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Likelihood ratio tests for the structural change of an AR(p) model to a Threshold AR(p) model

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This article considers the likelihood ratio (LR) test for the structural change of an AR model to a threshold AR model. Under the null hypothesis, it is shown that the LR test converges weakly to the maxima of a two-parameter vector Gaussian process. Using the approach in Chan and Tong (1990) and Chan (1991), we obtain a parameter-free limiting distribution when the errors are normal. This distribution is novel and its percentage points are tabulated via a Monte Carlo method. Simulation studies are carried out to assess the performance of the LR test in the finite sample and a real example is given.

Keywords: AR(p); likelihood ratio test; structure change; threshold AR(p); two-parameter Gaussian process.

1. INTRODUCTION

It is important to know if a statistical or econometric model is stable to some possible events, such as the great depression/expansion, oil price shocks and abrupt policy intervention. Due to the well need for model stability, a lot of literatures have focused on the tests of model stability. The earliest work dates back to Chow (1960) and Quandt (1960). For the history and more early results, we refer to Horváth (1993, 1995) and Csörgoő and Horváth (1997). Andrews (1993) studied the likelihood ratio (LR) test, Wald test and Lagrange multiplier test for a change point in a general model. Bai and Perron (1998) and Bai (1999) studied the Wald/LR test for the multiple structure change in the linear regression model. In the field of time series, Davis *et al.* (1995) studied the LR test for a change in the autoregressive (AR) models, Bai and Perron (1998) studied the LR test for the I(0) and I(1) multi-variate time series, Ling (2007) studied the Wald test for the long-memory fractional ARIMA model, and many others.

Unlike the previous structure change problem in which the changing is in the time horizon, the structure of a model may be changed from one state to others (i.e. the regimes of observations). Chan (1990, 1991) and Chan and Tong (1990) formulated this problem into the LR test for an AR model against a threshold AR (TAR) model, i.e.,

$$y_t = \psi_0 + \sum_{i=1}^{p} \psi_i y_{t-i} + I(y_{t-d} \le r)\left(\phi_0 + \sum_{i=1}^{p} \phi_i y_{t-i}\right) + \varepsilon_t, \tag{1}$$

where $\{\varepsilon_t\}$ are the i.i.d. random variables with mean 0 and variance $\sigma^2 > 0$, $I(\cdot)$ is the indicator function, r is the threshold parameter, $d(\ge 1)$ is the delay parameter. Under the null hypothesis, they showed that the LR test converges weakly to the maxima of a multi-parameter Brownian bridge. This kind of LR tests have been studied for many other models, e.g., Wong and Li (1997, 2000) for the TAR-ARCH/DTAR models, Hansen (1999, 2000) for the AR/regression models, Ling and Tong (2005) for the TMA models, and Li and Li (2008) for the TMA-GARCH models, among many others. Except Chan and Tong (1990) for the AR(1) models and Chan (1991) for the AR(p) models with the normal errors, the limiting distributions of these LR tests depend on the nuisance parameters and the error distributions, and hence ones have to simulate their critical values case by case.

In practice, the structure of a model may be changed in terms of both time horizon and states, see the real examples in Li and Lam (1995) and Wong and Li (1997). In this case, the tests for the change point and for the threshold are not the LR tests any more and should be less powerful. It motivates us to look for a more powerful test. Berkes *et al.* (2010) first studied the LR test for the change of an AR(1) model to a TAR(1) model. Under the null hypothesis, they showed that the LR test converges weekly to the maxima of a two-parameter Gaussian process. In particular, the limiting distribution in Berkes *et al.* (2010) is parameter- and distribution-free and can be tabulated. However, this kind of nice distributions cannot be derived when the null model is a higher-order AR model.

In this article, we consider the LR test for the structural change of an AR(p) model to a TAR(p) model. The TAR model was first investigated by Tong (1978, 1983) and Tong and Lim (1980), see also Tsay (1989), Chan (1993) and Chan and Tsay (1998). Under the null hypothesis, that is, $\phi_0 = \phi_1 = \cdots = \phi_p = 0$, the limiting distribution of our LR test is the maxima of a two-parameter vector Gaussian process. Using the approach in Chan and Tong (1990) and Chan (1991), we obtain a parameter-free limiting distribution when the errors are normal. This distribution is novel and its percentage points are tabulated via a Monte Carlo method. Simulation studies are carried out to assess the performance of the LR test in the finite sample and a real example is given.

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This article is organized as follows. Section 2 states the test statistic and presents our main results. The simulation studies and a real example are given in Section 3. All the proofs are presented in the Appendix.

2. LIKELIHOOD RATIO TEST AND MAIN RESULTS

To set up our problem, let $\{y_0, \ldots, y_N\}$ be $N + 1$ consecutive observations from the model

$$y_t = \begin{cases} \psi_0 + \sum_{i=1}^{p} \psi_i y_{t-i} + \varepsilon_t, & \text{if } t = l, \ldots, k, \\ \psi_0 + \sum_{i=1}^{p} \psi_i y_{t-i} + I(y_{t-d} \le r)(\phi_0 + \sum_{i=1}^{p} \phi_i y_{t-i}) + \varepsilon_t, & \text{if } t = k+1, \ldots, N, \end{cases} \tag{2}$$

where $l = \max(p,d)$, k is the break point, $\{\varepsilon_t\}$ are defined as in eqn (1), and all the roots of the characteristic equation $x^p - \psi_1 x^{p-1} - \cdots - \psi_p = 0$ lie inside the unit circle. Here, we assume that both p and d are known. Without loss of generality, we assume that $d \le p$ if $p \ge 1$, because we can set $p = d$ with $\psi_{p+1} = \cdots = \psi_d = 0$ and $\phi_{p+1} = \cdots = \phi_d = 0$ in eqn (2) when $d > p \ge 1$. We consider the following hypotheses:

$$\begin{cases} H_0 : k = N \text{ (no change has occurred)}, \\ H_1 : k = k^* < N \text{ (a change has occurred at time } k^*\text{) and } \phi \ne 0, \end{cases} \tag{3}$$

where $\phi = (\phi_0, \ldots, \phi_p)'$. Model (2) is an AR(p) model under H_0 and it is changing to a TAR(p) model after the time k^* under H_1. When $r = \infty$, eqn (3) is for testing the parameter change in the AR(p) model, for which the LR test was studied by Andrews (1993) and Davis *et al.* (1995). When $k = l - 1$, eqn (3) is for testing the threshold in the TAR(p) model, which was studied by Chan (1990, 1991). Under H_0, both k^* and r are absent, which is the main difficulty in our problem.

We consider the LR test for eqn (3). Under H_0, the quasi-likelihood function can be written as

$$L_{0n}(\psi, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=l}^{N}(y_t - \psi' Y_t)^2\right\},$$

where $\psi = (\psi_0, \ldots, \psi_p)'$, $Y_t = (1, y_{t-1}, \ldots, y_{t-p})'$ and $n = N - l + 1$ is the effective number of observations. Similarly, under H_1, assuming that k and r are both known, the quasi-likelihood function can be written as

$$L_{1n}(\psi, \phi, \sigma^2) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{n}{2}} \times \exp\left[-\frac{1}{2\sigma^2}\left\{\sum_{t=l}^{k}(y_t - \psi' Y_t)^2 + \sum_{t=k+1}^{N}(y_t - (\psi' + I_r(y_{t-d})\phi')Y_t)^2\right\}\right],$$

where $I_r(y_t) = I(y_t \le r)$. Clearly, we should reject H_0 in favour of H_1 if the likelihood ratio

$$\sup_{\psi,\sigma^2} L_{0n}(\psi, \sigma^2) \Big/ \sup_{\psi,\phi,\sigma^2} L_{1n}(\psi, \phi, \sigma^2)$$

is small. Furthermore, denote $Y = (y_l, \ldots, y_N)'$, $\varepsilon = (\varepsilon_l, \ldots, \varepsilon_N)'$, $\hat{I}_{kr}(y_t) = I(t > k, y_{t-d} \le r)$ and $\hat{X}_{kr} = (X, X_{kr})$, where

$$X = (Y_l, Y_{l+1}, \ldots, Y_N)',$$
$$X_{kr} = \mathrm{diag}\{\hat{I}_{kr}(y_t); t = l, \ldots, N\}X.$$

Then, via a long but elementary calculation, we can show that −2 times the log-likelihood ratio is

$$\mathrm{LR}_n(k, r) \triangleq n\{\log \hat{\sigma}_n^2 - \log \hat{\sigma}_n^2(k, r)\}, \tag{4}$$

where

$$\hat{\sigma}_n^2 = \frac{1}{n}\{Y'Y - (Y'X)(X'X)^{-1}(X'Y)\}, \tag{5}$$

$$\hat{\sigma}_n^2(k, r) = \frac{1}{n}\{Y'Y - (Y'\hat{X}_{kr})(\hat{X}_{kr}'\hat{X}_{kr})^{-1}(\hat{X}_{kr}'Y)\}. \tag{6}$$

Since the exact values of k and r are unknown under H_0, it is natural to construct our test by using the maxima of $\mathrm{LR}_n(k, r)$ on the ranges of k and r. Its limiting distribution involves the weak convergence of a two-parameter empirical process in Berkes *et al.* (2010), for which we need two assumptions as follows.

Assumption 1. *$E|\varepsilon_0|^{6+\delta} < \infty$ for some $\delta > 0$.*

Assumption 2. *$|P(a \le y_0 \le b)| \le C|a-b|^{\alpha}$ with some $C > 0$ and $0 < \alpha \le 1$.*

We denote $W(s, u)$, $0 \le s, u < \infty$, be a two-parameter Wiener process, i.e., W is a continuous Gaussian process with $EW(s,u) = 0$ and $E\{W(s, u)W(t, v)\} = \min(s, t)\min(u, v)$, see Csörgoő and Révész (1981). Also, we choose $0 \le \pi_1 < 1$ and $r_1 < r_2$, and let $k_1 = \lfloor n\pi_1 \rfloor + l - 1$ and $s_1 = 1 - \pi_1$, where $\lfloor a \rfloor$ is the integer part of a. Our first result is given in Theorem 1 and its proof is provided in the Appendix.

Theorem 1. *If Assumptions 1 and 2 hold, then under H_0, we have*

$$\max_{l-1\le k\le k_1}\sup_{r_1\le r\le r_2}\mathrm{LR}_n(k,r)\to^d \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\xi'_{\pi r}(\Sigma_{\pi r}-\Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r})^{-1}\xi_{\pi r},$$

as n *to* ∞, where $\to^d$ denotes the convergence in distribution, $\Sigma = E(Y_tY'_t)$, $\Sigma_{\pi r} = (1-\pi)E\{I_r(y_{t-d})Y_tY'_t\}$ and $\xi_{\pi r}$ be a Gaussian process with zero mean function and covariance kernel:

$$\mathrm{cov}(\xi_{\pi r},\xi_{wv}) = \Sigma_{\max(\pi,w)\min(r,v)} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{wv}.$$

If we do not cut the range of k and r, then the maxima of $\mathrm{LR}_n(k, r)$ on the full range $(k, r) \in [l - 1,N]$ *times* $[-\infty, \infty]$ converges to ∞ in probability, as n *to* ∞, see eqn (5) of Berkes *et al.* (2010). When $p = d = 1$ and without drift terms ψ_0 and ϕ_0, by Theorem 1, we can show that under H_0,

$$\max_{0\le k\le k_1}\sup_{r_1\le r\le r_2}\mathrm{LR}_n(k,r)\to^d \sup_{s_1\le s\le 1}\sup_{u_1\le u\le u_2}\frac{\{suW(1,1)-W(s,u)\}^2}{su(1-su)}, \tag{7}$$

as n *to* ∞, where $u_i = H(r_i)$ for $i = 1, 2$, and $H(r) = E\{y_t^2I_r(y_t)\}/\sigma_y^2$ with $\sigma_y^2 = Ey_t^2$. The limiting distribution in eqn (7) was first obtained by Berkes *et al.* (2010). Moreover, they constructed a weighted test which has a parameter- and distribution-free limiting distribution. However, this kind of limiting distribution cannot be derived when the null model is a higher-order AR model.

Assume $\varepsilon_t \sim N(0, \sigma^2)$. Following Chan and Tong (1990) and Chan (1991), we now derive an equivalent representation of the limiting distribution in Theorem 1. Without loss of generality, we assume $\psi_0 = 0$, because $I_r(y_t) = I_{r-c}(y_t - c)$ for any real constant c. Let $\Sigma_r = E\{I_r(y_{t-d})Y_tY'_t\}$. Then, by Theorem 1 of Chan (1991) and eqn (C.11) in Chan and Tong (1990), there exists a continuous invertible matrix function $\{Q_r; r_1 \le r \le r_2\}$ such that, for all r, $Q_r\Sigma Q'_r$ is an identity matrix and $Q_r\Sigma_rQ'_r = diag\{\lambda_i(r); i = 1,\cdots,p+1\}$ is a diagonal matrix, where

$$\lambda_i(r) = E\{I_r(y_t)\},\qquad \text{for } i = 1,\ldots,p-1,$$

and $\lambda_p(r)$ and $\lambda_{p+1}(r)$ are the roots of $x^2 - bx + c = 0$ with

$$b = E\{(1+y_t^2/\sigma_y^2)I_r(y_t)\},\ \ c = \frac{1}{\sigma_y^2}\left[E\{I_r(y_t)\}E\{y_t^2I_r(y_t)\} - E^2\{y_tI_r(y_t)\}\right],$$

where $\sigma_y^2 = Var(y_t)$. Moreover, $\lambda_i(r)$ are chosen so that they are continuous and strictly increasing in r. Since $\Sigma_{\pi r} = (1 - \pi)\Sigma_r$, it follows that

$$Q_r\Sigma_{\pi r}Q'_r = \mathrm{diag}\{(1-\pi)\lambda_i(r);\ \ i = 1,\ldots,p+1\}.$$

Further, let $Q_r\xi_{\pi r} = (B_{1,\pi r}, \ldots, B_{p+1,\pi r})'$. Then, by Theorem 1, it follows that $B_{i,\pi r}$'s are independent Gaussian processes with mean zero and covariance kernel:

$$\mathrm{cov}(B_{i,\pi r},B_{i,wv}) = \{1-\max(\pi,w)\}\lambda_i\{\min(r,v)\} - (1-\pi)(1-w)\lambda_i(r)\lambda_i(v).$$

Let $s = 1 - \pi$. It is not hard to verify that

$$\{B_{i,\pi r}; 0\le\pi\le\pi_1, r_1\le r\le r_2\}\stackrel{D}{=}\{s\lambda_i(r)W(1,1) - W(s,\lambda_i(r)); s_1\le s\le 1, r_1\le r\le r_2\},$$

where the notation $\xi\stackrel{D}{=}\eta$ means that random variables ξ and η have the same distribution. Thus, we can show that

$$\begin{aligned}\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\xi'_{\pi r}(\Sigma_{\pi r}-\Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r})^{-1}\xi_{\pi r} &= \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}(Q_r\xi_{\pi r})'\{(Q_r\Sigma_{\pi r}Q'_r) - (Q_r\Sigma_{\pi r}Q'_r)^2\}^{-1}(Q_r\xi_{\pi r})\\ &= \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\sum_{i=1}^{p+1}\frac{B_{i,\pi r}^2}{(1-\pi)\lambda_i(r) - \{(1-\pi)\lambda_i(r)\}^2}\stackrel{D}{=}\sup_{s_1\le s\le 1}\sup_{r_1\le r\le r_2}\sum_{i=1}^{p+1}\frac{\{s\lambda_i(r)W(1,1) - W(s,\lambda_i(r))\}^2}{s\lambda_i(r) - \{s\lambda_i(r)\}^2}.\end{aligned} \tag{8}$$

Particularly when $p = 0$, $\lambda_1(r) = E\{I_r(y_t)\}$. When $s_1 \equiv 1$, eqn (8) reduces to the asymptotic null distribution of the LR test for the TAR model in Chan and Tong (1990) and Chan (1991). In general, the distribution of y_t / σ_y depends on the parameter ψ. Thus, the limiting distribution of eqn (8) is not parameter-free.

However, when $\varepsilon_t \sim N(0,\sigma^2)$, under H_0, we have $y_t / \sigma_y \sim \Phi_0(x)$, where

$$\Phi_0(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt.$$

Let r be the τ%-quantile of $\{y_t\}$. Then, r / σ_y is the τ%-quantile of N(0,1), i.e., $\tau = \Phi_0(r / \sigma_y)$. Set $\tilde{\lambda}_i(\tau) \triangleq \lambda_i\{\sigma_y \Phi_0^{-1}(\tau)\}$, for $i = 1, \ldots, p + 1$. Then,

$$\tilde{\lambda}_i(\tau) = \Phi_0\{\Phi_0^{-1}(\tau)\} = \tau, \text{for } i = 1, \ldots, p - 1.$$

Similarly, we can show that $\tilde{\lambda}_p(\tau)$ and $\tilde{\lambda}_{p+1}(\tau)$ are the roots of $x^2 - \tilde{b}x + \tilde{c} = 0$ with

$$\tilde{b} = \tau + \Phi_2\{\Phi_0^{-1}(\tau)\},$$
$$\tilde{c} = \tau\Phi_2\{\Phi_0^{-1}(\tau)\} - [\Phi_1\{\Phi_0^{-1}(\tau)\}]^2,$$

where

$$\Phi_1(x) = \int_{-\infty}^{x} \frac{t}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt,$$
$$\Phi_2(x) = \int_{-\infty}^{x} \frac{t^2}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right) dt.$$

Let r_1 and r_2 be the π_0%-quantile and $(1 - \pi_0)$%-quantile of $\{y_t\}$, respectively, and $\pi_1 = 1 - \pi_0$, where $0 < \pi_0 \leq 1 / 2$. Then, by eqn (8), it follows that

$$\begin{aligned} &\sup_{0 \leq \pi \leq \pi_1} \sup_{r_1 \leq r \leq r_2} \xi'_{\pi r}(\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r})^{-1}\xi_{\pi r} \\ &= \sup_{\pi_0 \leq s \leq 1} \sup_{\pi_0 \leq \tau \leq 1-\pi_0} \sum_{i=1}^{p+1} \frac{\left\{s\tilde{\lambda}_i(\tau)W(1,1) - W(s, \tilde{\lambda}_i(\tau))\right\}^2}{s\tilde{\lambda}_i(\tau) - \{s\tilde{\lambda}_i(\tau)\}^2} \\ &\triangleq \sup_{\pi_0 \leq s \leq 1} \sup_{\pi_0 \leq \tau \leq 1-\pi_0} \sum_{i=1}^{p+1} \zeta_i(s, \tau), \end{aligned} \qquad (9)$$

where $\zeta_1(s, \tau), \cdots, \zeta_{p+1}(s, \tau)$ are independent Gaussian processes. Finally, we summarize our main results as follows:

Theorem 2. *Assume $\varepsilon_t \sim N(0,\sigma^2)$. Let $\{\zeta_i(s,\tau); i = 1, \cdots, p + 1\}$ be defined as in eqn (9). Then, for any $0 < \pi_0 \leq 1 / 2$, under H_0, we have*

$$\max_{l-1 \leq k \leq k_1} \sup_{r_1 \leq r \leq r_2} \mathrm{LR}_n(k, r) \to^d \sup_{\pi_0 \leq s \leq 1} \sup_{\pi_0 \leq \tau \leq 1-\pi_0} \sum_{i=1}^{p+1} \zeta_i(s, \tau) \triangleq \mathrm{LR}(\pi_0),$$

as n to ∞, where $k_1 = \lfloor n(1 - \pi_0) \rfloor + l - 1$, and r_1 and r_2 are the π_0%-quantile and $(1 - \pi_0)$%-quantile of $\{y_t\}$ respectively.

For convenience, we have restricted the search on both time horizon and states to the same π_0%-quantile. In practice, we can choose $\pi_0 = 0.10$ or 0.15, although so far how to choose an optimal π_0 remains unclear. The distribution of LR(π_0) now is parameter-free. The approximate upper percentage point of $\sup_{\pi_0 \leq \tau \leq 1-\pi_0} \sum_{i=1}^{p+1} \zeta_i(1, \tau)$ was given in Chan (1991) by the Poisson clumping heuristic method. Here, we use the Monte Carlo method. The percentage points c_α satisfying $P(\mathrm{LR}(\pi_0) > c_\alpha) = \alpha$ are given in Tables 1 and 2, when $\alpha = .01, .05, .10$ and $p = 1, 2, \ldots, 14, 16, 18, 20$ for an array of π_0 values between .05 and .50. The percentage points c_α are obtained by (i) approximating the distribution of LR(π_0) by the maximum of $\sum_{i=1}^{p+1} \zeta_i(s, \tau)$ over a fine grid of points $\Pi(K)$, where

$$\begin{aligned} \Pi(K) = &[\pi_0, 1] \cap \{\pi = j/K : j = 0, 1, \ldots, K\} \\ &\times [\pi_0, 1 - \pi_0] \cap \{\pi = j/K : j = 0, 1, \ldots, K\}, \end{aligned}$$

with $K = 500$, and (ii) simulating the distribution of LR(π_0) via 10,000 independent realizations. A single realization from LR(π_0) was obtained by simulating a two-parameter Wiener process at the discrete points in $\Pi(K)$ and computing $\max_{(s,\tau) \in \Pi(K)} \sum_{i=1}^{p+1} \zeta_i(s, \tau)$.

3. SIMULATION AND A REAL EXAMPLE

In this section, we first compare the performance of our test (LR_n) in finite samples with those of Chan's (1991) test ($\widehat{\mathrm{LR}}_n$) and Andrews's (1993) test ($\widetilde{\mathrm{LR}}_n$). We use the sample size $N = 200$ and 400 and the significance level $\alpha = 0.05$. The samples are generated from model eqn (2) with $p = 2$, $d = 1$, $r = 0$,

Table 1. Percentage points of $LR(\pi_0)$

	$p=0$			$p=1$			$p=2$		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.50	5.37	6.87	10.16	8.44	10.19	13.69	11.53	13.26	17.41
0.49	6.09	7.75	11.47	9.37	11.17	14.85	12.05	13.83	17.86
0.48	6.49	8.16	11.81	9.88	11.69	15.54	12.44	14.23	18.62
0.47	6.82	8.43	12.11	10.30	12.13	16.02	12.78	14.57	18.84
0.45	7.38	9.04	12.65	10.92	12.80	16.59	13.25	15.20	19.54
0.40	8.32	10.10	13.93	12.09	13.96	17.85	14.35	16.09	20.85
0.35	9.15	10.89	14.48	12.89	14.66	18.81	15.33	17.20	21.98
0.30	9.81	11.54	15.21	13.58	15.54	19.63	15.99	17.98	22.70
0.25	10.39	12.27	15.92	14.20	16.30	20.49	16.65	18.69	23.16
0.20	10.99	12.90	16.48	14.80	16.90	21.13	17.34	19.45	24.00
0.15	11.61	13.50	17.03	15.62	17.59	21.86	18.08	20.14	24.71
0.10	12.31	14.14	18.05	16.39	18.33	22.52	18.92	21.02	25.20
0.05	13.23	14.93	18.91	18.54	20.91	27.12	20.88	23.11	28.78
		$p=3$			$p=4$			$p=5$	
0.50	12.03	14.03	18.37	13.86	15.87	20.41	15.44	17.52	22.52
0.49	13.24	15.17	19.97	15.10	17.16	21.77	16.81	18.81	23.95
0.48	13.86	15.82	20.68	15.67	17.82	22.53	17.43	19.51	24.71
0.47	14.31	16.34	21.00	16.18	18.35	23.11	17.87	20.08	25.16
0.45	15.08	17.05	21.93	16.93	19.23	23.88	18.65	20.92	26.25
0.40	16.37	18.44	23.23	18.33	20.72	25.07	20.16	22.39	27.66
0.35	17.40	19.55	24.24	19.33	20.72	25.07	21.25	23.43	28.90
0.30	18.29	20.40	24.89	20.29	22.38	26.95	22.25	24.44	29.48
0.25	19.06	21.14	25.60	21.08	23.12	27.72	23.06	25.32	30.08
0.20	19.85	21.88	26.02	21.81	23.99	28.63	23.89	26.19	30.88
0.15	20.50	22.67	26.86	22.61	24.78	29.54	24.80	26.93	31.86
0.10	21.28	23.34	27.71	23.48	25.56	30.65	25.65	27.88	32.52
0.05	22.98	25.27	30.80	25.04	27.48	33.21	27.18	29.54	34.75
		$p=6$			$p=7$			$p=8$	
0.50	17.13	19.36	24.12	18.57	20.88	26.12	20.20	22.55	27.61
0.49	18.53	20.81	25.89	20.07	22.30	28.01	21.66	24.22	29.52
0.48	19.17	21.50	26.64	20.76	22.98	28.78	22.39	24.86	30.33
0.47	19.69	22.00	26.81	21.32	23.55	29.33	23.00	25.48	30.71
0.45	20.45	22.77	27.62	22.08	24.45	29.93	23.89	26.47	31.50
0.40	22.05	24.29	29.46	23.72	26.16	31.48	25.45	28.07	33.56
0.35	23.07	25.36	30.61	24.86	27.36	32.33	26.75	29.21	34.73
0.30	24.03	26.42	31.62	26.01	28.39	33.38	27.81	30.17	35.52
0.25	24.96	27.10	32.45	26.79	29.40	34.42	28.73	31.11	36.20
0.20	25.87	28.07	33.26	27.77	30.16	35.19	29.58	32.07	37.17
0.15	26.63	28.85	33.78	28.67	30.99	35.93	30.53	32.94	38.13
0.10	27.50	29.78	34.67	29.53	31.87	36.73	31.43	33.68	39.01
0.05	29.24	31.62	37.32	31.12	33.55	38.54	33.01	35.50	41.53

$$(\psi_0, \psi_1, \psi_2) = (1.0, -0.3, 0.04) \text{ and } (\phi_0, \phi_1, \phi_2) = \kappa(1.0, 1.0, 1.0),$$

where $\kappa = \{-0.8, -0.6, \ldots, 0.6, 0.8\}$, and the break point occurs at time $k = N/5$, $N/2$ and $4N/5$ respectively. The critical values of our test are taken from Tables 1 and 2. The empirical power and sizes of three tests are reported in Tables 3 and 4 when $\pi_0 = 0.10$ and 0.25 respectively. Their sizes correspond to the cases when $\kappa = 0.0$.

From Tables 3 and 4, it is clear that the sizes of these tests are close to their nominal ones. LR_n and $\widehat{\mathrm{LR}}_n$ have almost the same power when the break occurs early. This is reasonable since both of them have almost the same alternative in this case. $\widetilde{\mathrm{LR}}_n$ performs worse for these early breaks. When the break occurs in the middle or the end of the sample, LR_n is the most powerful one. In particular, when $k = 4N/5$ and κ is small, the power of LR_n is much larger than those of $\widehat{\mathrm{LR}}_n$ and $\widetilde{\mathrm{LR}}_n$. When $k = N/2$, the power of $\widehat{\mathrm{LR}}_n$ is slightly larger than those of $\widetilde{\mathrm{LR}}_n$. When $k = 4N/5$, the power of $\widetilde{\mathrm{LR}}_n$ is slightly larger than those of $\widehat{\mathrm{LR}}_n$. Overall, LR_n performs well in the finite sample and can always be the first choice to detect the break point, especially when both the sample size and κ are small.

We now apply our test to the weekly average closing stock price of China Construction Bank (CCB) Corporation in Hong Kong stock market from 27th October 2005 to 7th June 2010. The sample size is 241. We plot the series of CCB closing price $\{x_t\}$ and its log-return $\{y_t\}$ in Figure 1.

We use LR_n, $\widehat{\mathrm{LR}}_n$ and $\widetilde{\mathrm{LR}}_n$ with $\pi_0 = 0.10$ to test if $\{y_t\}$ has a change from an AR(1) to a threshold AR(1) model. The values of these test statistics are 19.32, 9.82 and 9.35 respectively. Our test rejects the null hypothesis at the significance level $\alpha = 0.05$, while other two tests $\widehat{\mathrm{LR}}_n$ and $\widetilde{\mathrm{LR}}_n$ cannot detect the change. We use model eqn (2) to fit the data and obtain the following model:

$$y_t = \begin{cases} -0.0002 + 0.1421 y_{t-1} + \varepsilon_t, & \text{if } t = 1, \ldots, 156, \\ -0.0002 + 0.1421 y_{t-1} - I(y_{t-1} \le -0.0579)(0.0049 + 0.8142 y_{t-1}) + \varepsilon_t, & \text{if } t = 157, \ldots, 240. \end{cases} \tag{10}$$

Table 2. Percentage points of $LR(\pi_0)$ (con't)

	$p = 9$			$p = 10$			$p = 11$		
π_0	10%	5%	1%	10%	5%	1%	10%	5%	1%
0.50	21.62	24.08	29.37	23.23	25.67	31.14	24.60	27.32	33.50
0.49	23.21	25.73	31.24	24.84	27.28	32.97	26.27	28.96	35.50
0.48	23.93	26.51	32.00	25.58	28.16	33.79	27.05	29.77	36.40
0.47	24.56	27.13	32.49	26.22	28.60	34.18	27.69	30.43	36.98
0.45	25.46	28.17	33.48	27.18	29.60	35.35	28.68	31.46	37.64
0.40	27.18	29.64	35.33	28.81	31.38	37.17	30.37	33.32	39.17
0.35	28.38	31.17	36.59	30.08	32.57	38.05	31.73	34.57	40.56
0.30	29.51	32.13	37.38	31.15	33.64	39.14	32.95	35.68	41.34
0.25	30.44	33.05	38.47	32.20	34.68	40.11	33.92	36.59	42.30
0.20	31.40	33.93	39.29	33.15	35.66	40.99	34.77	37.43	43.17
0.15	32.32	34.85	40.30	34.09	36.71	41.67	35.68	38.31	43.89
0.10	33.29	35.66	40.95	35.07	37.56	42.66	36.78	39.37	44.75
0.05	34.85	37.32	43.26	36.68	39.20	44.43	38.40	41.08	46.51
		$p = 12$			$p = 13$			$p = 14$	
0.50	26.22	28.90	34.34	27.56	30.03	35.61	29.00	31.73	37.90
0.49	27.88	30.69	36.26	29.33	31.96	37.38	30.80	33.68	39.49
0.48	28.75	31.52	36.90	30.14	32.75	38.38	31.59	34.35	40.60
0.47	29.29	32.11	37.59	30.73	33.43	38.94	32.15	35.05	41.33
0.45	30.34	33.16	38.78	31.79	34.56	39.95	33.28	36.04	42.16
0.40	31.99	34.80	41.40	33.59	36.32	41.95	35.14	37.89	44.40
0.35	33.30	36.05	42.34	34.98	37.78	43.52	36.49	39.52	45.52
0.30	34.46	37.00	43.36	36.18	38.89	44.78	37.67	40.54	46.70
0.25	35.33	38.13	44.26	37.13	40.02	45.73	38.78	41.70	47.31
0.20	36.30	39.12	44.90	38.17	40.81	46.82	39.84	42.69	48.41
0.15	37.32	40.03	45.46	39.16	41.63	47.39	40.88	43.68	49.57
0.10	38.40	41.15	46.36	40.21	42.64	48.39	41.91	44.70	50.51
0.05	40.31	43.00	48.14	41.76	44.53	50.08	43.69	46.22	52.00
		$p = 16$			$p = 18$			$p = 20$	
0.50	31.65	34.44	40.13	34.29	37.36	43.28	36.92	40.08	46.32
0.49	33.63	36.43	42.00	36.17	39.28	45.32	38.78	42.21	48.77
0.48	34.47	37.21	43.03	37.21	40.33	46.25	39.76	42.97	49.99
0.47	35.15	38.01	43.68	37.96	40.95	46.89	40.51	43.83	50.64
0.45	36.22	39.07	44.56	38.98	41.96	48.06	41.64	44.81	52.24
0.40	38.28	41.10	47.04	41.19	43.92	50.25	43.76	46.99	54.09
0.35	39.76	42.48	48.45	42.54	45.42	52.02	45.29	48.43	55.35
0.30	40.99	43.57	49.28	43.76	46.77	52.92	46.76	49.86	56.49
0.25	42.01	44.63	50.42	44.85	47.75	53.87	47.95	50.85	57.77
0.20	42.94	45.73	51.65	46.06	48.95	54.60	49.09	52.00	58.57
0.15	43.89	46.65	52.34	47.21	50.06	55.54	50.28	53.39	59.63
0.10	45.07	47.97	53.32	48.42	51.05	56.34	51.40	54.52	60.63
0.05	46.98	49.77	55.05	50.06	52.81	58.28	53.16	56.13	61.89

Table 3. Empirical power when $\pi_0 = 0.10$

		$k = N/5$			$k = N/2$			$k = 4N/5$		
κ	N	LR_n	$\widehat{LR}_n$	$\widetilde{LR}_n$	LR_n	$\widehat{LR}_n$	$\widetilde{LR}_n$	LR_n	$\widehat{LR}_n$	$\widetilde{LR}_n$
−0.8	200	0.995	0.989	0.352	0.965	0.813	0.473	0.604	0.248	0.351
−0.6	200	0.924	0.880	0.192	0.740	0.470	0.302	0.344	0.135	0.231
−0.4	200	0.478	0.482	0.096	0.326	0.184	0.161	0.132	0.064	0.106
−0.2	200	0.127	0.138	0.065	0.098	0.091	0.076	0.081	0.064	0.069
0.0	200	0.040	0.049	0.051	0.035	0.047	0.052	0.033	0.040	0.050
0.2	200	0.169	0.138	0.072	0.120	0.082	0.077	0.117	0.050	0.071
0.4	200	0.695	0.715	0.155	0.539	0.350	0.226	0.272	0.098	0.189
0.6	200	0.986	0.994	0.347	0.946	0.855	0.582	0.622	0.334	0.430
0.8	200	0.981	1.00	0.797	0.989	0.990	0.901	0.884	0.692	0.725
−0.8	400	1.00	1.00	0.558	1.00	0.973	0.820	0.936	0.506	0.631
−0.6	400	1.00	0.992	0.330	0.990	0.798	0.585	0.720	0.217	0.415
−0.4	400	0.891	0.823	0.190	0.725	0.383	0.290	0.293	0.095	0.182
−0.2	400	0.223	0.242	0.069	0.168	0.114	0.102	0.073	0.051	0.077
0.0	400	0.044	0.040	0.049	0.042	0.049	0.048	0.041	0.048	0.059
0.2	400	0.239	0.256	0.078	0.199	0.126	0.144	0.087	0.065	0.090
0.4	400	0.979	0.967	0.245	0.910	0.632	0.485	0.461	0.171	0.325
0.6	400	0.999	1.00	0.670	1.00	0.990	0.893	0.900	0.534	0.706
0.8	400	0.998	1.00	0.979	1.00	1.00	0.994	0.986	0.897	0.936

Table 4. Empirical power when $\pi_0 = 0.10$

		$k = N/5$			$k = N/2$			$k = 4N/5$		
κ	N	LR_n	$\widehat{LR}_n$	$\widetilde{LR}_n$	LR_n	$\widehat{LR}_n$	$\widetilde{LR}_n$	LR_n	$\widehat{LR}_n$	$\widetilde{LR}_n$
−0.8	200	0.991	0.976	0.364	0.957	0.775	0.545	0.577	0.293	0.291
−0.6	200	0.915	0.872	0.154	0.794	0.451	0.335	0.291	0.105	0.154
−0.4	200	0.486	0.482	0.091	0.388	0.214	0.201	0.103	0.058	0.098
−0.2	200	0.097	0.121	0.072	0.093	0.087	0.094	0.051	0.067	0.064
0.0	200	0.044	0.064	0.040	0.033	0.051	0.038	0.035	0.043	0.049
0.2	200	0.106	0.122	0.068	0.096	0.082	0.098	0.054	0.065	0.078
0.4	200	0.597	0.574	0.135	0.478	0.317	0.292	0.186	0.101	0.145
0.6	200	0.970	0.959	0.358	0.920	0.772	0.616	0.525	0.304	0.330
0.8	200	0.999	1.00	0.765	0.992	0.975	0.898	0.823	0.618	0.612
−0.8	400	1.00	1.00	0.543	1.00	0.975	0.848	0.854	0.457	0.509
−0.6	400	0.998	0.995	0.283	0.985	0.781	0.621	0.585	0.188	0.290
−0.4	400	0.896	0.803	0.159	0.710	0.349	0.343	0.235	0.090	0.154
−0.2	400	0.233	0.228	0.067	0.160	0.100	0.117	0.082	0.063	0.074
0.0	400	0.053	0.067	0.044	0.039	0.054	0.053	0.033	0.044	0.059
0.2	400	0.254	0.246	0.079	0.218	0.108	0.139	0.082	0.066	0.068
0.4	400	0.935	0.897	0.232	0.851	0.556	0.510	0.375	0.144	0.252
0.6	400	0.999	0.997	0.558	1.00	0.971	0.910	0.857	0.527	0.615
0.8	400	1.00	1.00	0.929	1.00	0.999	0.997	0.983	0.886	0.902

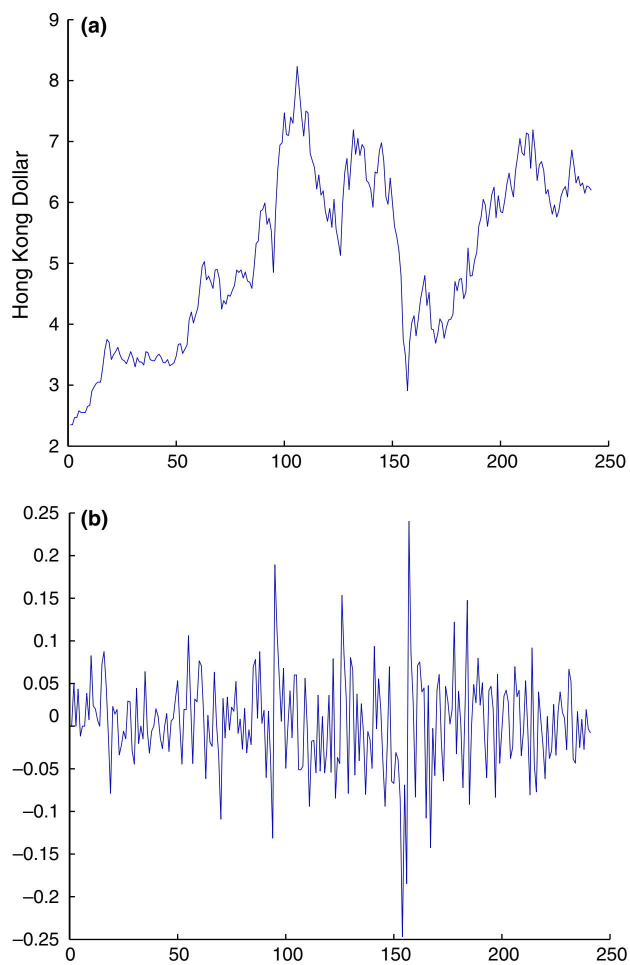

Figure 1. Weekly average closing price of CCB (top) and its log-return (bottom) from 27th October 2005 to 7th June 2010

The Figure 2 is the plot of residuals. The Ljung–Box test Q(20) shows that model (10) is adequate at the significance level $\alpha = 0.05$, and the Kolmogorov–Smirnov test cannot reject the normality hypothesis at the significance level $\alpha = 0.05$. In model eqn (10), we find that the change occurred in the first week in October 2008. It is well known that the bankruptcy of Lehman Brothers happened on 15th September 2008, and the credit crunch spread all of the world around this period. All the stock markets including the Hong

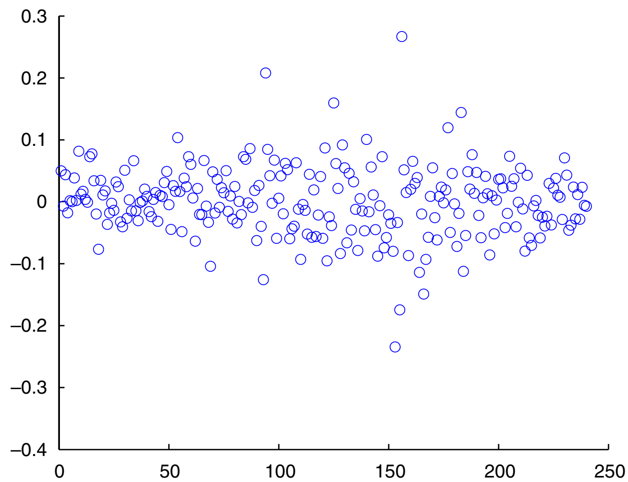

Figure 2. Residuals plot

Kong market experienced a huge change after that. The CCB Corporation is the third biggest constituent stock of Hang Seng Index. Its structure is certainly changed because of this financial tsunami. Our finding matches to this real event.

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APPENDIX

In this Appendix, we give the proof of Theorem 1. To be simple, we assume that $l = 1$ without loss of generality. By eqn (4) and a two-term Taylor expansion, we have

$$\mathrm{LR}_n(k,r) = \frac{1}{\hat{\sigma}_n^2} W_n(k,r) + \frac{1}{2n\xi_n^2(k,r)} W_n^2(k,r), \tag{A1}$$

where $W_n(k,r) = n\{\hat{\sigma}_n^2 - \hat{\sigma}_n^2(k,r)\}$ and $\xi_n^2(k,r)$ is between $\hat{\sigma}_n^2$ and $\hat{\sigma}_n^2(k,r)$. By eqns (5) and (6), we can show that

$$W_n(k,r) = T'_{kr} \left\{ \frac{X'_{kr}X_{kr}}{n} - \frac{X'_{kr}X}{n} \left(\frac{X'X}{n} \right)^{-1} \frac{X'X_{kr}}{n} \right\}^{-1} T_{kr}, \tag{A2}$$

where $T_{kr} = n^{-1/2}\{X'_{kr} - X'_{kr}X(X'X)^{-1}X'\}Y$. We now rescale the time axis by setting $k = \lfloor n\pi \rfloor$ with $\pi \in [0,1]$, and denote

$$T^*_{\pi r} \triangleq T_{\lfloor n\pi \rfloor, r} \text{ and } X^*_{\pi r} = X_{\lfloor n\pi \rfloor, r}.$$

The following lemma tells us that each component of $T^*_{\pi r}$ is a normalized sum of a martingale difference sequence.

Lemma A.1. *Let Σ and $\Sigma_{\pi r}$ be defined as in Theorem 1. Under H_0, we have*

(i) $$\sup_{0 \le \pi \le \pi_1} \sup_{r_1 \le r \le r_2} \left| \left\{ \frac{X^{*\prime}_{\pi r} X^*_{\pi r}}{n} - \frac{X^{*\prime}_{\pi r} X}{n} \left(\frac{X'X}{n} \right)^{-1} \frac{X'X^*_{\pi r}}{n} \right\}^{-1} - (\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r})^{-1} \right| = o_p(1),$$

(ii) $$\sup_{0 \le \pi \le \pi_1} \sup_{r_1 \le r \le r_2} \left| T^*_{\pi r} - \frac{1}{\sqrt{n}} X^{*\prime}_{\pi r}\varepsilon + \frac{1}{\sqrt{n}} \Sigma_{\pi r}\Sigma^{-1}X'\varepsilon \right| = o_p(1).$$

Proof. First, by Lemma 2.1 (i) and (iii) in Chan (1990), it follows that Σ is invertible, and for every $(\pi,r) \in [0,\pi_1] \times [r_1,r_2]$, $\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r}$ is positive definite. Second, by the ergodic theorem, it is easy to see that

$$\frac{X'X}{n} \to \Sigma \text{ almost surely, as } n \to \infty. \tag{A3}$$

Third, a similar argument as for Lemma 2 in Berkes *et al.* (2010), we can show that

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\left|\frac{X^{*}_{\pi r}{}'X}{n}-\Sigma_{\pi r}\right| = o_p(1), \tag{A4}$$

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\left|\frac{X^{*}_{\pi r}{}'X^{*}_{\pi r}}{n}-\Sigma_{\pi r}\right| = o_p(1). \tag{A5}$$

Note that $T^{*}_{\pi r} = n^{-1/2}\{X^{*}_{\pi r}{}' - X^{*}_{\pi r}{}'X(X'X)^{-1}X'\}\varepsilon$ if H_0 holds. Then, (i) and (ii) follow readily from eqns (A.3)–(A.5). This completes the proof. □QED

Let D_{p+1}([0,1] *times* $(-\infty,\infty)$) denote the function spaces of all functions, mapping R([0,1] *times* $[r_1,r_2]$) into R^{p+1}, that are right continuous and have left-hand limits. Typically, we equip D_{p+1}([0,1] *times* $(-\infty,\infty)$) with the topology of uniform convergence over compact sets, see Pollard (1984) for more details on these spaces. The following Lemma gives the weak convergence of $T^{*}_{\pi r}$.

Lemma A.2. *If Assumptions* 2.1 *and* 2.2 *hold, then under* H_0, $\{T^{*}_{\pi r}\}$ *converges weakly to* $\{\sigma\xi_{\pi r}\}$ *in* $D_{p+1}([0,1] \times ()\infty,\infty))$, *where* $\{\xi_{\pi r}\}$ *be defined as in Theorem* 1.

Proof. It is straightforward to show that the finite dimensional distribution of $\{T^{*}_{\pi r}\}$ converges to this of $\{\sigma\xi_{\pi r}\}$. By Assumptions 2.1 and 2.2 and using Theorem 3 of Berkes *et al.* (2010), each component of $\{T^{*}_{\pi r}\}$ is tight, and hence $\{T^{*}_{\pi r}\}$ is tight. Thus, the conclusion holds.

Proof of Theorem 1. *Note that* $k = \lfloor n\pi \rfloor$ *with* $\pi \in [0,1]$. *Then,*

$$\max_{0\le k\le k_1}\sup_{r_1\le r\le r_2} \mathrm{LR}_n(k,r) = \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\mathrm{LR}_n(\lfloor n\pi\rfloor, r). \tag{A6}$$

By Lemma A.2, we can see that

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} T^{*}_{\pi r} = O_p(1).$$

Combining it with eqn (A2) and Lemma A.1 (i), it follows that

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} W_n(\lfloor n\pi\rfloor, r) = \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} T^{*}_{\pi r}{}'(\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma_{\pi r}{}')^{-1}T^{*}_{\pi r} + o_p(1). \tag{A7}$$

Define the functional

$$L : x(\cdot) \in D_{p+1}([0,\pi_1]\times[r_1,r_2]) \to \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} x(\pi,r)'(\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r})^{-1}x(\pi,r).$$

Now, $\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r}$ is a continuous matrix function over $[0,\pi_1]\times[r_1,r_2]$, and $\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}|x(\pi,r)|<\infty$, thus L is a continuous functional. By Lemma A.2, it yields that $L(T^{*}_{\pi r})$ converges weakly to $L(\sigma\xi_{\pi r})$. Then, by eqn (A.7), it follows that

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} W_n(\lfloor n\pi\rfloor, r) \to^{d} \sigma^2 \sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} \xi'_{\pi r}(\Sigma_{\pi r} - \Sigma_{\pi r}\Sigma^{-1}\Sigma'_{\pi r})^{-1}\xi_{\pi r}, \tag{A8}$$

which entails that

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2} W_n(\lfloor n\pi\rfloor, r) = O_p(1). \tag{A9}$$

Thus, we can conclude that

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\left|\hat\sigma^2_n(\lfloor n\pi\rfloor, r) - \hat\sigma^2_n\right| = o_p(1).$$

Note that $\xi^2_n(\lfloor n\pi\rfloor, r)$ is between $\hat\sigma^2_n$ and $\hat\sigma^2_n(\lfloor n\pi\rfloor, r)$. Since $\hat\sigma^2_n$ *to* σ^2 in probability, see Brockwell and Davis (1991), by the previous result, we have

$$\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\frac{1}{\xi^2_n(\lfloor n\pi\rfloor, r)} = O_p(1). \tag{A10}$$

Therefore, by eqns (A.9) and (A.10), it follows that

$$\frac{1}{2n}\sup_{0\le\pi\le\pi_1}\sup_{r_1\le r\le r_2}\frac{1}{\xi_n^2(\lfloor n\pi\rfloor,r)}W_n^2(\lfloor n\pi\rfloor,r)=o_p(1). \tag{A11}$$

Now, the conclusion follows from eqns (A.1), (A.6), (A.8) and (A.11). This completes the proof. □

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