Sample Weights

Weighting Observations by Uniqueness (1/2)

- Two labels y_i and y_j are concurrent at *t* when both are a function of at least one common return, $r_{t-1,t} = \frac{p_t}{p_{t-1}} - 1$.

1. For each observation $t = 1, \dots, T$ we form a binary array, $\{1_{t,i}\}_{i=1,\dots,I}$, with $1_{t,i} \in \{0,1\}$, which indicates whether its outcome spans over return $r_{t-1,t}$.
2. We compute the number of labels concurrent at *t*, $c_t = \sum_{i=1}^{I} 1_{t,i}$.
3. The uniqueness of a label *i* at time t is $u_{t,i} = 1_{t,i} c_t^{-1}$.
4. The average uniqueness of label *i* is the average $u_{t,i}$ over the label's lifespan, $\bar{u}_i = \left(\sum_{t=1}^{T} u_{t,i}\right)\left(\sum_{t=1}^{T} 1_{t,i}\right)^{-1}$.

Weighting Observations by Uniqueness (2/2)

5. Sample weights can be defined as the sum of the attributed absolute log returns, $\left|r_{t_{i-1},t_i}\right|$, over the event's lifespan, $[t_{i,0}, t_{i,1}]$,

$$\widetilde{w}_i = \left| \sum_{t=t_{i,0}}^{t_{i,1}} \frac{r_{t-1,t}}{c_t} \right|_1$$

$$w_i = \widetilde{w}_i I \left(\sum_{j=1}^{I} \widetilde{w}_j \right)$$

- The rationale for this method is that we weight an observation as a function of the absolute log returns that can be attributed *uniquely* to it.
- We can use these weights for sequential bootstrap (section 4.5).

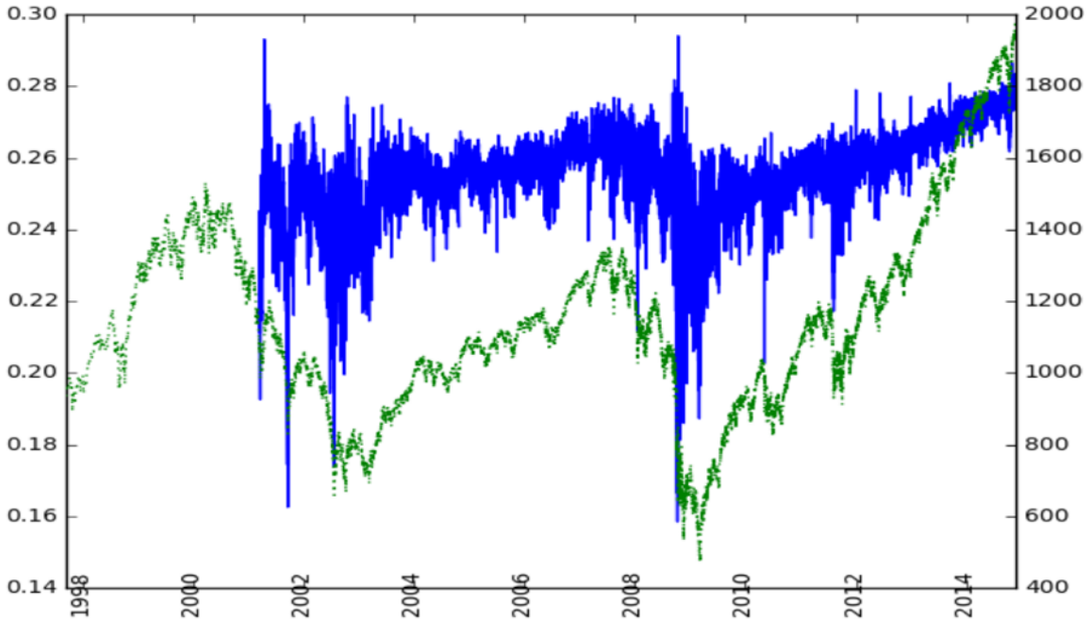
Fractionally Differentiated Features

The Stationary vs. Memory Dilemma

- In order to perform inferential analyses, researchers need to work with invariant processes, such as
  - returns on prices (or changes in log-prices)
  - changes in yield
  - changes in volatility
- These operations make the series stationary, at the expense of removing all memory from the original series.
- Memory is the basis for the model's predictive power.
  - For example, equilibrium (stationary) models need some memory to assess how far the price process has drifted away from the long-term expected value in order to generate a forecast.
- The dilemma is
  - returns are stationary however memory-less; and
  - prices have memory however they are non-stationary.

The Optimal Stationary-Memory Trade Off

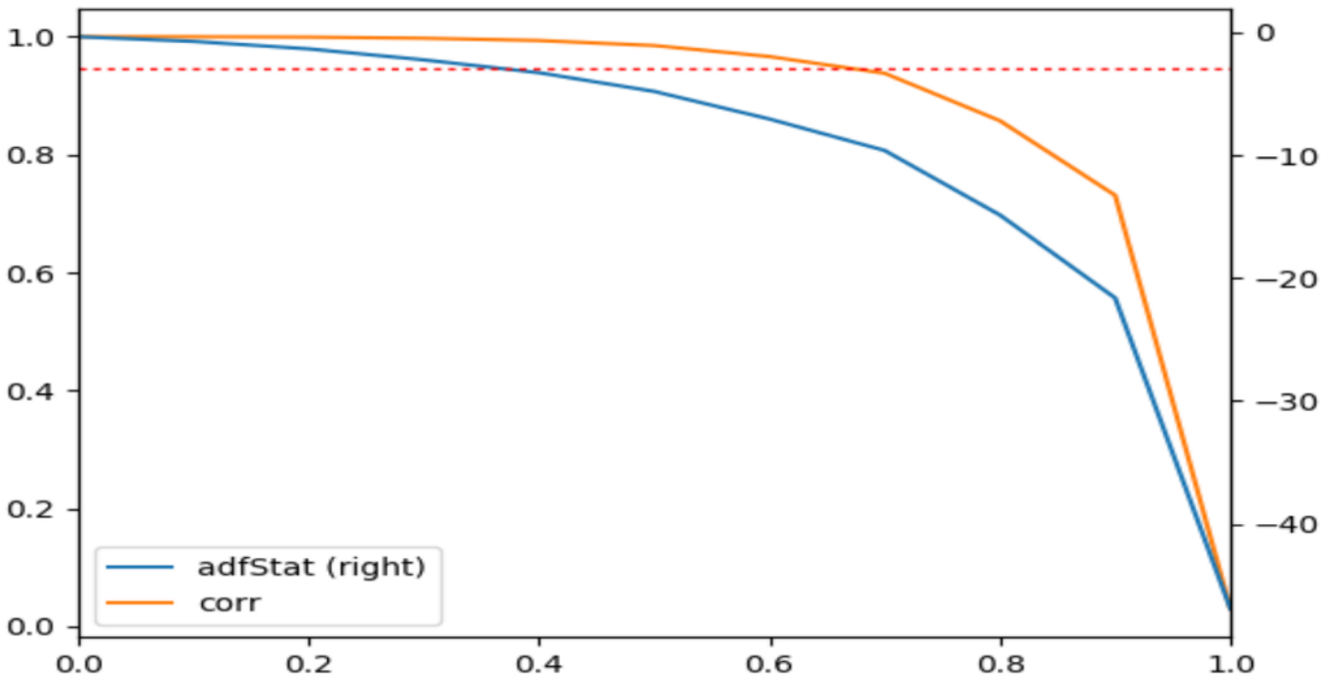
- Question: What is the minimum amount of differentiation that makes a price series stationary while preserving as much memory as possible?
- Answer: We would like to generalize the notion of returns to consider stationary series where not all memory is erased.
- Under this framework, returns are just one kind of (and in most cases suboptimal) price transformation among many other possible.



- Green line: E-mini S&P 500 futures trade bars of size 1E4
- Blue line: Fractionally differentiated ($d = .4$)
- Over a short time span, it resembles returns
- Over a longer time span, it resembles price levels

Example 1: S&P 500 Futures

- On the x-axis, the *d* value used to generate the series on which the ADF stat was computed.
- On the left y-axis, the correlation between the original series ($d = 0$) and the differentiated series at various *d* values.
- On the right y-axis, ADF stats computed on log prices.



The original series ($d = 0$) has an ADF stat of -0.3387, while the returns series ($d = 1$) has an ADF stat of -46.9114.

At a 95% confidence level, the test's critical value is -2.8623.

The ADF stat crosses that threshold in the vicinity of $d = 0.35$, where correlation is still very high (0.995).

Example 2: Optimal FradDiff Stationarity (1/2)

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
AD1 Curncy	-1.7253	-1.8665	-2.2801	-2.9743	-3.9590	-5.4450	-7.7387	-10.3412	-15.7255	-22.5170	-43.8281
BO1 Comdty	-0.7039	-1.0021	-1.5848	-2.4038	-3.4284	-4.8916	-7.0604	-9.5089	-14.4065	-20.4393	-38.0683
BP1 Curncy	-1.0573	-1.4963	-2.3223	-3.4641	-4.8976	-6.9157	-9.8833	-13.1575	-19.4238	-26.6320	-43.3284
BTS1 Comdty	-1.7987	-2.1428	-2.7600	-3.7019	-4.8522	-6.2412	-7.8115	-9.4645	-11.0334	-12.4470	-13.6410
BZ1 Index	-1.6569	-1.8766	-2.3948	-3.2145	-4.2821	-5.9431	-8.3329	-10.9046	-15.7006	-20.7224	-29.9510
C 1 Comdty	-1.7870	-2.1273	-2.9539	-4.1642	-5.7307	-7.9577	-11.1798	-14.6946	-20.9925	-27.6602	-39.3576
CC1 Comdty	-2.3743	-2.9503	-4.1694	-5.8997	-8.0868	-10.9871	-14.8206	-18.6154	-24.1738	-29.0285	-34.8580
CD1 Curncy	-1.6304	-2.0557	-2.7284	-3.8380	-5.2341	-7.3172	-10.3738	-13.8263	-20.2897	-27.6242	-43.6794
CF1 Index	-1.5539	-1.9387	-2.7421	-3.9235	-5.5085	-7.7585	-11.0571	-14.6829	-21.4877	-28.9810	-44.5059
CL1 Comdty	-0.3795	-0.7164	-1.3359	-2.2018	-3.2603	-4.7499	-6.9504	-9.4531	-14.4936	-20.8392	-41.1169
CN1 Comdty	-0.8798	-0.8711	-1.1020	-1.4626	-1.9732	-2.7508	-3.9217	-5.2944	-8.4257	-12.7300	-42.1411
CO1 Comdty	-0.5124	-0.8468	-1.4247	-2.2402	-3.2566	-4.7022	-6.8601	-9.2836	-14.1511	-20.2313	-39.2207
CT1 Comdty	-1.7604	-2.0728	-2.7529	-3.7853	-5.1397	-7.1123	-10.0137	-13.1851	-19.0603	-25.4513	-37.5703
DM1 Index	-0.1929	-0.5718	-1.2414	-2.1127	-3.1765	-4.6695	-6.8852	-9.4219	-14.6726	-21.5411	-49.2663
DU1 Comdty	-0.3365	-0.4572	-0.7647	-1.1447	-1.6132	-2.2759	-3.3389	-4.5689	-7.2101	-10.9025	-42.9012
DX1 Curncy	-1.5768	-1.9458	-2.7358	-3.8423	-5.3101	-7.3507	-10.3569	-13.6451	-19.5832	-25.8907	-37.2623
EC1 Comdty	-0.2727	-0.6650	-1.3359	-2.2112	-3.3112	-4.8320	-7.0777	-9.6299	-14.8258	-21.4634	-44.6452
EC1 Curncy	-1.4733	-1.9344	-2.8507	-4.1588	-5.8240	-8.1834	-11.6278	-15.4095	-22.4317	-30.1482	-45.6373
ED1 Comdty	-0.4084	-0.5350	-0.7948	-1.1772	-1.6633	-2.3818	-3.4601	-4.7041	-7.4373	-11.3175	-46.4487
EE1 Curncy	-1.2100	-1.6378	-2.4216	-3.5470	-4.9821	-7.0166	-9.9962	-13.2920	-19.5047	-26.5158	-41.4672
EO1 Comdty	-0.7903	-0.8917	-1.0551	-1.3465	-1.7302	-2.3500	-3.3068	-4.5136	-7.0157	-10.6463	-45.2100
EO1 Index	-0.6561	-1.0567	-1.7409	-2.6774	-3.8543	-5.5096	-7.9133	-10.5674	-15.6442	-21.3066	-35.1397
ER1 Comdty	-0.1970	-0.3442	-0.6334	-1.0363	-1.5327	-2.2378	-3.2819	-4.4647	-7.1031	-10.7389	-40.0407
ES1 Index	-0.3387	-0.7206	-1.3324	-2.2252	-3.2733	-4.7976	-7.0436	-9.6095	-14.8624	-21.6177	-46.9114
FA1 Index	-0.5292	-0.8526	-1.4250	-2.2359	-3.2500	-4.6902	-6.8272	-9.2410	-14.1664	-20.3733	-41.9705
FC1 Comdty	-1.8846	-2.1853	-2.8808	-3.8546	-5.1483	-7.0226	-9.6889	-12.5679	-17.8160	-23.0530	-31.6503
FV1 Comdty	-0.7257	-0.8515	-1.0596	-1.4304	-1.8312	-2.5302	-3.6296	-4.9499	-7.8292	-12.0467	-49.1508
G 1 Comdty	0.2326	0.0026	-0.4686	-1.0590	-1.7453	-2.6761	-4.0336	-5.5624	-8.8575	-13.3277	-42.9177
GC1 Comdty	-2.2221	-2.3544	-2.7467	-3.4140	-4.4861	-6.0632	-8.4803	-11.2152	-16.7111	-23.1750	-39.0715
GX1 Index	-1.5418	-1.7749	-2.4666	-3.4417	-4.7321	-6.6155	-9.3667	-12.5240	-18.6291	-25.8116	-43.3610
HG1 Comdty	-1.7372	-2.1495	-2.8323	-3.9090	-5.3257	-7.3805	-10.4121	-13.7669	-19.8902	-26.5819	-39.3267
HI1 Index	-1.8289	-2.0432	-2.6203	-3.5233	-4.7514	-6.5743	-9.2733	-12.3722	-18.5308	-25.9762	-45.3396
HO1 Comdty	-1.6024	-1.9941	-2.6619	-3.7131	-5.1772	-7.2468	-10.3326	-13.6745	-19.9728	-26.9772	-40.9824
IB1 Index	-2.3912	-2.8254	-3.5813	-4.8774	-6.5884	-9.0665	-12.7381	-16.6706	-23.6752	-30.7986	-43.0687
IK1 Comdty	-1.7373	-2.3000	-2.7764	-3.7101	-4.8686	-6.3504	-8.2195	-9.8636	-11.7882	-13.3983	-14.8391
IR1 Comdty	-2.0622	-2.4188	-3.1736	-4.3178	-5.8119	-7.9816	-11.2102	-14.7956	-21.6158	-29.4555	-46.2683
JA1 Comdty	-2.4701	-2.7292	-3.3925	-4.4658	-5.9236	-8.0270	-11.2082	-14.7198	-21.2681	-28.4380	-42.1937
JB1 Comdty	-0.2081	-0.4319	-0.8490	-1.4289	-2.1160	-3.0932	-4.5740	-6.3061	-9.9454	-15.0151	-47.6037
JE1 Curncy	-0.9268	-1.2078	-1.7565	-2.5398	-3.5545	-5.0270	-7.2096	-9.6808	-14.6271	-20.7168	-37.6954
JG1 Comdty	-1.7468	-1.8071	-2.0654	-2.5447	-3.2237	-4.3418	-6.0690	-8.0537	-12.3908	-18.1881	-44.2884
JO1 Comdty	-3.0052	-3.3099	-4.2639	-5.7291	-7.5686	-10.1683	-13.7068	-17.3054	-22.7853	-27.7011	-33.4658
JY1 Curncy	-1.2616	-1.5891	-2.2042	-3.1407	-4.3715	-6.1600	-8.8261	-11.8449	-17.8275	-25.0700	-44.8394
KC1 Comdty	-0.7786	-1.1172	-1.7723	-2.7185	-3.8875	-5.5651	-8.0217	-10.7422	-15.9423	-21.8651	-35.3354
L 1 Comdty	-0.0805	-0.2228	-0.6144	-1.0751	-1.6335	-2.4186	-3.5676	-4.8749	-7.7528	-11.7669	-44.0349

These tables show ADF stats for the most liquid futures contracts worldwide.

One row per instrument, and one column per differentiation value.

Highlighted in green are ADF values that do not reject the null hypothesis of unit root.

Highlighted in red are ADF values that reject the null hypothesis of unit root.

Example 2: Optimal FradDiff Stationarity (2/2)

	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
LB1 Comdty	-2.0133	-2.2043	-2.7692	-3.7363	-4.9980	-6.8712	-9.5572	-12.5024	-17.7300	-23.1173	-31.9508
LC1 Comdty	-3.0977	-3.2487	-4.0104	-5.1441	-6.8472	-9.1425	-12.4560	-16.0186	-21.8070	-27.1929	-34.2574
LH1 Comdty	-2.4059	-2.5980	-2.6847	-3.0616	-3.7269	-4.8461	-6.6899	-8.8143	-13.3179	-18.6747	-34.4944
MFS1 Index	-1.8618	-2.4061	-3.0316	-4.2111	-5.6544	-8.2728	-11.3954	-14.2083	-19.2276	-23.7318	-29.9174
NG1 Comdty	-1.2022	-1.2278	-1.2971	-1.5259	-1.9188	-2.5619	-3.5688	-4.7757	-7.4392	-11.2455	-41.3164
NI1 Index	-1.0865	-1.4354	-2.1171	-3.0946	-4.3528	-6.1476	-8.8056	-11.7667	-17.6428	-24.6738	-43.8325
NK1 Index	-0.8467	-1.1964	-1.8390	-2.7349	-3.8871	-5.5119	-7.9025	-10.5570	-15.8085	-22.0688	-38.7505
NQ1 Index	0.0153	-0.2883	-0.7985	-1.5227	-2.3900	-3.5965	-5.3719	-7.4372	-11.7580	-17.5718	-47.7300
NX1 Index	-1.2749	-1.6410	-2.3648	-3.4331	-4.8169	-6.8106	-9.7514	-13.0195	-19.3190	-26.5442	-43.2635
O 1 Comdty	-1.9643	-2.3536	-3.1711	-4.4057	-6.0102	-8.3139	-11.6484	-15.2893	-21.7540	-28.5592	-39.9112
OAT1 Comdty	-2.1234	-1.9151	-2.2928	-2.9948	-3.9627	-5.3126	-7.0749	-8.8556	-11.2388	-13.2080	-15.0069
OE1 Comdty	0.1688	-0.0863	-0.4587	-0.8500	-1.3174	-2.0411	-2.9760	-4.0461	-6.4504	-9.8420	-44.0898
PA1 Comdty	-1.4237	-1.6949	-2.2550	-3.1287	-4.2748	-5.9456	-8.4346	-11.2251	-16.6076	-22.8823	-37.8283
PE1 Curncy	-1.7713	-2.1928	-3.0869	-4.3894	-6.0523	-8.4218	-11.9137	-15.7241	-22.6601	-30.1037	-43.8788
PT1 Index	-1.9088	-2.2753	-3.0047	-4.1548	-5.6979	-7.9456	-11.2588	-14.8504	-21.5933	-28.9158	-43.4395
QS1 Comdty	-0.2084	-0.4919	-0.9675	-1.6192	-2.4490	-3.6160	-5.3075	-7.2161	-11.0838	-15.9596	-32.1660
RR1 Comdty	-0.0657	-0.4432	-0.9827	-1.6856	-2.5403	-3.7445	-5.4592	-7.4618	-11.4360	-16.4247	-33.0067
RTA1 Index	-0.4991	-0.8450	-1.4518	-2.2701	-3.3347	-4.8131	-7.0163	-9.4859	-14.4313	-20.5139	-38.4632
RX1 Comdty	0.3374	0.0368	-0.3370	-0.8033	-1.3293	-2.0307	-3.1201	-4.2717	-6.8379	-10.4035	-43.1525
S 1 Comdty	-2.3905	-2.5632	-3.0364	-3.8647	-5.0057	-6.7561	-9.4036	-12.4148	-18.2529	-24.9520	-39.1747
SB1 Comdty	-1.3895	-1.7489	-2.4806	-3.5180	-4.9204	-6.9044	-9.7911	-12.8777	-18.5958	-24.6554	-35.9220
SF1 Curncy	-2.4335	-2.8967	-3.8496	-5.3187	-7.2411	-9.9945	-13.9627	-18.2641	-25.8117	-33.5334	-46.1841
SI1 Comdty	-1.6435	-1.9468	-2.6104	-3.6207	-4.9544	-6.8834	-9.7471	-12.9306	-18.9448	-25.6872	-39.6744
SM1 Comdty	-2.1197	-2.0686	-2.2593	-2.7314	-3.5152	-4.6986	-6.5691	-8.7911	-13.3516	-19.1866	-37.8627
SM1 Index	-1.4716	-1.7336	-2.3942	-3.3732	-4.6921	-6.5834	-9.3968	-12.5018	-18.5601	-25.5175	-42.7253
SP1 Index	-0.5900	-0.9726	-1.6887	-2.6118	-3.7857	-5.4356	-7.8842	-10.6784	-16.4223	-23.8436	-50.2515
ST1 Index	-1.5957	-1.8926	-2.5130	-3.4803	-4.7593	-6.6294	-9.4127	-12.5153	-18.4786	-25.2546	-40.7900
TP1 Index	-1.2901	-1.6144	-2.2911	-3.3049	-4.5946	-6.4768	-9.2514	-12.3480	-18.5256	-25.9865	-46.2311
TU1 Comdty	-0.6340	-0.6768	-0.8529	-1.1306	-1.5256	-2.1951	-3.2065	-4.2674	-6.8060	-10.4758	-48.7361
TW1 Index	-1.1854	-1.5331	-2.2852	-3.3336	-4.6677	-6.5776	-9.3678	-12.4932	-18.5628	-25.6502	-42.5179
TY1 Comdty	-0.8208	-0.9876	-1.2585	-1.6069	-2.1026	-2.8142	-4.0467	-5.4328	-8.6137	-13.1678	-48.6412
UB1 Comdty	-0.3052	-0.5418	-0.9441	-1.4744	-2.1400	-3.0797	-4.4703	-6.0749	-9.4466	-13.9063	-36.3328
US1 Comdty	-0.8071	-1.1082	-1.5195	-2.0586	-2.8385	-4.0023	-5.7401	-7.7040	-12.0160	-18.0689	-47.9605
VG1 Index	-1.9920	-2.4127	-3.3269	-4.7189	-6.5700	-9.1847	-13.0116	-17.1131	-24.4105	-31.9086	-44.9058
VH1 Index	-1.5805	-1.9248	-2.7044	-3.8438	-5.3480	-7.5449	-10.7841	-14.3586	-21.2567	-29.0585	-46.5168
W 1 Comdty	-0.6236	-0.9148	-1.3959	-2.1267	-3.0507	-4.3849	-6.3497	-8.6538	-13.3216	-19.3053	-41.4181
XB1 Comdty	-2.2352	-2.4744	-2.9506	-3.7092	-4.9733	-6.7217	-9.4858	-12.5086	-18.3777	-25.0316	-39.5784
XG1 Comdty	-2.0082	-2.0972	-2.3756	-3.0026	-3.9027	-5.3023	-7.5000	-10.0158	-15.1353	-21.6376	-41.2603
XM1 Comdty	-0.9140	-1.1841	-1.8967	-2.8240	-4.0056	-5.6936	-8.2092	-11.0940	-17.0495	-24.7002	-51.5154
XP1 Index	-1.5053	-1.7699	-2.4437	-3.4436	-4.7258	-6.6019	-9.3891	-12.5294	-18.8368	-26.5249	-48.0102
YM1 Comdty	-1.1028	-1.1658	-1.6422	-2.3731	-3.3197	-4.6849	-6.7878	-9.1765	-14.2354	-20.9065	-49.2648
YS1 Comdty	-1.9101	-2.1735	-2.8727	-3.8500	-5.2679	-7.2488	-10.2821	-13.6430	-19.9992	-27.0788	-41.5913
Z 1 Index	-1.3096	-1.7242	-2.6045	-3.7736	-5.3196	-7.5241	-10.7341	-14.2851	-21.0992	-28.7746	-45.6802

Most financial series can be made stationary with a fractional differentiation of order $d < 0.5$.

However, most financial studies are based on returns, where $d = 1$.

The implication is that for decades most financial research has been based on **over-differentiated (memory-less) series**, leading to spurious forecasts and overfitting.