Introduction to Modal Logic

MACT4133: Formal and Mathematical Logic Project

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1 Introduction

A division of formal logic known as modal logic deals with the ideas of necessity and possibility. It is employed to support arguments for claims that hold in certain possible worlds but not others. In addition to the known propositional logic operators, the main operators in modal logic are the box $\Box$ and diamond $\Diamond$ operators, which are used to express necessity and possibility, respectively. We shall talk about the syntax and semantics of modal logic, which were first formalized by Saul Kripke -an authoritative figure in modal logic who eventually proved the completeness of one of its systems at the age of 18. We will also discuss proof trees (in systems K, M, and S5) and how to prove the validity and invalidity of formulas in modal logic.

2 Syntax of Modal Logic

As mentioned before, modal logic uses all the symbols in propositional logic as well as the box and diamond operators. The box operator $\Box$ followed by a well formed formula φ means that φ is necessary while $\diamond\varphi$ means that the formula is possible. In addition to the rules of inference in propositional logic, modal logic involves two rules of inference involving the two new operators.

Rule 1: $\Box\ A = \neg\Diamond\neg A$
This practically means that if something is necessary then its falsehood is not possible.
Rule 2: $\Diamond\ A = \neg\Box\neg A$
This practically means that if something is possible, then its falsehood is not necessary

We know that $\{\neg,\wedge\}$ are a complete set of operators in propositional logic. We can then add our new operators and say that $\{\neg,\wedge,\ \Box,\ \Diamond\}$ are a complete set of operators in modal logic. But then, since we know from rules one and two that we can express the new operators in terms of each other and the negation symbol, then the sets $\{\neg,\wedge,\Box\}$ and $\{\neg,\wedge,\Diamond\}$ are complete sets of operators in modal logic.

2.1 Definitions:

Definition 1- Impossibility: We say that φ is impossible iff $\neg\Diamond\varphi$ which is equivalent to saying that $\Box\neg\varphi$
Definition 2- Analyticity: We say that φ is Analytic iff $\Box\varphi \vee \Box\neg\varphi$
Definition 3- Contingency: We say that φ is contingent iff $\Diamond\varphi \wedge \Diamond\neg\varphi$

Figure 1 explains the relationship between Necessity, Possibility, Contingency, Analyticity, and Impossibility. The solid arrow resembles an implication. The dotted arrow resembles the implication of a disjunction. The solid line terminated with boxes on each end resembles a contradiction.

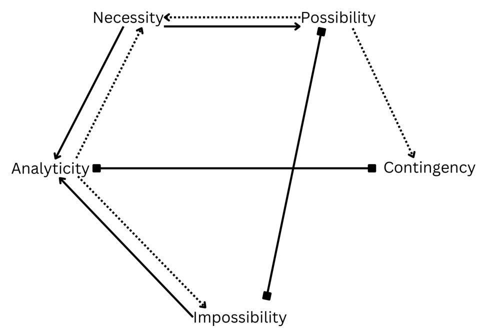


Figure 1: The relation between Necessity, Possibility, Contingency, Analyticity, and Impossibility

The reader should understand from this figure that:
necessity implies possibility and analyticity.
Impossibility implies analyticity.
Possibility implies necessity or contingency.
Analyticity implies necessity or impossibility.
Possibility and impossibility are contradicting.
Analyticity and contingency are contradicting.

2.2 Well Formed Formulae in Modal Logic WFF

We will use the same approach in defining WFF in modal logic as we used in propositiol logic.
1- All propositional variables are WFF
2- $\neg\varphi$ is a WFF iff φ is a WFF
3- $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$ are instances of WFF iff φ and ψ are WFF
4- $\Box\varphi$ and $\Diamond\varphi$ are WFF iff φ is a WFF
5- Nothing else is a WFF

3 Semantics and Validity in Modal Logic

We know how a formula in propositional logic can be true or false. We first define a truth assignment δ which assigns 0 or 1 to every propositional variable, which can then be uniquely extended to assign 0 or 1 to every other formula. The rules for extending the assignment depend on how we think and make inferences. There is little controversy for example that the formula $(p \wedge q)$ can be true (assigned a value of 1) if and only if p is true and q is true; it is false otherwise. This is just how we reason, and our process of deductive reasoning is clear to us. However, when it comes to reasoning about the possibility and necessity of propositions, matters become mush less clear. For example, if P is true, then must it be true that P is possibly necessary? That is, should our proof system, in order to be sound, prove a formula like $(p \rightarrow \Diamond\Box p)$? Moreover, is this implication necessary? Should p imply $\Box(p \rightarrow \Diamond\Box p)$ instead?

In the beginning of the 20th century, philosophers used axiomatic proof systems, supplying them with the most intuitive schemas of formulas, such as $(\Box\Box p \rightarrow p)$. However, it was still unclear whether formulas like these are obviously true and must be regarded as axioms, and for this reason modal logic and the associated concepts of necessity and possibility seemed vague and were heavily criticized, especially by the philosopher W. V. O. Quine, who argued for the abandonment of modal talk completely.

3.1 Possible World Semantics

The most useful way to think of the truth of modal proposition lies in the concept of Possible Worlds. To construct the concept intuitively we have to use our mental capability of imagining or hypothesizing the existence of worlds in which certain events has happened, didn't happen, couldn't have happened, etc. For example: World War II ended in 1945 is a true proposition in our world, however in another possible world, to say that WWII ended in 1946 is the true statement and not the previous one. The set of possible worlds is represented in the set W, which consists of $\{w_0, w_1, w_2, ...\}$.

To be more precise, it is not merely the ability to imagine a scenario or hypothesize regarding it which qualify a world to be a possible world. It is important to distinguish between logical and nomological possibilities when considering possible worlds. The realm of logical possibilities eliminates logical contradictions such as $\varphi \wedge \neg\varphi$, while the realm of nomological possibilities eliminates logical contradictions as well as events that are not logical in our world such as events that defy the laws of physics. The speed of light could be slightly different in a different possible world, but $\varphi \wedge \neg\varphi$ is not true in any possible world.

This, however, doesn't quite capture the full picture as well. How are worlds accessed either logically or nomologically? This will take us to the accessibility relation R which is what defines what worlds are accessible from which ones.

3.2 The Accessibility Relation R and the Truth Assignment Function a

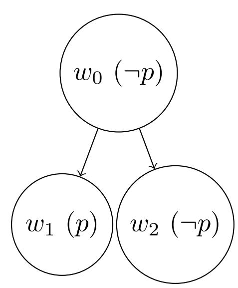
The accessibility relation R is a binary relation that operates on the set of possible worlds W which is $\{w_0, w_1, w_2, ...\}$, we say $w_0 R w_1$ if and only if w_1 is accessible from w_0. R is subject to the interpretation of a K-model (i.e. how System K interprets R) which assigns to it some properties (either reflexivity, transitivity, or symmetry) and assigns which world are related to each other.

Similar to δ in propositional logic, a is a function that maps all propositions in modal logic to either 1 or 0. The following two (most fundamental) axioms will illustrate how the concepts of possible worlds, accessibility relations, and truth assignments come together. They are as follows:
1.$\Diamond q$ holds in world w_0 belonging to W if and only if there exists a world w_i in W which is accessible to w_0 (i.e. $w_0 R w_i$ under K-interpretation of the accessibility relation R) within which $a_{w_i}(q)$=1 where $i = 1, 2, ..., n$.
2.$\Box q$ holds in world w_0 belonging to W if and only if for all worlds w_i in W which are accessible to w_0 (i.e. $w_0 R w_i$ under K-interpretation of the accessibility relation R) $a_{w_i}(q)$=1 where $i = 1, 2, ..., n$.

However, this is very basic and we will need to understand how arguments can be satisfiable or valid. For this reason one has to find a proof system to help arrive at the validity or invalidity of a modal logic argument. We will introduce, therefore, in this paper what is referred to as "Truth Trees for K," where K refers to the System K, which Saul Kripke came up with in his aforementioned work. The following section will discuss this in detail.

3.3 System K

We can describe System K to be a K-Model of $\langle W, R, a\rangle$, where it interprets the accessibility relation (it is irreflexive, non-symmetric, and non-transitive in System K) and the truth function in all the possible worlds in W. This following example will illustrate what is meant by what is discussed earlier. Let $W = \{w_0, w_1, w_2\}$, and let $R^k = \{w_0 R w_1, w_0 R w_2\}$. Let p=pigs can fly, and $a_{w_0}(p) =_k 0$, $a_{w_1}(p) =_k 1$, and $a_{w_2}(p) =_k 0$.



Now let's iterate the two most basic and intuitive understanding of modal operators and apply them in this example:
1.$\Diamond q$ holds in world w_0 belonging to W if and only if there exists a world w_i in W which is accessible to w_0 (i.e. $w_0 R w_i$ under K-interpretation of the accessibility relation R) within which $a_{w_i}(q)$=1 where $i = 1, 2, ..., n$.
2.$\Box q$ holds in world w_0 belonging to W if and only if for all worlds w_i in W which are accessible to w_0 (i.e. $w_0 R w_i$ under K-interpretation of the accessibility relation R) $a_{w_i}(q)$=1 where $i = 1, 2, ..., n$. Back to the example, what is the truth value of $\Diamond p$ and $\Box p$ in w_0? It is clear that $a_{w_0}(\Diamond p)$=1, because there exists a

world accessible to w_0 in which p is true. It is also clear that $a_{w_0}(\Box p)$=0, because there exists a world w_2 accessible to w_0 within which p doesn't hold. Suppose that $a_{w_2}(p)$=1, then (counter-intuitively) $a_{w_0}(\Box p)$=1, because in all worlds accessible to w_0 we find pigs flying (i.e. p is true) even though pigs do not fly in the actual world w_0. This weird conclusion is because w_0 is not accessible to itself, therefore different systems which include different definitions of R are needed and this will be discussed later.

3.4 Validity and Invalidity of arguments in System K

Formally speaking, an argument φ_1,φ_2,...,φ_n $\models_k \psi$, where φ_i and ψ are WFF and $i = 1, 2, ..., n$, is said to be K-valid if and only if for every K-model $\langle W, R, a\rangle$, and world w in W, $a_w(\varphi)$=1. To be specific, $\models_k A$ is the equivalent of saying that the statement A is K-valid (or K-logically true). On the other hand, this argument is said to have a K-counterexample if and only if there exists a K-model $\langle W, R, a\rangle$ and world w in W, for which $a_w(\varphi_1)$=1 and $a_w(\varphi_2)$=1 and ... and $a_w(\varphi_n)$=1 but $a_w(\psi)$=0. This means, if there exists a world w in the set W within which $a_w(\psi)$=0 and all the premises of the argument are true: $a_w(\varphi_i)$=1 under a certain K-interpretation of the accessibility relation R, then the argument is said to be invalid. This means that K-invalidity is arrived at only through finding what is called a K-counterexample.

How do we go about finding the counterexample or concluding that there doesn't exist any counterexample? For this end, we will use Truth Trees in System K. It is important to give a brief introduction to the concept of a Truth Tree in system K (from now on will be referred to as a K-Tree). A K-Tree is a tree-like structure that is used to represent the possible worlds and their accessibility relations in System K. Each node in a K-tree represents a possible world, and each edge represents an accessibility relation between two possible worlds.

First of all, we will need to present the branching rules which K-Trees share with the Truth Trees for propositional logic. Given that p and q are propositional variables, their Truth Tree rules are as follows:

(1) (p∧q)
(2) p
(3) q

(1) (p→q)
(2) ¬p q

(1) ¬(p∨q)
(2) ¬p
(3) ¬q

(1) (p∨q)
(2) p q

(1) ¬(p∧q)
(2) ¬p ¬q

(1) ¬(p→q)
(2) p ¬q

Now we include the definitive features of K-Trees, which are: world signifiers and tree formation rules dealing with the modal logic operators $\Box$ and $\Diamond$. World signifiers are the index or name of each world of which the formula is said, this

will be illustrated in the K-Tree diagrams later. The tree formation rules are concerning four cases, which are (for some WFF φ): $\Box\varphi$, $\Diamond\varphi$, $\neg\Box\varphi$, and $\neg\Diamond\varphi$.
1. In the case of $\Diamond\varphi$, the tree is allowed to open another world different from the world $\Diamond\varphi$ is true within such that φ will be true in this new world.
2. In the case of $\Box\varphi$, whenever a new world is opened, φ is included in this world only in so far is it is accessible from the world within which the formula $\Box\varphi$. So, if $\Box\varphi$ is valid in w_0, and $w_0 R w_i$, where $i = 1, 2, ..., n$, it must be the case that $a_{w_i}(\varphi)$=1.
3. In the case of $\neg\Box\varphi$, it can be rewritten as $\Diamond\neg\varphi$ which means that what applies in case 1 will apply here with one difference: instead of φ being the formula which will hold in the newly opened world, it is $\neg\varphi$ which will hold.
4. In the case of $\neg\Diamond\varphi$, it can be rewritten as $\Box\neg\varphi$ which means that what applies in case 2 will apply here with one difference: instead of φ being true of all other accessible worlds it will be $\neg\varphi$ which will hold in all other accessible worlds.

Now It is important to illustrate the steps of constructing a K-Tree. Given the following argument φ_1, φ_2, ... , $\varphi_n \models_k \psi$, assume we are trying to prove the validity of ψ. First, we start with w_0 and write all the formulas φ_i in the given argument and write the negation of ψ. Why do we do this? It is because we are trying to prove the validity of ψ by proving that its negation will lead to contradictions in all accessible possible worlds, thus proving the definition of validity which is that for all accessible possible worlds ψ is true. Secondly, we start constructing branches to the K-Tree based on the rules of branching mentioned above. Finally, for ψ to be valid we have to reach contradictions in all branches of the tree as mentioned. If one or more branches are still open (i.e. no contradiction is found), it means that the negation of ψ is true in an accessible possible world which is the K-counterexample we try to establish invalidity with. As a result, there would be a world w in W where $a_w(\psi)$=0 under the K-model, which is the definition of invalidity.

To reiterate for the sake of clarity, a K-Tree is constructed for the purpose of testing whether or not there exists a world w in W such that the formula we are trying to prove is false given the K-model. If not, then it is valid. Otherwise, it is invalid. The algorithm is as follows:

1. The root node represents the current possible world. It includes the premises of the argument and the negation of the conclusion.

2. Apply the rules of branching, whereby possibilities open new worlds and necessities fill in those worlds.

3. A world is closed if a contradiction is found, and a formula is only valid if all the worlds opened are closed by a contradiction.

4. If a world remains open, this means that there exists a world accessible to the initial world within which the opposite of the formula holds, which means that the formula itself doesn't hold there. Given the definition of

K-invalidity, the formula becomes K-invalid and is written as:
$\varphi_1, \varphi_2, \ldots, \varphi_n \nvDash_k \psi$ where the φ_i's are the premises of the argument and ψ is the formula we are trying to prove valid or invalid.

Now we need to give an example. Here is a fairly simple tree showing the K-validity of $\Box(p \to q), \Box p \models \Box q$:

$$
\begin{array}{lc}
\Box q & \\
(1) & \Box(p \to q) \;\;| w0 \\
(2) & \Box p \;\;| w0 \\
(3) & \neg\Box q \;\;| w0 \\
(4) & \neg q \;\;| w1 \\
(5) & p \to q \;\;| w1 \\
(6) & p \;\;| w1 \\
(7) & \neg p \;\;| w1 \qquad q \;\;| w1 \\
(8) & \bot \qquad\qquad\quad \bot
\end{array}
$$

Here is another more challenging tree showing the K-validity of $\Box(p \wedge q) \models (\Box p \wedge \Box q)$.

$$
\begin{array}{lcc}
(\Box p \wedge \Box q) & & \\
(1) & \multicolumn{2}{c}{\Box(p \wedge q) \;\;| w0} \\
(2) & \multicolumn{2}{c}{\neg(\Box p \wedge \Box q) \;\;| w0} \\
(3) & \neg\Box p \;\;| w0 & \neg\Box q \;\;| w0 \\
(4) & \neg p \;\;| w1 & \neg q \;\;| w2 \\
(5) & (p \wedge q) \;\;| w1 & (p \wedge q) \;\;| w2 \\
(6) & p \;\;| w1 & p \;\;| w2 \\
(7) & q \;\;| w1 & q \;\;| w2 \\
(8) & \bot & \bot
\end{array}
$$

Here is another example of a tree showing the invalidity (K-invalidity) of $\Diamond(p \to q), \Diamond p \models \Diamond q$:

$$
\begin{array}{lcc}
\Diamond q & & \\
(1) & \multicolumn{2}{c}{\Diamond(p \to q) \;\;| w0} \\
(2) & \multicolumn{2}{c}{\Diamond p \;\;| w0} \\
(3) & \multicolumn{2}{c}{\neg\Diamond q \;\;| w0} \\
(4) & \multicolumn{2}{c}{p \to q \;\;| w1} \\
(5) & \multicolumn{2}{c}{\neg q \;\;| w1} \\
(6) & \neg p \;\;| w1 & q \;\;| w1 \\
(7) & p \;\;| w2 & \bot \\
(8) & \neg q \;\;| w2 &
\end{array}
$$

4 A Strange Property of System K

System K fails to establish some of the intuitive conclusions that we make in modal logic. For example: while the modal proposition $\Box P \rightarrow P$ is intuitive to us, in that if a proposition is necessary, then it is true, system K fails to establish it as shown below:

$\Box p \rightarrow p$

(1)	$\neg(\Box p \rightarrow p) \mid w0$
(2)	$\Box p \mid w0$
(3)	$\neg p \mid w0$

This is as far as we can go because there are no possibilities to open new worlds. For us to be able to prove such a trivial proposition, we must develop a new proof system with a modified accessibility relation. The reader might have understood by now that the interpretation of the accessibility relation R in system K is irreflexive, asymmestric and not necessarily transitive. So, if we modify this relation so that it becomes reflexive, symmetric, and tansitive, we will be able to prove such trivial propositions.

5 System M

Let us start by only adding the reflexivity property to R, and let's call the new system that has this new property M.

Let us prove the proposition $\Box P \rightarrow P$ in our new system.

$\Box p \rightarrow p$

(1)	$\neg(\Box p \rightarrow p) \mid w0$
(2)	$\Box p \mid w0$
(3)	$\neg p \mid w0$
(4)	$p \mid w0$
(5)	$\perp$

Since R became reflexive in M, then w0Rw0, which means that any necessary proposition φ in w0 is true in w0, which is how we reach that p is true in w0 and hence we reach a contradiction that proves our proposition.

6 System B

If we add symmetry and reflexivity to system K we get system B. System B also helps us prove other trivial propositions such as $P \rightarrow \Box\Diamond P$, which essentialy means that if a proposition is true then its possibility is necessary. Below is its proof in system B:

$p \to \Box\Diamond p$

(1)	$\neg(p \to \Box\Diamond p) \mid w0$
(2)	$p \mid w0$
(3)	$\neg\Box\Diamond p \mid w0$
(4)	$\Diamond\neg\Diamond p \mid w0$
(5)	$\neg\Diamond p \mid w1$
(6)	$\Box\neg p \mid w1$
(7)	$\neg p \mid w0$
(8)	$\perp$

We reach step seven by establishing the symmetric relation w0Rw1, then w1Rw0. As we reached a contradiction in the root world (p and $\neg p$), then the whole branch is closed and the conclusion is valid.

7 System S4

If we modify the accessibility relation to add reflexivity and transitivity, we get system S4. As always, here is a trivial proposition that is unprovable in system K: $\Box p \to \Box\Box p$. And here is an S4 tree that utilizes reflexivity and transitivity of the accessibility relation to prove it:

$\Box p \to \Box\Box p$

(1)	$\neg(\Box p \to \Box\Box p) \mid w0$
(2)	$\Box p \mid w0$
(3)	$\neg\Box\Box p \mid w0$
(4)	$\Diamond\neg\Box p \mid w0$
(5)	$\neg\Box p \mid w1$
(6)	$\Diamond\neg p \mid w1$
(7)	$\neg p \mid w2$
(8)	$p \mid w2$
(9)	$\perp$

Since we opened w2 from w1, and w1 was opened from w0, then by transitivity, w2 is accessible from w0, which is why any necessary proposition in w0 like p is true in w2, which is where we get our contradiction.

8 System S5

If we modify the accessibility relation to add reflexivity, symmetry and transitivity, we get system S5, which is the most powerful system. It was powerful enough that Saul Kripke proved its completeness. As always, here is a trivial proposition that is unprovable in system K: $\Diamond p \to \Box\Diamond p$. And here is an S5 tree that utilizes the reflexivity, symmetry, and transitivity of the accessibility relation to prove it:

$\Diamond p \to \Box\Diamond p$

(1)	$\neg(\Diamond p \to \Box\Diamond p) \;\mid w0$
(2)	$\Diamond p \;\mid w0$
(3)	$\neg\Box\Diamond p \;\mid w0$
(4)	$\Diamond\neg\Diamond p \;\mid w0$
(5)	$p \;\mid w1$
(6)	$\neg\Diamond p \;\mid w2$
(7)	$\Box\neg p \;\mid w2$
(8)	$\neg p \;\mid w1$
(9)	$\perp$

Step by step solution: In steps 1, 2, and 3, we put the negation of the required to prove proposition in w0 followed by affirming the antecedent and denying the consequent. In step 4, we convert $\neg\Box\Diamond p$ to $\Diamond\neg\Diamond p$ by applying rule 1 and canceling the double negation. In steps 5 and 6, we open new worlds w1 and w2 using the possibility of p in w0 and the possibility of $\neg\Diamond p$ in w0 respectively. In step 7, we convert $\neg\Diamond p$ to $\Box\neg\Diamond p$ by applying rule 1 and canceling the double negation. Finally, in step 8 we use the symmetric property to say that w2Rw0 since w0Rw2, and then we use the transitivity property to say that w2Rw1 since w2Rw0 and w0Rw1. By establishing that w2Rw1, then any necessary proposition such as $\neg p$ in w2 will be true in w1, hence step 8 is valid, and it makes us arrive at a contradiction with step 5 showing that the argument is valid.

9 Conclusion

In conclusion, this paper has provided a comprehensive overview of modal logic, focusing on its syntax and semantics. We have explored the fundamental concepts of modal logic, including the box and diamond operators that express necessity and possibility, respectively. The paper has presented the syntax of modal logic formulas, emphasizing the use of well formed formulas and some complete sets of operators.

Furthermore, we delved into the semantics of modal logic, where possible worlds play a central role. We discussed the notion of possible worlds as complete descriptions of how things could be, and the accessibility relation that captures the relationships between these worlds. Additionally, the truth assignment function was introduced, enabling us to evaluate the truth value of formulas in different possible worlds.

To further illustrate the practical application of modal logic, this paper introduced proof trees in several modal logic systems, including K, M, B, S4, and S5. These proof trees provided a systematic and intuitive approach to proving the validity and invalidity of arguments within each system. By following the branches of the proof trees and considering the accessibility relation, we were able to derive conclusions based on given premises.

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