Solutions

Q1 a.

i)(**6 marks**)

Causality
For a causal system, its output at time index *n* depends only on the values of the input at *n* and earlier time instants.
(**1 mark**)
Stability
For every bounded input sequence, the system produces a bounded output sequence.
A bounded input x[n]: $|x[n]| \leq B_x < \infty$; a bounded output y[n]: $|y[n]| \leq B_y < \infty$, where B_x and B_y are fixed with a finite positive value.
(**1 mark**)
Linearity
It is defined by the principle of superposition which has two requirements for the system to meet. Suppose the sequence $y_1[n]$ is the response of the system to the input sequence $x_1[n]$, and $y_2[n]$ the response to $x_2[n]$, then we have
additivity property:
$$T\{x_1[n] + x_2[n]\} = T\{x_1[n]\} + T\{x_2[n]\} = y_1[n] + y_2[n]$$
(**1 mark**)
scaling property:
$$T\{ax_1[n]\} = aT\{x_1[n]\} = ay_1[n]$$
(**1 mark**)
Time invariance
A time shift or delay of the input sequence causes a corresponding shift in the output sequence. Given the input and output sequence x[n] and y[n], for all n_0, the input sequence with values $x_1[n]=x[n-n_0]$ produces the output sequence with values $y_1[n]=y[n-n_0]$.
(**2 marks**)

ii) (**4 marks**)
Suppose the impulse response of the system is h[n]. Then given the input $x[n]=4^n$, its output y[n] is given by
$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]4^{(n-k)}$$
$$= 4^n \sum_{k=-\infty}^{+\infty} h[k]4^{-k} = \alpha x[n]$$
where $\alpha = \sum_{k=-\infty}^{+\infty} h[k]4^{-k}$ is a scalar.
(**3 marks**)
So 4^n is the eigenfunction of the system.
(**1 mark**)

Q1 b.

i) (**4 marks**)

1) $x[n+N]=e^{j2(n+N)\pi/3}=e^{j2n\pi/3}e^{j2N\pi/3}$

To be periodic, 2Nπ/3=2kπ, where k and N are integers. We can choose k=1 and N=3 to satisfy the equation. So it is periodic and its period is N=3.

(**2 marks**)

2) $x[n+N]=\cos((n+N)\pi/\sqrt{2})=\cos(n\pi/\sqrt{2}+N\pi/\sqrt{2})$

To be periodic, $N\pi/\sqrt{2}=2k\pi$, where k and N are integers. However, we can not find any pair of k and N to satisfy that equation. So it is not periodic.

(**2 marks**)

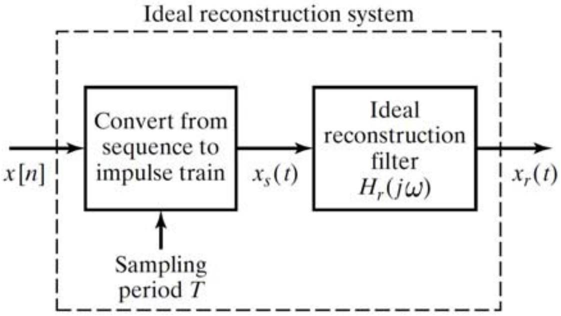
ii) (**3 marks**)

We need to know the sampling period *T* and whether it is sampled according to the requirement of the Nyquist sampling theorem. If it is not sampled according to the requirement of the Nyquist sampling theorem, we will not be able to recover the original continuous-time signals.

(**3 marks**)

iii) (**3 marks**)

A block diagram representation of the process is given below



(**2 marks**)

The ideal lowpass filter $H_r(j\omega)$ has a gain of *T* and a cutoff frequency ω_c, which should be larger than or equal to the highest frequency ω_N of the original continuous-time signal. The output $x_r(t)$ will be the original signal.

(**1 mark**)

Q2 a.

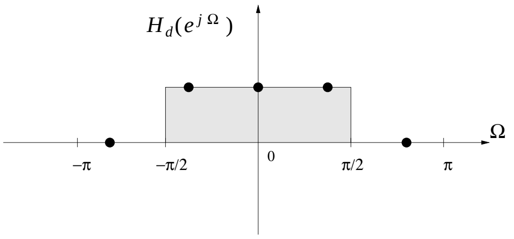
i) (**4 marks**)

The passband edge 0.5KHz corresponds to the normalised frequency

$\Omega_p=0.5*2\pi/2=\pi/2$

(**1 mark**)

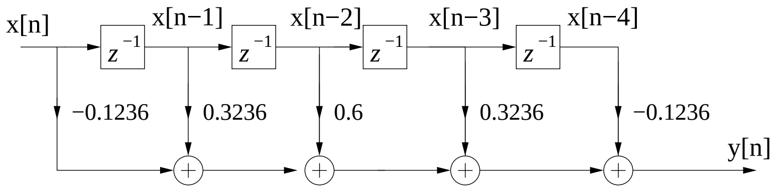
Then the ideal frequency response is given by



(**1 mark**)
We approximate the ideal one by 5 equally spaced samples, each $2\pi/5=1.26$ rad apart. Using inverse DFT we have

h[-2]=-0.1236, h[-1]=0.3236, h[0]=0.6, h[1]=0.3236, h[2]=-0.1236.
(**2 marks**)
We can shift the response by two positions to obtain a causal filter.

ii) (**2 marks**)



iii) (**2 marks**)
y[n]=-0.1236x[n]+0.3236x[n-1]+0.6x[n-2]+0.3236x[n-3]-0.1236x[n-4]

Q2 b. (6 marks)

If the system is causal, it follows that its impulse response must be a right-sided sequence, and the ROC must be outside of the outermost pole.
(**1 mark**)
Suppose the original causal system H(z) has zeros at c_k, k=1, …, M, then its inverse systems will be causal if and only if we associate the ROC
$|z|>\max|c_k|$
with the inverse system $H_i(z)$.
(**1 mark**)
If we also require that the inverse be stable, then the ROC of $H_i(z)$ must include the unit circle. Therefore it must be true that
$\max|c_k|<1$,
i.e. all zeros of H(z) must be inside the unit circle.
(**1 mark**)
As H(z) is also the inverse of $H_i(z)$, for H(z) to be both stable and causal, all of the zeros of $H_i(z)$ (which are the poles of H(z)) should also be inside the unit circle.
(**2 marks**)
Consequently, an LTI system is stable and causal and also has a stable and causal inverse if and only if both the poles and zeros of H(z) are inside the unit circle.
(**1 mark**)

Q2(c)

i) (**4 marks**)
Its transfer function H(z) can be obtained by placing the zeros in the numerator and the poles in the denominator:

$$H(z) = C\frac{z(z-0.9j)(z+0.9j)}{(z-0.6-0.6j)(z-0.6+0.6j)(z+0.6)}$$

$$= C\frac{z^3+0.81z}{(z^2-1.2z+0.72)(z+0.6)} = C\frac{z^3+0.81z}{z^3-0.6z^2+0.432}$$

$$= C\frac{1+0.81z^{-2}}{1-0.6z^{-1}+0.432z^{-3}}$$

where C is an unknown non-zero constant and can not be determined by the pole-zero plot.

(**3 marks**)

Its ROC is |z|>$0.6\sqrt{2} = 0.85$

(**1 mark**)

ii) (**2 marks**)

The transfer function $H_i(z)$ of its inverse system is given by:

$$H_i(z) = \frac{(z-0.6-0.6j)(z-0.6+0.6j)(z+0.6)}{Cz(z-0.9j)(z+0.9j)} = \frac{1-0.6z^{-1}+0.432z^{-3}}{C(1+0.81z^{-2})}$$

(**1 mark**)

Its ROC is |z|>0.9

(**1 mark**)

Q3 a. (4 marks)

The DFT of a sequence x[n] with length N is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j(2\pi/N)kn}$$

(**2 marks**)

For x[n]={1, 2, 2, 1}, N=4, then

$$X[k] = \sum_{n=0}^{3} x[n]e^{-jkn\frac{\pi}{2}} = 1 + 2e^{-jk\frac{\pi}{2}} + 2e^{-jk\pi} + e^{-jk\frac{3\pi}{2}}$$

(**1 mark**)

X[0]=6, X[1]=-1-j, X[2]=0, X[3]=-1+j

(**1 mark**)

Q3 b. (6 marks)

Using linear convolution, the third sequence $x_3[n]$ is given by

$$x_3[n] = \sum_{m=-\infty}^{\infty} x_1[m]x_2[n-m]$$

(**1 mark**)

The product $x_1[m]x_2[n-m]$ is zero for all *m* whenever n<0 and n>L+P-2. Therefore, (L+P-1) is the maximum length of the sequence $x_3[n]$.

(**1 mark**)

To calculate $x_3[n]$ using DFT, we first form the N-point sequence $\hat{x}_1[n]$ by adding N-L zeros to the L-points sequence $x_1[n]$ and the N-point sequence $\hat{x}_2[n]$ by adding N-P zeros to the P-points sequence $x_2[n]$ (N=L+P-1).

(**1 mark**)

Then we calculate the DFT $X_1[k]$ and $X_2[k]$ of $\hat{x}_1[n]$ and $\hat{x}_2[n]$ for k=0, 1, …, N-1. The product of the two DFTs is given by $X_3[k] = X_1[k]X_2[k]$ with a length of N.

(**2 marks**)

Applying the inverse DFT to $X_3[k]$, we then obtain the desired sequence $x_3[n]$

(**1 mark**)

Q3 c.

i) (**5 marks**)

Impulse invariance method:

Inverse Laplace transform

$h_a(t) = 50e^{-50t}$

(**1 mark**)

At sampling instants nT (T=1/80 sec), we have

$h_a(nT) = 50e^{-\frac{5}{8}n} = 50 \times 0.535^n$

(**2 marks**)

From z-transform table, we have

$$H_d(z) = \frac{50z}{z - 0.535}$$

(**2 marks**)

ii) (**5 marks**)

Bilinear transform method

The normalised cutoff frequency is

Wc=50/80=5/8 rad

(**1 mark**)

Using the relationship $\omega = \frac{2}{T}\tan(\frac{\Omega}{2})$, we have

$$\omega_b = \frac{2}{1/80}\tan(\frac{5/8}{2}) = 51.7 \text{ rad/sec}$$

(**1 mark**)

Then we have

$$\frac{Y(s)}{X(s)} = \frac{51.7}{s + 51.7}$$

(**1 mark**)

With

$$s = \frac{2(z-1)}{T(z+1)}$$

we have

$$H_d(z) = \frac{51.7}{\frac{2(z-1)}{0.0125(z+1)} + 51.7} = \frac{51.7z + 51.7}{211.7z - 108.3}$$

(**2 marks**)

Q4 a. (8 marks)

x[n] can be expressed as a linear combination of complex exponentials:

$$x[n] = A\cos(\Omega_0 n + \phi) = \frac{A}{2}(e^{j\phi}e^{j\Omega_0 n} + e^{-j\phi}e^{-j\Omega_0 n})$$

(**1 mark**)

The system's response to $x_1[n] = \frac{A}{2}e^{j\phi}e^{j\Omega_0 n}$ is $y_1[n] = \frac{A}{2}H(e^{j\Omega_0})e^{j\phi}e^{j\Omega_0 n}$;

its response to $x_2[n] = \frac{A}{2}e^{-j\phi}e^{-j\Omega_0 n}$ is $y_2[n] = \frac{A}{2}H(e^{-j\Omega_0})e^{-j\phi}e^{-j\Omega_0 n}$.

(**2 marks**)

The total output is then $y[n] = \frac{A}{2}H(e^{j\Omega_0})e^{j\phi}e^{j\Omega_0 n} + \frac{A}{2}H(e^{-j\Omega_0})e^{-j\phi}e^{-j\Omega_0 n}$.

(**1 mark**)

If h[n] is real-valued, then we have $H(e^{-j\Omega_0}) = H^*(e^{j\Omega_0})$

(**1 mark**)

and the output will be

$$y[n] = A|H(e^{j\Omega_0})|\cos(\Omega_0 n + \phi + \theta)$$

(**2 marks**)

where $\theta = \angle H(e^{j\Omega_0})$ is the phase of the system function at frequency Ω_0

(**1 mark**)

Q4 b.

i) (**4 marks**)

It is z-transform is given by

$$H(z) = \sum_{k=0}^{\infty}(\frac{1}{2})^k z^{-k} == \frac{1}{1-0.5z^{-1}} = \frac{z}{z-0.5} \qquad |z| > \frac{1}{2}$$

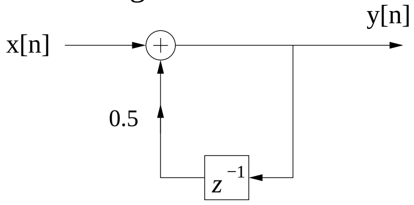
(**2 marks**)

Then the LCCD equation is

$y[n] = x[n] + 0.5y[n-1]$

(**1 mark**)

Block diagram:



(**1 mark**)

ii) (**4 marks**)

Its frequency response is given by

$H(e^{j\Omega}) = \frac{1}{1-0.5e^{-j\Omega}}$

(**1 mark**)

The input signal has two components 10 and 2sin(nπ/2).

The first is at DC, then $H(e^{j\Omega})\,|_{\Omega=0}=2$.

(**1 mark**)

The second has a frequency of π/2, then

$$H(e^{j\Omega})\,|_{\Omega=\pi/2}=\frac{1}{1+0.5j}$$

Its angle is -0.46 rad and its magnitude is 0.89.

(**1 mark**)

Then the output is given by

y[n]=20+1.78sin(nπ/2-0.46)

(**1 mark**)

Q4 c. (4 marks)

The following points constitute a comparison between IIR and FIR filters, based on a particular specification of filter and a pre-defined fixed sample rate:

i) FIR filters are always stable, while stability has to be carefully assessed when designing IIR filters.
ii) Finite word-length effects can cause a theoretically stable IIR filter to be unstable when implemented, while FIR filter performance is generally not as sensitive to finite word-length effects.
iii) For a given magnitude specification, FIR filters generally need to be higher order, i.e. require more coefficients, hence more storage space and more multiplications, compared to IIR filters.
iv) FIR filters can be designed with a perfectly linear phase-shift characteristic, which is particularly advantageous when minimum distortion is required, e.g. audio applications.

(**1 mark for each point**)