The Asymmetric Travelling Salesman Problem in Sparse Digraphs*

Łukasz Kowalik[†]

Konrad Majewski[‡]

Abstract

ASYMMETRIC TRAVELLING SALESMAN PROBLEM (ATSP) and its special case DIRECTED HAMILTONICITY are among the most fundamental problems in computer science. The dynamic programming algorithm running in time $\mathcal{O}^*(2^n)$ developed almost 60 years ago by Bellman, Held and Karp, is still the state of the art for both of these problems.

In this work we focus on *sparse* digraphs.

First, we recall known approaches for Undirected Hamiltonicity and TSP in sparse graphs and we analyse their consequences for Directed Hamiltonicity and ATSP in sparse digraphs, either by adapting the algorithm, or by using reductions. In this way, we get a number of running time upper bounds for a few classes of sparse digraphs, including $\mathcal{O}^*(2^{n/3})$ for digraphs with both out- and indegree bounded by 2, and $\mathcal{O}^*(3^{n/2})$ for digraphs with outdegree bounded by 3.

Our main results are focused on digraphs of bounded average outdegree d. The baseline for ATSP here is a simple enumeration of cycle covers which can be done in time bounded by $\mathcal{O}^*(\mu(d)^n)$ for a function $\mu(d) \leq (\lceil d \rceil!)^{1/\lceil d \rceil}$. One can also observe that DIRECTED HAMILTONICITY can be solved in randomized time $\mathcal{O}^*((2-2^{-d})^n)$ and polynomial space, by adapting a recent result of Björklund [ISAAC 2018] stated originally for UNDIRECTED HAMILTONICITY in sparse bipartite graphs. We present two new deterministic algorithms for ATSP: the first running in time $\mathcal{O}(2^{0.441(d-1)n})$ and polynomial space, and the second in exponential space with running time of $\mathcal{O}^*(\tau(d)^{n/2})$ for a function $\tau(d) \leq d$.





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[†]Institute of Informatics, University of Warsaw, Poland (kowalik@mimuw.edu.pl)

[‡]Faculty of Mathematics, Informatics, and Mechanics, University of Warsaw, Poland (km371194@students.mimuw.edu.pl)

1 Introduction

In the DIRECTED HAMILTONICITY problem, given a directed graph (digraph) G one has to decide if G has a Hamiltonian cycle, i.e., a simple cycle that visits all vertices. In its weighted version, called ATSP, we additionally have integer weights on edges $w: E \to \mathbb{Z}$, and the goal is to find a minimum weight Hamiltonian cycle in G.

The ATSP problem has a dynamic programming algorithm running in time and space $\mathcal{O}^*(2^n)$ due to Bellman [4] and Held and Karp [27]. Gurevich and Shelah [26] obtained the best known polynomial space algorithm, running in time $\mathcal{O}(4^n n^{\log n})$. It is a major open problem whether there is an algorithm in time $\mathcal{O}^*((2-\varepsilon)^n)$ for an $\varepsilon > 0$, even for the unweighted case of DIRECTED HAMILTONICITY. However, there has been a significant progress in answering this question in variants of DIRECTED HAMILTONICITY. Namely, Björklund and Husfeldt [8] showed that the parity of the number of Hamiltonian cycles in a digraph can be determined in time $O(1.619^n)$ and Cygan, Kratsch and Nederlof [16] solved the bipartite case of DIRECTED HAMILTONICITY in time $\mathcal{O}(1.888^n)$, which was later improved to $\mathcal{O}^*(3^{n/2}) = O(1.74^n)$ by Björklund, Kaski and Koutis [11].

Undirected graphs. Even more is known in the undirected setting, where the problems are called Underected Hamiltonicity and TSP. Björklund [5] shows that Underected HAMILTONICITY can be solved in time $\mathcal{O}(1.66^n)$ in general and $\mathcal{O}^*(2^{n/2}) = O(1.42^n)$ in the bipartite case. Very recently, Nederlof [33] showed that the bipartite case of TSP admits an algorithm in time $O(1.9999^n)$, assuming that square matrices can be multiplied in time $O(n^{2+o(1)})$. Finally, there is a number of results for UNDIRECTED HAMILTONICITY and TSP restricted to graphs that are somewhat sparse. An early example is an algorithm of Eppstein [19] for TSP in graphs of maximum degree 3, running in time $O^*(2^{n/3}) = O(1.26^n)$. This result has been later improved and generalized to larger values of maximum degree, we refer the reader to Table 1 for details (Δ denotes the maximum degree). Perhaps the most general measure of graph sparsity is the average degree d. Cygan and Pilipczuk [18] showed that whenever d is bounded, the 2^n barrier for TSP can be broken, although only slightly. More precisely, they proved the bound $0^*(2^{(1-\varepsilon_d)n})$, where $\varepsilon_d = 1/(2^{2d+1} \cdot 20d \cdot e^{e^{20d}})$. We note that although their result was stated for undirected graphs, the same reasoning can be made for digraphs of average total degree (sum of indegree and outdegree). For small values of d, more significant improvements are possible. Namely, by combining the algorithms for UNDIRECTED HAMILTONICITY and TSP parameterized by pathwidth [13, 16] with a bound on pathwidth of sparse graphs [30] we get the upper bound of $O(1.12^{dn})$ and $O(1.14^{dn})$, respectively. For Underected Hamiltonicity, if the input graph is additionally bipartite, Björklund [6] shows the $O^*((2-2^{1-d})^{n/2})$ upper bound.

Directed sparse graphs: hidden results The goal of this paper is to investigate DIRECTED HAMILTONICITY and ATSP in sparse directed graphs. Quite surprisingly, not much results in this topic are stated explicitly. In fact, we were able to find just a few references of this kind: Björklund, Husfeldt, Kaski and Koivisto [9] describe an algorithm for digraphs with total degree bounded by D that works in time $0^*((2 - \varepsilon'_D)^n)$, for $\varepsilon'_D = 2 - (2^{D+1} - 2D - 2)^{1/(D+1)}$. Second, Cygan et al. [17] describe an algorithm for DIRECTED HAMILTONICITY running in time $6^t n^{O(1)}$, where t is the treewidth of the input graph. Finally, Björklund and Williams [12] show a deterministic algorithm which counts Hamiltonian cycles in directed graphs of average degree d in time $2^{n-\Omega(n/d)}$ and exponential space. Very recently,

Graph class	UNDIRECTED HA	MILTONICITY	TRAVELLING SALESMAN PROBLEM		
general	1.66^n	[5]	$ \begin{array}{ c c c c } \hline & 2^n \\ & 4^n n^{\log n} \end{array} $	(4,27) [26]	
bipartite	1.42^n	[5]	$ \begin{array}{c c} 2^n \\ 4^n \end{array} $	(4,27) [34]	
$\Delta = 3$	$1.16^n \qquad \blacksquare \qquad \qquad 1.24^n$	[16] [38]	$ \begin{array}{c c} 1.22^n \\ 1.24^n \end{array} $	[13] [38]	
$\Delta = 4$	$1.51^n \qquad \blacksquare$ $1.59^n \qquad \blacksquare$	[16]+[20] [10]	1.63^{n} 1.70^{n}	[13]+[20] [39]	
$\Delta = 5$	1.63^n	[10]	1.88^{n} 2.35^{n}	[13]+[20] [40]	
any Δ			$(2 - \varepsilon_{\Delta}')^n$	(9)	
$\operatorname{avgdeg} \leq d$	1.12 ^{dn} ● E	[16]+[30]	$\begin{array}{ c c c }\hline & 1.14^{dn} \\ & 2^{(1-\varepsilon_d)n} \\ \hline \end{array}$	• [13]+[30] • [18]	
bipartite avgdeg $\leq d$	$(2-2^{1-d})^{n/2}$	[6]			
pathwidth	3.42 ^{pw} ● ■	1 [16]	4.28 ^{pw}	(13]	
treewidth	4 ^{tw} 🍎 🗓	[17]	9.56 ^{tw}	(13]	

Table 1: Running times (with polynomial factors omitted) of algorithms for *undirected* graphs. Rows marked with denote exponential space algorithms, rows marked with denote Monte Carlo algorithms.

Björklund [7], using a different approach, obtained the same running time for the decision DIRECTED HAMILTONICITY problem, but lowering the space to polynomial, at the cost of using randomization. The authors of these two works have not put an effort to optimize the constants hidden in the Ω notation. By following the analysis in each of these papers as-is, we get the saving term in the exponent at least n/(111d) (for a faster, randomized algorithm) and n/(500d), respectively.

However, one cannot say that nothing more is known, because many results for undirected graphs imply some running time bounds in the directed setting. We devote the first part of this work to investigating such implications. In some cases, the implications are immediate. For example, Gebauer [24,25] shows an algorithm running in time $O^*(3^{n/2}) = O^*(1.74^n)$ that solves TSP in graphs of maximum degree 4. It uses the meet-in-the-middle approach and can be sketched as follows: guess two opposite vertices of the solution cycle, generate a family of paths of length n/2 from each of them (of size at most $3^{n/2}$) and store one of the families in a dictionary to enumerate all complementary pairs of paths in time $O^*(3^{n/2})$. This algorithm,

Graph class	DIRECTED HAMILTONICITY			ASYMMETRIC TRAVELLING SALESMAN PROBLEM		
general	2^n		[3,29,31]	$\begin{array}{ c c } & 2^n \\ & 4^n n^{\log n} \end{array}$		[4, 27] [26]
bipartite	1.74^n	:::	[11]	$ \begin{array}{c c} 2^n \\ 4^n \end{array} $	•	[4, 27] [34]
(2,2)-graphs	1.26^n		(Corollary 2.9)	1.26^{n}		(Corollary 2.9)
$\Delta^+ = 3$	1.74^n	•	(Corollary 2.12)	1.74^n	•	(Corollary 2.12)
$\Delta = 3$	1.13^n		(Corollary 2.11)	1.13^{n}		(Corollary 2.11)
any Δ	$(2-2^{-\Delta/2})^n$!!!	(Theorem 2.14)	$(2-\varepsilon_{\Delta}')^n$	•	[9]
average outdeg $\leq d$	$\begin{array}{ c c c c }\hline & \mu(d)^n \\ 2^{0.441(d-1)n} \\ \hline & \sqrt{\tau(d)}^n \\ & (2-2^{-d})^n \\ & 2^{(1-\Omega(1/d))n} \\ \hline \end{array}$		(Corollary 2.7) (Theorem 1.1) (Theorem 1.2) (Theorem 2.14) [7]	$ \begin{array}{c c} \mu(d)^n \\ 2^{0.441(d-1)n} \\ \sqrt{\tau(d)}^n \\ 2^{(1-\varepsilon_{2d})n} \end{array} $		(Corollary 2.7) (Theorem 1.1) (Theorem 1.2) [18]
treewidth	6 ^{tw}		[17]			

Table 2: Running times (with polynomial factors omitted) of the algorithms for directed graphs. We preserve the notation from Table 1. By Δ^+ we denote maximum outdegree and Δ denotes maximum total degree. Treewidth refers to the underlying undirected graph.

without a change, can be used for ATSP in digraphs of maximum outdegree 3, with the same running time bound (see Theorem 2.12).

The other implications that we found rely on a simple reduction from ATSP to a variant of TSP in bipartite undirected graphs (see Lemma 2.1): replace each vertex v of the input digraph G by two vertices v^{out} , v^{in} joined by an edge of weight 0, and for each edge $(u,v) \in E(G)$ create an edge $u^{\text{out}}v^{\text{in}}$ of the same weight. Then find a lightest Hamiltonian cycle that contains the matching $M = \{v^{\text{out}}v^{\text{in}} \mid v \in V(G)\}$. By applying this reduction to a digraph with both outdegrees and indegrees bounded by 2, which we call a (2,2)-graph, and using Eppstein's algorithm [19] we get the running time of $\mathcal{O}^*(2^{n/3}) = \mathcal{O}^*(1.26^n)$, see Corollary 2.9. Another consequence is an algorithm running in time $\mathcal{O}^*(2^{n/6})$ for digraphs of maximum total degree 3, see Corollary 2.11. These two simple classes of digraphs were studied by Plesník [35], who showed that DIRECTED HAMILTONICITY remains NP-complete when restricted to them.

We can also apply the reduction to an arbitrary digraph of average outdegree d. A naive approach would be then to enumerate all perfect matchings in the bipartite graph induced by edges $\{u^{\text{out}}v^{\text{in}} \mid (u,v) \in E(G)\}$. Indeed, each such matching corresponds to a cycle cover in the input graph, so we basically enumerate cycle covers and filter-out the disconnected ones. Thanks to Bregman-Minc inequality [14] which bounds the permanent in sparse matrices the

resulting algorithm has running time $\mathcal{O}^*(\mu(d)^n)$, where

$$\mu(d) = (\lfloor d \rfloor!)^{\frac{\lfloor d \rfloor + 1 - d}{\lfloor d \rfloor}} (\lceil d \rceil!)^{\frac{d - \lfloor d \rfloor}{\lceil d \rceil}} \leq (\lceil d \rceil!)^{1/\lceil d \rceil}.$$

See Corollary 2.7 for details.

Yet another upper bound for digraphs of average outdegree d is obtained by using the reduction described above and next applying Björklund's algorithm for sparse bipartite graphs [6] with a slight modification to force the matching M in the Hamiltonian cycle (see Theorem 2.14). The resulting algorithm has running time $O^*((2-2^{-d})^n)$.

Directed sparse graphs: main results The simple consequences that we describe above are complemented by two more technical results.

The first algorithm runs in polynomial space and realizes the following idea. Assume d < 3. Then many of the vertices of the input graph have outdegree at most 2, and we can just branch on vertices of outdegree at least 3, and solve the resulting (2,2)-graph using the fast $O^*(2^{n/3})$ -time algorithm mentioned before. This idea can be boosted a bit in the case when the initial branching is too costly, i.e., there are many vertices of high outdegree: then we observe that in such an unbalanced graph one can apply the simple cycle cover enumeration which then runs faster than in graphs of the same density but with balanced outdegrees. After a technical analysis of the running time we get the following theorem.

Theorem 1.1. ATSP restricted to digraphs of average outdegree at most d can be solved in time $\mathfrak{O}^*(2^{\alpha(d-1)n})$ and polynomial space, where $\alpha = \frac{7}{12} - \frac{1}{12(\log_2 3 - 1)} < 0.44088$.

The second algorithm generalizes Gebauer's meet-in-the-middle approach to digraphs of average outdegree d. (We note that it uses exponential space.)

Theorem 1.2. ATSP restricted to digraphs of average outdegree at most d can be solved in time $O^*(\tau(d)^{n/2})$ and the same space, where

$$\tau(d) = \lfloor d \rfloor^{\lfloor d \rfloor + 1 - d} (\lfloor d \rfloor + 1)^{d - \lfloor d \rfloor} \le d$$

Which algorithm is the best? Figure 1 compares four algorithms for solving ATSP and DIRECTED HAMILTONICITY in digraphs of average outdegree d described above:

- enumco: enumerating cycle covers (Corollary 2.7),
- Björklund: adaptation of Björklund's bipartite graphs algorithm (Theorem 2.14),
- branch+: branching boosted by enumerating cycle covers (Theorem 1.1).
- mim: meet in the middle (Theorem 1.2),

The choice of the best (in terms of the asymptotic worst-case running time) algorithm depends on d, on whether we can afford exponential space, and on whether we solve ATSP or just DIRECTED HAMILTONICITY. We can conclude the following.

• ATSP in polynomial space: for d < 2.746 use branch+, for $d \in [2.746, 8.627]$ use enumcc, and for d > 8.627 use the general algorithm of Gurevich and Shelah [26].

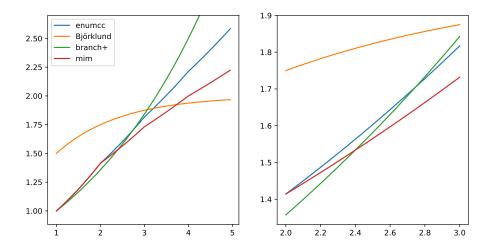


Figure 1: Comparison of the running times of algorithms for solving ATSP (enumcc, branch+, mim) and DIRECTED HAMILTONICITY (Björklund) in sparse digraphs. Horizontal axis: average degree d, vertical axis: base b from the running time bound of the form $O^*(b^n)$.

- ATSP in exponential space: for d < 2.398 use branch+, for $d \in [2.398, 3.999]$ use mim, and for d > 3.999 use the algorithm of Cygan and Pilipczuk [18].
- DIRECTED HAMILTONICITY in polynomial space: for d < 2.746 use branch+, for $d \in [2.746, 3.203]$ use enumce, for d > 3.203 use Björklund, and for sufficiently large d use the algorithm of Björklund [7].
- DIRECTED HAMILTONICITY in exponential space: for d < 2.398 use branch+, for $d \in [2.398, 3.734]$ use mim, for d > 3.734 use Björklund, and for sufficiently large d use the algorithm of Björklund and Williams [12].

2 Reductions from undirected graphs

The objective of this section is to recall two reductions from the ATSP to the (forced) TSP. Then, we will discuss existing methods of solving Underected Hamiltonicity and TSP, and present their implications for corresponding problems in directed graphs. The summary of this section is presented in Tables 1 and 2.

2.1 General reductions

We recall that in the FORCED TRAVELLING SALESMAN PROBLEM [19,36,38,39], we are given an undirected graph G, a weight function $w: E(G) \to \mathbb{Z}$, and a subset $F \subseteq E(G)$. We say that a Hamiltonian cycle H is admissible, if $F \subseteq H$. The goal is to find an admissible Hamiltonian cycle of the minimum total weight of the edges (or, report that there is no such cycle). Moreover, we define the BIPARTITE FORCED MATCHING TSP (BFM-TSP) as a special case of the FORCED TSP, where graph G is bipartite, and the edges of F form a perfect matching in G.

The following lemma provides the relationship between the BFM-TSP and the ATSP.

Lemma 2.1. For every instance (G, w) of ATSP, where G is a digraph on n vertices, there is an equivalent instance $(\widehat{G}, \widehat{w}, M)$ of BFM-TSP such that \widehat{G} is a graph on 2n vertices.

Moreover, if both outdegrees and indegrees of G are bounded by D, then \widehat{G} has maximum degree D. Similarly, if G has average outdegree d, then \widehat{G} has average degree d+1.

Proof. Let (G, w) be an instance of ATSP. Let $V^{\mathsf{out}} = \{v^{\mathsf{out}} \mid v \in V(G)\}$ and $V^{\mathsf{in}} = \{v^{\mathsf{in}} \mid v \in V(G)\}$. We define \widehat{G} as a bipartite graph on the vertex set $V(\widehat{G}) = V^{\mathsf{out}} \cup V^{\mathsf{in}}$ with edges $E(\widehat{G}) = \{u^{\mathsf{out}}v^{\mathsf{in}} \mid (u,v) \in E(G)\} \cup M$, where M is the perfect matching $M = \{v^{\mathsf{in}}v^{\mathsf{out}} \mid v \in V(G)\}$. The edges of $E(\widehat{G}) \setminus M$ inherit the weight from G, i.e. for $(u,v) \in E(G)$ we set $\widehat{w}(u^{\mathsf{out}}v^{\mathsf{in}}) = w(uv)$. Edges of M have weight 0.

We claim that $(\widehat{G}, \widehat{w}, M)$ is the desired instance of BFM-TSP. Indeed, \widehat{G} has 2n vertices, and given a Hamiltonian cycle $C := (v_1, \ldots, v_n)$ in G, we can construct a perfect matching $M' \subseteq E(\widehat{G})$, where $M' = \{v_i^{\mathsf{out}} v_{i+1}^{\mathsf{in}} \mid i = 1, \ldots, n\}$ (we set $v_{n+1} := v_1$). Then, $M \cup M'$ forms a Hamiltonian cycle in \widehat{G} of the same weight as C. Conversely, consider a Hamiltonian cycle \widehat{H} in \widehat{G} such that $\widehat{H} = M \cup M'$ for a matching M'. Then $M' \subseteq E(\widehat{G}) \setminus M$. Hence, after orienting edges of M' from V^{out} to V^{in} and contracting each edge $v^{\mathsf{in}}v^{\mathsf{out}} \in M$ to a single vertex v, we get a Hamiltonian cycle H in G of weight $\widehat{w}(\widehat{H})$.

Lemma 2.1 implies, in particular, that if there is an algorithm for BFM-TSP running in time $\mathcal{O}^*(f(n))$, then there is an algorithm for ATSP running in time $\mathcal{O}^*(f(2n))$.

When we solve an ATSP instance (G, w), in some cases it is easier to work with an equivalent instance of TSP (without forced edges).

Lemma 2.2. For every instance (G, w) of ATSP, where G is a digraph on n vertices, there is an equivalent instance $(\widetilde{G}, \widetilde{w})$ of TSP such that \widetilde{G} is an undirected graph on 3n vertices.

Proof. This is a classic result. Given an instance (G, w) of ATSP, we start with constructing an equivalent instance $(\widehat{G}, \widehat{w}, M)$ of BFM-TSP by applying Lemma 2.1. Then, we substitute in \widehat{G} every edge $v^{\mathsf{in}}v^{\mathsf{out}} \in M$ with a simple path of length 2: $(v^{\mathsf{in}}, v^{\mathsf{mid}}, v^{\mathsf{out}})$, where new edges $v^{\mathsf{in}}v^{\mathsf{mid}}$ and $v^{\mathsf{mid}}v^{\mathsf{out}}$ have weight 0. We see that \widetilde{G} has 3n vertices, and every Hamiltonian cycle \widetilde{H} in \widetilde{G} corresponds to a Hamiltonian cycle \widehat{H} in \widehat{G} such that $M \subseteq \widehat{H}$, and $\widehat{w}(\widehat{H}) = \widetilde{w}(\widetilde{H})$.

2.2 Enumerating cycle covers

Let $(\widehat{G}, \widehat{w}, M)$ be an instance of BFM-TSP, and let \mathcal{M} be a family of all perfect matchings in $\widehat{G} - M$. We observe that every cycle cover in \widehat{G} which contains all edges of M is of the form $M \cup M'$, where $M' \in \mathcal{M}$. Hence, our goal is to find a matching $M' \in \mathcal{M}$ such that $M \cup M'$ is a Hamiltonian cycle in \widehat{G} , and the weight of M' is minimum possible. One way to do it is to list all the perfect matchings $M' \in \mathcal{M}$, and choose the best one among these which form with M a Hamiltonian cycle in \widehat{G} . We will investigate the complexity of such an approach in sparse graphs.

It is known that all perfect matchings in bipartite graph \widehat{G} can be listed in time $|\mathcal{M}|n^{\mathcal{O}(1)}$ and polynomial space [22]. Hence, it is enough to provide a bound on the size of \mathcal{M} in sparse graphs. We start with recalling a classic result of Bregman together with its standard application.

Theorem 2.3 (Bregman-Minc inequality [14,37]). Let A be an $n \times n$ binary matrix, and let r_i denote the number of ones in the i-th row. Then

$$\operatorname{per} A \le \prod_{i=1}^{n} (r_i!)^{1/r_i}.$$

Corollary 2.4 ([1]). Let H be a bipartite digraph on $V^{\text{out}} \cup V^{\text{in}}$, where $|V^{\text{out}}| = |V^{\text{in}}| = n$, and let d_1, \ldots, d_n denote the degrees of vertices of V^{out} . Then, the number of perfect matchings in H can be bounded by

$$\prod_{i=1}^{n} (d_i!)^{1/d_i}.$$

Corollary 2.5. ATSP restricted to digraphs of outdegree bounded by D can be solved in time $(D!)^{n/D}n^{O(1)}$ and polynomial space.

Proof. Given an instance (G, w) of ATSP, we use Lemma 2.1 to obtain an equivalent instance $(\widehat{G}, \widehat{w}, M)$ of BFM-TSP. Then, $H := \widehat{G} - M$ is a bipartite graph on $V^{\mathsf{out}} \cup V^{\mathsf{in}}$, and all vertices of V^{out} in H have degree at most D. By Corollary 2.4, there are at most $(D!)^{n/D}$ perfect matchings in H. Hence, according to our initial observation, the instance $(\widehat{G}, \widehat{w}, M)$ can be solved in time $(D!)^{n/D} n^{\mathcal{O}(1)}$.

To the best of our knowledge, Corollary 2.5 provides the fastest polynomial space algorithm for $D \in \{3, 4, ..., 8\}$. The Bregman-Minc inequality is also useful for digraphs with bounded average outdegree. First, we need to quote an analytic lemma.

Lemma 2.6 ([2]). For a function $f(d) := (d!)^{1/d}$, and numbers $d_1, d_2 \in \mathbb{N}$, where $d_1 < d_2$, the following inequality holds:

$$f(d_1)f(d_2) \le f(d_1+1)f(d_2-1).$$

Corollary 2.7. ATSP restricted to digraphs of average outdegree d can be solved in time $\mu(d)^n n^{O(1)}$ and polynomial space, where

$$\mu(d) = (|d|!)^{\frac{\lfloor d\rfloor + 1 - d}{\lfloor d\rfloor}} (\lceil d\rceil !)^{\frac{d - \lfloor d\rfloor}{\lceil d\rceil}}$$

In particular, for integral values of d, the running time is bounded by $(d!)^{n/d}n^{O(1)}$.

Proof. As before, we start by constructing an equivalent instance (G, \widehat{w}, M) of BFM-TSP. Let d_1, \ldots, d_n denote the degrees of vertices of V^{out} in $\widehat{G} - M$. Note that their average is equal to d. By Corollary 2.4, $\widehat{G} - M$ has at most $\prod_{i=1}^n (d_i!)^{1/d_i}$ perfect matchings. Lemma 2.6 implies that this value is maximized if for all i, we have $d_i = \lfloor d \rfloor$, or $d_i = \lceil d \rceil$. Then, we claim that there must be $(\lfloor d \rfloor + 1 - d)n$ vertices of degree $\lfloor d \rfloor$ and $(d - \lfloor d \rfloor)n$ vertices of degree $\lceil d \rceil$. Indeed, this is true for $d \in \mathbb{N}$, and for $d \notin \mathbb{N}$, if we denote the number of vertices of degree $\lceil d \rceil$ by γ , then we have $\gamma \lceil d \rceil + (n - \gamma) \lfloor d \rfloor = nd$, and hence $\gamma = \gamma(\lceil d \rceil - \lfloor d \rfloor) = n(d - \lfloor d \rfloor)$. It follows that there are at most $\mu(d)^n$ perfect matchings in $\widehat{G} - M$.

2.3 Branching algorithms

One of the most common techniques which is used for solving NP-hard problems in sparse graphs is branching (bounded search trees). It is based on optimizing exhaustive search algorithms by bounding the size of the recursion tree. In case of TSP, the first result of this kind is due to Eppstein [19]. He showed a branching algorithm for subcubic graphs running in time $O^*(2^{n/3})$. Actually, he proved a stronger result in his work.

Theorem 2.8 ([19]). Forced TSP restricted to subcubic graphs can be solved in time $2^{(n-|F|)/3}n^{O(1)}$ and polynomial space.

Corollary 2.9. ATSP restricted to digraphs with all out- and indegrees at most 2 can be solved in time $O^*(2^{n/3})$ and polynomial space.

Proof. Let (G, w) be an instance of ATSP, where G is a digraph with all out- and indegrees at most 2. We apply Lemma 2.1 to obtain an equivalent instance $(\widehat{G}, \widehat{w}, M)$ of BFM-TSP. We know that \widehat{G} has 2n vertices, and is subcubic. Moreover, $(\widehat{G}, \widehat{w}, M)$ is an instance of FORCED TSP with |M| = n forced edges. Hence, we can use Theorem 2.8 to solve it in time $O^*(2^{(2n-n)/3}) = O^*(2^{n/3})$.

We should note here that since the work of Eppstein, faster algorithms for TSP in subcubic graphs were developed [28, 32, 38]. However, all of them run still in time $O^*(2^{n/3})$ when we apply them to the 2n-vertex subcubic graphs resulting from digraphs with all out- and indegrees at most 2 (as described in the proof of Corollary 2.9).

Since Lemma 2.1 allows us to transfer some of the results for subcubic instances of TSP to its version in digraphs with all out- and indegrees at most 2, one may also ask whether there is a relationship between subcubic instances of undirected TSP and instances of ATSP with maximum total degree at most 3. (Recall that total degree of a vertex is the sum of its indegre and outdegree.) The following lemma (implicit in Plesník [35]) answers this question indirectly.

Lemma 2.10 ([35]). There is an algorithm for ATSP restricted to digraphs of maximum total degree 3, and working in time $O^*(f(n))$ if and only if there is an algorithm for ATSP restricted to digraphs with out- and indegrees at most 2, and working in time $O^*(f(2n))$.

Proof. (\Longrightarrow) Let G be a digraph on n vertices with all out- and indegrees at most 2. Let $(\widehat{G}, \widehat{w}, M)$ be the instance of BFM-TSP defined in the proof of Lemma 2.1. We construct a weighted digraph G' by orienting the edges of $E(\widehat{G}) \setminus M$ from V^{out} to V^{in} , and edges of M from V^{in} to V^{out} (the weights stay the same). We see that G' has 2n vertices, has all total degrees at most 3, and Hamiltonian cycles in G' correspond to Hamiltonian cycles in G of the same weight.

(\iff) Let G_3 be a digraph on 2n vertices with maximum total degree at most 3. We may assume that all indegrees and outegrees in graph G_3 equal to 1 or 2, because otherwise G_3 has no Hamiltonian cycle. Since the total degree of each vertex is at most 3, each vertex has exactly one incoming edge or exactly one outgoing edge. We see that every Hamiltonian cycle in G_3 must contain all such edges, hence they can be contracted. When we contract an edge (u,v) we also remove edges of the form $(u,_)$ and $(_,v)$. Let us denote the remaining graph by G'. We claim that G' has all out- and indegrees at most 2. Indeed, consider a contraction of an edge (u,v) to a new vertex x. Since we remove all other edges of the form $(u,_)$ and

 $(_,v)$, there is a one-to-one correspondence between the edges entering (resp. leaving) x and the edges entering u (resp. leaving v). Hence, a single contraction does not increase the maximum out- or indegree, which implies that after all contractions all out- and indegrees are still at most 2. Moreover, every vertex from G_3 takes part in at least one edge contraction, and thus $|V(G')| \leq |V(G_3)|/2 = n$.

By combining Lemma 2.10 with Corollary 2.9 we obtain the following.

Corollary 2.11. ATSP restricted to digraphs of maximum total degree 3 can be solved in time $O^*(2^{n/6})$ and polynomial space.

2.4 Meet in the middle technique

Another approach for solving TSP in sparse graphs was suggested by Gebauer [24]. Although it was originally presented for undirected graphs of maximum degree 4, we recall it here for digraphs with outdegrees bounded by D, since the same method can be applied to them.

Theorem 2.12 ([24]). ATSP restricted to digraphs with outdegrees bounded by D can be solved in time $O^*(D^{n/2})$ and exponential space.

The idea of this algorithm can be sketched as follows. We guess a pair of vertices (u_1, u_2) which divide a hypothetical Hamiltonian cycle into two (almost) equal parts. Next, we run a branching procedure to generate all the paths \mathcal{P}_1 from u_1 to u_2 of length $\lfloor n/2 \rfloor$, and all the paths \mathcal{P}_2 from u_2 to u_1 of length $\lceil n/2 \rceil$. Finally, we try to combine such paths into one Hamiltonian cycle by memorizing \mathcal{P}_1 in a dictionary and iterating over paths $P_1 \in \mathcal{P}_1$.

For a detailed description, we refer to the original work of Gebauer [24], and to Section 4, where we generalize this result to digraphs of bounded *average* outdegree.

2.5 Algebraic methods

Björklund [6] shows the following result.

Theorem 2.13 ([6]). There is a Monte Carlo algorithm which solves Underected Hamiltonicity restricted to bipartite graphs of average degree at most d in time $O^*((2-2^{1-d})^{n/2})$ and polynomial space.

It turns out that the proof of Theorem 2.13 can be modified to get the following Theorem. The idea is to use the reduction of Lemma 2.1 to get a sparse bipartite graph and modify the construction of Theorem 2.13 so that a relevant forced matching is a part of the resulting Hamiltonian cycle.

Theorem 2.14. There is a Monte Carlo algorithm which solves DIRECTED HAMILTONICITY restricted to digraphs of average outdegree at most d in time $O^*((2-2^{-d})^n)$ and polynomial space.

Proof. We assume that the reader is familiar with the proof of Theorem 2.13. We apply Lemma 2.1 and we get a bipartite undirected graph $\widehat{G} = (I \cup J, \widehat{E})$ and a perfect matching $F \subseteq \widehat{E}$. Recall that \widehat{G} has 2n vertices and average degree at most d+1. The goal is to decide whether \widehat{G} has a Hamiltonian cycle H that contains F.

Similarly as in [6] we define a polynomial matrix M with rows indexed by the vertices of I, and columns indexed by the vertices of J, as follows.

$$M(a,x,z)_{i,j} = \begin{cases} \sum_{k \in I \setminus \{i\}} z_{i,j} z_{j,k} (a_{j,k} + x_k) & \text{when } ij \in F, \\ z_{i,j} z_{j,k} (a_{j,k} + x_k) & \text{when } ij \notin F, \text{ but } jk \in F. \end{cases}$$

These polynomials have three types of variables: x_i for every $i \in I$, $a_{j,i}$ for every edge $ji \in \hat{E}$, $j \in J$, $i \in I$. The third type of variable is somewhat special. Pick a fixed edge $e^* = i^*j^* \in F$. For every edge $ij \in \hat{E} \setminus \{e^*\}$ there is one variable with two names $z_{i,j}$ and $z_{j,i}$; there are also two different variables z_{i^*,j^*} and z_{j^*,i^*} . Then we define a polynomial over a large enough field of characteristic two:

$$\phi = \sum_{x \in \{0,1\}^{n/2}} \det(M(a, x, z))$$

Now we should prove that thanks to cancellation in a field of characteristic two, $\phi = \sum_{H \in \mathcal{H}} \prod_{ij \in H} z_{i,j}$, where \mathcal{H} is the set of all Hamiltonian cycles in \widehat{G} which contain F. Björklund (Lemma 3 in [6]) shows this equality for the original polynomial using three observations: 1) after cancellation, the surviving terms do not contain a-variables, 2) each surviving term corresponds to a unique cycle cover in the graph, and 3) terms corresponding to non-Hamiltonian cycle covers pair-up and cancel-out, because if we reverse the lexicographically first cycle that does not contain e^* , then we get exactly the same term (and if we reverse a Hamiltonian cycle we get a different term, because of the asymmetry in defining z variables). The arguments used in [6] for proving 1)-3) still hold for the new polynomial, essentially for the same reasons.

The second ingredient of Björklund's construction is an upper bound on probability that none of the columns of M(a, x, z) is identically zero, where $x \in \{0, 1\}^{n/2}$ is a fixed assignment, z is the vector of all $z_{i,j}$ variables, and $a \in \{0, 1\}^{n/2}$ is a random assignment. The calculation relies on the observation that if for a vertex $j \in J$ we have $a_{j,i} + x_i \equiv 0 \pmod{2}$ for all $ij \in \widehat{E}$, then the column of j is identically zero. Note that this observation still holds for our new design. It follows that the probability bounds derived in [6] apply also in our case.

The third ingredient is efficient identification of assignments $x \in \{0,1\}^{n/2}$, for which $\det(M(a,x,z))$ is non-zero (for fixed, random, values of a). This is done by creating a Boolean variable w_v corresponding to every variable x_v and building a CNF formula such that its satisfying assignments correspond to a superset of all assignments of x_v variables that result in non-zero $\det(M(a,x,z))$. Again, the fact that the resulting formula is in CNF follows from the fact that the j-th column is non-zero if for some $i \in I$ we have $a_{j,i} + x_i \equiv 1 \pmod{2}$, which is also true in our design. Finally, Björklund [6] shows how to enumarate all satisfying assignments of the CNF formula efficiently, what is not altered in any way by our changes in the design of polynomial ϕ .

2.6 Dynamic programming on pathwidth decompositions

There are many works [20,21,30] which show that the pathwidth of sparse undirected graphs is relatively small, and which provide a polynomial time algorithm for computing the corresponding decomposition. (For a definition of pathwidth, see [15], section 7.2.) These results, combined with algorithms working on a path decomposition of the input graph [13,16], often lead to the fastest algorithms for sparse undirected graphs (see Table 1).

A natural question that arises here is whether these methods can be transferred to the corresponding problems in sparse directed graphs. There are two natural strategies for that: either use the path decomposition of the underlying undirected graph, or the path decomposition of the graph resulting from the reduction of Lemma 2.1 or Lemma 2.2. Although in this way one can get algorithms faster than $O^*(2^n)$ for some classes of sparse digraphs, it does not help to improve any of the bounds in Table 2, at least by combining currently known results. For completeness, in the remainder of this section we provide calculations that support this claim.

Let us try the direct approach first. We can use the following result of Cygan et al.

Theorem 2.15 ([17]). There is a Monte Carlo algorithm which, given a graph G with a tree decomposition of its underlying undirected graph of width tw, solves DIRECTED HAMILTONICITY for G in time $6^{tw}n^{O(1)}$ and exponential space.

Consider a (2, 2)-graph, i.e., a digraph with both out- and indegrees bounded by 2. The undirected graph underlying a (2, 2)-graph has maximum degree 4, and hence it has pathwidth at most n/3 + o(n), according to Theorem 2.16 below.

Theorem 2.16 ([20]). For every $\varepsilon > 0$, there exists an integer N_{ε} such that for every undirected graph G on $n \geq N_{\varepsilon}$ vertices the inequality

$$pw(G) \le \frac{1}{6}n_3 + \frac{1}{3}n_4 + \frac{13}{50}n_5 + n_{\ge 6} + \varepsilon n$$

holds, where n_k is the number of vertices of degree k in G, and $n_{\geq 6}$ is the number of vertices of degree at least 6. Moreover, a path decomposition which witnesses the above inequality can be computed in polynomial time.

This, combined with Theorem 2.15 gives an algorithm for DIRECTED HAMILTONICITY running in time $\mathcal{O}(6^{n/3+o(n)}) = \mathcal{O}(1.82^n)$, much slower than in Corollary 2.9.

Now consider a digraph of average outdegree degree d. Then, the underlying undirected graph has average degree 2d, and we can bound its pathwidth using the following result.

Theorem 2.17 ([30]). Let G be an n-vertex undirected graph of average degree d. Then

$$pw(G) \le \frac{dn}{11.538} + o(n).$$

Moreover, a path decomposition which witnesses the above inequality can be computed in polynomial time.

It follows that the algorithm from Theorem 2.15 applied on a digraph of average outdegree d has running time of $\mathcal{O}(6^{2dn/11.538+o(n)}) = \mathcal{O}(1.365^{dn})$, which can be seen to be slower than, say, enumerating cycle covers (Corollary 2.7) for all values of d.

Now let us focus on the reduction approach. We can use the following two results.

Theorem 2.18 ([16]). There is a Monte Carlo algorithm which, given a graph G with its path decompositions of width pw, solves Underected Hamiltonicity for G in time $(2 + \sqrt{2})^{pw} n^{O(1)}$ and exponential space.

Moreover, if G is subcubic, the running time is bounded by $(1+\sqrt{2})^{pw}n^{O(1)}$ instead.

Theorem 2.19 ([13]). There is an algorithm which, given a graph G with its path decomposition of width pw, solves TSP for G in time $(2 + 2^{\omega/2})^{pw} n^{O(1)}$ and exponential space, where ω is the matrix multiplication exponent.

Moreover, if G is subcubic, the running time is bounded by $(1+2^{\omega/2})n^{O(1)}$ instead.

Theorems 2.18 and 2.19 combined with Theorem 2.16 give, in particular, $\mathcal{O}(1.16^n)$ and $\mathcal{O}(1.22^n)$ algorithms for Underected Hamiltonicity and TSP in subcubic graphs, respectively. For undirected graphs of average degree at most d we can combine the above theorems with with Theorem 2.17 to obtain algorithms in time $\mathcal{O}(1.12^{dn})$ and $\mathcal{O}(1.14^{dn})$ for Underected Hamiltonicity and TSP, respectively.

Now we turn to digraphs again. First consider (2,2)-graphs, i.e., digraphs with out- and indegrees bounded by 2. Let (G,w) be an instance of ATSP, where G is such a digraph. We use Lemma 2.2 to obtain an equivalent instance $(\widetilde{G},\widetilde{w})$ of (undirected) TSP. From the construction of \widetilde{G} we see that \widetilde{G} has 3n vertices of which at most 2n have degree 3, and the remaining ones have degree 2. Hence, by Theorem 2.16 we have $\operatorname{pw}(\widetilde{G}) \leq \frac{2n}{6} + o(3n) = \frac{n}{3} + o(n)$. Therefore, Theorems 2.18 and 2.19 give respectively the algorithms running in time $(1+\sqrt{2})^{n/3+o(n)}$ and $(2+2^{\omega/2})^{n/3+o(n)}$ for DIRECTED HAMILTONICITY and ATSP in G. Both results are worse than the running time of the algorithm from Corollary 2.9.

Again, consider digraphs with bounded average outdegree d. Let (G, w) be an instance of ATSP, where G is such a digraph. We use Lemma 2.2 to obtain an equivalent instance $(\widetilde{G}, \widetilde{w})$ of TSP. Then, \widetilde{G} has 2n vertices of average degree d+1, and n vertices of degree 2. The latter ones can increase the pathwidth only by 1 in total, hence $\operatorname{pw}(\widetilde{G}) \leq \frac{2(d+1)n}{11.538} + o(3n) = \frac{(d+1)n}{5.769} + o(n)$, and consequently, DIRECTED HAMILTONICITY and TSP in G can be solved in time $(2+\sqrt{2})^{(d+1)n/5.769+o(n)}$ and $(2+2^{\omega/2})^{(d+1)n/5.769+o(n)}$, respectively. Both results are worse than the algorithm enumerating cycle covers described in Subsection 2.2.

3 Polynomial space algorithm

This section is devoted to the proof of Theorem 1.1. We begin with introducing some additional notions, then we provide a branching algorithm which will be later used as a subroutine, and finally we describe and analyse an algorithm for digraphs of average outdegree at most d.

3.1 Preliminaries

Interfaces and switching walks. Let G be a directed graph (digraph). For a vertex v a set I_v^{in} of all incoming edges to v or a set I_v^{out} of all outgoing edges from v will be called an *interface* of v. We define the type of an interface of v so that $\mathsf{type}(I_v^{\text{in}}) = \mathsf{in}$ and $\mathsf{type}(I_v^{\text{out}}) = \mathsf{out}$.

Consider a sequence of distinct edges $\pi = e_1, \ldots, e_k$ in G such that if we forget about the orientation of edges, then we get a walk v_1, \ldots, v_{k+1} in the underlying undirected graph, where for $i=1,\ldots,k$ edge e_i is an orientation of v_iv_{i+1} . Assume additionally that for every $i=2,\ldots,k$ either both edges e_{i-1} and e_i enter v_i or both leave v_i , in other words, the orientation of edges on the walk alternates. Now, let I_1,\ldots,I_{k+1} be the consecutive interfaces visited by π , i.e., for every $j=1,\ldots,k+1$ we have that I_j is an interface of v_j and for every $j=1,\ldots,k$, we have $e_j \in I_j \cap I_{j+1}$. If $|I_1|,|I_k|>2$ and $|I_j|=2$, for $j=2,\ldots,k-1$, the sequence π will be called a switching walk. Similarly, if $|I_j|=2$ for $j=1,\ldots,k$, and $v_1=v_{k+1}$, i.e., the walk v_1,\ldots,v_{k+1} is closed, then π will be called a switching circuit. In both cases, length of π is defined as k. The sequence v_1,\ldots,v_{k+1} is called the vertex sequence of π . Abusing the notation slightly, we will refer to π as a set, when it is convenient. The motivation for introducing the notions of switching walks and circuits is given by the following lemma.

Lemma 3.1. Let $\pi = \{e_1, \ldots, e_k\}$ be a switching walk or a switching circuit in a digraph G. Let $H \subseteq E(G)$ be a Hamiltonian cycle in G. Then, $H \cap \pi = \{e_{2i-1} \mid i = 1, \ldots, \lfloor \frac{k+1}{2} \rfloor \}$, or $H \cap \pi = \{e_{2i} \mid i = 1, \ldots, \lfloor \frac{k}{2} \rfloor \}$.

Proof. Let us assume that π is a switching walk. (For a switching circuit the proof is analogous.) Consider two consecutive edges $e_i, e_{i+1} \in \pi$. By the definition of a switching walk, there is a vertex v with an interface I of size 2 such that $I = \{e_i, e_{i+1}\}$. Since the cycle H passes through v, we obtain that H must contain exactly one of the edges e_i and e_{i+1} , and the lemma easily follows.

In some cases it is convenient to study switching walks and circuits in the language of an auxiliary bipartite graph. Let $V^{\text{out}} = \{v^{\text{out}} \mid v \in V(G)\}$ and $V^{\text{in}} = \{v^{\text{in}} \mid v \in V(G)\}$. The interface graph of G is the bipartite graph I_G such that $V(I_G) = V^{\text{out}} \cup V^{\text{in}}$ and $E(I_G) = \{u^{\text{out}}v^{\text{in}} \mid (u,v) \in E(G)\}$. Clearly, there is a one-to-one correspondence between interfaces in G and vertices of I_G , and the degree of a vertex in I_G is the size of the corresponding interface. Moreover, if $\pi = e_1, \ldots, e_k$ is a switching walk in G with a vertex sequence v_1, \ldots, v_{k+1} and interface sequence I_1, \ldots, I_{k+1} , then π corresponds to a simple path $I(\pi) = v_1^{\text{type}(I_1)}, \ldots, v_{k+1}^{\text{type}(I_{k+1})}$ in G with endpoints of degree larger than 2, and all inner vertices of degree 2. Similarly, a switching circuit π corresponds to a simple cycle $I(\pi)$ in I_G with all vertices of degree 2 in I_G , i.e., $I(\pi)$ forms a connected component in I_G . Observe that both in the case of path and cycle above, the edges $I(\pi)$ are exactly the edges of I_G corresponding to the edges of π . Using the equivalence described in this paragraph, the following lemma is immediate.

Lemma 3.2. Edges of every digraph can be uniquely partitioned into switching walks and circuits. Moreover, the partition can be computed in linear time.

Proof. Let G be a digraph. Recall that by the definition of I_G , there is a one-to-one correspondence between edges of G and edges of I_G . It is clear that edges of I_G can be uniquely partitioned into (1) cycles with all vertices of degree 2 and (2) paths with both endpoints of degree at least 3 and all inner vertices of degree 2. The corresponding switching circuits and switching walks form the desired partition of E(G). An algorithm which constructs the partition is straightforward.

Another view on Corollary 2.9. The run of the algorithm from Corollary 2.9 can be interpreted using the introduced notions as follows. We apply Lemma 3.2 to partition E(G) into switching walks and circuits. If there is a switching circuit π of length at least 6, we guess the intersection of π with a hypothetical Hamiltonian cycle in G. By Lemma 3.1 there are two possibilities for this intersection (in both cases it consists of at least 3 edges of G). We consider both cases by marking chosen edges as forced, and recursively calling on the remaining graph. If there is no switching circuit of length at least 6, it turns out that the remaining instance can be solved by finding a minimum spanning tree in an auxiliary graph (see [19] for the details). Hence, the size of the recursion tree of this algorithm can be bounded by $2^{n/3}$.

3.2 Branching subroutine

Let us consider a digraph G. By $t_i(G)$ we will denote the number of vertices of G with outdegree equal to i. Let $k = n - t_1(G)$ be the number of vertices of G with outdegree at

```
Pseudocode 1: AtspBranching(G, weight)
  Input: G – a digraph on n > 2 vertices,
          weight – a function E(G) \to \mathbb{Z}
  Output: the minimum weight of a Hamiltonian cycle in G,
            or \infty if there is no such cycle
  Function AtspForcedEdge(G, weight, e):
      Let e = (u, v)
      G_1 \leftarrow G with removed edges of the form (v, u), (u, x) and (x, v) for x \in V(G)
      G' \leftarrow G_1 with contracted vertices u and v
      weight' \leftarrow weights of E(G') inherited from G appropriately
      return weight(e) + AtspBranching(G', weight')
  Function AtspBranching(G, weight):
      if G has exactly two vertices u and v then
                                                                                                          (a)
       return weight((u, v)) + weight((v, u)) if (u, v), (v, u) \in E(G), or \infty otherwise
      if there is an empty interface in G i.e. a vertex of out- or indegree 0 then
                                                                                                          (b)
      if there is an interface I = \{e\} of size 1 then
                                                                                                          (c)
       return AtspForcedEdge(G, weight, e)
      Use Lemma 3.2 to partition E(G) into switching walks and circuits
      if there is a switching walk \pi which begins and ends at the same interface I then
                                                                                                          (d)
          G' \leftarrow G with removed edges of I \setminus \pi
          return AtspBranching(G', weight)
      if there is a switching walk \pi of even length then
                                                                                                          (e)
          Let \pi = (e_1, \dots, e_{2k})
          return min(AtspForcedEdge(G, weight, e_1), AtspForcedEdge(G, weight, e_{2k}))
      if there is no interface of size at least 3 then
                                                                                                          (f)
          Apply Corollary 2.9 to G and return the weight of the solution, or \infty
      else
                                                                                                          (g)
          Let I = \{e_1, \dots, e_s\} be an out-interface of size s \geq 3
          for i=1,\ldots,s do
              result \leftarrow min(result, AtspForcedEdge(G, weight, e_i))
          return result
```

least 2, and let s_1, \ldots, s_k be the sequence of these outdegrees. Then, let us denote the sum $\sum_{i=1}^k (s_i - 2)$ by S(G). An analogous sum for indegrees will be denoted by $S^-(G)$. Note that if G has no vertex of out- or indegree 1, then by the handshaking lemma $S(G) = S^-(G)$.

Theorem 3.3. ATSP can be solved in time $O^*(2^{(n-t_1(G))/3} + \beta S(G))$ and polynomial space, where $\beta = \log_2 3 - 1 < 0.585$.

Proof. The idea behind this algorithm is to branch on interfaces of size greater than 2, reducing the initial problem to the case of (2,2)-graphs, and then to apply Corollary 2.9. A detailed description is presented in Pseudocode 1. Our algorithm consists of two functions: AtspBranching(G, weight) – the main one, which solves ATSP in G, and an auxiliary function AtspForcedEdge(G, weight, e) that returns the minimum weight of a Hamiltonian cycle H in G such that $e \in H$ (or ∞ if there is no such cycle). Note that AtspForcedEdge modifies the input digraph G, and calls AtspBranching on the new digraph G'. We observe that every Hamiltonian cycle in G' of weight w corresponds to a Hamiltonian cycle in G of weight w + weight(e)

and containing edge e, and vice versa.

Given a digraph G with a function weight : $E(G) \to \mathbb{Z}$, AtspBranching starts by considering a number of trivial cases (a)-(c), where either G has only 2 vertices, or there is a vertex with out- or indegree at most 1. Next, we apply Lemma 3.2 to decompose E(G) into switching walks and circuits, and we deal with a situation when there is a switching walk $\pi=(e_1,\ldots,e_{2k})$ of even length in G (cases (d)-(e) in Pseudocode 1). Denote by I, respectively I', the interface which π starts, respectively ends, at. Consider a Hamiltonian cycle H in G. By Lemma 3.1 we obtain that either $e_1 \in H \cap \pi$, or $e_{2k} \in H \cap \pi$. We consider the following two cases.

- If I = I', then we have $H \cap I \in \{e_1, e_{2k}\}$, and thus all edges of $I \setminus \pi$ can be safely removed as they cannot be extended to a Hamiltonian cycle in G. This is realized in step (d) of the pseudocode. Note that if a switching walk π starts and ends at the same interface, then it must be of even length, since orientation of edges on π alternates.
- If $I \neq I'$, we branch by guessing if $e_1 \in H \cap \pi$, or $e_{2k} \in H \cap \pi$ (step (e) of the pseudocode).

If none of the above cases holds, we check whether all interfaces consist of at most 2 edges (cases (f) - (g) in Pseudocode 1). If so, then G is a (2,2)-graph, and we can solve ATSP for G by applying Corollary 2.9. If not, we choose an out-interface I of size at least 3, and we branch on it, by guessing which of the edges of I to pick as a part of a Hamiltonian cycle. Note that since G has no interface of size 1, then it has an interface of size at least 3 if and only if it has an out-interface of size at least 3.

Time complexity analysis. We begin with providing a few simple facts concerning the properties of our algorithm.

Claim 1. During execution of algorithm AtspBranching, the value of S(G) cannot increase.

Proof. Clearly, removing an edge cannot increase the value of S(G). Moreover, whenever we contract an edge (u, v) (call the resulting vertex x), we remove edges of the form (v, u), $(u, _)$, $(_, v)$. Hence, the out-interface, respectively the in-interface, of x is a subset of the out-interface of v, respectively the in-interface of v, and the other interfaces remain unchanged. \Box

Claim 2. During execution of algorithm AtspBranching, graph G is simple, i.e. does not contain two edges of the same head and tail.

Proof. Without loss of generality, we may assume that the input graph is simple, for otherwise we just discard the lighter edge. Moreover, before contracting an edge (u, v) (call the resulting vertex x), we remove edges of the form $(v, u), (u, _), (_, v)$. Hence, after contraction there is no loop (x, x), every edge outgoing from x corresponds to an edge outgoing from v, and every edge incoming to x corresponds to an edge incoming to v.

Claim 3. Let $\pi = (e_1, \ldots, e_k)$ be a switching walk in G. Assume that during the run of our algorithm we decided to take an edge e_1 by calling AtspForcedEdge(G, weight, e_1). Then, by exhaustively applying rule (c) of AtspBranching to the resulting digraph, we will remove from G all edges of the form e_{2i} , and contract all edges of the form e_{2i+1} . An analogous statement can be made if we start with discarding edge e_1 instead of contracting it.

Denote $f(n, S) = 2^{n/3+\beta S}$, where β is the constant from Theorem 3.3. We need to prove that the running time of our algorithm is bounded by $f(n - t_1(G), S(G))n^{O(1)}$. We proceed by induction on $t_1(G) + S(G)$.

If $t_1(G) > 0$, then our algorithm starts by choosing edges which form interfaces of size 1, what leads to a digraph with at most $\max(2, n - t_1(G))$ vertices. Hence, by the induction hypothesis the running time is bounded by $f(n - t_1(G), S(G))n^{O(1)}$.

In what follows we assume $t_1(G) = 0$. If G satisfies condition (a) or (b), then our algorithm runs in polynomial time. Similarly, we can assume that G does not satisfy conditions (c) and (d), as applying the corresponding reductions exhaustively takes only polynomial time and does not increase the value of S(G), according to Claim 1.

From now on, we assume that conditions (a) - (d) do not hold for G. If S(G) = 0, then our algorithm executes the algorithm from Corollary 2.9 and therefore its running time is bounded by $O^*(2^{n/3})$, as desired. Now, assume S(G) > 0. It remains to analyse cases (e) and (g) of AtspBranching.

Case (e). Let us assume that there is a switching walk $\pi = (e_1, \ldots, e_{2k})$ of even length in G which starts at interface I of size $s \geq 3$, and ends at interface $I' \neq I$ of size $s' \geq 3$. Let G' be a digraph obtained from G by running AtspForcedEdge $(G, weight, e_1)$ and exhaustively applying rules (a) - (d) to the resulting digraph.

Since edge e_1 is contracted in AtspForcedEdge, we have $|V(G')| \leq |V(G)| - 1$. We claim that $S(G') \leq S(G) - 2$. Assume $\mathsf{type}(I) = \mathsf{type}(I') = \mathsf{out}$. By Claim 3, for all $i = 1, \ldots, k$, edge e_{2i-1} was contracted, and edge e_{2i} was removed. We observe that contracting edge e_1 results in removing interface I from the graph, and discarding edge e_{2k} decreases the size of I' by 1. By Claim 1 operations performed on edges e_2, \ldots, e_{2k-1} do not increase the value of S(G). Hence, $S(G) - S(G') \geq (s-2) + 1 \geq 2$, as desired. If $\mathsf{type}(I) = \mathsf{type}(I') = \mathsf{in}$, then by the same reasoning, we obtain $S^-(G') \leq S(G) - 2$ but since there are no interfaces of size 1 in G', we have $S(G') = S^-(G')$, and the claim follows.

Hence, by the induction hypothesis, the running time of our algorithm applied to G' is bounded by f(n-1,S(G)-2). To obtain the desired bound for digraph G we need to show that $2f(n-1,S(G)-2) \leq f(n,S(G))$, or, equivalently $\log_2(2f(n-1,S(G)-2)) \leq \log_2 f(n,S(G))$. We obtain

$$\log_2(2f(n-1,S(G)-2)) = 1 + \frac{n-1}{3} + \beta(S(G)-2) = \frac{n}{3} + \beta S(G) + \frac{2}{3} - 2\beta$$

$$\leq \frac{n}{3} + \beta S(G) = \log_2 f(n,S(G)).$$

Case (g). Now, we assume that G does not satisfy conditions (a)-(f). Let I be an out-interface of size $s \geq 3$, and consider an edge $e \in I$. Let G' be a digraph obtained from G after choosing edge e by running AtspForcedEdge(G, weight, e), and let G'' be a digraph obtained from G' by the subsequent exhaustive application of rules (a)-(d) by AtspBranching. Define $\Delta n = |V(G)| - |V(G'')|$, and $\Delta S = S(G) - S(G'')$.

Claim 4. It holds that $\Delta n \ge 1$, $\Delta S \ge s - 2 \ge 1$, and $\Delta n + \Delta S \ge s + 1$.

Proof. For a digraph G we denote n(G) = |V(G)|. First, we analyse a direct impact of calling $\mathsf{AtspForcedEdge}(G, \mathsf{weight}, e)$. All edges of I are removed from G, hence by Claim 1 we have $\Delta S \geq S(G) - S(G') \geq s - 2 \geq 1$. Moreover, edge e gets contracted, and thus $\Delta n \geq n(G) - n(G') = 1$. We are left with proving that $(n(G') - n(G'')) + (S(G') - S(G'')) \geq 2$, since then we will have

$$\Delta n + \Delta S = (n(G) - n(G') + n(G') - n(G'')) + (S(G) - S(G') + S(G') - S(G''))$$

$$= (n(G) - n(G')) + (S(G) - S(G')) + (n(G') - n(G'') + S(G') - S(G''))$$

$$> 1 + (s - 2) + 2 = s + 1.$$

Let π be the switching walk which starts with edge e. Let e' be the last edge of π (it is possible that π has length 1 and e' = e). We recall that at step (g) every switching walk in G is of odd length. Take an in-interface I' such that $e' \in I'$. By the definition of switching walk, $|I'| \geq 3$, so let e', e'_1, e'_2 be three different edges of I'. For j = 1, 2 denote by π_j the switching walk which ends with edge e'_j . Let e_j be the first edge of π_j , and let I_j be an out-interface such that $e_j \in I_j$.

Let $F, R \subseteq E(G)$ be edges of G which correspond to the edges that were taken (and hence, contracted) and removed, respectively, during the run of our algorithm which leads from digraph G to digraph G''. We have $e \in F$. By Claim 3 applied to π , we obtain $e' \in F$. Therefore, $e'_1, e'_2 \in R$, and again by Claim 3 applied to π_1 and π_2 , we obtain $e_1, e_2 \in R$. Now, we consider a few cases.

- If I, I_1, I_2 are pairwise different out-interfaces, then during processing of digraph G' we removed edges e_1, e_2 from different out-interfaces of size at least 3. Therefore, $S(G') S(G'') \ge 2$.
- If $I = I_1 = I_2$, then among switching walks π, π_1, π_2 there are least two of length greater than 1 (hence, of length at least 3), because otherwise the graph is not simple, contradicting Claim 2. Let us assume that these are walks π and π_1 (the other cases are analogous). Then, by Claim 3, during processing of digraph G' we contracted edge e' and the second edge of π_1 . Therefore, $n(G') n(G'') \geq 2$.
- If $I_1 = I_2 \neq I$, or $I = I_1 \neq I_2$, or $I = I_2 \neq I_1$, then at least one switching walk among π, π_1, π_2 is of length at least 3, and there is another interface apart from I that gets smaller. Hence, we obtain in a similar way as before that $n(G'') n(G') \geq 1$, and $S(G'') S(G') \geq 1$.

┙

Since $\Delta S \geq 1$, we have S(G'') < S(G), and thus by the induction hypothesis the running time of our algorithm applied to G'' is bounded by $f(n(G''), S(G'')) = f(n - \Delta n, S(G) - \Delta S)$. In step (g) of AtspBranching we branch into s such subcases, hence we need to prove that $s \cdot f(n - \Delta n, S(G) - \Delta S) \leq f(n, S(G))$. We will show the equivalent $\log_2(s \cdot f(n - \Delta n, S(G) - \Delta S)) \leq \log_2 f(n, S(G))$. Indeed,

$$\log_{2}(s \cdot f(n - \Delta n, S(G) - \Delta S)) = \log_{2} s + \frac{n - \Delta n}{3} + \beta(S(G) - \Delta S)$$

$$= \frac{n}{3} + \beta S(G) + \log_{2} s - \frac{\Delta n}{3} - \beta \Delta S$$

$$\leq \frac{n}{3} + \beta S(G) + \log_{2} s - \frac{s + 1 - \Delta S}{3} - \beta \Delta S \qquad \text{(Claim 4)}$$

$$= \frac{n}{3} + \beta S(G) + \log_{2} s - \frac{s + 1}{3} - (\beta - \frac{1}{3})\Delta S$$

$$\leq \frac{n}{3} + \beta S(G) + \log_{2} s - \frac{s + 1}{3} - (\beta - \frac{1}{3})(s - 2) \qquad \text{(Claim 4)}$$

$$= \frac{n}{3} + \beta S(G) + \log_{2} s - 1 - \beta(s - 2)$$

$$\leq \frac{n}{3} + \beta S(G)$$

$$= \log_{2} f(n, S(G)). \qquad (\triangle)$$

where inequality (\triangle) follows from the fact that the function $x \mapsto \frac{\log_2 x - 1}{x - 2}$ is decreasing on $[3, \infty)$, and thus it can be bounded by the value at x = 3 which is equal to β . Consequently, the inequality $\log_2 s \le 1 + \beta(s - 2)$ holds for $s \ge 3$.

3.3 General algorithm

The idea behind our general algorithm is to run in parallel two algorithms: our branching algorithm from Theorem 3.3 (which we will refer to as Algorithm A), and enumerating cycle covers from Subsection 2.2 (Algorithm B). We finish when one of these algorithms terminates. Our goal is to prove that the time complexity of such an approach is bounded by $\textcircled{O}^*(2^{\alpha(d-1)n})$ if we apply it to digraphs of average outdegree at most d, where α is the constant from Theorem 1.1.

Note that when implementing this algorithm, one may also compare the values of $\frac{n}{3} + \beta S(G)$ and $\alpha(d-1)n$, and, depending on the result, run either Algorithm (A), or Algorithm (B).

Let G be a digraph on n vertices, of average outdegree d. We may assume that d > 1, for otherwise ATSP in G can be solved in polynomial time. Let $t_i := t_i(G)$ for i = 1, ..., n - 1, and denote $\mathbf{t} = (t_1, ..., t_{n-1})$. The numbers $t_1, ..., t_{n-1}$ satisfy the conditions

$$\begin{cases} \sum_{i=1}^{n-1} t_i &= n\\ \sum_{i=1}^{n-1} t_i \cdot i &= dn \end{cases}$$
 (\delta)

According to Theorem 3.3 and Corollary 2.4, the running time of Algorithm A and Algorithm B for G, up to a polynomial factor, can be bounded by functions

$$f(\mathbf{t}) := 2^{\frac{n-t_1}{3} + \beta(\sum_{i=2}^{n-1} t_i(i-2))}$$

and

$$g(\mathbf{t}) := \prod_{i=2}^{n-1} (i!)^{t_i/i},$$

respectively. Define $h(\mathbf{t}) = \min(f(\mathbf{t}), g(\mathbf{t}))$. Our task can be rephrased in the following way: we want to find the maximum value of $h(\mathbf{t})$ over vectors \mathbf{t} which satisfy conditions (\Diamond). From now on, we assume that the numbers t_1, \ldots, t_{n-1} satisfy these conditions and maximize the value of $h(\mathbf{t})$. If there are many such valuations, then we choose the one which is lexicographically minimal.

Claim 5. If $t_i, t_j \neq 0$, for $i, j \geq 2$, then $|i - j| \leq 1$. In particular, among numbers t_2, \ldots, t_n at most two are nonzero.

Proof. Suppose that $t_i, t_j > 0$ where $i < i + 2 \le j$. We define a vector \mathbf{t}' such that $t_k' = t_k$ for all k except for k = i, i + 1, j - 1, j where t_k' is equal to $t_i - 1, t_{i+1} + 1, t_{j-1} + 1, t_j - 1$, respectively. (However, we set $t_{i+1}' = t_{i+1} + 2$ if i + 1 = j - 1.) We observe that conditions (\diamondsuit) still hold for \mathbf{t}' . Moreover, we have $f(\mathbf{t}') = f(\mathbf{t})$, and by Lemma 2.6 $g(\mathbf{t}') \ge g(\mathbf{t})$. Hence, $h(\mathbf{t}') \ge h(\mathbf{t})$, and $\mathbf{t}' <_{\text{lex}} \mathbf{t}$ which contradicts the choice of vector \mathbf{t} .

In further analysis we denote $r_i := t_i/((d-1)n)$ for i = 1, ..., n-1. By Claim 5 it is enough to consider the following two cases.

Case 1. $t_i = 0$ for $i \notin \{1, 2, 3\}$.

Then, conditions (\Diamond) take the form of $t_1 + t_2 + t_3 = n$ and $t_1 + 2t_2 + 3t_3 = dn$. Hence, $t_2 + 2t_3 = (d-1)n$, or, equivalently, $t_2 + 2t_3 = 1$. Thus, we have

$$\frac{1}{(d-1)n}\log_2 f(\mathbf{t}) = \frac{1}{3}(r_2 + r_3) + \beta r_3 = \frac{1}{3}(1 - 2r_3 + r_3) + \beta r_3 = \frac{1}{3} + \left(\beta - \frac{1}{3}\right)r_3$$
$$\frac{1}{(d-1)n}\log_2 g(\mathbf{t}) = \frac{1}{2}r_2 + \frac{\log_2 6}{3}r_3 = \frac{1}{2}(1 - 2r_3) + \frac{\log_2 6}{3}r_3 = \frac{1}{2} - \left(1 - \frac{\log_2 6}{3}\right)r_3.$$

Hence, we obtain that $f(\mathbf{t}) = 2^{\frac{1}{3}(d-1)n}$ if $r_3 = 0$, and $f(\mathbf{t})$ is increasing as a function of r_3 . Similarly, $g(\mathbf{t})$ is a decreasing function of r_3 with $g(\mathbf{t}) = 2^{\frac{1}{2}(d-1)n}$ for $r_3 = 0$. Therefore, the minimum of $f(\mathbf{t})$ and $g(\mathbf{t})$ can be upper bounded by the value of f at point $\mathbf{t_0}$ such that $f(\mathbf{t_0}) = g(\mathbf{t_0})$. In our case this equality holds if

$$\frac{1}{3} + (\beta - \frac{1}{3}) r_3 = \frac{1}{2} - (1 - \frac{\log_2 6}{3}) r_3$$

from which we obtain

$$r_3 = \frac{\frac{1}{2} - \frac{1}{3}}{\beta - \frac{1}{3} + 1 - \frac{\log_2 6}{3}} = \frac{\frac{1}{6}}{\beta - \frac{1}{3} + 1 - \frac{\log_2 3 + 1}{3}} = \frac{\frac{1}{6}}{\beta - \frac{\log_2 3 - 1}{3}} = \frac{\frac{1}{6}}{\beta - \frac{\beta}{3}} = \frac{1}{4\beta}.$$

To obtain a bound on $h(\mathbf{t})$ it remains to plug the above value into the formula for $f(\mathbf{t})$:

$$\frac{1}{(d-1)n}\log_2 h(\mathbf{t}) \le \frac{1}{3} + \left(\beta - \frac{1}{3}\right) \frac{1}{4\beta} = \frac{7}{12} - \frac{1}{12\beta} = \alpha$$

Hence, $h(\mathbf{t}) \leq 2^{\alpha(d-1)n}$, as desired.

Case 2. $t_i = 0$ for $i \notin \{1, s, s + 1\}$, where $s \ge 3$.

We will show that in this case $g(\mathbf{t}) = o(2^{0.431(d-1)n}) = o(2^{\alpha(d-1)n})$. Conditions (\lozenge) take now the form of $t_1 + t_s + t_{s+1} = n$ and $t_1 + st_s + (s+1)t_{s+1} = dn$. Hence, we obtain $(s-1)t_s + st_{s+1} = (d-1)n$, or, equivalently, $(s-1)r_s + sr_{s+1} = 1$. Therefore

$$\frac{1}{(d-1)n}\log_2 g(\mathbf{t}) = \frac{\log_2 s!}{s}r_s + \frac{\log_2(s+1)!}{s+1}r_{s+1} = \frac{\log_2 s!}{s(s-1)}(1 - sr_{s+1}) + \frac{\log_2(s+1)!}{s+1}r_{s+1} \\
= \frac{\log_2 s!}{s(s-1)} - \left(\frac{\log_2 s!}{s-1} - \frac{\log_2(s+1)!}{s+1}\right)r_{s+1}.$$

One can verify that the inequality $\frac{\log_2 s!}{s-1} > \frac{\log_2(s+1)!}{s+1}$ holds for $s \ge 2$, and that the sequence $\left(\frac{\log_2 s!}{s(s-1)}\right)_{s>3}$ is decreasing. Therefore

$$\frac{1}{(d-1)n}\log_2 g(\mathbf{t}) \le \frac{\log_2 6}{6} < 0.431.$$

Hence, $g(\mathbf{t}) < 2^{0.431(d-1)n}$, which ends the proof.

4 Exponential space algorithm

In this section we establish Theorem 1.2.

Let G be a digraph with n vertices and m = dn edges. For simplicity, we assume in this section that n is even, for otherwise we can pick an arbitrary vertex v, and split it into two vertices v^{in} and v^{out} with edges inherited from v appropriately and with one additional edge $(v^{\text{in}}, v^{\text{out}})$ – this operation adds one vertex to the graph but does not increase the average outdegree. We will say that a simple path P in G is (l, D)-light if the length of P is l, and the sum of outdegrees of inner vertices of P is bounded by D. For a vertex $v \in V(G)$, and positive integers l, D, by $\mathcal{P}_{v,l,D}$ we will denote the family of all (l, D)-light paths in G which start at vertex v.

Our algorithm relies on the following two lemmas.

Lemma 4.1. Let H be a Hamiltonian cycle in G. Then, the edges of H can be partitioned into two (n/2, m/2)-light paths.

Lemma 4.2. For a digraph G, a vertex v, and integers l, D, the family $\mathfrak{P}_{v,l,D}$ can be computed in time $\tau(D/(l-1))^{l-1}n^{\mathfrak{O}(1)}$ where the function τ is defined as in Theorem 1.2.

Before we proceed to the proofs of above lemmas, let us see how to derive Theorem 1.2 from them. Given a digraph G, the algorithm starts by iterating over all pairs of distinct vertices u_1 and u_2 . For each such a pair we use Lemma 4.2 to obtain the families $\mathcal{P}_1 = \mathcal{P}_{u_1,n/2,m/2}$ and $\mathcal{P}_2 = \mathcal{P}_{u_2,n/2,m/2}$. By filtering them, we may assume that all paths from \mathcal{P}_1 end at u_2 , and all paths from \mathcal{P}_2 end at u_1 . Next, we create a dictionary \mathcal{D} with an entry $\{\text{key}: V(P_1), \text{value}: \text{weight}(P_1)\}$ for every path $P_1 \in \mathcal{P}_1$. (In case there is more than one path on the same set of vertices we keep only one entry with the minimum weight.) Then, we iterate over all paths $P_2 \in \mathcal{P}_2$, and we look up in \mathcal{D} a subset $V'(P_2) := (V(G) \setminus V(P_2)) \cup \{u_1, u_2\}$. For every hit we calculate the sum: weight $(P_2) + \mathcal{D}[V'(P_2)]$, and we return the minimum of these values.

The correctness of this procedure is a direct corollary from Lemma 4.1. Moreover, the running time of the algorithm is dominated, up to a polynomial factor, by the running time of the algorithm from Lemma 4.2, which in our case is bounded by

$$\tau \left(\frac{\frac{m}{2}}{\frac{n}{2}-1}\right)^{n/2-1} n^{\mathcal{O}(1)} = \tau \left(\frac{d}{1-\frac{2}{n}}\right)^{n/2-1} n^{\mathcal{O}(1)} = \tau(d)^{n/2} n^{\mathcal{O}(1)}$$

where the last equality follows from the fact that when d is fixed, then for sufficiently large n we have $\lfloor d/(1-\frac{2}{n})\rfloor = \lfloor d\rfloor$. Note that we implement the dictionary \mathcal{D} as a balanced tree, so each lookup takes time $\mathcal{O}(\log |\mathcal{D}|) = \mathcal{O}(n)$.

Proof of Lemma 4.1. Let k = n/2, and let $d_0, d_1, \ldots, d_{2k-1}$ be the outdegrees of consecutive vertices on H. Denote $S_i = d_i + d_{i+1} + \ldots + d_{i+k-1}$. (In this proof indices are understood modulo 2k.) We need to prove that for some index j both expressions $S_j - d_j$ and $S_{j+k} - d_{j+k}$ do not exceed m/2.

Let $R_i := S_i - S_{i+k}$. We observe that $R_k = S_k - S_0 = -R_0$. Hence, there exists an index $j \in \{0, \ldots, k-1\}$ such that $R_j \cdot R_{j+1} \leq 0$. Without loss of generality, we may assume that $R_j \leq 0$ (equivalently, $S_j \leq S_{j+k}$), for otherwise we can just shift all indices by k. Then, $R_{j+1} \geq 0$ (equivalently, $S_{j+1+k} \leq S_{j+1}$). Thus we obtain

$$S_j - d_j \le S_j \le \frac{1}{2}(S_j + S_{j+k}) = \frac{m}{2}$$

$$S_{j+k} - d_{j+k} \le S_{j+k+1} \le \frac{1}{2}(S_{j+k+1} + S_{j+1}) = \frac{m}{2}.$$

This ends the proof.

Before we proceed to the proof of Lemma 4.2, we state a technical lemma.

Lemma 4.3. Let a_1, \ldots, a_k be integers with an average bounded by \bar{a} . Then, $a_1 \cdot \ldots \cdot a_k \leq \tau(\bar{a})^k$.

Proof. First, we see that if the product of numbers a_i is maximum, then for all i we have $a_i = \lfloor \bar{a} \rfloor$ or $a_i = \lceil \bar{a} \rceil$, for otherwise we could either increase some number by 1, or replace some two numbers a_i , a_j ($a_i < a_j$) with numbers $a_i + 1$, $a_j - 1$. Assume that among numbers a_1, \ldots, a_k there are k_1 numbers equal to $\lfloor \bar{a} \rfloor$ and $k - k_1$ numbers equal to $\lceil \bar{a} \rceil$. Moreover, assume that $\bar{a} \notin \mathbb{Z}$, and consequently $\lceil \bar{a} \rceil = |\bar{a}| + 1$. Then

$$k_1|\bar{a}| + (k-k_1)(|\bar{a}|+1) = a_1 + \ldots + a_k \le k\bar{a}$$

Pseudocode 2: GeneratePaths(G, path, l, D)Input: G – a digraph, path – a sequence of vertices forming a path in G, l, D – positive integers Output: A collection of all simple paths of the form: $\text{path}\#(v_1,\ldots,v_l)$ such that $\sum_{i=1}^{l-1} \text{outdeg}(v_i) \leq D$ $u \leftarrow \text{the last vertex on path;}$ for $vertex\ v_1\ such\ that\ (u,v_1) \in E(G)\ \text{and}\ v_1 \not\in \text{path do}$ if $l=1\ \text{then}$ print $path\#v_1$; else if $\text{outdeg}(v_1)+(l-2)\leq D\ \text{then}$ GeneratePaths $(G, \text{path}\#v_1,\ l-1,\ D-\text{outdeg}(v_1))$;

and thus $k_1 \geq k(|\bar{a}| + 1 - \bar{a})$. Hence

$$a_1 \cdot \ldots \cdot a_k = |\bar{a}|^{k_1} \lceil \bar{a} \rceil^{k-k_1} \le |\bar{a}|^{k(\lfloor \bar{a} \rfloor + 1 - \bar{a})} (|\bar{a}| + 1)^{k(\bar{a} - \lfloor \bar{a} \rfloor)} = \tau(\bar{a})^k.$$

To finish the proof, we observe that if $\bar{a} \in \mathbb{Z}$, then $\tau(\bar{a}) = \bar{a}$, so we obtain a bound \bar{a}^k , as desired.

Proof of Lemma 4.2. We apply a simple branching procedure which starts at vertex v, and at each step guesses the next vertex on a path by considering all reasonable possibilities. A detailed description of the algorithm can be found in Pseudocode 2. (To compute the family $\mathcal{P}_{v,l,D}$ we call the function GeneratePaths with the arguments $(G, \{v\}, l, D)$.) Note that before appending a vertex to the current path we check whether the sum of outdegrees on the new path is not too large (line marked with (\checkmark) in the Pseudocode). More precisely, we check whether appending a sequence of vertices of outdegree 1 to the new path would give us a correct (l, D)-path.

The correctness of such a procedure is straightforward. It remains to estimate its time complexity. Let $\mathfrak{T}(G)$ be a search tree representing execution of this algorithm. We claim that $\mathfrak{T}(G)$ contains at most $\tau(D/(l-1))^{l-1}n$ leaves, where $\tau(d) = \lfloor d \rfloor^{\lfloor d \rfloor + 1 - d} (\lfloor d \rfloor + 1)^{d - \lfloor d \rfloor} \leq d$

We will say that a directed rooted tree \mathcal{T} has the property (\star) if for any path u_0, \ldots, u_h from the root of \mathcal{T} to some leaf u_h we have $h \leq l$, and the sum $\sum_{i=1}^{h-1} \text{outdeg}(u_i)$ is bounded by D - (l - h). From the description of the algorithm we see that $\mathcal{T}(G)$ has the property (\star) . Indeed, let u_0, \ldots, u_h be a path from the root to a leaf in $\mathcal{T}(G)$. Before the algorithm entered the vertex u_{h-1} the following condition was checked:

outdeg
$$(u_{h-1}) + (l - (h-2) - 2) \le D - \sum_{i=1}^{h-2} \text{outdeg}(u_i)$$

which is equivalent to $\sum_{i=1}^{h-1} \text{outdeg}(u_i) \leq D - (l-h)$. Hence it is enough to prove the following claim. (A similar statement appears in the work of Gebauer [23].)

Claim 6. Any tree \Im with the property (\star) has at most $\tau(D/(l-1))^{l-1}n$ leaves.

Given a tree \mathcal{T} with the property (\star) we modify it so that the property (\star) is preserved and the number of leaves in it does not increase. First, we may assume that all leaves in \mathcal{T}

are at depth exactly l. Indeed, let u_0, \ldots, u_h be a path from the root of \mathfrak{T} to some leaf u_h at depth h < l. Then, we may append to it a path $u_h, u_{h+1}, \ldots, u_l$ – this operation does not change the number of leaves, and the property (\star) is preserved because

$$\sum_{i=1}^{l-1} \operatorname{outdeg}(u_i) = \sum_{i=1}^{h-1} \operatorname{outdeg}(u_i) + (l-h) \le D$$

Next, we modify \mathcal{T} iteratively. Let $\mathcal{T}_1 := \mathcal{T}$. At *i*-th step, for $i = 1, \ldots, l-1$, we consider the family \mathcal{S}_i of all subtrees in \mathcal{T}_i with a root at depth *i*. Let $S_i \in \mathcal{S}_i$ be a subtree with the maximum number of leaves. We create a tree \mathcal{T}_{i+1} by substituting in \mathcal{T}_i all subtrees from \mathcal{S}_i with S_i . We observe that for every $i = 1, \ldots, l-1$ tree \mathcal{T}_i has depth l, the number of leaves in \mathcal{T}_i is bounded by the number of leaves in \mathcal{T}_{i+1} , and all vertices in \mathcal{T}_i at the same depth $j \leq i-1$ have the same outdegree. Combining the latter property with the fact that the condition (\star) holds for leaves in the subtree S_i , we obtain inductively that every tree \mathcal{T}_i still has the property (\star) .

Now, we consider the tree \mathcal{T}_l . For i = 0, ..., l-1 let d_i be the outdegree of any vertex at depth i in \mathcal{T}_l . Then we may bound the number of leaves in \mathcal{T}_l by

$$d_0 \cdot \prod_{i=1}^{l-1} d_i \le n \left(\frac{\sum_{i=1}^{l-1} d_i}{l-1} \right)^{l-1} \le n \left(\frac{D}{l-1} \right)^{l-1}$$

To obtain a tighter bound on the size of \mathcal{T}_l we observe that in the above estimation we obtain an equality only if $d_i = D/(l-1)$ for i = 1, ..., l-1. However, this is impossible unless expression D/(l-1) is integral. After applying Lemma 4.3 we get the tighter bound which proves the claim of Lemma 4.2.

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