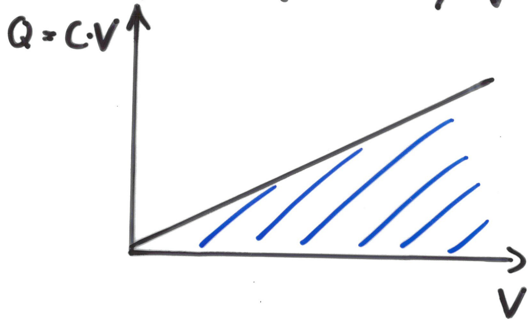
Energy in the electromagnetic field – the Poynting vector

1. energy in the electric field

Consider charging a capacitor. The energy stored is the work W done by voltage V trying to store charge Q on the plates.



$$W_{electric} = \underset{\uparrow \atop P = \frac{dW}{dt}}{=} \int P\,dt \underset{\uparrow \atop P = V\cdot I}{=} \int V I\,dt \underset{\uparrow \atop I = \frac{\partial Q}{\partial t}}{=} \int V\,dQ \underset{\uparrow \atop Q = C\cdot V}{=} \frac{1}{C}\int Q\,dQ = \frac{1}{2C} Q^2 = \frac{1}{2} C V^2$$

where

$$C = \frac{\varepsilon_0 \varepsilon_r w l}{d}$$

and

$$\underline{E} = -\operatorname{grad} V \Rightarrow V = -\int \underline{E}\,d\underline{x} = -|\underline{E}|\cdot d$$

$$\Rightarrow W_{electric} = \frac{1}{2} \frac{\varepsilon_0 \varepsilon_r w l}{\cancel{d}} |\underline{E}|^2 d^{\cancel{2}}$$

$$= \frac{1}{2} \varepsilon_0 \varepsilon_r \underbrace{w l d}_{=\text{volume between plates}} |\underline{E}|^2$$

$\Rightarrow$ energy <u>density</u> of electric field: $\boxed{\tilde{w}_e = \frac{1}{2}\varepsilon_0 \varepsilon_r E^2 = \frac{1}{2} E D}$

2. energy in the magnetic field

$$W_{magnetic} = \int P\,dt = \int V I\,dt \underset{\uparrow \atop V = L\cdot\frac{\partial I}{\partial t}}{=} L \int \frac{\partial I}{\partial t} I\,dt = L\int I\,dI = \frac{1}{2} L I^2$$

is the energy stored in an inductor

where

$$L = \mu_0 \mu_r \frac{d l}{w}$$ (from p. 33)

and

$I = \oint \underline{H}\, d\underline{r} = Hw$ (p. 33/34)

$\Rightarrow W_{magnetic} = \frac{1}{2}\mu_0\mu_r \frac{d\ell}{\cancel{w}} H^2 w^{\cancel{2}}$

$= \frac{1}{2}\mu_0\mu_r \underbrace{d\ell w}_{\text{volume}} |\underline{H}|^2$

$\Rightarrow$ energy density of magnetic field: $\boxed{\tilde{w}_m = \frac{1}{2}\mu_0\mu_r H^2 = \frac{1}{2}HB}$

3. total energy in the electromagnetic field per unit volume:

$$\boxed{\tilde{w}_{total} = \tilde{w}_e + \tilde{w}_m = \frac{1}{2}(\varepsilon_0\varepsilon_r E^2 + \mu_0\mu_r H^2) = \frac{1}{2}(ED + HB)}$$

Note that both fields travel as plane waves between the parallel plates of the transmission line, with $\underline{E} \perp \underline{H} \perp \underline{k}$.

4. power per unit area between the parallel plates:

$$\boxed{\tilde{P} = \text{power density}} = \frac{P}{A} = \frac{W}{t}\frac{1}{A} = \frac{W}{t}\frac{1}{wd} = \underbrace{\frac{W}{w\ell d}}_{\text{volume}} \cdot \underbrace{\frac{\ell}{t}}_{\text{speed}}$$

$$= \tilde{w}_{total} \cdot v$$

with above $\tilde{w}_{total}$ and $v = \frac{1}{\sqrt{\mu_0\mu_r\varepsilon_0\varepsilon_r}}$ as velocity of the wave

$$= \frac{1}{2}\left(\sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}E^2 + \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}H^2\right)$$

remember that

$Z_0 = \frac{E}{H} = \sqrt{\frac{\mu_0\mu_r}{\varepsilon_0\varepsilon_r}}$ is the impedance in matter, so

$$\tilde{P} = \frac{1}{2}\left(\frac{H}{\cancel{E}} \cdot E^{\cancel{2}} + \frac{E}{\cancel{H}} H^{\cancel{2}}\right)$$

$$= \boxed{EH}$$

Note that because of the symmetry of both terms in the brackets, the electric and the magnetic field have the same power density!

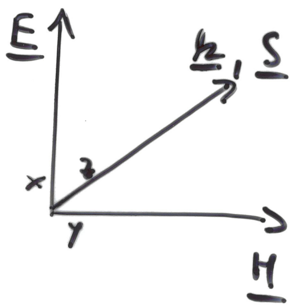
alternative deduction:

$P = V \cdot I$ with $V = dE$ and $I = Hw$, hence

$P = EH \cdot w \cdot d$

$\Rightarrow \tilde{P} = \frac{P}{wd} = EH$ as above definition of the Poynting vector : $\boxed{\underline{S} := \underline{E} \times \underline{H}}$

$\underline{S}$ is a vector pointing into the direction of radiated energy flow, with a magnitude of $|\underline{S}| = |\underline{E}|\,|\underline{H}| \sin \sphericalangle (\underline{E}, \underline{H})$ that gives the magnitude of the power density $\mathcal{P}$ projected in the direction $\tilde{k}$ of flow.



application example:

A semiconductor laser emits 100 mW from a cleaved facet through an active region of size $2\mu m \times 5\mu m$. Find the rms value of the electric field at the facet.

$$S = EH = \frac{E^2}{Z_0} = E^2 \sqrt{\frac{\varepsilon_0 \cancel{\varepsilon_r}}{\mu_0 \cancel{\mu_r}}} = \frac{E^2}{377\,\Omega}$$

$$\overset{''}{\frac{P}{A}} = \frac{100\text{ mW}}{2\mu m \cdot 5\mu m}$$

$$\Rightarrow \quad E \approx 1.9 \cdot 10^6 \frac{V}{m} \approx 2 MV/m \quad (!)$$

absorption in lossy materials

Absorption means power dissipation in the material.

Example: transmission line with lossy dielectric of conductivity σ

conductance: $G = \sigma \cdot \frac{lw}{d}$

$\Rightarrow$ specific conductance: $\boxed{G^* = \frac{G}{l} = \frac{\sigma w}{d}}$

Recall approximation:

$e^{j(\omega t - \tilde{k}x)}$ with $\tilde{k}^2 = \omega^2 L^* C^* \left(1 - j \frac{G^*}{\omega C^*}\right)$

and $C^* = \varepsilon_0 \varepsilon_r \frac{w}{d}$

$$\Rightarrow \boxed{\tilde{k} = \omega \underbrace{\sqrt{L^* C^*}}_{\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}} \sqrt{1 - j \underbrace{\frac{G^*}{\omega C^*}}_{\frac{\sigma}{\omega \varepsilon_0 \varepsilon_r}}}}$$

now has a negative imaginary part, which describes damping!

1. case: weakly absorbing materials : $\frac{\sigma}{\omega \varepsilon_0 \varepsilon_r} \ll 1$

example: light in glass

use Binomial expression $\sqrt{1-x} \approx 1 - \frac{1}{2}x$ for small x

$$\tilde{k} \approx \omega \sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r} \left(1 - \frac{j\sigma}{2\omega \varepsilon_0 \varepsilon_r}\right)$$

$$= \underbrace{\omega \sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r}}_{k_0 = \frac{\omega}{v}} - \underbrace{\frac{j\sigma}{2} \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}}}_{j\beta}$$

with absorption length $\boxed{\beta = \frac{\sigma}{2} \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}}}$

For the EM wave we now have

$$E = E_0 e^{j(\omega t - \tilde{k}x)} = E_0 e^{j(\omega t - k_0 x)} e^{-\beta x}$$

and

$$H = H_0 e^{j(\omega t - \tilde{k}x)} = \underbrace{H_0 e^{j(\omega t - k_0 x)}}_{\text{standard waves}} \underbrace{e^{-\beta x}}_{\text{absorption}}$$

⇒ power and power density decays as

$S = EH \propto e^{-2\beta x}$; i.e. falls to $1/e$ fraction of its original value after travelling a distance $\frac{1}{2\beta} = \frac{1}{\sigma} \sqrt{\frac{\varepsilon_0 \varepsilon_r}{\mu_0 \mu_r}}$

application example:

A multimode optical fibre has a core refractive index of $n = 1.5$ at free space wavelength of $\lambda = 1.4\ \mu m$. The effective AC conductivity is $\sigma = 2 \cdot 10^{-5}\ \Omega^{-1} m^{-1}$.
This is due to OH^- radicals in the fibre that have been absorbed during fibre growth. Find the absorption loss in dB/km.

$$n = \frac{c}{v} = \frac{1}{\sqrt{\mu_0 \varepsilon_0}} \cdot \sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r} = \sqrt{\varepsilon_r \mu_r}$$

$\mu_r \approx 1$ (unmagnetic) $\Rightarrow \varepsilon_r = n^2 = 1.5^2 = 2.25$

$\frac{\sigma}{2\pi f \varepsilon_0 \varepsilon_r} \approx 7 \cdot 10^{-10} \ll 1$, hence approximation above valid

$\Rightarrow \beta = \frac{\sigma}{2} \sqrt{\frac{\mu_0 \mu_r}{\varepsilon_0 \varepsilon_r}} \approx 0.0025\ m^{-1}$

$\Rightarrow$ in 1 km distance the intensity falls by $e^{-2 \cdot 1000 m \cdot 0.0025 m^{-1}}$
$= 0.0067 \approx 10^{-2.2}$; i.e. the loss is ≈ 22 dB/km.