ICS 2020 Problem Sheet #4

Problem 4.1:

a)

Show that $\leq$ is a partial order:

let $p \in \Sigma*$

$(p,p) \in \leq$

we conclude that $\leq$ is reflexive.

let $p,w \in \Sigma*$

$p \leq w$ and $p \leq w \Rightarrow pw \in \leq$ and $wp \in \leq$

$\Rightarrow p=w$

we conclude that $\leq$ is antisymmetric.

let $a,b,c \in \Sigma*$

$((a, b) \in \leq \wedge (b, c) \in \leq) \Rightarrow (a,c) \in \leq$

we conclude that $\leq$ is transitive.

Then $\leq$ is a partial order.

b)

Show that $<$ is a strict partial order:

let $p \in \Sigma*$

(p , p) in $w = pq : p = pq$

since p is a proper prefix :$(p , p) \notin <$

then $<$ is irreflexive.

let $p,w \in \Sigma*$

$(p, w) \in < \Rightarrow w=pq; p \neq w / q,p,w \in \Sigma*$

$(w, p) \in < \Rightarrow p=wq' ; p \neq w / q',p,w \in \Sigma*$

both expressions are not equivalent ;

we conclude that $(p, w) \in < \Rightarrow (w, p) \notin <$

then $<$ is asymmetric

let $a, b, c \in \Sigma*$

$((a, b) \in < \wedge (b, c) \in <) \Rightarrow (b=aq; a \neq b / q,a,b \in \Sigma*) \wedge (c=bq' ; b \neq c / q',c,b \in \Sigma*)$

$\Rightarrow$ a is a proper prefix and c is a word $(c \in \Sigma*)$

then $(a,c) \in <$

which means that $<$ is transitive

we conclude that $<$ is a strict partial order .

c)
$(a, b) \in \leq \lor (b, a) \in \leq$
$\leq$ is a partial order relation, it is therefore total.

since $<$ is asymmetric, it is not total (since : $\forall a, b \in \Sigma* \;/\; (a, b) \in < \Rightarrow (b, a) \notin <$)

Problem 4.2:

a)
we know that a function $f : X \to Y$ is called bijective if every element of the codomain Y is mapped to by exactly one element of the domain X.

with $x \in A$, $y \in C$
$g \circ f$ is bijective $\Leftrightarrow$ $g(f(x))$ is bijective $\Leftrightarrow$ $\forall x \in A, . \exists! y \in C : g(f(x))=y$

let's assume that g is not surjective :
g not surjective $\Leftrightarrow$ $\exists y \in C$: there is no f(x) in B that realize $g(f(x))=y$
which is wrong; We conclude that g is surjective

let $x,x' \in A$
$x=x' \Rightarrow g(f(x))=g(f(x'))$
$\Rightarrow f(x)=f(x')$(since g bijective)
then $\exists x,x' \in A : x=x' \Rightarrow f(x)=f(x')$
We conclude that g is surjective

b)
f injective $\Leftrightarrow$ $\forall x, y \in A \;/\; f(x) = f(y) \Rightarrow x = y$
g surjective $\Leftrightarrow$: $\forall y \in Y . \exists x \in X . f(x) = y$
for
f: A {a, b, c} $\to$ B {f(a), f(b),f(c), d}
g: B {f(a), f(b),f(c), d} $\to$ C {g(f(a)), g(f(b)), g(f(c)), g(d)}
In this example $g \circ f$ is not bijective because not C had 4 elements and A only had 3 which means $g \circ f$ is not bijective.

c)
$g(f(x))$ is bijective $\Leftrightarrow$ $\forall x \in A, . \exists! y \in C : g(f(x))=y$
for:
f not surjective $\Leftrightarrow$ $\exists b \in B$: there is no x in A that realize $f(x)=b$
and g not injective
f: A {a, b, c} $\to$ B {f(a), f(b),f(c), b}
g: B {f(a), f(b),f(c), b} $\to$ C {g(f(a)), g(f(b)), g(f(c))}
with $g(f(c))= g(b)$ (since g not injective)
since in this case $g \circ f$ mapps every element of A to C, it is then bijective.

We conclude that $g \circ f$ is bijective even though f is not surjective and g is not injective.

Problem 4.3: