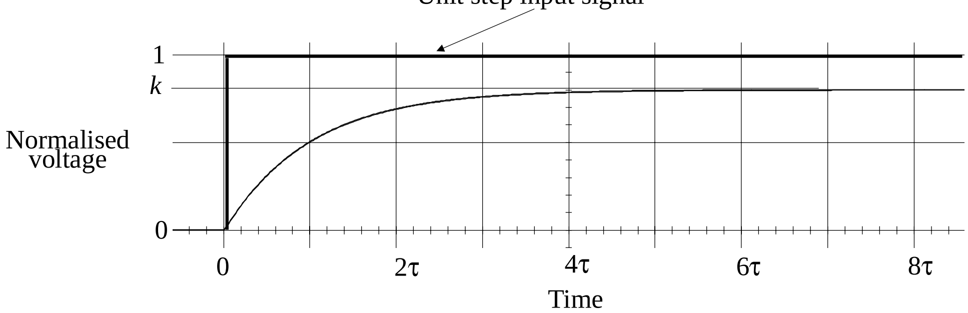
Type (3) is the sum of a low pass (1) and a high pass (2) response with different frequency independent gains. The responses for (1), (2) and (3) are given first and the transfer functions then worked out.

Low Pass

Frequency response:

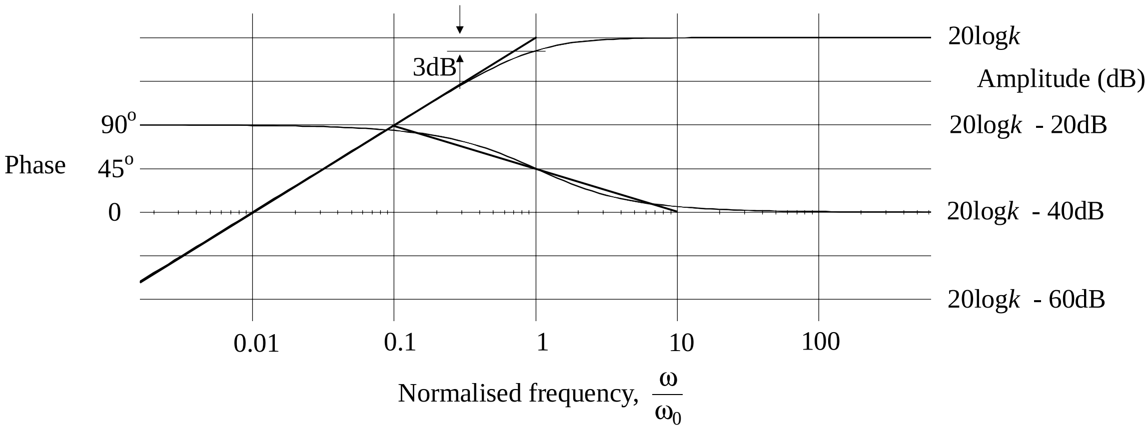


Transient response:

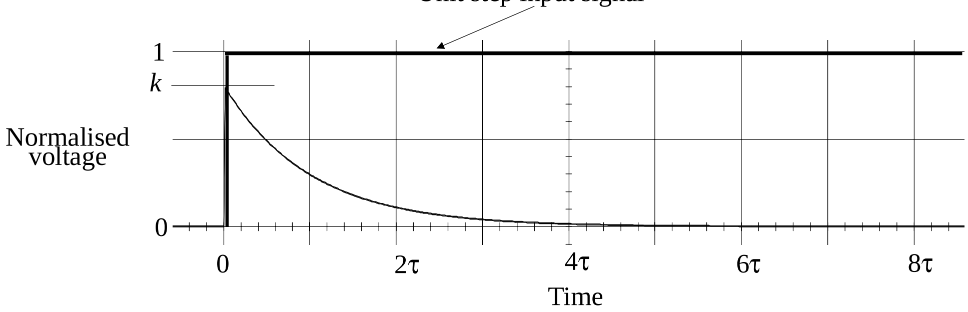


High Pass

Frequency response:



Transient response:



For both high pass and low pass, the pass band gain is given by 20log*k* and the corner frequency by ω_0. Notice how good the straight line approximations (also known as Bode approximations) are to both the amplitude and phase responses. Once the transfer function has been reduced to a standard form, ω_0, *k* and the response type completely specify the response shape.

The transient responses are easily drawn once the response type (high pass or low pass) has been identified. A circuit described by a low pass transfer function will always have a unit step (transient) response of the form $v(t) = k\,(1 - \exp\,(t/\tau)\,)$ whereas one described by a high pass transfer function will always have a unit step response of the form $v(t) = k \exp\,(t/\tau)$. In both cases $\tau = 1/\omega_0$. In the low pass case the frequency independent gain *k* determines the aiming level of the exponential rise while in the high pass case *k* determines the initial height of the exponential waveshape. If the amplitude of the input step is V_S, the initial height and aiming levels become kV_S instead of *k*.

Pole - Zero

The pole-zero case is in fact a linear sum of high pass and low pass responses, each being multiplied by a different frequency independent gain. Equation (3) can be expanded as follows:

$$\frac{v_o}{v_i} = k\,\frac{1 + j\,\dfrac{\omega}{\omega_1}}{1 + j\,\dfrac{\omega}{\omega_0}} = k\,\frac{1}{1 + j\,\dfrac{\omega}{\omega_0}} + k\,\frac{j\,\dfrac{\omega}{\omega_1}}{1 + j\,\dfrac{\omega}{\omega_0}} = k\,\frac{1}{1 + j\,\dfrac{\omega}{\omega_0}} + \frac{k\,\omega_0}{\omega_1}\,\frac{j\,\dfrac{\omega}{\omega_0}}{1 + j\,\dfrac{\omega}{\omega_0}} \qquad (4)$$

The form of equation (3) is most useful from the point of view of frequency responses but the final form of equation (4) is most useful from the point of view of transient responses. The gain multiplying the low pass part of equation (4) - the low frequency gain - can in general be larger than or smaller than the gain multiplying the high pass part of equation (4) - the high frequency gain. In some cases one is always higher than the other.

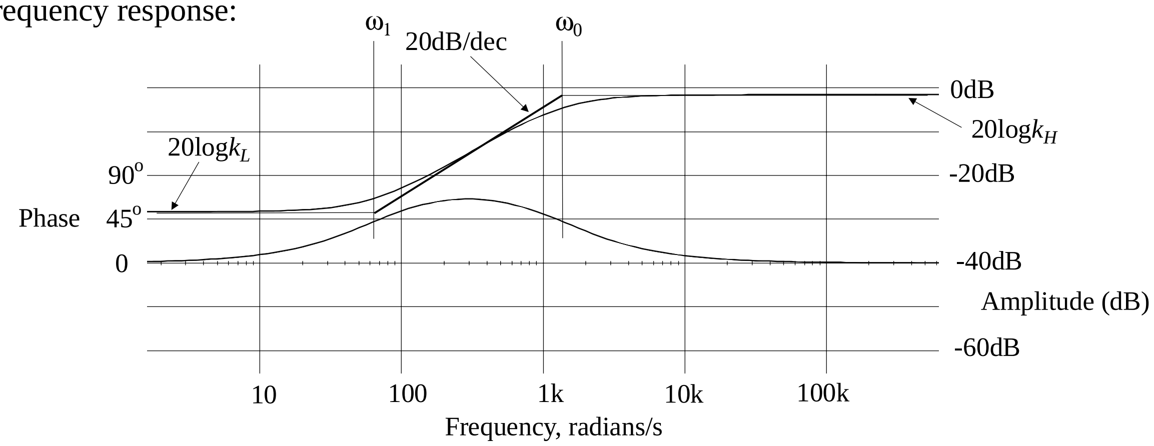
The pole-zero response below is drawn for a circuit where the low frequency gain is lower than the high frequency gain or, in other words, in terms of equation (3), $\omega_1 < \omega_0$. The high frequency gain, k_H, is $k_L\omega_0/\omega_1$ as in the final form of equation (4). Notice the parts ω_0, ω_1, k_L and k_H play in the response shapes - once again they provide the coordinates needed for a straight line approximation to the real curve and define the initial and aiming levels of the transient response.

The phase response appears here as a positive going hump. It gets close to 90^o only if ω_0 and ω_1 are widely spaced (say two orders of magnitude or more). It is not easy to make an accurate sketch of the phase response for pole-zero circuits without evaluating some points on the phase-frequency graph. This shape of phase response is useful as compensation to ensure stability in feedback systems and is commonly used in such applications.

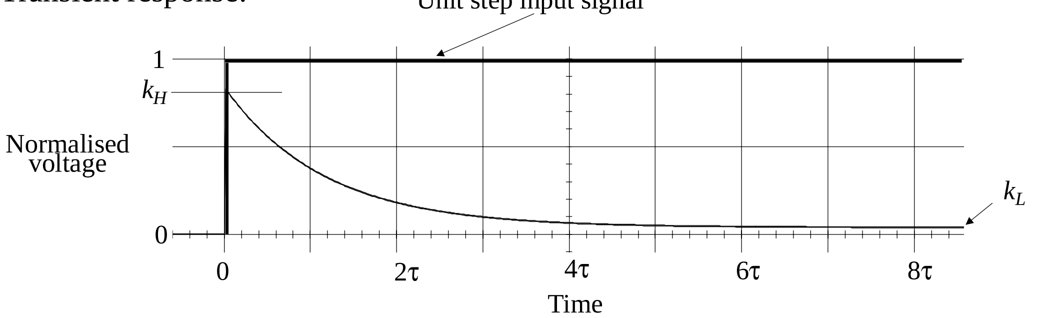
The step reponse moves exponentially from its initial value to its aiming level with a time constant of $1/\omega_0$. Notice that it is always the denominator of the transfer function that gives the circuit time constant.

If $\omega_0 < \omega_1$ the high frequency gain is lower than the low frequency gain, so the gain falls between ω_0 and ω_1. The phase is a negative going hump - essentially an upside down version of the shape shown. The transient response becomes a rising exponential starting with an initial step of k_H (for a unit step input) and aiming for a level of k_L with a time constant of $1/\omega_0$.

Frequency response:

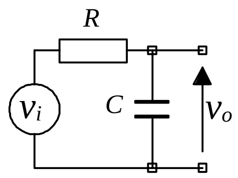


Transient response:



1

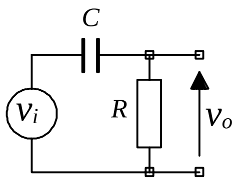
$$\frac{v_o}{v_i} = \frac{\dfrac{1}{j\,\omega C}}{R + \dfrac{1}{j\,\omega C}} = \frac{1}{1 + j\,\omega CR} = \frac{1}{1 + j\,\dfrac{\omega}{\omega_0}}$$



where $\omega_0 = 1/RC$ and $k = 1$. This is a "prototype" low pass circuit.

2

$$\frac{v_o}{v_i} = \frac{R}{R + \dfrac{1}{j\,\omega C}} = \frac{j\,\omega CR}{1 + j\,\omega CR} = \frac{j\,\dfrac{\omega}{\omega_0}}{1 + j\,\dfrac{\omega}{\omega_0}}$$

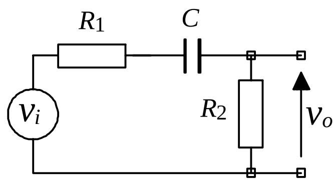


where $\omega_0 = 1/RC$ and $k = 1$. This is a "prototype" high pass circuit.

3

$$\frac{v_o}{v_i} = \frac{R_2}{R_1 + R_2 + \dfrac{1}{j\,\omega C}} = \frac{j\,\omega C R_2}{1 + j\,\omega C\,(R_1 + R_2)}$$

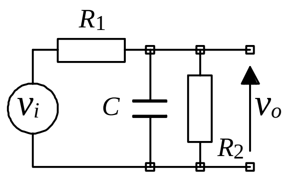
$$= \frac{R_2}{R_1 + R_2}\;\frac{j\,\omega C\,(R_1 + R_2)}{1 + j\,\omega C\,(R_1 + R_2)} = k\;\frac{j\,\dfrac{\omega}{\omega_0}}{1 + j\,\dfrac{\omega}{\omega_0}}$$



where $\omega_0 = 1/C(R_1 + R_2)$ and $k = R_2/(R_1 + R_2)$. This is a high pass circuit.

4

$$\frac{v_o}{v_i} = \frac{\dfrac{\dfrac{R_2}{j\,\omega C}}{R_2 + \dfrac{1}{j\,\omega C}}}{R_1 + \dfrac{\dfrac{R_2}{j\,\omega C}}{R_2 + \dfrac{1}{j\,\omega C}}} = \frac{\dfrac{R_2}{1 + j\,\omega C\,R_2}}{R_1 + \dfrac{R_2}{1 + j\,\omega C\,R_2}}$$



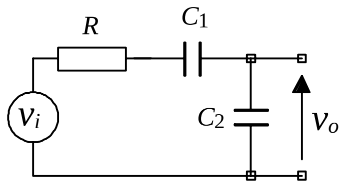
$$= \frac{R_2}{R_1 + j\,\omega C\,R_2 R_1 + R_2} = \frac{R_2}{R_1 + R_2}\;\frac{1}{1 + j\,\omega\,\dfrac{C\,R_2 R_1}{R_1 + R_2}} = k\;\frac{1}{1 + j\,\dfrac{\omega}{\omega_0}}$$

where $\omega_0 = (R_1 + R_2)/CR_1R_2$ and $k = R_2/(R_1 + R_2)$. This is a low pass circuit.

5

$$\frac{v_o}{v_i} = \frac{\dfrac{1}{j\,\omega C_2}}{R + \dfrac{1}{j\,\omega C_1} + \dfrac{1}{j\,\omega C_2}} = \frac{1}{j\,\omega C_2 R + \dfrac{C_2}{C_1} + 1}$$

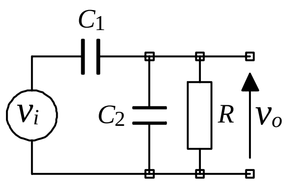
$$= \frac{C_1}{C_1 + C_2}\,\frac{1}{1 + j\,\omega\,\dfrac{C_1 C_2 R}{C_1 + C_2}} = k\,\frac{1}{1 + j\,\dfrac{\omega}{\omega_0}}$$



where $\omega_0 = (C_1 + C_2)/RC_1C_2$ and $k = C_1/(C_1 + C_2)$. This is a low pass circuit.

6

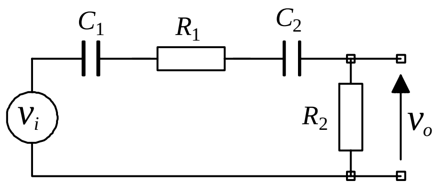
$$\frac{v_o}{v_i} = \frac{\dfrac{\dfrac{R}{j\,\omega C_2}}{R + \dfrac{1}{j\,\omega C_2}}}{\dfrac{1}{j\,\omega C_1} + \dfrac{\dfrac{R}{j\,\omega C_2}}{R + \dfrac{1}{j\,\omega C_2}}} = \frac{\dfrac{R}{1 + j\,\omega C_2 R}}{\dfrac{1}{j\,\omega C_1} + \dfrac{R}{1 + j\,\omega C_2 R}} = \frac{j\,\omega C_1 R}{1 + j\,\omega C_2 R + j\,\omega C_1 R}$$



$$= \frac{j\,\omega C_1 R}{1 + j\,\omega (C_1 + C_2) R} = \frac{C_1}{C_1 + C_2}\,\frac{j\,\omega (C_1 + C_2) R}{1 + j\,\omega (C_1 + C_2) R} = k\,\frac{j\,\dfrac{\omega}{\omega_0}}{1 + j\,\dfrac{\omega}{\omega_0}}$$

where $\omega_0 = 1/R\,(C_1 + C_2)$ and $k = C_1/(C_1 + C_2)$. This is a high pass circuit.

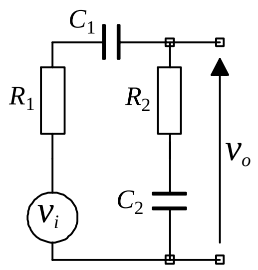
7 This circuit will give the same responses as (3) since C_1 and C_2 can be combined into a single component which will perform the function of C in (3). The response is thus a high pass response of the form:



$$\frac{v_o}{v_i} = k\,\frac{j\,\dfrac{\omega}{\omega_0}}{1 + j\,\dfrac{\omega}{\omega_0}}$$ where $\omega_0 = (C_1 + C_2)/C_1C_2(R_1 + R_2)$ and $k = R_2/(R_1 + R_2)$

8

$$\frac{v_o}{v_i} = \frac{R_2 + \dfrac{1}{j\,\omega C_2}}{R_1 + R_2 + \dfrac{1}{j\,\omega C_1} + \dfrac{1}{j\,\omega C_2}} = \frac{j\,\omega C_2 R_2 + 1}{j\,\omega C_2 (R_1 + R_2) + \dfrac{C_2}{C_1} + 1}$$

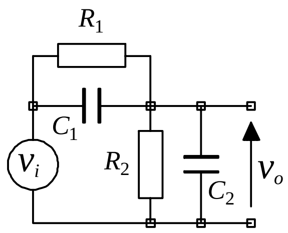


$$= \frac{C_1}{C_1+C_2}\ \frac{1+j\,\omega C_2R_2}{1+j\,\omega\,\dfrac{C_1C_2(R_1+R_2)}{C_1+C_2}} = k\ \frac{1+j\,\dfrac{\omega}{\omega_1}}{1+j\,\dfrac{\omega}{\omega_0}}$$

where $\omega_0 = (C_1+C_2)/C_1C_2(R_1+R_2)$, $\omega_1 = 1/C_2R_2$ and $k = C_1/(C_1+C_2)$. The circuit is a pole - zero circuit in which ω_0 can be greater than or less than ω_1.

9

$$\frac{v_o}{v_i} = \frac{\dfrac{\dfrac{R_2}{j\,\omega C_2}}{R_2+\dfrac{1}{j\,\omega C_2}}}{\dfrac{\dfrac{R_1}{j\,\omega C_1}}{R_1+\dfrac{1}{j\,\omega C_1}}+\dfrac{\dfrac{R_2}{j\,\omega C_2}}{R_2+\dfrac{1}{j\,\omega C_2}}} = \frac{\dfrac{R_2}{1+j\,\omega C_2R_2}}{\dfrac{R_1}{1+j\,\omega C_1R_1}+\dfrac{R_2}{1+j\,\omega C_2R_2}}$$



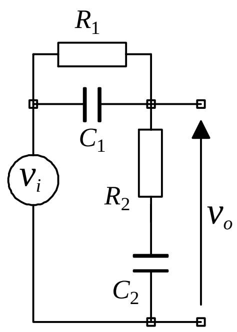
$$= \frac{R_2(1+j\,\omega C_1R_1)}{R_1(1+j\,\omega C_2R_2)+R_2(1+j\,\omega C_1R_1)} = \frac{R_2(1+j\,\omega C_1R_1)}{R_1+R_2+j\,\omega(C_1+C_2)R_1R_2}$$

$$= \frac{R_2}{R_1+R_2}\ \frac{1+j\,\omega C_1R_1}{1+j\,\omega\,\dfrac{(C_1+C_2)R_1R_2}{R_1+R_2}} = k\ \frac{1+j\,\dfrac{\omega}{\omega_1}}{1+j\,\dfrac{\omega}{\omega_0}}$$

where $\omega_0 = (R_1+R_2)/R_1R_2(C_1+C_2)$, $\omega_1 = 1/C_1R_1$ and $k = R_2/(R_1+R_2)$. The circuit is a pole - zero circuit in which ω_0 can be greater than or less than ω_1. This particular circuit is important because it represents the equivalent circuit of a dividing probe - oscilloscope input combination. R_2 and C_2 represent the input impedance of the oscilloscope and R_1 and C_1 the probe impedance. C_2 usually includes an adjustable component that is part of the probe and can be adjusted to make $\omega_1 = \omega_0$. Under this condition all frequency dependent effects vanish and the circuit is said to be balanced. The same ideas are used in all wideband attenuators.

10

$$\frac{v_o}{v_i} = \frac{R_2+\dfrac{1}{j\,\omega C_2}}{\dfrac{\dfrac{R_1}{j\,\omega C_1}}{R_1+\dfrac{1}{j\,\omega C_1}}+R_2+\dfrac{1}{j\,\omega C_2}} = \frac{R_2+\dfrac{1}{j\,\omega C_2}}{\dfrac{R_1}{1+j\,\omega C_1R_1}+R_2+\dfrac{1}{j\,\omega C_2}}$$

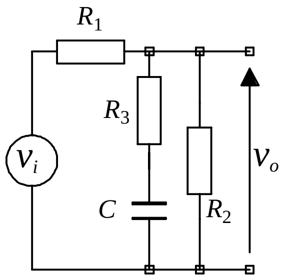


$$= \frac{(j\,\omega C_2R_2+1)(1+j\,\omega C_1R_1)}{j\,\omega C_2R_1+j\omega C_2R_2+(j\,\omega)^2C_1C_2R_1R_2+1+j\,\omega C_1R_1}$$

$$= \frac{1 + j\,\omega(C_1R_1 + C_2R_2) + (j\,\omega)^2 C_1C_2R_1R_2}{1 + j\,\omega(C_1R_1 + C_2R_1 + C_2R_2) + (j\,\omega)^2 C_1C_2R_1R_2}$$

This is a second order transfer function because the denominator contains terms of $(j\,\omega)^2$.

11



$$\frac{v_o}{v_i} = \frac{\dfrac{R_2\left(R_3 + \dfrac{1}{j\,\omega C}\right)}{R_2 + R_3 + \dfrac{1}{j\,\omega C}}}{R_1 + \dfrac{R_2\left(R_3 + \dfrac{1}{j\,\omega C}\right)}{R_2 + R_3 + \dfrac{1}{j\,\omega C}}} = \frac{\dfrac{R_2\,(1 + j\,\omega C\,R_3)}{1 + j\,\omega C\,(R_2 + R_3)}}{R_1 + \dfrac{R_2\,(1 + j\,\omega C\,R_3)}{1 + j\,\omega C\,(R_2 + R_3)}}$$

$$= \frac{R_2\,(1 + j\,\omega C\,R_3)}{R_1 + j\,\omega C\,R_1(R_2 + R_3) + R_2 + j\,\omega C\,R_1R_3} = \frac{R_2}{R_1 + R_2}\;\frac{1 + j\,\omega C\,R_3}{1 + \;j\,\omega\,\dfrac{C\,(R_1R_2 + R_1R_3 + R_2R_3)}{R_1 + R_2}}$$

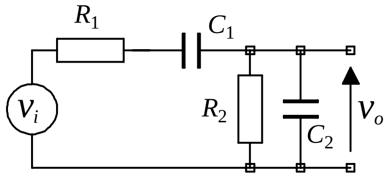
$$= k\,\frac{1 + j\,\dfrac{\omega}{\omega_1}}{1 + j\,\dfrac{\omega}{\omega_0}} \quad \text{where } \omega_0 = \frac{R_1 + R_2}{C\,(R_1R_2 + R_1R_3 + R_2R_3)}\,, \quad \omega_1 = \frac{1}{C\,R_3} \quad \text{and} \quad k = \frac{R_2}{R_1 + R_2}$$

In this circuit ω_0 is always less than ω_1 and there are various ways of showing that this is the case. Inspection of the circuit reveals that the high frequency gain is always lower than the low frequency gain implying that the pole frequency is always lower than the zero frequency. An alternative is to examine the sign of $\omega_0 - \omega_1$: if $\omega_0 > \omega_1$ the answer is positive but if $\omega_0 < \omega_1$ the answer is negative. In the cases where $\omega_0 > \omega_1$ or $\omega_0 < \omega_1$ may be true the sign of $\omega_0 - \omega_1$ will depend on circuit component values.

12

$$\frac{v_o}{v_i} = \frac{\dfrac{\dfrac{R_2}{j\,\omega C}}{R_2 + \dfrac{1}{j\,\omega C}}}{R_1 + \dfrac{1}{j\,\omega C_1} + \dfrac{\dfrac{R_2}{j\,\omega C}}{R_2 + \dfrac{1}{j\,\omega C}}} = \frac{\dfrac{R_2}{1 + j\,\omega C\,R_2}}{R_1 + \dfrac{1}{j\,\omega C_1} + \dfrac{R_2}{1 + j\,\omega C\,R_2}}$$

$$= \frac{j\,\omega C_1R_2}{1 + j\,\omega(C_1R_1 + C_1R_2 + C_2R_2) + (j\,\omega)^2 C_1C_2R_1R_2}$$



This is a second order circuit. The intermediate steps between line one and line two are left as an exercise for you!