7. Numerical methods for the solution of electromagnetic field problems.

Analytical methods of the type described earlier in this course provide a mathematically elegant means of calculating electrostatic, magneto-static and time varying magnetic fields. However, although they often provide 'exact' solutions in which the influence of certain parameters can sometimes be readily evident, they are often limited to rather simple geometries and source distributions. As a consequence, the problem domain is often simplified to achieve a solution. Moreover, magnetic saturation can play an important role in many devices. This non-linear behaviour (i.e. B and H are not related by a single permeability) is extremely difficult to include.

As a result of these limitations, many practical problems in design and analysis make use of numerical methods for solving fields. The range and capability of numerical methods have grown enormously in the past 30+ years as the computational power available has grown exponentially. It is now possible to solve problems of complexity that seemed impossible only a few years ago, e.g. full 3D non-linear, transient, time-stepped model of an electrical machine coupled to a complex mechanical load, although the solution time may still be many hours even on a powerful PC.

There are many commercial numerical electromagnetic field solution software packages available which are able to solve many different types of electromagnetic problems. These range from fairly straightforward 2D static packages (FEMM) through to suites of 2D / 3D modelling tools with complex solid modelling and CAD interfaces (FLUX, ANSYS, MAXWELL, OPERA). Although these employ a wide range of different user interfaces and mathematical models, at their core they are all based on the same basic idea of representing the field distribution using a very large number of localised simplified field representations and coupling these localised approximations together to provide an overall field distribution. Although they are able to provide very detailed field distributions, it should always be remembered that they are based on approximations and that they are only as good as the problem specification provided by the user.

Iterative solution of two-dimensional Laplace's equation by finite difference method

An interesting example of a numerical method, which provides a useful starting point for understanding the nature of numerical methods, is the solution of the two-dimensional Laplace's equation for an electrostatic field using a so-called finite difference approximation. The basic equation in Cartesian coordinates is:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

Rather than solve this equation as a continuous mathematical function of x and y, numerical methods calculate the field at a number (often a very large number) of discrete points within the problem domain. We can start the process of deriving a numerical approximation for

solving this two-dimensional form of Laplace's equation by considering the one-dimensional series of 3 points shown in Figure 1.

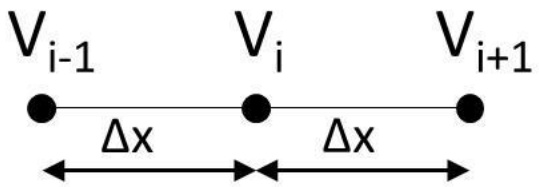

Figure 1 One-dimensional series of discrete points in x-direction

In order to solve the governing field equation, we need to make use of a Taylor series to represent the potential at one node in terms of the potential at other nodes and the spatial derivatives of the field between them. The Taylor series (or sometimes called the Taylor expansion) of a function is a representation of a function as an infinite series of terms that are calculated from its derivatives at a particular point. In the general case of a function f(x), the Taylor series can be expressed an infinite series based on its derivatives at fixed point x=a:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

Clearly, the Taylor expansion is based on a summation up to infinity. In practice, we terminate the series at some practical value of n (noting that the n! term in the denominator means that successive terms tend to diminish in magnitude with increasing n). When we terminate the Taylor series at a particular value of n, we express the remaining terms in terms of the 'big-O' notation which in essence provides an indication of the growth rate or order of the truncated terms.

Returning to Figure 1 and taking the point i-1 which is has an x-coordinate which is –Δx from point i, then the potential V_{i-1} can be expressed by a Taylor series approximation

$$V_{i-1} = V_i + \frac{1}{1!}\frac{dV}{dx}\bigg|_i (-\Delta x) + \frac{1}{2!}\left(\frac{d^2V}{dx^2}\right)\bigg|_i (-\Delta x)^2 + \frac{1}{3!}\left(\frac{d^3V}{dx^3}\right)\bigg|_i (-\Delta x)^3 + O(-\Delta x)^4$$

$$= V_i - \frac{dV}{dx}\bigg|_i \Delta x + \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)\bigg|_i \Delta x^2 - \frac{1}{6}\left(\frac{d^3V}{dx^3}\right)\bigg|_i \Delta x^3 + O\Delta x^4$$

Applying a similar procedure to point i+1 yields

$$V_{i+1} = V_i + \frac{dV}{dx}\bigg|_i \Delta x + \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)\bigg|_i \Delta x^2 + \frac{1}{6}\left(\frac{d^3V}{dx^3}\right)\bigg|_i \Delta x^3 + O\Delta x^4$$

Adding the equations for V_{i-1} and V_{i+1} yields:

$$V_{i-1} + V_{i+1} = 2V_i + \left(\frac{d^2V}{dx^2}\right)\bigg|_i \Delta x^2 + O\Delta x^4$$

This can be rearranged to give:

$$\left(\frac{d^2V}{dx^2}\right)\Big|_i = \frac{V_{i-1} - 2V_i + V_{i+1}}{\Delta x^2} + O\Delta x^4$$

A similar procedure can be applied in the y-direction to get an approximation for $\frac{d^2V}{dy^2}$ for the 3 points shown in Figure 2.

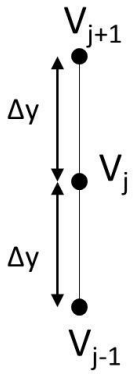

Figure 2 One-dimensional series of discrete points in y-direction

$$\left(\frac{d^2V}{dy^2}\right)\Big|_j = \frac{V_{j-1} - 2V_j + V_{j+1}}{\Delta y^2} + O\Delta y^4$$

If we consider a general point C which is surrounded by 4 equidistant points (i.e. $\Delta x = \Delta y = h$) labelled 1,2,3 and 4 as shown in Figure 3.

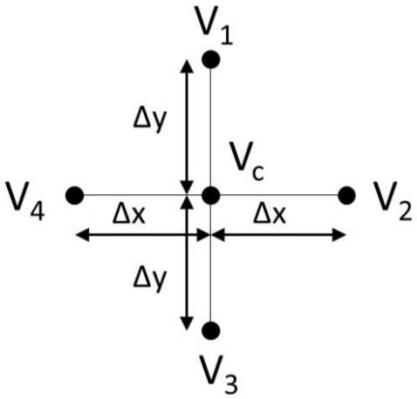

Figure 3 Two dimensional array of points

We can use the approximations derived above to provide an approximate solution to the two-dimensional Laplace's equation.

$$\left(\frac{d^2V}{dx^2}\right) + \left(\frac{d^2V}{dy^2}\right)\Big|_C = \frac{V_4 - 2V_C + V_2}{h^2} + \frac{V_3 - 2V_C + V_1}{h^2} = 0$$

$$V_4 - 2V_C + V_2 + V_3 - 2V_C + V_1 = 0$$

And hence, the potential at point C is given by:

$$V_c = \frac{1}{4}(V_1 + V_2 + V_3 + V_4)$$

This is a remarkably simple expression for an approximation to the potential at point C in terms of the surrounding potential at 4 equidistant points. Although the end result looks like the simple average of the individual potentials of the surrounding 4 points, it is important to recognise that is comes from a rigorous approximation procedure for Laplace's equation. It is also possible to derive corresponding expressions for non-uniform grids, e.g. for a regular grid with different increments in x and y, the potential at point C is given by:

$$V_c = \frac{V_1 + V_2 + \left(\frac{\Delta x}{\Delta y}\right)^2 (V_3 + V_4)}{2\left(1 + \left(\frac{\Delta x}{\Delta y}\right)^2\right)}$$

which is consistent with equation for a uniform grid when $\Delta x = \Delta y$.

Application to a practical problem

We can populate a problem domain with a grid of regularly spaced points, and apply the equations derived above at each point in the grid. However, it will be apparent that with the exception of points with fixed known potentials (generally those on a boundary) each potential depends on the surrounding potential and vice-versa. The way out of this apparently circular problem is to adopt an iterative process in which we repeatedly calculate the potential at each successive point using updated values of potentials at the surrounding points. In well bounded problems, this will generally lead to an acceptable level of convergence (which it is important not to confuse with accuracy – more of that later).

Taking the example of the trough considered previously in section 3 of the notes for which an analytical solution was shown (but not solved per se in the notes). Supposing the dimensions d and b are set to 1.5m and 1.2m respectively and we define an array of 30 equi-spaced points which cover the trough as shown in Figure 4. Eighteen of these points sit on the walls of trough and hence have fixed and known potentials defined by the boundary potential. The 12 remaining points (labelled as 1-12) have unknown potentials which depend on the surrounding potentials.

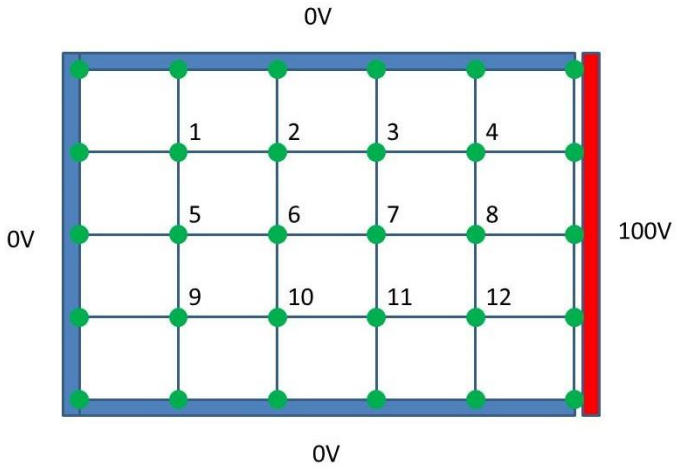

Figure 4 Trough test-problem

We can readily tabulate the individual equations which can be used to calculate the potential at each of the 12 points of unknown potential. In order to start the calculation of the unknown potentials, we need to start with some initial estimates for the unknown potentials as they might be required for calculation of the potential at another point before the calculation procedure is applied to the point itself. In principle, we could start with any set of values, e.g. last week's winning lottery numbers, and the iteration process will generally progress to a converged solution. However, the number of iterations required to achieve some prescribed level of convergence can be reduced by choosing 'sensible' values, in some cases informed by a sense of the likely values. However, as will be illustrated in this example, we can simply start with all unknown initial potentials set to zero.

The table below contains the equations which define the unknown potentials, the initial estimates and the results from 7 iteration cycles. The procedure for calculating each value is based on using the most up to date estimates available and applying the equations for each point in succession. It is interesting to note that the absolute dimensions of the trough do not appear explicitly in the equation. Indeed, providing the ratio of the dimensions a and b are the same, the solution simply scales with the absolute size, e.g. a trough 1.5km x 1.2km would give rise to the same potentials at the discrete points if the same number of grid divisions was used.

As will be apparent, the solutions start to 'settle down' as the successive iterations are completed. We define this level of 'settling down' as the difference in potential at a point between successive iterations. By way of example, if we take point 4, the difference between successive iterations reduces from 6.3V between iterations 1 and 2 down to 0.8V between iterations 6 and 7. Clearly we need to define some criterion by which we judge it is time to stop which provides an acceptable trade-off between the computation time and the degree of convergence achieved. Supposing we set a criterion that the solutions produced between successive iterations should differ by <1V, then we would terminate the process after 7 iterations.

Point		Initial estimate	Iteration						
			1st	2nd	3rd	4th	5th	6th	7th
1	$=\frac{0+V_2+V_5+0}{4}$	**0**	0	0.0	0.0	0.4	0.8	1.3	1.7
2	$=\frac{0+V_3+V_6+V_1}{4}$	**0**	0	0.0	1.6	2.7	4.2	5.1	6.0
3	$=\frac{0+V_4+V_7+V_2}{4}$	**0**	0	6.3	9.4	12.5	14.1	15.6	16.5
4	$=\frac{0+100+V_8+V_3}{4}$	**0**	25	31.3	35.9	37.9	39.6	40.5	41.3
5	$=\frac{V_1+V_6+V_9+0}{4}$	**0**	0	0.0	0.0	0.4	1.2	1.7	2.4
6	$=\frac{V_2+V_7+V_{10}+V_5}{4}$	**0**	0	0.0	1.6	3.9	5.4	7.1	8.0
7	$=\frac{V_3+V_8+V_{11}+V_6}{4}$	**0**	0	6.3	12.5	15.6	18.8	20.3	22.0
8	$=\frac{V_4+100+V_{12}+V_7}{4}$	**0**	25	37.5	42.2	46.1	47.9	49.5	50.3
9	$=\frac{V_5+V_{10}+0+0}{4}$	**0**	0	0.0	0.0	0.4	0.8	1.3	1.7
10	$=\frac{V_6+V_{11}+0+V_9}{4}$	**0**	0	0.0	1.6	2.7	4.2	5.1	6.0
11	$=\frac{V_7+V_{12}+0+V_{10}}{4}$	**0**	0	6.3	9.4	12.5	14.1	15.6	16.5
12	$=\frac{V_8+100+0+V_{11}}{4}$	**0**	25	31.3	35.9	37.9	39.6	40.5	41.3

It is important to recognise that the degree of convergence achieved is **NOT** the same as the 'accuracy' of the potential calculation. Since the method is based on a spatial approximation, the accuracy is determined to a large extent by how fine a grid is used – in general the solution will improve in accuracy when more points are used within the problem domain. However, again there is a trade-off between computation time and accuracy.

Returning to the analytical expression derived for the trough problem, it is interesting to compare the numerical approximations to the exact solution of the analytical solution (recognising that the analytical solution was based on a truncated series expansion). The potential at a point (x,y) in the trough is given by:

$$V(x,y)=\frac{4V_0}{\pi}\sum_{m=1,odd}^{\infty}\frac{1}{m}\frac{\sinh(\frac{m\pi x}{b})}{\sinh(\frac{m\pi d}{b})}\sin\frac{m\pi y}{b}$$

Looking at point 4 which is located at (1.2m,0.9m) relative to the origin at the lower left hand corner of the trough, the analytical expression (taking m up to 13) predicts a potential of 43.4V as compared to 41.3V for a numerical calculation terminated after 7 iterations. It is

interesting to note that 20 iterations of the numerical solution yields a potential of 43.1V (with a value 43.1V after 19 iterations).

It is worth noting that whereas the analytical method can be used in a one-off manner to calculate the potential as a specific point, the numerical method is required to calculate the field throughout the full problem domain, even if only the potential at a single point is required.

This method can be applied to a range of different geometries, providing the geometry is a good fit to a rectangular grid, noting that non-uniform increments of distance between points in the grid can be used, albeit with more complex expressions for potential.

Taking the example of two parallel plates at a sharp corner shown in Figure 5. We could solve this problem as it stands with 45 unknown potentials and 40 known potential. However, we can exploit the symmetry in the problem domain to reduce the problem to fewer unknown (which will make it quicker to solve). The reduced problem domain which exploits this symmetry is shown in Figure 6. This has 24 unknowns, i.e. a computational saving of almost 50%. Points 1-18 have exactly the same equations for their potential based on the four surrounding points. However the points on the diagonal line of symmetry, i.e. 25, 35 and 45 do not have 4 surrounding points. However, we can use the symmetry to fill in for the absent points in their equations. By way of example, in the full problem domain, the expression for the potential at point 25 would have been:

$$V_{25} = \frac{1}{4}(V_{16} + V_{26} + V_{34} + V_{24})$$

However, the symmetry of **both** the geometry and the sources is such that the potential at point 24 is the same as point 16, and 34 is the same as point 26. Hence, the expression for point 25 in the reduced problem domain is:

$$V_{25} = \frac{1}{4}(2V_{16} + 2V_{26})$$

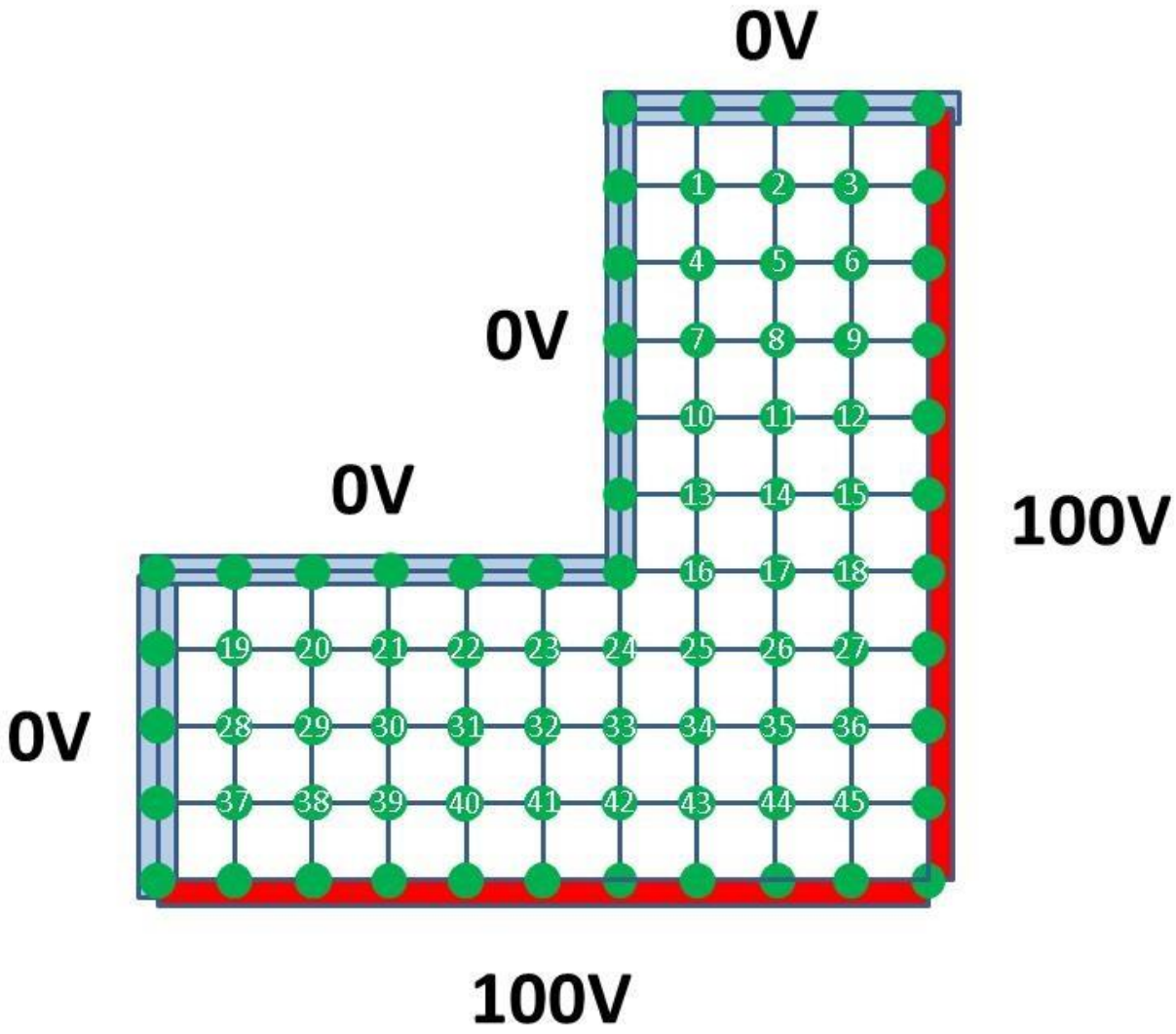

Figure 5 Sharp corner test problem

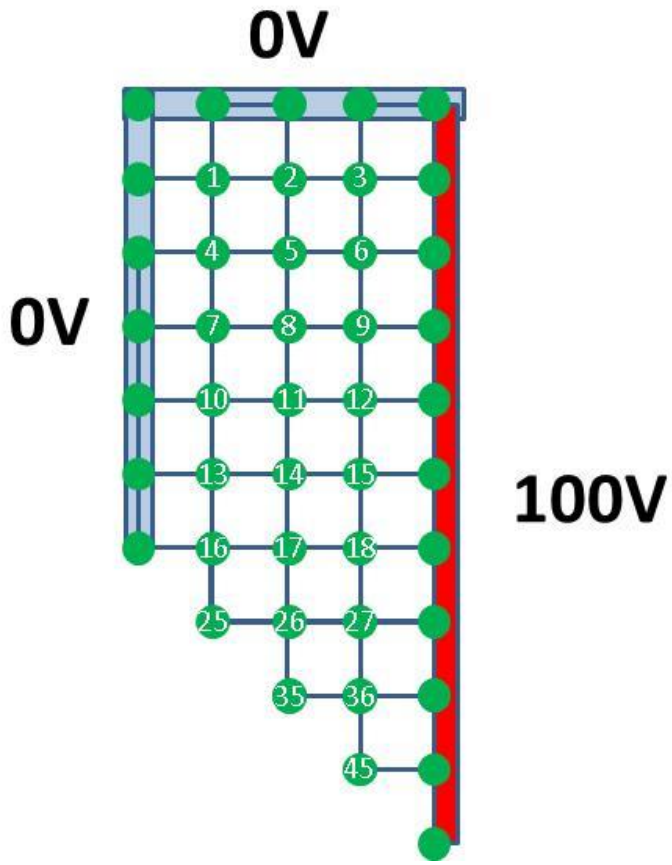

Figure 6 Reduced problem domain for corner test problem