definition of general series impedance (Z^*) and shunt admittance (Y^*)

$$-\frac{\partial V}{\partial x} = R^* I + L^* \frac{\partial I}{\partial t} \overset{(\dot{I} = j\omega I)}{=} (\underbrace{R^* + j\omega L^*}_{\text{= series impedance } Z^* \text{ per length}}) I$$

$$-\frac{\partial I}{\partial x} = G^* V + C^* \frac{\partial V}{\partial t} \underset{(\dot{V} = j\omega V)}{=} (\underbrace{G^* + j\omega C^*}_{\text{= shunt admittance } Y^* \text{ per length}}) V$$

With $\boxed{Z^* = R^* + j\omega L^*}$

and $\boxed{Y^* = G^* + j\omega C^*}$

above equations can be written as

$$\frac{\partial V}{\partial x} = -Z^* I \Rightarrow \frac{\partial^2 V}{\partial x^2} = -Z^* \frac{\partial I}{\partial x} = Z^* Y^* V \quad \text{(i)}$$

$$\frac{\partial I}{\partial x} = -Y^* V \Rightarrow \frac{\partial^2 I}{\partial x^2} = -Y^* \frac{\partial V}{\partial x} = Z^* Y^* I \quad \text{(ii)}$$

$\Rightarrow$ Both $V = V_0 e^{j(\omega t - \tilde{k}x)}$ and $I = I_0 e^{j(\omega t - \tilde{k}x)}$

obey wave equations with

$$\boxed{Z^* \cdot Y^*} = (R^* + j\omega L^*)(G^* + j\omega C^*)$$
$$= -\omega^2 L^* C^* + R^* G^* + j\omega (R^* C^* + L^* G^*)$$
$$= \boxed{-\tilde{k}^2}$$

As (i) and (ii) are time independent, $e^{j\omega t}$ component can be ignored. Also, wave may travel forward or backward in space; ie get „telegrapher's equations"

$$V = V(x) = V_0^+ e^{-j\tilde{k}x} + V_0^- e^{+j\tilde{k}x}$$
$$I = I(x) = I_0^+ e^{-j\tilde{k}x} + I_0^- e^{+j\tilde{k}x}$$

forwards　　backwards

travelling waves

Note:

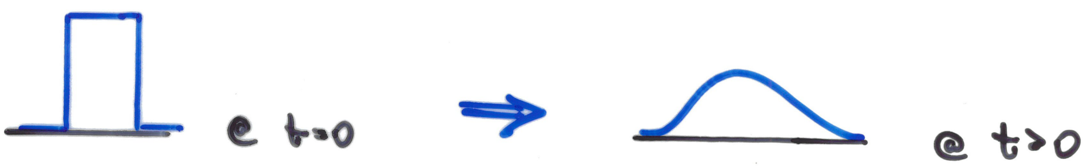
$$\tilde{k} = \sqrt{Z^* \cdot Y^*}$$
$$= \omega \sqrt{L^* C^*} \sqrt{1 - \frac{R^* G^*}{\omega^2 L^* C^*} - j\left(\frac{G^*}{\omega C^*} + \frac{R^*}{\omega L^*}\right)}$$
$$\approx \omega \sqrt{L^* C^*} \left[1 - \frac{j}{2\omega}\left(\frac{G^*}{C^*} + \frac{R^*}{L^*}\right)\right] \quad \text{for small } R^*, G^*$$

implications: dispersion and attenuation

a) complex, where imaginary part describes attenuation

b) both, exact solution of real part and imaginary part are having components non-linear in ω, i.e.
speed $v = \frac{\omega}{\tilde{k}}$ becomes frequency-dependent:
dispersion $\quad v = v(\omega)$

this means a pulse spreads out while it travels:



characteristic impedance of transmission line

$$\frac{\partial V}{\partial x} = -Z^* I \quad \text{and} \quad \frac{\partial I}{\partial x} = -Y^* V$$

with

$$V = V_0^+ e^{-j\tilde{k}x} + V_0^- e^{+j\tilde{k}x}$$

$$I = I_0^+ e^{-j\tilde{k}x} + I_0^- e^{+j\tilde{k}x}$$

$$\rightarrow \underline{-j\tilde{k} V_0^+ e^{-j\tilde{k}x}} + \underline{j\tilde{k} V_0^- e^{+j\tilde{k}x}} = \underline{-Z^* I_0^+ e^{-j\tilde{k}x}} \underline{-Z^* I_0^- e^{+j\tilde{k}x}}$$

$$\rightarrow V_0^+ = \frac{Z^*}{j\tilde{k}} I_0^+ \qquad V_0^- = -\frac{Z^*}{j\tilde{k}} I_0^-$$

$$\rightarrow \boxed{V_0^\pm = \pm Z_0 I_0^\pm} \text{ with characteristic impedance}$$

$$\boxed{\text{where } Z_0 = \frac{Z^*}{j\tilde{k}}} \text{ of the transmission line}$$

($=: Z_0$)

note: the same can be shown for the current:

$$\frac{\partial I}{\partial x} = -Y^* V$$

$$\rightarrow \underline{-j\tilde{k} I_0^+ e^{-j\tilde{k}x}} \underline{+j\tilde{k} I_0^- e^{+j\tilde{k}x}} = \underline{-Y^* V_0^+ e^{-j\tilde{k}x}} \underline{-Y^* V_0^- e^{+j\tilde{k}x}}$$

$$\rightarrow I_0^+ = \frac{Y^*}{j\tilde{k}} V_0^+ \qquad I_0^- = -\frac{Y^*}{j\tilde{k}} V_0^-$$

$$I_0^\pm = \pm \frac{1}{Z_0} V_0^\pm$$

A) lossless transmission line : $R^* = 0 = G^*$

$$\tilde{k}^2 = -Z^* Y^* = \omega^2 L^* C^*$$

$$\rightarrow \tilde{k} = \omega \sqrt{L^* C^*}$$

$$\Rightarrow Z_0 = \frac{Z^*}{j\tilde{k}} = \frac{\cancel{R^*} + j\omega L^*}{j\tilde{k}} = \frac{\cancel{j\omega} L^*}{\cancel{j\omega}\sqrt{L^* C^*}} = \sqrt{\frac{L^*}{C^*}}$$

is real-valued and frequency independent so that $V(x)$ and $I(x)$ are always in phase!

note: dielectrics are usually chosen so that $Z_0 = 50\,\Omega$, $75\,\Omega$ or sometimes $300\,\Omega$

B) wave reflection can occur:

feed in V, I and consider load @ $t=0$ and position $x=0$ (move origin to load for ease of calculation) and use $V_0^\pm = \pm Z_0 I_0^\pm$

$$\Rightarrow \frac{V_0^+ + V_0^-}{Z_L} \underset{x=0}{\overset{\uparrow}{=}} \frac{V(0)}{Z_L} = I(0) = I_0^+ + I_0^- = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

$$= \frac{V_0^+ - V_0^-}{Z_0}$$

$$\Rightarrow \boxed{\Gamma := \frac{V_0^-}{V_0^+} = \frac{Z_L - Z_0}{Z_L + Z_0}}$$ is voltage reflection coefficient of load

special cases:

$Z_L = Z_0 \Rightarrow \Gamma = 0$

$\Rightarrow$ no reflection: all power is delivered to load in case of matched impedance

$Z_L = \infty \Rightarrow \Gamma = +1$

$Z_L = 0 \Rightarrow \Gamma = -1$

general:

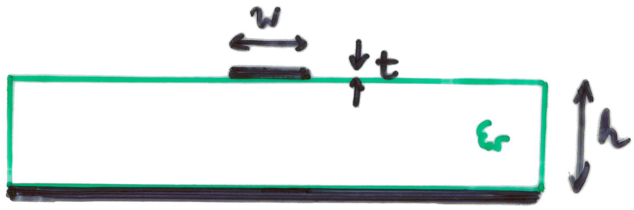
Z_L or Z_0 can be complex

$\Rightarrow \Gamma$ has imaginary part where phase angle describes phase change of reflected wave

examples from printed circuit boards (PCBs)

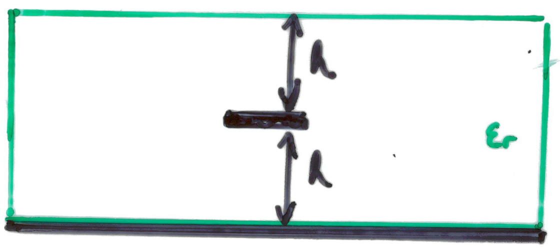
given:
- dimensions: $w = 100\,\mu m$, $t = 15\,\mu m$, $h = 140\,\mu m$
- material: $\varepsilon_r = 4.5$ (glass, or SiO_2); $Z_L = 50\,\Omega$ (standard load)

A: microstrip on top



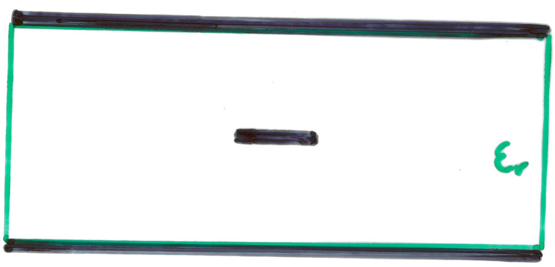
$Z_0 = 58\,\Omega \Rightarrow \Gamma = 7.5\,\%$

B: embedded microstrip



$Z_0 = 54\,\Omega \rightarrow \Gamma = 4.0\,\%$

C: embedded stripline



$Z_0 \approx 50\,\Omega \rightarrow \Gamma \approx 0$

formula from HA Wheeler, IEEE Trans. Microwave Theory and Techn. 13:2 (1965) p. 172

$$Z_0 \approx \frac{87}{1.41 + \sqrt{\varepsilon_r}} \ln \frac{5.98\,h}{0.8\,w + t} \approx 53.6\,\Omega \text{ for above}$$