Approximation of the Double Travelling Salesman Problem with Multiple Stacks

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Abstract

The Double Travelling Salesman Problem with Multiple Stacks, DTSPMS, deals with the collect and delivery of n commodities in two distinct cities, where the pickup and the delivery tours are related by LIFO constraints. During the pickup tour, commodities are loaded into a container of k rows, or stacks, with capacity c. This paper focuses on computational aspects of the DTSPMS, which is NP-hard.

We first review the complexity of two critical subproblems: deciding whether a given pair of pickup and delivery tours is feasible and, given a loading plan, finding an optimal pair of pickup and delivery tours, are both polynomial under some conditions on k and c.

We then prove a (3k)/2 standard approximation for the Min Metric k DTSPMS, where k is a universal constant, and other approximation results for various versions of the problem.

We finally present a matching-based heuristic for the 2 DTSPMS, which is a special case with k=2 rows, when the distances are symmetric. This yields a 1/2-o(1), 3/4-o(1) and 3/2+o(1) standard approximation for respectively Max 2 DTSPMS, its restriction Max 2 DTSPMS -(1,2) with distances 1 and 2, and Min 2 DTSPMS -(1,2), and a 1/2-o(1) differential approximation for Min 2 DTSPMS and Max 2 DTSPMS.

1 Introduction and problem statement

The Double Travelling Salesman Problem with Multiple Stacks, DTSPMS, was introduced in [37]. It was initially motivated by the request of a transportation company that owns a fleet of vehicles and a depot in m distinct cities. For any pair (C, C') of cities, the company handles orders from the suppliers of city C to the customers of city C', proceeding as follows. In city C, a single vehicle picks up all the orders to be delivered to a customer in C', storing them into a single container. Then the container is sent to C', where a local vehicle delivers the commodities. The operator is thus faced with two levels of transportation: the local routing inside the cities, and the long-haul transportation between the cities.

The DTSPMS addresses the local routing problem: given a set of orders from suppliers of C to customers of C', it aims at finding an optimal pair of tours (T_P, T_D) , where T_P is a pickup tour on C and T_D is a delivery tour on C'. The value of a solution (T_P, T_D) is the sum of the costs of tours T_P and T_D . The problem specificity relies on the way the containers are managed: a container consists of a given number of rows that can be accessed only by their front side, and no reloading plan is allowed. The rows of the container are thus subject to Last In First Out (LIFO) rules that constrain the delivery tour. Namely, for a given row, the latter has no other choice but to handle the goods stored in this row in the precise reverse order than the one performed by the pickup tour.

Note that the *Double Travelling Salesman Problem with Multiple Stacks* is also known as the *Multiple Stacks Travelling Salesman Problem* (STSP) in the litterature, [42, 40, 41].

The DTSPMS has strong connections with the Traveling Salesman Problem (TSP). Given a node set V and a distance function $d:V^2\to\mathbb{Q}^+$, the TSP consists in finding a tour T that visits every node $i\in V$ exactly once, and whose total distance $d(T)=\sum_{e\in T}d(e)$ is minimized, or maximized in some cases.

Formally, an instance $I=(n,k,c,d_P,d_D,\operatorname{opt})$ of the DTSPMS consists of a number n of orders, a number k of rows of the container, a capacity $c \geq \lceil n/k \rceil$ which is the same for all rows of the container, two distance functions $d_P,d_D:V^2\to\mathbb{Q}^+$ where $V=\{0,1,\ldots,n\}$, and an optimization goal opt $\in \{\min,\max\}$. Indices $i=1,\ldots,n$ refer to the orders (and thus to the location of the associated supplier in city C and the associated customer in city C'), whereas index 0 refers to the depot (and thus to the location of the depot in cities C and C'). Functions d_P,d_D satisfy $d_P(i,i)=d_D(i,i)=0, i\in V$. We consider instances $I_P=(V,d_P),I_D=(V,d_D)$ and $I_\Sigma=(V,d_\Sigma)$ of the TSP, where d_Σ is defined as

$$d_{\Sigma}(i,j) = d_P(i,j) + d_D(j,i), \quad i,j \in V$$

and the goal on I_P, I_D, I_{Σ} is to minimize iff the goal is to minimize on I.

On the one hand, a pair (T_P, T_D) of pickup and delivery tours is feasible for the DTSPMS on I iff there exists a loading plan of the commodities into the rows of the container that satisfies the following property:

Property 1. For every pair of orders (i, j) such that T_P handles i before j in the pickup tour, then either T_D handles j before i, or commodities i and j are loaded in two distinct rows.

Note that such a loading plan always exists when k=n and in that case the problem consists of solving two independent TSP. On the other hand, given a tour $T=(0,i_1,\ldots,i_n,0)$, let T^- denotes its reverse tour, i.e., $T^-=(0,i_n,\ldots,i_1,0)$. Then the pair (T,T^-) of pickup and delivery tours is a feasible solution for the DTSPMS on I for all k,c. For example, one feasible loading plan consists of putting commodities i_1,\ldots,i_c in the first row, then commodities i_{c+1},\ldots,i_{2c} in the second row, and so on.

Hence, solving the TSP independently on I_D and I_P can be seen as a relaxation of the DTSPMS on I, and solving DTSPMS on I provides a relaxation of the TSP on I_{Σ} . In particular, when k=n, the DTSPMS on I reduces to solving independently the TSP on I_D and I_P , and when k=1, the DTSPMS on I reduces to the TSP on I_{Σ} . Therefore the DTSPMS is $\mathbf{NP}-\mathbf{hard}$.

Notice that DTSPMS is different from the *Pickup and Delivery Travelling Salesman Problem* (PDTSP), where pickups and deliveries are operated during the same tour. Indeed, the PDTSP is a *Travelling Salesman Problem with Precedence Constraints* (PCTSP) where the set of precedence constraints consists of a perfect matching on $V \setminus \{0\}$, while in the DTSPMS, the computation of an optimal pickup tour (or of an optimal delivery tour) when the loading plan is given is a PCTSP where the precedence constraints partition $V = \{1, \ldots, n\}$ into k strict orders (see section 2.2).

Before closing this introduction by the organization of the paper, we now define some variants of the DTSPMS. When dealing with routing problems, various assumptions can be made on the distance functions. They may be symmetric, or satisfy the triangular inequalities (metric case), or take values in $\{a,b\}$ for two reals $a \neq b$. A distance d on a vertex set V is symmetric when $d(u,v) = d(v,u), u,v \in V$. It is metric provided that $d(u,v) \leq d(u,w) + d(w,v), u,v,w \in V$. The corresponding restrictions of the DTSPMS are denoted by Symmetric DTSPMS, Metric DTSPMS and DTSPMS - (a, b), respectively.

We denote by kDTSPMS the restriction of the DTSPMS where the number k of rows is a universal constant. We call the instances when $c \geq n$ as uncapacitated, and tight when $c = \lceil n/k \rceil$.

Many algorithms have already been proposed for the DTSPMS: see [37, 19, 20, 43] for metaheuristic approaches, [27] (where the authors solve the 2 DTSPMS by means of the k best pickup tours and the k best delivery tours), [36, 28, 2, 4] (branch and cut algorithms for the DTSPMS) or [10] (a branch and bound algorithm for the 2 DTSPMS) for exact approaches. Several methods have also been proposed for some generalizations of the problem such as the Pick-up and delivery TSP with multiple stacks [39] and VRP (multiple tours) with multiple stacks [23, 12]. In this paper we focus on the computational aspects of the DTSPMS. The paper is organized as follows:

• Section 2 reviews complexity aspects of two subproblems. Solutions of the DTSPMS consist of a pair of pickup and delivery tours on V and a loading plan on $\{1,\ldots,n\}$. If one fixes the former, then deciding whether it admits a feasible loading plan or not consists of an instance of the bounded coloring problem, BC in permutation graphs (section 2.1). If one fixes the latter, then optimizing the pickup or delivery tour with respect to the considered loading plan consists of an instance of the TSP with precedence constraints, PCTSP where the precedence constraints partition $\{1,\ldots,n\}$ into at most k linear orders (section 2.2). These two subproblems of the DTSPMS may turn to be tractable, depending on k and c.

- Section 3 provides approximation results with the standard ratio for the DTSPMS with a general number k of rows, exploring connections to the TSP. We first compare the optimal value of the DTSPMS to optimal values of the TSP with distances d_P , d_D and $d_P + d_D$, in the general and metric cases. We then derive both positive and negative approximability results for the DTSPMS, in particular we show that Hamiltonian cycles that are 3/2-standard approximate for the Min Metric TSP and the distance $d_P + d_D$ are 3-standard approximate for the Min Metric DTSPMS.
- Section 4 provides standard approximation ratios for the symmetric 2 DTSPMS, *i.e.* k=2 and distances are symmetric. We present a matching-based heuristic which yields a standard approximation ratio of 1/2 o(1), 3/4 o(1) and 3/2 + o(1) for respectively Max 2 DTSPMS, Max 2 DTSPMS -(1,2) and Min 2 DTSPMS -(1,2).
- Section 5 is strictly devoted to differential approximation. The main result is a differential approximation ratio of 1/2 o(1) for the 2 DTSPMS, obtained by adapting the matching heuristic of the previous section.

Section 6 finally concludes the paper with some perspectives.

2 Complexity of DTSPMS subproblems

A solution of the DTSPMS consists of two parts: the loading plan, and the pair of pickup and delivery tours. We here investigate the subproblems obtained when fixing either the loading plan, or the pair of pickup and delivery tours.

2.1 Deciding feasibility of a pair of tours

A pair of tours T_P and T_D is a feasible pair of pickup and delivery tours iff every pair of nodes (i,j) that are visited in the same order in T_P and T_D can be loaded in two distinct rows (Property 1). Equivalently, let $G' = (V \setminus \{0\}, E')$ be the graph where E' consists of the pairs of commodities i,j such that T_P and T_D visit i,j in the same order; then (T_P,T_D) is a feasible pair of pickup and delivery tours $iff\ V \setminus \{0\}$ can be partitioned into k independent sets of size at most c in G'. Indeed, assume that there exists such a partition W_1,\ldots,W_k of $V \setminus \{0\}$. We note the pick-up tour $T_P = (0,i_1,\ldots,i_n,0)$. One can build a feasible loading plan the following way: iteratively for $p = 1,\ldots,n$, load the node i_P at the end of the row r such that $i_P \in W_r$. By definition of E', such a loading plan admits T_D as a delivery tour compatible with T_P : when two nodes $i \neq j \in V \setminus \{0\}$ are loaded in a same row r, either T_P visits i before j and T_D visits j before i, or T_P visits j before i and T_D visits i before j. Hence, every non-empty row $r \in \{1,\ldots,k\}$ consists of a subsequence (i'_1,\ldots,i'_ℓ) of (i_1,\ldots,i_n) that is visited in the reverse order (i'_l,\ldots,i'_ℓ) in T_D .

Now G' is a permutation graph (consider the ordering either i_1, \ldots, i_n or j_1, \ldots, j_n of $V \setminus \{0\}$) and thus, finding a coloring of G' using at most k colors,

if such a coloring exists, requires a $O(n \log n)$ time (see [38]). Consequently, deciding the feasibility of a pair of tours for the uncapacited DTSPMS is in **P**. Furthermore, the *Bounded Coloring problem*, BC consists, given a graph G and two integers k, c such that $kc \geq n$, in assigning to every node $i \in V(G)$ a color $f(v) \in \{1, \ldots, k\}$ in such a way that:

- 1. f is a coloring, that is, two nodes $i, j \in V(G)$ such that $(i, j) \in E(G)$ are assigned a different color;
- 2. every color $r \in \{1, \dots, k\}$ is assigned to at most c vertices of V(G).

The BC problem is also known as the Mutual Exclusion Scheduling problem in the literature. This problem was proved to be $\mathbf{NP}-\mathbf{complete}$ in permutation graphs given any universal constant $c \geq 6$, [24]. Bonomo et al. extended the BC problem to the "Capacitated Coloring problem", CC where a maximum size c_r is specified for each color r. They proposed a $O(n^{k^2+k+1}k^3)$ -time algorithm to solve the CC problem in co-comparability graphs. Consequently, deciding the feasibility of a pair of tours for the kDTSPMS is in \mathbf{P} . Proposition 2.1 summarizes the former discussion.

Proposition 2.1 ([11, 42, 9]). Given a pair (T_P, T_D) of tours, deciding whether it admits or not a consistent loading plan and designing such a loading plan if the answer is YES is:

- 1. in **P** when c = n (regardless of k) [11, 42] or k is a universal constant [9];
- 2. NP complete when c is a universal constant greater than 5 [9].

2.2 Optimizing the tours for a given loading plan

Let \mathcal{P} be a loading plan of $\{1,\ldots,n\}$. A tour T on V is a consistent pickup tour with respect to \mathcal{P} iff for any pair of nodes (i,j) such that \mathcal{P} loads i at a lower position than j in a same row r, T visits i before j. Symmetrically, T on V is a delivery tour consistent with \mathcal{P} iff for any two distinct nodes i,j such that, in some row r, \mathcal{P} loads i before j, T visits j before i. Hence, the problem of computing an optimal tour with respect to opt and d_P (resp., d_D), among the pickup and delivery tours that are consistent with \mathcal{P} , is an instance of the Travelling Salesman Problem with Precedence Constraints, PCTSP, where the precedence constraints partition $\{1,\ldots,n\}$ into k strict orders.

The PCTSP consists, given a binary relation B on $\{1, \ldots, n\}$ encoding precedence constraints between pairs of nodes, in finding an optimal tour among those tours T that satisfy:

$$T \text{ visits } i \text{ before } j, \qquad (i,j) \in B$$
 (1)

This problem also is know as "the Minimum Setup Scheduling" problem in the literature. Colbourn and Pulleyblank proposed in [15] a dynamic programming procedure which runs within polynomial time when the precedence constraints

define a partial order of bounded width, *i.e.*, when the maximum number of pairwise incomparable nodes is bounded above by a universal constant.

Here, the set of pairs of nodes subject to precedence constraints for the pickup (resp., delivery) tour instance induced by a loading plan \mathcal{P} , is denoted by B_P (resp., B_D):

$$B_P = \{(i, j) \mid \mathcal{P} \text{ loads } i \text{ in the same row as } j \text{ at a lower position than } j\}$$

 $B_D = \{(j, i) \mid (i, j) \in B_P\}$

 B_P and B_D encode two partial orders of same width at most k. Consequently,

Proposition 2.2 ([11, 42]). The computation of a best compatible pair (T_P, T_D) of pickup and delivery tours with respect to a given loading plan can be done within a $O(k^2|V|^k)$ time. This problem therefore is in **P** for all constant positive integer k when considering the kDTSPMS.

Let us comment a bit further this fact. The dynamic procedure proposed in [15] can be seen as some application of the dynamic programming procedure for the TSP [22]. Let T^* refer to an optimal tour on V. Moreover, given a subset $W \subseteq V \setminus \{0\}$ together with a node $i \in W$, let $P^*_{W,i}$ refer to an optimal Hamiltonian path from 0 to i on $W \cup \{0\}$. Then the optimal tour expresses as

$$T^* = \arg \operatorname{opt}_{i \in V \setminus \{0\}} \left\{ d(P_{V \setminus \{0\}, i}^*) + d(i, 0) \right\}$$
 (2)

for a collection $\{P_{W,i}^* \mid W \subseteq V \setminus \{0\}, i \in W\}$ of paths that satisfies the system:

$$P_{\{i\},i}^* = (0,i), i \in V \setminus \{0\} (3)$$

$$P_{W,i}^* = \operatorname{arg} \operatorname{opt}_{j \in W \setminus \{i\}} \left\{ d(P_{W \setminus \{i\},j}^*) + d(j,i) \right\}, \qquad i \in W \subset V \setminus \{0\}$$
 (4)

Any algorithm that implements this principle will generate $\sum_{p=2}^n p\binom{n}{p} = \Theta(n2^n)$ paths $P^*_{W,i}$, while the computation of a given path $P^*_{W,i}$ requires the comparizon of |W|-1 values.

Now, when considering a set B of precedence constraints, not any pair $(W \subseteq V \setminus \{0\}, i \in W)$ is relevant, as not any elementary path with starting vertex 0 can be completed into a tour on V that is consistent with B. Let $\pi(W)$ denote the set of vertices i such that $(i,j) \in B$ for some vertex $j \in W$; then, on the one hand, W must satisfy $\pi(W) \subseteq W$. On the other hand, i must satisfy $i \in W \setminus \pi(W)$. Equivalently, we shall restrict to the pairs $(W \cup \pi(W), i)$ where W is an antichain of the considered partial order and $i \in W$. If k refers to the width of the partial order, then the number of such pairs is bounded above by $\sum_{p=1}^k p\binom{n}{p} \le kn^k$. In particular when $B = B_D$ (resp., $B = B_P$), the antichains to consider correspond to the choice of at most one element per row and the dynamic procedure runs within $O(k^2n^k)$ -time.

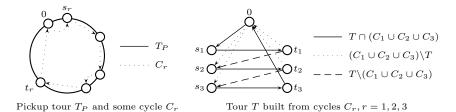


Figure 1: The tour T of Lemma 3.1: illustration when h=3

3 Standard approximation of the DTSPMS using reductions to the TSP

We study the relative complexity of DTSPMS in regards to TSP. The question is to know in what extent DTSPMS is harder to solve than the TSP, and how far the optimal value of DTSPMS is from the optimal value of related TSP.

3.1 Comparison of optimal values

In what follows, we describe a loading plan as a collection $\mathcal{P} = (P_1, \dots, P_k)$ of node-disjoint paths on $V \setminus \{0\} = \{1, \dots, n\}$. Such a collection actually corresponds to a loading plan iff(i) it partitions $V \setminus \{0\}$, i.e., $\bigcup_{r=1}^k V(P_r) = V \setminus \{0\}$ and (ii) each path P_r connects at most c nodes. The loading plan loads item i in position p of row r iff node i is in position p in path P_r .

Let $I=(n,k,c,d_P,d_D,\text{opt})$ be an instance of the DTSPMS. We consider the three instances I_P,I_D,I_Σ of the TSP that were introduced in section 5.1. The following relations are trivially satisfied:

$$OPT(I_P) + OPT(I_D) \succ OPT(I) \succ OPT(I_{\Sigma})$$
 (5)

Now assume that distance functions d_P and d_D are metric, and the goal on I is to minimize. Since both d_D and d_P satisfy the triangular inequalities, so does d_{Σ} , and thus I_{Σ} is metric. The metric case provides a more accurate comparison of the optimal values on I and I_{Σ} than in the general case:

Lemma 3.1. For all instances $I = (n, k, c, d_P, d_D)$ of the Min Metric DTSPMS, I and its related instance I_{Σ} of the Min Metric TSP satisfy

$$OPT(I_{\Sigma}) \le k \, OPT(I)$$
 (6)

Furthermore, relation (6) is asymptotically tight.

Proof. Let (\mathcal{P}, T_P, T_D) be any feasible solution on I, and let h refer to the number of rows \mathcal{P} actually uses. We establish inequality (6) by deriving from (\mathcal{P}, T_P, T_D) a solution T of the TSP on I_{Σ} with value:

$$d_{\Sigma}(T) \le h \left(d_P(T_P) + d_D(T_D) \right) \tag{7}$$

We assume w.l.o.g that in \mathcal{P} , rows 1 to h are the non-empty ones. Each row $r \in \{1, \ldots, h\}$ consists of a path P_r from some node s_r to some node t_r . We define the following cycles:

$$C_r = \{(0, s_r)\} \cup P_r \cup \{(t_r, 0)\}, \qquad r \in \{1, \dots, h\}$$

For any row $r \in \{1, ..., h\}$, T_P and P_r visit the nodes of $V(P_r)$ in the same order, whereas T_D visits these nodes in the reverse order (see Figure 1). We deduce from this observation and the fact that d_P, d_D satisfy the triangular inequality:

$$d_P(C_r) \le d_P(T_P), \qquad r \in \{1, \dots, h\} \tag{8}$$

$$d_D(C_r^-) \le d_D(T_D), \qquad r \in \{1, \dots, h\}$$
 (9)

where C^- denotes cycle C in the reverse order. Summing relations (8) and (9) over r = 1, ..., h, we obtain:

$$\sum_{r=1}^{h} d_{\Sigma}(C_r) \le h \left(d_P(T_P) + d_D(T_D) \right) \tag{10}$$

Now consider the tour T defined as (see Figure 1):

$$T = \left(\bigcup_{r=1}^{h} C_r \setminus \bigcup_{r=1}^{h-1} \{(t_r, 0), (0, s_{r+1})\}\right) \bigcup_{r=1}^{h-1} \{(t_r, s_{r+1})\}$$

By construction, we have:

$$d_{\Sigma}(T) = \sum_{r=1}^{h} d_{\Sigma}(C_r) - \sum_{r=1}^{h-1} \left(d_{\Sigma}(t_r, 0) + d_{\Sigma}(0, s_{r+1}) - d_{\Sigma}(t_r, s_{r+1}) \right)$$
(11)

Since d_{Σ} is metric, for all $r \in \{1, ..., h-1\}$, quantity $d_{\Sigma}(t_r, 0) + d_{\Sigma}(0, s_{r+1}) - d_{\Sigma}(t_r, s_{r+1})$ is non-negative. It thus follows from relations (11) and (10) that the proposed tour T indeed satisfied inequality (7). Relation (6) then is a straightforward consequence of inequalities $OPT(I_{\Sigma}) \leq d_{\Sigma}(T)$, $d_{\Sigma}(T) \leq h(d_P(T_P) + d_D(T_D))$ and $h \leq k$.

It remains us to establish that this relation is asymptotically tight. To do so, we associate with any real number $\lambda > 0$ and any two integers k, c such that $c \geq k \geq 2$ an instance $I(\lambda, k, c)$ of the Min Metric DTSPMS on kc + 1 vertices (including the depot vertex) with k rows, each with capacity c. Let T_P and T_D be the tours defined by (indices are taken modulo kc):

$$T_P = \{(i, i+1) \mid i \in \{0, \dots, kc\}\} \quad T_D = \{(i, i+c) \mid i \in \{0, \dots, kc\}\}$$

Then on $I(\lambda, k, c)$, d_P takes value λ over T_P , d_D takes value 1 T_D , d_P and d_D are defined by metric closure anywhere else.

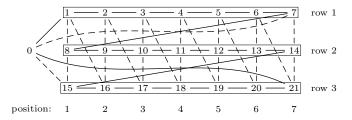


Figure 2: Tighness of relation (6): illustration when k=3 and c=7. Plain lines indicate edges of distance λ for d_P whereas dashed lines indicate edges of distance 1 for d_D .

By construction, T_P and T_D are optimal on respectively I_P and I_D . Moreover, the pair (T_P, T_D) is feasible for the DTSPMS on $I(\lambda, k, c)$, considering the loading plan $\mathcal{P} = (P_1, \ldots, P_k)$ defined as (see Figure 2 for some illustration):

$$P_r = ((r-1)c+1, (r-1)c+2, \dots, rc), \quad r \in \{1, \dots, k\}$$

With respect to \mathcal{P} , T_P follows paths P_1, P_2, \ldots, P_k in that order, whereas T_D visits the nodes according to their position in the container, from position c to position 1. The pair (T_P, T_D) therefore defines an optimal solution on $I(\lambda, k, c)$, with value:

$$OPT(I(\lambda, k, c)) = (kc + 1) \times (\lambda + 1) \tag{12}$$

Now consider instance $I_{\Sigma}(\lambda,k,c)$ of the Min Metric TSP. Let i,j be two distinct nodes from $\{0,\ldots,kc\}$. By definition of T_P and T_D , if i and j are at distance lesser than c in T_P , then they are at distance at least k in T_D and thus, $d_{\Sigma}(i,j) \geq 1 \times \lambda + k \times 1$. Otherwise, we have $d_{\Sigma}(i,j) \geq c \times \lambda + 1 \times 1$. Furthermore, if i and j are adjacent in T_P (resp., in T_D), then they are at distance exactly k (resp., c) in T_D (resp., in T_P). We deduce that either T_P or T_D is optimal on $I_{\Sigma}(\lambda,k,c)$, depending on $c\lambda + 1$ versus $\lambda + k$. The optimal value on $I_{\Sigma}(\lambda,k,c)$ therefore satisfies

$$OPT(I_{\Sigma}(\lambda, k, c)) = (kc+1) \times \min\{\lambda + k, \lambda c + 1\}$$
(13)

In particular if $\lambda + k \leq \lambda c + 1$ iff $\lambda \geq (k-1)/(c-1)$, then we have:

$$\frac{OPT(I_{\Sigma}(\lambda, k, c))}{OPT(I(\lambda, k, c))} = \frac{\lambda + k}{\lambda + 1}$$

This ratio tends to k when $\lambda = (k-1)/(c-1)$ and c tends to $+\infty$, what concludes the proof.

3.2 Approximation results

Approximation theory aims at providing approximate solutions of good quality for optimization problems that are hard to solve. Although we recall some

definitions, we assume that the reader is familiar with the main concepts of approximation theory. If not, we refer to, eg., [3, 16] for standard and differential approximation, respectively. In what follows, since one manipulates both maximization and minimization goals, we use notations \succeq , \succ , opt, \overline{opt} instead of \geq , >, max, min (resp., \leq , <, min, max) if the goal is to maximize (resp., to minimize).

Let $\Pi \in \mathbf{NPO}$ be an optimization problem and \mathcal{I}_{Π} its set of instances. Given an instance $I \in \mathcal{I}_{\Pi}$, the value of an optimal solution on I is denoted by OPT(I). Finally, let \mathcal{A} be an algorithm that provides feasible solutions for Π ; then APX(I) refers to the value of the solution output by \mathcal{A} on $I \in \mathcal{I}_{\Pi}$.

The standard approximation ratio compares the value APX(I) of the approximate solution to the optimal value OPT(I). If Π is a maximation problem, then \mathcal{A} is said to be ρ -approximate for some function $\rho: \mathcal{I}_{\Pi} \to]0,1]$ iff

$$APX(I) \ge \rho(I) \times OPT(I),$$
 $I \in \mathcal{I}_{\Pi}$

If the goal on Π is to minimize, \mathcal{A} is said to be ρ -approximate for some function $\rho: \mathcal{I}_{\Pi} \to [1, +\infty[$ iff

$$APX(I) \le \rho(I) \times OPT(I), \qquad I \in \mathcal{I}_{\Pi}$$

 Π is said to be approximable within factor ρ iff it admits a polynomial time ρ -approximation algorithm.

For instance, within the standard approximation framework, the Symmetric Max TSP is approximable within factor 61/81-o(1), [13]. By contrast, the Symmetric Min TSP is not approximable within any constant factor. However, when restricting to metric instances, the Symmetric Min Metric TSP is approximable within a factor of 3/2 (by means of the famous Christofides algorithm [14]).

Preliminary observe that the TSP naturally reduces to the DTSPMS. Namely, let I be an instance of the TSP on vertex set V, and pick any $v \in V$. Given any two non-negative integers k,c such that $kc \geq |V|-1$, we can associate with I an instance I' of DTSPMS with k rows of capacity c, on which vertex v represents the depot vertex, and a tour T on V takes distances $d_P(T) = d(T)/2$ and $d_D(T) = d(T^-)/2$. Given any tour T on V, (T,T^-) is a feasible pair of pickup and delivery tours on I', with value d(T). Conversely, any feasible pair (T_P,T_D) of pickup and delivery tours on I' brings two feasible solutions T_P and T_D^- of I whose value satisfy:

$$\min (d(T_P), d(T_D^-)) \le (d_P(T_P) + d_D(T_D))/2 \le \max (d(T_P), d(T_D^-))$$

Instance I' of the DTSPMS can be seen as some generalization of instance I of the TSP in that I admits more solutions and possibly more solution values than I does. However, one can with each such solution (T_P, T_D) associate a solution T of I – and thus, a solution (T, T^-) of I' – with value at least $d_P(T_P) + d_D(T_D)$ if the goal is to maximize, at most $d_P(T_P) + d_D(T_D)$ if the goal is to minimize. We conclude that the two instances I and I' are equivalent to approximate.

Thereby, known inapproximability bounds for the TSP also hold for the DTSPMS. In particular, the symmetric MinTSP is not approximable within

ratio $2^{-p(|V|)}$ for all polynomials p unless $\mathbf{P} = \mathbf{NP}$ (folklore, but see e.g., [44]). The Symmetric Min DTSPMS therefore is $\mathbf{NP} - hard$ to approximate within a ratio of $O(2^{|V|})$. We similarly deduce from [5] that Max Metric DTSPMS is Max SNP-hard.

DTSPMS reduces to TSP by means of a polynomial-time reduction that preserves standard approximation up to some factor for the maximization case, as well as for the bivalued case, as described in the proposition below:

Proposition 3.2. The DTSPMS reduces to the TSP by means of a polynomial time reduction that maps ρ -standard approximate solutions of the TSP onto solutions of the DTSPMS with a standard approximation guarantee of

- (i) $\rho/2$ for Max DTSPMS,
- (ii) $(\rho + a/b)/2$ for Max DTSPMS (a, b),
- (iii) $(\rho + b/a)/2$ for Min DTSPMS (a, b).

The reduction preserves the distance properties of the input instance. It thus notably maps symmetric, metric, (a,b)-valued instances of the DTSPMS to respectively symmetric, metric, (a,b)-valuated instances of the TSP.

Proof. Let $I=(n,k,c,d_P,d_D,\text{opt})$ be an instance of DTSPMS. Compute a tour T_{α} on I_{α} , $\alpha \in \{P,D\}$. Pick the pair (T_P,T_P^-) of pickup and delivery tours if $d_P(T_P)+d_D(T_P^-) \succeq d_P(T_D^-)+d_D(T_D)$ and the pair (T_D^-,T_D) otherwise.

Assume that T_{α} is ρ -approximate for the TSP on I_{α} , $\alpha \in \{P, D\}$. By construction, the value APX(I) of the approximate solution on I satisfies:

$$APX(I) \succeq 1/2 \times (d_P(T_P) + d_D(T_P^-) + d_P(T_D^-) + d_D(T_D))$$

$$\succeq \rho/2 \times (OPT(I_P) + OPT(I_D)) + 1/2 \times (d_D(T_P^-) + d_P(T_D^-))$$

$$\succeq \rho/2 \times OPT(I) + 1/2 \times (d_D(T_P^-) + d_P(T_D^-))$$

The result is straightforward for (i), considering $d_D(T_P^-) + d_P(T_D^-) \ge 0$. As for (ii) and (iii), simply observe that both quantities $d_D(T_P^-) + d_P(T_D^-)$ and OPT(I) express as the sum of 2(n+1) edge distances, that all belong to $\{a,b\}$ where $0 \le a < b$. In the maximization case, we deduce that we have:

$$d_D(T_D^-) + d_P(T_D^-) > 2(n+1)a > a/b \times 2(n+1)b > a/b \times OPT(I)$$

The argument for the minimization case is symmetrical.

As for the metric case, Lemma 3.1 yields the following conditional approximation result:

Proposition 3.3. For all constant positive integer k, Min Metric k DTSPMS reduces to the Min Metric TSP by means of a polynomial time reduction that preserves the standard approximation ratio up to a multiplicative factor of k. The reduction maps symmetric instances of the DTSPMS to symmetric instances of the TSP.

Proof. Given an instance I of the Min Metric DTSPMS, first associate with I instance I_{Σ} of Min Metric TSP. Then associate with any tour T on I_{Σ} the pair (T, T^{-}) of pickup and delivery tours on I. If T is ρ -approximate on I_{Σ} , then the value APX(I) of the approximate solution on I satisfies

$$APX(I) = d_{\Sigma}(T) \le \rho \times OPT(I_{\Sigma}) \le k\rho \times OPT(I)$$

where the right-hand side inequality follows from (6).

We derive from Propositions 3.2 and 3.3 the following positive approximation results for the DTSPMS:

Theorem 3.4. The following bounds hold for the standard approximability of DTSPMS:

Restriction	Ratio	Reference	
Max DTSPMS	3/8	(i) of Prop. 3.2 & [34]	
$Symmetric \; Max DTSPMS$	7/18	(i) of Prop. 3.2 & [35]	
Max Metric DTSPMS	35/88	(i) of Prop. 3.2 & [26]	
$Symmetric \; Max Metric DTSPMS$	7/16	(i) of Prop. 3.2 & [25]	
$Symmetric \; MaxDTSPMS - (0,1)$	3/7	(i) of Prop. 3.2 & [7]	
Min Metric DTSPMS	$(2k\log V)/3$	Prop. 3.3 & [18]	
$Symmetric \ Min \ Metric \ DTSPMS$	(3k)/2	Prop. 3.3 & [14]	
MinDTSPMS-(1,2)	13/8	(iii) of Prop. 3.2 & [8]	
$Symmetric \ Min \ DTSPMS - (1,2)$	11/7	(iii) of Prop. 3.2 & [7, 1]	

4 Standard approximation of the symmetric 2 DTSPMS

Thereafter, we assume that the distance functions are symmetric, the container has two rows, each of which can receive (at least) $\lceil n/2 \rceil$ commodities.

When facing routing problems, it is rather natural to manipulate optimal matchings, as they somehow bring "one half" of the optimum value. We already know from Proposition 3.4 that the Minimum metric case of 2 DTSPMS is approximable within standard factor 3. We here present a matching-based heuristic that provides standard approximation ratios of 1/2 - o(1) for the Max 2 DTSPMS, $1/2 \times (1 + a/b) - o(1)$ for the Max 2 DTSPMS - (a, b), and $1/2 \times (1 + b/a) + o(1)$ for the Min 2 DTSPMS - (a, b).

4.1 The matching-based algorithm

Let I be an instance of the 2DTSPMS and let V denote the vertex set that is considered in I. Algorithm 1 runs in three steps. First, it computes for $\alpha \in \{P, D\}$ a (near-) perfect matching M_{α} on V that is optimal with respect to d_{α} and opt, which is well known to require a low $(\mathcal{O}(|V|^3))$ polynomial time.

Second, it builds a loading plan $\mathcal{P}=(P_1,P_2)$, considering the connected components of the multigraph $(V,M_P\cup M_D)$ one after each other, starting with the connected component that contains the depot vertex 0. Let W_0,\ldots,W_h

```
Algorithm 1: APX_2DTSPMS
 Input: A vertex set V = \{0, 1, ..., n\} where n > 0, two symmetric
            distance functions d_P, d_D: V^2 \to \mathbb{Q}^+, an optimization goal opt
  Output: A balanced 2-rows loading plan \mathcal{P} of V\setminus\{0\} and an optimal
               pair (T_P^*(\mathcal{P}), T_D^*(\mathcal{P})) of pickup and delivery tours on V with
               respect to d_P, d_D, opt and \mathcal{P}
  foreach \alpha in \{P, D\} do
      Compute a (near-) perfect matching M_{\alpha} on V that is optimal with
        respect to d_{\alpha} and opt;
 \mathcal{P} \longleftarrow (\emptyset, \emptyset);
  foreach Connected component W of the multi-edge set M_P \cup M_D do
      if W induces a cycle (0, v_1, \ldots, v_{2m-1}, 0) then
          Insert (v_1, \ldots, v_m) at the beginning of row 1 in \mathcal{P};
       Insert (v_{2m-1}, \ldots, v_{m+1}) at the beginning of row 2 in \mathcal{P};
      else if W induces a chain (0, v_1, \ldots, v_{2m}, 0) \setminus \{(v_j, v_{j+1})\} then
          Insert (v_1, \ldots, v_m) at the beginning of row 1 in \mathcal{P};
       Insert (v_{2m}, \ldots, v_{m+1}) at the beginning of row 2 in \mathcal{P};
      else if W induces a cycle (v_1, \ldots, v_{2m}, v_1) then
       Add (v_1, \ldots, v_m) at the end of row 1 in \mathcal{P};
Add (v_{2m}, \ldots, v_{m+1}) at the end of row 2 in \mathcal{P};
      else if W induces a chain (v_1, \ldots, v_{2m+1}) then
          Add (v_1, \ldots, v_m) at the end of row 1 in \mathcal{P};
          Add (v_{2m+1}, \ldots, v_{m+1}) at the end of row 2 in \mathcal{P};
  foreach \alpha in \{P, D\} do
      Compute a tour T^*_{\alpha}(\mathcal{P}) on V that is consistent with \mathcal{P} and optimal
        with respect to d_{\alpha} and opt;
```

return $(\mathcal{P}, T_P^*(\mathcal{P}), T_D^*(\mathcal{P}))$;

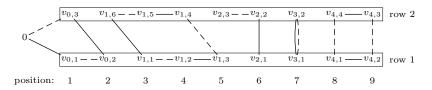


Figure 3: The approximate loading plan given the two (near-)perfect matchings M_P (in plain lines) and M_D (in dashed lines) on V.

denote these components. If |V| is even, then every component W_s induces on $M_P \cup M_D$ an elementary cycle of even length. Otherwise, W_s induces an elementary chain on an odd number of vertices for a single index $s \in \{0,\ldots,h\}$. We assume w.l.o.g that $0 \in W_0$; thus vertex set W_0 induces either a cycle $(v_{0,0}=0,v_{0,1},\ldots,v_{0,m_0},0)$, or a chain which consists of a cycle $(v_{0,0}=0,v_{0,1},\ldots,v_{0,m_0},0)$ minus some edge $(v_{0,j},v_{0,j+1})$ for a single index $j \in \{0,\ldots,m_0\}$ (index j+1 is taken modulo m_0+1). For W_0 , the heuristic inserts the sequences

$$\begin{array}{ll} & (v_{0,1},\dots,v_{0,\lceil m_0/2\rceil}) & \text{at the beginning of } P_1 \\ \text{and} & (v_{0,m_0},\dots,v_{0,\lceil m_0/2\rceil+1}) & \text{at the beginning of } P_2. \end{array}$$

Any other component W_s induces either a cycle $(v_{s,1}, \ldots, v_{s,m_s}, v_{s,1})$ or a chain $(v_{s,1}, \ldots, v_{s,m_s})$, where the nodes $v_{s,1}, \ldots, v_{s,m_s}$ all belong to $V \setminus \{0\}$. The heuristic inserts the sequences

$$\begin{array}{ll} & (v_{s,1},\ldots,v_{s,\lfloor m_s/2\rfloor}) & \text{at the end of } P_1 \\ \text{and} & (v_{s,m_s},\ldots,v_{s,\lfloor m_s/2\rfloor+1}) & \text{at the end of } P_2. \end{array}$$

Figure 3 provides an illustration of the obtained loading plan. In any case, the obtained approximate packing \mathcal{P} satisfies the capacity constraints: row 1 receives plus one vertex vs. row 2 when loading vertices from W_0 if W_0 induces a cycle, whereas it receives minus one vertex vs. row 2 when loading vertices from W_s for some index $s \in \{1, \ldots, h\}$ if |V| is odd and W_s is the single component that induces a chain.

Third, it computes the best pair $(T_P^*(\mathcal{P}), T_D^*(\mathcal{P}))$ of pickup and delivery tours with respect to \mathcal{P} and opt. According to Proposition 2.2, this last step requires a $\mathcal{O}(|V|^2)$ computation time. The overall complexity of Algorithm 1 is therefore polynomial.

4.2 Approximation analysis

Theorem 4.1. Algorithm 1 provides within polynomial time a standard approximation guarantee of

(i)
$$1/2 - 1/(2|V|)$$
 for Max DTSPMS,

(ii)
$$1/2 \times (1+a/b) - 1/(2|V|) \times (1-a/b)$$
 for $\operatorname{Max} \mathsf{DTSPMS} - (\mathsf{a},\mathsf{b}),$

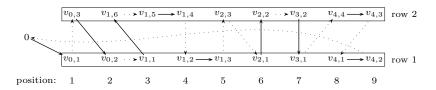


Figure 4: A feasible pickup tour that uses all the edges in M_P . Edges of M_P are depicted in plain lines.

(iii)
$$1/2 \times (1 + b/a) + 1/(2|V|) \times (b/a - 1)$$
 for Min DTSPMS – (a, b).

Moreover, all these approximation ratios are tight.

Proof. Let $APX = d_P(T_P^*(\mathcal{P})) + d_D(T_D^*(\mathcal{P}))$ denote the value of the approximate solution. By construction, given $\alpha \in \{P, D\}$, M_{α} is consistent with the approximate loading plan \mathcal{P} . Said equivalently, there exist two edge sets A_P, A_D such that $(M_P \cup A_P, M_D \cup A_D)$ defines a feasible pair of pickup and delivery tours with respect to \mathcal{P} (see Figure 4 for some illustration). Since Algorithm 1 returns the best pair of such tours, the approximate value satisfies:

$$APX \succeq d_P(M_P) + d_D(M_D) + d_P(A_P) + d_D(A_D)$$
 (14)

If |V| is even, then any tour T on V is the union of two perfect matchings on V. Since M_{α} , $\alpha \in \{P, D\}$ are optimal-weight matchings, we deduce:

$$d_{\alpha}(M_{\alpha}) \succeq \frac{1}{2}OPT(I_{\alpha}), \qquad \alpha \in \{P, D\}$$
 (15)

We deduce from relations (14),(15) and (5):

$$APX \succeq \frac{1}{2}OPT(I) + d_P(A_P) + d_D(A_D)$$
(16)

This enables to conclude result (i) for the Max 2 DTSPMS, considering $d_P(A_P) \ge 0$ and $d_D(A_D) \ge 0$. For the bivalued case given two real numbers 0 < a < b, considering that $d_P(A_P) + d_D(A_D)$ and OPT(I) express as the sum of respectively |V| and 2|V| edge distances, we have:

$$\frac{d_P(A_P) + d_D(A_D)}{OPT(I)} \ge \frac{|V|a}{2|V|b} = \frac{a}{2b}, \qquad \frac{d_P(A_P) + d_D(A_D)}{OPT(I)} \le \frac{|V|b}{2|V|a} = \frac{b}{2a}$$

This leads to (ii) and (iii) for respectively the maximimization and the minimization cases.

When |V| is odd, given any Hamiltonian cycle T on V and any edge $e \in T$, the edge set $T \setminus \{e\}$ consists of the union of two near-perfect matchings on V. Given $\alpha \in \{P, D\}$ and a tour T_{α}^* on V that is of optimal with respect to d_{α} and

opt, we denote by e_{α} an arc of T_{α}^{*} having maximum distance d_{α} if the goal is to minimize, and minimum distance d_{α} otherwise. M_{α} and e_{α} therefore satisfy $2d_{\alpha}(M_{\alpha}) \succeq (d_{\alpha}(T_{\alpha}^{*}) - d_{\alpha}(e_{\alpha}))$ and $d_{\alpha}(e_{\alpha}) \preceq d_{\alpha}(T_{\alpha}^{*})/|V|$. As a consequence,

$$d_{\alpha}(M_{\alpha}) \succeq \frac{1}{2} \left(1 - \frac{1}{|V|} \right) OPT(I_{\alpha}), \qquad \alpha \in \{P, D\}$$
 (17)

We derive (i) for the Max 2 DTSPMS, considering again $d_{\alpha}(A_{\alpha}) \geq 0$, $\alpha \in \{P, D\}$. As for results (ii) and (iii) for the 2 DTSPMS – (a, b), similarly to the even case, we observe:

$$\frac{d_P(A_P) + d_D(A_D)}{OPT(I)} \ge \frac{(|V| + 1)a}{2|V|b} = \frac{a}{2b} \left(1 + \frac{1}{|V|} \right)$$
$$\frac{d_P(A_P) + d_D(A_D)}{OPT(I)} \le \frac{(|V| + 1)b}{2|V|a} = \frac{b}{2a} \left(1 + \frac{1}{|V|} \right)$$

We deduce from the above relations together with relations (14) and (17) a standard approximation ratio of $1/2 \times (1-1/|V|) + a/(2b) \times (1+1/|V|) = 1/2 + a/(2b) - o(1)$ when the goal is to maximize, of $1/2 \times (1-1/|V|) + b/(2a) \times (1+1/|V|) = 1/2 + b/(2a) + o(1)$ when the goal is to minimize.

In order to establish the tightness of the analysis, we consider bivaluated instances $I(\lambda,\mu,n),\,n\in\mathbb{N}^*,\,\lambda\neq\mu\in\mathbb{Q}^+$ of the Symmetric 2 DTSPMS. Given an integer n and two reals $\lambda,\mu,\,I(\lambda,\mu,n)=(4n,2n,d_P,d_D,\text{opt})$ where opt = max if $\lambda>\mu$ and min otherwise, and distances d_P,d_D take value μ on all edges, but along the cycle $(0,1,\ldots,4n,0)$. We denote by $V_n=\{0,1,\ldots,4n\}$ the vertex set in $I(\lambda,\mu,n)$. For this instance, the pair $(T_P=(0,1,\ldots,4n,0),T_D=T_P^-)$ of pickup and delivery tours is optimal, with value

$$OPT(I(\lambda, \mu, n)) = 2|V_n|\lambda \tag{18}$$

Now assume that when running Algorithm 1 on $I(\lambda, \mu, n)$, both M_P and M_D pick edges $\{2i-1, 2i\}$, $i \in \{1, ..., 2n\}$. Additionally assume that the loading plan $\mathcal{P} = (P_1, P_2)$ built from M_P, M_D is the following:

$$P_1 = (4, 8, \dots, 4n, 1, 5, \dots, 4n - 3)$$

 $P_2 = (3, 7, \dots, 4n - 1, 2, 6, \dots, 4n - 2)$

Observe that the edges of the cycle $(0,1,\ldots,4n,0)$ that are consistent with \mathcal{P} precisely are the edges of $M_P=M_D$. Accordingly, Algorithm 1 returns a solution with value

$$APX(I(\lambda, \mu, n)) = (|V_n| - 1)\lambda + (|V_n| + 1)\mu$$
(19)

Combining (18) and (19), one gets:

$$\begin{split} \frac{APX(I(\lambda,\mu,n))}{OPT(I(\lambda,\mu,n))} &= \frac{1}{2} \left(1 - \frac{1}{|V_n|} \right) + \frac{1}{2} \left(1 + \frac{1}{|V_n|} \right) \frac{\mu}{\lambda} \\ &= \begin{cases} &\frac{1}{2} \left(1 - \frac{1}{|V_n|} \right) & \text{if } (\lambda,\mu) = (1,0) \\ &\frac{1}{2} \left(1 + \frac{a}{b} \right) - \frac{1}{2|V_n|} \left(1 - \frac{a}{b} \right) & \text{if } (\lambda,\mu) = (b,a) \\ &\frac{1}{2} \left(1 + \frac{b}{a} \right) + \frac{1}{2|V_n|} \left(\frac{b}{a} - 1 \right) & \text{if } (\lambda,\mu) = (a,b) \end{cases} \end{split}$$

Families $I(1,0,n), n \in \mathbb{N}^*$, $I(b,a,n), n \in \mathbb{N}^*$ and $I(a,b,n), n \in \mathbb{N}^*$ thus establish the tightness of the analysis for respectively Max DTSPMS, Max DTSPMS - (a, b) and Min DTSPMS - (a, b).

5 Differential approximation results

In this section, we provide approximation results for the differential approximation ratio, which offers a complementary view of approximation vs the standard ratio, as we shall see further. The differential ratio is the ratio of |APX(I) - WOR(I)| by the instance diameter |OPT(I) - WOR(I)|, where WOR(I) is the value of a worst solution. In that differential framework, \mathcal{A} is said to be ρ -approximate for some $\rho : \mathcal{I}_{\Pi} \to]0,1]$ iff

$$\frac{APX(I) - WOR(I)}{OPT(I) - WOR(I)} \ge \rho(I), \qquad I \in \mathcal{I}_{\Pi}$$

i.e., $APX(I) \geq \rho(I)OPT(I) + (1 - \rho(I))WOR(I), I \in \mathcal{I}_{\Pi}$ if the goal is to maximize, $APX(I) \leq \rho(I)OPT(I) + (1 - \rho(I))WOR(I), I \in \mathcal{I}_{\Pi}$ otherwise.

As for standard approximation, Π is said to be approximable within factor ρ with the differential ratio *iff* it admits a polynomial time ρ -approximation algorithm. For more insights about the differential approximation measure, we invite the reader to refer to [16].

Many differential approximation results have been provided for routing and TSP related problems [29, 30, 31, 33, 32, 6, 17, 21]. For example the symmetric TSP is approximable within differential factor $3/4 - \varepsilon$ [17].

5.1 Properties of differential vs standard ratio for the TSP

The TSP has the interesting property that the minimization, maximization and metric cases are all equivalent as regards to differential approximation, which is illustrated in what follows.

The restriction of the TSP to metric instances is denoted by Metric TSP. Furthermore, Min TSP refers to the TSP where the goal is to minimize, whereas Max TSP refers to the TSP where the goal is to maximize. Let I = (V, d) be an instance of the Symmetric Min TSP, characterized by:

$$OPT(I) = \min\{d(T) \mid T \in \mathcal{T}_V\}$$

where \mathcal{T}_V denotes the set of Hamiltonian tours on V. Let us note $d_{max} = \max_{i,j \in V: i \neq j} \{d(i,j)\}$ and $d_{min} = \min_{i,j \in V: i \neq j} \{d(i,j)\}$ the maximum and the minimum distances between any pair of nodes, and consider the two instances I_1, I_2 defined as:

$$\begin{aligned} OPT(I_1) &= \max\{d_1(T) \,|\, T \in \mathcal{T}_V\} \quad \text{where } d_1 = d_{max} - d \\ OPT(I_2) &= \min\{d_2(T) \,|\, T \in \mathcal{T}_V\} \quad \text{where } d_2 = d + d_{max} - 2d_{min} \end{aligned}$$

Distances d_1 , d_2 are non-negative. Furthermore, one can easily check that d_2 satisfies the triangle inequalities. I_1 and I_2 therefore are instances of the Max TSP and of the Min Metric TSP, respectively, that can be equivalently expressed as:

$$\begin{array}{ll} OPT(I_1) &= |V| d_{max} - \min\{d(T) \,|\, T \in \mathcal{T}_V\} \\ OPT(I_2) &= |V| (d_{max} - 2d_{min}) + \min\{d(T) \,|\, T \in \mathcal{T}_V\} \end{array}$$

Hence, the three instances I, I_1, I_2 of the TSP correspond to the same optimization problem, up to an affine transformation of their objective function. Accordingly, these instances are equivalent to differentially approximate. Indeed, observe that for all $T \in \mathcal{T}_V$, we have:

$$\frac{d(T) - WOR(I)}{OPT(I) - WOR(I)} = \frac{d_1(T) - WOR(I_1)}{OPT(I_1) - WOR(I_1)} = \frac{d_2(T) - WOR(I_2)}{OPT(I_2) - WOR(I_2)}$$

Hence, and in contrast with the standard approximation framework, the Min TSP, the Max TSP and their restriction to the metric case are strictly equivalent to differentially approximate. The symmetric case of these problems notably all are approximable within a differential factor of $3/4 - \varepsilon$, $\varepsilon > 0$, [17].

Using similar arguments, Min DTSPMS, Max DTSPMS, Min Metric DTSPMS and Max Metric DTSPMS are equivalent with respect to their differential approximability.

5.2 Differential approximation of the general DTSPMS

In Section 3, we derived standard approximation results for the DTSPMS from connections between the optimal values of a given instance I of the DTSPMS and of instances I_P , I_D and I_Σ of the TSP. Such connections between the extremal values on I and I_Σ similarly allow to derive differential approximation results for DTSPMS from differential approximation results for TSP.

First, symmetrically to (5), the worst solution values on instances I_P, I_D, I_{Σ} of the TSP and I of the DTSPMS obviously satisfy:

$$WOR(I_P) + WOR(I_D) \leq WOR(I) \leq WOR(I_{\Sigma})$$
 (20)

Now let (T_P, T_D) refer to an optimal solution of I. On the one hand, T_P and T_D^- both are feasible solutions of I_{Σ} . Therefore, we have:

$$OPT(I_{\Sigma}) \geq 1/2 \times (d_{P}(T_{P}) + d_{D}(T_{P}^{-}) + d_{P}(T_{D}^{-}) + d_{D}(T_{D}))$$

$$\geq 1/2 \times (OPT(I) + d_{P}(T_{D}^{-}) + d_{D}(T_{P}^{-}))$$

On the other hand, (T_D^-, T_P^-) is a feasible pair of pickup and delivery tour on I. Accordingly, we have $d_P(T_D^-) + d_D(T_P^-) \succeq WOR(I)$. The following Proposition thus holds:

Lemma 5.1. Given any instance I of the DTSPMS, I and its related instance I_{Σ} of the TSP satisfy:

$$OPT(I_{\Sigma}) \succeq (WOR(I)) + OPT(I))/2$$
 (21)

Relation (21) indicates that the optimal value of I_{Σ} provides a 1/2-differential approximation of OPT(I). It also yields a rather simple differential approximation preserving reduction from DTSPMS to TSP.

Proposition 5.2. The (Symmetric) DTSPMS reduces to the (Symmetric) TSP by means of a polynomial time reduction that maps ρ -differential approximate solutions of the TSP onto solutions of the DTSPMS with a differential approximation guarantee of $\rho/2$.

Proof. Let I be an instance of DTSPMS. Given any tour T on V, (T, T^-) is a feasible pair of pickup and delivery tours on I, with value $d_{\Sigma}(T)$. In particular if T is ρ -approximate for the TSP on I_{Σ} , then we have:

$$d_{\Sigma}(T) \succeq \rho OPT(I_{\Sigma}) + (1 - \rho)WOR(I_{\Sigma})$$

$$\succeq \rho (OPT(I) + WOR(I)) / 2 + (1 - \rho)WOR(I) \quad \text{using (20) & (21)}$$

$$= \rho / 2 \times OPT(I) + (1 - \rho / 2)WOR(I)$$

Solution (T, T^-) therefore is $\rho/2$ -approximate on I.

The theorem below is a straightforward consequence of Proposition 5.2 and the result of [17].

Theorem 5.3. The Symmetric DTSPMS is approximable within differential ratio $3/4 - \varepsilon$, $\varepsilon > 0$.

5.3 Differential approximation of the 2 DTSPMS

Some adaptation of the heuristic of Section 4 enables to reach a differential approximation ratio of 1/2 - o(1) for the 2 DTSPMS. In the proposed heuristic, the computation of optimal matchings brings "one half" of the optimal value OPT(I), which allows to establish a standard approximation guarantee of 1/2 - o(1) for the maximization case. Obtaining such a guarantee with respect to the differential approximation measure additionally requires the comparison of the remaining part of the approximate solution – namely, completions A_P and A_D of matchings M_P and M_D – to the worst solution value WOR(I). This comparison to the worst solution value captures the specificity of differential approximation, and may make it hard to establish differential approximation guarantees.

Consider an instance I of the 2DTSPMS where $|V|=2\nu+2$ is even (we will speak later of the case when |V| is odd). We seek perfect matchings that complement the matchings M_P and M_D . With a given balanced loading plan $\mathcal{P}=((i_1,\ldots,i_{\nu+1}),(j_1,\ldots,j_{\nu}))$ of $V\backslash\{0\}$, we associate the two perfect matchings

$$A(\mathcal{P}) = \{(i_p, j_p) \mid p = 1, \dots, \nu\} \cup \{(i_{\nu+1}, 0)\}$$

$$A'(\mathcal{P}) = \{(i_1, i_2)\} \cup \{(i_p, j_{p-2}) \mid p = 3, \dots, \nu + 1\} \cup \{(j_{\nu}, 0)\}$$

on V. These matching are depicted in Figure 5 in case when $\nu = 7$. Furthermore, we denote by \mathcal{P}' the loading plan obtained from \mathcal{P} when exchanging the storage of the two nodes that are loaded at position 1 of rows 1 and 2.

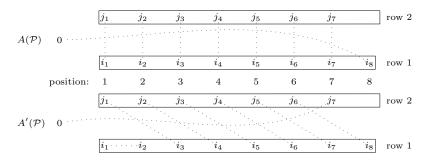


Figure 5: The perfect matchings $A(\mathcal{P})$ and $A'(\mathcal{P})$ given a loading plan \mathcal{P} (both matchings are drawn in dotted lines).

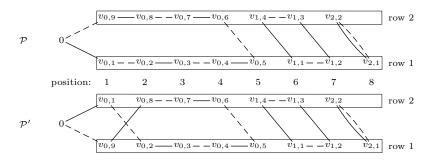


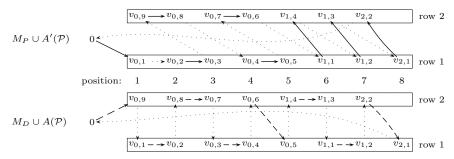
Figure 6: Loading plans $\mathcal{P} = \mathcal{P}(M_P, M_D)$ and $\mathcal{P}' = \mathcal{P}'(M_P, M_D)$ given two perfect matchings M_P (in plain lines) and M_D (in dashed lines).

Thereafter, we consider a pair $(\mathcal{P}, \mathcal{P}')$ of loading plans where $\mathcal{P} = \mathcal{P}(M_P, M_D)$ refers to the approximate loading plan of Section 4. Figure 6 depicts the loading plans \mathcal{P} and \mathcal{P}' given two perfect matchings M_P and M_D . The following Lemma holds:

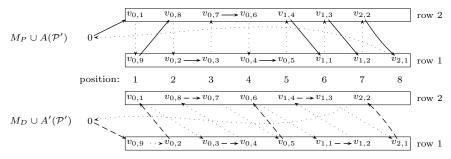
Lemma 5.4. Let $A = A(\mathcal{P}) = A(\mathcal{P}')$. Then,

- (i) $M_P \cup A$ is a feasible pickup tour with respect to \mathcal{P} and \mathcal{P}' , and $M_D \cup A$ is a feasible delivery tour with respect to \mathcal{P} and \mathcal{P}' .
- (ii) For $\alpha \in \{P, D\}$, if M_{α} links the depot to the vertex which is loaded in \mathcal{P} at position 1 of row 1, then $M_{\alpha} \cup A'(\mathcal{P})$ is a feasible tour with respect to \mathcal{P} ; symmetrically, if M_{α} links vertex 0 to the first vertex in row 1 of \mathcal{P}' , then $M_{\alpha} \cup A'(\mathcal{P}')$ is a feasible tour with respect to \mathcal{P}' .
- (iii) $(A \cup A'(\mathcal{P}), A \cup A'(\mathcal{P}'))$ is a feasible pair of pickup and delivery tours on V for the 2 DTSPMS with tight capacity.

Figures 7 and 8 propose some illustration of these facts.



A feasible pair of pickup and delivery tours with respect to the loading plan \mathcal{P}



A feasible pair of pickup and delivery tours with respect to the loading plan \mathcal{P}'

Figure 7: The approximate solutions $(\mathcal{P}, M_P \cup A'(\mathcal{P}), M_D \cup A)$ and $(\mathcal{P}', M_P \cup A), M_D \cup A'(\mathcal{P}')$: M_P is depicted in plain lines, M_D is depicted in dashed lines, $A, A'(\mathcal{P})$ and $A'(\mathcal{P}')$ are depicted in dotted lines.

Proof. (i) and (ii) Let $Q = ((i_1, \ldots, i_{\nu+1}), (j_1, \ldots, j_{\nu})) \in \{\mathcal{P}, \mathcal{P}'\}$. First consider completion A. By construction of \mathcal{P} and \mathcal{P}' , each of the two perfect matchings M_P, M_D connects vertex 0 to either i_1 or j_1 , a single vertex in $\{i_p, j_p\}$ to a single vertex in $\{i_{p+1}, j_{p+1}\}$ for each position $p \in \{1, \ldots, \nu-1\}$, and either i_{ν} or j_{ν} to vertex $i_{\nu+1}$. $M_P \cup A$ and $M_D \cup A$ therefore both define Hamiltonian cycles on V, and these cycles induce feasible pickup and delivery tours with respect to Q.

Similarly for $A'(\mathcal{Q})$, for all $\alpha \in \{P, D\}$ such that $(0, i_1) \in M_{\alpha}$, M_{α} connects: 0 to i_1 ; i_2 to $\{i_3, j_1\}$; $\{i_p, j_{p-2}\}$ to $\{i_{p+1}, j_{p-1}\}$ for every position $p \in \{3, \ldots, \nu\}$, and $\{i_{\nu+1}, j_{\nu-1}\}$ to j_{ν} . We deduce that $M_{\alpha} \cup A'(\mathcal{Q})$ induces a feasible tour with respect to \mathcal{Q} provided that $(0, i_1) \in M_{\alpha}$.

(iii) Let $\mathcal{P} = ((i_1, \dots, i_{\nu+1}), (j_1, \dots, j_{\nu}))$. By definition of A and $A'(\mathcal{P})$, $A \cup A'(\mathcal{P})$ can be viewed as the tour either

$$(0, j_{\nu}, i_{\nu}, j_{\nu-2}, i_{\nu-2}, \dots, j_2, i_2, i_1, j_1, i_3, j_3, i_5, j_5, \dots, i_{\nu-1}, j_{\nu-1}, i_{\nu+1}, 0)$$

or
$$(0, j_{\nu}, i_{\nu}, j_{\nu-2}, i_{\nu-2}, \dots, j_3, i_3, j_1, i_1, i_2, j_2, i_4, j_4, \dots, i_{\nu-1}, j_{\nu-1}, i_{\nu+1}, 0)$$

on V, depending on ν mod 2. Let T refer to this tour. Furthermore, by definition of \mathcal{P}' , $A \cup A'(\mathcal{P}')$ can be obtained from $A \cup A'(\mathcal{P})$ by substituing with the two edges (i_1, i_2) and (j_1, i_3) the edges (j_1, i_2) and (i_1, i_3) . Therefore, $A \cup A'(\mathcal{P}')$

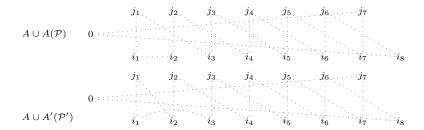


Figure 8: The two tours $A \cup A'(\mathcal{P})$ and $A \cup A'(\mathcal{P}')$ given a loading plan $\mathcal{P} = ((i_1, \ldots, i_{\nu+1}), (j_1, \ldots, j_{\nu}))$

induces on V a tour T' which just the same as T, but swapping the two vertices i_1 and j_1 . We deduce that the pair (T', T^-) of tours defines a feasible pair of pickup and delivery tours on V, considering e.g. the loading plans

$$((j_{\nu}, i_{\nu}, j_{\nu-2}, i_{\nu-2}, \dots, j_2, i_2, i_1), (j_1, i_3, j_3, i_5, j_5, \dots, i_{\nu-1}, j_{\nu-1}, i_{\nu+1}))$$
 and
$$((j_{\nu}, i_{\nu}, j_{\nu-2}, i_{\nu-2}, \dots, j_3, i_3, j_1), (i_1, i_2, j_2, i_4, j_4, \dots, i_{\nu-1}, j_{\nu-1}, i_{\nu+1}))$$

for respectively the even and the odd cases.

Algorithm 2: DAPX_2DTSPMS_EVEN

Input: A vertex set $V = \{0, ..., n\}$ where n is odd, two symmetric distance functions $d_P, d_D : V^2 \to \mathbb{Q}^+$, an optimization goal opt

Output: A balanced 2-rows loading plan \mathcal{P} of $V\setminus\{0\}$ and an optimal pair $(T_P^*(\mathcal{P}), T_D^*(\mathcal{P}))$ of pickup and delivery tours on V with respect to d_P, d_D , opt and \mathcal{P}

 $(\mathcal{P}, T_P, T_D) \leftarrow \text{APX_2DTSPMS_EVEN}(n, d_P, d_P, \text{opt});$

 $\mathcal{P}' \longleftarrow \mathcal{P};$

Exchange in \mathcal{P}' the nodes that are stored at position 1 in rows 1 and 2; foreach α in $\{P, D\}$ do

Compute a tour T'_{α} on V that is consistent with \mathcal{P}' and optimal with respect to d_{α} and opt;

else

return $(\mathcal{P}', T_P', T_D')$;

Theorem 5.5. The Symmetric 2 DTSPMS is approximable within a differential factor of 1/2 - o(1).

Proof. In case when |V| is even, we show that Algorithm 2 provides a 1/2-differential approximation for the Symmetric 2 DTSPMS, *i.e.* for any instance I Algorithm 2 returns a solution with value $APX \succeq OPT(I)/2 + WOR(I)/2$. We assume without loss of generality that the goal on I is to maximize. Let m_0 denote the number of vertices that lie in $M_P \cup M_D$ on the cycle that contains 0. We separate the proof in two parts, depending on whether $m_0 = 2$ or $m_0 \ge 4$.

Let $\mathcal{P} = ((i_1, \ldots, i_{\nu+1}), (j_1, \ldots, j_{\nu}))$. When $m_0 = 2$, the perfect matchings M_P and M_D both contain edge $(0, i_1)$. It thus follows from Lemma 5.4 that $M_P \cup A$ and $M_P \cup A'(\mathcal{P})$ on the one hand, $M_D \cup A$ and $M_D \cup A'(\mathcal{P})$ on the other hand, are feasible pickup and delivery tours with respect to \mathcal{P} . Since Algorithm 2 returns a best pair of pickup and delivery tours with respect to \mathcal{P} or \mathcal{P}' , we deduce that the value APX of the solution returned by the Algorithm satisfies:

$$APX \geq \max \{ d_P(M_P \cup A) + d_D(M_D \cup A), \\ d_P(M_P \cup A'(P)) + d_D(M_D \cup A'(P)) \} \\ \geq d_P(M_P) + d_D(M_D) + d_D(A \cup A'(P)) / 2$$

We already know that quantity $d_P(M_P)+d_D(M_D)$ is bounded below by OPT(I)/2. Now, since $A \cup A'(P)$ is a Hamiltonian tour on V, we also have $d_{\Sigma}(A \cup A'(P)) \ge WOR(I)$. This concludes the proof for the case when $m_0 = 2$.

When $m_0 \geq 4$, either $(0, i_1) \in M_P$ and $(0, j_1) \in M_D$, or $(0, j_1) \in M_P$ and $(0, i_1) \in M_D$. Assume w.l.o.g. that the former occurs. Lemma 5.4 in this case ensures that $(\mathcal{P}, M_P \cup A'(\mathcal{P}), M_D \cup A)$ and $(\mathcal{P}', M_P \cup A, M_D \cup A'(\mathcal{P}'))$ are feasible solutions on I. Similarly to the preceding case, we deduce from the fact that Algorithm 2 returns a best pair of tours with respect to \mathcal{P} or \mathcal{P}' that we have:

$$APX \geq \max \{ d_P (M_P \cup A'(P)) + d_D(M_D \cup A), \\ d_P(M_P \cup A) + d_D (M_D \cup A'(P')) \} \\ \geq OPT(I)/2 + (d_P (A \cup A'(P)) + d_D (A \cup A'(P')))/2$$

Now we know from Lemma 5.4 that $(A \cup A'(\mathcal{P}), A \cup A'(\mathcal{P}'))$ defines a feasible pair of pickup and delivery tours, which concludes the proof.

In case when |V| is odd, the algorithm mostly consists in computing a loading plan for each $x \in V$, each based on the computation of a pair (M_P^x, M_D^x) of optimal perfect matchings on $V \setminus \{x\}$. Since the proof is technical and brings no new insights on the problem, we put it in a separated appendix.

6 Conclusion

We have provided many approximation results for the Double TSP with Multiple Stacks or its restriction with two stacks, for several kinds of distances. Among them, Min Metric k DTSPMS with tight capacities is approximable within standard factor (3/2)k, whereas 2 DTSPMS is approximable within differential factor 1/2-o(1). Also, Max 2 DTSPMS, Max 2 DTSPMS -(1,2) and Min 2 DTSPMS -(1,2) with tight capacities are approximable within standard factor 1/2-o(1), 3/4-o(1) and 3/2+o(1), respectively. Most of our positive approximation results

on the general problem are obtained from reductions from the TSP. For the problem with two stacks, we designed a dedicated algorithm based on optimal matchings and suitable completions that can be compared to the best and worst tours. The analysis is non trivial and provides interesting approximation results, in both cases of standard and differential approximation. An open problem is to design tailored algorithms for the case with more than two stacks, which could improve the approximation ratios found with TSP reductions. The VRP generalization is also interesting to study, although its complexity would make a real challenge to find approximation results.

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A APPENDIX: Differential approximation of the DTSPMS on an odd number of vertices

A.1 The general idea of the proof

Let I be an instance of the 2 DTSPMS on a node set V such that |V| is odd. We assume w.l.o.g. that the goal on I is to maximize. In what follows, T_* denotes

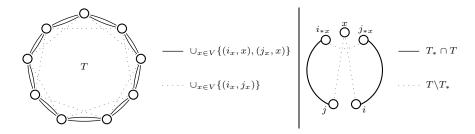


Figure 9: On the left: illustration of inequality (24) (proof of Lemma A.1). On the right: illustration of inequality (28) (proof of Theorem A.3)

a worst solution on I_{Σ} , i.e., T_* is a tour of minimum distance d_{Σ} . Furthermore, $(\mathcal{P}^*, T_P^*, T_D^*)$ denotes an optimal solution on I. Given $x \in V$, we denote by V_x the vertex set $V \setminus \{x\}$. Morever, given an index $\alpha \in \{P, D\}$, M_{α}^x refers to a maximal perfect matching on V_x with respect to opt and d_{α} . Finally, given two nodes $i, j \in V_x$, $\delta_{\alpha}^x(i, j)$ refers to the quantity $d_{\alpha}(i, x) + d_{\alpha}(x, j) - d_{\alpha}(i, j)$. By extension, given a tour T on V, $\delta_{\alpha}^x(T)$ refers to $\delta_{\alpha}^x(i, j)$ for the two vertices i and j that are adjacent to x in T. We make some observation on the extremal values:

Lemma A.1. $M_P^x, M_D^x, x \in V$, OPT(I), WOR(I) satisfy:

$$\forall x \in V, \ d_P(M_P^x) + d_D(M_D^x) \ge 1/2 \times (OPT(I) - \delta_P^x(T_P^*) - \delta_D^x(T_D^*))$$
 (22)

$$\sum_{x \in V} \left(\delta_P^x(T_P^*) + \delta_D^x(T_D^*) - \delta_P^x(T_*) - \delta_D^x(T_*) \right) \le 4 \left(OPT(I) - WOR(I) \right) \tag{23}$$

Proof. Relation (22). Given a vertex $x \in V$ and an index $\alpha \in \{P, D\}$, for the two nodes i, j such that $(i, x), (x, j) \in T_{\alpha}^*$, $T_{\alpha}^* \setminus \{(i, x), (x, j)\} \cup \{(i, j)\}$ defines a tour on V_x . Since M_{α}^x is a maximal perfect matching on V_x , $d_{\alpha}(M_{\alpha}^x)$ is at least one half of the value of this tour. Thus we have $2d_{\alpha}(M_{\alpha}^x) \geq d_{\alpha}(T_{\alpha}^*) - \delta_{\alpha}^x(T_{\alpha}^*)$, $\alpha \in \{P, D\}$, while $\sum_{\alpha = P, D} d_{\alpha}(T_{\alpha}^*) = OPT(I)$.

Relation (23). Given a tour $T = (0, v_1, \dots, v_n, 0)$ on V, let T^2 be the set of arcs (i, j) such that i and j are at distance 2 in T, *i.e.*,

$$T^2 = \{(i,j) \mid \exists h, \ (i,h), (h,j) \in T\}$$

If |V| is odd iff n is even, then T^2 is the tour $(0, v_2, v_4, \ldots, v_n, 1, 3, \ldots, v_{n-1}, 0)$ on V. One one hand, for $\alpha \in \{P, D\}$ the quantities $\delta_{\alpha}^{x}(T), x \in V$ satisfy

$$\sum_{x \in V} \delta_{\alpha}^{x}(T) = 2d_{\alpha}(T) - d_{\alpha}(T^{2})$$
(24)

(see Figure 9). Hence, any tour T on V satisfies

$$\sum_{x \in V} \left(\delta_P^x(T) + \delta_D^x(T) \right) = 2d_{\Sigma}(T) - d_{\Sigma}(T^2) \ge 2WOR(I) - OPT(I)$$

On the other hand, (T, T^{2-}) is a feasible pair of pickup and delivery tours: consider e.g. the loading plan $((v_1, v_3, \ldots, v_n), (v_2, v_4, \ldots, v_{n-1}))$. Hence, any feasible pair (T_P, T_D) of pickup and delivery tours satisfies:

```
\begin{split} &\sum_{x \in V} \left( \delta_P^x(T_P) + \delta_D^x(T_D) \right) \\ &= & 2d_P(T_P) + 2d_P(T_D) - d_P(T_P^2) - d_D(T_D^2) \\ &= & 2\left( d_P(T_P) + d_P(T_D) \right) - \left( d_P(T_P^2) + d_D(T_P) \right) - \left( d_P(T_D) + d_D(T_D^2) \right) \\ &+ \left( d_D(T_P) + d_P(T_D) \right) \\ &\preceq & 3OPT(I) - 2WOR(I) \end{split}
```

which concludes the lemma.

```
Algorithm 3: Appproximate loading plan for the DTSPMS
```

```
Input: An instance I = (n, 2, \lceil n/2 \rceil, d_P, d_D, \text{opt}) of the DTSPMS on
        vertex set V such that |V| is odd
Output: An approximate loading plan of V \setminus \{0\}
for x \in V do
   Compute a perfect matching M_{\alpha}^{x} on V \setminus \{x\} that is of optimum weight
     with respect to d_{\alpha} and opt;
   /* LOADING_PLAN_1, LOADING_PLAN_2, LOADING_PLAN_3,
       LOADING_PLAN_4 return a loading plan of V \setminus \{0\} that
       admits a pickup tour that contains M_P^x as well as a
       delivery tour that contains {\cal M}_D^x
   if x \neq 0 then
      /* The multi-edge set M_P^x \cup M_D^x consists of h_x+1 cycles
          of even length on vertex set W_s, s \in \{0,\dots,h_x\} . The
          depot vertex belongs to W_0.
      if |W_0| = 2 then
      /* The multi-edge set {\cal M}^0_P \cup {\cal M}^0_D consists of h_0 cycles of
          even length on vertex set W_s, s \in \{1, \dots, h_0\}.
      return \operatorname{arg} \operatorname{opt}_{\{\mathcal{P} \mid x \in V\}} \left\{ \sum_{\alpha = P, D} d_{\alpha} \left( T_{\alpha}(\mathcal{P}) \right) \right\};
```

We adapt the heuristic for the even case to the odd case. The adaptation mainly consists in computing a loading plan per vertex $x \in V$, instead of a

single loading plan. The algorithm computes for any $x \in V$ a pair (M_P^x, M_D^x) of optimal perfect matchings on V_x and builds a loading plan \mathcal{P}_x that is consistent with both M_P^x and M_D^x . Let $APX_x = d_P(T_P^*(\mathcal{P}_x)) + d_D(T_D^*(\mathcal{P}_x))$ denote the value of loading plan \mathcal{P}_x , $x \in V$; the algorithm then returns the loading plan among $\{\mathcal{P}_x \mid x \in V\}$ that achieves the best value APX_x . Hence, the value APX of the solution returned by the algorithm clearly satisfies:

$$APX \ge \frac{1}{|V|} \sum_{x \in V} APX_x \tag{25}$$

Before providing the approximability result for the odd case, we need the following lemma. As the proof is long with multiple cases, it is given in a separated section.

Lemma A.2. For any node $x \in V$, there exists a subset F_x of $V_x \times V_x$ such that

(i) For all $(i, j) \in F_x$, there exists a pair $(\mathcal{P}_x, M_P^x \cup N_P^x, M_D \cup N_D^x), (\mathcal{P}'_x, M_P^x \cup N_P^x, M_D^x \cup N_D^x)$ of feasible solutions on I such that

$$\begin{array}{c} \left(\left(N_P^x \cup N_P'^x\right) \backslash \left\{(x,i),(x,j)\right\} \cup \left\{(i,j)\right\}, \\ \left(N_D^x \cup N_D'^x\right) \backslash \left\{(x,i),(x,j)\right\} \cup \left\{(i,j)\right\}\right) \end{array}$$

is a feasible pair of pickup and delivery tours on I

(ii) F_x intersects all Hamiltonian cycles on V_x

Theorem A.3. Algorithm 3 is a 1/2-differential approximation for the Symmetric 2DTSPMS, i.e. for any instance I of the Symmetric 2DTSPMS where |V| is odd, Algorithm 3 returns a solution with value $APX \geq OPT(I)/2 + WOR(I)/2$.

Proof. We first show that for all $x \in V$,

$$APX_x \ge d_P(M_P^x) + d_D(M_D^x) + \frac{1}{2}(WOR(I) + \delta_P^x(T_*) + \delta_D^x(T_*))$$
 (26)

Then we successively deduce from relations (25),(26),(22) and (23) that the approximate value satisfies

$$\begin{array}{ll} APX & \geq \frac{1}{|V|} \sum_{x \in V} APX_x \\ & \geq \frac{1}{|V|} \sum_{x \in V} \left(d_P(M_P^x) + d_D(M_D^x) + \frac{1}{2} \left(WOR(I) + \delta_P^x(T_*) + \delta_D^x(T_*) \right) \right) \\ & \geq \frac{1}{2|V|} \sum_{x \in V} \left(OPT(I) + WOR(I) - \sum_{\alpha \in \{P,D\}} \left(\delta_\alpha^x(T_\alpha^*) - \delta_\alpha^x(T_*) \right) \right) \\ & \geq \left(\frac{1}{2} - \frac{2}{|V|} \right) OPT(I) + \left(\frac{1}{2} + \frac{2}{|V|} \right) WOR(I) \end{array}$$

which ends the proof of the 1/2-differential ratio.

Now, let us prove relation (26). Consider the edge set F_x of Lemma A.2 and some edge $(i_x, j_x) \in F_x$ optimizing $\delta_P^x(i, j) + \delta_D^x(i, j)$ over this set. According to

Lemma A.2, there exist two edge sets N_x , N_x' such that $\{i_x, x\}$, $\{x, j_x\} \in N_x \cup N_x'$ and $(\mathcal{P}_x, M_P^x \cup N_x, M_D^x \cup N_x)$, $(\mathcal{P}_x', M_P^x \cup N_x', M_D^x \cup N_x')$ are feasible solutions on I. Since Algorithm 3 considers both solutions $(\mathcal{P}_x, T_P^*(\mathcal{P}_x), T_D^*(\mathcal{P}_x))$ and $(\mathcal{P}_x', T_P^*(\mathcal{P}_x'), T_D^*(\mathcal{P}_x'))$, we deduce:

$$APX_x \ge \frac{1}{2} \left(2d_P(M_P^x) + 2d_D(M_D^x) + d_{\Sigma}(N_x) + d_{\Sigma}(N_x') \right)$$

Moreover, since Lemma A.2 additionally indicates that $(N_x \cup N'_x) \setminus \{\{x, i_x\}, \{x, j_x\}\} \cup \{\{i_x, j_x\}\}$ is a Hamiltonian cycle on V, we get that the value of this tour with respect to d_{Σ} is better than WOR(I). Hence,

$$APX_x \ge d_P(M_P^x) + d_D(M_D^x) + \frac{1}{2}WOR(I) + \frac{1}{2}(\delta_P(i_x, j_x) + \delta_D(i_x, j_x))$$
 (27)

Let i_{*x} and j_{*x} respectively denote the predecessor and the successor of x in T_* and let T_{*x} refer to the tour $T_* \setminus \{(i_{*x}, x), (x, j_{*x})\} \cup \{(i_{*x}, j_{*x})\}$ on V_x . On the one hand, since T_* is a tour on V of worst value with respect to d_{Σ} and the optimization goal, $d_{\Sigma}(T) \geq d_{\Sigma}(T_*)$ for all tour T on V obtained from T_* by first removing edges $(i_{*x}, x), (x, j_{*x})$ as well as some other edge $(i, j) \in T_*$, and then adding the edges $(i, x), (x, j), (i_{*x}, j_{*x})$ (see Figure 9). Equivalently,

$$\delta_P^x(i,j) + \delta_D^x(i,j) \ge \delta_P^x(i_{*x}, j_{*x}) + \delta_D^x(i_{*x}, j_{*x}), \qquad (i,j) \in T_{*x}$$
 (28)

On the other hand, since by (A.2) F_x intersects any Hamiltonian cycle on V_x , the tour T_{*x} on V_x intersects F_x on some arc (i, j). The optimality of (i_x, j_x) over F_x ensures that (i, j) and (i_x, j_x) satisfy:

$$\delta_P^x(i_x, j_x) + \delta_D^x(i_x, j_x) \ge \delta_P^x(i, j) + \delta_D^x(i, j) \tag{29}$$

We deduce from the two previous inequalities that edge (i_x, j_x) satisfies

$$\delta_P^x(i_x, j_x) + \delta_D^x(i_x, j_x) \ge \delta_P^x(T_*) + \delta_D^x(T_*) \tag{30}$$

Together with inequality (27), we obtain expression (26).

A.2 Proof of Lemma A.2

In what follows, given $x \in V$, one considers two perfect matchings M_P^x, M_D^x on V_x and the connected components W_0, W_1, \ldots, W_h of the perfect 2-matching $M_P^x \cup M_D^x$ on V_x , where W_0 refers to the component that contains the depot vertex provided that $x \neq 0$. Each component W_s induces on $(V_x, M_P^x \cup M_D^x)$ a cycle of even length. We describe these cycles by $\{0, v_{0,1}, \ldots, v_{0,2m_0+1}, 0\}$ if $x \neq 0$ and s = 0, by $\{v_{s,1}, \ldots, v_{s,2m_s}, v_{s,1}\}$ otherwise.

A.2.1 Case $x \neq 0$ and $|W_0| = 2$

In this case, W_0 induces the cycle $\{0, v_{0,1}, 0\}$, and $h \ge 1$. Since any tour on V_x links vertex 0 to some vertex in $V_x \setminus \{0, v_{0,1}\}$, we define F_x as

$$F_x = \{0\} \times V_x \setminus \{0, v_{0,1}\}$$

and pick some edge $e_x \in F_x$ that maximizes $\delta_P^x(e) + \delta_D^x(e)$. We assume w.l.o.g. that e_x is the edge $(0, v_{h,m_h})$. We define $\mathcal{P}_x, N_P^x, N_D^x$ and $\mathcal{P}_x', N_P'^x, N_D'^x$ as follows:

- let $\mathcal{P}_x = \mathcal{P}'_x$ be the loading plan obtained from $\mathcal{P}(M_P^x, M_D^x)$ by loading x at the end of row 2;
- define $N_P^x = N_D^x$ as the edge set obtained from $A(\mathcal{P})$ by substituing for the edge $(v_{h,m_h}, 0)$ the chain $(v_{h,m_h}, x, 0)$;
- define $N_P'^x = N_D'^x$ as the edge set obtained from $A'(\mathcal{P})$ by substituing for the edge $(v_{h,m_h+1},0)$ the chain $(v_{h,m_h+1},x,0)$.

The fact that $\mathcal{P}_x, N_P^x, N_D^x, \mathcal{P}_x', N_P'^x, N_D'^x$ and e_x satisfy condition (i) of Lemma A.2 is straightforward from Lemma 5.4.

A.2.2 Case $x \neq 0$ and $|W_0| \geq 4$

In this case, W_0 induces on $M_P^x \cup M_D^x$ the cycle $(0, v_{0,1}, \dots, v_{0,2m_0+1}, 0)$ where $2m_0 + 1 \geq 3$. We introduce a new family $\mathcal{Q} = \mathcal{Q}(M_P^x, M_D^x)$ of loading plans given two matchings M_P^x, M_D^x on V_x for some $x \neq 0$, as well as new families $B(\mathcal{Q})$ and $B'(\mathcal{Q})$ of matchings given a loading plan \mathcal{Q} .

 $\mathcal{Q}(M_P^x, M_D^x)$ is obtained from $\mathcal{P}(M_P^x, M_D^x)$ by exchanging for each even position p in $\{1, \ldots, m_0\}$ the vertex loaded at position p in row 1 with the vertex loaded at position p in row 2. Observe that vertices of $V_x \setminus W_0$ are loaded in $\mathcal{Q}(M_P^x, M_D^x)$ just as the same as in $\mathcal{P}(M_P^x, M_D^x)$.

We describe rows 1 and 2 of \mathcal{Q} by respectively $(i_1, \ldots, i_{\nu+1})$ and (j_1, \ldots, j_{ν}) . Furthermore, we introduce the cycle $\Gamma = (i_1, i_2, \ldots, i_{m_0}, j_{m_0}, j_{m_0-1}, \ldots, j_1, i_1)$. We then build two perfect matchings $B(\mathcal{Q})$ and $B'(\mathcal{Q})$ on V_x as follows:

- half of the edges of Γ , including (i_1, j_1) , into $B(\mathcal{Q})$, and the other half into $B'(\mathcal{Q})$;
- add into B(Q) edges $(i_p, j_p), p \in \{m_0 + 1, \dots, (n-3)/2\}$ and $(i_{(n-1)/2}, 0)$;
- add into $B'(\mathcal{Q})$ edges $(i_{p+2}, j_p), p \in \{m_0 + 1, \dots, (n-5)/2\}, (i_{m_0}, i_{m_0+1})$ and $(j_{(n-3)/2}, 0)$.

Using similar arguments as in Lemma 5.4, it is nit too hard to see that the following facts hold:

- Fact 1. (i) $M_P^x \cup B(Q)$ and $M_P^x \cup B'(Q)$ are feasible pickup tours with respect to Q on V_x ;
- (ii) $M_D^x \cup B(Q)$ and $M_D^x \cup B'(Q)$ are feasible delivery tours with respect to Q on V_x ;
- (iii) $B(Q) \cup B'(Q)$ is the union of Γ and some other cycle Γ' over $V_x \setminus W_0 \cup \{0, v_{0,2m_0+1}\}$.

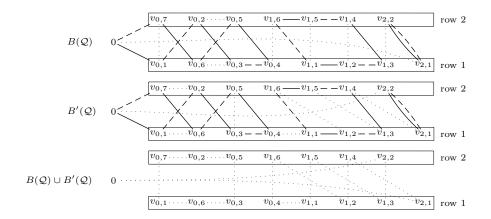


Figure 10: Loading plan $Q = Q(M_P^x, M_D^x)$ of $V_x \setminus \{0\}$ and completions B(Q), B'(Q) (in dotted lines) given two perfect matchings M_P^x (in plain lines) and M_D^x (in dashed lines) on V_x

We omit the proof, but invite the reader to refer to Figure 10.

Since any tout on V_x connects a vertex of $V(\Gamma)$ to a vertex of $V_x \setminus V(\Gamma)$, we define F_x as

$$F_x = V(\Gamma) \times (V_x \backslash V(\Gamma))$$

and consider an edge e_x that maximizes $\delta_P^x(e) + \delta_D^x(e)$ over F_x . By construction, e_x is incident to some vertex $v_{0,j} \in \{v_{0,1}, \dots, v_{0,2m_0}\}$. We assume w.l.o.g. $j \leq m_0$.

We build a first loading plan \mathcal{P}_x and matchings N_P^x and N_D^x as follows:

- set $\mathcal{P}_x = \mathcal{Q}(M_P^x, M_D^x)$;
- insert x in row 2 at position p=2 if j=1, at position p=j otherwise;
- define $N_P^x = N_D^x$ as the edge set obtained from B(Q) if j is odd, from B'(Q) otherwise, by substituting for the edge (j_p, j_{p+1}) the chain (j_p, x, j_{p+1}) .

We build a second loading plan \mathcal{P}'_x and matchings $N_P'^x$ and $N_D'^x$ as follows, depending on e_x . Starting with $\mathcal{P}'_x = \mathcal{Q}(M_P^x, M_D^x)$, if $e_x = (v_{0,j}, 0)$, then:

- insert x at position n/2 in row 2;
- if p is odd, then define $N_P'^x = N_D'^x$ as the edge set obtained from B'(Q) by substituing for the edge $(i_{n/2}, 0)$ the chain $(i_{n/2}, x, 0)$; otherwise, $N_P'^x$ and $N_D'^x$ are obtained from B(Q) by substituing for the edge $(j_{n/2-1}, 0)$ the chain $(j_{n/2-1}, x, 0)$.

If $e_x = (v_{0,j}, v_{0,m_0+1})$, then:

• insert x at position $m_0 + 1$ in row 2;

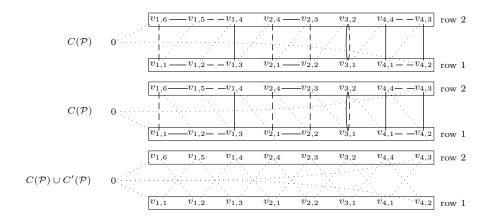


Figure 11: Loading plan $\mathcal{P} = \mathcal{P}(M_P^0, M_D^0)$ of $V \setminus \{0\}$ and completions $C(\mathcal{P}), C'(\mathcal{P})$ (in dotted lines) given two perfect matchings M_P^0 (in plain lines) and M_D^0 (in dashed lines) on V_0

• if p is odd, then define $N_P'^x = N_D'^x$ as the edge set obtained from B'(Q) by substituing for the edge $(v_{0,m_0+1},x,v_{1,1})$ the chain $(v_{0,m_0+1},x,v_{1,1})$; otherwise, $N_P'^x$ and $N_D'^x$ are obtained from B(Q) by substituing for the edge $(v_{0,m_0+1},x,v_{1,2m_1})$ the chain $(v_{0,m_0+1},x,v_{1,2m_1})$.

It remains us to consider the case when e_x is incident to a vertex in $V_x \setminus W_0$. We assume w.l.o.g. that $e_x = (v_{0,j}, v_{h,m_h})$, then:

- insert x at position n/2 in row 2;
- if p is odd, then define $N_P'^x = N_D'^x$ as the edge set obtained from $B'(\mathcal{Q})$ by substituting for the edge $(i_{n/2}, 0)$ the chain $(i_{n/2}, x, 0)$; otherwise, define $N_P'^x = N_D'^x$ as the edge set obtained from $B(\mathcal{Q})$ by substituting for the edge $(j_{n/2-1}, 0)$ the chain $(j_{n/2-1}, x, 0)$.

The fact that $\mathcal{P}_x, \mathcal{P}'_x, e_x$ and the considered matchings satisfy condition (i) of Lemma A.2 is straightforward from Fact 1.

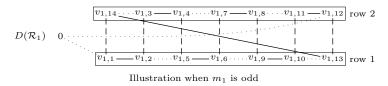
A.2.3 Case x = 0 and $h \ge 2$

Likewise the previous case when $m_0 = 0$, we consider for F_0 the edge set

$$F_0 = \bigcup_{1 \le s \le t \le h} W_s \times W_t$$

Let e_0 be an edge in F_0 that maximizes $\delta_P^0(e) + \delta_D^0(e)$. We may assume w.l.o.g. that e_0 is the edge $(v_{1,1}, v_{h,m_h+1})$ if $|V_0|/2$ is odd, and $(v_{1,1}, v_{h,m_h})$ otherwise.

We consider the approximate loading plan $\mathcal{P}_0 = \mathcal{P}(M_P^0, M_D^0)$ Algorithm 1 returns on V_0 . Furthermore, similarly to completions $A(\mathcal{P})$ and $A'(\mathcal{P})$, we



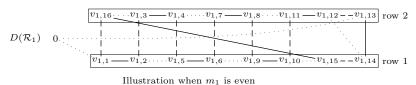


Figure 12: Alternate loading plan $\mathcal{R}_i = \mathcal{R}_i(M_P^0, M_D^0)$ of $V \setminus \{0\}$ an completion $D(\mathcal{R}_i)$ (in dotted lines) given two perfect matchings M_P^0, M_D^0 on V_0 such that $M_P^0 \cup M_D^0$ is a Hamiltonian cycle on V_0

associate with a loading plan $\mathcal{P} = ((i_1, \dots, i_{(n-1)/2}), (j_1, \dots, j_{(n-1)/2}))$ of V_0 the two perfect matchings $C(\mathcal{P})$ and $C(\mathcal{P})$ on V_0 defined by:

$$C(\mathcal{P}) = \{(i_p, j_{p+1}) \mid p = 1, \dots, (n-3)/2\} \cup \{(i_{(n-1)/2}, 0)\}$$

$$C'(\mathcal{P}) = \{(j_p, i_{p+1}) \mid p = 1, \dots, (n-3)/2\} \cup \{(j_{(n-1)/2}, 0)\}$$

Observe that the edge sets $C(\mathcal{P}_0) \cup M_P$, $C'(\mathcal{P}_0) \cup M_P$, $C(\mathcal{P}_0) \cup M_D$ and $C'(\mathcal{P}_0) \cup M_D$ all induce on V feasible pickup and delivery tours with respect to \mathcal{P} . Moreover, if (n-1)/2 is odd, then $C(\mathcal{P}_0) \cup C'(\mathcal{P}_0)$ induces on V_0 the Hamiltonian cycle

$$(i_1, j_2, i_3, j_4, \dots, i_{(n-1)/2}, j_1, i_2, j_3, i_4, \dots, j_{(n-1)/2}, i_1)$$

Otherwise, $C(\mathcal{P}_0) \cup C'(\mathcal{P}_0)$ is the union of the two cycles

and
$$(i_1, j_2, i_3, j_4, \dots, i_{(n-1)/2}, j_{(n-1)/2}, i_1)$$

 $(j_1, i_2, j_3, i_4, \dots, j_{(n-1)/2}, j_{(n-1)/2}, j_1).$

We deduce that \mathcal{P}_0 , $C(\mathcal{P})$, $C'(\mathcal{P})$ and e_0 satisfy (see Figure 11 for some illustration):

- $(\mathcal{P}_0, M_P^0 \cup C(\mathcal{P}_0), M_D^0 \cup C(\mathcal{P}_0))$ and $(\mathcal{P}_0, M_P^0 \cup C'(\mathcal{P}_0), M_D^0 \cup C'(\mathcal{P}_0))$ are feasible solutions on I;
- $C(\mathcal{P}_0)$, $C'(\mathcal{P}_0)$ and e_0 satisfy condition (i) of Lemma A.2.

A.2.4 Case x = 0 and h = 1

Let $\{v_1, \ldots, v_m\}$ denote the Hamiltonian cycle $M_P^0 \cup M_D^0$ on V_0 . We consider two families $\mathcal{P}(i), i \in [m]$ and $\mathcal{Q}(i), i \in [m]$ of loading plans on V_0 , together with their associated completions $N_i^{\mathcal{P}}$ and $N_i^{\mathcal{Q}}, i \in [m]$. The first family consists of

the basic loading plan on V_0 and its associated completion N_2 , but fixing vertex i on coordinates (1,1); namely (indexes are taken mod m):

$$\mathcal{P}(i) = ((v_i, \dots, v_{i+m/2-1}), (v_{i-1}, \dots, v_{i-m/2})), \qquad i \in [m]$$
(31)

$$N^{\mathcal{P}}(i) = \bigcup_{r=1}^{c-2} \left\{ \{v_{i-r}, v_{i+r}\} \right\} \cup \left\{ \{v_{i+m/2}, v_i\} \right\}, \qquad i \in [m]$$
 (32)

The loading plans Q(i) of the second family basically consists, starting with vertex v_i , in loading two consecutive vertices of the cycle $\{v_1, \ldots, v_m, v_1\}$ into alternatively row 1 and row 2. Precisely, given $i \in [m]$, let $E_3(i)$ denote the following perfect matching on V_0 :

$$E_3(i) = \{\{v_j, v_{j+3}\} \mid 1 \le j \le n - 1, \ j \bmod 2 \ne i \bmod 2\}$$
 (33)

Then, if m/2 is odd (iff $m \mod 4 = 2$), the loading plan Q(i) and its associated completion are defined as (indexes are taken mod m):

$$Q(i) = \left\{ \begin{array}{ll} (v_i, v_{i+1}, v_{i+4}, v_{i+5}, \dots, v_{i-6}, v_{i-5}, v_{i-2}), \\ (v_{i-1}, v_{i+2}, v_{i+3}, \dots, v_{i-8}, v_{i-7}, v_{i-4}, v_{i-3}) \end{array} \right\}$$
(34)

$$N^{\mathcal{Q}}(i) = E_3(i) \tag{35}$$

Otherwise (thus $m \mod 4 = 0$), Q(i) and $N^{Q}(i)$ are defined as:

$$Q(i) = \left\{ \begin{array}{l} (v_i, v_{i+1}, v_{i+4}, v_{i+5}, \dots, v_{i-8}, v_{i-7}, v_{i-2}, v_{i-3}), \\ (v_{i-1}, v_{i+2}, v_{i+3}, \dots, v_{i-10}, v_{i-9}, v_{i-6}, v_{i-5}, v_{i-4}) \end{array} \right\}$$
(36)

$$N^{\mathcal{Q}}(i) = E_3(i) \setminus \{\{v_{i-3}, v_i\}, \{v_{i-5}, v_{i-2}\}, \{v_{i-7}, v_{i-4}\}\}$$

$$\cup \{\{v_{i-5}, v_{i-3}\}, \{v_{i-7}, v_{i-2}\}, \{v_{i-4}, v_i\}\}$$

$$(37)$$

The loading plans Q(i) and their completion $N^{Q}(i)$ are depicted in Figures 12.

We know from the previous analysis that the triple $(\mathcal{P}(i), M_D^0 \cup N^{\mathcal{P}}(i), M_D^0 \cup N^{\mathcal{P}}(i))$ is a feasible solution for the DTSPMS₀ on V_0 , $i \in [m]$. Furthermore, similar arguments enable to establish that $(\mathcal{Q}(i), M_D^0 \cup N^{\mathcal{Q}}(i), M_D^0 \cup N^{\mathcal{Q}}(i))$ also is a feasible solution of the DTSPMS₀ on V_0 , $i \in [m]$. As a consequence, if we define, given $i \in [m]$, $\mathcal{P}_0(i)$, $\mathcal{Q}_0(i)$, $N_0^{\mathcal{P}}(i)$, $N_0^{\mathcal{Q}}(i)$ as

- $\mathcal{P}_0(i) = \mathcal{P}(i)$ and $\mathcal{Q}_0(i) = \mathcal{Q}(i)$ if x = 0, $\mathcal{P}_0(i)$ and $\mathcal{Q}_0(i)$ are obtained from respectively $\mathcal{P}(i)$ and $\mathcal{Q}(i)$ by inserting x at rank c in row 1 otherwise;
- $N_0^{\mathcal{P}}(i)$ is obtained from $N^{\mathcal{P}}(i)$ by inserting x on the edge $\{v_{i+m/2}, v_i\}$, $N_0^{\mathcal{Q}}(i)$ is obtained from $N^{\mathcal{Q}}(i)$ by inserting x on the edge $\{v_{i-3}, v_i\}$ if m/2 is odd, $\{v_{i-4}, v_i\}$ otherwise;

then we deduce from the feasibility of the solutions that are considered on V_0 that $(\mathcal{P}_0(i), M_P^0 \cup N_0^{\mathcal{P}}(i), M_D^0 \cup N_0^{\mathcal{P}}(i))$ and $(\mathcal{Q}_0(i), M_P^0 \cup N_0^{\mathcal{Q}}(i), M_D^0 \cup N_0^{\mathcal{Q}}(i))$ are feasible solutions on $I, i \in [m]$. We thus are interested in triples $(\mathcal{P}_0(i), \mathcal{P}_0(j), (i', j')), (\mathcal{P}_0(i), \mathcal{Q}_0(j), (i', j'))$ or $(\mathcal{Q}_0(i), \mathcal{Q}_0(j), (i', j'))$ that satisfy condition (i) of Lemma A.2. We establish:

Claim 1. Let $i \neq j$ in [m]. In any of the following cases, N, N' and $\{i, j\}$ satisfy condition (i) of Lemma A.2:

	$m,\;i,j$	N	N'
(1)	j-i is prime with m	$N_0^{\mathcal{P}}(i)$	$N_0^{\mathcal{P}}(j)$
(2)	$m/2 \equiv 3 \mod 6$ and $i \not\equiv j \mod 3$	$N_0^{\mathcal{P}}(i)$	$N_0^{\mathcal{Q}}(j)$
(3)	$m/2 \mod 6 \in \{1,5\}$ and $i \not\equiv j \mod 2$	$N_0^{\mathcal{Q}}(i)$	$N_0^{\mathcal{Q}}(j)$
(4)	$m/2 \equiv 0 \mod 6$ and $i \not\equiv j \mod 2$		
(5)	$m/2 \equiv 4 \mod 6, i \not\equiv j \mod 2$		
	and $(j-i) \not\equiv \pm 1 \bmod m$		
(6)	$m/2 \equiv 2 \mod 6$ and $(j-i) \equiv \pm 1 \mod m$		
(7)	$m/2 \equiv 2 \mod 6$ and $i \not\equiv j \mod 2$	$N_0^{\mathcal{Q}}(i+4)$	$N_0^{\mathcal{Q}}(j)$
	and $((j-i) \bmod m) \bmod 6 \in \{1,5\}$		
(8)	$m/2 \equiv 2 \mod 6$ and $i \not\equiv j \mod 2$	$N_0^{\mathcal{Q}}(i)$	$N_0^{\mathcal{Q}}(j+4)$
	and $((j-i) \bmod m) \bmod 6 \in \{3,5\}$		

Proof. All along the argument, indexes in [m] are taken mod m. Preliminary note that N, N' and $\{i, j\}$ satisfy condition (i) of Lemma A.2 $iff(N \cup N') \setminus \{\{i, x\}, \{x, j\}\} \cup \{\{i, j\}, \{i', j'\}\}$ is a Hamiltonian cycle on V $iff(N \cup N') \setminus \{\{i, x\}, \{x, j\}, \{i', x\}, \{x, j'\}\} \cup \{\{i, j\}, \{i', j'\}\}$ is a Hamiltonian cycle on V_0 for the two vertices $i' \neq i$ and $j' \neq j$ such that $\{i', x\} \in N$ and $\{j', x\} \in N'$.

• (1): preliminary note that $N^{\mathcal{P}}(1) = N_2$ and $N^{\mathcal{P}}(m) = N_1$. More generally, assume w.l.o.g. that $i = m \equiv 0 \mod m$ and let j be some integer in [m]. In $N^{\mathcal{P}}(0)$, a vertex $v_h \in V_0$ is adjacent to v_{-h} if $h \notin \{0, m/2\}$, to $v_{h+m/2}$ otherwise. In $N^{\mathcal{P}}(j)$, $v_h = v_{j+(h-j)}$ is adjacent to $v_{j-(h-j)} = v_{2j-h}$ if $h \notin \{j, j+m/2\}$, to $v_{h+m/2}$ otherwise. Hence, starting from $v_0, v_{m/2}, v_j$ and $v_{j+m/2}, N^{\mathcal{P}}(0) \cup N^{\mathcal{P}}(j)$ generates the sequences:

$$v_0, v_{2j}, v_{-2j}, v_{4j}, v_{-4j}, \dots$$
 $v_{m/2}, v_{2j-m/2}, v_{-2j+m/2}, v_{4j-m/2}, v_{-4j+m/2}, \dots$ $v_j, v_{-j}, v_{3j}, v_{-3j}, v_{5j}, \dots$ $v_{j+m/2}, v_{-j-m/2}, v_{3j+m/2}, v_{-3j+m/2}, v_{5j-m/2}, \dots$

Since j is prime with m/2, $2rj \equiv 0 \mod m$ for some $r \in \mathbb{N}^*$ iff $rj \equiv 0 \mod (m/2)$ iff r is a multiple of m/2. Now, we may have either $((0 \equiv m/2) \not\equiv (j \equiv j + m/2)) \mod 2$, or $((0 \equiv j) \not\equiv (m/2 \equiv j + m/2)) \mod 2$, or $((0 \equiv j + m/2) \not\equiv (m/2 \equiv j)) \mod 2$. If the first or the third case occur (iff j is odd), then $(m/2)j \equiv m/2 \mod m$ whereas if the second case occurs (iff j is even), then $(m/2)j \equiv 0 \mod m$. $N^{\mathcal{P}}(0) \cup N^{\mathcal{P}}(j)$ therefore takes the following expression depending on the partity of m/2 and j:

$N^{\mathcal{P}}(0) \cup N^{\mathcal{P}}(j)$	m/2	j
$\{v_0, v_{2j}, v_{-2j}, v_{4j}, \dots, v_{(m/2)j} = v_{m/2}, v_0\}$ $\cup \{v_j, v_{-j}, v_{3j}, \dots, v_{(m/2-1)j}, v_{-(m/2-1)j} = v_{j+m/2}, v_j\}$	even	odd
$ \{v_0, v_{2j}, v_{-2j}, v_{4j}, \dots, v_{(m/2-1)j}, v_{-(m/2-1)j} = v_j, v_{j+m/2}, v_{-j-m/2}, v_{3j+m/2}, \dots, v_{(m/2)j+m/2} = v_{m/2}, v_0\} $	odd	even
$ \{v_0, v_{2j}, v_{-2j}, v_{4j}, \dots, v_{(m/2-1)j}, v_{-(m/2-1)j} = v_{j+m/2}, v_j, v_{-j}, v_{3j}, \dots, v_{(m/2)j} = v_{m/2}, v_0\}. $	odd	odd

Hence, if j is odd, then the set $(N^{\mathcal{P}}(0) \cup N^{\mathcal{P}}(j)) \setminus \{\{v_0, v_{m/2}\}, \{v_j, v_{j+m/2}\}\} \cup \{\{v_0, v_j\}, \{v_{m/2}, v_{j+m/2}\}\}$ is a Hamiltonian cycle on V_0 .

• (2): assume w.l.o.g. that i=m and thus, $N^{\mathcal{P}}(0)=\{\{v_h,v_{-h}\},h=1,\ldots,m/2-1\}\cup\{\{v_0,v_{m/2}\}\}$. Let $j\in[m]$ such that $j\not\equiv i \bmod 3$. In E(j), v_h is adjacent to v_{h+3} if $h\not\equiv j \bmod 2$ and to v_{h-3} otherwise. Note that, since $m \bmod 2 = m \bmod 3 = 0$, $h \equiv -h \bmod 2$ and $h \equiv -h \bmod 3$, $h \in [m]$; by contrast, $h \pm 3 \not\equiv h \bmod 2$, $h \in [m]$. $N^{\mathcal{P}}(0) \cup E(j)$ is the union of the two cycles

$$\begin{cases} \{v_0, v_{-3}, v_3, v_6, v_{-6}, \dots, v_{-m/2} = v_{m/2}, v_0\} & \text{if } j \text{ is even,} \\ \{v_0, v_3, v_{-3}, v_{-6}, v_6, \dots, v_{m/2}, v_0\} & \text{if } j \text{ otherwise} \\ \text{and } \{v_j, v_{-j}, v_{-j-3}, v_{j+3}, v_{j+6}, \dots, v_{-j-3(m/3-1)}, v_{j+3(m/3-1)} = v_{j-3}, v_j\}. \end{cases}$$

The edge set $(N^{\mathcal{P}}(0) \cup N^{\mathcal{Q}}(j)) \setminus \{\{v_0, v_{m/2}\}, \{v_{j-3}, v_j\}\} \cup \{\{v_0, v_j\}, \{v_{m/2}, v_{j-3}\}\}$ therefore is a Hamiltonian cycle on V_0 .

- (3): The set E_3 defined as $E_3 = \{\{v_h, v_{h+3}\} \mid 1 \leq h \leq m\} \equiv (\mathbb{Z}/m\mathbb{Z}, \pm 3)$ is a Hamiltonian cycle on [m] iff $m \not\equiv 0 \mod 3$ iff $m/2 \not\equiv 0 \mod 3$. When m/2 is odd and $i \not\equiv j \mod 2$, then $N^{\mathcal{Q}}(i) \cup N^{\mathcal{Q}}(j) = E_3$ and thus, $N^{\mathcal{Q}}(i) \cup N^{\mathcal{Q}}(j) \setminus \{\{v_{i-3}, v_i\}, \{v_{j-3}, v_j\}\} \cup \{\{v_{i-3}, v_{j-3}\}, \{v_i, v_j\}\}$ therefore is a Hamiltonian cycle on V.
- (4), (5), (6), (7), (8): assume w.l.o.g. that $i = m \equiv 0 \mod m$. Let thus j be some odd index in [m]. First consider the set

$$E' = E_3 \setminus \{\{v_{-7}, v_{-4}\}, \{v_{-5}, v_{-2}\}, \{v_{-3}, v_0\}\} \cup \{\{v_{-7}, v_{-2}\}, \{v_{-5}, v_{-3}\}, \{v_{-4}, v_0\}\}$$

The following Table provides an explicit description of the two edge sets E_3 and E' and also identifies three paths C_0, C_1, C_2 in E', depending on $m \mod 6$:

$$\begin{array}{c} m \equiv 0 \bmod 6 \\ E_3: \ \{v_0, v_3, \ldots, v_{-3}, v_0\}, \{v_1, v_4, \ldots, v_{-5}, v_{-2}, v_1\}, \{v_{-1}, v_2, \ldots, v_{-7}, v_{-4}, v_{-1}\} \\ E': \ \{\underbrace{v_0, v_3, \ldots, v_{-3}, v_{-5}, \ldots, v_4, v_1, v_{-2}, v_{-7}, \ldots, v_2, v_{-1}, v_{-4}, v_0}_{C_0} \\ \hline m \equiv 2 \bmod 6 \\ E_3: \ \{v_0, v_3, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-3}, v_0\} \\ E': \ \{\underbrace{v_0, v_3, \ldots, v_{-5}, v_{-3}, \ldots, v_2, v_{-1}, v_{-4}, v_0}_{C_0}, \{\underbrace{v_{-2}, v_1, v_4, \ldots, v_{-7}, v_{-2}}_{C_1}\}_{C_0} \\ \hline m \equiv 4 \bmod 6 \\ \hline E_3: \ \{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0\} \\ E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E_3: \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_0}_{C_2} \\ \hline E': \ \{\underbrace{v_0, v_3, \ldots, v_{-7}, v_{-4}, v_{-1}, v_2, v_5, \ldots, v_{-5}, v_{-2}, v_1, v_4, \ldots, v_{-3}, v_{-5}, \ldots, v_{-5}, v_{-2}, v_{-1}, v_{-4}, v_0}_{C_2} \\ \end{bmatrix}$$

Now consider $N^{\mathcal{Q}}(0) \cup N^{\mathcal{Q}}(j)$, that is, the edge set

$$E' \quad \setminus \{\{v_{j-7}, v_{j-4}\}, \{v_{j-5}, v_{j-2}\}, \{v_{j-3}, v_{j}\}\} \\ \cup \{\{v_{j-7}, v_{j-2}\}, \{v_{j-5}, v_{j-3}\}, \{v_{j-4}, v_{j}\}\}$$

Although the three edges $\{v_{j-7}, v_{j-4}\}, \{v_{j-5}, v_{j-2}\}, \{v_{j-3}, v_j\}$ always lie on the paths C_0, C_1, C_2 , their location and orientation depend on $m \mod 6$ and j; the following Table locates these arcs on C_0, C_1, C_2 depending on $m \mod 6$ and j:

	C_0	C_1	C_2			
$m \equiv 0 \bmod 6$						
$j \equiv 3 \mod 6 \Leftrightarrow j \in \{3, 9, \dots, -3\}$	(j-3,j)	(j-2, j-5)	(j-4,j-7)			
$j \equiv 1 \mod 6 \Leftrightarrow j \in \{1, 7, \dots, -5\}$	(j-7,j-4)	(j, j - 3)	(j-2,j-5)			
$m \equiv 2 \bmod 6$						
$j \equiv 3 \bmod 6 \Leftrightarrow j \in \{3, 9, \dots, -5\}$	(j-3,j)	(j-5, j-2)				
$j \equiv 1 \bmod 6, j \neq -1, 1$	(j-7, j-4)	(j-3,j)	(j-2, j-5)			
$\Leftrightarrow j \in \{7, 13, \dots, -7\}$						
$m \equiv 4 \bmod 6$						
$j \equiv 3 \bmod 6, j \neq -1$	(j-3,j)	(j-5, j-2)	(j-4,j-7)			
$\Leftrightarrow j \in \{3, 9, \dots, -7\}$						
j = 1		(j-3,j), (j-7,j-4)	(j-2, j-5)			
		(j-7, j-4)				

Considering for each of these six cases the way edges $\{v_{j-7}, v_{j-2}\}, \{v_{j-5}, v_{j-3}\}, \{v_{j-4}, v_j\}$ reconnect the subchains generated by the removal of $\{v_{j-7}, v_{j-4}\}, \{v_{j-5}, v_{j-2}\}, \{v_{j-3}, v_j\}$ from E', we eventually obverse that $N^{\mathcal{Q}}(0) \cup N^{\mathcal{Q}}(j)$ consists of:

- a cycle of the shape $\{v_0,\ldots,v_{j-4},v_j,\ldots,v_{-4},v_0\}$ when $m\equiv 0 \mod 6$ and $j\equiv 3 \mod 6$, or $m\equiv 2 \mod 6$ and $j\neq \pm 1\equiv 1 \mod 6$, the union of two cycles such that the two edges $\{v_0,v_{-4}\}$ and $\{v_j,v_{j-4}\}$ do not belong to the same cycle when $m\equiv 0 \mod 6$ and $j\equiv 1 \mod 6$, or $m\equiv 2 \mod 6$ and $j\equiv 3 \mod 6$, or $m\equiv 4 \mod 6$ and $j\equiv 1$; in both the two cases, $N^{\mathcal{Q}}(0)\cup N^{\mathcal{Q}}(j)\setminus\{\{v_{-4},v_0\},\{v_{j-4},v_j\}\}\cup\{\{v_{-4},v_{j-4}\},\{v_0,v_j\}\}$ is a Hamiltonian cycle on V_0 .
- a cycle of the shape $\{v_0, \ldots, v_j, v_{j-4}, \ldots, v_{-4}, v_0\}$ when $m \equiv 4 \mod 6$ and $j \neq 1 \equiv 1 \mod 6$; in this case, $N^{\mathcal{Q}}(0) \cup N^{\mathcal{Q}}(j) \setminus \{\{v_{-4}, v_0\}, \{v_{j-4}, v_j\}\} \cup \{\{v_{-4}, v_j\}, \{v_0, v_{j-4}\}\}$ is a Hamiltonian cycle on V.

In order to conclude, finally observe that, given two indexes $i, j \in [m]$, when $m \equiv 0 \mod 6$, $(j-i) \equiv 5 \mod 6$ iff $(i-j) \equiv 1 \mod 6$; when $m \equiv 2 \mod 6$, $(j-i) \equiv 5 \mod 6$ iff $(i-j) \equiv 3 \mod 6$; eventually, when $m \equiv 4 \mod 6$, $(j-i) \equiv 1 \mod 6$ iff $(i-j) \equiv 3 \mod 6$ and:

- if $(j-i) \equiv 1 \mod 6$ and $j-i \neq 1$, then $(j-(i+4)) \equiv 3 \mod 6$; otherwise, (i+4)-j=3.
- if $(j-i) \equiv 3 \mod 6$ and $j-i \neq -1$, then $((j+4)-i) \equiv 1 \mod 6$; otherwise, (j+4)-i=3.
- if $(j-i) \equiv 5 \mod 6$, then $(j-(i+4)) \equiv 1 \mod 6$ and $((j+4)-i) \equiv 3 \mod 6$.