Fresnel's equations for reflection and transmission of planar waves

situation: planar wave onto smooth and straight interface between two media with different optical refractive indices n_1, n_2

definition: $R = \left(\frac{|\underline{E}_r|}{|\underline{E}_i|}\right)^2$ and $T = \left(\frac{|\underline{E}_t|}{|\underline{E}_i|}\right)^2$

reflection and transmission of wave

where

$\underline{E}_i$ = electric field of incident wave

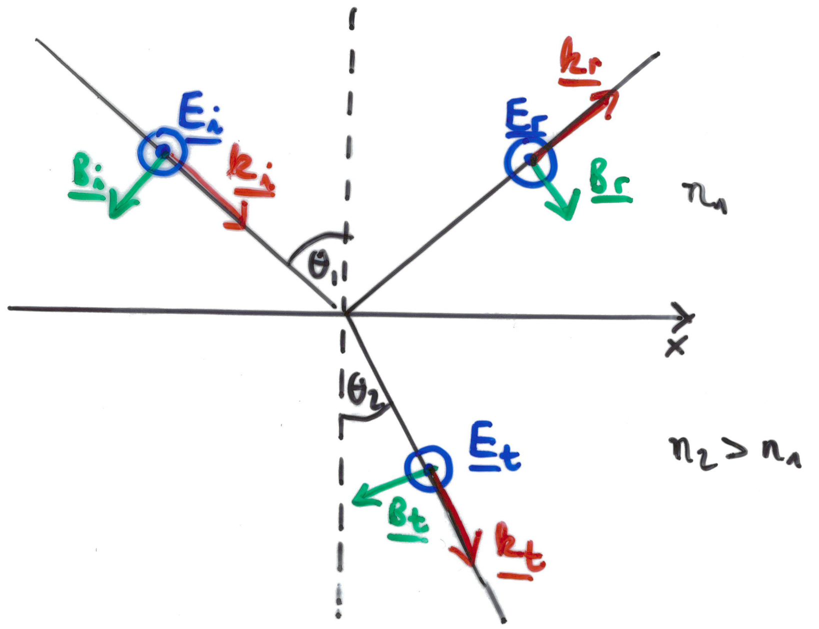
$\underline{E}_r$ = " " " reflected "

$\underline{E}_t$ = " " " transmitted "

need: Maxwell equations plus condition that the tangential (in-plane) components of both electrical and magnetic field vectors are continuous; hence need to distinguish

polarisation directions
- $\perp$ interface
- $\parallel$ interface

sketch:



$\underline{k}$, $\underline{E}$ and $\underline{B} = \mu_0 \mu_r \underline{H} = \frac{1}{\omega} \underline{k} \times \underline{E}$

form an orthogonal system (right-handed).

$k = |\underline{k}| = \frac{2\pi}{\lambda} = \frac{\omega}{v}$

continuity of tangential component of $\underline{E}$: $E_i + E_r = E_t$ (i)

continuity of tangential component of $\underline{H}$:

$$-H_i \cos\theta_1 + H_r \cos\theta_1 = -H_t \cos\theta_2 \qquad \text{(ii)}$$

Use $\underline{H} = \frac{1}{\mu_0\mu_r}\underline{B} = \frac{1}{\omega\mu_0\mu_r}\underline{k}\times\underline{E} = \sqrt{\frac{\varepsilon_0\varepsilon_r}{\mu_0\mu_r}}\frac{\underline{k}}{|\underline{k}|}\times\underline{E}$

to get

$$H_i = \sqrt{\frac{\varepsilon_0\varepsilon_1}{\mu_0\mu_1}}\,E_i \,,$$

$$H_r = \sqrt{\frac{\varepsilon_0\varepsilon_1}{\mu_0\mu_1}}\,E_r \,,$$

$$H_t = \sqrt{\frac{\varepsilon_0\varepsilon_2}{\mu_0\mu_2}}\,E_t$$

Insert into (ii) and divide by $-\sqrt{\frac{\varepsilon_0}{\mu_0}}$:

$$\sqrt{\frac{\varepsilon_1}{\mu_1}}\,(E_i - E_r)\cos\theta_1 = \sqrt{\frac{\varepsilon_2}{\mu_2}}\,E_t \cos\theta_2$$

Now use $\mu_1 \approx \mu_2$ and $n_i \approx \sqrt{\varepsilon_i}$

$$n_1 \,(E_i - E_r)\cos\theta_1 = n_2\, E_t \cos\theta_2 \qquad \text{(iii)}$$

Now combine (i) and (iii) to get

$$R_\perp = \left(\frac{E_r}{E_i}\right)^2 = \left(\frac{n_1\cos\theta_1 - n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2}\right)^2$$

$$T_\perp = \left(\frac{E_t}{E_i}\right)^2 = \left(\frac{2n_1\cos\theta_1}{n_1\cos\theta_1 + n_2\cos\theta_2}\right)^2$$

Now substitute n_i by complex $\underline{n_i} = n_i + j\,\varkappa_i$

$$R_\perp = \frac{(n_1\cos\theta_1 - n_2\cos\theta_2)^2 + (\varkappa_1\cos\theta_1 - \varkappa_2\cos\theta_2)^2}{(n_1\cos\theta_1 + n_2\cos\theta_2)^2 + (\varkappa_1\cos\theta_1 + \varkappa_2\cos\theta_2)^2}$$

$$T_\perp = \frac{(2n_1\cos\theta_1)^2 + (2\varkappa_1\cos\theta_1)^2}{(n_1\cos\theta_1 + n_2\cos\theta_2)^2 + (\varkappa_1\cos\theta_1 + \varkappa_2\cos\theta_2)^2}$$

And analogous for the parallel polarisation components:

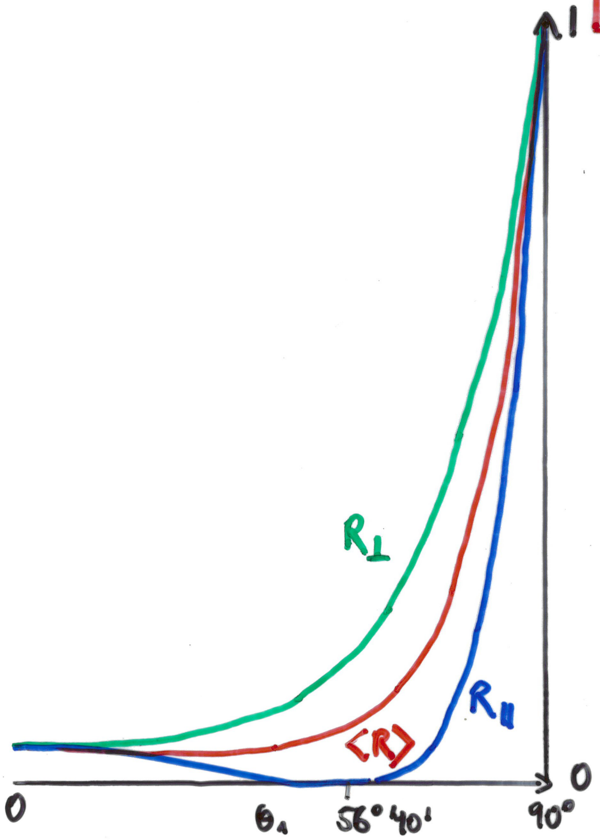
$$R_\parallel = \frac{(n_2\cos\theta_1 - n_1\cos\theta_2)^2 + (\varkappa_2\cos\theta_1 - \varkappa_1\cos\theta_2)^2}{(n_2\cos\theta_1 + n_1\cos\theta_2)^2 + (\varkappa_2\cos\theta_1 + \varkappa_1\cos\theta_2)^2}$$

$$T_\parallel = \frac{(2n_1\cos\theta_1)^2 + (2\varkappa_1\cos\theta_1)^2}{(n_2\cos\theta_1 + n_1\cos\theta_2)^2 + (\varkappa_2\cos\theta_1 + \varkappa_1\cos\theta_2)^2}$$

for unpolarised light: $\boxed{R = \langle R \rangle = \frac{1}{2}(R_\perp + R_\parallel)}$

Because of Snell's Law $\frac{n_1}{n_2} = \frac{\sin\Theta_2}{\sin\Theta_1}$ we can plot R and T vs Θ_1 for a given material with $n_1, n_2, \varkappa_1, \varkappa_2$.

Note that for $\varkappa_1 = \varkappa_2 = 0$ that $R_\perp(\Theta_1)$ grows monotonically while $R_\parallel(\Theta_1)$ has a zero at the **Brewster-angle** $\boxed{\Theta_1 = \arctan\frac{n_2}{n_1}}$



At Brewster angle:

$$R_\parallel = 0 \Leftrightarrow n_2 \cos\Theta_1 = n_1 \cos\Theta_2$$

$$\Leftrightarrow \tan\Theta_1 = \frac{\sin\Theta_1}{\cos\Theta_1} = \frac{\cos\overbrace{(90° - \Theta_1)}^{\Theta_2}}{\cos\Theta_1} = \frac{\cos\Theta_2}{\cos\Theta_1} = \frac{n_2}{n_1}$$

For metals, $\varkappa_1 > 0$ and $\varkappa_2 > 0$ and the curves are shifted upwards.

special case of vertical incidence

$\theta_1 = 0 = \theta_2$

$$\rightarrow \quad R = R_\perp = R_\parallel = \boxed{\frac{(n_1 - n_2)^2 + (\varkappa_1 - \varkappa_2)^2}{(n_1 + n_2)^2 + (\varkappa_1 + \varkappa_2)^2}}$$

$$T = T_\perp = T_\parallel = \boxed{\frac{4\,(n_1^2 + \varkappa_1^2)}{(n_1 + n_2)^2 + (\varkappa_1 + \varkappa_2)^2}}$$

note on absorption in metals

consider light incident on a metal surface $(n_2 = n,\ \varkappa_2 = \varkappa)$ from vacuum $(n_1 = 1,\ \varkappa_1 = 0)$

$$R = \frac{(n-1)^2 + \varkappa^2}{(n+1)^2 + \varkappa^2} \rightarrow 1 \quad , \text{ if}$$

a) $n \gg 1$, $\varkappa$ arbitrary (e.g. diamond: $R \approx 0.16$)
$n = 2.3$, $\varkappa \approx 0$

or

b) $\varkappa \gg 1$, n arbitrary (e.g. gold: $R \approx 0.95$)
$n = 0.15$, $\varkappa = 3.22$