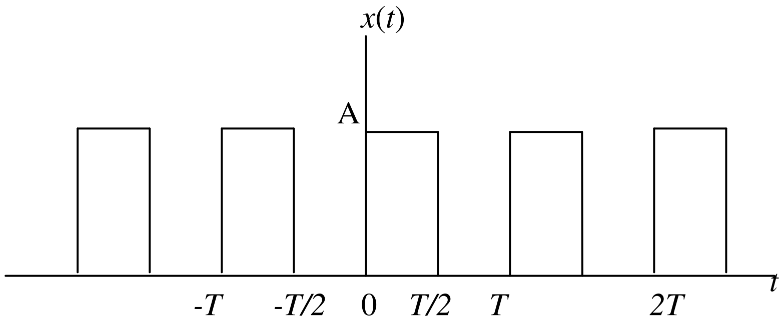
Tutorial 2: Solutions

1. Determine the (i) complex Fourier Series and (ii) trigonometric Fourier Series approximation of the signal shown below



(i) The complex Fourier Series coefficient,

$$c_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_o t}dt = \frac{1}{T}\int_0^{T/2} Ae^{-jn\omega_o t}dt = -\frac{A}{jn\omega_o T}e^{-jn\omega_o t}\Big|_0^{T/2}$$

$$c_n = \frac{A}{jn\left(\frac{2\pi}{T}\right)T}\left[1-e^{-jn\omega_o T/2}\right] = \frac{A}{j2n\pi}\left[1-e^{-jn(2\pi/T)T/2}\right] = \frac{A}{j2n\pi}\left[1-e^{-jn\pi}\right]$$

$$c_n = \begin{cases} 0 & n = even \\ \frac{A}{jn\pi} & n = odd \end{cases}.$$

$$c_n = \begin{cases} 0 & n = even \\ -j\frac{A}{n\pi} & n = odd \end{cases}$$

Note that $e^{-jn\pi} = \cos(n\pi) - j\sin(n\pi) = 1$ for n = even and $e^{-jn\pi} = \cos(n\pi) - j\sin(n\pi) = -1$ for n = odd.

The dc component,

$$c_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{1}{T}\int_0^{T/2} Adt = \frac{A}{T}t\Big|_0^{T/2} = \frac{A}{2}.$$

For n>0 $\angle c_n = \tan^{-1}(-\infty) = -\frac{\pi}{2}$

For n<0 $\angle c_n = \tan^{-1}(\infty) = \frac{\pi}{2}$

Finally we have,

$$x(t) = \sum_{n=-\infty}^{\infty}|c_n|e^{j(n\omega_o t+\angle c_n)} = c_0 + \sum_{n=1}^{\infty}|c_n|e^{j(n\omega_o t+\angle c_n)} + \sum_{n=-\infty}^{-1}|c_n|e^{j(n\omega_o t+\angle c_n)}.$$

Since $c_n = 0$ when n = even number, we have

$$x(t) = \frac{A}{2} + \sum_{m=1}^{\infty}\left|\frac{A}{(2m-1)\pi}\right|e^{j((2m-1)\omega_o t-\pi/2)} + \sum_{m=-\infty}^{-1}\left|\frac{A}{(2m+1)\pi}\right|e^{j((2m+1)\omega_o t+\pi/2)}.$$

(ii) The trigonometric Fourier Series coefficients are,

$$a_o = c_o = \frac{A}{2}.$$

We know that a_n = 2Re[c_n] and b_n = -2Im[c_n]. Since Re[c_n] = 0 and c_n = 0 for n = even, therefore a_n = 0.

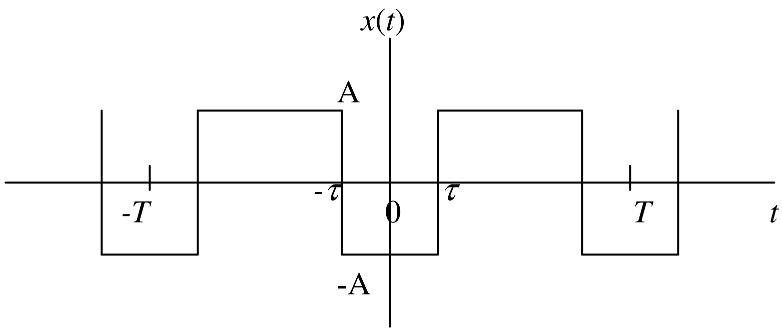
$$b_n = -2\mathrm{Im}[c_n] = -2\,\mathrm{Im}\left[\frac{-jA}{n\pi}\right] = \frac{2A}{n\pi} \text{ for } n = \text{odd}.$$

As before if we let $n = 2m - 1$, we have

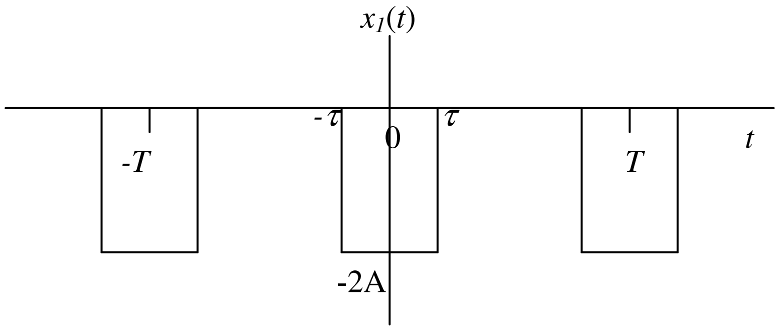
$$x(t) = \frac{A}{2} + \sum_{m=1.}^{\infty} \frac{2A}{\pi}\left(\frac{1}{2m-1}\right)\sin\left((2m-1)\omega_o t\right)$$

$$x(t) = \frac{A}{2} + \frac{2A}{\pi}\left(\sin(\omega_o t) + \frac{1}{3}\sin(3\omega_o t) + \frac{1}{5}\sin(5\omega_o t) + \ldots\right).$$

2. Find and sketch the magnitude spectrum of the periodic signal shown below for (i) $\tau = T/4$ and (ii) $\tau = T/8$ (assume T = 1s and A = 1/2).



Consider a signal $x_1(t)$ shown below,



Note that $x(t) = x_1(t) + A$. It is easier to compute the Fourier Series coefficient for $x_1(t)$ since $x_1(t) = 0$ for $\tau < t < T - \tau$.

The complex Fourier Series coefficient for $x_1(t)$,

$$c_0 = \frac{1}{T}\int_{-\tau}^{\tau}(-2A)dt = \frac{-2A}{T}t\Big|_{-\tau}^{\tau} = -\frac{4A\tau}{T}.$$

$$c_n = \frac{1}{T}\int_{-\tau}^{\tau}(-2A)e^{-jn\omega_o t}dt = \frac{2A}{jn\omega_o T}e^{-jn\omega_o t}\Big|_{-\tau}^{\tau} = \frac{2A}{jn\omega_o T}(e^{-jn\omega_o \tau} - e^{jn\omega_o \tau}) = -\frac{4A}{n\omega_o T}\left(\frac{e^{jn\omega_o \tau} - e^{-jn\omega_o \tau}}{j2}\right)$$

$$c_n = -\frac{4A\tau}{T}\frac{\sin(n\omega_o\tau)}{(n\omega_o\tau)}.$$

Therefore we have

$$x(t) = A + x_1(t) = A + \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t} = A - \frac{4A\tau}{T} + \sum_{n=1}^{\infty}\left(-\frac{4A\tau}{T}\frac{\sin(n\omega_o\tau)}{(n\omega_o\tau)}\right)e^{jn\omega_o t} + \sum_{n=-\infty}^{-1}\left(-\frac{4A\tau}{T}\frac{\sin(n\omega_o\tau)}{(n\omega_o\tau)}\right)e^{jn\omega_o t}$$

.

Note that $c_n = 0$ when $\sin(n\omega_o\tau) = 0$, that is when,

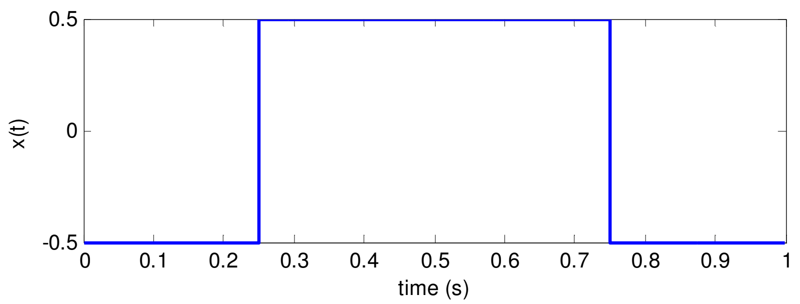
$n\omega_o\tau = m\pi \quad m = 0, \pm1, \pm2, \ldots$

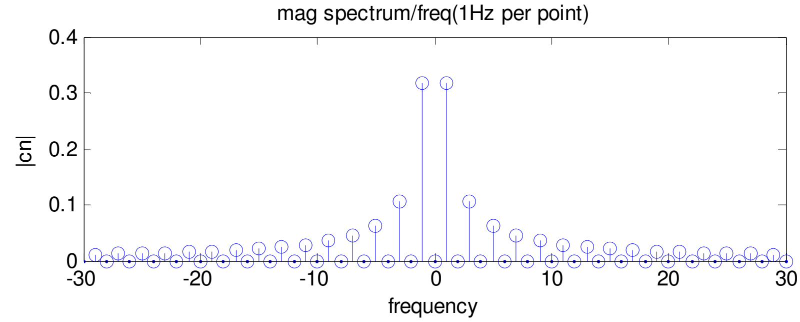
(i) Let $\omega_n = n\omega_o$. If $\tau = T/4 = 1/4$,

$|c_n| = 0$ when $\omega_n = n\omega_o = m\pi/\tau = \pm4\pi$ rad/s, $\pm8\pi$ rad/s…… or when $f = \pm2$ Hz, ±4 Hz, ±6 Hz….

$$|c_n| = \frac{4A}{T}\left(\frac{T}{4}\right)\left|\frac{\sin(\omega_n/4)}{(\omega_n/4)}\right| = \frac{1}{2}\left|\frac{\sin(\omega_n/4)}{(\omega_n/4)}\right|.$$

The d.c value for $x(t) = A - \frac{4A\tau}{T} = A - \frac{4A}{T}\left(\frac{T}{4}\right) = 0$.





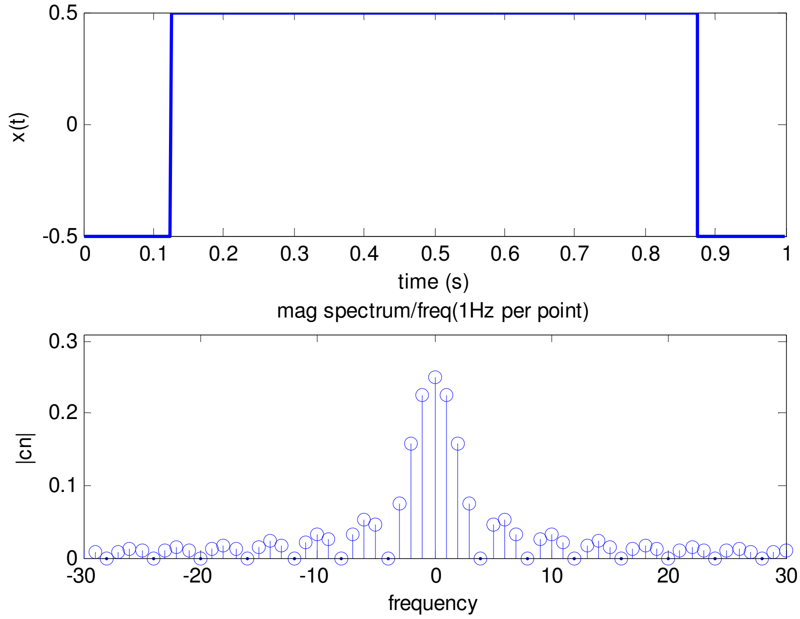
n	f(Hz)	$\lvert c_n \rvert$
1.0000	1.0000	0.3183
2.0000	2.0000	0
3.0000	3.0000	0.1061
4.0000	4.0000	0
5.0000	5.0000	0.0637
6.0000	6.0000	0
7.0000	7.0000	0.0455
8.0000	8.0000	0
9.0000	9.0000	0.0354
10.0000	10.0000	0

(ii) If $\tau = T/8 = 1/8$,

$|c_n| = 0$ when $\omega_n = n\omega_o = m\pi/\tau = \pm 8\pi$ rad/s, $\pm 16\pi$ rad/s…… or when $f = \pm 4$ Hz, ± 8 Hz, ± 12 Hz….

$$|c_n| = \frac{4A}{T}\left(\frac{T}{8}\right)\left|\frac{\sin(\omega_n/8)}{(\omega_n/8)}\right| = \frac{1}{4}\left|\frac{\sin(\omega_n/8)}{(\omega_n/8)}\right|.$$

The d.c value for $x(t) = A - \frac{4A\tau}{T} = A - \frac{4A}{T}\left(\frac{T}{8}\right) = \frac{A}{2} = \frac{1}{4}$.



n	f(Hz)	$|c_n|$
1.0000	1.0000	0.2251
2.0000	2.0000	0.1592
3.0000	3.0000	0.0750
4.0000	4.0000	0
5.0000	5.0000	0.0450
6.0000	6.0000	0.0531
7.0000	7.0000	0.0322
8.0000	8.0000	0
9.0000	9.0000	0.0250
10.0000	10.0000	0.0318

3. (a) Determine the average power in the signal $y(t) = \cos(4(t+3)) + \cos(7(t+3))$.

The average power $= \dfrac{1}{T}\int_{<T>} |y(t)|^2 dt$.

However we can also use the Parseval's theorem to find the average power.

$$y(t) = \frac{e^{j4(t-3)} + e^{-j4(t-3)}}{2} + \frac{e^{j7(t-3)} + e^{-j7(t-3)}}{2}$$

$$= \frac{e^{j4t}e^{-j12}}{2} + \frac{e^{-j4t}e^{j12}}{2} + \frac{e^{j7t}e^{-j21}}{2} + \frac{e^{-j7t}e^{j21}}{2} .$$

Parseval's theorem states:

The average power $P_{ave} = \sum_{n=-\infty}^{\infty} |c_n|^2$.

We have, $|c_{-7}| = |c_7| = \dfrac{1}{2}$ and $|c_{-4}| = |c_4| = \dfrac{1}{2}$, since $|e^{j21}| = |e^{-j21}| = 1$ and $|e^{j12}| = |e^{-j12}| = 1$. Therefore, $P_{ave} = |c_{-7}|^2 + |c_{-4}|^2 + |c_4|^2 + |c_7|^2 = 4\times\left(\dfrac{1}{2}\right)^2 = 1$.

4. Verify that a periodic signal can be represented by

$$x(t) = a_o + \sum_{n=1}^{\infty}(a_n \cos n\omega_o t + b_n \sin n\omega_o t),$$

where a_o is the dc components, a_n and b_n are the trigonometric Fourier series coefficients. Hence show that

$$a_n = \frac{2}{T}\int_{<T>} x(t)\cos n\omega_o t dt \text{ and } b_n = \frac{2}{T}\int_{<T>} x(t)\sin n\omega_o t dt .$$

A periodic signal with a fundamental frequency ω_o can be represented by

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$$

$$= c_o + \sum_{n=1}^{\infty}\left(c_n e^{jn\omega_o t} + c_{-n}e^{-jn\omega_o t}\right)$$

$$= c_o + \sum_{n=1}^{\infty}\left(c_n(\cos(n\omega_o t) + j\sin(n\omega_o t)) + c_{-n}(\cos(n\omega_o t) - j\sin(n\omega_o t))\right)$$

$$= c_o + \sum_{n=1}^{\infty}\left((c_n + c_{-n})(\cos(n\omega_o t)) + (c_n - c_{-n})(j\sin(n\omega_o t))\right)$$

$$= a_o + \sum_{n=1}^{\infty} a_n \cos(n\omega_o t) + b_n \sin(n\omega_o t),$$

where $a_o = c_o$, $a_n = c_n + c_{-n} = 2\mathrm{Re}[c_n]$ and $b_n = j(c_n - c_{-n}) = -2\mathrm{Im}[c_n]$.

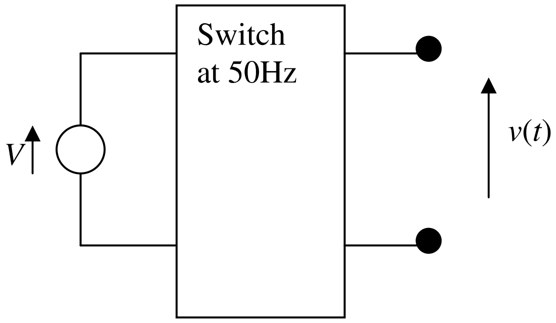
We know that $c_n = \dfrac{1}{T}\int_{<T>} x(t)e^{-jn\omega_o t}dt = \dfrac{1}{T}\int_{<T>} x(t)(\cos(n\omega_o t) - j\sin(n\omega_o t))dt$.

$\mathrm{Re}[c_n] = \dfrac{1}{T}\int_{<T>} x(t)\cos(n\omega_o t)dt$ and $\mathrm{Im}[c_n] = -\dfrac{1}{T}\int_{<T>} x(t)\sin(n\omega_o t)dt$.

Therefore $a_n = 2\mathrm{Re}[c_n] = \dfrac{2}{T}\int_{<T>} x(t)\cos(n\omega_o t)dt$

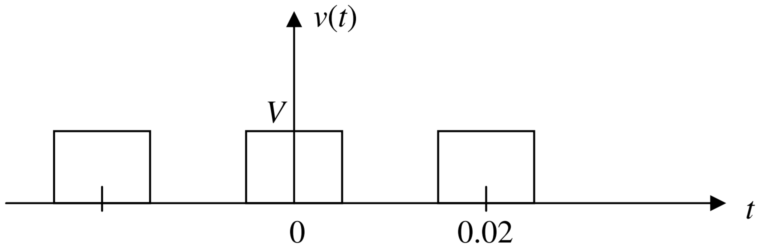
and $b_n = -2\mathrm{Im}[c_n] = \dfrac{2}{T}\int_{<T>} x(t)\sin(n\omega_o t)dt$.

5. Consider a simple dc to ac converter shown below, in which the conversion is achieved by switching the switch at 50Hz.



Sketch and label the output signal of the converter. Calculate the conversion efficiency, defined as $\dfrac{power \quad out}{power \quad in}$. [Assume that all the harmonics, except the fundamental, are removed by low pass filtering]

The output looks like this



First we need to obtain the complex Fourier Series of $v(t)$.

$$c_o = \frac{1}{T}\int_{-T/4}^{T/4} V dt = \frac{V}{T}\left[\frac{T}{2}\right] = \frac{V}{2}.$$

$$c_n = \frac{1}{T}\int_{-T/4}^{T/4} V e^{-jn\omega_o t} dt = -\frac{V}{jn\omega_o T}\left[e^{-jn\omega_o t}\right]_{-T/4}^{T/4} = -\frac{V}{j2n\pi}\left[e^{-jn\pi/2} - e^{jn\pi/2}\right] = \frac{V}{n\pi}\sin\left(\frac{n\pi}{2}\right).$$

$$c_n = \begin{cases} \dfrac{V}{n\pi} & if \quad n = 1,5,9.. \\ -\dfrac{V}{n\pi} & if \quad n = 3,7,11... \\ 0 & if \quad n = even \end{cases}$$

After low pass filtering $v(t) = \frac{V}{2} + \frac{V}{\pi}e^{j\omega_o t} + \frac{V}{\pi}e^{-j\omega_o t}$. Therefore the conversion efficiency is $\frac{power \quad out}{power \quad in} = \frac{\left(\frac{V}{2}\right)^2 + 2\left(\frac{V}{\pi}\right)^2}{V^2} = \frac{1}{4} + \frac{2}{\pi^2}$ [Use Parserval's theorem].