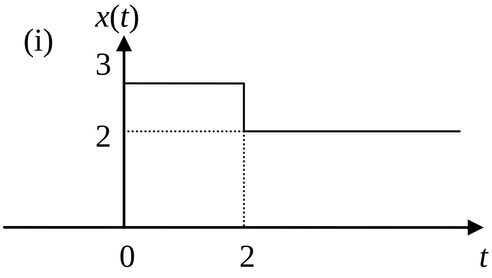
<u>Tutorial 1: Solutions</u>

1. How is the unit step function $u(t)$ related to (i) $\delta(t)$ and (ii) ramp function $r(t)$?

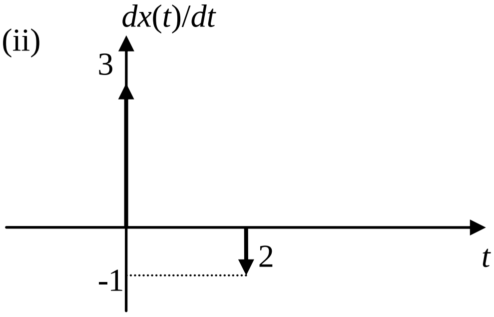
$$\text{(i) } u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \text{ or } \delta(t) = \frac{du(t)}{dt}. \quad \text{(ii) } r(t) = \int_{-\infty}^{t} u(\tau)d\tau = \int_{0}^{t} u(\tau)d\tau.$$

2. For a signal $x(t) = 3u(t) - u(t\text{-}2)$, sketch and label

(i)



(ii)



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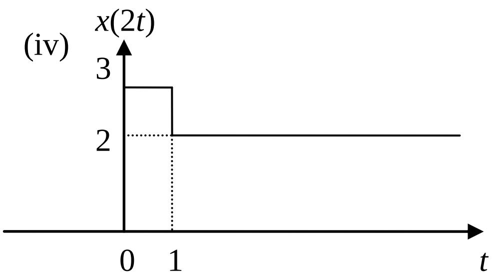
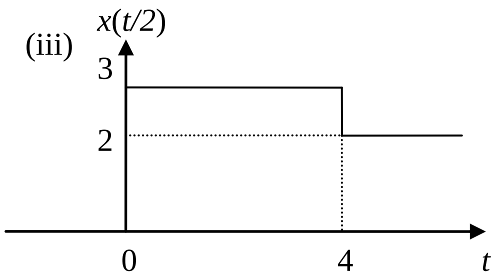
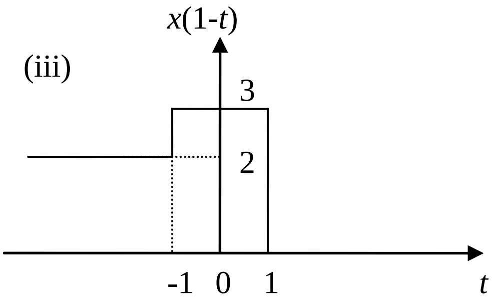


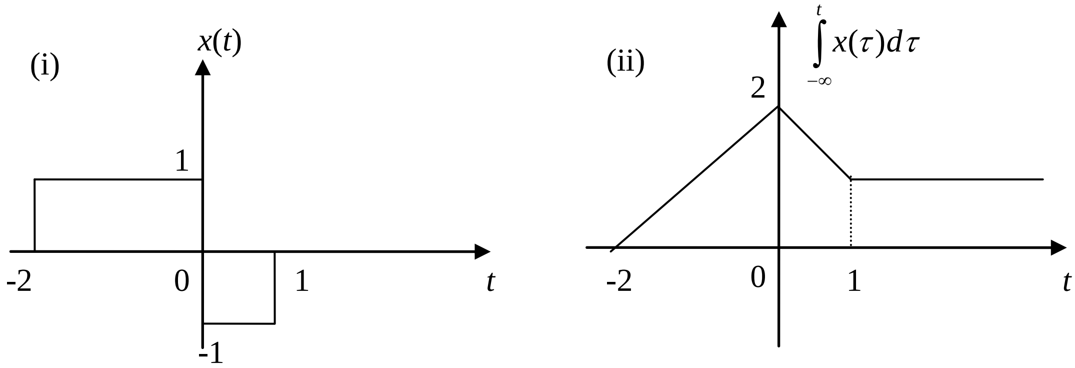
(iii)



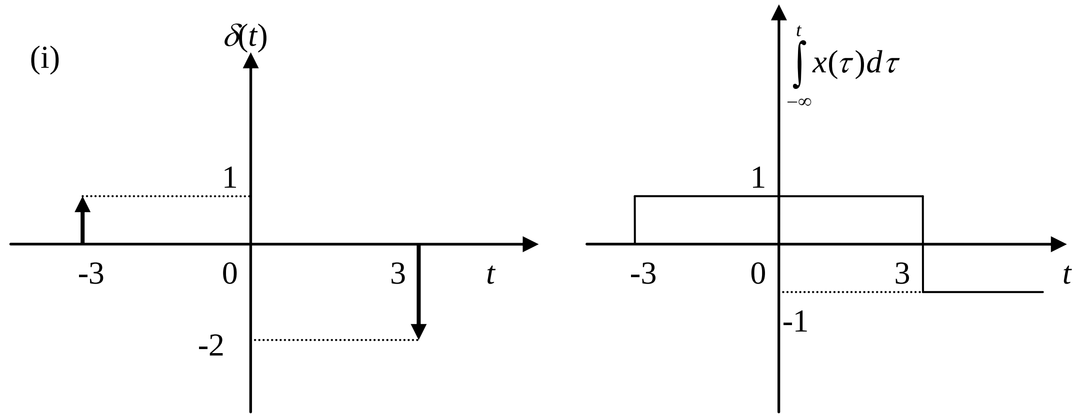
(iii)



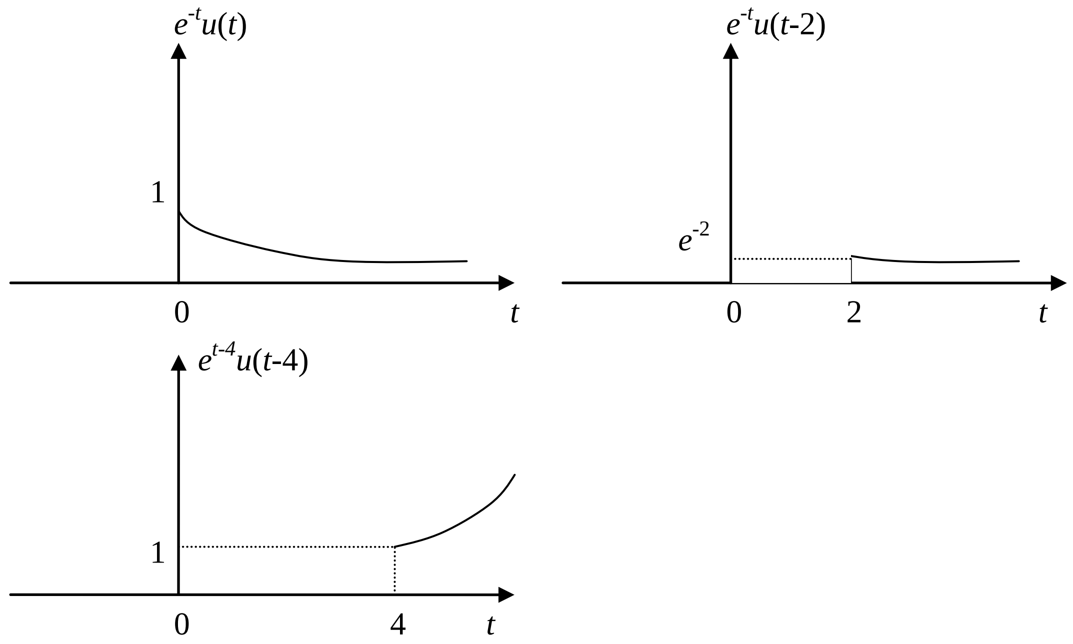
3. For $x(t) = u(t+2) - 2u(t) + u(t-1)$, sketch and label

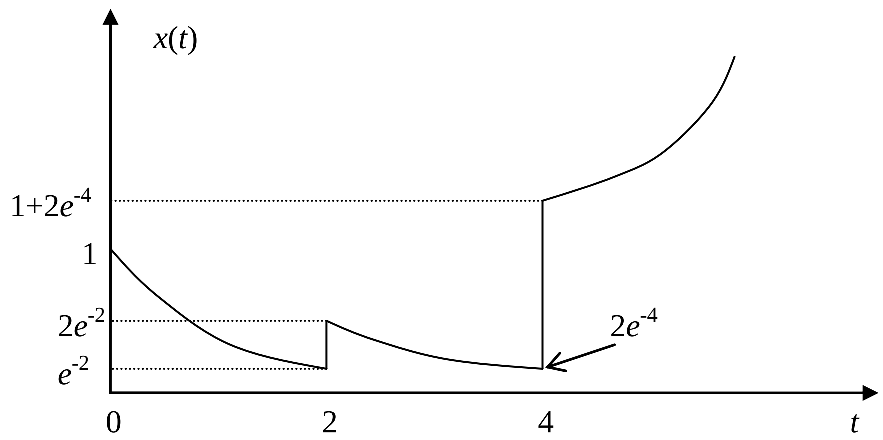


4. For $x(t) = \delta(t+3) - 2\delta(t-3)$, sketch and label



5. Sketch and label $x(t) = e^{-t}u(t) + e^{-t}u(t\text{-}2) + e^{t-4}u(t\text{-}4)$.





6. Are the following systems with or without memory, causal of noncausal?

(i) $y(t) = 2u(t)$: without memory, causal

(ii) $y(t) = \sin(u(t))$: without memory, causal

(iii) $y(t) = \sin(u(t+1))$: with memory, noncausal

(iv) $y(t) = e^{t-2}u(t\text{-}2)$: with memory, causal

7. Is the system represented by $y(t) = 1/x(t)$ linear and time-invariant?

The system output-input is described by $y(t) = 1/x(t)$.

If the input is $x_1(t)$ then the output will be $y_1(t) = 1/x_1(t)$.

If the input is $x_2(t)$ then the output will be $y_2(t) = 1/x_2(t)$.

However if the input is $ax_1(t) + bx_2(t)$ then the output will be

$\dfrac{1}{ax_1(t) + bx_2(t)} \neq ay_1(t) + by_2(t)$. Therefore the system is nonlinear.

If the input is $x(t-t_o)$ then the output will be $y(t-t_o) = 1/x(t-t_o)$. Hence the system is time invariant.

8. An RC high-pass circuit has a step response $g(t)=u(t)\exp(-t/RC)$. Sketch and derive an equation for the impulse response.

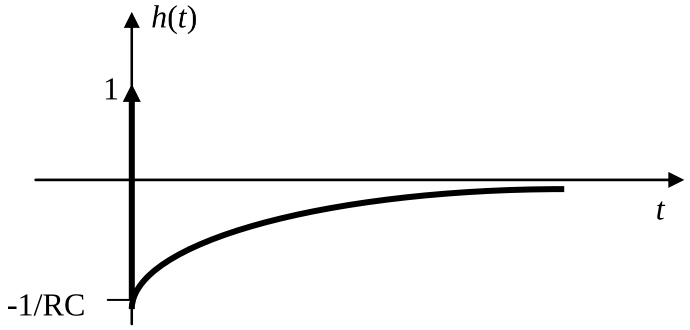
We know that impulse response $= \frac{d}{dt}$ (step response).

Therefore the impulse response

$$h(t) = \frac{d}{dt}\left[g(t)\right] = \frac{d}{dt}\left[u(t)\exp(-t/RC)\right]$$

$$= \exp(-t/RC)\frac{d}{dt}\left[u(t)\right] + u(t)\frac{d}{dt}\left[\exp(-t/RC)\right]$$

$$= \exp(-t/RC)\delta(t) + u(t)\left[-\frac{1}{RC}\exp(-t/RC)\right] = \delta(t)\exp(-t/RC) - \frac{u(t)}{RC}\exp(-t/RC).$$



9. A system has an impulse response $h(t)=\exp(-t)u(t)$. Find the unit step response of this system.

The unit step response is

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau = \int_{-\infty}^{t}\exp(-\tau)u(\tau)d\tau = \int_{0}^{t}\exp(-\tau)d\tau = -\exp(-\tau)\Big|_{o}^{t} = 1-\exp(-t).$$

Alternatively we can also use the convolution technique to compute the step response as follows

$$s(t) = h(t)*u(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty}\exp(-\tau)u(\tau)u(t-\tau)d\tau.$$

Since $u(\tau)u(t-\tau)$ only has value between 0 and t as shown below, we have

$$s(t) = \int_{0}^{t}\exp(-\tau)d\tau = -\exp(-\tau)\Big|_{0}^{t} = 1-\exp(-t).$$

10. Compute and sketch $y[n]=x[n]*z[n]$ where:

$x[n]$ =1,-1,2 for n =0,1,2

$z[n]$ = 1,2,3,-1 for n = -1,0,1,2

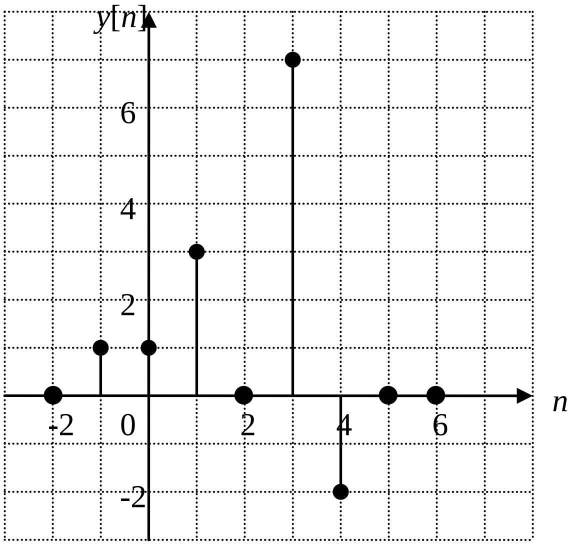
assume that each signal is zero elsewhere.

We can compute $y[n]$ using a table as follows

	k	-3	-2	-1	0	1	2	3	4	5
	$x[k]$	0	0	0	1	-1	2	0	0	0
n =-1	$z[-1-k]$	-1	3	2	1	0	0	0	0	0
n = 0	$z[-k]$	0	-1	3	2	1	0	0	0	0

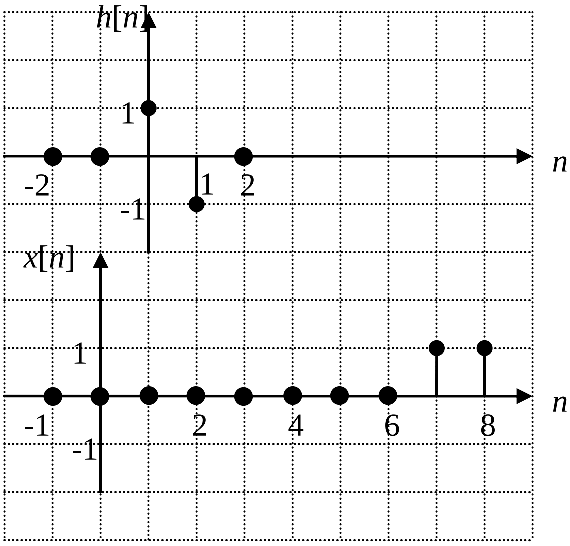
$n = 1$	$z[1-k]$	0	0	-1	3	2	1	0	0	0
$n = 2$	$z[2-k]$	0	0		-1	3	2	1	0	0
$n = 3$	$z[3-k]$	0	0	0	0	-1	3	2	1	0
$n = 4$	$z[4-k]$	0	0	0	0	0	-1	3	2	1
$n = 5$	$z[5-k]$	0	0	0	0	0	0	-1	3	2

	$y[n] = \sum x[k]z[n-k]$
$n = -1$	$1\times1=1$
$n = 0$	$(2\times1)+(1\times(-1))=1$
$n = 1$	$(3\times1)+(2\times(-1))+(1\times2)=3$
$n = 2$	$((-1)\times1)+(3\times(-1))+(2\times2)=0$
$n = 3$	$((-1)\times(-1))+(3\times2)=7$
$n = 4$	$((-1)\times2=-2$
$n = 5$	0



$y[n] = x[n] * z[n]$.

11. The impulse response of a system is given by $h[n]=-\delta[n-1] + \delta[n]$. By considering the input signal $x[n] = u[n-7]$, show that the system acts as an edge detector.

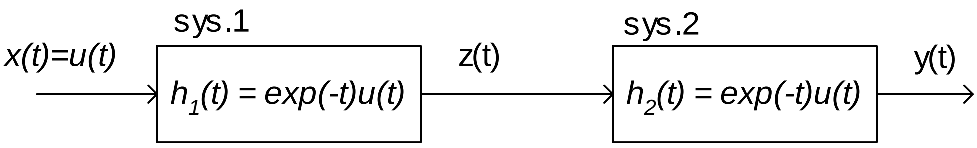


The response of the system can be obtained by performing a convolution between $x[n]$ and $h[n]$ as below:

	k	3	4	5	6	7	8	9	$y[n] = \sum x[k]h[n-k]$
	$x[k]$	0	0	0	0	1	1	1	
$n = 6$	$h[6-k]$	0	0	-1	1	0	0	0	0
$n = 7$	$h[7-k]$	0	0	0	-1	1	0	0	$1\times1 = 1$
$n = 8$	$h[8-k]$	0	0	0	0	-1	1	0	$(-1\times1) + (1\times1) = 0$
$n = 9$	$h[9-k]$	0	0	0	0	0	-1	1	$(-1\times1) + (1\times1) = 0$

$y[n] = x[n]*h[n]$ is zero everywhere except when $n = 7$. This shows that the system acts as an edge detector as it only has value at $n = 7$.

12. Find the output $y(t)$ for the system shown below when a unit-step input, $u(t)$ is applied.



$$z(t) = h_1(t) * x(t) = \int_{-\infty}^{\infty} h_1(\tau)u(t-\tau)d\tau = \int_{-\infty}^{\infty} \exp(-\tau)u(\tau)u(t-\tau)d\tau$$

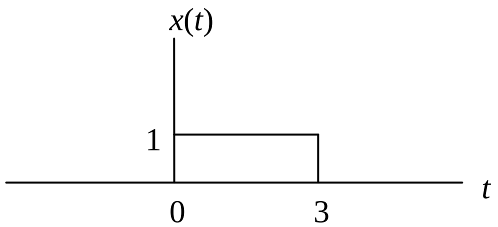
$$= \int_0^t \exp(-\tau)d\tau = 1 - \exp(-t)\text{, for } t \geq 0 \text{ or } [1-\exp(-t)]u(t).$$

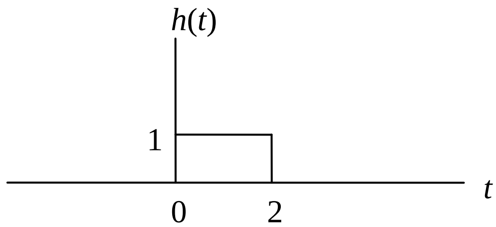
$$y(t) = h_2(t) * z(t) = \int_{-\infty}^{\infty} h_2(\tau)z(t-\tau)d\tau = \int_{-\infty}^{\infty} \exp(-\tau)u(\tau)[1-\exp(-(t-\tau))]u(t-\tau)d\tau$$

$$= \int_0^t \exp(-\tau)[1-\exp(-(t-\tau)]d\tau = \int_0^t [\exp(-\tau) - \exp(-t)]d\tau$$

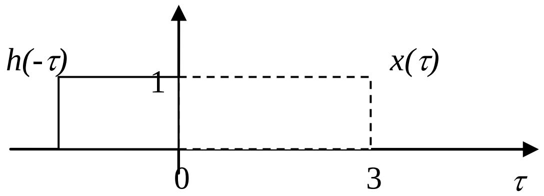
$$= -\exp(-\tau)\Big|_0^t - \tau\exp(-t)\Big|_0^t = 1 - \exp(-t) - t\exp(-t) = 1 - \exp(-t)(1+t).$$

13. Consider the signals $x(t)$ and $h(t)$ shown below. Compute $y(t) = x(t)*h(t)$ using (i) the graphical method (ii) the analytical method and write down the analytical expressions for $y(t)$.

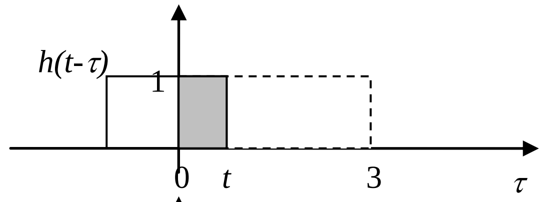




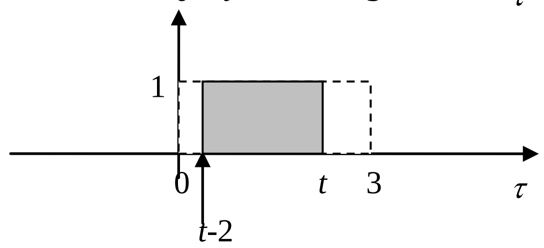
(i) Graphical method



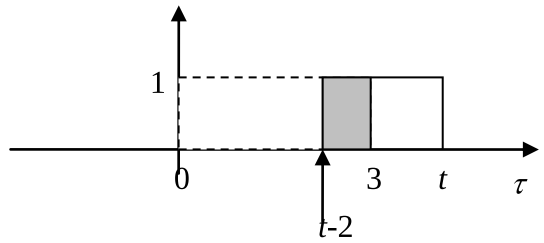
Interval I: For $t \leq 0$, no area overlap, $y(t) = 0$.



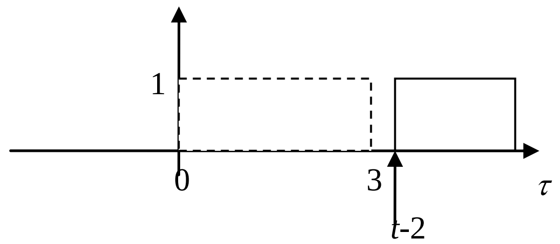
Interval II: For $0 < t \leq 2$ shaded area = $1\times t = t$, $y(t) = t$.



Interval III: For $2 < t \leq 3$, shaded area = $1\times 2 = 2$, $y(t) = 2$.



Interval IV: For $3 < t \leq 5$,
shaded area = $1\times(3-(t-2)) = 5-t$,
$y(t) = 5-t$.



Interval V: For $t > 5$, no area overlap, $y(t) = 0$.

In summary $$y(t) = \begin{cases} 0 & t \leq 0 \\ t & 0 < t \leq 2 \\ 2 & 2 < t \leq 3 \\ 5-t & 3 < t \leq 5 \\ 0 & t > 5 \end{cases}.$$

(ii) **Analytical method**

Consider the following intervals:

Interval I: For $t \leq 0$, $x(\tau)h(t-\tau) = 0$, $y(t) = 0$.

Interval II: For $0 < t \le 2$, $x(\tau)h(t-\tau) = 1$, $y(t) = \int_0^t x(\tau)h(t-\tau)d\tau = \int_0^t 1 d\tau = t$.

Interval III: For $2 < t \le 3$, $x(\tau)h(t-\tau) = 1$,

$$y(t) = \int_{t-2}^{t} x(\tau)h(t-\tau)d\tau = \int_{t-2}^{t} 1 d\tau = t-(t-2) = 2 \ .$$

Interval IV: For $3 < t \le 5$, $x(\tau)h(t-\tau) = 1$,

$$y(t) = \int_{t-2}^{3} x(\tau)h(t-\tau)d\tau = \int_{t-2}^{3} 1 d\tau = 3-(t-2) = 5-t \ .$$

Note that the upper integration limit is 3 as shown in the diagram above.

Interval V: For $3 < t \le 5$, $x(\tau)h(t-\tau) = 0$, $y(t) = 0$.

In summary $y(t) = \begin{cases} 0 & t \le 0 \\ t & 0 < t \le 2 \\ 2 & 2 < t \le 3 \\ 5-t & 3 < t \le 5 \\ 0 & t > 5 \end{cases}$.

14. Consider a signal $y[n] = 3x[n] + x[n-2]$. Obtain the impulse response and evaluate the response of the system to an input

$$x_1[n] = \begin{cases} 1 & n=0 \\ 1 & n=1 \\ 2 & n=2 \\ 0 & otherwise \end{cases} .$$

To obtain the impulse response $h[n]$ substituting $x[n] = \delta[n]$ gives

$h[n] = 3\delta[n] + \delta[n-2]$ or

$$h[n] = \begin{cases} 3 & n=0 \\ 0 & n=1 \\ 1 & n=2 \\ 0 & otherwise \end{cases} .$$

To compute the response due to $x_1[n]$, express $x_1[n]$ as a sum of weighted impulses, i.e

$x_1[n] = \delta[n] + \delta[n-1] + 2\delta[n-2]$.

Now the response is $y_1[n] = h[n] + h[n-1] + 2h[n-2]$

$n = 0$: $y_1[0] = h[0] + h[-1] + h[-2] = 3$

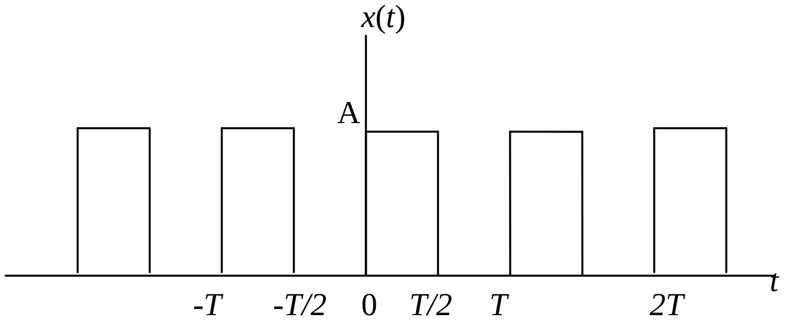
$n = 1$: $y_1[1] = h[1] + h[0] + 2h[-1] = 3$

$n = 2$: $y_1[2] = h[2] + h[1] + 2h[0] = 1 + 6 = 7$

$n = 3$: $y_1[3] = h[3] + h[2] + 2h[1] = 1$

$n = 4$: $y_1[4] = h[4] + h[3] + 2h[2] = 2$

15. Determine the Fourier Series approximation of the signal shown below



$$a_0 = \frac{1}{T}\int_0^{T/2} A\,dt = \frac{A}{2}$$

$$a_n = \frac{2}{T}\int_0^{T/2} A\cos\left[\frac{2\pi nt}{T}\right]dt$$

$$a_n = \frac{2A}{T}\int_0^{T/2} \cos\left[\frac{2\pi nt}{T}\right]dt$$

$$a_n = \frac{2A}{T}\frac{T}{2\pi n}\left[\sin\left[\frac{2\pi nt}{T}\right]\right]_0^{T/2}$$

$$a_n = \frac{A}{n\pi}\left[\sin[\pi n] - \sin(0)\right] = 0$$

$$b_n = \frac{2}{T}\int_0^{T/2} A\sin\left[\frac{2\pi nt}{T}\right]dt$$

$$b_n = \frac{2A}{T}\int_0^{T/2} \sin\left[\frac{2\pi nt}{T}\right]dt$$

$$b_n = \frac{-2A}{T}\frac{T}{2\pi n}\left[\cos\left[\frac{2\pi nt}{T}\right]\right]_0^{T/2}$$ For odd value of n. bn=0 for even n

$$b_n = \frac{-A}{n\pi}\left[\cos[\pi n] - \cos(0)\right]$$

$$b_n = \frac{2A}{n\pi}$$

$$x(t) = \frac{A}{2} + \sum_{n=1,3,5..}^{\infty} \frac{2A}{n\pi}\sin\left[\frac{2\pi nt}{T}\right]$$