The Paradise of Georg Cantor

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"No one will drive us from the paradise which Cantor created for us."
David Hilbert

Abstract.

This paper consists of two parts. the first part is a summary of the life of the mathematician Georg Cantor, including his early life, research interests, and mental health. The second part is a proof of Cantor's theorem. This is quite a self-contained paper, in that almost all terminology used in it are defined beforehand. After Cantor's theorem is proven, a couple of corollaries dependent on his theorem are established as well.

1 Cantor's Life

In this section we will present the life story of Georg Cantor based on the article "*The Nature of Infinity*" by JÃ¸rgen Veisdal [1].

Georg Cantor was born in 1845 in Saint Petersburg to Danish parents. He showed mathematical talent in his early teenage years. His father, who was a rich business man, provided him with education until 1863, when he died of tuberculosis. Cantor was doing his university studies at Hoheren Gewerbschule at the time. After his father's death, Cantor inherited half a million marks, and transferred to the University of Berlin.

Cantor's first research interest was number's theory, which he was inspired to pursue by his lecturers in Berlin, mainly Leopold Kronecker. After recieving his PhD. from Berlin, he worked as a private lecturer at Halle University.

Cantor's first paper titled *On a Theorem Concerning the Trignometric Series* was published in 1870. It was claimed that this paper hinted Cantor's deep interest in infinite sequences, the topic that he would go on to excel at. His second paper landed him a promotion to associate professor at Halle University in 1872.

Later in 1872, Cantor would meet a very important person, who he would update on his findings consistently. His name was Richard Dedekind, a professor of mathematics at the Technische Hochschule. Cantor had realized Dedekind's interest in the same area of research that he was interested in, and so, began a long correspondence between the two.

In one of the letters sent by Cantor to Dedekind in 1873, Cantor first mentioned the problem that he is most famous for, the nonexistence of a one-to-one relation between the natural numbers and the reals. Cantor mentioned it as a side problem that provoked his thoughts, after which Dedekind replied that it was of no practical importance and that he should move on. Cantor agreed at first, but the problem kept provoking his thoughts, and so he brought it up again many more times in later letters.

Later in 1873, the aforementioned problem became Cantor's primary area of research. He was able to formally prove it in 1874. Between 1874 and 1884, Cantor produced the work that is regarded as the origin of set theory.

In 1878, Cantor started working on what he called the continuum hypothesis (CH), in which he tried to prove that no infinite set had a number of elements strictly in between the number of elements of the set of all natural numbers and the set of reals. Proving the continuum hypotheses was the toughest challenge in Cantor's academic career. He even kept alternating on a daily basis between finding a proof of its truth and a proof of its falsehood.

Unfortunately, his obsession with proving the continuum hypothesis along with other factors were the cause of his mental health deterioration. His first mental breakdown was in the summer of 1884. Historians believe that one of the major reasons of Cantor's poor mental health at the time was the fierce opposition that his former mentor Leopold Kronecker posed towards Cantor's theory. Kronecker's opposition brought Cantor all kinds of distress not just because it was publicly announced, but also because there was a risk of Cantor's work not being published. Kronecker's opposition to Cantor was so intense that Kronecker called Cantor a "corrupter of youth" who "must be stopped."

In 1899, the tragic death of Cantor's youngest son induced another episode of mental illness that led him to return to the sanatorium, after which he lost his passion for mathematics.

In 1903, a paper was published in an attempt to disprove Cantor's theory, which offended Cantor deeply that it induced another spiral of mental illness for him and a loss of faith in God.

Cantor stayed in his position at Halle University till his death. He suffered from chronic depression in the last two decades of his life. During which he was occupied by defending the validity of his proofs. During World War I, he lived as a poor malnourished man until he went to the sanatorium for the last time in 1917. Sadly, he did not survive the last illness and died from a heart attack on January 6th 1918.

His continuum hypothesis would be disputed for a considerably long time after his death, until The Austro-Hungarian logician Kurt Godel established that the continuum hypothesis cannot be disproved from the other axioms in 1940.

2 Cantor's Theorem

We start by defining a set.

Definition 2.1. A *set* is a collection of objects.

One numeric example of sets is the set {1 , 2, 3}. Set elements are not necessarily numeric. For example, {London, Paris, Berlin} is an example of a set of Capitals, {square, circle, triangle} is an example of a set of geometrical shapes.

Getting back to numerical sets, another example is the set of all natural numbers, denoted by $\mathbb{N}$, which are {0 , 1 , 2, 3,....}

The set of all integers $\mathbb{Z}$ can be listed as {0, 1, -1, 2, -2, 3, -3,.....}.

The set of rational numbers $\mathbb{Q}$ includes any number that can be expressed as a fraction such that the numerator and denominator are integers, and that the denominator is different from zero.

The set of reals $\mathbb{R}$ is the set of all numbers on the number line, including all the integers ,rational numbers, and numbers that can be represented as an infinite decimal expansion (e.g. π)

Another important set is the empty set, denoted as $\emptyset$ or {} . The empty set is a set that has no elements in it.

Definition 2.2. Let A and B be sets. We say that A is a *subset* of B, and write $A \subseteq B$ if all the elements in A are also in B

Example 2.3. The following are examples of subsets.

1. The set of natural numbers is a subset of the set of integers, i.e., $\mathbb{N} \subseteq \mathbb{Z}$.
2. The set of natural numbers is a subset of the set of the reals, i.e., $\mathbb{N} \subseteq \mathbb{R}$.
3. the set {x, y, z} is a subset of the alphabet
4. the empty set $\emptyset$ is a subset of the set {1, 2, 3}

We next introduce the notion of a power set.

Definition 2.4. The *power set* $\mathcal{P}(S)$ of a set S is the set containing all of the subsets of S .

Example 2.5. *1. if S is the set $\{$ 1, 2, 3$\}$, then $\mathcal{P}(S)$ is the set $\{\emptyset$, $\{$1$\}$, $\{$2$\}$, $\{$3$\}$, $\{$1, 2$\}$, $\{$1, 3$\}$, $\{$2, 3$\}$, $\{$1, 2, 3$\}\}$*

2. if S is the set $\{\emptyset$ $\}$, then $\mathcal{P}(S)$ is $\{\emptyset$, $\{\emptyset$ $\}\}$

3. if S is the set $\{$Cantor, Dedekind$\}$, then $\mathcal{P}(S)$ is the set $\{\emptyset$, $\{$Cantor$\}$, $\{$Dedekind$\}$, $\{$Cantor, Dedekind$\}\}$

Let us now introduce the concept of a function and some of its properties.

Definition 2.6. Let A and B be sets.

- A *function* $f : A \to B$ from A to B is a function if and only if every element in A has exactly one image in B
- A function $f : A \to B$ is called *injective* if and only if there are no two distinct elements in A that have the same image in B.
- A function $f : A \to B$ is called *surjective* if and only if each element in B has at least one preimage in A.
- A function $f : A \to B$ is called *bijective* if and only if there are no two distinct elements in A that have the same image in B, and each element in B has exactly one preimage in A. In other words, A function is bijective if and only if it is injective and surjective..

Below, we explore some necessary conditions for the composition of two functions to be bijective.

Theorem 2.7. *Assume $f : A \to B$ and $g : B \to A$ are functions such that $f \circ g = \mathrm{id}_B$. Then g is injective and f is surjective.*

Proof. We will pursue a proof by contradiction. So, we will assume that g is not injective. This means that there can be two distinct elements of the set B called b_1 and b_2 such that $g(b_1) = g(b_2)$. In other words, they have the same image in the set A. Let this image be called a. However, since $f \circ g$ is an identity function ,when we apply the composite function $f \circ g$, it should map the domain of f back to its original values before g was applied, And since g was not injective, then when f is applied, a will have two images $b_1$and b_2 , which contradicts the definition of a function. Since we assumed that f is a function, we arrive at a contradiction. Thus g must be injective.

To prove that f must be surjective, we will also use a proof by contradiction. Suppose that f is not surjective. This means that there are elements in the co-domain B that have no preimage. When we apply the composite function $f \circ g$, then we map every element in B to exactly one image in A since g is injective. Then every element in A is mapped back to its original value in B. This means that every element in B will have a preimage since they all had images in A. This contradicts our assumption that f is not surjective, in that there are no elements in B with no preimage. Thus f is surjective. ∎

Next, we explain how functions are used to compare the sizes of sets.

Definition 2.8. Let A and B be any sets (finite or infinite).

- We say that the cardinality of A is equal to the cardinality of B, and write $|A| = |B|$, if there exists a a bijective function between A and B
- We say that the cardinality of A is less than or equal to the cardinality of B, and write $|A| \leq |B|$, if there exists an injective function $f : A \to B$
- We say that the cardinality of A is strictly less than the cardinality of B, and write $|A| < |B|$, if there exists an injective function $f : A \to B$, and there does not exist a bijective function $g : A \to B$.
- A set S is *countably infinite* if there exists at least one bijection $f : \mathbb{N} \to S$.

Observe that a countably infinite set is an infinite set for which we can enumerate *all* of its elements in a sequence indexed by the natural numbers. Of course the set of natural numbers itself is a countably infinite set. We will show below another example of a countably infinite set.

Theorem 2.9. *The set* $\{\alpha, \beta\} \times \mathbb{N} \times \{w, z\}$ *is countably infinite.*

Proof. We know that if an infinite set can be listed in a sequence indexed by the natural numbers, then the set is countable. The above Cartesian product can be listed using the following sequence:

{0- (α,0,ω) 1- (α,0,z) 2- (β,0,ω) 3- (β,0,z) 4- (α,1,ω) 5- (α,1,z) 6- (β,1,ω) 7- (β,1,z)........}

Thus, the above Cartesian product is countably infinite

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We will now present the main theorem of the article, Cantor's Theorem, which states that it is impossible to have a bijection between any set and its power set.

Theorem 2.10 (Cantor's Theorem)**.** *Let X be any (finite or infinite) set. Then $|X| < |\mathcal{P}(X)|$.*

Proof. Let X be any arbitrary set. By definition, to show that $|X| < |\mathcal{P}(X)|$ we need to prove that $|X| \leq |\mathcal{P}(X)|$ and $|X| \neq |\mathcal{P}(X)|$. In other words, we need to construct an injective function from X to $\mathcal{P}(X)$, and we need to show that it is impossible to have a bijection from X to $\mathcal{P}(X)$.

First, we will show that $|X| \leq |\mathcal{P}(X)|$ by constructing an injective function $g : X \rightarrow \mathcal{P}(X)$.

The simplest injective function between a set and its corresponding powerset can be be described as follows:

For any element a in the arbitrary set A , f (a) is defined by the set containing only a i.e. {a}. In this way, and by the definition of a powerset, every element in A has only one image in the powerset. Thus, there exists an injective function between any set (finite or infinite) and its corresponding powerset. (1)

Second, we will prove that $|X| \neq |\mathcal{P}(X)|$, meaning that it is impossible to have a bijection from X to its powerset. For the sake of contradiction, assume that there is a bijection $h : X \rightarrow \mathcal{P}(X)$.

Recall that the existence of a bijection between two sets implies the existence of a surjection between them by the definition of bijection. Thus if we can prove that there is no surjection, it will follow that there is no bijection. To continue our proof by contradiction, let's assume that there is a surjection k : $X \rightarrow \mathcal{P}(X)$, where every element x in X is mapped to a certain subset of X called s(x)

The next part of the proof is adapted from the article "On Cantor's important proofs" by *W. Mueckenheim* [2]

If we consider X to be an infinite set , there might be an arbitrary element m in X that has an image in $\mathcal{P}(X)$ equal to s(m) such that $m \notin$ s(m). Let us call the likes of number m as non-generators. It follows from the definition of a power set that $\mathcal{P}(X)$ includes an element M such that M is the set of all non-generators in X.

Using our assumption that k is surjective, we deduce that M has a preimage, let it be called t, in X. At this point:

1- $t \in M$ or

2- $t \notin M$

if $t \in M$, then t $\in$ s(t), which contradicts our previously defined concept of non-generators

if $t \notin M$, then $t \notin$ s(t), which makes t a non- generator according to the definition of non-generators, and since we defined M as a set of all non-generators, then $t \in M$, which contradicts $t \notin M$.

Since both cases yield contradictions, whence our assumption of an existence of such surjection is false for infinite sets. It then follows that there are no bijections between a set and its powerset if the set is infinite (2). Whence, from (1) and (2), the cardinality of an infinite set is strictly less than the cardinality of its power set(3).

In the case of finite sets, we will use a proof by induction to show that for any finite set A , the cardinality of $\mathcal{P}(A)$ is strictly greater than the cardinality of A. We will do that by trying to prove that if $|A| =$ n, then $|\mathcal{P}(A)| = 2^n$. We will call this predicate Q(A)

First, we will establish a base case of $n = 0$. We know that if this is the case then A is the empty set $\emptyset$. Then, by example 2.5.2, we know that $\mathcal{P}(A) = \{\emptyset\ \}$, which makes its cardinality equal to 1. Thus, $|\mathcal{P}(A)| = 2^n = 2^0 = 1$, and the base case holds.

Next is the induction step: For any set A, if Q(A) holds, then Q(B) also holds such that $|B| = |A| + 1 = n + 1$ and $|\mathcal{P}(B)| = 2^{n+1}$.

Q(A) is our induction hypothesis.

Consider the set B' such that $B' = B - \{$x$\}$, where x is any arbitrary element in B. we can conclude that $|B'| = |B| - 1 = n + 1 - 1 = n$, consequently, $|\mathcal{P}(B')| = 2^n$. If we adjoin {x} to all the subsets in B', then upon counting $\mathcal{P}(B)$ we will have all the subsets in B' and all the subsets in B' with x adjoined to them, which makes the count $2^n + 2^n = 2 \times 2^n = 2^{n+1}$

Thus the induction step holds and for any finite set A of cardinality n , $|\mathcal{P}(A)| = 2^n$ (4).

From number theory, we know that if $n \in \mathbb{N}$, then $n < 2^n$. Using this fact and (4), $|X| < |\mathcal{P}(X)|$ for all finite sets(5).

This proof by induction was adapted from the article "The Power Set Has $2^n Elements." by Matheus P and others.$[3]

Therefore, from (3) and (5) we have shown that $|X| < |\mathcal{P}(X)|$ for any set. ■

Definition 2.11. A set is *uncountable* if its elements cannot be listed in a sequence indexed by the natural numbers, which means that it has a cardinality strictly greater than $\mathbb{N}$.

From Cantor's Theorem we can deduce the following consequences.

Corollary 2.12. *The power set of the natural numbers is uncountable.*

Proof. From Cantor's theorem, we know that the cardinality of any set (finite or infinite) is strictly less than the cardinality of its power set. Thus, the cardinality of the power set of $\mathbb{N}$ is strictly greater than the cardinality of $\mathbb{N}$.

From definition 2.11, since the cardinality of the power set of $\mathbb{N}$ is strictly greater than the cardinality of $\mathbb{N}$, then $\mathcal{P}(\mathbb{N})$ is uncountably infinite.

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The following is another consequence of Cantor's Theorem.

Corollary 2.13. *There are infinitely many infinite sets $A_0, A_1, A_2, A_3, \ldots$ such that for each $i \in \mathbb{N}$ we have that $|A_i| < |A_{i+1}|$. That is,*

$$|A_0| < |A_1| < |A_2| < |A_3| < \cdots$$

In other words, there is an infinite hierarchy of infinities.

Proof. Let the set A_0 be the set of natural numbers $\mathbb{N}$. Let any set A_i such that i ≥ 1, be defined as $\mathcal{P}(A_{i-1})$. This implies that $A_1 = \mathcal{P}(\mathbb{N})$, and $A_2 = \mathcal{P}(\mathcal{P}(\mathbb{N}))$, and so on.

Recall that Cantor's theorem states that every set has a cardinality strictly less than the cardinality of its power set. Since every element in the above sequence is the power set of the its predecessor, then by Cantor's theorem, every element has a strictly greater cardinality than its predecessor's (1). Considering (1) and the fact that $\mathbb{N}$ is infinite, then all sets in the above sequence are infinite. Since that we can always take a powerset without restriction, this means that the above sequence can be an infinite sequence, which proves that there are infinitely many infinite sets that have cardinalities greater than one another. In other words, there is an infinite hierarchy of infinities.

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References

[1] Jorgen Veisdal. *The Nature of Infinity - and Beyond*. Medium, 2018.

[2] W.Mueckenheim. On Cantor's important proofs.

[3] Lobo, Matheus P. "The Power Set Has $2^n Elements."OSF Preprints$