1 Overview

The idea is to set a small integer S as a threshold for the size of subarrays. Once the size of a subarray in a recursive call of Mergesort is less than or equal to S, the algorithm will switch to Insertion Sort, which is efficient for small-sized input.

Its size is defined as

$$\frac{S}{2} < \frac{n}{2^c} \leq S$$

Where c is the number of times an array is halved

2 Time complexity of Insertion Sort for S

Define m such that:

$$m = \frac{n}{2^c}$$

Average number of key comparisons per iteration:

$$\frac{1}{i}\sum_{j=1}^{i} j$$

There are m-1 iterations, so the total number of key comparisons is:

$$\begin{aligned}
W(m) &= \sum_{i=1}^{m-1} \frac{1}{i} \sum_{j=1}^{i} j \\
&= \sum_{i=1}^{m-1} \frac{1}{i} \left(\frac{i(i+1)}{2}\right) \\
&= \frac{1}{2} \sum_{i=1}^{m-1} (i+1) \\
&= \frac{1}{2} \times \frac{(m+2)(m-1)}{2} \\
&= \frac{m^2}{4} + \frac{m}{4} - \frac{1}{2} \\
&= \frac{(\frac{n^2}{2^c})^2}{4} + \frac{(\frac{n^2}{2^c})}{4} - \frac{1}{2} \\
&= \frac{n^2}{2^{2c+2}} + \frac{n}{2^{c+2}} - \frac{1}{2}
\end{aligned}$$

3 Time complexity of Hybrid Algorithm

Let $n = 2^k$ such that $\frac{n}{2^c} = 2^{k-c}$ and $\frac{S}{2} < 2^{k-c} \leq S$:

$$\begin{aligned}
W(n) &= W(\frac{n}{2}) + W(\frac{n}{2}) + n - 1 \\
&= 2W(\frac{n}{2}) + n - 1 \\
&= 2^1 \times W(2^{k-1}) + 2^k - 1 \\
&= 2(2W(2^{k-2}) + 2^{k-1} - 1) + 2^k - 1 \\
&= 2^2 \times W(2^{k-2}) + 2(2^k) - (1+2) \\
&= 2^2(2W(2^{k-3}) + 2^{k-2} - 1) + 2(2^k) - (1+2) \\
&= 2^3 \times W(2^{k-3}) + 3(2^k) - (1+2+4) \\
&\dots \\
&= 2^c \times W(2^{k-c}) + c(2^k) - (1+2+4+\dots+2^{c-1}) \\
&= 2^c \times W(2^{k-c}) + nc - (2^c - 1)
\end{aligned}$$

As seen above, the time complexity for $W(2^{k-c})$ when the array goes into insertion sort is given by:

$$W(2^{k-c}) = \frac{n^2}{2^{2c+2}} + \frac{n}{2^{c+2}} - \frac{1}{2}$$

Therefore:

$$\begin{aligned}
W(n) &= 2^c \times (\frac{n^2}{2^{2c+2}} + \frac{n}{2^{c+2}} - \frac{1}{2}) + nc - (2^c - 1) \\
&= \frac{n^2}{2^{c+2}} + \frac{n}{2^2} - 2^{c-1} + nc - (2^c - 1)
\end{aligned}$$

Let $\frac{n}{2^c} = S$, $2^c = \frac{n}{S}$, $c = log(\frac{n}{S})$:

$$\begin{aligned}
W(n) &= \frac{nS}{4} + \frac{n}{4} - \frac{n}{2S} + nlog(\frac{n}{S}) - \frac{n}{S} + 1 \\
&= \mathcal{O}(nlog(n))
\end{aligned}$$