Useful Proofs

Proof 1: $A\sin(\omega t+\alpha)+B\sin(\omega t+\beta)=C\sin(\omega t+\gamma)$

$A\sin(\omega t+\alpha)=A\sin\omega t\cos\alpha+A\cos\omega t\sin\alpha.$

$B\sin(\omega t+\beta)=B\sin\omega t\cos\beta+B\cos\omega t\sin\beta.$

$C\sin(\omega t+\gamma)=C\sin\omega t\cos\gamma+C\cos\omega t\sin\gamma.$

Let $X = A\cos\alpha + B\cos\beta$ and $Y = A\sin\alpha + B\sin\beta$. By inspection we have,

$(A\cos\alpha + B\cos\beta)\sin\omega t = C\cos\gamma\sin\omega t.\ \therefore X = C\cos\gamma$

$(A\sin\alpha + B\sin\beta)\cos\omega t = C\sin\gamma\cos\omega t.\ \therefore Y = C\sin\gamma$

Therefore $\tan\gamma=\dfrac{Y}{X}$ and $C=\sqrt{X^2+Y^2}$.

Proof 2: $a_n = 2\,\mathrm{Re}[c_n]$ **and** $b_n = -2\,\mathrm{Im}[c_n]$

Let $x(t)=\sum_{n=0}^{\infty}x_n(t)$. The n^{th} term is given by

$$x_n(t) = a_n\cos n\omega_o t + b_n\sin n\omega_o t$$

$$=\frac{a_n}{2}\left(e^{jn\omega_o t}+e^{-jn\omega_o t}\right)+\frac{b_n}{2j}\left(e^{jn\omega_o t}-e^{-jn\omega_o t}\right)$$

$$=\frac{(a_n-jb_n)}{2}e^{jn\omega_o t}+\frac{(a_n+jb_n)}{2}e^{-jn\omega_o t}=c_n e^{jn\omega_o t}+c_{-n}e^{-jn\omega_o t}.$$

$$c_n=\frac{a_n-jb_n}{2} \text{ and } c_{-n}=\frac{a_n+jb_n}{2}.$$

$$|c_n|=|c_{-n}| \text{ and } \angle c_n=-\angle c_{-n}.$$

For $n > 0$, $\mathrm{Re}[c_n]=\dfrac{a_n}{2}\Rightarrow a_n=2\,\mathrm{Re}[c_n]$ and $\mathrm{Im}[c_n]=\dfrac{-b_n}{2}\Rightarrow b_n=-2\,\mathrm{Im}[c_n]$.

Summing all the harmonics we have $x(t)=c_0+\sum_{n=1}^{\infty}c_n e^{jn\omega_o t}+\sum_{n=-\infty}^{-1}c_n e^{jn\omega_o t}=\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_o t}$.

Proof 3: Parseval's theorem for calculation of average power

The signal energy over one period T is

$$E=\int_0^T x(t)x^*(t)dt.$$

The average power over one period is

$$P_{av}=\frac{1}{T}\int_0^T x(t)x^*(t)dt=\frac{1}{T}\int_0^T |x(t)|^2 dt.$$

If $x(t)=\left(\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_o t}\right)$ then $x^*(t)=\left(\sum_{n=-\infty}^{\infty}c_n e^{jn\omega_o t}\right)^*=\sum_{n=-\infty}^{\infty}c_n^{\ *}e^{-jn\omega_o t}=\sum_{n=-\infty}^{\infty}c_{-n}e^{-jn\omega_o t}$,

when $x(t)$ is real.

$$P_{av} = \frac{1}{T}\int_0^T \left(\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} \right) x^*(t)dt = \sum_{n=-\infty}^{\infty} c_n \left(\frac{1}{T}\int_0^T x^*(t)e^{jn\omega_o t}dt \right).$$

The term in the bracket is c_n^*.
So we have

$$P_{av} = \sum_{n=-\infty}^{\infty} c_n c_n^* = \sum_{n=-\infty}^{\infty} |c_n|^2$$

since $c_n^* = c_{-n}$ and $|c_n| = |c_{-n}|$.

Proof 4: Fourier Transform: Time shift property

$$\mathcal{F}[x(t\text{-}t_o)] = \int_{-\infty}^{\infty} x(t-t_o)e^{-j\omega t}dt = e^{-jwt_o}\int_{-\infty}^{\infty} x(t-t_o)e^{-j\omega(t-t_o)}dt.$$

Let $\tau = t - t_o$ and we have,

$$\mathcal{F}[x(t\text{-}t_o)] = e^{-j\omega t_o}\int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}d\tau = X(\omega)e^{-j\omega t_o}.$$

Proof 5: Fourier Transform: Frequency shift property

$$\mathcal{F}[x(t)e^{j\omega o t}] = \int_{-\infty}^{\infty} x(t)e^{j\omega_o t}e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j(\omega-\omega_o)t}dt = X(\omega-\omega_o).$$

$$\mathcal{F}[x(t)\cos\omega_o t] = \mathcal{F}\left[x(t)\frac{e^{j\omega_o t}+e^{-j\omega_o t}}{2} \right] = \frac{1}{2}[X(\omega+\omega_o)+X(\omega\text{-}\omega_o)].$$

Proof 6: Fourier Transform: Time scaling

$$\mathcal{F}[x(at)] = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(at)e^{-j(\omega/a)at}dt.$$ Replacing $\tau = at$,

$$\mathcal{F}[x(at)] = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau = \frac{1}{a}X\left(\frac{\omega}{a}\right).$$

Proof 7: Fourier Transform: Differentiation property

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = \int_{-\infty}^{\infty} \frac{dx(t)}{dt}e^{-j\omega t}dt.$$ Let $u = e^{-j\omega t}$ and $dv/dt = dx(t)/dt$. Integrating by parts we have,

$$\mathcal{F}\left[\frac{dx(t)}{dt}\right] = e^{-j\omega t}x(t)\Big|_{t=-\infty}^{t=\infty} - \int_{-\infty}^{\infty} x(t)(-j\omega)e^{-j\omega t}dt = j\omega X(\omega),$$

since $x(t) \to 0$ as $t \to \pm\infty$.

Proof 8: Fourier Transform: Parseval's theorem for calculating total energy

The total energy of a signal $x(t)$ is

$$E = \int_{-\infty}^{\infty} x(t)x^*(t)dt .$$

If $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$ then $x^*(t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right)^* = \frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega$.

$$E = \int_{-\infty}^{\infty} x(t)x^*(t)dt = \int_{-\infty}^{\infty} x(t)\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X^*(\omega)e^{-j\omega t}d\omega\right)dt$$

Rearranging gives,

$$E = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt\right)\int_{-\infty}^{\infty} X^*(\omega)d\omega .$$

The term in the bracket is $X(\omega)$. Hence

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega .$$