FOURIER TRANSFORM

The Fourier Series representation is applicable to periodic signals with infinite duration but many practical signals are non-periodic (or aperiodic) and have finite duration. We shall modify the Fourier Series so that it is applicable to aperiodic signals as well. The signal $x(t)$ in figure 1 can be expressed as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t},$$

where $c_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jn\omega_o t}dt$ and $\omega_o = 2\pi/T$.

$$c_n = \frac{1}{T}\int_{-\tau}^{\tau}(1)e^{-jn\omega_o t}dt = \frac{1}{jn\omega_o T}\left(e^{jn\omega_o\tau} - e^{-jn\omega_o\tau}\right) = \frac{2\tau}{T}\frac{\sin(n\omega_o\tau)}{(n\omega_o\tau)}$$

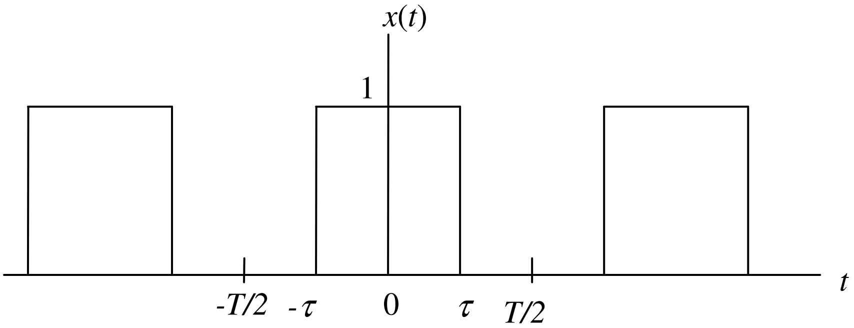


Figure 1: A square wave with period T.

c_n has magnitude described by an envelope function of $\frac{\sin\alpha}{\alpha}$ where $\alpha = n\omega_o\tau$ and a peak magnitude of $2\tau/T$, as shown in figure 2. Note that the function $\frac{\sin\alpha}{\alpha}$ is sampled every ω_o rad/s (i.e the frequency of the harmonics). The function $\frac{\sin\alpha}{\alpha}$ is a sinc function and it has a peak magnitude of 1 at $\alpha = 0$. (Use l'Hopital's rule: $\lim_{\alpha\to 0}\frac{\sin\alpha}{\alpha} = \lim_{\alpha\to 0}\frac{\cos\alpha}{1} = 1$).

Nulls of $\frac{\sin\alpha}{\alpha}$ occur when $\sin\alpha = 0$, that is when $\alpha = m\pi$ where m is integer to denote the nulls. Hence, the nulls are at $\omega = m\pi/\tau$ and we have the 1st null at π/τ, the 2nd null at $2\pi/\tau$ and so on.

Note: $\mathrm{sinc}(\alpha) = \begin{cases} \frac{\sin\alpha}{\alpha} & \alpha \neq 0 \\ 1 & \alpha = 0 \end{cases}$

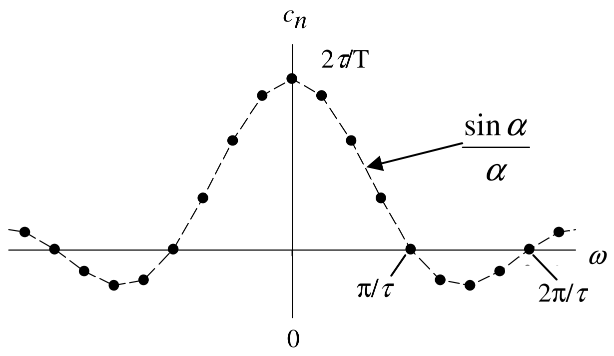


Figure 2: c_n with an envelope of $\frac{\sin\alpha}{\alpha}$.

Now, consider the envelope function $Tc_n = \left.\frac{2\tau\sin\omega\tau}{\omega\tau}\right|_{\omega=n\omega_o}$ when T is increased as illustrated in figure 3. We see that when T increases or equivalently when $\omega_o = 2\pi/T$ decreases, the envelope is sampled with an increasingly closer spacing. As $T \to \infty$, the original periodic square wave approaches a rectangular pulse and Tc_n becomes very closely spaced sample of the envelope that it appears at every frequency as shown in figure 3. We can therefore think of an aperiodic signal as the limit of a periodic signal when the period approaches infinity.

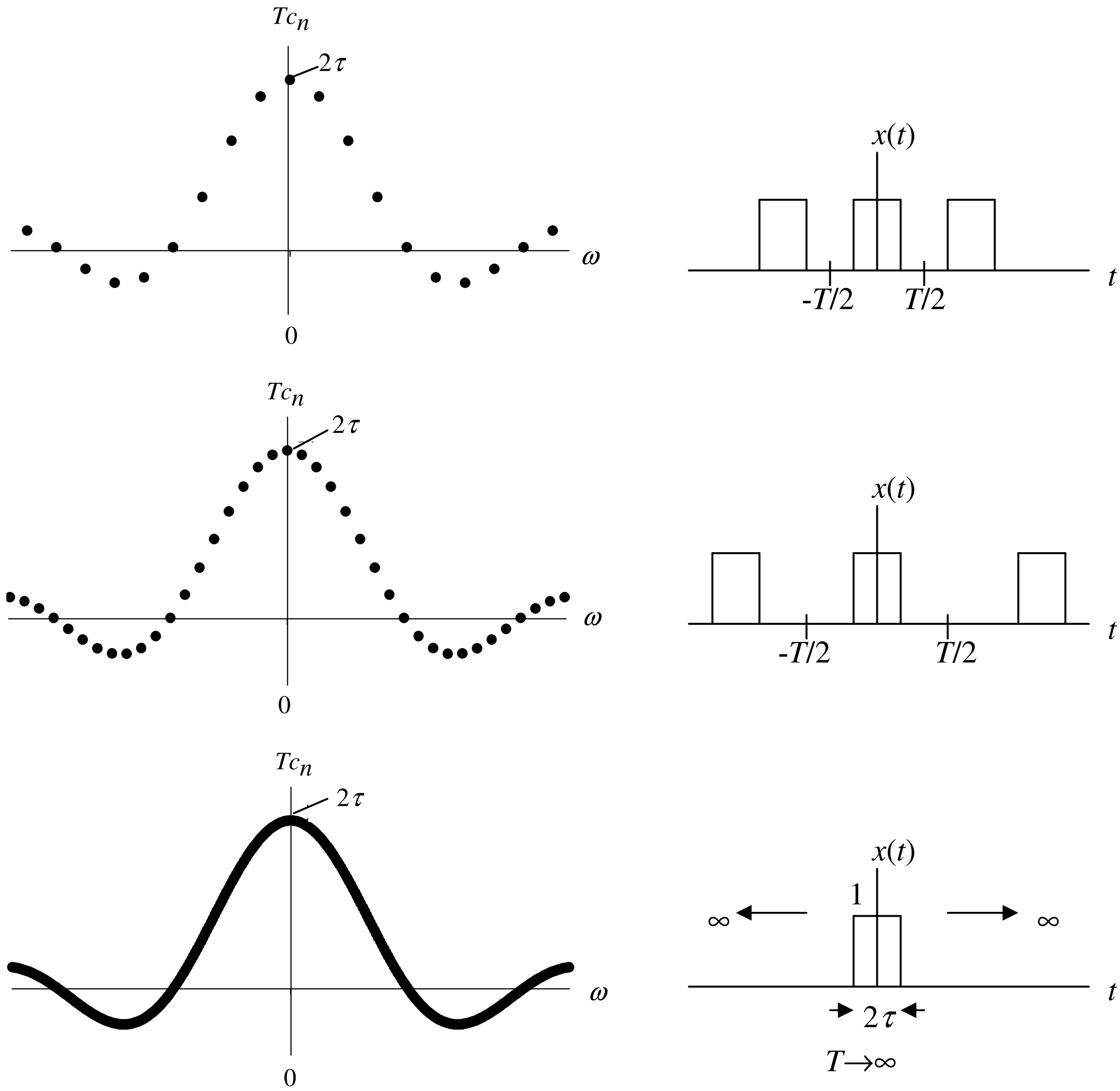


Figure 3: Fourier Series coefficients and their envelope for a periodic square wave with different period.

We shall now develop the Fourier Transform of a rectangular pulse $x(t)$. Let $X(\omega) = Tc_n = \int_{-T/2}^{T/2} x(t)e^{-j\omega t}dt$. We know that

$$x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} Tc_n e^{jn\omega_o t} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} X(\omega)e^{jn\omega_o t}\omega_o .$$

As $T\to\infty$, $\omega_o \to 0$ so that ω becomes a continuum and ω_o can be written as $d\omega$. The summation becomes an integration and hence we have,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

and

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt .$$

This is the Fourier Transform pair. $X(\omega)$ is called the Fourier Transform of $x(t)$ and $x(t)$ the inverse Fourier Transform of $X(\omega)$. More details can be found in (p.287, Oppenheim), (p.164, Kamen) and (p.127, Chen). If the symmetry of the signal $x(t)$ is known we can simplify the Fourier Transform integral to

$$X(\omega) = 2\int_0^\infty x(t)\cos\omega t dt$$

if $x(t)$ has an even symmetry and

$$X(\omega) = -j2\int_0^\infty x(t)\sin\omega t dt$$

if $x(t)$ has an odd symmetry. We will use $\mathcal{F}$ to denote the Fourier Transform operation, that is $\mathcal{F}[x(t)] = X(\omega)$.

Examples:

1. Obtain the Fourier Transform of the rectangular window function in figure 4.

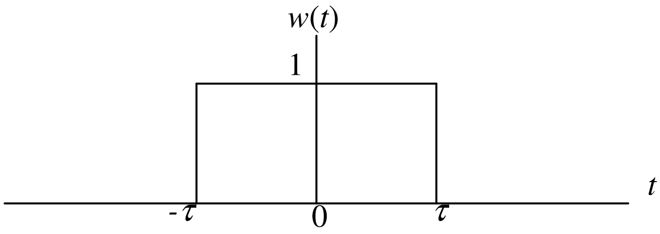


Figure 4: A rectangular window function with a duration of 2τ.

The peak amplitude of $X(\omega)$ is 2τ when $\omega = 0$ and the nulls occur at $\omega = n\pi/\tau$, where $n = 1, 2, 3\ldots$. From figure 5 we can see that as τ increases the peak amplitude of $X(\omega)$ increases and the width of the main lobe becomes narrower. In the limit as $\tau \rightarrow \infty$, $x(t) \rightarrow 1$ and $X(\omega) \rightarrow 2\pi\delta(\omega)$. In the other limit as $\tau \rightarrow 0$, $x(t) \rightarrow \delta(t)$ and $X(\omega) \rightarrow 1$.

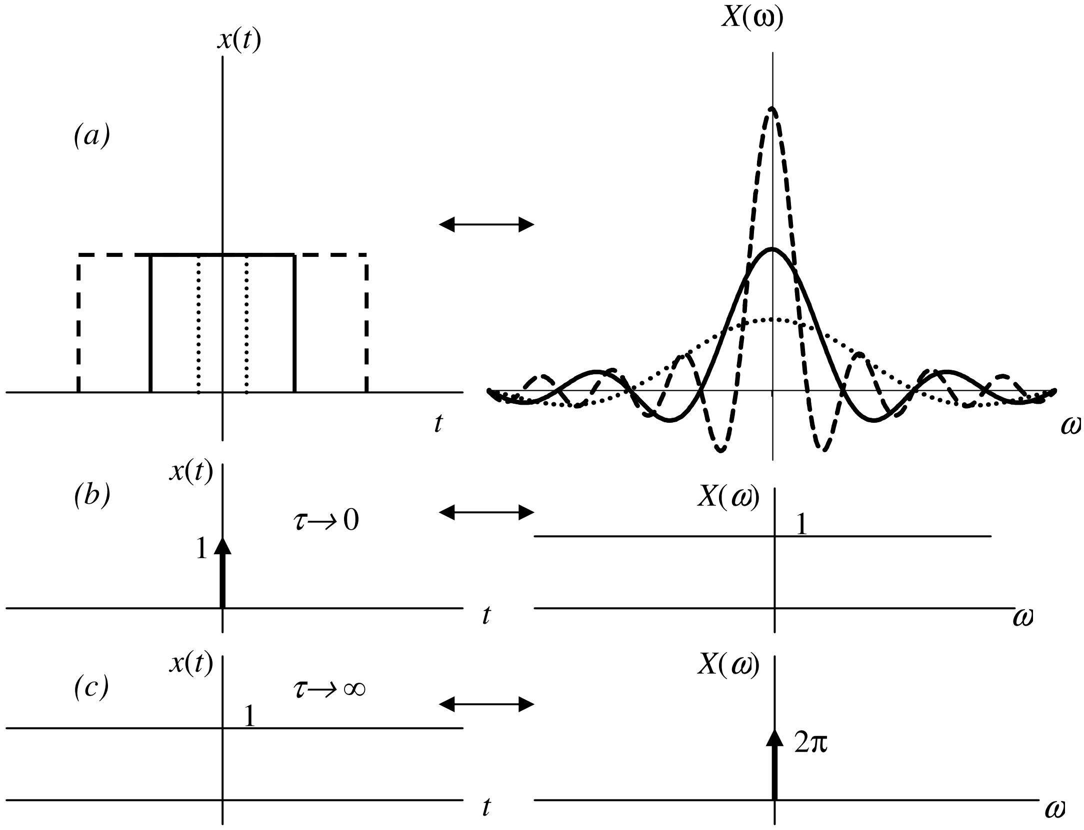


Figure 5: (a) Fourier Transform pairs for rectangular pulse with duration of τ (dotted line), 2τ (solid line) and 4τ (dashed line). (b) At the limit $\tau \rightarrow 0$. (c) At the limit $\tau \rightarrow \infty$.

2. Compute the time function that has the frequency spectrum (the positive half of the spectrum, $0 \le \omega \le \omega_c$, is an ideal low pass filter) shown in figure 6.

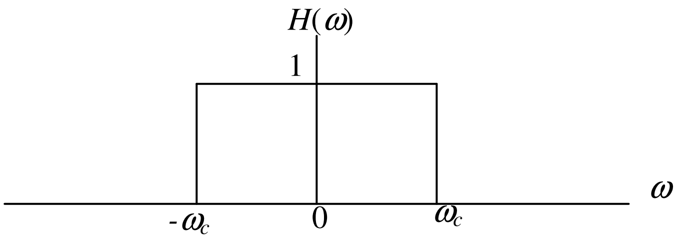
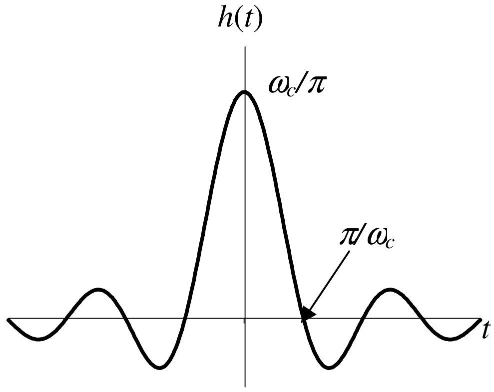
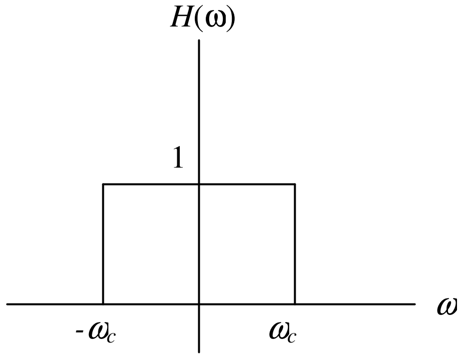


Figure 6: A rectangular spectrum defined by $H(\omega) = 1$ *for* $|\omega| \le \omega_c$ *and zero otherwise..*





3. Verify the Fourier Transform pair $x(t) = e^{-at}u(t) \leftrightarrow X(\omega) = \dfrac{1}{a + j\omega}$, $a > 0$.

Properties of Fourier Transform

Properties of Fourier Transform are important in analysis of signals and systems as they can simplify calculations and are applied in many practical applications. We will now look at some of the most important properties of Fourier Transform.

Linearity

If $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$

Then $ax_1(t) + bx_2(t) \leftrightarrow aX_1(\omega) + bX_2(\omega)$.

Time Shift

If $x(t) \leftrightarrow X(\omega)$ then $x(t- t_o) \leftrightarrow X(\omega)e^{-j\omega to}$.

Example:

Obtain the Fourier Transform of the signal in figure 7 using the time shift property and the Fourier Transform of the signal in figure 4.

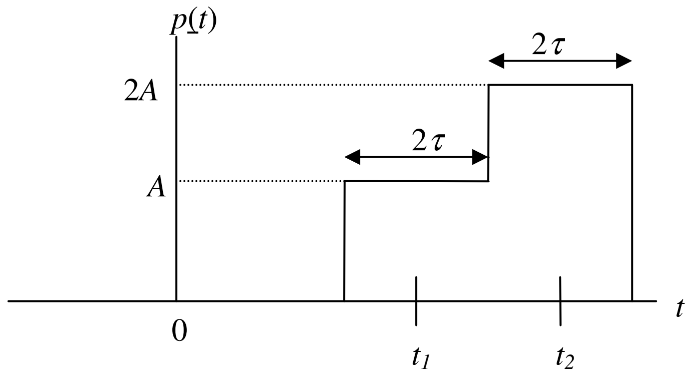


Figure 7: Signal p(t).

Frequency Shift

If $x(t) \leftrightarrow X(\omega)$ then $x(t)e^{j\omega_o t} \leftrightarrow X(\omega - \omega_o)$. The frequency spectrum of $x(t)$ has been shifted to ω_o. If $x(t)$ is multiplied by a sinusoidal signal we have,

$$x(t)\cos\omega_o t \leftrightarrow \frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$$

and

$$x(t)\sin\omega_o t \leftrightarrow \frac{j}{2}[X(\omega+\omega_o)-X(\omega-\omega_o)].$$

Time Scaling

If $x(t) \leftrightarrow X(\omega)$ then $x(at) \leftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$. If $a > 1$ $x(t)$ is time compressed. If $0 < a < 1$ $x(t)$ is time expanded.

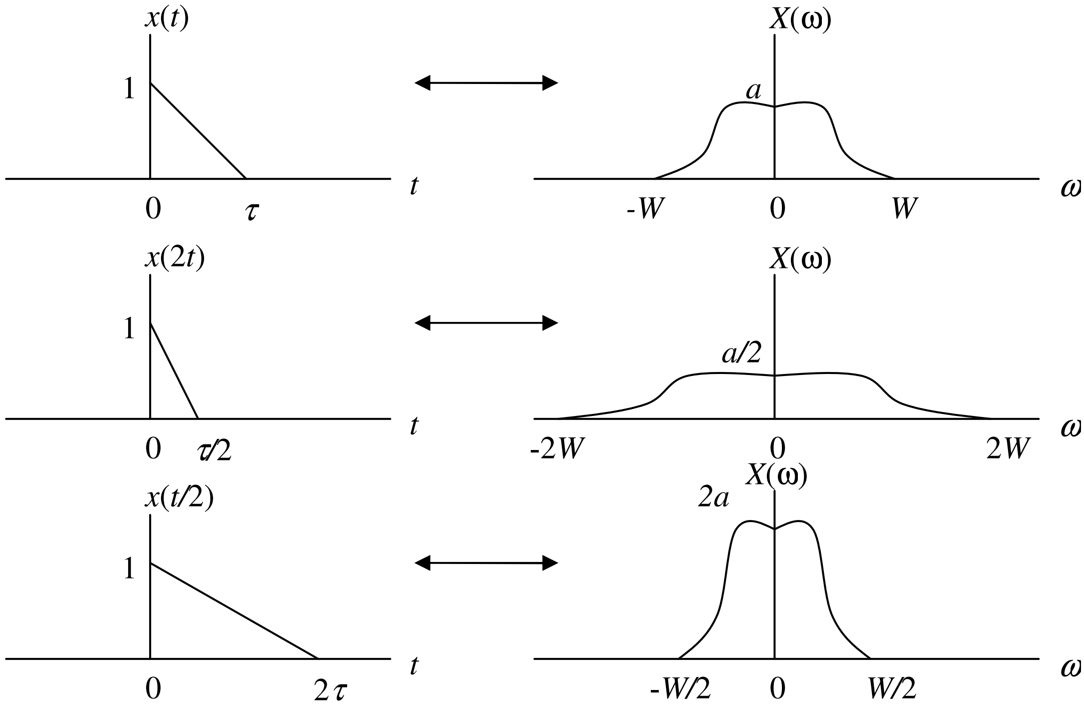


Figure 8: Time signal and its frequency spectrum.

Note: Time compression $\leftrightarrow$ frequency expansion

Time expansion $\leftrightarrow$ frequency compression

Differentiation and Integration

If $x(t) \leftrightarrow X(\omega)$ then $\frac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$.

Differentiation in time domain is replaced by $j\omega$ in frequency domain.

The integration property of Fourier Transform is described by

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega} X(\omega) + \pi X(0)\delta(\omega).$$

Example:

1. Obtain the Fourier Transform of the unit step $u(t)$, making use of the integration property of Fourier Transform.

Let $g(t) = \delta(t)$. We know that $g(t) = \delta(t) \leftrightarrow G(\omega) = 1$ and $u(t) = \int_{-\infty}^{t} g(\tau)d\tau$. Using the integration property we have,

$$X(\omega) = \mathcal{F}\left[\int_{-\infty}^{t} g(\tau)d\tau\right] = \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

We can recover the Fourier Transform of $\delta(t)$ by using the differentiation property.

$$G(\omega) = \mathcal{F}\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega) = j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1,$$

since $\omega\delta(\omega) = 0$.

2. Compute the Fourier Transform of a triangular signal shown in figure 9.

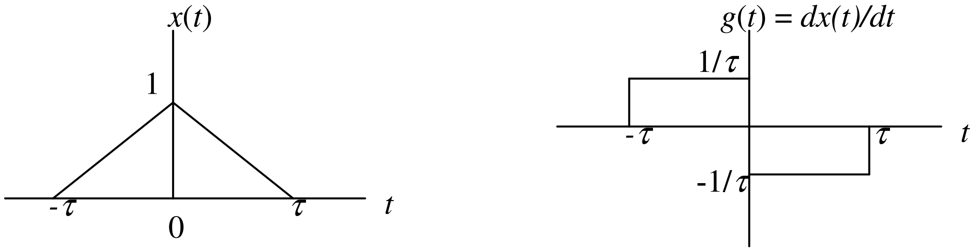


Figure 9: A triangular signal x(t)and g(t)=dx(t)/dt.

We know that the Fourier Transform of a rectangular pulse with duration of τ and amplitude of 1 is $\tau\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$. Using the time shift property,

$$G(\omega) = \tau\left(\frac{1}{\tau}\right)\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}e^{j\omega\tau/2} - \tau\left(\frac{1}{\tau}\right)\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}e^{-j\omega\tau/2}$$

$$= \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}(e^{j\omega\tau/2} - e^{-j\omega\tau/2})$$

$$= \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}(2j\sin(\omega\tau/2)) = j\omega\tau\left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right]^2.$$

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega).$$

$X(\omega) = G(\omega)/j\omega$ since $G(0) = 0$.

Finally we have, $X(\omega) = \tau\left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right]^2 = \tau\,\mathrm{sinc}^2(\omega\tau/2)$.

Duality

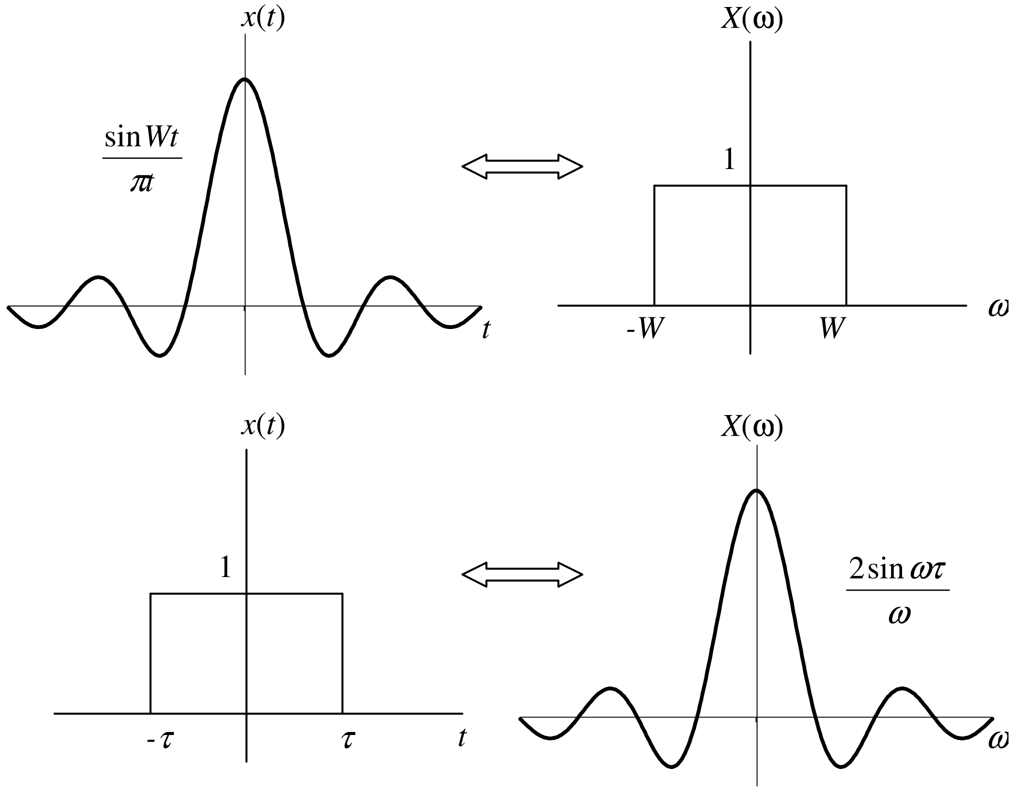


Figure 10: Fourier Transform pairs from examples 1 and 2.

The symmetry exhibited by examples 1 and 2 is referred to as duality. For any Fourier Transform pairs, there is a dual pair with the time and frequency variables interchanges.

Convolution

The output of a linear time invariant system, $y(t)$, can be obtained from the convolution of the input signal $x(t)$ with the system impulse response $h(t)$.

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

This convolution process in the time domain is equivalent to a multiplication process in the frequency domain, i.e. $x(t)*h(t) \leftrightarrow X(\omega).H(\omega)$.

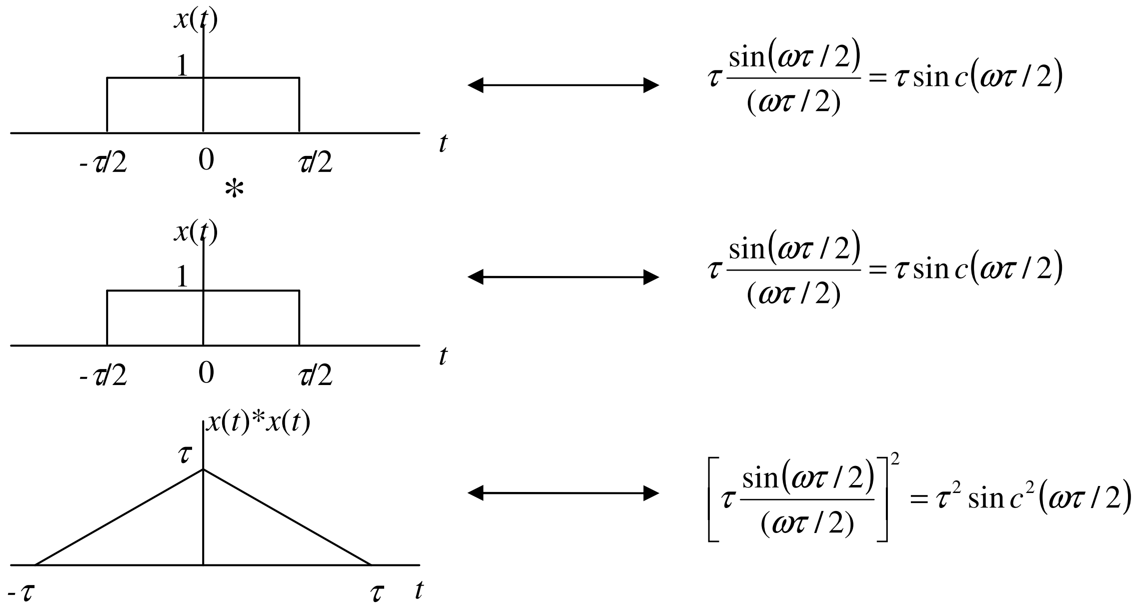


Figure 11: Convolution in time domain is equivalent to multiplication in frequency domain.

Multiplication

$$x(t).h(t) \leftrightarrow \frac{1}{2\pi}\left(X(\omega) * H(\omega)\right)$$

$$x(t).h(t) \leftrightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)H(\omega-\lambda)d\lambda .$$

Multiplication in time domain is equivalent to convolution in frequency domain.

Parseval's Theorem

Total energy of a signal $x(t) = E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, d\omega$. The energy contained within the frequency range $[\omega_1, \omega_2]$ is therefore given by

$$E = \frac{1}{2\pi} \int_{\omega_1}^{\omega_2} |X(\omega)|^2 \, d\omega .$$

Example:
Find the energy contained within the frequency range [0,2] in rad/s for the signal $x(t) = e^{-t}.u(t)$. The frequency spectrum is

$$X(\omega) = \int_0^{\infty} e^{-t} e^{-j\omega t} dt = \int_0^{\infty} e^{-(j\omega+1)t} dt = \frac{1}{(j\omega+1)} \left[-e^{-(j\omega+1)t} \right]_0^{\infty} = \frac{1}{(j\omega+1)} .$$

The energy within the frequency range [0,2] in rad/s is

$$E = \frac{1}{2\pi} \int_0^2 \left| \frac{1}{(j\omega+1)} \right|^2 d\omega = \frac{1}{2\pi} \int_0^2 \frac{1}{\omega^2+1} d\omega = \frac{1}{2\pi} \tan^{-1}(2) =$$

Fourier Transform Pairs

Signal	Fourier Transfrom
$\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_o t}$	$2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - n\omega_o)$
$e^{j\omega_o t}$	$2\pi\delta(\omega - \omega_o)$
$\cos\omega_o t$	$\pi[\delta(\omega + \omega_o) + \delta(\omega - \omega_o)]$
$\sin\omega_o t$	$j\pi[\delta(\omega + \omega_o) - \delta(\omega - \omega_o)]$
1	$2\pi\delta(\omega)$
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_o)$	$e^{-j\omega t_o}$
$e^{-at}u(t),\ a > 0$	$\frac{1}{a + j\omega}$
$x(t) = \begin{cases} 1, & \|t\| < \tau \\ 0, & \|t\| > \tau \end{cases}$	$\frac{2\sin\omega\tau}{\omega} = 2\tau \sin c(\omega\tau)$
$\frac{\sin\omega_c t}{\pi t} = \frac{\omega_c}{\pi} \sin c(\omega_c t)$	$X(\omega) = \begin{cases} 1, & \|\omega\| < \omega_c \\ 0, & \|\omega\| > \omega_c \end{cases}$
$\sum_{n=-\infty}^{\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$

Properties of Fourier Transform

Property	Aperiodic signal, $x(t)$	Fourier Transfrom, $X(\omega)$
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(\omega) + bX_2(\omega)$
Time Shifting	$x(t - t_o)$	$e^{-j\omega t_o} X(\omega)$
Frequency Shifting	$e^{j\omega_o t} x(t)$	$X(\omega - \omega_o)$
Time Scaling	$x(at)$	$\frac{1}{a} X\left(\frac{\omega}{a}\right)$
Differentiation in Time	$\frac{dx(t)}{dt}$	$j\omega X(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{dX(\omega)}{d\omega}$
Integration in time	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$
Convolution	$x(t)*h(t)$	$X(\omega).H(\omega)$
Multiplication in time	$x(t).h(t)$	$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\lambda)H(\omega-\lambda)d\lambda$
Parseval's Theorem	$E = \int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lvert X(\omega)\rvert^2 d\omega$	

NYQUIST SAMPLING THEOREM

Consider a CT signal $x(t)$ with frequency spectrum $X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$. If we sample the signal $x(t)$ every T seconds we have a sampled version of $x(t)$,

$$x_s(t) = \sum_{k=-\infty}^{\infty} x(kT)\delta(t-kT).$$

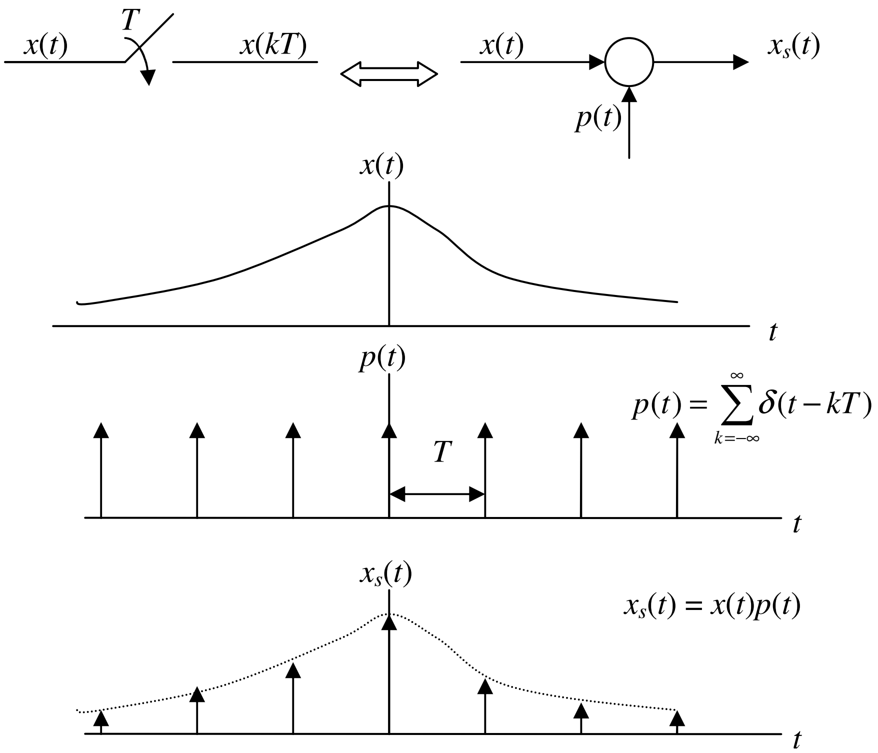
The discrete Fourier Transform is

$$X_s(\omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\omega t}dt = \sum_{k=-\infty}^{\infty} x(kT)e^{-j\omega kT}.$$

Recall that $x(kT)\delta(t\text{-}kT) = x(t)\delta(t\text{-}kT)$. We have,

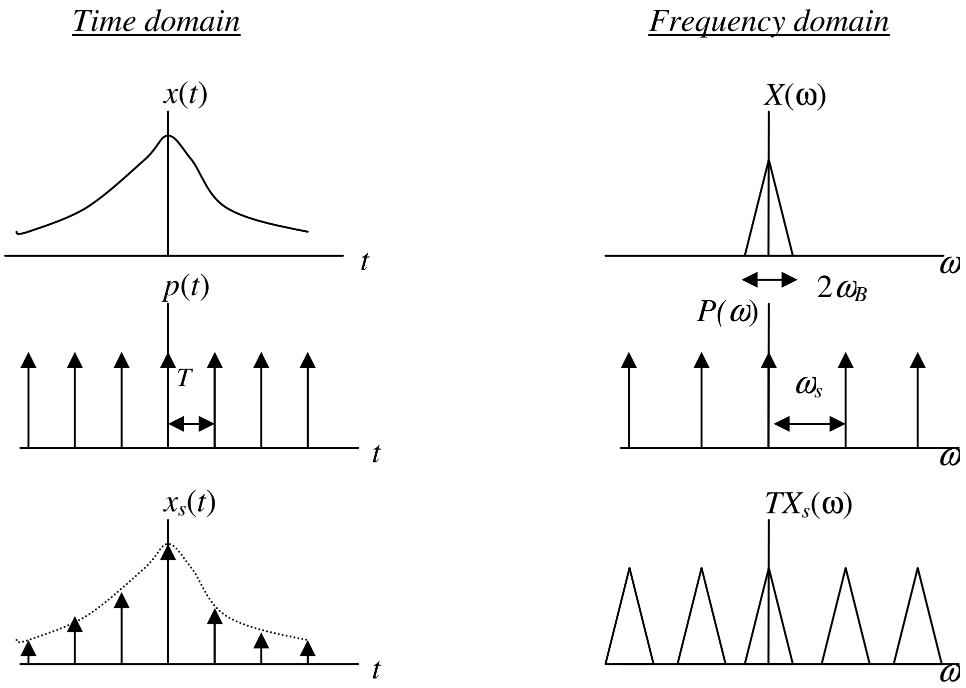
$$x_s(t) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-kT) = x(t)\sum_{k=-\infty}^{\infty}\delta(t-kT) = x(t)p(t),$$

where $p(t)$ is the sampling function so that the process of sampling can be considered as a modulation process.



The Fourier Series coefficient of $p(t)$ is $c_n = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)e^{-jn\omega_s t}dt = \frac{1}{T}$ so that we can write $p(t)$ as $p(t) = \sum_{n=-\infty}^{\infty}\frac{1}{T}e^{jn\omega_s t}$ where $\omega_s = 2\pi/T$ is the sampling frequency. The Fourier Transform of the sampling function is

$$P(\omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_s)$$



The signal $x(t)$ is said to be band-limited to ω_B if $[-\omega_B, \omega_B]$ is the smallest frequency band that contains all the nonzero spectrum $X(\omega)$ of $x(t)$, i.e $X(\omega) = 0$, for $|\omega| > \omega_B$. The spectrum $X_s(\omega)$ is given by

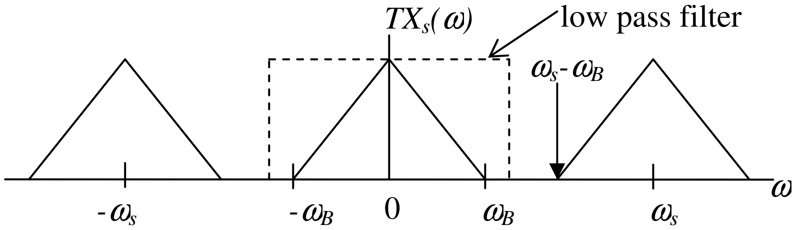
$$x_s(t) = x(t).p(t) \leftrightarrow X_s(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$$

$$X_s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) P(\omega - \lambda) d\lambda$$

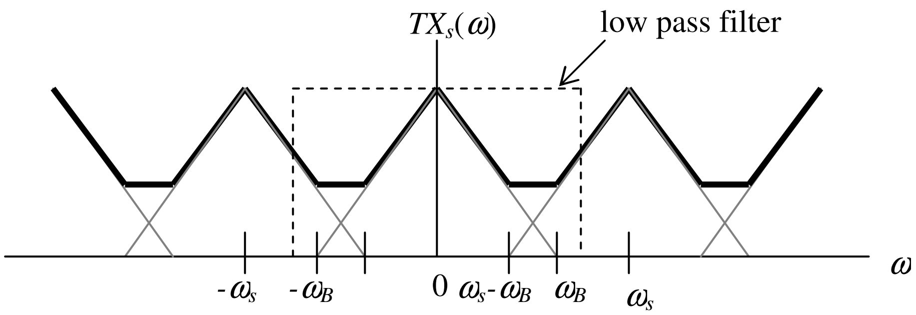
$$X_s(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\lambda) \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s - \lambda) d\lambda = \frac{1}{T} \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)$$

$$TX_s(\omega) = \sum_{n=-\infty}^{\infty} X(\omega - n\omega_s)\ .$$

This shows that a replica of $X(\omega)$ has been produced at $\pm\omega_s$, $\pm 2\omega_s$,… $\pm n\omega_s$. If $\omega_s - \omega_B > \omega_B$ we have,

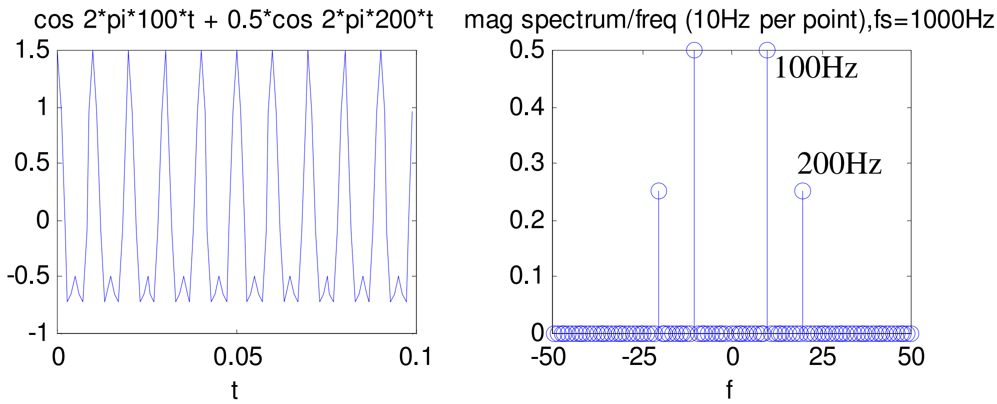


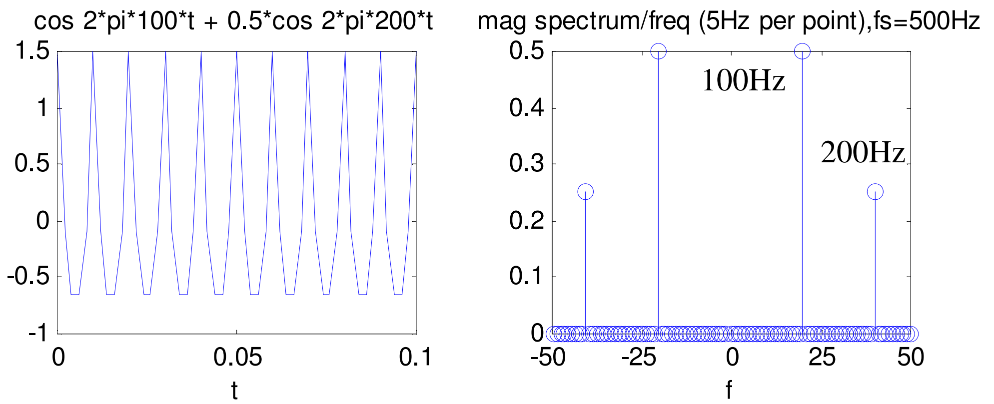
and we can recover the spectrum of $x(t)$ by using a low pass filter. However if $\omega_s - \omega_B < \omega_B$ we have,



and the spectrum of $x(t)$ is no longer replicated in $TX_s(\omega)$ and therefore is no longer recoverable by low pass filtering. It is clear that different components of $TX_s(\omega)$ overlap and this effect is referred to as ***aliasing***.

Example:
Consider a signal $x(t) = \cos 2\pi f_1 t + 0.5\cos 4\pi f_1 t$. We have,
$x(t) = \frac{1}{2}\left(e^{j\omega_1 t} + e^{-j\omega_1 t}\right) + \frac{1}{4}\left(e^{j2\omega_1 t} + e^{-j2\omega_1 t}\right)$ so that the Fourier Series coefficients are $c_1 = 1/2$, $c_{-1} = 1/2$, $c_2 = 1/4$ and $c_{-2} = 1/4$. Figure 12 shows the effect of changing the sampling frequency f_s on the spectrum of $x(t)$ when $f_1 = 100\text{Hz}$.





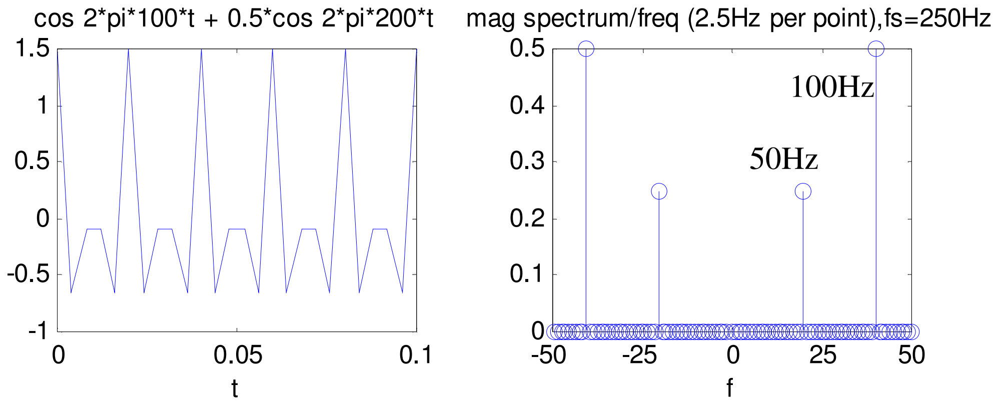


Figure 12: Effect of sampling at $f_s > 2f_1$ and $f_s < 2f_1$.

The maximum frequency f_{max} in the signal $x(t)$ is $2f_1$ = 200Hz. Clearly, the signal $x(t)$ can be recovered from $X_s(\omega)$ when f_s = 1000Hz and 500Hz since $f_s > 2f_{max}$ and thus, $X_s(\omega)$ has the correct frequency components. However the frequency spectra shows components at frequencies ±50Hz when f_s = 250Hz due to aliasing effects and therefore the signal $x(t)$ can no longer be reconstructed accurately.

If $x(t)$ is not band-limited, there will always be aliasing regardless of the sampling frequency. In practice aliasing effects are minimised by sampling at the highest possible frequency so that the aliased components do not distort the reconstructed signal significantly.

Nyquist Sampling Theorem.
For a CT signal $x(t)$ band-limited to f_{max} (in Hz) or $\omega_{max} = 2\pi f_{max}$ (in rad/s), its frequency spectrum $X(\omega) = 0$, for $|\omega| > \omega_{max}$. x(t) can be reconstructed from its sampled sequence $x_s(t)$ if the sampling frequency $f_s > 2f_{max}$.

MODULATION

Modulation is important in communication to shift the spectrum of a signal $x(t)$ to a higher frequency range so that it is possible to achieve good transmission. For instance an antenna with dimension of at least 1.5×10^5m is required to transmit a frequency of 200Hz (human voice lies in the frequency range 200Hz to 4kHz). There are many modulation techniques but we will only discuss the amplitude modulation.
Consider a signal $x(t)$ with a spectrum $X(\omega)$ band limited to W, i.e $X(\omega) = 0$ for $|\omega| > W$. In amplitude modulation (AM), the amplitude of a carrier signal $x_c(t) = \cos\omega_c t$ is modulated by the signal $x(t)$ where ω_c is assumed to be greater than W. The modulated signal is obtained by multiplying $x(t)$ by $x_c(t)$ to give

$$x_m(t) = x(t)\cos\omega_c t.$$

Figure 13 shows the modulated signal for $x(t) = e^{-0.1t}\cos t$ and $x_c(t) = \cos 30t$.

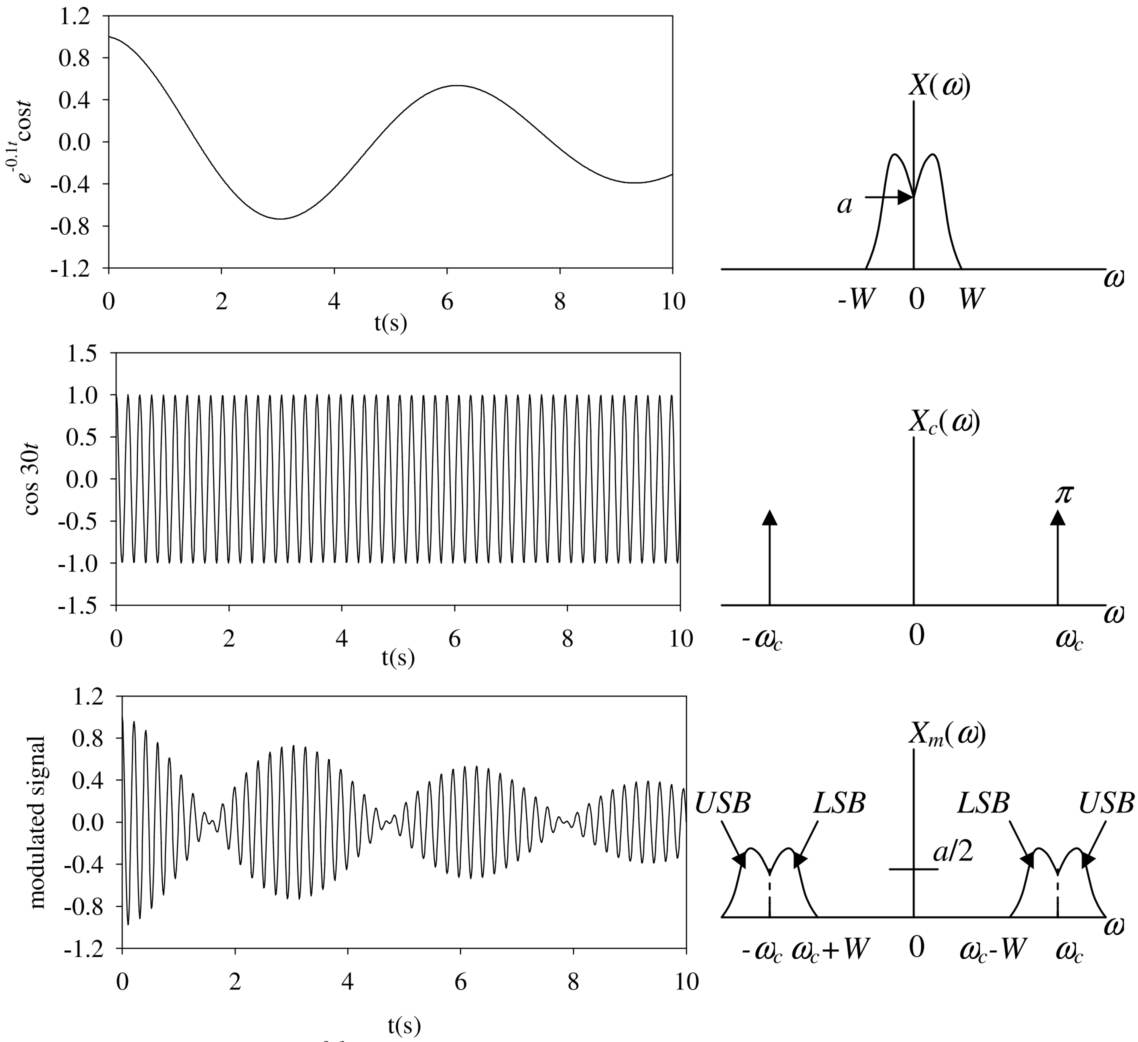


Figure 13: Top: $x(t) = e^{-0.1t}\cos t$. Middle: $x_c(t) = \cos 30t$. Bottom: modulated signal.

The spectrum of the modulated signal is $X_m(\omega) = \frac{1}{2}\left[X(\omega+\omega_c) + X(\omega-\omega_c)\right]$ (frequency shift property of Fourier Transform). The spectrum within $[\omega_c, \omega_c + W]$ is called the upper side band (USB) while the spectrum within $[\omega_c - W, \omega_c]$ is called the lower side band (LSB). The two impulses due to the carrier signal do not appear in

$X_m(\omega)$. The modulation scheme in figure 13 is therefore called double sideband-suppressed carrier (DSB-SC). This modulation scheme does not require the carrier signal to be transmitted leading to low power consumption. However it requires synchronisation between the transmitter and receiver for demodulation of the DSB-SC signals.

To recover the signal $x(t)$ from $x_m(t)$ we can use a synchronous demodulator which multiplies $x_m(t)$ by $\cos\omega_c t$. The output of the demodulator is

$$y(t) = x_m(t)\cos\omega_c t$$

and the Fourier Transform of $y(t)$ is

$$Y(\omega) = \frac{1}{2}\left[X_m(\omega+\omega_c) + X_m(\omega-\omega_c)\right] = \frac{1}{2}\left[\frac{1}{2}\left(X(\omega+2\omega_c) + X(\omega)\right) + \frac{1}{2}\left(X(\omega) + X(\omega-2\omega_c)\right)\right]$$

$$Y(\omega) = \frac{1}{2}X(\omega) + \frac{1}{4}X(\omega+2\omega_c) + \frac{1}{4}X(\omega-2\omega_c).$$

Next we use a low pass filter with a gain of 2 and a cutoff frequency greater than W but less than $2\omega_c$-W, to recover $x(t)$.

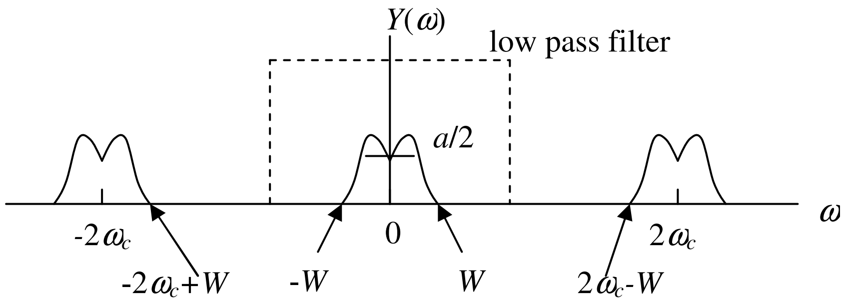


Figure 14: The signal x(t) is recovered from Y(ω) by using a low pass filter.

In another form of AM scheme, the modulated signal is given by

$$x_m'(t) = [A + x(t)]\cos\omega_c t$$

where A is selected so that $A + x(t) > 0$. The modulated signal is shown in figure 15 for $A = 1$.

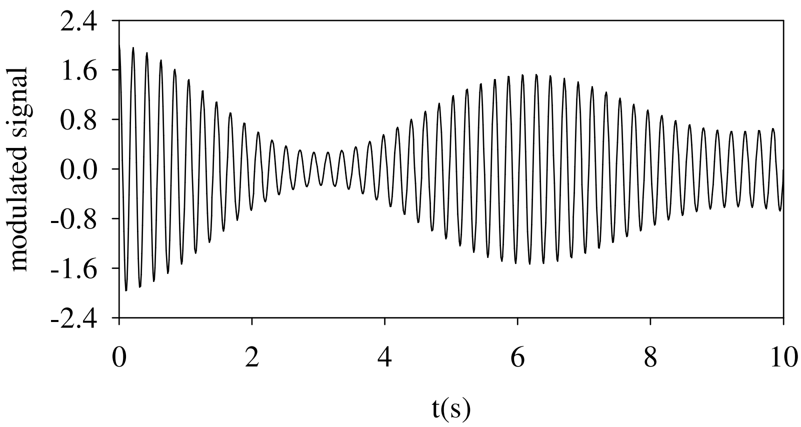


Figure 15: Modulated signal in double side band AM scheme with A = 1.

The frequency spectrum of $x_m'(t)$ is

$$X_m'(\omega) = A\pi[\delta(\omega+\omega_c) + \delta(\omega-\omega_c)] + 0.5[X(\omega+\omega_c) + X(\omega-\omega_c)].$$

This modulation scheme is referred to as double sideband (DSB) AM. The transmitted signal $x_m'(t)$ contains the carrier signal and both the USB and the LSB. In this scheme $x(t)$ can be reconstructed using an envelope detector which consists of a diode, a resistor and a capacitor, as shown in figure 16. If the values of R and C are properly selected the output of the demodulator will be close to the upper envelope of $x_m'(t)$.

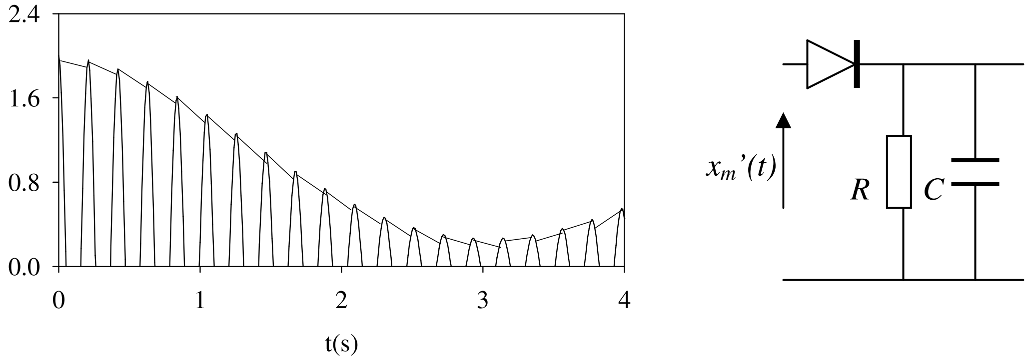


Figure 16: Output of demodulator for properly selected RC value.

In communication several signals can be transmitted simultaneously by using frequency division multiplexing in which each signal occupies a different portion of the radio spectrum. For instance the frequency band 540-1600kHz for AM radio and 87.5-108MHz for FM radio.

Note