Exercise 1.3

1. Let $f : \{1,3,4\} \to \{1,2,5\}$ and $g : \{1,2,5\} \to \{1,3\}$ be given by:

$$f = \{(1,2),(3,5),(4,1)\}, \quad g = \{(1,3),(2,3),(5,1)\}$$

Write down $g \circ f$.

2. Let f, g, h be functions from $\mathbb{R}$ to $\mathbb{R}$. Show that:

$$(f+g) \circ h = f \circ h + g \circ h$$

$$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

3. Find $g \circ f$ and $f \circ g$ if:

(i) $f(x) = |x|$, $g(x) = |5x - 2|$

(ii) $f(x) = 8x^3$, $g(x) = \frac{1}{x^3}$

4. If $f(x) = \frac{4x+3}{6x-4}$, $x \neq \frac{2}{3}$, show that $f \circ f(x) = x$.

5. Determine whether the following functions have an inverse:

(i) $f : \{1,2,3,4\} \to \{10\}$ with $f = \{(1,10),(2,10),(3,10),(4,10)\}$

(ii) $g : \{5,6,7,8\} \to \{1,2,3,4\}$ with $g = \{(5,4),(6,3),(7,4),(8,2)\}$

(iii) $h : \{2,3,4,5\} \to \{7,9,11,13\}$ with $h = \{(2,7),(3,9),(4,11),(5,13)\}$

6. Show that $f : [-1,1] \to \mathbb{R}$, given by $f(x) = \frac{x}{x+2}$, is one-one. Find the inverse of the function $f : [-1,1] \to \text{Range } f$. (Hint: For $y \in \text{Range } f$, solve $y = \frac{x}{x+2}$ for x, i.e., $x = \frac{2y}{1-y}$)

7. Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 4x + 3$. Show that f is invertible. Find the inverse of f.

8. Consider $f : \mathbb{R}_+ \to [4, \infty)$ given by $f(x) = x^2 + 4$. Show that f is invertible with the inverse f^{-1} of f given by $f^{-1}(y) = \sqrt{y-4}$, where $\mathbb{R}_+$ is the set of all non-negative real numbers.

9. Consider $f : \mathbb{R}_+ \to [-5, \infty)$ given by $f(x) = 9x^2 + 6x - 5$. Show that f is invertible with:

$$f^{-1}(y) = \frac{\sqrt{y+6} - 1}{3}$$

10. Let $f : X \to Y$ be an invertible function. Show that f has a unique inverse. (Hint: Suppose g_1 and g_2 are two inverses of f. Then for all $y \in Y$, $f \circ g_1(y) = 1_Y(y) = f \circ g_2(y)$. Use the one-one property of f.)

11. Consider $f : \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a, f(2) = b, f(3) = c$. Find f^{-1} and show that $(f^{-1})^{-1} = f$.

12. Let $f : X \to Y$ be an invertible function. Show that the inverse of f^{-1} is f, i.e., $(f^{-1})^{-1} = f$.

13. If $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (3 - x^3)^{1/3}$, then $f \circ f(x)$ is:

$$\text{(A) } \frac{1}{x^3} \quad \text{(B) } x^3 \quad \text{(C) } x \quad \text{(D) } (3 - x^3)$$

14. Let $f : \mathbb{R} - \left\{-\frac{4}{3}\right\} \to \mathbb{R}$ be a function defined as $f(x) = \frac{4x}{3x+4}$. The inverse of f is the map $g : \text{Range } f \to \mathbb{R} - \left\{-\frac{4}{3}\right\}$ given by:

$$\text{(A) } g(y) = \frac{3y}{3-4y} \quad \text{(B) } g(y) = \frac{4y}{4-3y} \quad \text{(C) } g(y) = \frac{4y}{3-4y} \quad \text{(D) } g(y) = \frac{3y}{4-3y}$$