

# *Exact extended formulation* of the linear assignment problem (LAP) polytope for solving the traveling salesman and quadratic assignment problems

Moustapha Diaby

OPIIM Department; University of Connecticut; Storrs, CT 06268  
moustapha.diaby@uconn.edu

Mark H. Karwan

Department of Industrial and Systems Engineering; SUNY at Buffalo; Amherst, NY 14260  
mkarwan@buffalo.edu

Lei Sun

Department of Industrial and Systems Engineering; SUNY at Buffalo; Amherst, NY 14260  
leisun@buffalo.edu

*Abstract:* We present an  $O(n^6)$  linear programming model for the traveling salesman (TSP) and quadratic assignment (QAP) problems. The basic model is developed within the framework of the TSP. It does not involve the city-to-city variables-based, traditional TSP polytope referred to in the literature as “*the* TSP polytope.” We do not model explicit Hamiltonian cycles of the cities. Instead, we use a time-dependent abstraction of TSP tours and develop a direct *extended formulation* of the linear assignment problem (LAP) polytope. The model is *exact* in the sense that it has integral extreme points which are in one-to-one correspondence with LAP assignment solutions and TSP tours. It can be solved optimally using any linear programming (LP) solver, hence offering a new (incidental) proof of the equality of the computational complexity classes “ $P$ ” and “ $NP$ .” The extensions of the model to the time-dependent traveling salesman problem (TDTSP) as well as the quadratic assignment problem (QAP) are straightforward. The reasons for the non-applicability of existing negative *extended formulations* results for “*the* TSP polytope” to the model in this paper as well as our software implementation and the computational experimentation we conducted are briefly discussed.

*Keywords:* Linear Programming; Assignment Problem; Traveling Salesman Problem; TSP; Time-Dependent Traveling Salesman Problem (TDTSP); Quadratic Assignment Problem (QAP); Combinatorial Optimization; Computational Complexity; “ $P$  vs.  $NP$ .”

## 1 Introduction

The model developed in this paper is applicable to the traveling salesman problem (TSP) as well as the quadratic assignment problem (QAP). For the sake of simplicity and clarity of exposition, we first focus on the TSP and then briefly discuss the extension to the QAP.

The traveling salesman problem (TSP) has been one of the most-studied problems over the past six-to-seven decades. Books that have been written on the problem and its variants include Lawler *et al.* (1985), Reinelt (1994), Guten and Punen (2002), Applegate *et al.* (2007), and Diaby and Karwan (2016). Review papers include Balas and Toth (1985), Padberg and Song (1991), Fischetti *et al.* (2002), Öncan *et al.* (2009), D’Ambrosio *et al.* (2010), and Roberti and Toth (2012). The modeling we use in this paper falls within the general class of the so-called “time-dependent” models introduced in the seminal paper of Picard and Queyranne (1978). Reviews of time-dependent models include Gouveia and Voss (1992), Abeledo *et al.* (2013), Godinho *et al.* (2014), and Gendreau *et al.* (2015). The  $O(n^6)$  model in this paper is focused on the “standard” TSP for the sake of simplifying the presentation. However, it applies readily to the time-dependent traveling salesman problem. It can also be extended in a straightforward manner to the quadratic assignment problem (QAP) and many of its variations (see Pardalos *et al.* (1994); Hahn *et al.* (2010); Abdel-Basset *et al.* (2018); Furini and Traversi (2019); among others).

The model in this paper is the result of our long-continued efforts to simplify the  $O(n^9)$  models in Diaby (2007) and Diaby and Karwan (2016). It is the same as that in its previous (unpublished) version (Diaby *et al.* (2019)). The removal of classes of constraints -which we found- were redundant in that previous version has enabled a better proof and much clearer exposition in this paper. The proposed model is a more direct *extended formulation* of the linear assignment problem (LAP) polytope, compared to the  $O(n^9)$  models in Diaby (2007) and Diaby and Karwan (2016). It does not model arcs explicitly, but it incidentally fits closely within the “generic flow based formulations” classification of Godinho *et al.* (2011; pp. 2-3) for asymmetric TSP models. It is an *exact model* for the TSP and the QAP in the sense that it has integral extreme points which are in one-to-one correspondence with LAP assignment solutions and TSP tours. It can be solved optimally using any linear programming (LP) solver, hence offering a new (and incidental) proof of the equality of the computational complexity classes “ $P$ ” and “ $NP$ .” Both the model and its proof-of-integrality are much simpler than those for our  $O(n^9)$  models in Diaby (2007) and Diaby and Karwan (2016).

Yannakakis (1991, pp. 444-445) discusses, essentially as an unexplained paradox, the fact that Edmonds (1970)’s formulation of the Minimum Spanning Tree Problem (MSTP) is of exponential size (due to its having “TSP subtour elimination”-like constraints), while Martin (1991)’s formulation which includes Edmonds’ variables is of polynomial size. This paradox is resolved in the recent papers of Diaby and Karwan (2017) and Diaby, Karwan, and Sun (2021) which are focused on the issue of scope and applicability/non-applicability of *extended formulations* (EF) work pertaining to model sizes in general. In those papers, it is shown that if two polytopes are (or can be) described in terms of sets of variables which are disjoint, then the *extension relations* which can be established between them by the introduction of redundant variables and constraints are only degenerate ones from which no valid inferences can be made as to model sizes. This fact is then used (in those papers) to develop multi-level refutations (including numerical counter-examples) of the recent “unconditional impossibility” claims with respect to the modeling of NP-complete problems as LPs (Fiorini *et al.* (2015)) in particular (see Diaby and Karwan (2017) and Diaby, Karwan, and Sun (2021)). The reasons for the non-applicability of existing negative

results for “*the* TSP polytope” to our developments in this paper are the same as those for the case of Martin (1991)’s formulation of the MSTP as it relates to Edmonds’ formulation of the MSTP, as developed in Diaby and Karwan (2017) and Diaby, Karwan, and Sun (2021).

The plan of this paper is as follows. We will conclude this section with some basic, foundational assumptions for our modeling of the TSP. Then, we will provide an overview of our LAP solution abstraction of TSP tours in section 2. The formulation of our proposed LP model will be developed in section 3. The integrality of the model will be discussed in section 4. Some immediate extensions (including to the QAP) will be discussed in section 5. The computational experimentation we conducted will be discussed in section 6. Finally, some concluding remarks will be offered in section 7, and a brief overview of our software implementation of the model will be given in the Appendix.

**Assumption 1** We assume without loss of generality (w.l.o.g.) that:

1. The number of cities is greater than 5;
2. The TSP graph is complete. (Arcs on which travel is not permitted can be handled in the optimization model by associating large (“Big- $M$ ”) costs to them);
3. City “0” has been designated as the starting and ending point of the travels.

**Definition 2** Like in the Miller-Tucker-Zemlin (1960)’s classical formulation, we refer to the order in which a given city is visited after city 0 in a given TSP tour as the “time-of-travel” of that city in that TSP tour. In other words, if city  $i$  is the  $r^{th}$  city to be visited after city 0 in a given TSP tour, then we will say that the *time-of-travel* of city  $i$  in the given tour is  $r$ .

## 2 LAP solution abstraction of TSP tours

The graph which underlies our modeling consists of isolated nodes corresponding to (city, *time-of-travel*) pairs. The term “layered graph” has been used to refer to this graph (Abeledo et al. (2013, pp. 3-4)) and also variants of it (Godinho et al. (2011, p. 6); for example). In this paper, we will draw from the terminology used in Diaby (2007) and Diaby and Karwan (2016) and refer to this graph as the “TSP Assignment Graph (TSPAG).” Also, we will refer to the set of nodes of this graph corresponding to a given city of the TSP as a “level” of the graph, and to the set of nodes corresponding to a given *time-of-travel* of the TSP as a “stage” of the graph. Our overall modeling approach consists of formulating the TSP as an *extended formulation* of the linear assignment problem (LAP) polytope over this graph. A formal statement and an illustration of the graph are given below.

**Notation 3 (TSPAG formalisms)**

1.  $n$  : Number of cities;
2.  $m := n - 1$  (Numbers of *levels* and *stages* respectively, of the TSPAG);

3.  $\{0, \dots, m\}$  : Index set for the cities;
4.  $L := \{1, \dots, m\}$  (Index set for the *levels* of the TSPAG);
5.  $S := \{1, \dots, m\}$  (Index set for the *stages* of the TSPAG);
6.  $N := \{(l, s) \in (L, S)\}$  (Set of nodes of the TSPAG. We will, henceforth, write  $(l, s) \in N$  as “ $\lceil l, s \rceil$ ” (with no “ceiling” or “floor” meaning attached to “ $\lceil$ ” and “ $\rceil$ ”) in order to distinguish it from other doublets).

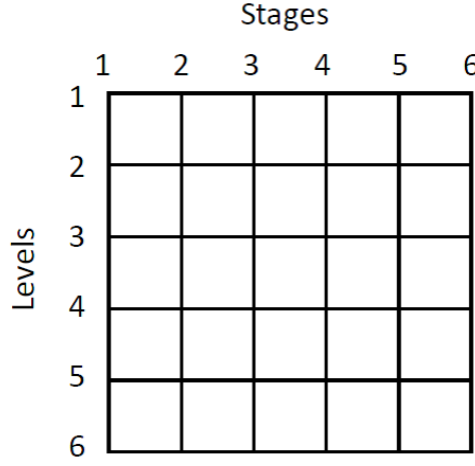


Figure 1: Illustration of the TSP Assignment Graph (TSPAG)

Our modeling and proofs (discussed in sections 3 and 4) rest on abstractions of the “path structures” defined below.

**Definition 4 (“TSP paths”)**

1. We refer to a set of nodes at consecutive *stages* of the TSPAG with exactly one node at each *stage* in the set involved as a *path* of the TSPAG. In other words, for  $(r, s) \in R^2 : s > r$ , we refer to  $\{\lceil u_p, p \rceil \in N, p = r, \dots, s\}$  as a *path* of the TSPAG.
2. We refer to a *path* of the TSPAG which (simultaneously) spans the *stages* and the *levels* of the TSPAG as a *TSP path* (of the TSPAG). In other words, letting  $m$  be the number of *stages* of the TSPAG, we refer to  $\{\lceil u_p, p \rceil \in N, p = 1, \dots, m : (\forall (p, q) \in R^2 : p \neq q, u_p \neq u_q)\}$  as a *TSP path* (of the TSPAG).

A *TSP path* is illustrated in Figure 2.

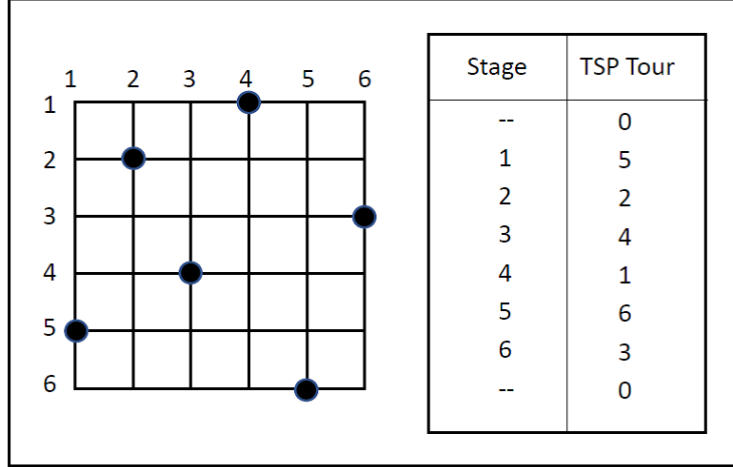


Figure 2: Illustration of *TSP paths*

**Remark 5**

1. The *TSP paths* of the TSPAG are in a one-to-one correspondence with the TSP tours of the TSP (subject to each TSP tour being uniquely represented in whatever scheme is being used to represent the TSP tours in terms of the TSP nodes).
2. Any extreme point of the Linear Assignment Problem (LAP) polytope defined over the TSPAG (i.e., the polytope induced by constraints (7)-(8) and (14) of section 3.2 below) can represent a TSP tour. Our *extended formulation* of the LAP polytope into a higher dimension is required in order to capture the cost function of the TSP problem only, not to represent/form the traditional city-to-city, edge-based TSP polytope.

We close this section with the following general conventions and initial developments.

**Notation 6 (General notations)**

1.  $Ext(\cdot)$  : Set of extreme points of  $(\cdot)$ .
2.  $(\cdot)^t$  : Transpose of matrix  $(\cdot)$ .
3.  $\mathbb{R}$  : Set of real numbers.
4.  $\mathbb{N}_+$  : Set of positive natural numbers.
5.  $V^\alpha$  :  $\alpha^{th}$  Cartesian power of  $V$ . ( $V^\alpha = \{(v_1, \dots, v_\alpha) : v_i \in V \text{ for each } i \in \{1, \dots, \alpha\}\}$ .)
6. Let  $\mathbf{u}$  be a vector of variables on  $[0, 1]$  in  $N^\alpha$  space. The characteristic  $\mathbf{u}$ -vector of  $V \subseteq N^\alpha$  is denoted  $\hat{\mathbf{u}}^V$ . (In order to obtain  $\hat{\mathbf{u}}^V$ , each component of  $\mathbf{u}$  indexed in  $V$  is set to “1,” and each component of  $\mathbf{u}$  not indexed in  $V$  is set to “0.”)

**Definition 7 (“Exhaust” Notion)**

Throughout the paper, we will use two (standard) notions of *exhaustion* which are as follows:

1. Let  $A$  be a set. Let  $B_1, \dots, B_a$  ( $a \in \mathbb{N}_+$ ) be subsets of  $A$  (i.e.,  $B_k \subseteq A$  for each  $k \in \{1, \dots, a\}$ ). We will say that  $B_1, \dots, B_a$  “exhaust”  $A$  iff  $\bigcup_{k \in \{1, \dots, a\}} B_k = A$ .
2. Let  $F \neq \emptyset$  be the feasible set of the system of linear constraints  $A_i^t y \leq a_i$ ,  $i = 1, \dots, \beta$  (where:  $A_i \in \mathbb{R}^\alpha$  for each  $i \in \{1, \dots, \beta\}$ ;  $y \in \mathbb{R}^\alpha$ ;  $a_i \in \mathbb{R}$  for each  $i \in \{1, \dots, \beta\}$ ; and  $\alpha, \beta \in \mathbb{N}_+$ ). We will say that  $z \in F$  “exhausts” (constraint)  $k$  ( $k \in \{1, \dots, \beta\}$ ) iff  $A_k^t z = a_k$ .

The following result pertains to the special case of systems of equality constraints.

**Lemma 8** Let  $F := \{y \in \mathbb{R}^\beta : Ay = a; a \neq \mathbf{0}\}$  (where  $A \in \mathbb{R}^{\alpha \times \beta}$ ;  $a \in \mathbb{R}^\alpha$ ). Let  $\bar{y} \in F$ .

Then,  $\forall z \in \mathbb{R}^\beta$  :

- (a)  $((\bar{y} + z) \in F \text{ or } (\bar{y} - z) \in F) \implies z \notin F$ .
- (b)  $z \in F \implies ((\bar{y} + z) \notin F \text{ and } (\bar{y} - z) \notin F)$ .

**Proof.** Observe that the two statements of the lemma are equivalent. Hence, we will provide the proof for Statement (a) only.

1.  $(\bar{y} + z) \in F \implies z \notin F$ .

$(\bar{y} + z) \in F$  implies:

$$A(\bar{y} + z) = a. \tag{1}$$

Also, we have:

$$A(\bar{y} + z) = A\bar{y} + Az = a + Az. \tag{2}$$

(1) and (2) imply:

$$Az = \mathbf{0} \neq a. \tag{3}$$

(3) implies  $z \notin F$ .

2.  $(\bar{y} - z) \in F \implies z \notin F$ .

$(\bar{y} - z) \in F$  implies:

$$A(\bar{y} - z) = a. \tag{4}$$

Also, we have:

$$A(\bar{y} - z) = A\bar{y} - Az = a - Az. \tag{5}$$

(4) and (5) imply:

$$Az = \mathbf{0} \neq a. \tag{6}$$

(6) implies  $z \notin F$ .

■

From a (mathematical programming) resource consumption perspective, the combination of Definition 7 and Lemma 8 essentially says that any given feasible solution to a model which has equality constraints only (except for nonnegativity constraints) *exhausts* each of the constraints of the model, and also that no “residual” which is feasible for the model can be generated from the given solution by addition to or subtraction from it.

### 3 Formulation of the linear programming (LP) model

#### 3.1 Model variables

We use two classes of variables defined in terms of the nodes of the TSPAG in our modeling. These variables have no restrictions other than the ones implied by our modeling constraints given in section 3.2. They are specified in the following notation.

##### Notation 9 (Modeling variables)

1.  $\forall [i, r] \in N$ ,  $w_{[i, r]}$  : Variable indicating the assignment of *level*  $i$  to *stage*  $r$ .
2.  $\forall ([i, p], [j, r], [k, s]) \in N^3$ ,  $x_{[i, p][j, r][k, s]}$  : Variable indicating the simultaneous assignments of *levels*  $i$ ,  $j$ , and  $k$  to *stages*  $p$ ,  $r$ , and  $s$ , respectively.
3.  $\forall ([i_\alpha, \alpha], [i_\beta, \beta], [i_\gamma, \gamma]) \in N^3$ ,  $\bar{x}([i_\alpha, \alpha], [i_\beta, \beta], [i_\gamma, \gamma])$  : Function that returns an  $x$ -variable with the (*level*, *stage*) pairs of indices arranged in increasing order of the *stage* indices. Specifically:

$$\forall ([i_\alpha, \alpha], [i_\beta, \beta], [i_\gamma, \gamma]) \in N^3,$$

$$\bar{x}([i_\alpha, \alpha], [i_\beta, \beta], [i_\gamma, \gamma]) := \begin{cases} x_{[i_\alpha, \alpha][i_\beta, \beta][i_\gamma, \gamma]} & \text{if } \alpha < \beta < \gamma; \\ x_{[i_\alpha, \alpha][i_\gamma, \gamma][i_\beta, \beta]} & \text{if } \alpha < \gamma < \beta; \\ x_{[i_\beta, \beta][i_\alpha, \alpha][i_\gamma, \gamma]} & \text{if } \beta < \alpha < \gamma; \\ x_{[i_\beta, \beta][i_\gamma, \gamma][i_\alpha, \alpha]} & \text{if } \beta < \gamma < \alpha; \\ x_{[i_\gamma, \gamma][i_\alpha, \alpha][i_\beta, \beta]} & \text{if } \gamma < \alpha < \beta; \\ x_{[i_\gamma, \gamma][i_\beta, \beta][i_\alpha, \alpha]} & \text{if } \gamma < \beta < \alpha; \\ 0 & \text{Otherwise.} \end{cases}$$

( $\bar{x}(\cdot)$  is used for the purpose of simplifying the exposition only.)

**Definition 10 (“Connectedness”)**

1. A pair of TSPAG nodes,  $[i, r]$  and  $[j, s]$ , are said to be “connected” in a given feasible solution to our LP constraints set *iff* there exists a third node,  $[u, p]$ , of the TSPAG such that  $\bar{x}([i, r], [j, s], [u, p])$  is greater than zero in the solution.
2. A given node of the TSPAG is said to be *connected* to a given *level (stage)* of the TSPAG in a given feasible solution to our model if it is *connected* to at least one node of the given *level (stage)* in the solution.

## 3.2 Model constraints

### 3.2.1 Statement of the constraints

The constraints of our model are as follows.

- **Linear Assignment Problem (LAP) constraints.**

$$\sum_{r=1}^m w_{[i,r]} = 1; \quad i = 1, \dots, m \quad (7)$$

$$\sum_{i=1}^m w_{[i,r]} = 1; \quad r = 1, \dots, m \quad (8)$$

- **Linear Extension (LE) constraints.**

$$w_{[i,r]} - \sum_{j=1; j \neq i}^m \sum_{k=1; k \neq i, j}^m \bar{x}([i, r], [j, s], [k, p]) = 0; \quad i, r = 1, \dots, m; \\ s = 1, \dots, m-1; \quad s \neq r; \quad p = s+1, \dots, m; \quad p \neq r \quad (9)$$

$$w_{[i,r]} - \sum_{s=1; s \neq r}^m \sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k, p]) = 0; \quad i, r = 1, \dots, m; \\ j = 1, \dots, m-1; \quad j \neq i; \quad k = j+1, \dots, m; \quad k \neq i \quad (10)$$

- **Connectivity Consistency (CC) constraints.**

$$\sum_{k=1; k \neq i, j}^m \bar{x}([i, r], [j, s], [k, p]) - \sum_{k=1; k \neq i, j}^m \bar{x}([i, r], [j, s], [k, p + \sigma_{rsp}]) = 0; \\ i, j = 1, \dots, m; \quad i \neq j; \quad r = 1, \dots, m-1; \quad s = r+1, \dots, m; \quad p = 1, \dots, m-1; \\ p \neq r, s; \quad p + \sigma_{rsp} \leq m; \quad \sigma_{rsp} := \arg \min_{q \in \{1, \dots, m-p+1\}} \{p+q : (p+q) \notin \{r, s\}\}. \quad (11)$$



$$\sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k, p]) - \sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k + \lambda_{ijk}, p]) = 0;$$

$$i, j = 1, \dots, m; i \neq j; r = 1, \dots, m-1; s = r+1, \dots, m; k = 1, \dots, m-1;$$

$$k \neq i, j; k + \lambda_{ijk} \leq m; \lambda_{ijk} := \arg \min_{l \in \{1, \dots, m-k+1\}} \{k+l : (k+l) \notin \{i, j\}\}. \quad (12)$$

- “Implicit-Zeros (IZ)” constraints.

$$x_{[i, r][j, s][k, p]} = 0 \quad \text{if } (!(r < s < p) \text{ or } !(i \neq j \neq k)) \quad (13)$$

- Nonnegativity (NN) constraints.

$$w_{[i, r]} \geq 0 \quad \forall i, r = 1, \dots, m \quad (14)$$

$$x_{[i, r][j, s][k, p]} \geq 0 \quad \forall i, r, j, s, k, p = 1, \dots, m. \quad (15)$$

Constraints (7) and (8) are the standard LAP constraints.

Constraints (9) and (10) establish initial *connectednesses* between a given node  $[i, r] \in N$  with a positive *weight* (i.e., with  $w_{[i, r]} > 0$ ) and nodes at all the other *stages* and *levels* of the TSPAG, respectively. They are illustrated in Figure 3.

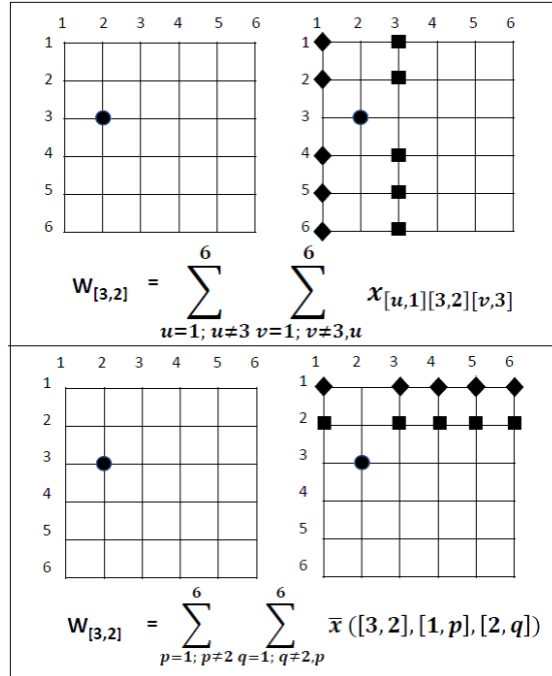


Figure 3: Illustration of the *LE Constraints*

Constraints (11) and (12) stipulate that the *connectedness* between two given nodes of the TSPAG must be “recognized” consistently across all the *stages* and all the *levels* of the graph, respectively.

In constraints (11),  $(p + \sigma_{rsp})$  is the index of the first *stage* after  $p$  which is distinct from  $r$  and  $s$  respectively. Hence, these constraints say that the total *connectedness* of a given node pair,  $([i, r], [j, s])$ , to a given *stage*,  $p$ , is equal to the total *connectedness* of the node pair to the first *stage* after  $p$ , excluding *stages*  $r$  and  $s$ . Note that if there exists no *stage* greater than  $p$  which is distinct from (both)  $r$  and  $s$ , then  $(p + \sigma_{rsp})$  would be equal to  $(m + 1)$ , so that there would be no constraint (11) for  $p$  and the given  $([i, r], [j, s])$  pair.

Similarly,  $(k + \lambda_{ijk})$  in constraints (12) is the index of the first *level* greater than  $k$  which is distinct from  $i$  and  $j$  respectively. Hence, these constraints say that the total *connectedness* of a given node pair,  $([i, r], [j, s])$ , to a given *level*,  $k$ , is equal to the total *connectedness* of the node pair to the first *level* after  $k$ , excluding *levels*  $i$  and  $j$ . If there exists no *level* greater than  $k$  which is distinct from  $i$  and from  $j$ , then  $(k + \lambda_{ijk})$  would be equal to  $(m + 1)$ , so that there would be no constraint (12) for  $k$  and the given  $([i, r], [j, s])$  pair.

The two sets of constraints ((11)-(12)) are illustrated in Figure 4.

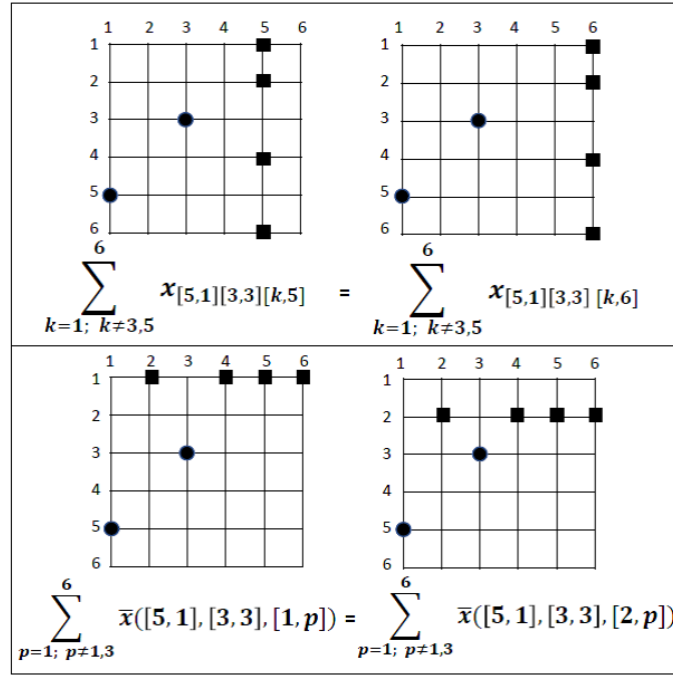


Figure 4: Illustration of the *CC Constraints*

The *IZ Constraints* (13) serve a dual purpose. They ensure that the *connectedness* among the nodes in a given triplet is modeled by a unique  $x$ -variable. They also preclude the *self-connectedness* of a node being “built into” a given  $x$ -variable. Finally, (14)-(15) are the usual nonnegativity constraints on the modeling variables.

The following notations and remarks will be needed in the development of our proposed proof of integrality.

**Remark 11**

1. It follows directly from constraints (7)-(10) and (14)-(15) that  $w \in [0, 1]^{m^2}$  and  $x \in [0, 1]^{m^6}$ .
2. It follows directly from constraints (13) that  $\bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p \rceil) > 0$  implies  $i \neq j \neq k$  and  $r \neq s \neq p$ .
3. The nonnegativity constraints (14)-(15) will be implicitly assumed to hold throughout the discussion in the remainder of this paper.
4. The fact that  $\bar{x}(\cdot)$  is invariant with respect to the order in which its arguments are listed will be used without being explicitly stated throughout the remainder of this paper whenever that is convenient and does not cause ambiguity.

**Notation 12 (Solution Instance Notations)**

1.  $Q := \left\{ (w^t \ x^t)^t \in [0, 1]^{m^2+m^6} : (w^t \ x^t)^t \text{ satisfies (7) -- (13)} \right\}.$
2.  $W(x) : \forall (w^t \ x^t)^t \in Q, W(x) := \{ \lceil i, r \rceil \in N : w_{\lceil i, r \rceil} > 0 \}.$
3.  $X(w) : \forall (w^t \ x^t)^t \in Q,$   
 $X(w) := \{ (\lceil i_1, r_1 \rceil, \lceil i_2, r_2 \rceil, \lceil i_3, r_3 \rceil) \in N^3 : \bar{x}(\lceil i_1, r_1 \rceil, \lceil i_2, r_2 \rceil, \lceil i_3, r_3 \rceil) > 0 \}.$
4.  $Y(w, x) : \forall (w^t \ x^t)^t \in Q, Y(w, x) := \{ (\lceil i_1, r_1 \rceil, \lceil i_2, r_2 \rceil) \in N^2 : (\exists \lceil i_3, r_3 \rceil \in N : (\lceil i_1, r_1 \rceil, \lceil i_2, r_2 \rceil, \lceil i_3, r_3 \rceil) \in X(w)) \}.$   
 (Set of *connected* node-pairs (Definition 10) to each other.)
5.  $AP(i, r, j, s) : \forall (w^t \ x^t)^t \in Q, \forall (\lceil i, r \rceil, \lceil j, s \rceil) \in N^2,$   
 $AP(i, r, j, s) := \{ \lceil u, p \rceil \in N : (\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) \in X(w) \}.$   
 (Set of nodes to which  $\lceil i, r \rceil$  and  $\lceil j, s \rceil$  are jointly *connected* (see Definition 10).)
6.  $\mathcal{A} := \left\{ w \in \mathbb{R}^{m^2} : w \text{ satisfies (7), (8), (14)} \right\}$
7.  $\bar{\mathcal{A}} := \left\{ w \in \mathbb{R}^{m^2} : \left( \exists x \in \mathbb{R}^{m^6} : (w^t \ x^t)^t \in Q \right) \right\}.$

**Remark 13** The proofs of the following statements are omitted because they are trivial:

1.  $\forall (w^t \ x^t)^t \in Q, \forall (\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) \in N^3, (\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) \in X(w) \iff$   
 $((\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x); (\lceil i, r \rceil, \lceil u, p \rceil) \in Y(w, x); \text{ and } (\lceil j, s \rceil, \lceil u, p \rceil) \in Y(w, x)).$

2.  $\forall (w^t \ x^t)^t \in Q, \forall (r, s) \in S^2,$   

$$\bigcup_{(i,j) \in L^2: (\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x)} (\{\lceil i, r \rceil, \lceil j, s \rceil\} \cup AP(i, r, j, s)) =$$

$$\bigcup_{(i,j) \in L^2: (\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x)} \{\lceil i, r \rceil, \lceil j, s \rceil\} \bigcup_{(i,j) \in L^2: (\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x)} AP(i, r, j, s) = W(x).$$
3.  $\mathcal{A}$  is the Linear Assignment Problem (LAP) polytope (see Burkard *et al.* (2009)).
4.  $\overline{\mathcal{A}} = \mathcal{A}$ . (i.e.,  $Q$  is an *extended formulation* of  $\mathcal{A}$ .)
5. Each integral point of  $Q$  is an extreme-point of  $Q$ .
6. There is a one-to-one correspondence between any two among the following:
  - (a) TSP tours;
  - (b) *TSP paths* of the TSPAG;
  - (c) LAP solutions over the TSPAG;
  - (d) Extreme points of  $\mathcal{A}$ ;
  - (e) Integral points of  $Q$ .

**Lemma 14 (“Structure” of Integral Solutions)**  $(w^t \ x^t)^t \in Q$  is integral *iff* there exists a unique  $\{\lceil v_p, p \rceil \in N, p = 1, \dots, m : (\forall (p, q) \in S^2 : p \neq q, v_p \neq v_q)\}$  such that the components of  $(w^t \ x^t)^t$  can be stated as follows:

- (a)  $\forall \lceil i, r \rceil \in N, w_{\lceil i, r \rceil} = \begin{cases} 1 & \text{if } i = v_r; \\ 0 & \text{otherwise.} \end{cases}$
- (b)  $\forall (\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) \in N^3,$   

$$\overline{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) = \begin{cases} 1 & \text{if } r \neq s \neq p \text{ and } (i, j, u) = (v_r, v_s, v_p); \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Let  $(w^t \ x^t)^t \in Q$ .

1. ( $\implies$  :)

Assume  $(w^t \ x^t)^t$  is integral. Then:

- (a) Statement (a) of the lemma follows directly (see Burkard et al. (2009); Bazaraa et al. (2010; pp. 513-546)) from the combination of constraints (7), (8), and (14).

(b) Observe that constraints (9)-(10) imply:

$$\begin{aligned} \forall(\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) \in N^3, \\ (w_{\lceil i, r \rceil} = 0 \text{ or } w_{\lceil j, s \rceil} = 0 \text{ or } w_{\lceil u, p \rceil} = 0) \implies \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) = 0. \end{aligned} \quad (16)$$

In other words (since  $(w^t \ x^t)^t$  is integral by premise),

$$\begin{aligned} \forall(\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) \in N^3, \ \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) > 0 \implies \\ \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil u, p \rceil) = 1 \implies (w_{\lceil i, r \rceil} = 1 \text{ and } w_{\lceil j, s \rceil} = 1 \text{ and } w_{\lceil u, p \rceil} = 1). \end{aligned} \quad (17)$$

Statement (b) of the lemma follows directly from (17).

2. ( $\Leftarrow$  :) Trivial.

■

### 3.3 Abstraction of the *TSP Paths* in terms of the modeling variables

The “global” perspective we take in developing our proof-of-integrality of  $Q$  is the converse of the conventional/natural one used in Combinatorial Optimization and Integer Programming work in general. When dealing with issues of integrality of an optimization model, the standard approach has been/is to focus on the extreme points of the model with the idea that *if* every extreme point of a model is integer, *then* every non-extreme point of that model is a convex combination of integral points of the model. Our approach in this paper is to focus on the interior points instead, with the idea that *if* every interior point of a model can be represented as a non-trivial convex combination of mathematical “objects” with a specific structure, *then* every extreme point of that model must be one of those “objects”/abstract structures.

In this paper, we abstract TSP tours into a specific kind of mathematical “object” defined in terms of our modeling variables. Then, we show that every feasible solution to our model is a convex combination of “objects” of this kind, hence leading to the conclusion that each extreme point of the model must be one of this kind of “object.” In the discussions to follow, we use the terminology “*Permutation Matrix Support-in- $x$*  (PMS $x$ )” to designate the mathematical “objects” into which we abstract the TSP tours in terms of our modeling variables.

**Definition 15 (“*Permutation-Matrix-Support-in- $x$* ”)** Let  $(w^t \ x^t)^t \in Q$ .

1. We refer to  $\{\lceil u_h, h \rceil \in W(x), \ h = 1, \dots, m\}$  as a “Permutation Matrix Support-in- $x$  (PMS $x$ )” if  $(\lceil u_g, g \rceil, \lceil u_p, p \rceil, \lceil u_q, q \rceil) \in X(w)$  for each  $(g, p, q) \in S^3$  which is such that  $g \neq p \neq q$ .

2. We will denote the set of all the  $PMSx$ 's of  $(w^t \ x^t)^t$  by  $\bar{\Pi}(w, x)$ .
3. We will denote the  $k^{th}$  member of  $\bar{\Pi}(w, x)$  (assuming  $\bar{\Pi}(w, x) \neq \emptyset$ ) by  $\Pi^k(w, x)$ .

The following result establishes the correspondence between  $PMSx$ 's and integral points of  $Q$ .

**Lemma 16** Let  $(w^t \ x^t)^t \in Q$ . Assume  $\bar{\Pi}(w, x) \neq \emptyset$ . For convenience, for each  $\Pi^k(w, x) \in \bar{\Pi}(w, x)$  ( $k \in \{1, \dots, |\bar{\Pi}(w, x)|\}$ ), define :

$$T^k(w, x) := \{(\lceil u_r^k, r \rceil, \lceil u_p^k, p \rceil, \lceil u_q^k, q \rceil) \in (\Pi^k(w, x))^3 : r < p < q\}.$$

Then, for each  $\Pi^k(w, x) \in \bar{\Pi}(w, x)$  ( $k \in \{1, \dots, |\bar{\Pi}(w, x)|\}$ ),  $\begin{pmatrix} \hat{w}^{\Pi^k(w, x)} \\ \hat{x}^{T^k(w, x)} \end{pmatrix}$  is an extreme point of  $Q$ .

**Proof.** Observe that from definitions (Notations 6.6), we have that for each  $\Pi^k(w, x) \in \bar{\Pi}(w, x)$  ( $k \in \{1, \dots, |\bar{\Pi}(w, x)|\}$ ), Characteristic Vector  $\begin{pmatrix} \hat{w}^{\Pi^k(w, x)} \\ \hat{x}^{T^k(w, x)} \end{pmatrix}$  is integral.

Observe also that  $\begin{pmatrix} \hat{w}^{\Pi^k(w, x)} \\ \hat{x}^{T^k(w, x)} \end{pmatrix}$  satisfies every constraint of  $Q$ .

The lemma follows directly from these observations above (see Remark 13.6). ■

**Corollary 17** Let  $(w^t \ x^t)^t \in Q$ . Each  $PMSx$  (we will show that there exists at least one) corresponds to exactly one integral point of  $Q$  (and therefore, according to Remark 13.6, to exactly one extreme point of  $\mathcal{A}$ , and to exactly one TSP tour), and vice versa.

$PMSx$ 's and Corollary 17 are illustrated in Figure 5.

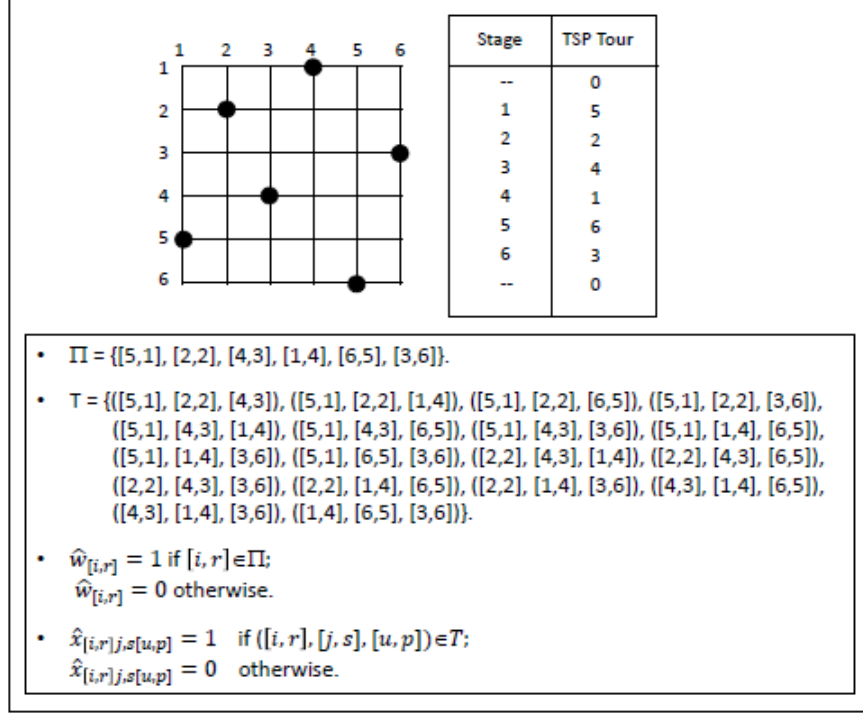


Figure 5: Illustration of a *PMSx* and associated *characteristic vectors*.

### 3.4 Model objective

The development of our objective function is predicated on our model being integral (i.e., on the model having integral extreme points). A wide variety of alternatives exists for this, along the lines discussed in Diaby and Karwan (2016; pp. 85-90). The cost function we use in this paper is shown in the following theorem.

**Theorem 18** Let  $(w^t \ x^t)^t \in Q$ . Assume  $Q$  is integral. Define  $\tilde{c} \in \mathbb{R}^6$  in terms of the TSP travel costs, as follows:

$$\tilde{c}_{[i,p][j,r][k,s]} := \begin{cases} c_{0i} + c_{ij} + c_{jk} & \text{if } (p = 1; r = 2; s = 3); \\ c_{jk} + c_{k0} & \text{if } (p = 1; r = m - 1; s = m); \\ c_{jk} & \text{If } (p = 1; 3 \leq r \leq m - 2; s = r + 1); \\ 0 & \text{Otherwise.} \end{cases} \quad (18)$$

Then,

$$\begin{aligned}\mathcal{V}(w, x) &:= \begin{pmatrix} \mathbf{0}^t & x^t \end{pmatrix} \cdot \begin{pmatrix} w^t & x^t \end{pmatrix}^t \\ &= \sum_{(\lceil i, p \rceil, \lceil j, r \rceil, \lceil k, s \rceil) \in N^3} \tilde{c}_{\lceil i, p \rceil, \lceil j, r \rceil, \lceil k, s \rceil} x_{\lceil i, p \rceil, \lceil j, r \rceil, \lceil k, s \rceil}\end{aligned}$$

is the (correct) weighted sum of the costs of the TSP tours represented by the integral points of  $Q$  which have positive weights in the extreme-point representation of  $\begin{pmatrix} w^t & x^t \end{pmatrix}^t$ .

**Proof.**  $Q$  being integral implies that every extreme-point representation of any given (arbitrary)  $\begin{pmatrix} w^t & x^t \end{pmatrix}^t \in Q$  is a convex combination of integral points of  $Q$ . Each integral point in such a representation is the characteristic vector associated with a  $PMSx$  of  $\begin{pmatrix} w^t & x^t \end{pmatrix}^t$  (Corollary 17). Hence, it is sufficient to show that  $\tilde{c}$  correctly accounts the total TSP travel cost associated with each  $PMSx$  of  $\begin{pmatrix} w^t & x^t \end{pmatrix}^t$ .

Consider a  $PMSx$  of  $\begin{pmatrix} w^t & x^t \end{pmatrix}^t$ ,  $\Pi^k(w, x) = \{ \lceil u_h^k, h \rceil \in W(x), h = 1, \dots, m \}$ , as specified in Definition 15. The (unique) TSP tour corresponding to  $\Pi^k(w, x)$  is  $0 \rightarrow u_1^k \rightarrow \dots \rightarrow u_m^k \rightarrow 0$ . Let “ $TCost$ ” denote the cost of this tour. Then, we have:

$$TCost = c_{0, u_1^k} + c_{u_m^k, 0} + \sum_{q=1}^{m-1} c_{u_q^k, u_{q+1}^k}. \quad (19)$$

Now, consider the total cost associated to  $\Pi^k(w, x)$ . This cost is the sum of the costs attached to the variables indexed in  $T^k(w, x)$ . Hence, using (18), the total cost associated to  $\Pi^k(w, x)$  is as follows:

$x_{\lceil u_p^k, p \rceil, \lceil u_r^k, r \rceil, \lceil u_s^k, s \rceil}$	$\tilde{c}_{\lceil u_p^k, p \rceil, \lceil u_r^k, r \rceil, \lceil u_s^k, s \rceil}$
$p = 1; r = 2; s = 3$	$c_{0, u_1^k} + c_{u_1^k, u_2^k} + c_{u_2^k, u_3^k}$
$p = 1; r = m - 1; s = m$	$c_{u_{m-1}^k, u_m^k} + c_{u_m^k, 0}$
$p = 1; r = 3; s = 4$	$c_{u_3^k, u_4^k}$
$\vdots$	$\vdots$
$p = 1; r = m - 2; s = m - 1$	$c_{u_{m-2}^k, u_{m-1}^k}$
Total cost associated to $\Pi^k(w, x) =$	$c_{0, u_1^k} + c_{u_m^k, 0} + \sum_{q=1}^{m-1} c_{u_q^k, u_{q+1}^k}$

Comparing the results above, we observe that the total cost associated to  $\Pi^k(w, x)$  is equal to the total cost of the TSP tour represented by  $\Pi^k(w, x)$ . ■

### 3.5 Computational complexity order of the model size

The following theorem establishes the polynomial size of the model.

**Theorem 19**



1. The computational complexity order of the number of *non-implicitly-zero* variables in the system (7) – (15) is  $O(n^6)$ .
2. The computational complexity order of the number of constraints which must be explicitly expressed in a linear programming (LP) optimization problem over the system (7) – (15) is  $O(n^5)$ .

**Proof.**

1. The possible total number of variables in the system (7)–(15) is equal to  $(m^6 + m^4 + m^2)$ , and the number of *implicitly-zero* variables in the system is greater than zero. Hence, letting  $\bar{n}_v$  denote the number of *non-implicitly-zero* variables in the system, we must have:

$$\bar{n}_v < m^6 + m^2 < 2m^6 = 2(n-1)^6. \quad (20)$$

Hence,  $\bar{n}_v$  is bounded by a 6<sup>th</sup>-degree polynomial function of  $n$ . Statement (1) of the theorem follows from this directly.

2. Consider the three classes of constraints which must be explicitly stated in the system (7) – (15). We have:

Constraint Class/Type	Bound on Total Count
$LAP$	$2m$
$LE$	$2m^4$
$CC$	$2m^5$

Hence, letting  $\bar{n}_c$  denote the number of constraints which must be explicitly expressed in (7) – (15), we must have:

$$\bar{n}_c < 2(m^5 + m^4 + m) < 6m^5 = 6(n-1)^5 \quad (21)$$

Hence,  $\bar{n}_c$  is bounded by a 5<sup>th</sup>-degree polynomial function of  $n$ . Statement (2) of the theorem follows from this directly.

■

## 4 Integrality of the LP model

Our proof approach consists of showing that there exist *PMSx* “patterns” in any given solution to our model ( $Q$ ), and that the given solution is a convex combination of the integral points of  $Q$  corresponding to those *PMSx*’s (as specified in Lemma 16 above).

The following lemma shows that there are LAP structures induced by constraints (11)–(12) over the nodes of the TSPAG.

**Lemma 20** Let  $(w^t \ x^t)^t \in Q$ . Let  $(\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x)$ . There must exist  $a_{irjs} \in (0, 1]$  such that the following two conditions are true:

$$\forall p \in S \setminus \{r, s\}, \quad \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p \rceil) = a_{irjs}. \quad (22)$$

$$\forall k \in L \setminus \{i, j\}, \quad \sum_{p=1; p \neq r, s}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p \rceil) = a_{irjs}. \quad (23)$$

**Proof.** From Definitions 12.4 and 12.5:

$$(\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x) \iff AP(i, r, j, s) \neq \emptyset. \quad (24)$$

Consider  $\lceil v, q \rceil \in AP(i, r, j, s)$ . We have the following.

1. Statement (22).

$\lceil v, q \rceil \in AP(i, r, j, s)$  implies:

$$\bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil v, q \rceil) > 0 \quad (25)$$

From (25), we get:

$$0 < \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil v, q \rceil) \leq \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, q \rceil). \quad (26)$$

Observe that constraints (11) imply:

$$\begin{aligned} & \forall (p_1, p_2) \in (S \setminus \{r, s\})^2, \\ & \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p_1 \rceil) = \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p_2 \rceil). \end{aligned} \quad (27)$$

Expression (1) of the lemma follows directly from the combination of (26) and (27).

2. Statement (23).

First, from statement (22), we have:

$$\sum_{p=1; p \neq r, s}^m \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p \rceil) = (m-2)a_{irjs}. \quad (28)$$

Now, using (24)-(25), we have:

$$0 < \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil v, q \rceil) \leq \sum_{p=1; p \neq r, s}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil v, p \rceil). \quad (29)$$

Observe that constraints (12) imply:

$$\begin{aligned} \forall(k_1, k_2) \in (S \setminus \{r, s\})^2, \\ \sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k_1, p]) = \sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k_2, p]). \end{aligned} \quad (30)$$

(29) and (30) imply:

$$\exists b_{irjs} > 0 : \sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k, p]) = b_{irjs} \quad \forall k \in L \setminus \{i, j\}. \quad (31)$$

From (31), we have:

$$\sum_{k=1; k \neq i, j}^m \sum_{p=1; p \neq r, s}^m \bar{x}([i, r], [j, s], [k, p]) = (m-2)b_{irjs}. \quad (32)$$

Combining (28) and (32), we must have:

$$b_{irjs} = \alpha_{irjs}. \quad (33)$$

Statement (23) follows from the combination of (31) and (33).

■

**Remark 21** Let  $(w^t \ x^t)^t \in Q$ . Let  $([i, r], [j, s]) \in Y(w, x)$ .

1. Constraints (22)-(23) have a LAP structure over the subgraph of the TSPAG which excludes *levels*  $i$  and  $j$  and *stages*  $r$  and  $s$  (see Burkard et al. (2009); Bazaraa et al. (2010; pp. 513-546)).
2. It follows from Statement (1) above and the integrality of the LAP polytope (Burkard et al. (2009); Bazaraa et al. (2010; pp. 513-546)) that the set of nodes *connected* to  $([i, r], [j, s])$  (Definition 10) can be re-grouped into collections of subsets each of which (subsets) corresponds to a LAP solution over the subgraph of the TSPAG which excludes *levels*  $i$  and  $j$  and *stages*  $r$  and  $s$ . Hence, each of the subsets in such a re-grouping/collection corresponds (see Remark 13.6) to a *TSP path* defined over the subgraph.
3. The augmentation of a given subset in a given re-grouping described in Statement (2) above with  $\{[i, r], [j, s]\}$  corresponds to a *TSP path* of the TSPAG, and therefore (according to Remark 13.6), to a LAP solution over the TSPAG, an extreme point of  $\mathcal{A}$ , and an integral point of  $Q$ , respectively.

4. By the convexity of the LAP polytope, alternate feasible re-groupings such as described in the statements (2) – (3) above in this remark are possible. (This is illustrated also in Figure 6 below.)

The discussions in the remark above (Remark 21) are formalized in the next definition below.

**Definition 22 (“Feasible Representation Groupings (FRG’s)”)** Let  $(w^t \ x^t)^t \in Q$ . Let  $(\lceil i, r \rceil, \lceil j, s \rceil) \in Y(w, x)$ .

1. We will refer to a re-grouping/collection of subsets of  $AP(i, r, j, s)$  corresponding to convex combination representations of a given solution of the LAP induced over the subgraph excluding *levels*  $i$  and  $j$  and *stages*  $r$  and  $s$  (as discussed in Remark 21 above) as a “Feasible Representation Grouping (FRG) (of  $AP(i, r, j, s)$ ).”
2. The number of possible FRG’s of  $AP(i, r, j, s)$  will be denoted by  $\mathfrak{g}_{irjs}(w, x)$ .
3. The index set for the FRG’s will be denoted  $\mathcal{G}_{irjs}(w, x) := \{1, \dots, \mathfrak{g}_{irjs}(w, x)\}$ .
4. We will refer to each of the subsets comprised in a given FRG as a “Joint Propagation Path in  $x$  ( $JPPx$ ) (for  $((\lceil i, r \rceil, \lceil j, s \rceil))$ ).”
5. The number of  $JPPx$ ’s comprising the  $g^{th}$  FRG will be denoted  $\psi_{irjs}^g$ .
6. The  $k^{th}$  of the  $JPPx$ ’s comprising the  $g^{th}$  FRG will be denoted  $\Psi_{irjs}^{g,k}$ .

In other words, denoting the *level* of the node included in  $\Psi_{irjs}^{g,k}$  at *stage*  $p \in S \setminus \{r, s\}$  by  $\mathbf{l}_p^{g,k}$  and the *stage* of the node included in  $\Psi_{irjs}^{g,k}$  at *level*  $u \in L \setminus \{i, j\}$  by  $\mathbf{s}_u^{g,k}$ , formal expressions for the  $JPPx$ ’s are as follows:

$$\forall g \in \mathcal{G}_{irjs}(w, x) = \{1, \dots, \mathfrak{g}_{irjs}(w, x)\}, \forall k \in \{1, \dots, \psi_{irjs}^g\},$$

$$\Psi_{irjs}^{g,k} := \{[\mathbf{l}_p^{g,k}, p] \in W(x) : \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, [\mathbf{l}_p^{g,k}, p]) > 0,$$

$$p = 1, \dots, m; \ p \notin \{r, s\}; \ \forall (q_1, q_2) \in (S \setminus \{r, s\})^2 : q_1 \neq q_2, \ \mathbf{l}_{q_1}^{g,k} \neq \mathbf{l}_{q_2}^{g,k};$$

$$= \{[u, \mathbf{s}_u^{g,k}] \in W(x) : \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, [u, \mathbf{s}_u^{g,k}]) > 0,$$

$$u = 1, \dots, m; \ u \notin \{i, j\}; \ \forall (v_1, v_2) \in (L \setminus \{i, j\})^2 : v_1 \neq v_2, \ \mathbf{s}_{v_1}^{g,k} \neq \mathbf{s}_{v_2}^{g,k}\}.$$

Definition 22 is illustrated in Figure 6. In the illustration,  $(\lceil 1, 1 \rceil, \lceil 2, 2 \rceil) \in Y(w, x)$ . Part (a) of the figure shows the nodes which are *connected* (Definition 10) to  $(\lceil 1, 1 \rceil, \lceil 2, 2 \rceil)$  in an arbitrary solution satisfying Lemma 20 for  $(\lceil 1, 1 \rceil, \lceil 2, 2 \rceil)$ , along with the number of possible FRG’s (which is 2 in the example). Part (b) and (c) of the figure show the two possible FRG’s. The reader may find it useful to put some emphasis on analyzing the positive  $x$ -variables discussed in the figure.

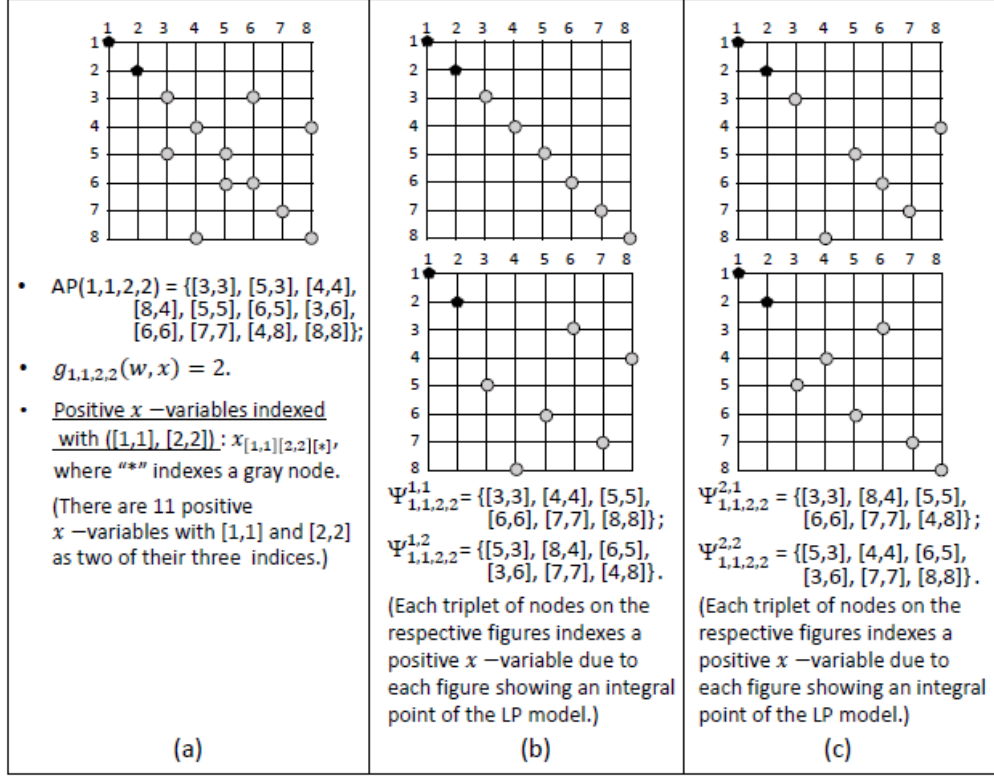


Figure 6: Illustration of the *Feasible Representation Groupings* (FRG's)

**Remark 23** Let  $(w^t \ x^t)^t \in Q$ . The truths of the statements below follow directly from definitions (Notation 12.4; Notation 12.5; Definition 22) and Lemma 20 (it may be useful to review Figure 6):

$$1. \forall ([i, r], [j, s]) \in N^2 :$$

$$(a) ([i, r], [j, s]) \in Y(w, x) \iff AP(i, r, j, s) \neq \emptyset \iff g_{irjs}(w, x) \geq 1.$$

(Essentially, if  $[i, r]$  and  $[j, s]$  are *connected* (Definition 10; Notation 12.4; Notation 12.5), then there must exist at least one *JPP*  $x$  for them, and vice versa.)

$$(b) \text{ If } ([i, r], [j, s]) \in Y(w, x), \text{ then for each } g \in \mathcal{G}_{irjs}(w, x), \bigcup_{k=1}^{\psi_{irjs}^g} \Psi_{irjs}^{g,k} = AP(i, r, j, s).$$

(The union of all the *JPP*'s in any given grouping/re-grouping of  $AP(i, r, j, s)$  *exhausts* (Definition 7)  $AP(i, r, j, s)$ .)

$$(c) \text{ From (b) above, if } ([i, r], [j, s]) \in Y(w, x), \text{ then } \bigcup_{g \in \mathcal{G}_{irjs}(w, x)} \bigcup_{k=1}^{\psi_{irjs}^g} \Psi_{irjs}^{g,k} = AP(i, r, j, s).$$

- (d)  $\forall [u, p] \in N, \bar{x}([i, r], [j, s], [u, p]) > 0 \iff \forall g \in \mathcal{G}_{irjs}(w, x), \exists k \in \{1, \dots, \psi_{irjs}^g\} : [u, p] \in \Psi_{irjs}^{g,k}.$

(A given node of the TSPAG indexes a positive  $x$ -variable along with two other given nodes if and only if the given node belongs to at least one  $JPPx$  in each FRG which is based on the other two given nodes.)

2.  $\forall (r, s) \in S^2,$

$$\begin{aligned} & \bigcup_{(i,j) \in L^2: ([i,r],[j,s]) \in Y(w,x)} \left( \{[i, r], [j, s]\} \bigcup_{g \in \mathcal{G}_{irjs}(w,x)} \bigcup_{k=1}^{\psi_{irjs}^g} \Psi_{irjs}^{g,k} \right) \\ &= \bigcup_{(i,j) \in L^2: ([i,r],[j,s]) \in Y(w,x)} (\{[i, r], [j, s]\} \cup AP(i, r, j, s)) \\ &= W(x). \end{aligned}$$

(The set of augmented  $JPPx$ 's for *connected* pairs of nodes (one for each) of two given *stages exhausts* (Definition 7) the set of nodes with positive *weights* in the solution at hand; see (1.c) above and Remark 13.2.)

We will now discuss our main theorem.

**Theorem 24**  $Q$  is integral, with a one-to-one correspondence between its extreme points and TSP tours.

**Proof.** We will show that every  $(w^t \ x^t)^t \in Q$  must resolve into a single  $PMSx$  or into a collection of  $PMSx$ 's. We will do this by essentially “tying together” what has been already done in the earlier parts of the paper.

Let  $(w^t \ x^t)^t \in Q$ .

As discussed in Remark 21, and using the terminology in Definition 22, Lemma 20 implies (see Remark 21.3) that each  $JPPx$  of  $([i, r], [j, s]) \in Y(w, x)$  augmented with  $\{[i, r], [j, s]\}$  corresponds to exactly one  $TSP$  path of the TSPAG (Definition 4), and to exactly one integral point of  $Q$ . Hence, we have that:

$$\begin{aligned} & \text{Each } JPPx \text{ of } ([i, r], [j, s]) \in Y(w, x) \text{ augmented with} \\ & \{[i, r], [j, s]\} \text{ is a } PMSx \text{ of } (w^t \ x^t)^t \text{ (see Definition 15).} \end{aligned} \tag{34}$$

Using Remark 23.2 and (34) above:

$$\begin{aligned} & \text{The set of all the nodes included in the augmented } JPPx\text{'s} \\ & \text{of } (w^t \ x^t)^t \text{ (hence equivalently, in the } PMSx \text{ of } (w^t \ x^t)^t) \\ & \text{exhausts (Definition 7.1) } W(x). \end{aligned} \tag{35}$$

Using (35) and Remark 13.1, we must have that:

$$\begin{aligned} & \text{The set all of the node-triplets respectively indexing} \\ & \text{characteristic } x\text{-variables associated to the } PMSx\text{'s} \\ & \text{of } (w^t \ x^t)^t \text{ exhausts (Definition 7.1) } X(w). \end{aligned} \quad (36)$$

By the convexity of  $Q$ , every convex combination of extreme points of  $Q$  is a feasible solution of  $Q$ . Hence, every convex combination of the extreme points corresponding to the  $\{[i, r], [j, s]\}$ -augmented  $PMSx$ 's of the  $([i, r], [j, s])$  pairs of  $Y(w, x)$  must be a feasible solution of  $Q$ . Hence (since each constraint of our model  $(Q)$  -except for the nonnegativity constraints- is an equality), the following must be true:

$$\begin{aligned} & \text{Every convex combination of the extreme points of} \\ & Q \text{ corresponding to the } \{[i, r], [j, s]\}\text{-augmented} \\ & PMSx\text{'s of the } ([i, r], [j, s]) \text{ pairs of } Y(w, x) \text{ exhausts} \\ & \text{(Definition 7.2) every constraint of } Q. \end{aligned} \quad (37)$$

From the combination of the fact that  $(w^t \ x^t)^t$  is a feasible solution to  $Q$ , Lemma 8, and the facts (35)-(37), we get that:

$$\begin{aligned} & (w^t \ x^t)^t \text{ must be a convex combination of the extreme} \\ & \text{points of } Q \text{ corresponding to the } \{[i, r], [j, s]\}\text{-augmented} \\ & PMSx\text{'s of the } ([i, r], [j, s]) \text{ pairs of } Y(w, x). \end{aligned} \quad (38)$$

Hence, in other words, after re-indexing the  $([i, r], [j, s]) \cup \Psi_{irjs}^{g,k}$   $PMSx$ 's of  $(w^t \ x^t)^t$  ( $([i, r], [j, s]) \in Y(w, x)$ ;  $g \in \mathcal{G}_{irjs}(w, x)$ ;  $k \in \{1, \dots, \psi_{irjs}^g\}$ ) into a single list (excluding repetitions) and re-labeling them using the notation in Lemma 16 (for convenience), the following must be true (possibly, after re-arrangement of the list):

$$\exists \left( 1 \leq \Delta_{wx} < \sum_{([i,r],[j,s]) \in Y(w,x)} \sum_{g \in \mathcal{G}_{irjs}(w,x)} \psi_{irjs}^g; \delta_\ell \in [0, 1], \ell = 1, \dots, \Delta_{wx} \right) :$$

$$(i) \sum_{\ell=1}^{\Delta_{wx}} \delta_\ell = 1;$$

$$(ii) \forall [i, r] \in N, w_{[i,r]} = \sum_{\ell=1}^{\Delta_{wx}} \delta_\ell \widehat{w}_{[i,r]}^{\Pi^\ell(w,x)};$$

$$(iii) \forall ([i, r], [j, s], [u, p]) \in N^3, x_{[i,r][j,s][u,p]} = \sum_{\ell=1}^{\Delta_{wx}} \delta_\ell \widehat{x}_{[i,r][j,s][u,p]}^{T^\ell(w,x)};$$

where  $\Delta_{wx}$  is the number of distinct  $PMSx$ 's of  $(w^t \ x^t)^t$ .

The theorem follows from this directly. ■

Hence, in retrospect, what our developments in this section have shown is that on one hand, constraints (9)-(10) “force” a unique representation of/“structure” for integral solutions of our model (which are extreme points of the model) in terms of our modeling variables (Lemma 14), while on the other hand, constraints (11)-(12) ensure that only convex combinations of those integral solutions (as expressed in terms of their representations (Lemma 20)) are considered.

**Corollary 25** The linear program (Problem TSPLP) below:

$$\begin{array}{|l}
 \text{Minimize :} \\
 \\
 \mathcal{V}(w, x) := (\mathbf{0}^t \quad x^t) \cdot (w^t \quad x^t)^t \\
 \\
 = \sum_{(\lceil i, p \rceil, \lceil j, r \rceil, \lceil k, s \rceil) \in N^3} \tilde{c}_{\lceil i, p \rceil \lceil j, r \rceil \lceil k, s \rceil} x_{\lceil i, p \rceil \lceil j, r \rceil \lceil k, s \rceil} \\
 \\
 \text{Subject to:} \\
 \\
 (w^t \quad x^t)^t \in Q
 \end{array}$$

correctly solves the TSP.

## 5 Some immediate extensions

In this section, we will discuss some extensions of our proposed model to some other well-studied problems. The extensions do not require any modifications to the constraints set ((8) – (15)) of our model. Hence, we refer to them as “immediate extensions.” The *proofs-of-correctness* for the objective functions that we apply for these extensions are similar to that of Theorem 18 and will therefore be omitted.

### 5.1 Time-dependent traveling salesman problem (TDTSP)

As indicated earlier, reviews of TDTSPs can be found in Gouveia and Voss (1992), Abeledo et al. (2013), Godinho et al. (2014), and Gendreau et al. (2015). In this section, we will consider the commonly-used form which was first introduced in the seminal Picard and Queyranne (1978) paper, and in which inter-city travel costs also depend on the *times-of-travel*.

Denote by  $d_{irj}$  the cost incurred when cities  $i$  and  $j$  are visited at *times*  $r$  and  $r + 1$ , respectively. Let  $d_{0i}$  be the travel cost from city “0” to city  $i$ , and  $d_{i0}$ , the cost of travel from



city  $i$  to city “0.” Then, the extension of our proposed LP model to the TDTSP consists of applying the objective function resulting from the costs below over the constraints set defined by (8) – (15) (i.e.,  $Q$ ):

$$\begin{aligned} & \forall ([i, p], [j, r], [k, s]) \in N^3, \\ & \bar{d}_{[i,p][j,r][k,s]} := \begin{cases} d_{0i} + d_{ipj} + d_{jrk} & \text{if } (p = 1; r = 2; s = 3); \\ d_{jrk} + d_{k0} & \text{if } (p = 1; r = m - 1; s = m); \\ d_{jrk} & \text{If } (p = 1; 3 \leq r \leq m - 2; s = r + 1); \\ 0 & \text{Otherwise.} \end{cases} \end{aligned} \quad (39)$$

## 5.2 Quadratic assignment problems

The quadratic assignment problem (QAP) is different from the LAP only in that its objective function consists of minimizing the sum of *assignment interaction* costs, plus *fixed costs* for the individual assignments. The QAP is one the most-extensively studied problems in operations research. The two best-recognized seminal papers are those by Koopmans and Beckmann (1957) and Lawler (1963), respectively. *NP-hardness* was established in the 1970’s (Sahni and Gonzales (1976)). Reviews can be found in Pardalos et al. (1994), Cela (1998), Anstreicher (2003), Loiola et al. (2007), Hahn et al. (2010), and Abdel-Basset et al. (2018), among others.

By letting  $L$  and  $S$  in Notations 3.4-3.5 stand for the two sets of objects to be assigned to each other, QAP and many of its variants can be solved as LPs over  $Q$ . We generically consider that there is a *fixed* cost,  $o_{ir}$ , which is incurred when  $i \in L$  is assigned to  $r \in S$ , and that an *interaction* cost,  $h_{irjs}$ , is incurred when  $i, j \in L$  are assigned to  $r, s \in S$ , respectively. The objective is to find an assignment which minimizes the total of these costs.

### 5.2.1 Generalized quadratic assignment problem (GQAP)

In the GQAP, the *assignment interaction* costs (the  $h_{irjs}$ 's) are arbitrary (see Hahn et al. (2010)). For this problem, the costs to attach to our  $x$ -variables are as shown in (40) below:

$$\bar{h}_{[i,p][j,r][k,s]} := \begin{cases} o_{ip} + h_{ipjr} + h_{ipks} & \text{if } (r = p + 1; r + 1 = s < m); \\ o_{ip} + o_{jr} + o_{ks} + h_{ipjr} + h_{ipks} + h_{jrks} & \text{if } (r = p + 1; r + 1 = s = m); \\ h_{ipks} & \text{If } (r = p + 1; r + 1 < s); \\ 0 & \text{Otherwise.} \end{cases} \quad (40)$$

### 5.2.2 “Standard” quadratic assignment problem (QAP)

In a facilities location/allocation context where the objective is to minimize the generic *material handling* costs (see Koopmans and Bechmanns (1957)), the GQAP reduces to the “standard” QAP. In this case, let  $L$  and  $S$  stand for the sets of “departments” and “sites,” respectively. Let  $f_{ij}$  ( $(i, j) \in L^2 : i \neq j$ ) denote the flow volume from *department*  $i$  to *department*  $j$ , and  $d_{rs}$  ( $(r, s) \in S^2 : r \neq s$ ), the cost of movement from *site*  $r$  to *site*  $s$ . Then, the *interaction* costs are *decomposable*, and (40) can be re-expressed using the following:

$$\begin{aligned} \forall (i, j) \in L^2 : i \neq j, \forall (r, s) \in S^2 : r \neq s, \\ h_{irjs} = f_{ij}d_{rs} + f_{ji}d_{sr}. \end{aligned} \quad (41)$$

### 5.2.3 Cubic assignment problem (CAP)

The CAP is an extension of the GQAP in which the *interaction* costs involve triplets (instead of doublets) of assignments (see Hahn et al. (2010)). Let the *interaction* cost of assigning  $i, j, k \in L$  to  $p, r, s \in S$  respectively, be denoted as  $e_{ipjrks}$ . The objective function coefficients to attach to our  $x$ -variables are  $\bar{h}_{[i,p][j,r][k,s]} = e_{ipjrks}$  for all  $([i, p], [j, r], [k, s]) \in N^3$ .

### 5.2.4 Relation to relaxation-linearization-technique (R-L-T) models

Our proposed model has a lot of similarity with the  $R$ - $L$ - $T$  models of Adams and Sherali (1986), Sherali and Adams (1999), Adams et al. (2007), and Hahn et al. (2012)). The classes of variables in our modeling are included in those of the *Level-2*  $R$ - $L$ - $T$  model. However, our constraint set strictly subsumes those of the *Level-2*  $R$ - $L$ - $T$  model.

## 6 Numerical experimentation

As shown in section 3.5 above, the numbers of variables and constraints of our proposed model are  $O(n^6)$  and  $O(n^5)$ , respectively. In order to get a sense of the actual size and the computational performance of the model (although we are aware that streamlined or large-scale optimization approaches will have to be eventually developed for the model to be useful in practice), we undertook a C# implementation of it (see the Appendix of this paper) and applied it to randomly-generated problems as well as some “test bank” problems from the literature.

### 6.1 Some implementation considerations

Our experimentation revealed that software/hardware-related numerical difficulties can arise when the differences between the city-to-city travels costs are very small. In such situations, the “behavior” of a LP solver may be unpredictable, including cases in which the LP procedure (we used CPLEX) fails to converge after more than 12 hours on 9-city problems, or terminates with indications of (meaningful/significant) primal or dual infeasibilities, or with a “basis singularity” error message. For example, with each cost equal to 10 for a 9-city problem, we know the optimal tour length is 90, but the interior point method (of CPLEX), even with crossover, oscillates between values such as 89.999999 and 90.000001 and results as described above. Using randomly generated costs between 9.999 and 10.001, we did not encounter this problem. In a more systematic way, we found that all these kinds of numerical issues consistently get resolved when the generalized flow conservation/*Kirchhoff* constraints below (which are redundant for the model) are added to the model.

$$\sum_{k=1; k \neq i, j}^m x_{[i, r][k, s-1][j, s]} - \sum_{k=1; k \neq i, j}^m x_{[i, r][j, s][k, s+1]} = 0; \\ i, j = 1, \dots, m; i \neq j; r = 1, \dots, m-3; s = r+2, \dots, m-1 \quad (42)$$

$$\sum_{k=1; k \neq i, j}^m x_{[k, r-1][i, r][j, s]} - \sum_{k=1; k \neq i, j}^m x_{[i, r][k, r+1][k, s+1]} = 0; \\ i, j = 1, \dots, m; i \neq j; r = 2, \dots, m-2; s = r+2, \dots, m \quad (43)$$

We suspect that there may be other types of such redundant constraints which may be effective in “helping along” LP solvers with respect to numerical issues as well. The following result shows that constraints (42) and (43) are indeed redundant for our model.

**Theorem 26** Constraints (42) and (43) are redundant for  $Q$ .

**Proof.** Let  $(w^t \ x^t)^t \in Q$ . Let  $(\lceil i, r \rceil, \lceil j, s \rceil) \in N^2$ .

From observation, constraints (11) imply:

$$\forall (p_1, p_2) \in (S \setminus \{r, s\})^2, \quad \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p_1 \rceil) = \sum_{k=1; k \neq i, j}^m \bar{x}(\lceil i, r \rceil, \lceil j, s \rceil, \lceil k, p_2 \rceil). \quad (44)$$

For  $r \in \{1, \dots, m-3\}$  and  $s \in \{r+2, \dots, m-1\}$ , constraints (42) are obtained from constraints (44) by letting  $p_1 = s-1$  and  $p_2 = s+1$  in constraints (44).

Similarly, for  $r \in \{2, \dots, m-2\}$  and  $s \in \{r+2, \dots, m\}$ , constraints (43) are obtained from constraints (44) by letting  $p_1 = r-1$  and  $p_2 = r+1$  in constraints (44). ■

## 6.2 Numerical results

For the purpose of assessing the actual model size, we ran “counting procedures” in our basic code (i.e., not including (42)-(43)) for TSPs with 7 to 25 cities. These runs were done on a Dell Precision T7610 workstation with dual-Intel Xeon E5-2605v2 processors (2.50 GHz each) and 512 GB of RAM. The results of these runs are shown in Table 1.

Number of Cities	Number of Variables	Number of Constraints
7	2,436	3,432
8	7,399	8,540
9	18,880	18,384
10	42,417	35,658
11	86,500	63,920
12	163,471	107,712
13	290,544	172,680
14	490,945	265,694
15	795,172	394,968
16	1,242,375	570,180
17	1,881,856	802,592
18	2,774,689	1,105,180
19	3,995,460	1,492,704
20	5,634,127	1,981,928
21	7,798,000	1,436,841
22	10,613,841	1,843,864
23	14,230,084	2,338,249
24	18,819,175	2,933,352
25	24,580,032	3,643,825

Table 1: TSP LP Size vs. Number of TSP Cities

We performed some regression on the results summarized in Table 1 above. The best fits we obtained are shown in Figure 7. While the complexity order of the number of variables and constraints of the model are  $O(n^6)$  and  $O(n^5)$  respectively (Theorem 19), the regressions appear to suggest actual (“practical”) size orders of  $O(n^4)$  and  $O(n^3)$ , respectively. These lower “practical” numbers are likely due to the many *implicitly-zero* variables in the model. We note that the fact that the number of constraints grows more slowly than the number of variables suggests that there may be an advantage to focusing on the primal problem in efforts aimed at developing streamlined simplex procedures for solving the model.

In solving our test problems, we used the “barrier method with no crossover” implementation of CPLEX 12.8, on a Dell OptiPlex 7050 MT computer with an Intel i7-7700 (3.6 GHz) processor and 64 GB of RAM. The correctness of the LP optima were verified using common/traditional integer programming TSP and QAP formulations and solving these using the branch-and-bound/cut procedures of CPLEX 12.8. The randomly-generated problems were for the TSP only. They were based on symmetric Euclidean distances. The cities were generated on a 100 by 100 grid and the Euclidean distances between them were modified by factors between 80%-120% and rounded to be integers. The largest problems we could solve under 60 hours of CPU time were 14-city problems. The results for these are summarized in Table 2. Each of the times shown in this table is the average of five (5) problems. Similar results were obtained using exact Euclidean distances, using uniform distributions for distances, and choices of symmetric/asymmetric/integer or non-integer values.

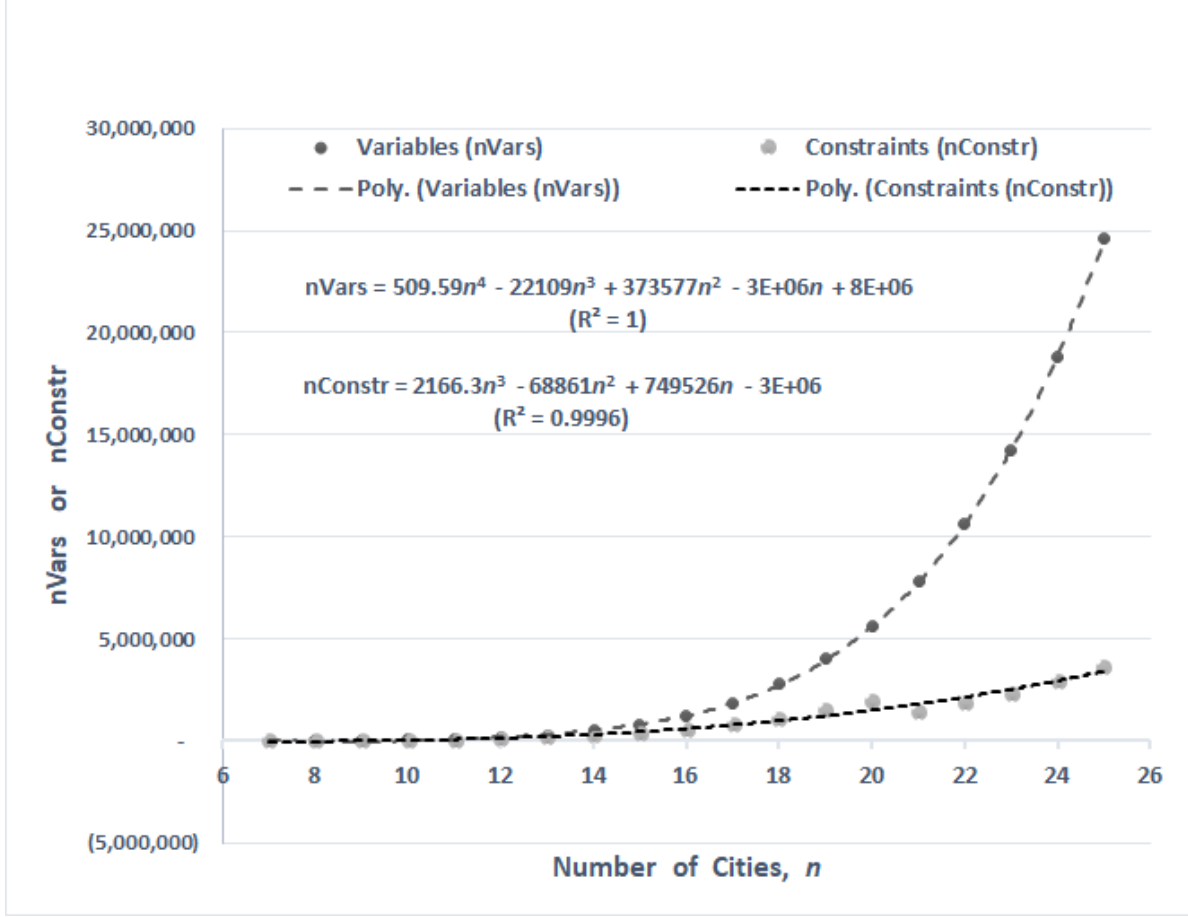


Figure 7: Number of Variables and Constraints vs. Number of TSP Cities

Number of Cities	CPU Time	
	Seconds	Minutes
7	1.13	0.02
8	8.75	0.15
9	37.31	0.62
10	202.91	3.38
11	963.63	16.06
12	5,123.64	85.39
13	22,289.41	371.49
14	66,890.66	1,114.84

Table 2: Computational Times vs. Number of TSP Cities

The regression we performed on the computational times in Table 2 are summarized in Figure 8. The fact that these times can be “well-fitted” by a polynomial function (of the number of cities) is consistent with the fact that both the size of our LP model and the

complexity of the the solution method we used are polynomial. The “practical” CPU time order which seems to be suggested by this regression is  $O(n^5)$ . We recall, as we indicated earlier in this section, that streamlined, large-scale-optimization, or efficient distributed-computing procedures for solving our proposed LP will need to be developed eventually, in order for the model to be useful in practice.

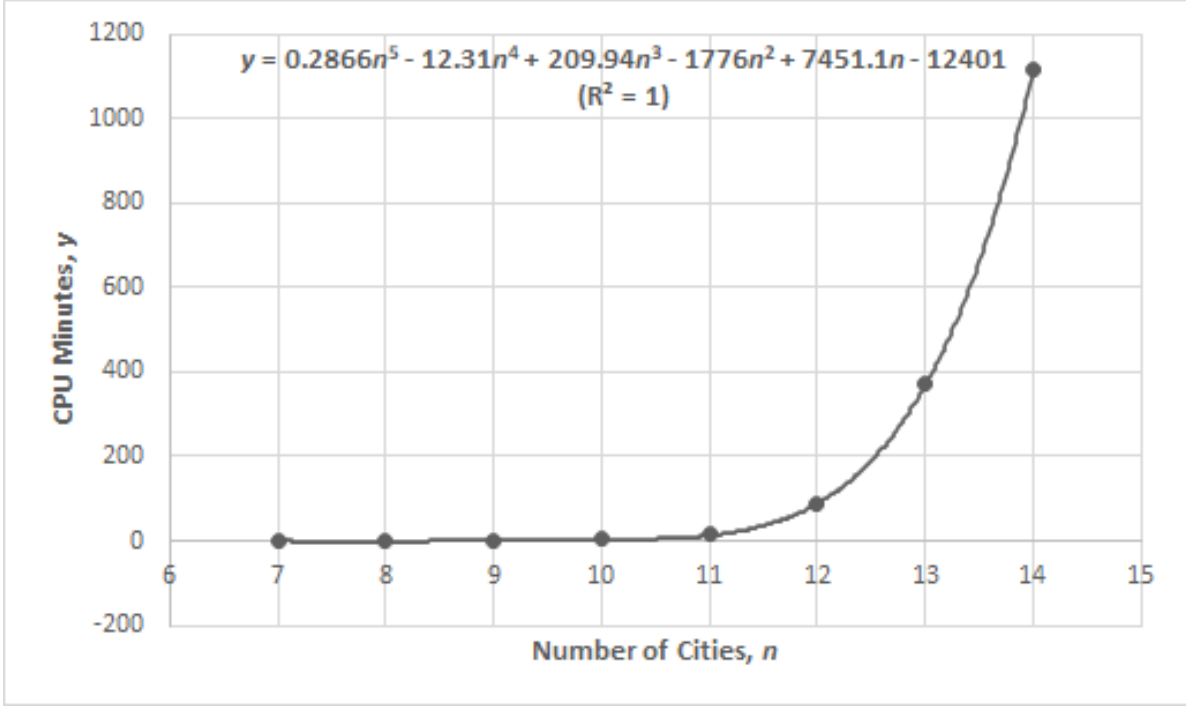


Figure 8: CPU Minutes vs. Number of TSP Cities

With respect to the “testbank” problems, we solved all of the TSP instances of the *SMAPO Library* (Reinelt (2010)). These involved 15,379 *10-city* problems, 192 *9-city* problems, 24 *8-city* problems, 6 *7-city* problems, and 4 *6-city* problems. We also solved the smallest (*12-department*) QAPs from the *QAPLIB Library* (Anjos (2018)), namely, Problems “Chr12a,” “Chr12b,” “Chr12c,” “Had12,” “Nug12,” “Rou12,” “Scr12,” “Tai112a,” and “Tai12b”. Our results for all of these (TSP and QAP) problems were similar to those for our randomly-generated problems, consistent with our expectations, based on our theoretical developments.

## 7 Conclusions

We have presented an *exact extended formulation* of the assignment problem polytope which solves the TSP and QAP and some of their generalizations as polynomial-sized linear programs (LPs). The model is an analog of the previous models developed by the first two

authors of this paper. However, it is much smaller and its proof is much simpler. Hence, we believe it represents a very significant improvement over those previous models. Our work complements our earlier affirmations resolving the important “ $P$  versus  $NP$ ” question. To paraphrase/quote from Diaby and Karwan (2016, pp. 5-7):

‘Our developments (and their incidental consequence of “ $P = NP$ ”) remove the exponential shift in complexity, but do not suggest a collapse of the “continuum of difficulty,” nor any change in the sequence along that continuum. In other words, our developments do not imply (or suggest) that all of the problems in the  $NP$  class have become equally “easy” to solve in practice. The suggestion is that, in theory, for  $NP$  problems, the “continuum of difficulty” actually ranges from low-degree-polynomial time complexity to increasingly-higher-degree-polynomial time complexities. However, from a theoretical perspective, we believe that these results make it necessary to reframe the computational complexity question away from: “Does there exist a polynomial algorithm for *Problem X*?” to (perhaps): “What is the smallest-dimensional space in which *Problem X* has a polynomial algorithm”’

In other words, since our work shows that every decidable problem which is solvable in polynomial time by a nondeterministic computer (i.e., every problem in the  $NP$  class) is tractable, the focus of Complexity Theory for class- $NP$  problems should be shifted to a new paradigm for “problem difficulty.” For example, Garey and Johnson (1979, p. 13) write:

‘As theoreticians continue to seek more powerful methods for proving problems intractable, parallel efforts focus on learning more about the ways in which various problems are interrelated with respect to their difficulty. As we suggested earlier, the discovery of such relationships between problems often can provide information useful for algorithm designers.’

Our suggestion is that, perhaps, the new paradigm could be a continuation or re-direction of current Complexity Theory in which classifications would not be independent of possible alternate encodings (or roughly, “modeling”) of a problem.

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# Appendix:

## Software Implementation

### • General Description and Interface

A software package, “TSP/QAP LP Solver,” has been developed to implement the model in this paper. The solver builds linear programming (LP) models for the traveling salesman and quadratic assignment problems and calls CPLEX 12.8 to solve them as LPs. The interface has been designed to run multiple replications of the chosen problem and run control settings at a time. With this tool, users can: (1) randomly generate or read a TSP or QAP input data in multiple ways; (2) directly solve the TSP or only build the LP models for them; (3) adjust CPLEX settings for different tests; (4) show solutions (optimal objective, variables, routes) in different formats. Standard integer programming (IP) models are incorporated and can be used for the purpose of verifying the correctness of the solutions obtained using our LP model. These are solved as IPs, and only their objective function values are displayed. A screenshot of the solver is shown in Figure 9 below.

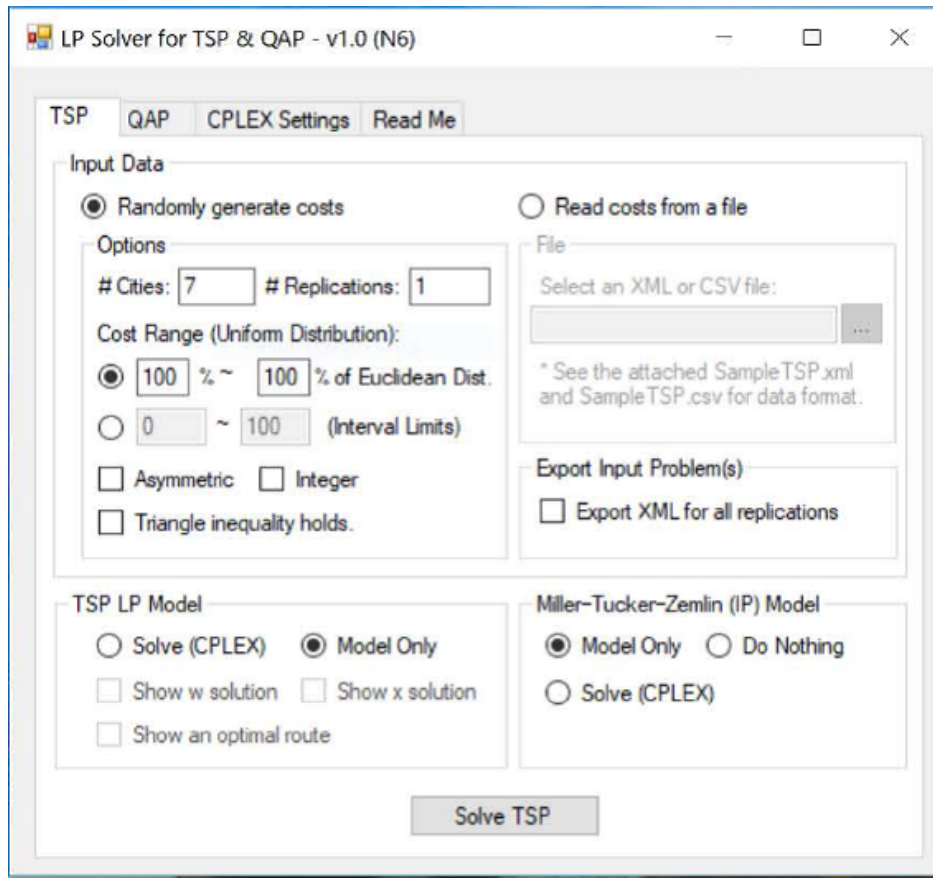


Figure 9: First Screen of the TSP/QAP LP Solver

- **Requirements**

TSP/QAP LP Solver is written in C# with .NET Framework 4 and calls CPLEX 12.8 to solve the TSP and QAP  $n^6$  LP models. For those with CPLEX 12.8, all functions in this program are available. Users who do not wish (or are not able) to use CPLEX 12.8 can choose the “Model Only” option in order to build .lp files for the LP or IP models, which they can then solve using the software of their choice. An advantage of solving the LP models endogeneously is that it gives the user an option to “parse” the LP solution in order to produce an optimal TSP tour or QAP assignment, thus removing the burden of having to interpret the optimal values of the modeling (the  $w$ - and  $x$ -) variables from the user. This is especially useful when the barrier method without crossover is used and it stops with a non-extreme-point, convex combination of alternate optimal solutions.

- **Data**

There are two ways to input data to the solver. They may be randomly generated or they may be read from files. For either way, users can check “Export all replications in XML format” to export input data to files in XML format for every replication. Randomly generating data supports the testing of multiple replications of a problem in a single run. Users input the number of cities (“# of Cities”) for the TSP or the number of departments (“# Depts/Sites”) for the QAP, and the number of replications (“# of Replications”) desired. For the TSP, cost values are generated based on either Euclidean distances or uniformly distributed random numbers. If the Euclidean distance option is chosen, the program will first randomly generate coordinates within a  $(0, 100) \times (0, 100)$  square plane, and then randomly generate costs within the given percentage range of Euclidean distances. If (absolute) interval limits is chosen, the program will randomly generate costs within the given range, not based on Euclidean distances. Other options include whether the cost matrix is asymmetric or not (checked or unchecked), whether the cost matrix is integer or not (checked or unchecked), and whether the triangle inequality holds or is not required (checked or unchecked). For the QAP, all the inter-departmental flows, inter-site distances, and fixed location assignment costs are generated from uniform distributions over the intervals specified by the user. Reading input data files supports XML and CSV formats as input file formats for the TSP, and CSV format only for the QAP. The required data format can be found in the included Sample.xml and Sample.csv. The XML data format for the TSP follows that of the classic TSPLIB.

- **Modelers and Solvers**

- **Modeler Settings** For the TSP, if the “Model Only” button is chosen, the program will build an .lp file without the requirement to use CPLEX. If the Miller-Tucker-Zemlin (MTZ) model is chosen to be solved, the program will call CPLEX to build and solve the MTZ IP model and display the optimal objective value for reference. If the  $n^6$  TSP LP model is chosen to be solved, the program will call CPLEX to build and solve the model and display the solution time, optimal

objective value and other solution information depending on which among the “Show  $w$  solution,” “Show  $x$  solution,” and “Show an optimal route” options are chosen. Similarly, for the QAP, if the “Model Only” button is chosen, the program will build an .lp file without the requirement to use CPLEX. If the standard QAP IP model is chosen to be solved, the program will call CPLEX to build and solve the IP model for QAP and display the optimal objective value for reference. If the QAP LP model is chosen to be solved, the program will call CPLEX to build and solve the model and display the solution time, optimal objective value and other solution information depending on which are chosen among the “Show  $w$  solution,” “Show  $x$  solution,” and “Show an optimal assignment” options.

- **CPLEX Settings** If users have the correct CPLEX version on their machines, they can adjust CPLEX parameters with this tool and solve the model with different algorithmic settings. For details of each adjustable parameter, please refer to a CPLEX Parameters Reference from IBM.
- **Results** All output files are located in the “Results/TSP” and “Results/QAP” subfolders of the folder containing the *TSP/QAP LP Solver* executable (“TSPsolvers.exe”), including the XML and CSV data files, .lp files, and solution text files. We note that the “parser” that is incorporated in the software is only heuristic. It is not, therefore, guaranteed to succeed in “retrieving” a TSP tour or QAP assignment when the barrier method without crossover stops with a non-extreme point solution. In such a case, the LP model will need to be re-solved either with primal simplex, dual simplex, or the barrier method with crossover to primal or dual simplex, if an extreme-point (integral) solution is desired. In our experimentation, the barrier method with dual crossover has consistently been the most efficient method for such cases.