Suggested Solutions to Select Exercises in Vector Calculus 6th ed., Marsden and Tromba

Brown University Honors Calculus (MATH 0350)

Spring 2021[1]

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Chapter 0

Introduction

This document provides suggested solutions to homework problems for the honors calculus course at Brown University in the spring semester of 2021. Problems are taken from the textbook *Vector Calculus 6th ed.*, by Marsden and Tromba [MT12]. Final answers or key points are highlighted in blue. However, in general, presenting the final answer or key points alone are insufficient for students obtain full credit for the problem.

Supplementary resources (code samples and images) can be found by visiting the github repository [Lim]. You may contact Josiah Lim[1] for:

- typos or errors in this document,
- suggestions or additions to the document, be they for the "common mistakes section" or the homework solutions themselves,
- PDFs of solutions by homework (instructors only, specific chapters to be included in each PDF can be easily modified).

0.0 Acknowledgements

A special thanks to all student collaborators, Hammad Izhar[2], and Professor Junehyuk Jung for helping in the writing of this document. Their hand typed solutions, proof reading, and template tex files significantly aided the completion of this document.

0.1 Mathematical Notation

Here, we define some notations used in this document that may not be obvious to the reader.

- $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard basis vectors in $\mathbb{R}^3$. In general, **boldface** variables like $\mathbf{c}$ and $\mathbf{s}$ represents vectors, not scalars.
- $\mathbf{0}$ is the zero vector in $\mathbb{R}^n$, where n should be clear from context.
- (ρ, θ, ϕ), in that order, are the spherical coordinates, where $\rho \in [0, +\infty)$, $\theta \in [0, 2\pi)$, $\phi \in [0, \pi)$.
- $\mathbf{D}f(\mathbf{x})$ is the derivative of f at the point the point $\mathbf{x}$.

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- $\frac{\partial(x,y)}{\partial(u,v)}$, $\frac{\partial(x,y,z)}{\partial(u,v,w)}$ are the Jacobian determinants.
- $Hf(\mathbf{x})$ is the Hessian *matrix* of f, the matrix of second partial derivatives. This is not the same as $Hf(\mathbf{x}_0)(\mathbf{h})$, the Hessian of f at x_0, which is a quadratic function, not a matrix.

  The relation between them, however, is that $Hf(\mathbf{x}_0)(\mathbf{h}) = \frac{1}{2}\mathbf{h}\, Hf(\mathbf{x}_0)\, \mathbf{h}^T$, a matrix multiplication, as shown in [MT12, p.172].
- $\cdot \,\Big|_{\mathbf{x}}$ denotes the expression $\cdot$ evaluated at $\mathbf{x}$, where the substitution of values to their respective variables should be clear from context. This is just multivariable shorthand for the more familiar one variable $\cdot \,\Big|_{x=c}$.

0.2 List of Known Typos

Some problems in the textbook have typos. They might mislead the student into solving the wrong problem. For important typos, instructors should forewarn students of them, less unnecessary confusion or time wastage occur.

- *4.1 Problem 14:* the trajectory should intersect the xz plane, not the yz plane, in order to get real-valued solutions.
- *4.4 Problem 37:* students may assume that $f : \mathbb{R}^3 \to \mathbb{R}$, hence, $f(x, y, z) = x^2y$.
- *5.1 Problem 37:* the angle in Figure 5.1.12 should be θ, not v.
- *6.1 Problem 2:* the domain should be $\mathbb{R}^3 \to \mathbb{R}^3$, not $\mathbb{R}^2 \to \mathbb{R}^2$.
- *7.6 Problem 14:* the integral should read $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, not $\iint_S \mathbf{F} \cdot \mathbf{n}\, dA$.
- *8.2 Problem 1:* the last point should be $(-1, 3, -2)$, not $(-1, 3, 2)$.

0.3 Misinterpretation/ambiguities of Problems

Some problems may be phrased in ways which could mislead students into solving the wrong problem or making a wrong assumption. Here are some of them.

- *1.1 Problem 32:* as far as possible, proofs should work with general vectors ($\mathbf{v}$, $\mathbf{w}$, etc). Students should not assume this vector is in $\mathbb{R}^2$ only.
- *1.4 Problem 8:* each "(variable)=constant" is a separate case. Hence, there should be 6 cases in total, 3 from part (a), 3 from part (b).
- *2.5 Problem 29:* the desired result does not follow immediately from the hint. Students should note that the limits in the hint are constants, while the upper limit in the desired result is in terms of x.
- *3.2 Problem 2:* the computed expansions should be general, around some arbitrary point $\mathbf{x}_0$, not just the origin. So, we want $\mathbf{L}(\mathbf{x}_0 + \mathbf{h})$, not just $\mathbf{L}(\mathbf{h})$.
- *3.4 Problem 25:* by "trim", we cut off only a very thin slice off the edge of the mirror. Imagine smoothing the edge with sand paper, not so much trimming away a rectangular piece of mirror from the edge.
- *3.4 Problem 28:*
  - figure 3.4.7, while helpful, can also add to confusion. See the alternative diagram shown in the suggested solutions, which can potentially be shared with students.

  - In this question, "*when* Snell's law holds" is not the logical "*if* Snell's law holds". So, do not attempt to show "Snell's law holds $\Rightarrow$ trip is in minimum time". Rather, the question is asking more of: is it true that when we have attained the minimum time, Snell's law is also true? This rephrased question, we can answer by Lagrange multipliers.

- *3.4 Problem 37:* students should solve this problem by method of Lagrange multipliers (even though direct integrate with respect to t is a lot faster). The point of the exercise is to understand better Lagrange multipliers. Hence, we should stick with the three variables x, y, z and convert the constraint equation (closed curve) from being in terms of t to x, y, z.

- *7.3 Problem 23:* tudents should additionally assume that $R > r$. Otherwise, if $R = r$, then the normal at the origin is 0; if $R < r$, the torus would not be a differentiable at the origin. See "Torus" on Wikipedia for picture.

- *7.5 Problem 17(c):* this problem is to be completed also, even though Problem 7.5.16 was itself not assigned.

- *8.1 Problem 24:*
  - for this problem, we want to work in polar coordinates. However, since the variable r is used to denote the boundary of the region, we need a new variable for the "radius variable" in polar coordinates. So, we have chosen ρ to represent what r would have typically represented in polar coordinates.
  - a "region in polar coordinates" can be assumed here to mean a ρ-simple region, where $\theta \in [a, b]$ and $\rho \in [0, r(\theta)]$ such that $r(\theta) \geq 0$ for all $\theta \in [a, b]$. In other words, the region is enclosed by the curve $\rho = r(\theta)$. See figure in the suggested solution for an illustration.

- *8.2 Problem 5:* for clarity, ∂S can also be described by $\{(x, y, z) : x^2 + y^2 = 1,\ z = 0\}$. The boundary is still in $\mathbb{R}^3$, even though the problem only states two variables.

0.4 Problem Solving Tips

General strategies

- when encountering a seemingly tedious integral, consider checking the table of integrals at the back of the textbook to save some time.

- where appropriate (or after Chapter 7.2) employ more often the corollary of Theorem 3 [MT12, p.367], which is also the result of Problem 7.2.16, which says: the integral of a gradient field around a closed curve C is 0. Knowing this often saves on a lot of computation.

- if stuck (after a long time) on gaining an intuition for a problem or figuring out whether a limit exists, consider plotting a graph. You can try Desmos, Geogebra, or MATLAB. MATLAB sample code is available in section 0.6.

Specific chapters and problems

- *2.2:* it helps to know the one-variable limit $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$ and techniques such as L'Hôpital's rule.

- *2.2 Problem 8:* By observing the denominator $x^2 + y^2$ $(= r^2)$, one can have a hint as to try this problem in polar coordinates.

  When it comes to taking limits at the origin, observe that the definition of a limit in [MT12, p.92] and the epsilon-delta definition of a limit in Theorem 6 [MT12, p.99] can both be re-expressed into a polar coordinate version.

In polar coordinates, being "closer" or being within a neighborhood of the origin $(0,0)$ is equivalent to having $r \to 0$. θ is not relevant since it does not make a point closer to the origin $(0,0)$. Hence, $(x,y) \to (0,0)$ is equivalent to $r \to 0$.

- *2.5 Problem 26:* Try to find a problem with the notation used in the argument. Write out the equation in full (derivatives explicitly evaluated) on some concrete f and g. To make it easier, take a lower dimensional, concrete example like $w = f(x,y) = x + y$ and $y = g(x) = x^2$. Once you see what is happening, try to generalize and make a theoretical diagnosis for why the argument in the problem is wrong.

- *2.5 Problem 29:* Let $F(u,v) = \int_0^v f(u,y)\, dy$. Apply the chain rule on this and see what you can get. To motivate this choice of $F(u,v)$, check the footnote in the suggested solution. Remember: y is a "dummy variable" that is used only in the integral, hence, $F(u,v)$ is independent of y. We could easily swap out y for another letter, say t, to get the equivalent $F(u,v) = \int_0^v f(u,t)\, dt$.

- *2.6 Problem 22:* The amount of temperature increase/decrease is very much related to the directional derivative. If $T(x,y,z)$ is the temperature, then the directional derivative tells us how much T (temperature) is changing by going in a particular direction.

- *3.4 Problem 37:* to convert the closed curve coordinates from being in terms of t to x, y, z instead, consider using double angle formulae. Then, try to relate the three coordinates into a single equation, which will give us an equation for the constraint set.

- *5.2 Problem 15:* being familiar with the definition of integrability in [MT12, p.271-272] will be very helpful.

- *5.3 Problem 4(b):* integrate by breaking up the integral into two, $-1 \leq x < 0$ and $0 \leq x \leq 1$. How does that help with the $|\cdot|$ sign?

- *7.4 Problem 21:* problem 7.4.21 the equation $x^2 + y^2 = x$ is a valid equation for a cylinder (and circle). Try completing the square. Consider using spherical coordinates and perform double integrals on the appropriate bounds.

- *7.5 Problem 24:* to argue that the area $A(\Phi) \leq J(\Phi)$, the AM-GM inequality is helpful, which says: "$\sqrt{xy} \leq \frac{x+y}{2}$ for all $x, y \geq 0$. Moreover, equality holds if and only if $x = y$."

- *8.2 Problem 26:* make use of formulae in [MT12, p.255] to make the solution more concise.

0.5 Common Mistakes in Student Solutions

Certain homework problems led to common (but crucial) mistakes. This list highlights some of them, organized by chapter.

Chapter 1

- *1.3:* the determinant and the absolute value of the determinant not the same. Since they both share the same $|\cdot|$ notation, be careful.

- *1.4:* cylindrical coordinates can have negative r too, so $r \in \mathbb{R}$. However, spherical coordinates must have $\rho \geq 0$.

Chapter 2

- *2.5 Problem 26:* many students will not correctly identify the mistake in notation. This is a good problem in solidifying a student's understanding of the chain rule.
- *2.6 Problem 22(c):* note that the temperature has to decrease, so the directional derivative must not only be small in magnitude but negative as well.

Chapter 3

- *3.2:* students should be careful in their choice of notation for the Taylor expansion. For example, when expanding at the origin $(0,0)$, while it is fine to call the expansion $f\big((0,0)+(x,y)\big)=f(x,y)=\cdots$, it would be notationally clearer to call the expansion $f\big((0,0)+(h_1,h_2)\big)=f(h_1,h_2)=\cdots$.

  The distinction is more important when we consider the expansion about some point (x_0,y_0), instead of $(0,0)$. If we went with the former notation, we get $f\big((x_0,y_0)+(x,y)\big)=f(x_0+x,y_0+y)=\cdots$; with the latter, we get $f\big((x_0,y_0)+(h_1,h_2)\big)=f(x_0+h_1,y_0+h_2)=\cdots$. The former expansion with (x,y) seems confusing, because is seems like we are trying to find $f(x,y)=f(x_0+x,y_0+y)$; while the latter is clearer in telling us that $f(x,y)=f(x_0+h_1,y_0+h_2)$, which is some distance away from (x_0,y_0).

  In all, it is about finding good notation that won't add unnecessary confusion/misconception to the students themselves.
- *3.3:* on longer word problems, checking whether critical points are maxima/minima should not be forgotten. Students tend to stop at merely computing the critical points and forget this key check when dealing with longer, more complex optimization problems.
- *3.3 Problem 23:* checking for degeneracy of Hessian $Hf=\begin{pmatrix}0&0\\0&2\end{pmatrix}$ *alone* is not sufficient. Just because the Hessian is not positive definite does not imply that it is not a relative minimum. As per [MT12, p.176], "if the determinant of the Hessian is zero, ... nothing can be said about the nature of the critical point without further analysis."
- *3.3 Problem 41:* extrema should be computed separately along (i) the boundary and (ii) the interior. This is based on the strategy given in [MT12, p.181].
- *3.4:* show the existence of an extrema before applying the method of Lagrange multipliers. Many, many students fail to justify why the desired extrema exists—otherwise, we may well be talking about the empty set! (That said, given the scope of MATH 0350, a rigorous proof for the existence of an extrema is not required. A convincing argument suffices.)

Chapter 4

- *4.3:* vector fields should show sufficient number of arrows to illuminate key features. These include, but are not limited to, vectors at the axes, stationary points, asymptotes, linear (straight line) trajectories, symmetries. (Even better would be vectors with length corresponding to the magnitude!)

Chapter 5

- *5.1 Problem 7:* by the statement of Cavalieri's principle, $a\le x\le b$, so the limits should be from $-r$ to r, not the other way round. This may fix certain students' answers that are of the opposite polarity.
- *5.2 Problem 15:* integrability is more concerned with the existence of a limit, as per definition in [MT12, p.271-272]. It is not so much about the discontinuities or non-differentiability. In fact, not

continuous $\not\Leftarrow$ not integrable. Consider the function $f(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$. Then $\displaystyle\int_{\mathbb{R}} f(x)\ dx = 1$, integrable but not continuous.

- *5.4 Problem 10:* students should show why their choice of m and M are indeed the smallest and largest possible values attained by the function. Merely stating that $m = \frac{1}{3}$ and $M = \frac{1}{2}$ is not satisfactory, since one could have worked backwards from the desired solution to obtain those m and M values.

Chapter 6

- *6.1 Problem 5:* a unit circle and a unit disk is different. The unit circle is a circular line ("ring"), while the unit disk is a filled in circle ("plate").

Chapter 7

- *7.2 Problem 18:* students should try to use Theorem 3 [MT12, p.367] to solve this problem, rather than observing that $f(x, y, z)$ could be guessed by "un-doing/integrating/reverse-engineering $\nabla f(x, y, z)$" through trial and error or a clever observation.

- *7.3 Problem 17(c):* students should be more explicit or meticulous that the every condition of being in the surface is fulfilled. That means checking both for r and θ, and being explicit about the fulfillment of every condition.

- *7.3 Problem 23(b):* a thorough analysis should be made to show why the normal vector is always nonzero. One approach is to show it by contradiction.

- *7.4 Problem 6:* students should try to use a change of variables to polar coordinates. The substitution $x = u$, $y = v$, $z = uv$ doesn't essentially perform a change of variables in a meaningful sense.

- *7.5 Problem 24:* if the student approaches the problem of showing $A(\Phi) \leq J(\Phi)$ by "Method 2" shown in the suggested solution, then they must be sure to mention the nonnegativity of norms (which says that $\|\mathbf{T}_u\|,\ \|\mathbf{T}_v\| \geq 0$). Without sufficient justification, the step $\|\mathbf{T}_u\|\|\mathbf{T}_v\| \sin\theta \leq \|\mathbf{T}_u\|\|\mathbf{T}_v\|$ is not valid.

  Consider the following counterexample without the "nonnegativity of norms" assumption. Let $a = -1(< 0),\ b = 1,\ \theta = \frac{3\pi}{2}$, then $ab \sin\theta = 1 \geq -1 = ab$.

Chapter 8

- *8.1 Problem 24:* students should not apply Green's theorem on the formula $\frac{1}{2}\int_{\partial D} x\ dy - y\ dx$ in Theorem 2 [MT12, p.433], for this amounts to "un-doing" the theorem and not using it at all. Rather, students should perform a change of variables directly to the equation $\frac{1}{2}\int_{\partial D} x\ dy - y\ dx$ in Theorem 2.

- *8.2 Problem 17:* this hemisphere does not have a base.

- *8.2 Problem 26:* the $\mathcal{C}^2$-ness of the f, g (which leads to the $\mathcal{C}^1$-ness of $\nabla f, \nabla g$) must be mentioned in the student's argument. This is a very important condition before Stokes' theorem can be invoked.

0.6 Code

Where helpful, instructors or students may (re)produce computer visualizations to the homework problems. Visualizations provide an intuition when tackling a problem and allow for an appreciation for the geometric perspective of a problem.

We provide a few simple templates for generating plots and diagrams on MATLAB. Code samples to all MATLAB generated plots are also available on github [Lim].

Curves

```
% % curve y = x^2
% create x values from -2 to 2, spaced 0.1 apart
x = -2:0.1:2;

% compute y = x^2.
% .^, rather than ^, is used when operating on lists/arrays
y = x.^2;

% plot curve
plot(x,y);          % for curves in 3D, use plot3(x,y,z) instead of plot(x,y)
```

Shaded Region

```
% % region bounded by y=2x, y=x, x=pi, and x= 2pi
% specify vertices of the region
x = [pi pi 2*pi 2*pi]; y = [pi 2*pi 4*pi 2*pi];

patch(x,y,'b');              % last argument 'b' is the color blue
xline(0); yline(0);          % axes lines
xlabel('x'); ylabel('y');    % axes labels
grid on; axis equal;         % grid lines and equal scaling on axes
axis([-2 8 -2 16]);          % customize the x and y limits of generated figure
```

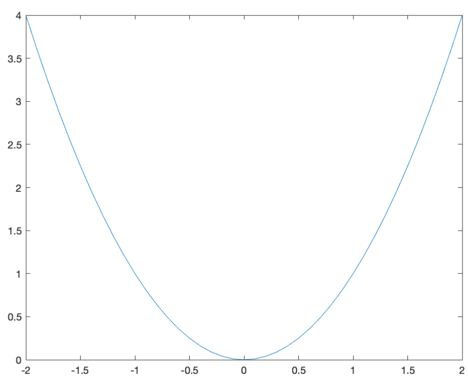
Vector Field

```
% % vector field F(x,y) = (-x,y)
% create x-y grid from -3 to 3 on each axis, with interval length 0.5
[X,Y] = meshgrid(-3:0.5:3,-3:0.5:3);

% compute vector field F(x,y) = (-x,y) = (u,v)
U = -X; V = Y;

% plot vector field
quiver(X,Y,U,V);
```

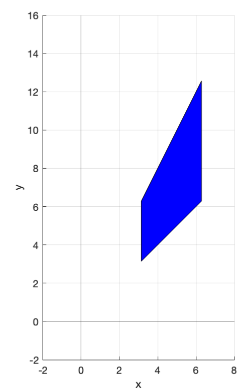
Surface

```
% % surface f(x,y) = sinx + cosy
% create a 20x20 grid of evenly spaced points from 0 to 2pi
% linspace is advised when spacing is hard to compute for limits like 2pi
[X,Y] = meshgrid(linspace(0,2*pi,20),linspace(0,2*pi,20));

% compute f(x,y), storing as Z
Z = sin(X) + cos(Y);

% plot surface
surf(X,Y,Z);
```

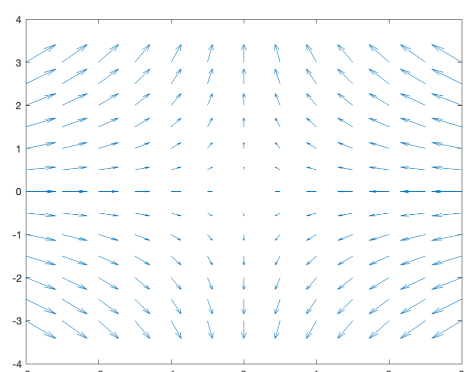
```
xlabel('x'); ylabel('y'); zlabel('z');              % axes labels
title('myTitle');                                    % title
view([-1 -1.5 1.5]); axis([0 2*pi 0 2*pi -2 2]);     % customized view perspective
```



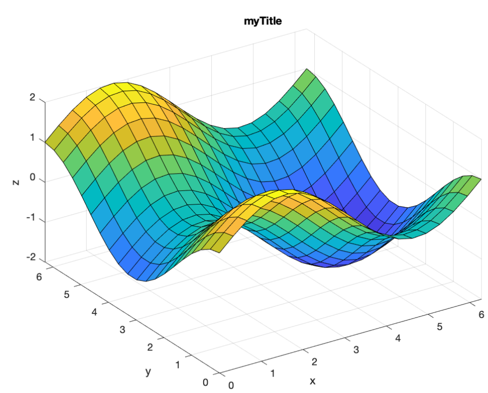
(a) Curve



(b) Region



(c) Vector field



(d) Surface

Above: Plots generated by the code samples.

0.7 List of Homework Problems

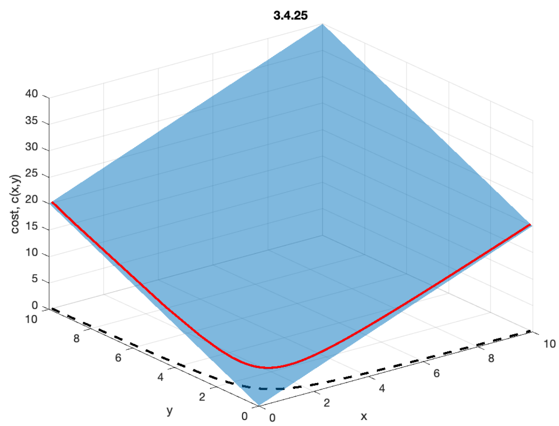
HW	Chapter	Problems
#1	1.1	1, 6, 8, 14, 16, 22, 23, 24, 28, 29, 32
	1.2	29 2, 3, 11, 15, 18, 20, 27, 31
	1.3	2(d), 3, 5, 7, 12, 15(b), 16(c), 26, 29, 31
#2	1.4	2, 3, 4, 6(b,d), 8, 10
	1.5	1, 2(a), 4, 7, 16, 17, 24
	2.1	2, 3, 6, 13, 19, 26, 32
	2.2	4, 8, 12(a,b), 14, 17, 25(a,b)
#3	2.3	1(a,b), 2(a,b), 3b, 5, 10(a,d), 11, 17, 19(c), 22, 24
	2.4	2, 4, 8, 18, 20, 23, 24
	2.5	3(b), 5, 15, 17(a,b), 26, 29, 35
	2.6	2(c), 3(a), 5, 6, 8(b), 10(b), 17(b), 20, 22, 25
#4	3.1	4, 7(b), 11(a,b), 19, 25, 28
	3.2	1, 2, 4, 6, 10
	3.3	1, 6, 11, 18, 20, 23, 34, 41, 52
	3.4	3, 4, 13, 25, 28, 37
#5	4.1	2, 9, 14, 20
	4.2	1, 2, 7, 12
	4.3	4, 7, 8, 9, 12, 13, 17, 21(a)
#6	4.4	1, 2, 8, 13, 14, 18, 24, 25, 34, 37
	5.1	1(a,c), 2(a,c), 3(d), 6, 7, 12
	5.2	1(a,c), 2(a,c), 4, 6, 8, 13, 15
#7	5.3	1, 2, 4(a,b,c), 8, 10, 11, 12
	5.4	1(a,c), 2, 4(b,c), 7, 10, 15, 16
	5.5	2, 4, 7, 11, 14, 16, 22, 30
#8	6.1	1(a,b), 2, 4, 5, 7, 10, 11
	6.2	2, 3, 4, 6, 15, 23, 29, 33
	7.1	2, 4, 10(a), 12(b)
	7.2	1, 3(b,d), 4(a,d), 11, 13, 16, 18
#9	7.3	2, 7, 10, 13(a), 17, 23
	7.4	1, 5, 6, 10, 21, 24
	7.5	2, 4, 6, 12, 17, 23, 24, 26
#10	7.6	2, 3, 7, 10, 14, 22
	8.1	2, 6, 9, 11(a), 14(a), 16, 20, 24, 27
	8.2	1, 4, 5, 7, 10, 13, 17, 18, 23, 25, 26, 27

Existence of minimum cost

The constraint is the fixed area

$$g(x, y) = xy = A.$$

Informally, we justify the existence of a minimum cost by the following reasoning. The graph of $c(x, y)$ is a plane with normal vector $(2p, 2q, -1)$. Since $p, q > 0$, the plane is upwards sloping in the positive xyc quadrant. The constraint set is the hyperbola $y = \frac{A}{x}$, which when projected onto the plane $c(x, y)$ has a minimum, because when we following the trajectory of the parabola (constraint) to $x, y \to \infty$, we see that $c(x, y) \to \infty$ too. So a minimum must exist somewhere "near" the origin.



Problem 3.4.25: Arguing for the existence of a minimum. Plot illustrates when $A = p = q = 1$. The blue surface represents $c(x, y)$, the dotted curve represents the constraint $g(x, y) = A$ on the xy plane, and the red curve represents the values of $c(x, y)$ along the constraint.

Dimensions of minimum cost

By the method of Lagrange multipliers, we seek solutions to

$$\nabla c(x, y) = \lambda \nabla g(x, y),$$
$$(2p, 2q) = \lambda(y, x).$$

Thus, $x = \frac{2q}{\lambda}$ and $y = \frac{2p}{\lambda}$, which gives us $A = \frac{4pq}{\lambda^2} \Rightarrow \lambda = \sqrt{\frac{4pq}{A}}$. We choose the $\lambda > 0$ since we require $x, y > 0$.

Thus, the dimensions are $x = \sqrt{\frac{qA}{p}}$ and $y = \sqrt{\frac{pA}{q}}$. $\square$

Problem 3.4.28. *A light ray travels from point A to point B crossing a boundary between two media (see Figure 3.4.7)* [11]*. In the first medium its speed is v_1, and in the second it is v_2. Show that the trip is made*

[11] Figure 3.4.7, while helpful, can also be misleading or adding to the confusion. In particular, it is shown in the figure, though not obvious, that the medium is separated by the x-axis, hence the solid line. The y-axis plays little role in the problem and should (more or less) be disregarded, hence the dotted line.

See the solution's figure instead for a more appropriate representation of the problem *in minimum time when*[12] *Snell's law holds:*

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}.$$

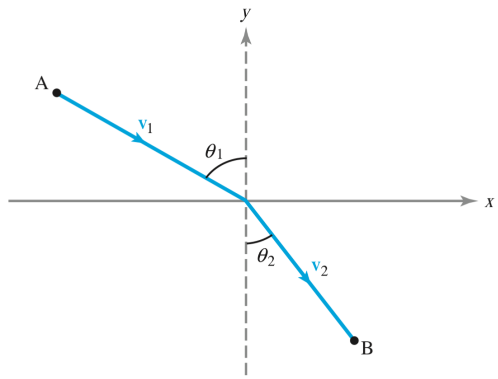
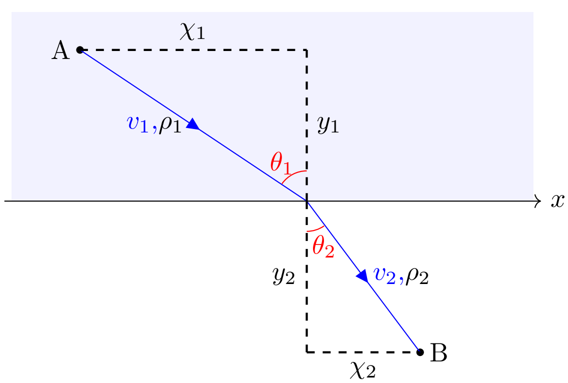
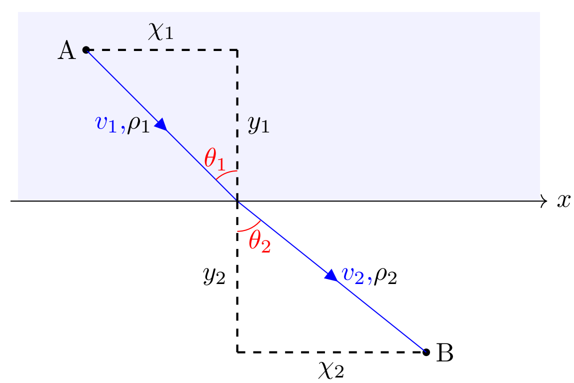


Figure 3.4.7: Snell's law of refraction

Solution. See the following diagram for an interpretation of the problem.





Problem 3.4.28: Examples showing two possible paths the light travels through. The different mediums are denoted by color. Red θ_i are the variables of this problem. χ_i and ρ_i are related to θ_i trigonometrically by $\chi_i(\theta_i) = y_i \tan\theta_i$ and $\rho_i(\theta_i) = \frac{y_i}{\cos\theta_i}$. The constraint of this problem is to keep $\chi_1(\theta_1) + \chi_2(\theta_2)$ constant. (Of course, we can solve the analogous problem by making the allowing the vertical distance to change with θ_i (hence swap y_i for $\gamma_i(\theta)$, constant to variable) and the horizontal distance to be fixed (swap $\chi_i(\theta)$ for x_i). The problem should still be solvable).

Let A be the point $(-x_1, y_1)$ and B be the point $(x_2, -y_2)$. (Hopefully, without confusion, we let the English letters be *constants* and Greek letters be *variables*.)

Then, we define the function to be minimized as

$$f(\theta_1, \theta_2) = t_1(\theta_1) + t_2(\theta_2), \qquad \theta_1, \theta_2 \in [0, \tfrac{\pi}{2}),$$

[12] In this question, "*when* Snell's law holds" is <u>not</u> the logical "*if* Snell's law holds". So, do not attempt to show "Snell's law holds $\Rightarrow$ trip is in minimum time". Rather, the question is asking more of: is it true that when we have attained the minimum time, Snell's law is also true? This rephrased question, we can answer by Lagrange multipliers.

Understanding this subtle point should help the student appreciate the nature of Lagrange multipliers in Theorem 8 [MT12, p.186], which is only a "necessary condition" ($\Rightarrow$). This point is reiterated at the end of Example 2 on [MT12, p.189].

$D = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1\right\}$, unit ball.

T is not one-to-one. By counterexample, $T(1, 0, 0) = (0, 0, 1) = T(1, 0, 2\pi)$.

There are many options, but one is to restrict the domain to $\rho \in (0, 1]$, $\phi \in (0, \pi)$, $\theta \in (0, 2\pi]$. □

6.2 The Change of Variables Theorem

Problem 6.2.2. *Suggest a substitution/transformation that will simplify the following integrands, and find their Jacobians.*

(a) $\iint_R (5x + y)^3 (x + 9y)^4 \, dA$

(b) $\iint_R x \sin(6x + 7y) - 3y \sin(6x + 7y) \, dA$

Solution. [3]

(a) Substitute $u = 5x + y,\ v = x + 9y$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 5 & 1 \\ 1 & 9 \end{vmatrix} = 44 \Longrightarrow \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{44}$$

Or, rearrange the variables to get $x = \frac{9}{44}u - \frac{1}{44}v,\ y = -\frac{1}{44}u + \frac{5}{44}v$ and compute the Jacobian determinant.

(b) Substitute $u = x - 3y,\ v = 6x + 7y$.

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} 1 & -3 \\ 6 & 7 \end{vmatrix} = 25 \Longrightarrow \frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{25}$$

Or, rearrange the variables to get $x = \frac{7}{25}u + \frac{3}{25}v,\ y = -\frac{6}{25}u + \frac{1}{25}v$ and compute the Jacobian determinant.

□

Problem 6.2.3. *Let D be the unit disk: $x^2 + y^2 \le 1$. Evaluate*

$$\iint_D exp(x^2 + y^2) \, dx \, dy$$

by making a change of variables to polar coordinates.

Solution. We use the change of variables $dx \, dy = r \, dr \ \ d\theta$, where $r^2 = x^2 + y^2$. Hence,

$$\begin{aligned}\iint_D e^{x^2+y^2} \, dx \, dy &= \int_0^{2\pi} \int_0^1 re^{r^2} \, dr \, d\theta \\ &= \int_0^{2\pi} \frac{e-1}{2} \, d\theta \\ &= \pi(e-1)\end{aligned}$$

□

[3] If the substitution is done in reverse, e.g. $u = x + 9y,\ v = 5x + y$ for part (a), then the Jacobian will be the negative of it, $-\frac{1}{44}$. Likewise for part (b).

Problem 6.2.4. *Let D be the region $0 \le y \le x$ and $0 \le x \le 1$. Evaluate*

$$\iint_D (x+y)\,dx\,dy$$

by making the change of variables $x = u+v$, $y = u-v$. Check your answer by evaluating the integral directly by using an iterated integral.

Solution. By the change of variables $x = u+v$ and $y = u-v$, we get

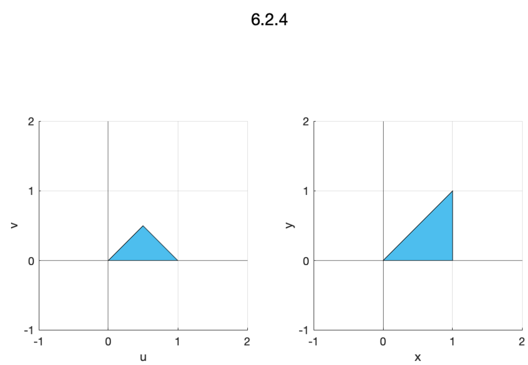
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2.$$

The new region is described by $0 \le u \le 1$, $u \le v \le 1-u$, which is v-simple. Thus, the integral after the change of variables is

$$\int_0^{1/2}\int_v^{1-v} (u+v+u-v)\,|-2|\,du\,dv = \int_0^{1/2} 2-4v\,dv = \frac{1}{2}.$$

We check this against the direct integration, which gives

$$\int_0^1\int_0^x (x+y)\,dy\,dx = \int_0^1 x^2 + \frac{x^2}{2}\,dx = \frac{1}{2}.$$



Problem 6.2.4: Left: D^*, Right: D

□

Problem 6.2.6. *Let $T(u,v) = (x(u,v),\ y(u,v))$ be the mapping defined by $T(u,v) = (u, v(1+u))$. Let $D*$ be the rectangle $[0,1]\times[1,2]$. Find $D = T(D*)$ and evaluate*

(a) $\iint_D xy\,dx\,dy$

(b) $\iint_D (x-y)\,dx\,dy$

by making a change of variables to evaluate them as integrals over $D*$.

Solution. First, we calculate the Jacobian used for both parts of the question

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & v \\ 0 & 1+u \end{vmatrix} = 1+u$$

In particular, $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 1+u$ since $u \in [0,1]$. Also, we get that the region $D = T(D*)$ is the quadrilateral with vertices $(0,1), (0,2), (1,2), (1,4)$.

(a) $\iint_S \mathbf{F} \cdot \, d\mathbf{S}$:

$$\mathbf{F}(\Phi(\theta,\phi)) = (\cos\theta\sin\phi, \sin\theta\sin\phi, 0) \implies \mathbf{F}(\Phi(\theta,\phi)) \cdot (\mathbf{T}_\phi \times \mathbf{T}_\theta) = \sin^3\phi.$$

Therefore, the surface integral is [9]

$$\iint_S \mathbf{F} \cdot \, d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/2} \sin^3\phi \, d\phi \, d\theta = \frac{4}{3}\pi.$$

$\iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S}$:

Now notice that $\mathbf{F}(x,y,z) = (x,y,0) = \nabla f$ where $f(x,y,z) = \frac{1}{2}x^2 + \frac{1}{2}y^2$. Since $\mathbf{F}$ can be written as the gradient vector field, its curl is 0. Therefore,

$$\iint_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \, d\mathbf{S} = 0.$$

$\int_C \mathbf{F} \cdot \, d\mathbf{s}$:

If we parametrize $\mathbf{c}(t) = (\cos t, \sin t, 0)$ with $t \in [0, 2\pi]$, we get

$$\begin{gathered} \mathbf{c}'(t) = (-\sin t, \cos t, 0), \quad \mathbf{F}(\mathbf{c}(t)) = (\cos t, \sin t, 0), \\ \Rightarrow \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = 0. \end{gathered}$$

Therefore,

$$\int_C \mathbf{F} \cdot \, d\mathbf{s} = 0.$$

(b) $\iint_S \mathbf{F} \cdot \, d\mathbf{S}$:

$$\mathbf{F}(\Phi(\theta,\phi)) = (\sin\phi\sin\theta, \sin\phi\cos\theta, 0), \Rightarrow \mathbf{F}(\Phi(\theta,\phi)) \cdot (\mathbf{T}_\phi \times \mathbf{T}_\theta) = \sin^3\phi\sin 2\theta.$$

Thus,

$$\iint_S \mathbf{F} \cdot \, d\mathbf{S} = \int_0^{\pi/2} \int_0^{2\pi} \sin^3\phi \sin 2\theta \, d\theta \, d\phi = \int_0^{\pi/2} 0 \, d\phi = 0.$$

$\iint_S (\nabla \times \mathbf{F}) \cdot \, d\mathbf{S}$:

Now notice that $\mathbf{F}(x,y,z) = (y,x,0) = \nabla f$ where $f(x,y,z) = xy$. Since $\mathbf{F}$ can be written as the gradient vector field, its curl is 0. Therefore,

$$\int\int_S (\boldsymbol{\nabla} \times \mathbf{F}) \cdot \, d\mathbf{S} = 0.$$

$\int_C \mathbf{F} \cdot \, d\mathbf{s}$:

By the same parameterization for $\mathbf{c}(t)$ as part (a), we get

$$\mathbf{F}(\mathbf{c}(t)) = (\sin t, \cos t, 0) \Rightarrow \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) = \cos^2 t - \sin^2 t = \cos 2t.$$

Therefore,

$$\int_0^{2\pi} \cos 2t \, dt = \left[\frac{1}{2}\sin 2t\right]_0^{2\pi} = 0.$$

$\square$

[9] The integral $\int \sin^n x \, dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x \, dx$ can be found in the table of integrals in [MT12]

Chapter 8

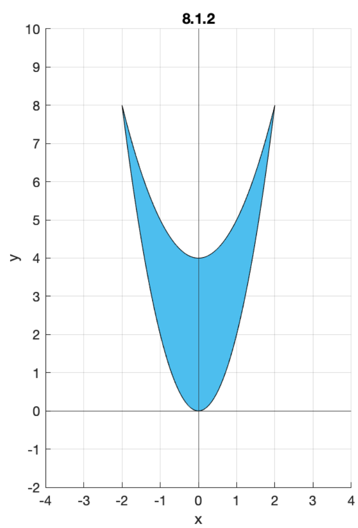
The Integral Theorems of Vector Analysis

8.1 Green's Theorem

Problem 8.1.2. *Let D be the region in the xy plane lying between the curves $y = x^2 + 4$ and $y = 2x^2$. Describe the boundary ∂D as a piecewise smooth curve, oriented counterclockwise.*

Solution. [1]

$$\mathbf{c}(t) = \begin{cases} (2-t, (2-t)^2+4) & 0 \le t \le 4, \\ (t-6, 2(t-6)^2) & 4 < t \le 8. \end{cases}$$



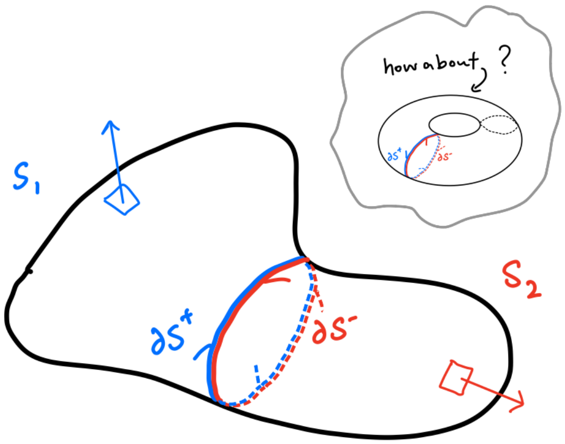
Problem 8.1.2: Region D and its boundary ∂D.

□

Problem 8.1.6. *Verify Green's theorem for the region D and boundary ∂D, and functions P and Q.*

$$D = \left[0, \tfrac{\pi}{2}\right] \times \left[0, \tfrac{\pi}{2}\right], \qquad P(x,y) = \sin x, \qquad Q(x,y) = \cos y.$$

[1] Start points may vary, in this solution, we traversed the top ($y = x^2 + 4$) before traversing the bottom ($y = 2x^2$).



Problem 8.2.25: Intuition (and an additional case to think about in your free time).

Solution.

(a) From [MT12, p.255], we know that

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}\mathbf{F} + \nabla f \times \mathbf{F}.$$

Let $\mathbf{F} = \nabla g$, a gradient vector field. We know that gradient vector fields have 0 curl. Thus the above simplifies to

$$\operatorname{curl}(f\nabla g) = \nabla f \times \nabla g.$$

Since g is a $\mathcal{C}^2$ function, ∇g is a $\mathcal{C}^1$ function. Hence, $f\nabla g$ is $\mathcal{C}^1$ also. Applying Stokes' theorem, we get

$$\int_C f\nabla g \cdot\, d\mathbf{s} = \int_{\partial D} f\nabla g \cdot\, d\mathbf{s} = \iint_D \operatorname{curl}(f\nabla g)\cdot\, d\mathbf{S} = \iint_D (\nabla f \times \nabla g)\cdot\, d\mathbf{S}.$$

(b) From [MT12, p.255], we know that

$$\nabla(fg) = f\nabla g + g\nabla f,$$

which is a $\mathcal{C}^1$ function. Since gradient vector fields always have 0 curl and by applying Stokes' theorem, we get

$$\int_C (f\nabla g + g\nabla f)\cdot\, d\mathbf{s} = \int_C \operatorname{curl}\nabla(fg)\cdot\, d\mathbf{s} = \int_C \mathbf{0}\cdot\, d\mathbf{s} = 0.$$

□

Problem 8.2.27.

(a) If C is a closed curve that is the boundary of a surface S, and $\mathbf{v}$ is a constant vector, show that

$$\int_C \mathbf{v}\cdot\, d\mathbf{s} = 0.$$

(b) Show that this is true even if C is not the boundary of a surface S.

Solution.

(a) Since $\mathbf{v}$ is constant, we know that $\operatorname{curl}\mathbf{v} = 0$. Then, by Stokes' theorem,

$$\int_C \mathbf{v} \cdot \, d\mathbf{s} = \iint_S \operatorname{curl}\mathbf{v} \cdot \, d\mathbf{S} = \iint_S 0 \cdot \, d\mathbf{S} = 0.$$

(b) We can no longer apply Stokes' theorem. But we can see that if $\mathbf{v}$ is constant, then we can always find an $\mathbf{F}$ such that $\mathbf{v}$ is the gradient vector field of $\mathbf{F}$, i.e. $\nabla\mathbf{F} = \mathbf{v}$ for some function $\mathbf{F}$.

We can then say that the integral of a gradient vector field $\mathbf{v}$ over any closed curve C is always 0. Hence, it is still true, even if C is not the boundary of a surface S.

$\square$

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[MT12] Jerrold E. Marsden and Anthony Tromba. *Vector Calculus*. North-Holland Mathematical Library. New York: W.H. Freeman, 2012. ISBN: 1-4292-1508-9.