Automata, Computability, and Complexity

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SHEET #6:

Exercise 1 :

a)Prove that the language $L_a = \{a^{(2^n)} \mid n \in N\}$ is not context-free:
We assume L_a that is Context-free, , the Pumping Lemma guarantees the existence of a pumping length p such that if s is any string in La of length at least p, then s may be divided into fives pieces:
s = uvxyz
for
s = $a^{(2^3)}$ = aaaaaaaa
s= (a)(aaa)(a)(aa)(a)
s = uvxyz
following the pumping lemma
$uv^ixy^iz \in L_a$ ($\forall$ i $\in$ N)
for i = 2:
uv^2xy^2z=(a)(aaaaaa)(a)(aaaa)(a) $\notin L_a$

For s $\in L_a$, | s | must at least be even:
Case 1:
|uxz| = 2k ($\forall$ k $\in$ N) and contains only as we should therefore have that $|v^i y^i|$ = 2k in order for s $\in L_a$, but we have
$|v^i y^i|$ = i*(|v|+|y|) either:

|v| = 2k+1 and |y| = 2k+1 : $|v^i y^i|$ = i*(4k'+2) even, for
s = (a)(aaa)()(aaa)(a) if i=2: (a)(aaaaaa)()(aaaaaa)(a) $\notin L_a$

or |v| = 2k and |y| = 2k : $|v^i y^i|$ = i * (4*k') even, for
s = ()(aaaa)()(aaaa)() if i=3 ()(a^{12})()(a^{12})()$\notin L_a$

or |v| = 2k+1 and |y| = 2k : $|v^i y^i|$ = i * (4k'+1) = 4ik' + i is odd or even depending on i's value, if i odd, |s| odd => s $\notin L_a$
if i is even, the only s we can find that the only s $\in L_a$ with |s|<p is:
s = ()(a)()()() if i=3 ()(a^3)()()()$\notin L_a$

or $|v| = 2k$ and $|y|= 2k+1$: $|v^i y^i| = i * (4k'+1) = 4ik' + i$ is odd or even depending on i's value, if i odd, $|s|$ odd => $s \notin L_a$
if i is even, the only s we can find that the only $s \in L_a$ with $|s|<p$ is:
s = ()()()(a)() if i=3 $()()()(a^3)() \notin L_a$

Case 2:
$|uxz| = 2k+1$ ($\forall$ k $\in$ N) and contains only as we should therefore have that $|v^i y^i| = 2k+1$ in order for $s \in L_a$, but we have
$|v^i y^i| = i*(|v|+|y|)$ either:

$|v| = 2k+1$ and $|y| = 2k+1$: $|v^i y^i| = i*(4k'+2)$ even, therefore there is no s $\in L_a$(since s odd), with $|uxz| = 2k+1$ and $|v^i y^i| = i*(4k'+2)$

or $|v| = 2k$ and $|y| = 2k$: $|v^i y^i| = i * (4*k')$ even, therefore there is no s $\in L_a$(since s odd), with $|uxz| = 2k+1$ and $|v^i y^i| = i * (4*k')$

or $|v| = 2k+1$ and $|y| = 2k$: $|v^i y^i| = i * (4k'+1) = 4ik' + i$ is odd or even depending on i's value, if i even, $|s|$ odd => $s \notin L_a$
if i is odd, for:
s = (aaa)(aaaaa)()()() if i=2 $(aaa)(a^{10})()()() \notin L_a$

or $|v| = 2k$ and $|y|= 2k+1$: $|v^i y^i| = i * (4k'+1) = 4ik' + i$ is odd or even depending on i's value, if i even, $|s|$ odd => $s \notin L_a$
if i is odd, for:
s = (aaa)()()(aaaaa)() if i=2 $(aaa)()()(a^{10})() \notin L_a$
We conclude that for every case we discussed, there is always a contradiction with one of the rules of the pumping lemma for context-free languages.
Therefore, L_a is not a context-free language.

b) Prove that for every $k \geq 1$ the language L = $\{a^n b^{kn} c^n \mid n \in N\}$ is not context-free
We assume that L is Context-free, , the Pumping Lemma guarantees the existence of a pumping length p such that if s is any string in L ss of length at least p, then s may be divided into fives pieces:
s = uvxyz

Case 1:
v only has a’s and y only has c’s:
s=(a)(aa)(bbbbbb)(cc)(c) for i = 2:

(a)(aaaa)(bbbbbb)(cccc)(c) $\notin$ L

Case 2:
v and y only has b’s:
s=(aa)(b)(bbb)(bb)(cc) for i = 2:

(aa)(bb)(bbb)(bbbb)(cc) $\notin$ L

Case 3:
v only has a’s followed b’s and y only has b’s followed c’s :
s=(a)(ab)(bbb)(bbc)(c) for i = 2:

(a)(abab)(bbb)(bbcbbc)(c) $\notin$ L

Case 4 :
Any other combination of v and y will contradict one of the rules of the pumping lemma.

We conclude that for every case we discussed, there is always a contradiction with one of the rules of the pumping lemma for context-free languages.
Therefore, L is not a context-free language.

Exercice 2 :

Exercice 2: Sequence of configurations:

q_1 101 # 101	x x q_2 1 # x 0 1	x x x # q_5 x x 1
x q_3 0 1 # 101	x x 1 q_2 # x 0 1	x x x # x q_5 x 1
x 0 q_3 1 # 101	x x 1 # q_4 x 0 1	x x x # x q_6 x x
x 0 1 q_3 # 101	x x 1 # x q_4 0 1	x x x # q_6 x x x
x 0 1 # q_5 101	x x 1 # q_6 x x 1	x x q_6 # x x x
x 0 1 q_6 # x 0 1	x x 1 q_6 # x x 1	x x q_7 x # x x x
x 0 q_7 1 # x 0 1	x x q_7 1 # x x 1	x x x q_1 # x x x
x q_7 0 1 # x 0 1	x x q_1 1 # x x 1	x x x # q_8 x x x
x q_1 0 1 # x 0 1	x x x q_3 # x x 1	x x x # x q_8 x x
		x x x # x x q_8 x
		x x x # x x x q_8 ␣
		accept.

Exercice 3 :

We are looking for an algorithm of a TM that accepts only the words w^n ∈ Σ = {0, 1}*.

The language consists of all strings that have equal numbers of 0’s and 1’s.

Our TM should compare the number of 0’s and 1’s, if they are equal accept otherwise reject. An algorithm doing this task could be:

1) Cross the first occurrence of 0, starting from the leftmost element and going to right, if tape contains only one 0, reject.
2) go to leftmost element
3) skip 0’s and x’s going to the left and cross the first occurance of 1, if only one 1 is found, accept. If no 1 is found reject
4) go to leftmost element and go to 1)