BENCHOP—High complexity
Problem descriptions and status of reference solutions

June 4, 2017

1 SABR stochastic-local volatility model

The *Stochastic Alpha Beta Rho (SABR)* model [6] is an established SDE system which is often used for interest rates and FX modeling in practice. The SABR model is based on a parametric local volatility component in terms of a model parameter, β. The formal definition of the SABR model reads

$$\begin{aligned} \mathrm{d}S(t) &= \sigma(t)S^{\beta}(t)\mathrm{d}W_S(t), & S(0) &= S_0 \exp\left(rT\right), \\ \mathrm{d}\sigma(t) &= \alpha\sigma(t)\mathrm{d}W_{\sigma}(t), & \sigma(0) &= \sigma_0. \end{aligned}$$

where $S(t) = \bar{S}(t)\exp\left(r(T-t)\right)$ denotes the forward value of the underlying asset $\bar{S}(t)$, with r the interest rate, S_0 the spot price and T the contract's final time. Quantity $\sigma(t)$ denotes the stochastic volatility, $W_S(t)$ and $W_{\sigma}(t)$ are two correlated Brownian motions with constant correlation coefficient ρ (i.e. $W_S W_{\sigma} = \rho t$). The open model parameters are $\alpha > 0$ (the volatility of the volatility), $0 \leq \beta \leq 1$ (the elasticity) and ρ (the correlation coefficient). The corresponding PDE for the valuation of options is given by:

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 S^{2\beta}\frac{\partial^2 v}{\partial S^2} + \rho\alpha S^{\beta}\sigma^2\frac{\partial^2 v}{\partial\sigma\partial S} + \frac{1}{2}\alpha^2\sigma^2\frac{\partial^2 v}{\partial\sigma^2} - rv = 0$$

for $S > 0$, $\sigma > 0$ and $0 \leq t < T$.

Two parameter sets:

- Set I ([4]): $T = 2$, $r = 0.0$, $S_0 = 0.5$, $\sigma_0 = 0.5$, $\alpha = 0.4$, $\beta = 0.5$ and $\rho = 0$.
- Set II ([2]): $T = 10$, $r = 0.0$, $S_0 = 0.07$, $\sigma_0 = 0.4$, $\alpha = 0.8$, $\beta = 0.5$ and $\rho = -0.6$.

European call option payoff $\max(S(T) - K_i(T), 0)$ with three strikes

$$\begin{aligned} K_i(T) &= S(0)\exp(0.1 \times T \times \delta_i), \\ \delta_i &= -1.0, 0.0, 1.0. \end{aligned}$$

Notes:

- Consider implied volatilities next to option prices, for comparison purposes.
- For $\rho = 0$, there is a formula for the exact simulation of the SABR model [8].
- The use of time discretization MC schemes can give a loss of the martingale property. A correction must be introduced then.

1.1 MATLAB interfaces

```
[U1,U2,U3]=SABReuCallI_MTH(param)
[U1,U2,U3]=SABReuCallII_MTH(param)
```

1.2 Reference solutions

The methods in [8] could be implemented. Not done. May take some work.

2 Quadratic local stochastic volatility model

In the following τ denotes forward time and t backward time.

See e.g. [10]:

$$\begin{cases} dS_\tau &= rS_\tau\, d\tau + \sqrt{V_\tau}\, f(S_\tau)\, dW^1_\tau, \\ dV_\tau &= \kappa(\eta - V_\tau)\, d\tau + \sigma\sqrt{V_\tau}\, dW^2_\tau, \end{cases}$$

with $f(s) = \frac{1}{2}\alpha s^2 + \beta s + \gamma$. Select

- Heston: $\alpha = 0$, $\beta = 1$, $\gamma = 0$.
- QLSV: $\alpha = 0.02$, $\beta = 0$, $\gamma = 0$.

PDE:

$$\frac{\partial u}{\partial t} = \tfrac{1}{2} f(s)^2 v \frac{\partial^2 u}{\partial s^2} + \rho\sigma f(s) v \frac{\partial^2 u}{\partial s \partial v} + \tfrac{1}{2}\sigma^2 v \frac{\partial^2 u}{\partial v^2} + rs\frac{\partial u}{\partial s} + \kappa(\eta - v)\frac{\partial u}{\partial v} - ru.$$

for $s > 0$, $v > 0$ and $0 < t \leq T$.

One parameter set (see [10]):

$$T = 1,\ r = 0,\ \kappa = 2.58,\ \eta = 0.043,\ \sigma = 1,\ \rho = -0.36.$$

Consider

- European call option payoff $\max(s - K, 0)$ with $K = 100$.
- Double-no-touch option paying 1 if $L < S_\tau < U$ (for all τ) and 0 else with $L = 50$, $U = 150$.

Three spot values: $(S_0, V_0) = (S_0, 0.114)$ for $S_0 = 75, 100, 125$.

Notes:

- Feller condition is violated.
- If $\alpha = 0$, $\beta = 1$, $\gamma = 0$, $\rho = 0$ there are semi-closed analytic formulas for both options.

2.1 MATLAB interfaces

```
[U1,U2,U3]=HSTeuCall_MTH(param)
[U1,U2,U3]=HSTdnTouch_MTH(param)

[U1,U2,U3]=QLSVeuCall_MTH(param)
[U1,U2,U3]=QLSVdnTouch_MTH(param)
```

2.2 Reference solutions

Exact results for European Call and Heston case.

$$\begin{aligned} U_1 &= 0.908502728459621 \\ U_2 &= 9.046650119220969 \\ U_3 &= 28.514786399298796 \end{aligned}$$

Timings and results on laptop for serial code double no touch Heston version.

Tol	Time	U_1	U_2	U_3
0.1	18 s	0.83175	0.08972	0.66065
0.01	18 s	0.8275	0.8956	0.65845
0.001	641 s	0.830740	0.897159	0.660219
0.001	641 s	0.8307	0.8976	0.6614
0.0001	18 h	0.8309	0.8974	0.6610

Timings and results on laptop for serial code QLSV European Call.

Tol	Time	U_1	U_2	U_3
0.1	7.7 s	0.5260	8.8546	28.9147
		0.5217	8.8474	28.9255
0.01	1401 s	0.5220	8.8480	28.9203
0.001	39 h 17 m	0.5235	8.8545	28.9284
0.0001	162 d 18 h			

Timings and results on laptop for serial code QLSV double no touch.

Tol	Time	U_1	U_2	U_3
0.1	43s	0.9355	0.9165	0.5951
0.01	43s	0.9300	0.9122	0.5895
		0.9344	0.9163	0.5934
		0.9346	0.9134	0.5955
0.001	1093 s	0.9330	0.9144	0.5919
0.0001	32 h 30 m	0.9338	0.9147	0.5928

3 Heston–Hull–White model

The Heston–Hull–White model is a hybrid asset price model combining the Heston stochastic volatility and Hull–White stochastic interest rate models, see e.g. [3, 5].

HHW SDE:

$$\begin{cases} dS_\tau &= R_\tau S_\tau \, d\tau + \sqrt{V_\tau}\, S_\tau \, dW_\tau^1, \\ dV_\tau &= \kappa(\eta - V_\tau)\, d\tau + \sigma_1 \sqrt{V_\tau}\, dW_\tau^2, \\ dR_\tau &= a(b(\tau) - R_\tau)\, d\tau + \sigma_2 \, dW_\tau^3. \end{cases}$$

HHW PDE:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \tfrac{1}{2} s^2 v \frac{\partial^2 u}{\partial s^2} + \tfrac{1}{2}\sigma_1^2 v \frac{\partial^2 u}{\partial v^2} + \tfrac{1}{2}\sigma_2^2 \frac{\partial^2 u}{\partial r^2} \\ &\quad + \rho_{12}\sigma_1 s v \frac{\partial^2 u}{\partial s \partial v} + \rho_{13}\sigma_2 s \sqrt{v} \frac{\partial^2 u}{\partial s \partial r} + \rho_{23}\sigma_1\sigma_2 \sqrt{v} \frac{\partial^2 u}{\partial v \partial r} \\ &\quad + rs \frac{\partial u}{\partial s} + \kappa(\eta - v) \frac{\partial u}{\partial v} + a(b(T-t) - r) \frac{\partial u}{\partial r} - ru \end{aligned}$$

for $s > 0$, $v > 0$, $-\infty < r < \infty$ and $0 < t \le T$.

Two parameter sets (cf. [1, 5]):

$T = 10,\ \kappa = 0.5,\ \eta = 0.04,\ \sigma_1 = 1,\ \sigma_2 = 0.09,\ \rho_{12} = -0.9,\ \rho_{13} = 0.6\ (0),\ \rho_{23} = -0.7\ (0),$
$a = 0.08$ and $b(\tau) \equiv 0.10$.

European call option payoff $\max(s - K, 0)$ with $K = 100$.

Three spot values: $(S_0, V_0, R_0) = (S_0, 0.04, 0.10)$ for $S_0 = 75, 100, 125$.

Notes:

- Feller condition is violated.
- For $\rho_{13} = \rho_{23} = 0$ there is semi-closed analytic formula akin to Heston.
- Transformation to 2D PDE with time-dependent coefficients is possible. Hence, if numerical PDE approach is followed, indicate which PDE is solved (2D or 3D).

3.1 MATLAB interfaces

```
[U1,U2,U3]=HHWeuCallI_MTH(param)
[U1,U2,U3]=HHWeuCallII_MTH(param)
```

3.2 Reference solutions

For the first parameter set it seems problemetic to get a solution. Standard Monte Carlo simulation seems to give a biased result.

For the second parameter set there is an exact solution that can be computed accurately.

$$\begin{aligned} U_1 &= 35.391128418768275 \\ U_2 &= 54.676160407431524 \\ U_3 &= 75.287150855775010 \end{aligned}$$

4 European spread option on two assets

European spread call option on two assets obtained by modifying a problem in BENCHOP [12]. The payoff reads

$$g(S) = \max\{S_1 - S_2 - K, 0\}$$

with $K = 5$. The assets follow correlated geometrical Brownian motions with the correlation $\rho = 0.5$. The interest rate is $r = 0.03$. The option is valuated for the asset values $S_i = 100$ with the expiry $T = 1$.

4.1 Constant volatility

The volatility for both assets is $\sigma_i = 0.15$.

4.2 Volatility function

The volatility for both assets is given by the function

$$\sigma_i(S_i, t) = 0.15 + 0.15(0.5 + 2t)\frac{(S_i/100 - 1.2)^2}{(S_i/100)^2 + 1.44}.$$

4.3 MATLAB interfaces

```
[U]=BSeuCallspreadU_MTH(param)
[Delta]=BSeuCallspreadDelta_MTH(param)
[Vega]=BSeuCallspreadVega_MTH(param)

[U]=BSeuCallspreadLocVolU_MTH(param)
[Delta]=BSeuCallspreadLocVolDelta_MTH(param)
[Vega]=BSeuCallspreadLocVolVega_MTH(param)
```

4.4 Reference solutions

I the case of constant volatility, the exact solution can be computed and we have

$$\begin{aligned} U &= 3.868705208134835 \\ \Delta_1 &= 0.401629412929925 \\ \Delta_2 &= -0.344947244888925 \\ \mathcal{V} &= 37.792689414732820 \end{aligned}$$

With MC simulation for the constant volatility we get

Tol	Time	U	Δ_1	Δ_2	$\mathcal{V}$
0.1	1–2 s	3.8703	0.4017	-0.3450	37.8006
0.01	1–7 s	3.8678	0.4017	-0.3450	37.7855
0.001	7–609 s	3.8684	0.4015	-0.3448	37.7925
0.0001	592– s	3.8688	0.4016	-0.3449	

Currently no solutions for the local volatility case.

5 American put on the minimum of two assets

Two asset American option considered by Zvan, Forsyth & Vetzal [13]. The payoff reads

$$g(S) = \max\{K - \min\{S_1,\, S_2\},\, 0\}$$

with the strike price $K = 40$. The assets follow correlated geometrical Brownian motions with the volatilities $\sigma_i = 0.3$ and the correlation $\rho = 0.5$. The option is valuated for the asset values $S_i = 40$ with the expiry $T = 0.5$. The interest rate is $r = 0.05$.

5.1 MATLAB interfaces

```
[U]=BSamPutminU_MTH(param)
[Delta]=BSamPutminDelta_MTH(param)
[Vega]=BSamPutminVega_MTH(param)
```

5.2 Reference solutions

Currently no solutions.

6 American put on the minimum of two assets with jumps

We use the same problem as above, but replace the Brownian motion with Merton jump diffusion dynmaics.

6.1 Merton asset dynamics

Underlying asset prices S_i are modeled by a multidimensional Merton model with contemporaneously jumps. The ith component S_i is given by

$$\frac{dS_i(t)}{S_i(t)} = (r - \lambda\kappa_i)dt + dB_i(t) + \left(e^{J_i(t)} - 1\right) dP(t),$$

where $[dB_i(t)]_{i=1,d}$ is a multidimensional Brownian motion with the covariance matrix defined by its elements $\sigma_i(S_i,t)\sigma_j(S_j,t)\rho_{ij}$ and without drift, $P(t)$ is a Poisson process with the arrival rate λ, the jump $[J_i(t)]_{i=1,d}$ follows multivariate normal distribution with the covariance matrix defined by its elements $\sigma_i^J \sigma_j^J \rho_{ij}^J$ and the mean values μ_i^J. The expected jump of the ith component is

$$\kappa_i = E\left[e^{J_i(t)} - 1\right] = \exp\left(\mu_i^J + \frac{1}{2}\sum_{j=1}^{d} \sigma_i^J \sigma_j^J \rho_{ij}^J\right) - 1.$$

The risk free interest rate is r. When the jump arrival rate λ is zero and the volatilities σ_i are constant, the above model reduces to the standard Black–Scholes model for d assets.

The corresponding PIDE-formulation is given by

$$\begin{aligned} &\frac{\partial u}{\partial t} + \frac{1}{2}\sum_{i=1}^{d}\sum_{j=1}^{d} \rho_{ij}\sigma_i\sigma_j s_i s_j \frac{\partial^2 u}{\partial s_i \partial s_j} + \sum_{i=1}^{d}(r - \lambda\kappa_i)s_i \frac{\partial u}{\partial s_i} - (r+\lambda)u \\ &\quad + \lambda \int_{\mathbb{R}^d} u(se^y, t)p(y)dy = 0, \end{aligned} \tag{6.1}$$

where $se^y = (s_1 e^{y_1}, \ldots, s_d e^{y_d})$ and $p(y)$ is the multivariate normal distribution describing the jumps.

6.2 MATLAB interfaces

```
[U]=MRTamPutminU_MTH(param)
[Delta]=MRTamPutminDelta_MTH(param)
[Vega]=MRTamPutminVega_MTH(param)
```

6.3 Reference solutions

Currently no solutions.

7 Arithmetic basket options on 3 and 10 assets

The arithmetic basket put option payoff reads

$$g(S) = \max\left\{K - \frac{1}{d}\sum_{i=1}^{d} S_i,\ 0\right\}.$$

The model parameters are the same as the ones used by Leitao & Oosterlee [9]. The assets follow correlated geometrical Brownian motions with the volatilities $\sigma_i = 0.2$. The strike price is $K = 40$. The option is valuated for the asset values $S_i = 40$ with the expiry $T = 1$. The interest rate is $r = 0.06$.

7.1 European option and low constant correlation

The correlation between assets is $\rho = 0.25$.

7.2 European option and high variable correlations

The correlation matrix has the entries $\rho_{ij} = a^{|i-j|}$, where $a = 0.9$.

7.3 American option and low constant correlation

The correlation between assets is $\rho = 0.25$.

7.4 American option and high variable correlations

The correlation matrix has the entries $\rho_{ij} = a^{|i-j|}$, where $a = 0.9$.

7.5 MATLAB interfaces

```
[U]=BSeuPut3DbasketLCCU_MTH(param)
[U]=BSeuPut3DbasketHVCU_MTH(param)
[U]=BSeuPut10DbasketLCCU_MTH(param)
[U]=BSeuPut10DbasketHVCU_MTH(param)
```

```
[U]=BSamPut3DbasketLCCU_MTH(param)
[U]=BSamPut3DbasketHVCU_MTH(param)
[U]=BSamPut10DbasketLCCU_MTH(param)
[U]=BSamPut10DbasketHVCU_MTH(param)
```

7.6 Reference solutions

For the European put options

	3D Low		3D High		10D Low		10D High	
Tol	Time	U	Time	U	Time	U	Time	U
0.1	2.1 s	1.2231	1.9 s	1.9362	5.6	0.84181	5.7	1.63686
0.01	1.9 s s	1.2231	1.8 s	1.9362	5.3	0.84177	5.6	1.63683
0.001	1.9 s	1.2230	1.8	1.9632	4.7	0.84182	5.0	1.63684
0.0001	2.1 s	1.2230	1.9	1.9362	4.6	0.84177	5.6	1.63684
0.00001	163 s	1.22309	5.2	1.936198	297	0.84180	106	1.636837

No solutions yet for the American options.

8 European arithmetic basket option on four assets

The arithmetic basket put option payoff reads

$$g(S) = \max\left\{K - \frac{1}{d}\sum_{i=1}^{d} S_i,\, 0\right\}.$$

The model parameters are the same as the ones used by Hendricks, Ehrhardt, & Günther [7]. The volatilities are $\sigma_i = 0.3$ and the correlation matrix is

$$\rho = \begin{pmatrix} 1 & 0.3 & 0.4 & 0.5 \\ 0.3 & 1 & 0.2 & 0.25 \\ 0.4 & 0.2 & 1 & 0.3 \\ 0.5 & 0.25 & 0.3 & 1 \end{pmatrix}.$$

The strike price is $K = 40$. The option is valuated for the asset values $S_i = 40$ with the expiry $T = 1$. The interest rate is $r = 0.06$.

8.1 MATLAB interfaces

```
[U]=BSeuPut4DbasketU_MTH(param)
```

8.2 Reference solutions

Tol	Time	U
0.1	2.3	2.23722
0.01	2.47	2.23741
0.001	2.7	2.23721
0.0001	10	2.237258
0.00001	802	2.237227

9 Arithmetic basket options on five assets

The weighted basket put option payoff reads

$$g(S) = \max\left\{K - \sum_{i=1}^{d} w_i S_i,\, 0\right\}.$$

The model parameters are the same as the ones used by Reisinger & Wittum [11] to approximate DAX index. The volatilities are $\sigma = [0.518, 0.648, 0.623, 0.570, 0.530]^T$ and the correlation matrix is

$$\rho = \begin{pmatrix} 1.00 & 0.79 & 0.82 & 0.91 & 0.84 \\ 0.79 & 1.00 & 0.73 & 0.80 & 0.76 \\ 0.82 & 0.73 & 1.00 & 0.77 & 0.72 \\ 0.91 & 0.80 & 0.77 & 1.00 & 0.90 \\ 0.84 & 0.76 & 0.72 & 0.90 & 1.00 \end{pmatrix}.$$

The strike price is $K = 1$ and the weights are $w = [0.381, 0.065, 0.057, 0.270, 0.227]^T$. The option is valuated for the asset values $S_i = K$ with the expiry $T = 1$. The interest rate is $r = 0.05$.

9.1 MATLAB interfaces

```
[U]=BSeuPut5DbasketU_MTH(param)
[Delta]=BSeuPut5DbasketDelta_MTH(param)
[Vega]=BSeuPut5DbasketVega_MTH(param)
```

9.2 Reference solutions

Tol	Time	U
0.1	3.0	0.175843
0.01	3.2	0.1758406
0.001	3.4	0.1758401
0.0001	3.5	0.17583891
0.00001	3.2	0.17583919

Tol	Time	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5
0.1	13	-0.1400706	-0.0227246	-0.02050892	-0.0949957	-0.0840740
0.01	15	-0.1400667	-0.0227276	-0.02050987	-0.0940018	-0.0840704
0.001	13	-0.1400332	-0.0227198	-0.02050569	-0.0939948	-0.0840571
0.0001	13	-0.1400564	-0.0227225	-0.02051091	-0.0940001	-0.0840703
0.00001	51	-0.1400500	-0.0227244	-0.02050790	-0.0939959	-0.0840693

Tol	Time	$\mathcal{V}_1$	$\mathcal{V}_2$	$\mathcal{V}_3$	$\mathcal{V}_4$	$\mathcal{V}_5$
0.1	19	0.1389	0.02044	0.01789	0.09692	0.07997
0.01	19	0.1389	0.02044	0.01790	0.09692	0.07997
0.001	20	0.13893	0.020442	0.017897	0.096916	0.079966
0.0001	20	0.13894	0.020439	0.017900	0.096914	0.079968
0.00001	854	0.138946	0.020441	0.017896	0.096917	0.079970

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