EEE225: Analogue and Digital Electronics

Lecture IX

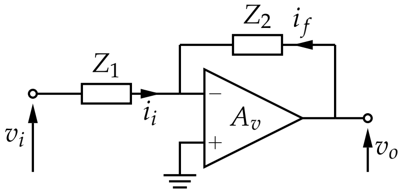
James E. Green

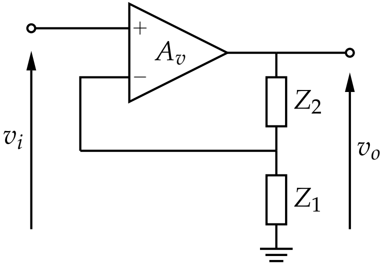
Department of Electronic Engineering
University of Sheffield
j.e.green@sheffield.ac.uk

This Lecture

Pole-Zero Circuits

Pole-zero circuits aim to adjust the magnitude and phase response of an analogue system. They are constructed from the standard amplifier blocks but with Z_1 or Z_2 having some frequency dependent components - almost always capacitors. Inductors are too imperfect[1]





[1] If an inductance is required, it may be manufactured with a capacitance and an opamp or two forming a gyrator, a kind of impedance transformer. See http://sound.westhost.com/articles/gyrator-filters.htm for examples.

Standard Forms

First order transfer functions fall into one of three standard forms, low pass,

$$\frac{v_o}{v_i} = k\,\frac{1}{1+s\,\tau} = k\,\frac{1}{1+j\,\frac{\omega}{\omega_0}} = k\,\frac{1}{1+j\,\frac{f}{f_0}} \tag{1}$$

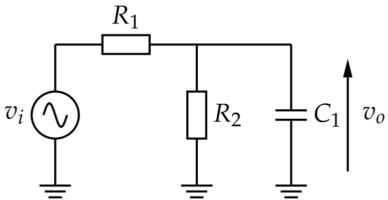
high pass,

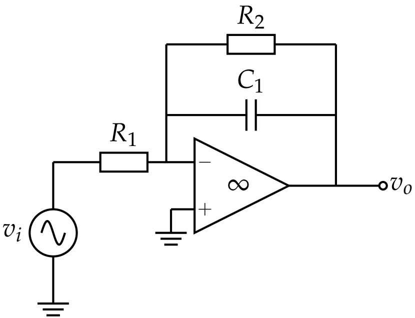
$$\frac{v_o}{v_i} = k\,\frac{s\,\tau}{1+s\,\tau} = k\,\frac{j\,\frac{\omega}{\omega_0}}{1+j\,\frac{\omega}{\omega_0}} = k\,\frac{j\,\frac{f}{f_0}}{1+j\,\frac{f}{f_0}} \tag{2}$$

and pole zero,

$$\frac{v_o}{v_i} = k\,\frac{1+s\,\tau_1}{1+s\,\tau_0} = k\,\frac{1+j\,\frac{\omega}{\omega_1}}{1+j\,\frac{\omega}{\omega_0}} = k\,\frac{1+j\,\frac{f}{f_1}}{1+j\,\frac{f}{f_0}} \tag{3}$$

Passive and Active First Order: Low Pass with 'k'





For the passive circuit:

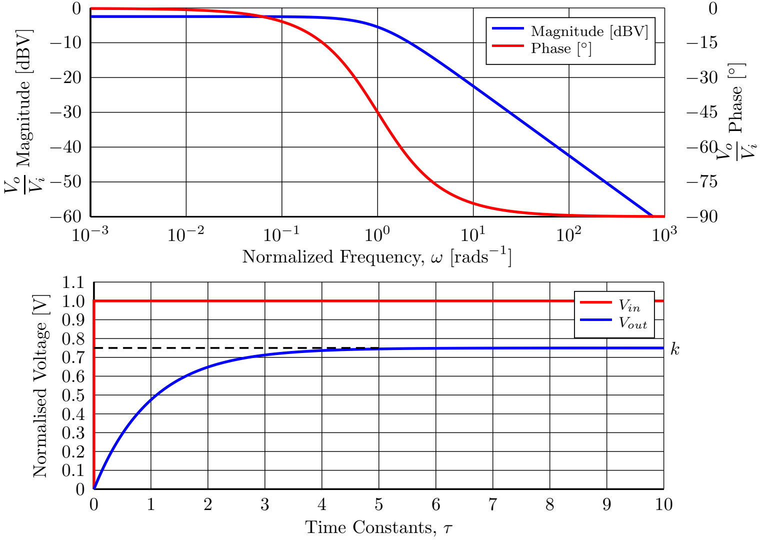
$$\frac{R_2}{R_1 + R_2} \cdot \frac{1}{s\, C_1\, (R_1 // R_2) + 1} \quad (4)$$

For the active circuit:

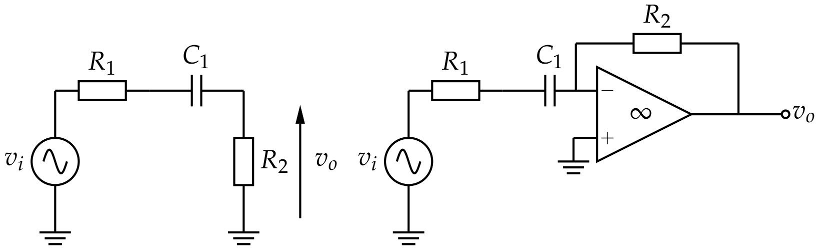
$$-\frac{R_2}{R_1} \cdot \frac{1}{s\, C_1\, R_2 + 1} \quad (5)$$

They are not identical! but they are similar in the shape of the frequency response.

Time and Frequency Domain Response (Passive Version)



Passive and Active First Order: High Pass with 'k'



For the passive circuit:

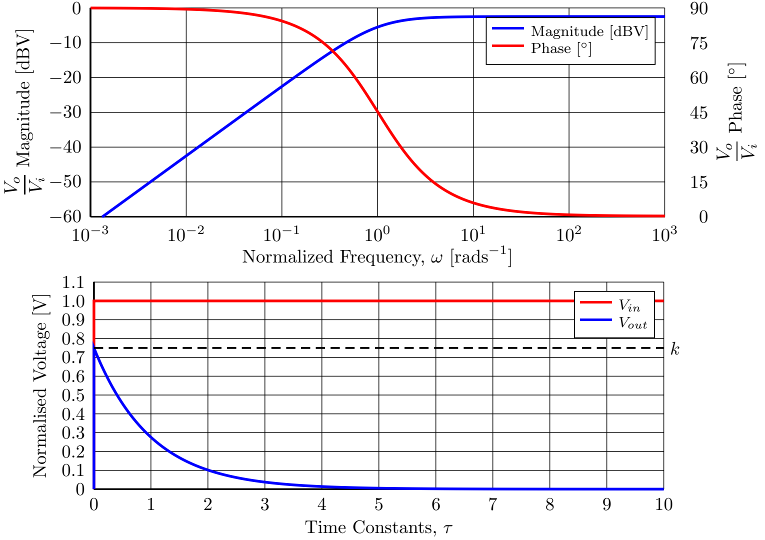
$$\frac{R_2}{R_1 + R_2} \cdot \frac{s\,C_1\,(R_1 + R_2)}{s\,C_1\,(R_1 + R_2) + 1} \tag{6}$$

For the active circuit:

$$-\frac{R_2}{R_1} \cdot \frac{s\,C_1\,R_1}{s\,C_1\,R_1 + 1} \tag{7}$$

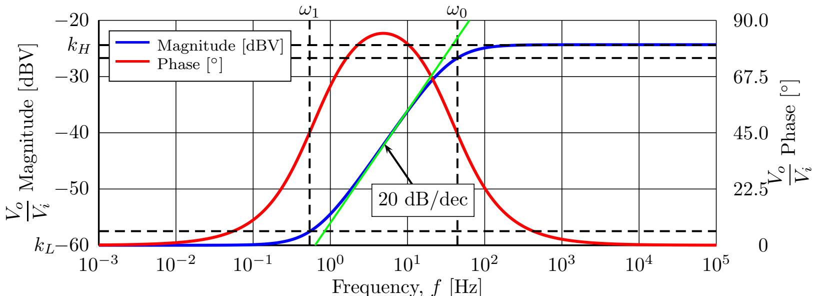
They are not identical! but they are similar in the shape of the frequency response.

Time and Frequency Domain Response (Passive Version)



Passive and Active First Order: Pole-Zero (or Zero-Pole)

- The PZ system is the linear sum of HP and LP
- There is one pole and one zero.
- The pole may appear at a lower or higher frequency than the zero. The circuit is called pole-zero regardless!
- The pole determines the time constant, τ
- Occasionally may be called lead or lag compensator in control systems discussion.



- There are two "gains" a low frequency (or DC, $f \to 0$) gain and a high frequency ($f \to \infty$) gain, k_L and k_H respectively.
- If zero frequency (ω_1) $<$ pole frequency (ω_0) then $k_L < k_H$ and phase "leads" (+ ve) between the pole and zero. This is the case in the last slide.
- If the zero frequency (ω_1) $>$ pole frequency (ω_0) then $k_L > k_H$ and phase "lags" (- ve) between the pole and zero.
- Magnitude slope tends to ± 20 dB/dec as the system is first order. Phase tends to +90 or -90 depending on PZ or ZP but often does not make it all the way.

Standard Forms:

frequency domain:

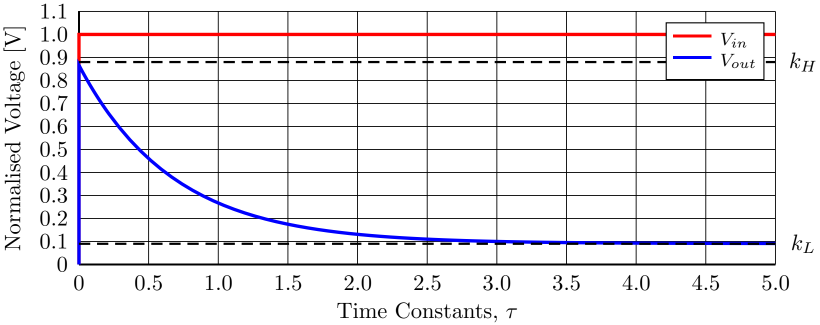
$$k\,\frac{1 + s\,\tau_1}{1 + s\,\tau_0} \tag{8}$$

Alternatively:

$$k \cdot \frac{1}{1 + s\,\tau_0} + k \cdot \frac{\tau_1}{\tau_0} \cdot \frac{s\,\tau_0}{1 + s\,\tau_0} \tag{9}$$

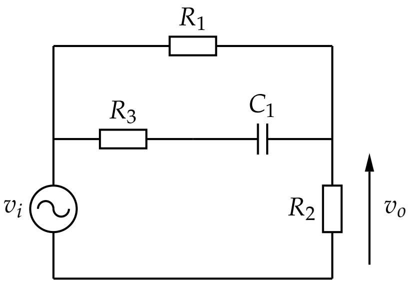
- The high frequency gain, $k_H = k \cdot \frac{\tau_1}{\tau_0}$ and $k = k_L$.

The step response depends on which of the pole or zero are at the lower frequency but for zero frequency < pole frequency we have something that is broadly HP looking but v_{out} does not fall to zero, it tends towards k_L. For zero frequency > pole frequency we have something broadly LP but also having a finite k_H.



Passive Pole-Zero Example

Find the transfer function of the following PZ circuit.



- Notice k is at the front and has no ω dependence.
- The s^0 (unity) coefficient is 1 in the numerator and denominator.
- The highest power of s is one.
- Always ask yourself, what is HF gain? what is LF gain? (good sanity check)...

$$k \cdot \frac{s\,\tau_1 + 1}{s\,\tau_0 + 1} = \frac{R_2}{R_1 + R_2} \cdot \frac{s\,C_1\,(R_1 + R_3) + 1}{s\,C_1 \left(\frac{R_2\,R_1 + R_2\,R_3 + R_1\,R_3}{R_1 + R_2} \right) + 1} \quad (10)$$

It's a potential divider with R_2 developing the output voltage,

$$v_o = \frac{R_2\, v_i}{R_2 + R_1 \,//\, \left(R_3 + \frac{1}{s\,C_1}\right)} \tag{11}$$

Expanding,

$$\frac{v_o}{v_i} = \frac{R_2}{R_2 + \dfrac{R_1\left(R_3 + \frac{1}{s\,C_1}\right)}{R_1 + R_3 + \frac{1}{s\,C_1}}} \tag{12}$$

Need to head towards $1 + s\,\tau$ on the bottom. Multiply top (numerator) and bottom (denominator) by $R_1 + R_3 + \frac{1}{s\,C_1}$

$$\frac{R_2\left(R_1 + R_3 + \frac{1}{s\,C_1}\right)}{R_2\left(R_1 + R_3 + \frac{1}{s\,C_1}\right) + R_1\left(R_3 + \frac{1}{s\,C_1}\right)} \tag{13}$$

Multiplying out the brackets (expanding),

$$\frac{R_2\,R_1 + R_3\,R_2 + \frac{R_2}{s\,C_1}}{R_2\,R_1 + R_3\,R_2 + \frac{R_2}{s\,C_1} + R_1\,R_3 + \frac{R_1}{s\,C_1}} \tag{14}$$

Multiplying top and bottom by $s\,C_1$,

$$\frac{(R_2\,R_1 + R_3\,R_2)\,s\,C_1 + R_2}{s\,C_1\,R_2\,(R_1 + R_3) + R_2 + R_1\,R_3\,s\,C_1 + R_1} \tag{15}$$

The unity term (coefficient of s^0) in the denominator is $R_1 + R_2$. So lets divide top and bottom by $R_1 + R_2$ to get $s\,\tau + 1$ on the bottom.

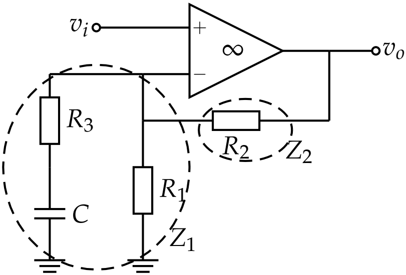
$$\frac{s\,C_1\,\frac{R_2(R_1+R_3)}{R_1+R_2} + \frac{R_2}{R_1+R_2}}{s\,C_1\,\frac{(R_2\,R_1+R_2\,R_3+R_1\,R_3)}{R_1+R_2} + 1} \tag{16}$$

Having found the desired form of the denominator we know the pole has a time-constant, $\tau_0 = C_1 \frac{(R_2\,R_1+R_2\,R_3+R_1\,R_3)}{R_1+R_2}$. The numerator is still not in the right form though as it must be $1+s\,\tau_1$. We need to divide the numerator by the numerator's present coefficients of s^0, which are $\frac{R_2}{R_1+R_2}$. We can't change the denominator though, it is already in the desired form, so we are unbalancing our expression. k, the frequency independent gain, will restore balance by becoming the unity coefficients of the numerator, $\frac{R_2}{R_1+R_2}$.

$$\frac{\frac{R_2}{R_1+R_2}\cdot\left(s\,C_1\cdot\frac{\frac{\cancel{R_2}(R_1+R_3)}{\cancel{R_1+R_2}}}{\frac{\cancel{R_2}}{\cancel{R_1+R_2}}}+\frac{\cancel{\frac{R_2}{R_1+R_2}}}{\cancel{\frac{R_2}{R_1+R_2}}}\right)}{s\,C_1\frac{(R_2\,R_1+R_2\,R_3+R_1\,R_3)}{R_1+R_2}+1} \tag{17}$$

Performing the cancellations in (17) and bringing k outside of the fraction yields (10).

Active Pole-Zero Example



HF gain: (at HF, C $\to$ 0 Ω)

$$\frac{v_o}{v_i} = \frac{R_2 + (R_1//R_3)}{R_1//R_3} \tag{18}$$

LF gain: (at LF, C $\to \infty$ Ω)

$$\frac{v_o}{v_i} = \frac{R_2 + R_1}{R_1} \tag{19}$$

This is a standard non-inverting amplifier which has the gain expression:

$$\frac{v_o}{v_i} = \frac{Z_2 + Z_1}{Z_1} = \frac{R_2 + R_1 \, // \, \left(R_3 + \frac{1}{j\,\omega\,C}\right)}{R_1 \, // \, \left(R_3 + \frac{1}{j\,\omega\,C}\right)} \tag{20}$$

$$\frac{R_2 + \frac{R_1\left(R_3+\frac{1}{j\,\omega\,C}\right)}{R_1+R_3+\frac{1}{j\,\omega\,C}}}{\frac{R_1\left(R_3+\frac{1}{j\,\omega\,C}\right)}{R_1+R_3+\frac{1}{j\,\omega\,C}}} \tag{21}$$

Multiply top and bottom by $j\,\omega\,C$,

$$\frac{R_2 + \frac{R_1\,(R_3\,j\,\omega\,C+1)}{1+j\,\omega\,C(R_1+R_3)}}{\frac{R_1\,(R_3\,j\,\omega\,C+1)}{1+j\,\omega\,C(R_1+R_3)}} \tag{22}$$

Multiply top and bottom by $1 + j\,\omega\,C\,(R_1 + R_3)$,

$$\frac{R_2\,(1 + j\,\omega\,C\,(R_1 + R_3)) + R_1\,\,(1 + j\,\omega\,C\,R_3)}{R_1\,(1 + j\,\omega\,C\,R_3)} \tag{23}$$

Collecting terms,

$$\frac{R_1 + R_2 + j\,\omega\,(R_2\,R_1 + R_2\,R_3 + R_1\,R_3)\;C}{R_1\,(1 + j\,\omega\,C\,R_3)} \tag{24}$$

Taking k outside, and comparing terms with the standard form,

$$\frac{R_1 + R_2}{R1} \cdot \frac{1 + j\,\omega\,C\left(\frac{R_2\,R_1 + R_2\,R_3 + R_1\,R_3}{R_1 + R_2}\right)}{1 + j\,\omega\,C\,R_3} \equiv k\frac{1 + j\,\omega\,\tau_1}{1 + j\,\omega\,\tau_0} \equiv k\,\frac{1 + j\frac{f}{f_1}}{1 + j\frac{f}{f_0}} \tag{25}$$

$$f_1 = \frac{R_1 + R_2}{2\,\pi\,C\,(R_1\,R_2 + R_2\,R_3 + R_1\,R_3)} \tag{26}$$

$$f_0 = \frac{1}{2\pi\,C\,R_3} \tag{27}$$

$$k = \frac{R_1 + R_2}{R_1} \tag{28}$$

when $\omega >> 2\pi f_1$ and $2\pi f_0$ (i.e. at high frequencies), the 1s are negligible compared to the f terms,

$$\left|\frac{v_o}{v_i}\right| = k \left|\frac{\cancel{1^2} + \left(\frac{f}{f_1}\right)^2}{\cancel{1^2} + \left(\frac{f}{f_0}\right)^2}\right|^{\frac{1}{2}} = k\,\frac{\frac{f}{f_1}}{\frac{f}{f_0}} = k\,\frac{f_0}{f_1} \tag{29}$$

$$k\,\frac{f_0}{f_1} = \frac{\cancel{R_1 + R_2}}{R_1} \cdot \frac{\frac{1}{\cancel{2\pi C}\,R_3}}{\frac{\cancel{R_1+R_2}}{\cancel{2\pi C}(R_1\,R_2+R_2\,R_3+R_1\,R_3)}} \tag{30}$$

$$\frac{R_1\,R_2 + R_2\,R_3 + R_1\,R_3}{R_1\,R_3} = \frac{R_1\,R_2 + R_2\,R_3}{R_1\,R_3} + 1 \tag{31}$$

$$R2\left(\frac{R_1 + R_2}{R_1\,R_3}\right) + 1 = \frac{R2}{R_1//R_3} + 1 = \frac{R_2 + R_1\,//\,R_3}{R_1\,//\,R_3} \tag{32}$$

Compare (32) with (18). At low frequencies, $\omega << 2\pi f_1$ and $2\pi f_0$, the 1s dominate the f terms, and gain $\rightarrow k$.

Review

- Revisited some EEE117 material on frequency and time domain response of first order LP and HP systems.
- Noted that the Pole-Zero circuit is a summation of the LP and HP first order circuits.
- Enumerated some key points about the pole zero circuit/system including:
  - There is one pole and one zero
  - The pole can be found at a lower frequency than the zero or *vice versa*.
  - The pole determines the time constant, τ.
  - sometimes called "lead/lag compensation circuits".
- Examined a passive network pole zero circuit similar to EEE117
- Examined an active, opamp based, pole zero circuit.