Automata, Computability, and Complexity

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SHEET #9:

Exercise 1:

We assume that B would be countable. Then, there would exist

f(x)= representation of x in the numerical system of base 2

Each number of our decimal system has a unique representation in an arbitrary numerical system (of base distinct of 10). This fact ensure the function is injective.

Exercise 2:

Let S = { <M> | M is a TM that accepts w^r whenever it accepts w }.

Assume that T is decidable. Then some TM M decides T.
We construct TM M′ as follows:

M′ = "On input x:

1.If x ≠ 01 and x ≠ 10, reject.
2. If x = 01, accept.
3. If x = 10 simulate M on w.
4. If M accepts w, accept. If M halts and rejects, reject."

If ⟨M,w⟩ ∈ ATM then M accepts w and L(M′)={01,10}, so M′∈T.

On the other hand, if ⟨M,w⟩ ∉ ATM then L(M′)={01}, so M′∉T.
Therefore, ⟨M,w⟩ ∈ ATM iff M′∈ T, so ATM ≤m T.

Therefore ATM is decidable. Since ATM is undecidable, it must be the case that our assumption that T is decidable is false, so T is undecidable.

Exercise 3:
Prove that A_{DEC} is undecidable:

Assume that TM R is a DECIDER for A_{DEC}.

Construct TM S to decide A_{TM}, as follows:

"On input <M, w>, an encoding of a TM M and a string w:

1- Run TM R on input <M, w> to see if M halts on w

2- If R accepts (ie M halts on w) simulate M on w until it halts

If the simulation halts in M's Accept state, then S
. enters its Accept state

If the simulation halts in M's Reject state, then S .
. enters its Reject state

3- If R rejects (ie M does not halt on w), then S enters its
. Reject state

Thus TM S decides A_{TM}, but this is a contradiction since A_{TM} is NOT DECIDABLE.

Thus, R can not exist and A_{DEC} is undecidable

Exercise 4:
Show that A is Turing-recognizable if and only if A ≤m ATM.

We only need to show recognizable and not decidable.
Suppose A is Turing-recognizable. Take machine MA recognizing A.

The following computable function is trivial:

f(w) = <MA, w>

- If w ∈ A, then f(w) ∈ ATM
- If f(w) ∈ ATM, then w ∈ A

Thus A ≤m ATM.

Suppose now that $A \leq_m ATM$ (want to show that A is recognizable).
Take the function f computed by M_f where we know:

$$w \in A \Longleftrightarrow f(w) \in ATM$$

We know that ATM is recognizable, so can compute f(w) recognize if f(w) $\in$ ATM and therefore $w \in A$.