Introduction to Probability Theory

Sample Space and Events: Set of all possible outcomes of an experiment is known as the *sample space* of the experiment and is denoted by *S*. For some examples are given below:

1. If the experiment consists of the flipping of a coin, then $S = \{H, T\}$ where H means the outcome of the toss is a head and T means it is a tail.
2. If the experiment consists of rolling a die, then the sample space is
   $S = \{1,2,3,4,5,6\}$.
3. If the experiments consists of flipping two coins, then the sample space consists of the following four points:
   $S = \{(H,H),(H,T),(T,H),(T,T)\}$.
4. If the experiment consists of rolling two dice, then the sample space consists of the following 36 points:
   $$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$
   where the outcome (*i* , *j*) is said to occur if *i* appears on the first die and *j* on the second die.

Any subset *E* of the sample space *S* is known as an *event*. Some examples are given below:

1. If $E = \{2,4,6\}$ then E would be the event that an even number appears on the roll of a die.
2. If $E = \{(H,H),(H,T)\}$, then E is the event that a head appears on the first coin.
3. If $E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$, then E is the event that the sum of the dice equals seven.

The event $E \cup F$ is often referred to as the *union* of the event *E* and the event *F*. For example, if $E = \{1,3,5\}$ and $F = \{1,2,3\}$ then $E \cup F = \{1,2,3,5\}$.

For any two events *E* and *F*, we may also define the new event *EF*, referred to as the *intersection* of *E* and *F*. For example, if $E = \{1,3,5\}$ and $F = \{1,2,3\}$ then $E \cap F = \{1,3\}$. If $EF = \varnothing$, then *E* and *F* are said to be *mutually exclusive* events.

For any event *E* we define the new event E^c, referred to as the *complement* of *E*, to consist of all outcomes in the sample space *S* that are not in *E*. That is E^c will occur if and only if *E* does not occur. For example, if $E = \{1,3,5\}$, then $E^c = \{2,4,6\}$.

<u>Probabilities Defined on Events:</u> Consider an experiment whose sample space is *S*. For each event *E* of the sample space *S*, we assume that a number *P*(*E*) is defined and satisfied the following three conditions:

- $0 \le P(E) \le 1$.
- $P(S) = 1$.
- For any sequence of events $E_1, E_2, \cdots$ that are mutually exclusive, $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$.

We refer to *P*(*E*) as the probability of the event *E*.

Since the events *E* and E^c are always mutually exclusive and since $E \cup E^c = S$, we get $1 = P(S) = P(E \cup E^c) = P(E) + P(E^c)$.
For any two not mutually exclusive events *E* and *F*, $P(E \cup F) = P(E) + P(F) - P(EF)$.
And if *E* and *F* are mutually exclusive events then, $P(E \cup F) = P(E) + P(F)$.

<u>Example 1:</u> Suppose that we toss two coins and assume that each of the four outcomes in the sample space, $S = \{(H,H),(H,T),(T,H),(T,T)\}$ is equally likely and hence has probability ¼. Let $E = \{(H,H),(H,T)\}$ and $F = \{(H,H),(T,H)\}$. That is *E* is the event that the first coin falls heads and *F* is the event that the second coin falls heads. The probability that either the first or second coin falls heads is given by:

$$P(E \cup F) = P(E) + P(F) - P(EF) = \frac{1}{2} + \frac{1}{2} = 1 - P(\{H,H\}) = 1 - \frac{1}{4} = \frac{3}{4} .$$

We can also calculate the probability that any one of the three events *E* or *F* or *G* occurs. This is done as follows:

$$\begin{aligned} P(E \cup F \cup G) &= P((E \cup F) \cup G) \\ &= P(E \cup F) + P(G) - P((E \cup F)G) \\ &= P(E) + P(F) - P(EF) + P(G) - P(EG \cup FG) \\ &= P(E) + P(F) - P(EF) + P(G) - P(EG) - P(FG) + P(EGFG) \\ &= P(E) + P(F) + P(G) - P(EF) - P(EG) - P(FG) + P(EFG) \end{aligned}$$

In fact, it can be shown by induction that, for any *n* events $E_1, E_2, \cdots E_n$

$$\begin{aligned} &P(E_1 \cup E_2 \cup \cdots \cup E_n) \\ &= \sum_i P(E_i) - \sum_{i<j} P(E_i E_j) + \sum_{i<j<k} P(E_i E_j E_k) - \sum_{i<j<k<l} P(E_i E_j E_k E_l) + \cdots + (-1)^n P(E_1 E_2 \cdots E_n) \end{aligned}$$

<u>Conditional Probabilities:</u> Suppose that we toss two dice and that each of the 36 outcomes is equally likely to occur and hence has probability 1/36. If we let *E* and *F* denote respectively the event that the sum of the dice is six and the event that the first die is four, then the probability just obtained is called the conditional probability that *E* occurs given that *F* has occurred and is denoted by $P(E \mid F) = \dfrac{P(EF)}{P(F)}$.

Example 2: Suppose cards numbered one through ten are placed in a hat, mixed up and then one of the cards is drawn. If we are told that the number on the drawn card is at least five, then what is the conditional probability that it is ten?

Solution: Let E denote the event that the number of the drawn card is ten and let F be the event that it is at least five. The desire probability is $P(E|F)=\dfrac{P(EF)}{P(F)}=\dfrac{1/10}{6/10}=\dfrac{1}{6}$.

Example 3: Peter can either take a course in computer or in chemistry. If Peter takes the computer course, then he will receive an A grade with probability $1/2$, while if he takes the chemistry course then he will receive an A grade with probability $1/3$. Peter decides to base his decision on the flip of a fair coin. What is the probability that Peter will get an A in chemistry?

Solution: If we let F be the event that Peter takes chemistry and E denote the event that he receives an A in whatever course he takes, then the desired probability is
$P(EF)=P(F)P(E|F)=\dfrac{1}{2}\cdot\dfrac{1}{3}=\dfrac{1}{6}$.

Example 4: Suppose an urn contains seven black balls and five white balls. We draw two balls from the urn without replacement. Assuming that each ball in the urn is equally likely to be drawn, what is the probability that both balls are black?

Solution: Let F and E denote respectively the events that the first and second balls drawn are black. Now, given that the first ball selected is black, there are six remaining black balls and five white balls and so $P(E|F)=\dfrac{6}{11}$ and $P(F)=\dfrac{7}{12}$, our desired probability is
$P(EF)=P(F)P(E|F)=\dfrac{7}{12}\cdot\dfrac{6}{11}=\dfrac{7}{22}$.

Independent Events: Two events E and F are said to be *independent* if $P(EF)=P(E)P(F)$.

Two events E and F are not independent are said to be *dependent*.

The events $E_1,E_2,\cdots,E_n$ are said to be independent if for every subset $E_1,E_2,\cdots,E_r$, $r\le n$ of these events $P(E_1,E_2,\cdots,E_r)=P(E_1)P(E_2)\cdots P(E_r)$.

Bayes' Formula: Let E and F be events. We may express E as $E=EF\cup EF^c$ because in order for a point to be in E, it must either be in both E and F or it must be in E and not in F. Since EF and EF^c are obviously mutually exclusive, we have that
$P(E)=P(EF)+P(EF^c)=P(E|F)P(F)+P(E|F^c)P(F^c)=P(E|F)P(F)+P(E|F^c)(1-P(F))$

Example 5: Consider two urns. The first contains two white and seven black balls and the second contains five white and six black balls. We flip a fair coin and then draw a ball from the first urn or the second urn depending on whether the outcome was heads or tails. What is the conditional probability that the outcome of the toss was heads given that a white ball was selected?

Solution: Let W be the event that a white ball is drawn and let H be the event that the coin comes up heads. The desire probability $P(H|W)$ may be calculated as follows:

$$P(H|W)=\frac{P(HW)}{P(W)}=\frac{P(W|H)P(H)}{P(W)}=\frac{\left(\frac{2}{9}\right)\cdot\left(\frac{1}{2}\right)}{\left(\frac{2}{9}\right)\cdot\left(\frac{1}{2}\right)+\left(\frac{5}{11}\right)\cdot\left(\frac{1}{2}\right)}=\frac{\left(\frac{1}{9}\right)}{\left(\frac{1}{9}\right)\cdot\left(\frac{5}{22}\right)}=\frac{1}{9}\cdot\frac{9\cdot 22}{(22+45)}=\frac{22}{67}$$

Example 6: A laboratory blood test is 95 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

Solution: Let D be the event that the tested person has the disease and E be the event that his test result is positive. The desire probability $P(D|E)$ is obtained by

$$P(D|E)=\frac{P(DE)}{P(E)}=\frac{P(E|D)P(D)}{P(E|D)P(D)+P(E|D^c)P(D^c)}=\frac{(0.95)(0.005)}{(0.95)(0.005)+(0.01)(0.995)}=\frac{95}{294}\approx 0.323$$

Suppose that $F_1, F_2, \cdots, F_n$ are mutually exclusive events such that $\bigcup_{i=1}^{n} F_i = S$. In other words, exactly on e of the events $F_1, F_2, \cdots, F_n$ will occur. By writing $E=\bigcup_{i=1}^{n} EF_i$ and using the fact that the events $EF_i,\ i=1,2,\cdots,n$ are mutually exclusive we obtain that

$P(E)=\bigcup_{i=1}^{n} P(EF_i)=\sum_{i=1}^{n} P(E|F_i)P(F_i)$. Suppose, E has occurred and we are interested in determining which one of the F_j also occurred. Thus, we have

$P(F_j|E)=\dfrac{P(EF_j)}{P(E)}=\dfrac{P(E|F_j)P(F_j)}{\sum_{i=1}^{n} P(E|F_i)P(F_i)}$ is known as *Bayes' Formula*.

☺Good Luck☺

***** You MUST solve more examples, such as the examples 1.1 to 1.9, 1.12 to 1.14 from Textbook (Sheldon Ross). Otherwise you may Fail !!! *****

