The University Of Sheffield.

Lecture content

- Properties of Fourier Transform
  - Linearity
  - Time Shift
  - Frequency Shift
  - Time Scaling
  - Differentiation and Integration

Properties of Fourier Transform

Linearity

If $x_1(t) \leftrightarrow X_1(\omega)$ and $x_2(t) \leftrightarrow X_2(\omega)$

Then $ax_1(t) + bx_2(t) \leftrightarrow aX_1(\omega) + bX_2(\omega)$.

Time Shift

If $x(t) \leftrightarrow X(\omega)$ then $x(t- t_o) \leftrightarrow X(\omega)\, e^{-j\omega t_o}$

Example:

Obtain the Fourier Transform of the signal in figure 7 using the time shift property and the Fourier Transform of the signal in figure 4.

Fourier Transform

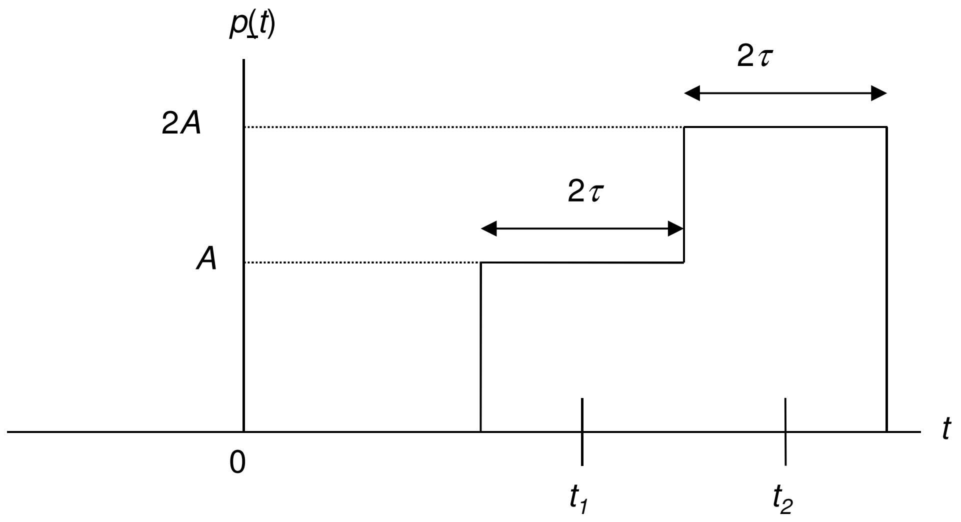


Figure 7: Signal p(t).

Properties of Fourier Transform

Frequency Shift

If $x(t) \leftrightarrow X(\omega)$ then $x(t)e^{j\omega_o t} \leftrightarrow X(\omega - \omega_o)$

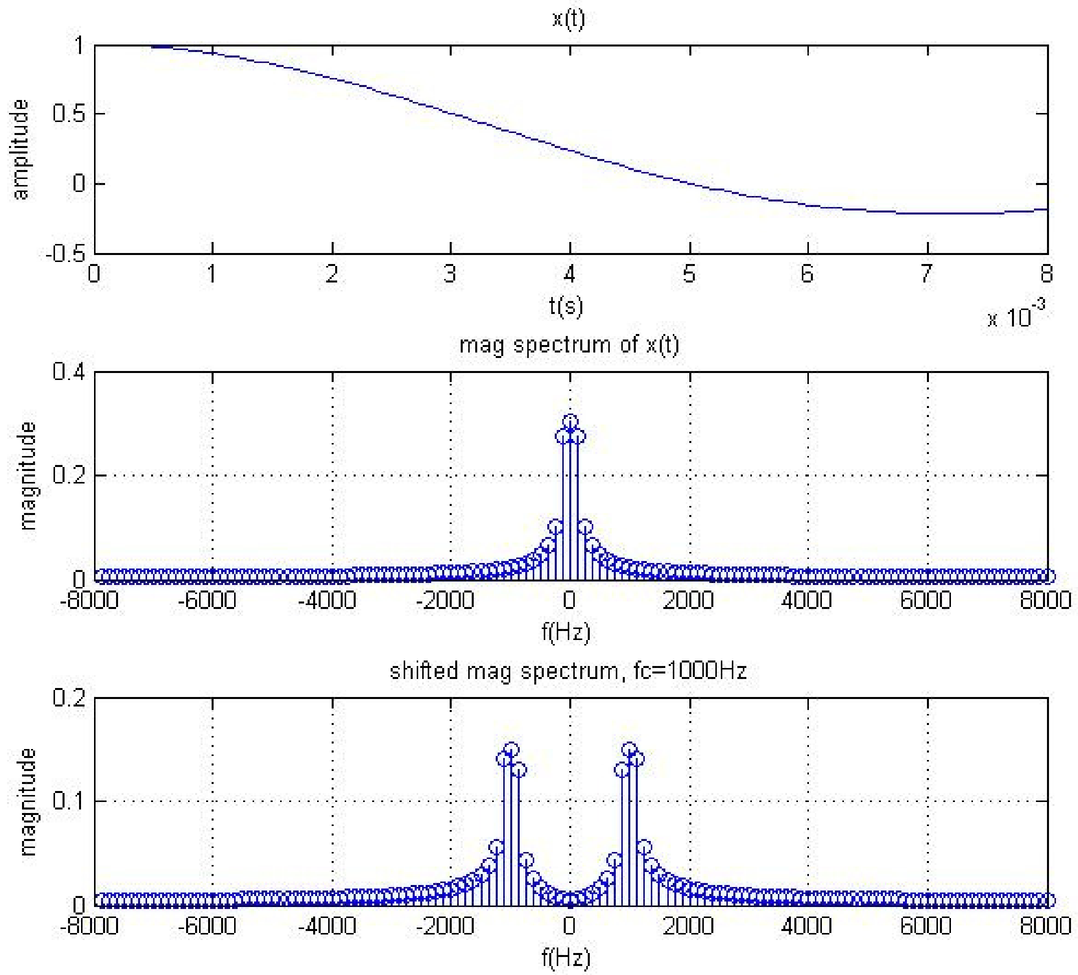
The frequency spectrum of $x(t)$ has been shifted to ω_o. If $x(t)$ is multiplied by a sinusoidal signal we have,

$$x(t)\cos\omega_o t \leftrightarrow \frac{1}{2}[X(\omega+\omega_o)+X(\omega-\omega_o)]$$

and

$$x(t)\sin\omega_o t \leftrightarrow \frac{j}{2}[X(\omega+\omega_o)-X(\omega-\omega_o)].$$

Frequency Shift




```
function FT_freq_shift(f,fc)
A=1;
fs=16000;
n=128;  %number of points
t=[1/fs:1/fs:n/fs];
x=A*(sin(2*pi*f*t))./(2*pi*f*t);    %generate a sinc function
y1=cos(2*pi*fc*t);
y=x.*y1;

%generate magnitude spectrum
k=[0:64 -63:-1];
Y=fft(y)/n;
mag_sig=abs(Y);
X=fft(x)/n;
mag_x=abs(X);
fa=(fs/n)*k;

%plot graphs
subplot(3,1,1),plot(t,x);
str1=['x(t)'];
title(str1);
xlabel('t(s)');
ylabel('amplitude');

subplot(3,1,2),stem(fa,mag_x);
str2=['mag spectrum of x(t)'];
title(str2);
grid;
xlabel('f(Hz)');
ylabel('magnitude');

subplot(3,1,3),stem(fa,mag_sig);   %phase in radian
str3=['shifted mag spectrum, fc=',num2str(fc),'Hz'];
title(str3);
grid;
xlabel('f(Hz)');
ylabel('magnitude');
```

Properties of Fourier Transform

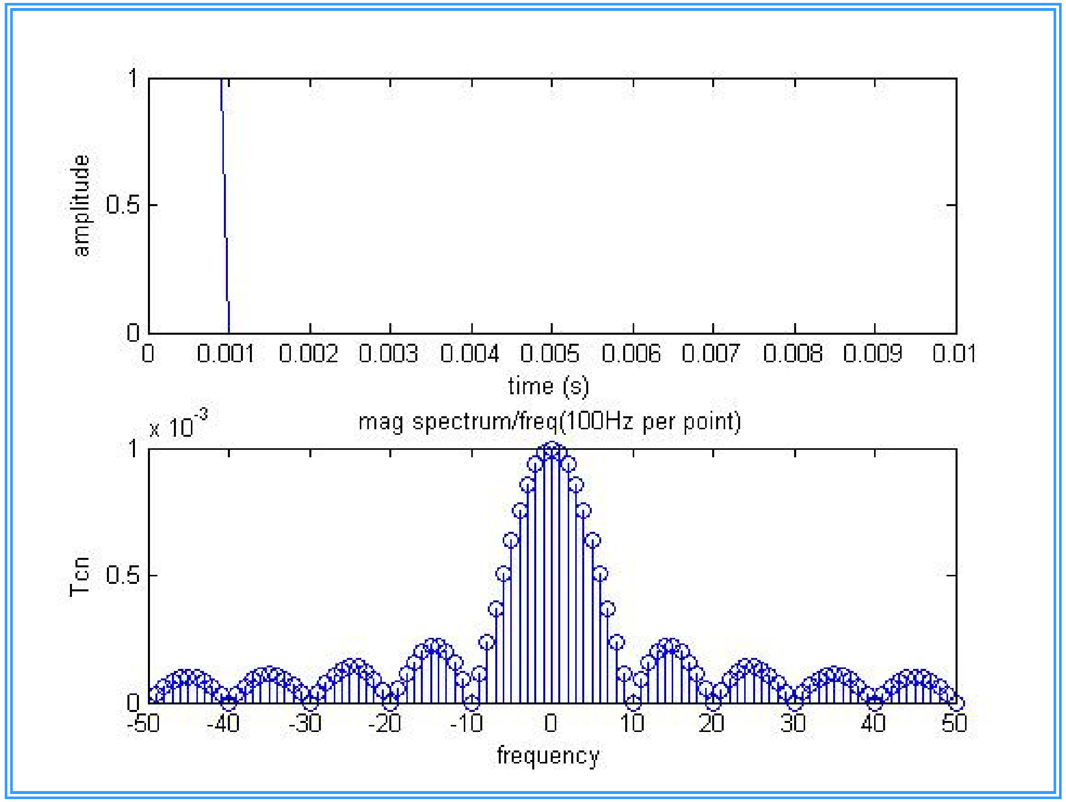
<u>Time Scaling</u>

If $x(t) \leftrightarrow X(\omega)$ then $x(at) \leftrightarrow \dfrac{1}{a}X\left(\dfrac{\omega}{a}\right)$

If $a > 1$ $x(t)$ is time compressed. If $0 < a < 1$ $x(t)$ is time expanded.

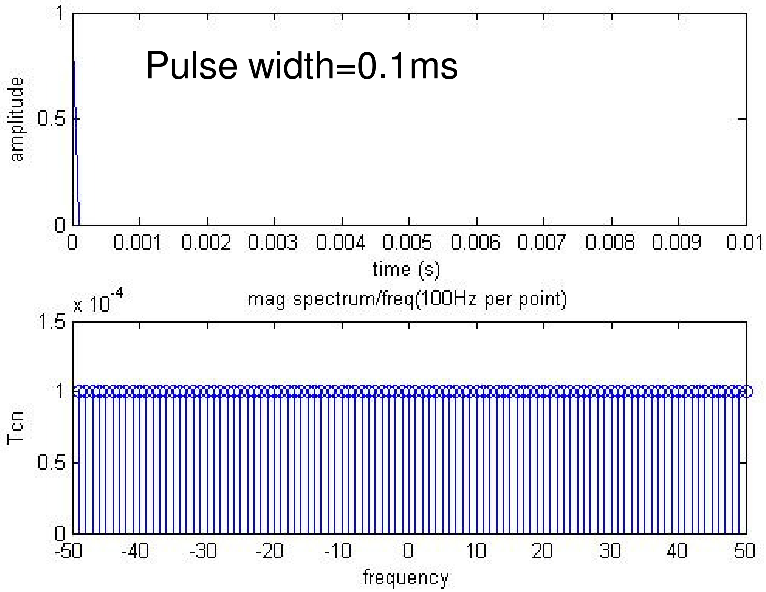
Note: Time compression $\leftrightarrow$ frequency expansion
Time expansion $\leftrightarrow$ frequency compression

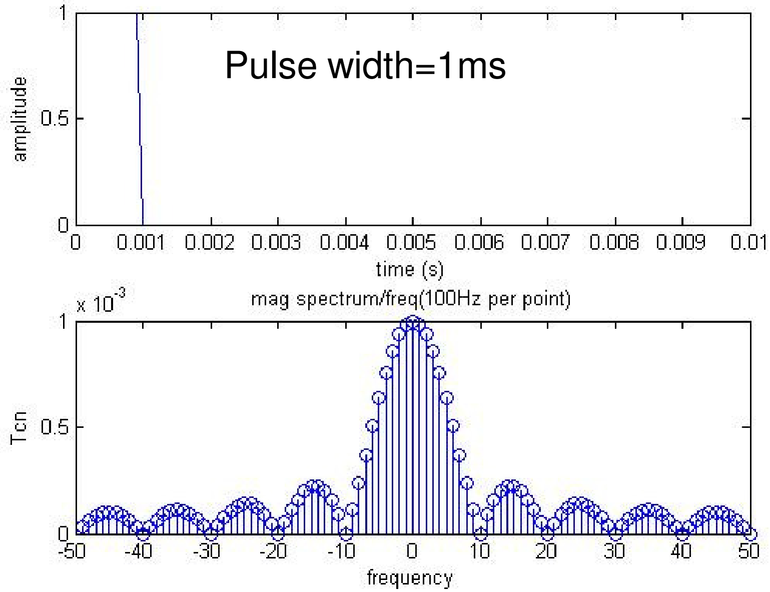
1ms pulse width

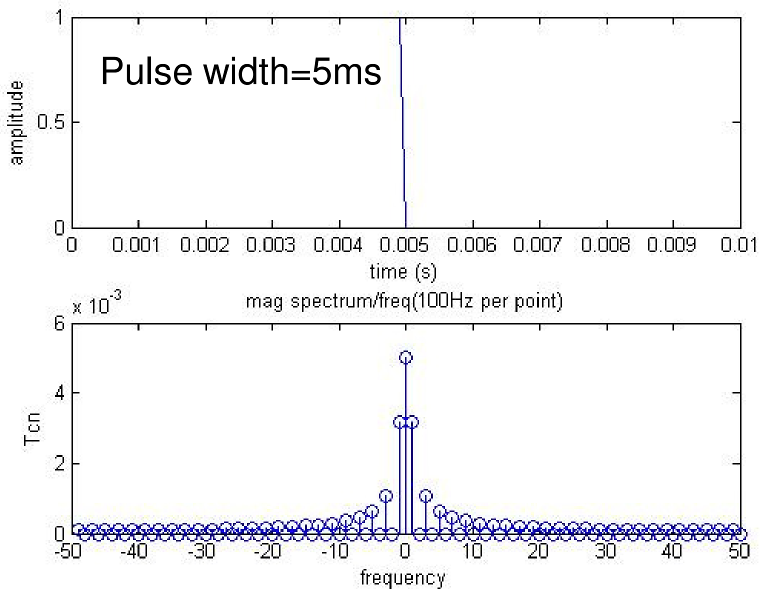


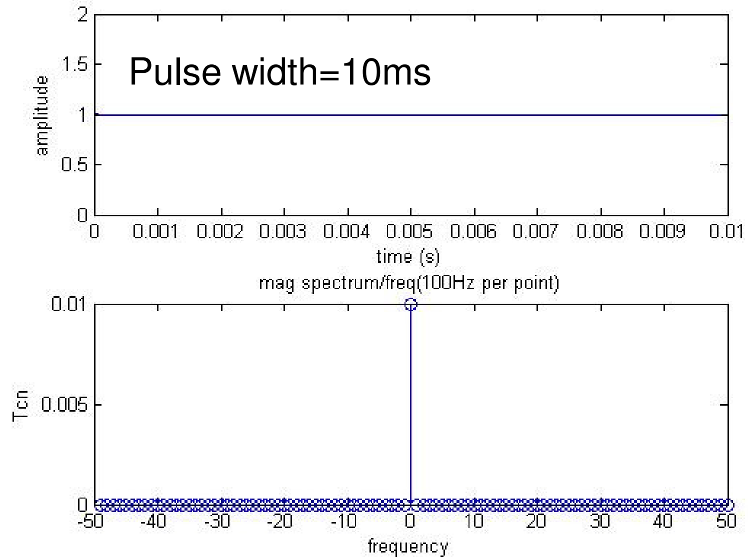

```
function pulse1(fs,T,tau)

t1=[0:1/fs:T-1/fs];
numpts=T*fs;
c1=tau*numpts/T;
c2=numpts-c1;
p1=[ones(1,c1) zeros(1,c2)];

subplot(2,1,1),plot(t1,p1,'b');
xlabel('time (s)');
ylabel('amplitude');
k1=[0:numpts/2 -(numpts/2-1):-1];
P1=fft(p1)/numpts;

mag=T*abs(P1)
subplot(2,1,2),stem(k1,mag);
f1=num2str(fs/numpts);
xlabel('frequency');
ylabel('Tcn');
str2=['mag spectrum/freq(',num2str(fs/numpts),'Hz per point)'];
title(str2);
```

Different pulse widths

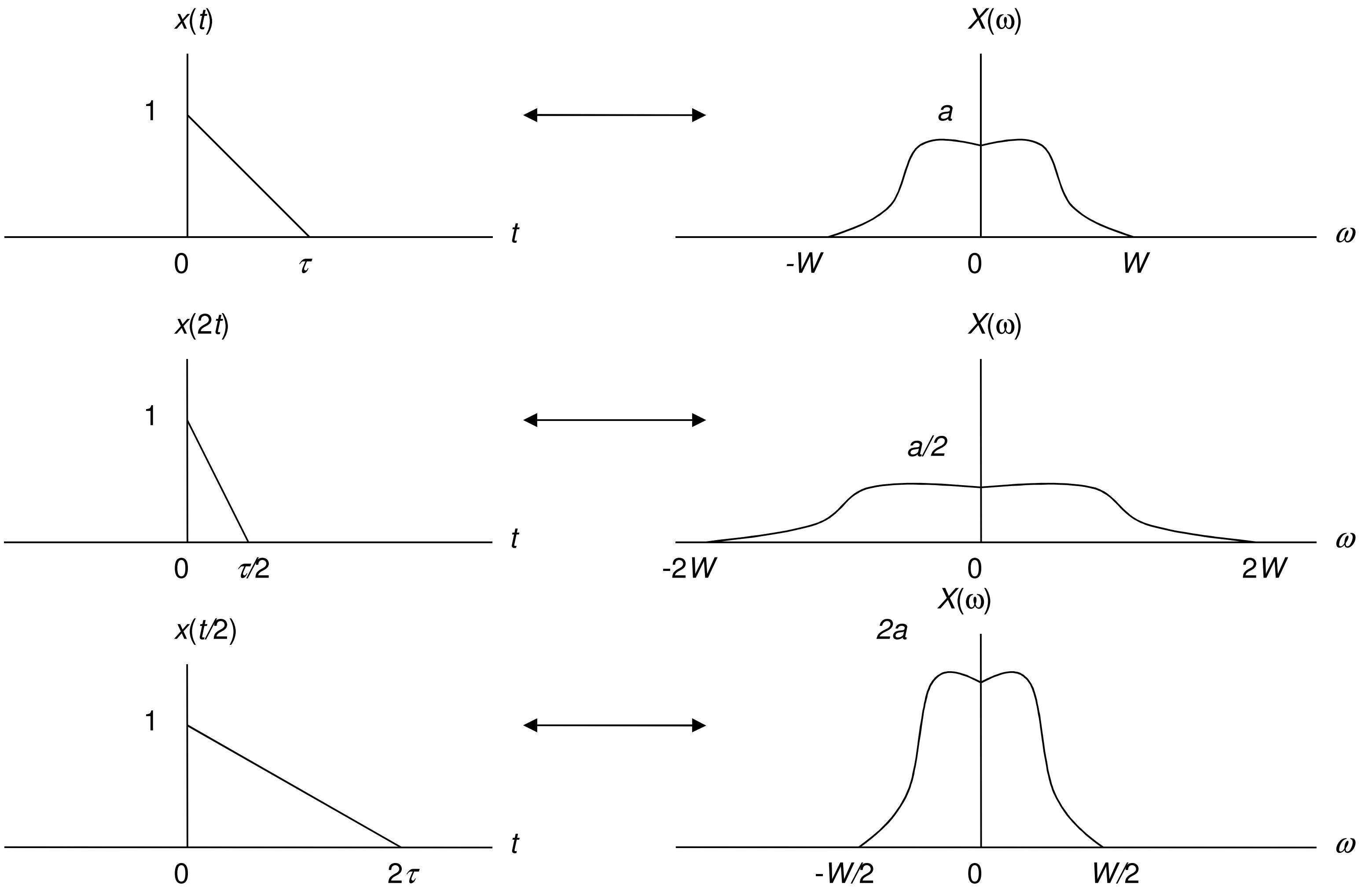








Properties of Fourier Transform



Properties of Fourier Transform

Differentiation and Integration

If $x(t) \leftrightarrow X(\omega)$ then $\dfrac{dx(t)}{dt} \leftrightarrow j\omega X(\omega)$.

Differentiation in time domain is replaced by $j\omega$ in frequency domain.

The integration property of Fourier Transform is described by

$$\int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{j\omega}X(\omega) + \pi X(0)\delta(\omega)$$

Properties of Fourier Transform

Example:

1. Obtain the Fourier Transform of the unit step $u(t)$, making use of the integration property of Fourier Transform.

2. Compute the Fourier Transform of a triangular signal shown in figure 9.

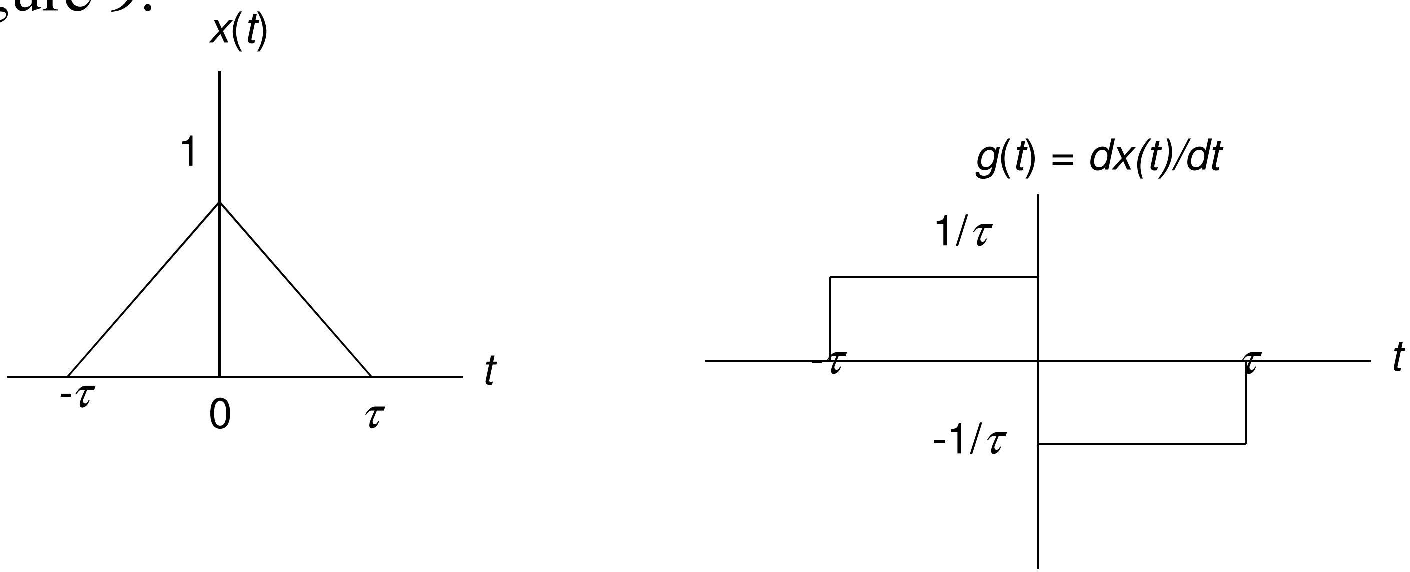


Figure 9: A triangular signal x(t)and g(t)=dx(t)/dt.

Example 1

Let $g(t) = \delta(t)$. We know that $g(t) = \delta(t) \leftrightarrow G(\omega) = 1$ and $u(t) = \int_{-\infty}^{t} g(\tau)d\tau$
Using the integration property we have,

$$X(\omega) = \mathrm{F}\left[\int_{-\infty}^{t} g(\tau)d\tau\right] = \frac{G(\omega)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Check:
We can recover the Fourier Transform of $\delta(t)$ by using the differentiation property.

$$G(\omega) = \mathrm{F}\left[\frac{dx(t)}{dt}\right] = j\omega X(\omega) = j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1,$$

since $\omega\delta(\omega) = 0$.

Example 2

We know that the Fourier Transform of a rectangular pulse with duration of τ and amplitude of 1 is $\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$. Using the time shift property,

$$G(\omega) = \tau\left(\frac{1}{\tau}\right)\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}e^{j\omega\tau/2} - \tau\left(\frac{1}{\tau}\right)\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}e^{-j\omega\tau/2}$$

$$= \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}(e^{j\omega\tau/2} - e^{-j\omega\tau/2})$$

$$= \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}(2j\sin(\omega\tau/2)) = j\omega\tau\left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right]^2$$

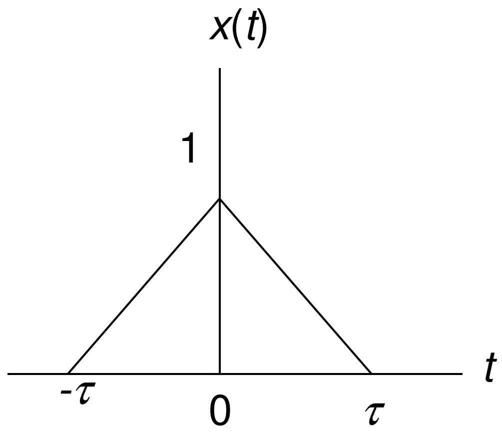
$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau \leftrightarrow \frac{1}{j\omega}G(\omega) + \pi G(0)\delta(\omega)$$

$X(\omega) = G(\omega)/j\omega$ since $G(0) = 0$.

Finally we have, $X(\omega) = \tau\left[\frac{\sin(\omega\tau/2)}{(\omega\tau/2)}\right]^2 = \tau \sin c^2(\omega\tau/2)$

Properties of Fourier Transform

Useful to remember



$$X(\omega) = \tau \operatorname{sinc}^2(\omega\tau/2)$$