Pricing of Basket Options Using Dimension Reduction and Adaptive Finite Differences in Space, and Discontinuous Galerkin in Time

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Abstract European basket options are priced by solving the multi-dimensional Black–Scholes–Merton equation. Standard numerical methods to solve these problems often suffer from the "curse of dimensionality". We tackle this by using a dimension reduction technique based on a principal component analysis with an asymptotic expansion. Adaptive finite differences are used for the spatial discretization. In time we employ a discontinuous Galerkin scheme. The efficiency of our proposed method to solve a five-dimensional problem is demonstrated through numerical experiments and compared with a Monte-Carlo method.

1 Introduction

Pricing of options is something that's going on daily in banks and financial institutes. For many options there exist no analytical solution to the pricing problem and fast and accurate numerical methods are of utmost importance.

We consider a Black–Scholes–Merton market [1, 8] with one risk free asset with price process $B(t)$ and d risky assets with processes $\mathbf{S}(t) = (S_1(t) \cdots S_d(t))$ given by the following dynamics

$$\begin{aligned} dB(t) &= rB(t)dt, \\ dS_1(t) &= \alpha_1 S_1(t)dt + \sigma_1 S_1(t)dW_1(t), \\ &\vdots \\ dS_d(t) &= \alpha_d S_d(t)dt + \sigma_d S_d(t)dW_d(t), \end{aligned}$$

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B. Karasözen et al. (eds.), *Numerical Mathematics and Advanced Applications ENUMATH 2015*, Lecture Notes in Computational Science and Engineering 112, DOI 10.1007/978-3-319-39929-4_58

where $r \in \mathbb{R}$ is the short rate of interest, and W_i are correlated Wiener processes with $\langle dW_i, dW_j \rangle = \rho_{ij} dt$. Finally $\alpha_i, \sigma_i \in \mathbb{R}$ are the local mean of return and volatility of S_i respectively. An option issued on $\mathbf{S}$ that at time of maturity T pays $\Phi(\mathbf{S})$ can be priced from

$$u = e^{-r(T-t)} \mathbb{E}^Q \left[\Phi(\mathbf{S}(T)) \right], \tag{1}$$

where $\mathbb{E}^Q[\cdot]$ denotes the expected value under the risk-neutral measure Q. A standard way to price multi-dimensional problems is to use a Monte-Carlo method, simulating paths of $S_i(t)$ combined with (1). This approach is known to converge very slowly.

In [1] and [8] it was independently shown that the price u of an option issued on the risky asset can be obtained from solving the Black–Scholes–Merton equation

$$\begin{aligned} \frac{\partial u}{\partial t} + \sum_{i=1}^{d} r s_i \frac{\partial u}{\partial s_i} + \frac{1}{2} \sum_{i,j=1}^{d} \sigma_i \sigma_j \rho_{ij} s_i s_j \frac{\partial^2 u}{\partial s_i \partial s_j} - ru = 0, \\ u(T, \mathbf{S}) = \Phi(\mathbf{S}). \end{aligned} \tag{2}$$

We consider European basket options that at time of maturity pays

$$\Phi(\mathbf{S}) = \max\left(\sum_{i=1}^{d} \mu_i s_i - K, 0\right), \tag{3}$$

where μ_i determines the fraction of asset S_i in the basket and K is the so called strike price of the option. A standard discretization of (2) leads to the "curse of dimensionality" – the number of degrees of freedom of the discretized problem grows exponentially in the number of dimensions d. We will therefore introduce a dimension reduction techique based on a principal component analysis (PCA) and asymptotic expansions.

The outline of the paper is as follows: In Sect. 2, the dimension reduction technique is presented. The discretization in space and time is discussed in Sects. 3 and 4 respectively. Section 5 is devoted to the presentation of numerical results and we give concluding remarks in Sect. 6.

2 Dimension Reduction

We follow [2, 3, 12] and make the following change of variables

$$\mathbf{x} = \mathbf{Q}^T ln\mathbf{S} + \bar{\mathbf{b}}\tau, \tag{4}$$

where $\tau = T - t$, $b_i = \sum_{j=1}^{d} q_{ji}(r - \frac{\sigma_j^2}{2})$ and $\mathbf{Q}$ is the eigenvector-matrix of the covariance matrix $\boldsymbol{\Sigma}$ with elements $\boldsymbol{\Sigma}_{i,j} = \sigma_i \sigma_j \rho_{ij}$. Implementing the changes to (2)

and (3) gives

$$\begin{aligned} \tfrac{\partial u}{\partial \tau} - \tfrac{1}{2}\textstyle\sum_{i=1}^{d} \lambda_i \tfrac{\partial^2 u}{\partial x_i^2} + ru = 0 \\ u(0,\mathbf{x}) = \max\left(\textstyle\sum_{i=1}^{d} \mu_i e^{\sum_{j=1}^{d} q_{ji} x_j} - K, 0\right), \end{aligned} \tag{5}$$

where λ_i are the eigenvalues of $\boldsymbol{\Sigma}$. Here $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_d|$.

The dimension reduction process is completed by an asymptotic expansion where we approximate each of the non-principal dimensions by a linear asymptotic expansion. Following [2, 3, 12] the asymptotic expansion of the solution is given by

$$u = u^{(1)} + \sum_{i=2}^{d} \lambda_i \left.\frac{\partial u}{\partial \lambda_i}\right|_{\bar{\lambda}=\bar{\lambda}^{(1)}} + \mathcal{O}(||\bar{\lambda} - \bar{\lambda}^{(1)}||^2) \tag{6}$$

where $u^{(1)}$ is the solution to the one-dimensional problem in the principal direction (corresponding to the largest eigenvalue), $\bar{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_d)$, and $\bar{\lambda}^{(1)} = (\lambda_1, 0, \ldots, 0)$. The derivatives in (6) can be approximated by a finite difference method

$$\left.\frac{\partial u}{\partial \lambda_i}\right|_{\bar{\lambda}=\bar{\lambda}^{(1)}} = \frac{u^{(1,i)} - u^{(1)}}{\lambda_i} + \mathcal{O}(\lambda_i^2) \tag{7}$$

where $u^{(1,i)}$ is the solution to the two-dimensional problem on the plane spanned by the principal axis x_1 and axis i corresponding the ith largest eigenvalue. Thus, the d-dimensional problem is broken down to one one-dimensional and $(d-1)$ two-dimensional problems. From (6) and (7) we see that if the eigenvalues λ_i, $i = 2, \ldots, d$ are small, the error introduced from the expansion is small.

3 Adaptive Finite Differences in Space

The PCA and asympotic expansions lead to the following PDEs to solve

$$\frac{\partial u}{\partial \tau} = \mathscr{L} u, \tag{8}$$

where the one- and two-dimensional spatial operators are defined respectively by

$$\mathscr{L} u = \frac{1}{2}\lambda_1 \frac{\partial^2 u}{\partial x_1^2} - ru, \tag{9a}$$

$$\mathscr{L} u = \frac{1}{2}\lambda_1 \frac{\partial^2 u}{\partial x_1^2} + \frac{1}{2}\lambda_i \frac{\partial^2 u}{\partial x_i^2} - ru, i = 2, \ldots, d. \tag{9b}$$

We apply a discretization in space using finite differences on a structured but possibly non-equidistant grid. The number of grid points in dimension i is N_i, $i = 1, \ldots, d$. The second derivative in direction i is approximated as

$$\frac{\partial^2 u(x_{ik})}{\partial x_k^2} \approx a_{ik} u(x_{i(k+1)}) + b_{ik} u(x_{ik}) + c_{ik} u(x_{i(k-1)}), \tag{10}$$

where $a_{ik} = \frac{2}{h_{ik}(h_{i,k-1}+h_{ik})}$, $b_{ik} = \frac{-2}{h_{i,k-1}h_{ik}}$, $c_{ik} = \frac{2}{h_{i,k-1}(h_{i,k-1}+h_{ik})}$. The approximation (10) is second-order accurate in space if there is a smooth variation of the grid such that $h_{i,k-1} = h_{i,k}(1 + \mathcal{O}(h_{i,k}))$.

We will use the computational domain $x_i^0 - 5 \leq x_i \leq x_i^0 + 5$, $i = 1, \ldots, d$, where $\mathbf{x}^0 = \mathbf{Q}^T ln\mathbf{S}^0 + \bar{\mathbf{b}}T$. At the boundaries of the computational domain we need to impose some boundary conditions. In the principal axis we use Dirichlet boundary conditions, while we approximate the solution across the boundaries in the other dimensions to be linear

$$u = 0, \, x_1 = x_1^0 - 5, \tag{11a}$$

$$u = \sum_{i=1}^{d} \mu_i e^{\sum_{j=1}^{n} q_{ij}(x_j - b_j \tau)} - Ke^{-r\tau}, \, x_1 = x_1^0 + 5, \tag{11b}$$

$$\frac{\partial^2 u}{\partial x_i^2} = 0, \, \begin{cases} x_i = x_i^0 - 5, \\ x_i = x_i^0 + 5, \end{cases} \, i = 2, \ldots, d. \tag{11c}$$

Next, we introduce spatial adaptivity for (8) with (9a), i.e. the 1D-problem in the principal axis. The outline of the adaptive algorithm is the same as in e.g.[9]:

1. Solve the PDE once using a coarse equidistant grid with N_1^c grid-points.
2. Create a new spatial grid aiming to fulfil the required accuracy.
3. Solve the PDE using the new adaptive grid with N_1 grid-points.

We will here only briefly discuss how to construct the adaptive grid, for a thorough explanation, see [5, 6, 9, 10, 16] . Assume that for the computed solution u_h using space-step h, it holds that $u_h = u + h^2 c(x) + \mathcal{O}(h^3)$. Using the second order accuracy also in the local discretization error ψ_h we get $\psi_h = h^2 \eta(x) + \mathcal{O}(h^3)$. Omitting higher order terms we obtain after some algebraic manipulations $\psi_h = (\delta_{2h} - \delta_h)/3$ where $\delta_h = A_h u_h$. Estimating $\psi_{\bar{h}}$ using space-steps $\bar{h}$ and $2\bar{h}$ gives $\eta(x) = \psi_{\bar{h}}(x)/\bar{h}^2(x)$ and we get $|\psi_h(x)| = \left|h^2(x) \cdot \psi_{\bar{h}}(x)/\bar{h}^2(x)\right|$. In order to control the local discretization error and keep $|\psi_h(x)| \leq \epsilon$ for some $\epsilon > 0$, we use $h(x) = \bar{h}(x) \left(\epsilon/|\psi_{\bar{h}}(x)|\right)^{1/2}$. Since the local discretization error varies over time and we want to have the same spatial grid for all $0 < \tau < T$, we will use $\psi_{\bar{h}}(x) = \max(|\psi_{\bar{h}}^{(\frac{T}{3})}(x)|, |\psi_{\bar{h}}^{(\frac{2T}{3})}(x)|, |\psi_{\bar{h}}^{(T)}(x)|)$.

For the two-dimensional problems (8), and (9b), we will in the principal axis use the adaptive grid computed. In the other axis we will use an equidistant grid

using $\sim N_1/5$ grid-points. For a small number of grid-points like this, steps 1. and 2. above cost relatively much and the usage of equidistant points is preferable.

4 Discontinuous Galerkin in Time

The spatial discretization defined in Sect. 3 results in d systems of ordinary differential equations

$$\frac{du(\tau)}{d\tau} = Au(\tau) + f(\tau), \quad 0 \le t \le T,\ u(0) = u_0 = \Phi, \tag{12}$$

where the vector $f(\tau)$ contains the boundary conditions. We will use a dG scheme to discretize (12), as in [4, 7, 11, 14, 15, 17, 18]. In [15] it was shown that for option pricing problems, dG is superior to other time-stepping schemes such as backwards differentiation formula of order 2, Crank-Nicolson and Rannacher time-stepping.

Divide $(0, T)$ into M subintervals $\{I_m = (\tau_{m-1}, \tau_m)\}_{m=1}^{M}$ of size $k = T/M$. Define $\mathscr{P}^r(I_m)$ as the space of polynomials of degree r or less on the interval I_m and $\mathbb{U} = \{U : U_m \in \mathscr{P}^r(I_m)\}$ to be the finite element space containing the piecewise polynomials. In the dG method the finite element solution U is continuous within each time interval I_m, but may be discontinuous at the nodes $\tau_1, \ldots, \tau_{M-1}$. We define the one-sided limits of a piecewise continuous function $u(\tau)$ as $u_m^+ := \lim_{s\to 0+} u(\tau_m + s)$, $u_m^- := \lim_{s\to 0+} u(\tau_m - s)$ and the jump across τ_m as $[u_m] := u_m^+ - u_m^-$.

The solution of (12) using a dG method of degree r (with order of accuracy $2r+1$) can be obtained by finding $U \in \mathbb{U}$ such that $\int_{I_m} (\dot{U} - AU)w(\tau)\,d\tau + [U_{m-1}]w(\tau_{m-1}) = \int_{I_m} fw(\tau)\,d\tau$, for $m = 1, \ldots, M$, all $w(\tau) \in \mathbb{U}$ and $U_0^- = u_0$.

Let $\{\varphi\}_{j=0}^{r}$ be a basis of the polynomial space $\mathscr{P}_r(-1, 1)$ and let time shape functions on time interval I_m be given by $\varphi_j \circ F_m^{-1}$, where the mapping F_m defines a linear mapping $F_m : (-1, 1) \to I_m$. Expanding U_m in $\mathscr{P}_r(I_m)$ $U_m = \sum_{j=0}^{r} u_{m,j}(\varphi_j \circ F_m^{-1})$ and using the basis $\{\varphi\}_{j=0}^{r}$ as test functions $w(t)$, we get after some algebraic manipulations

$$\left(\mathbf{C} \otimes \mathbf{I} - \frac{k}{2}\mathbf{G} \otimes \mathbf{A}\right)\mathbf{u}^m = \frac{k}{2}\mathbf{f}^1 + \mathbf{f}^2, \tag{13}$$

where $\otimes$ denotes the Kronecker product, $\mathbf{u}$ is the coefficient vector of U_m, $\mathbf{u}^m = \begin{pmatrix} u_{m,0} & \ldots & u_{m,r} \end{pmatrix}^T$, and

$$f_{m,i}^1 := \int_{-1}^{1} (f \circ F_m)\varphi_i\,d\tau, \qquad C_{ij} := \int_{-1}^{1} \varphi_j'\varphi_i\,d\tau + \varphi_j(-1)\varphi_i(-1),$$
$$f_{m,i}^2 := \varphi_i(-1)\sum_{j=0}^{r_m} \varphi_j(1)u_{m-1,j}, \quad G_{ij} = \int_{-1}^{1} \varphi_j\varphi_i\,d\tau.$$

The time-integration of the Dirichlet boundary condition at $x_1^0 + 5$ requires the integration of (11b). This is accomplished using the composite trapezoidal rule with 100 intervals within each time-step.

Equation (13) forms a linear system of size $(r+1)N$ for each time step. Using the temporal shape functions $\varphi_i(\tau) = (i+1/2)^{1/2} L_i(\tau)$, with L_i denoting the i-th Legendre polynomial on $(-1, 1)$, this system decouples into $r+1$ linear systems of size N, [14]. We get $\mathbf{G} = \mathbf{I}$ and

$$C_{ij} = v_{ij}\left(i+\frac{1}{2}\right)^{1/2}\left(j+\frac{1}{2}\right)^{1/2}, v_{ij} = \begin{cases} (-1)^{i+j} & \text{if } j < i \\ 1 & \text{otherwise} \end{cases}, i,j = 0,\ldots,r.$$

It can be shown that $\mathbf{C}$ is diagonalizable in $\mathbb{C}$ at least for $0 \le r \le 100$, [17]. Thus there exists a matrix $\mathbf{Q} \in \mathbb{C}^{(r+1)\times(r+1)}$ such that $\mathbf{Q}^{-1}\mathbf{C}\mathbf{Q} = \boldsymbol{\Delta} = \mathrm{diag}(\delta_0,\ldots,\delta_r)$. Multiplying (13) by $\mathbf{Q}^{-1} \otimes \mathbf{I}$ from the left gives a block-diagonal system that decouples into

$$\left(\delta_j \mathbf{M} - \frac{k}{2}\mathbf{A}\right)\mathbf{w}_j^m = \mathbf{g}_j, \quad j = 0,\ldots,r,$$

where $\mathbf{w}^m := (\mathbf{Q}^{-1} \otimes \mathbf{I})\mathbf{u}^m$ and $\mathbf{g} := (\mathbf{Q}^{-1} \otimes \mathbf{I})\left(\frac{k}{2}\mathbf{f^1} + \mathbf{f^2}\right)$. Hence, in each time step we have to solve $r+1$ linear systems of size N, which greatly reduces the time of computation and the usage of computer memory compared to solving the whole system (13) of size $(r+1)N$. The linear systems of equations are solved using restarted GMRES(6), [13], with an incomplete LU-factorization as preconditioner [9].

5 Numerical Results

We use an example from [12] and consider the highly correlated basket option defined by the parameters in Table 1. However, we consider a call option (3) while a put option is considered in [12]. We have $\bar{\lambda} =$

Table 1 Parameters for five-dimensional problem considered

r	T	K	Equity	$\bar{S}^0$	μ	$\bar{\sigma}$	ρ_{ij}
0.05	1	1	Deutsche Bank	$\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}0.381\\0.065\\0.057\\0.270\\0.227\end{pmatrix}$	$\begin{pmatrix}0.518\\0.648\\0.623\\0.570\\0.530\end{pmatrix}$	$\begin{pmatrix}1.00 & 0.79 & 0.82 & 0.91 & 0.84\\ 0.79 & 1.00 & 0.73 & 0.80 & 0.76\\ 0.82 & 0.73 & 1.00 & 0.77 & 0.72\\ 0.91 & 0.80 & 0.77 & 1.00 & 0.90\\ 0.84 & 0.76 & 0.72 & 0.90 & 1.00\end{pmatrix}$
			Hypo-Vereinsbank				
			Commerzbank				
			Allianz				
			Münchner Rück				

$(1.4089, 0.1124, 0.1006, 0.0388, 0.0213)$ for this problem. When we employ adaptivity we have used $N_1^c = 21$ in the coarse initial solution.

The implementation was made in MATLAB and run on an Apple MacBook Pro with 3.1 GHz Dual-Core Intel Core i7, Turbo Boost up to 3.4 GHz and 16 GB SDRAM.

We start by comparing the solution obtained using a Monte-Carlo method to price the option with the value obtained using our proposed method.

- Monte-Carlo method with 10^9 sampling paths: 0.22461.
- Our proposed method with $N_1 = 2594$ and 20 time-steps of dG with $r = 1$: 0.22443.

We see that the error introduced by truncating the asymptotic expansion is in the order of $\sim 10^{-4}$. We aim for a discretization that does not make the error in the final solution substantially larger.

First, we study the effect of using adaptivity in the principal axis x_1. In Fig. 1a we display the error in the solution as a function of the number of grid-points in the principal axis. The CPU-time as a function of error is presented in Fig. 1b. Note, that the somewhat erratic convergence behaviour is due to the fact that we study the point-wise error in $\mathbf{x}^0$ only. From Fig. 1a, b it is clear that the rate of convergence is close to the expected second-order for both the equidistant grids and adaptive grids. The error is smaller using adaptive grids using a certain number of grid-points and the CPU-time to reach a certain accuracy is considerably smaller.

Next, we study the error introduced by the discretization in time. In Fig. 2a we present the error as a function of number of adaptive grid-points in the principal axis using 6 and 20 time-steps respectively. It is clear that the error in the solution is not increased by using fewer amount of time-steps. Finally, in Fig. 2b we present the CPU-time it takes to compute the solution as a function of error. It is obvious that we gain by using the smaller number of time-steps. With adaptive grid-points

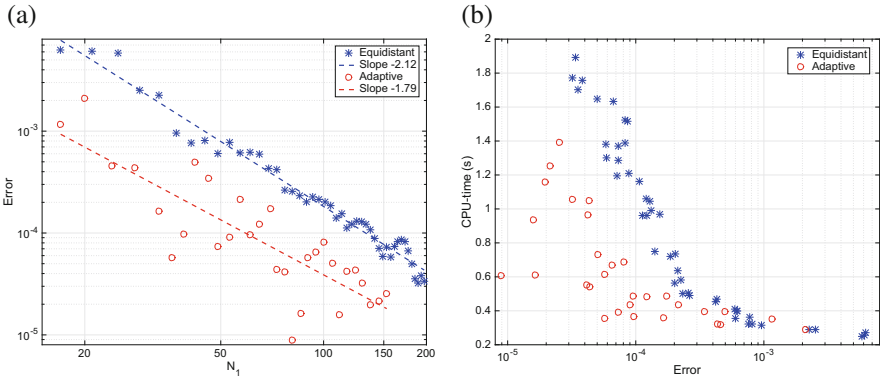

Fig. 1 Comparison of adaptive and equidistant grids. (**a**) Error as a function of N_1. (**b**) CPU-time as a function of error

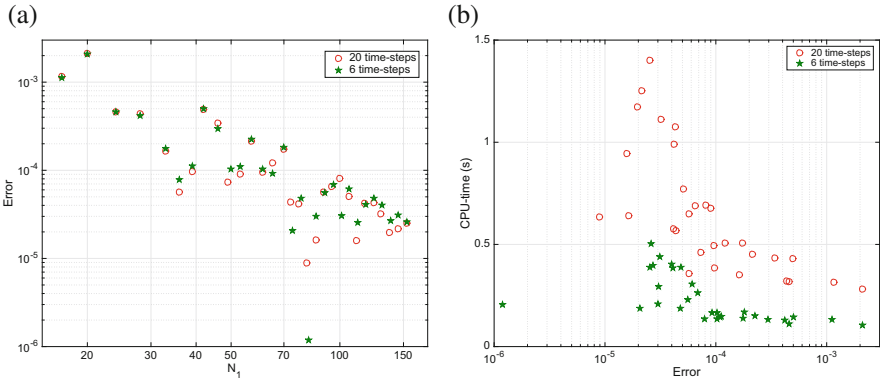

Fig. 2 Comparison of 6 and 20 time-steps with dG using $r = 1$. (**a**) Error as a function of N_1. (**b**) CPU-time as a function of error

in space and 6 time-steps of dG with $r = 1$ it takes less than 0.5 s to compute a solution that has an error $< 10^{-4}$ which is the accuracy that we aimed for. The time to compute a solution with the same accuracy level using a Monte-Carlo method is several minutes.

6 Conclusions

We consider the numerical solution of the multi-dimensional Black–Scholes–Merton equation to price basket options. A principal component analysis together with an asymptotic expansion is used to reduce the dimensionality of the underlying problem. The resulting PDEs are discretized in space with adaptive finite differences and a discontinuous Galerkin scheme in time. The efficiency of the proposed method is demonstrated for a five-dimensional basket option with highly correlated underlying assets.

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