Binomial Coefficients

<u>Basic Identities:</u> The symbol $\binom{n}{k}$ is a binomial coefficient, which we read as "n choose k". It is the number of ways to choose k-element subset from an n-element set. For example, from the set {1, 2, 3, 4}, we can choose two elements in six ways,

{1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4};

Thus, $\binom{4}{2} = 6$. We can use the following ways to express $\binom{n}{r}$,

$$\binom{n}{r} = {}^{n}C_{r} = \frac{n!}{(n-r)!r!} = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 1}{[r(r-1)\cdots 1].[(n-r)(n-r-1)\cdots 1]} = \frac{n(n-1)\cdots(n-r+1)}{r(r-1)\cdots 1} = \frac{n^{\underline{r}}}{r!}$$

For example, $\binom{4}{2} = \frac{4\cdot 3}{2\cdot 1} = 6$. We call n the *upper index* and r the *lower index*. The indices are restricted to be nonnegative integers by the combinatorial interpretation, because sets don't have negative or fractional numbers of elements. Thus,

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)\cdots(r-k+1)}{k(k-1)\cdots 1} = \dfrac{r^{\underline{k}}}{k!} & \text{, integer k } \geq 0 \\ 0 & \text{, integer k} < 0 \end{cases}$$

Now, we will see some small cases of binomial coefficients called *pascal's triangle*.

Table : Pascal's Triangle

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$	$\binom{n}{8}$	$\binom{n}{9}$	$\binom{n}{10}$
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10	5	1					
6	1	6	15	20	15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

The numbers in pascal's triangle satisfy some important identities.

a) *Symmetry Identity* : $\binom{n}{k} = \binom{n}{n-k}$, integer $n \geq 0$ and integer k.

R.H.S. = $\binom{n}{n-k} = \frac{n!}{(n-k)!(n-(n-k))!} = \frac{n!}{(n-k)!k!} = \binom{n}{k}$ = L.H.S. (Proved)

b) *Absorption Identity* : $\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$, integer $k \neq 0$

L.H.S. = $\binom{r}{k} = \frac{r!}{k!(r-k)!} = \frac{r}{k}\left(\frac{(r-1)!}{(k-1)!((r-1)-(k-1))!}\right) = \frac{r}{k}\binom{r-1}{k-1}$ = R.H.S. (Proved)

If we multiply both sides of absorption identity by k, then it works even for k = 0.

$$k\binom{r}{k} = r\binom{r-1}{k-1} \quad , \quad \text{integer } k.$$

This formula has a companion that keeps the lower index intact.

$$(r-k)\binom{r}{k} = r\binom{r-1}{k} \quad , \quad \text{integer } k.$$

L.H.S.= $(r-k)\binom{r}{k} = (r-k)\binom{r}{r-k}$; [Symmetry identity]

$= (r-k)\frac{r}{(r-k)}\binom{r-1}{r-k-1}$; [Absorption identity]

$= r\binom{r-1}{r-k-1} = r\binom{r-1}{(r-1)-(r-k-1)}$; [Symmetry identity]

$= r\binom{r-1}{k}$ = R.H.S. (Proved)

c) *Addition formula* : $\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$, integer k.

R.H.S.= $\binom{r-1}{k} + \binom{r-1}{k-1} = \frac{(r-1)!}{k!(r-1-k)!} + \frac{(r-1)!}{(k-1)!((r-1)-(k-1))!}$

$$= \frac{(r-1)!}{k!(r-k-1)!} + \frac{(r-1)!}{(k-1)!(r-k)!}$$

$$= \frac{(r-1)!}{(k-1)!(r-k-1)!}\left(\frac{1}{k} + \frac{1}{r-k}\right)$$

$$= \frac{(r-1)!}{(k-1)!(r-k-1)!}\left(\frac{r-k+k}{k(r-k)}\right)$$

$$= \frac{r(r-1)!}{k(k-1)!(r-k)(r-k-1)!} = \frac{r!}{k!(r-k)!} = \binom{r}{k} = \text{L.H.S.} \quad \text{(Proved)}$$

We can also get some new identities from addition formula. For example,

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$
$$= \binom{4}{3} + \binom{3}{2} + \binom{3}{1}$$
$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0}$$
$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \binom{1}{-1}$$

Since, $\binom{1}{-1} = 0$, that term disappears and we can stop.

This method yields the general formula,

$$\sum_{k \le n} \binom{r+k}{k} = \binom{r}{0} + \binom{r+1}{1} + \cdots + \binom{r+n}{n} = \binom{r+n+1}{n}, \qquad \text{integer } n.$$

If we unfold the recurrence in other way, then we get

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$
$$= \binom{3}{3} + \binom{3}{2} + \binom{4}{2}$$
$$= \binom{2}{3} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$
$$= \binom{1}{3} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$
$$= \binom{0}{3} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

Now, $\binom{0}{3}$ is zero (so are $\binom{0}{2}$ and $\binom{1}{2}$, but these make the identity nicer) and we can spot the general pattern:

$$\sum_{0 \le k \le n} \binom{k}{m} = \binom{0}{m} + \binom{1}{m} + \cdots + \binom{n}{m} = \binom{n+1}{m+1}, \text{ integer } m, n \ge 0$$

This identity which is call *summation on the upper index*, expresses a binomial coefficient as the sum of others whose lower indices are constant. This identity has an interesting combinatorial interpretation. If we want to choose (m + 1) tickets from a set of (n + 1) tickets numbered 0 through n, there are $\binom{k}{m}$ ways to do this when the largest ticket selected is number k.

We can prove last two identities by induction using addition formula, but we can also prove them from each other.

$$\text{L.H.S.} = \sum_{k \le n} \binom{m+k}{k} = \sum_{-m \le k \le n} \binom{m+k}{k} = \sum_{-m \le k \le n} \binom{m+k}{m+k-k} \quad ; \quad \text{[Symmetry identity]}$$

$$= \sum_{-m \le k \le n} \binom{m+k}{m}$$

$$= \sum_{0 \le k \le m+n} \binom{k}{m}$$

$$= \binom{m+n+1}{m+1} \quad ; \quad \text{[Summation on the upper index]}$$

$$= \binom{m+n+1}{(m+n+1)-(m+1)} \quad ; \quad \text{[Symmetry identity]}$$

$$= \binom{m+n+1}{n} = \text{R.H.S.} \quad \text{(Proved)}$$

Binomial coefficients get their name from the *binomial theorem*, which deals with powers of the binomial expression ($x + y$). Let's look at the smallest cases of this theorem:

$$(x+y)^0 = 1 \cdot x^0 y^0$$

$$(x+y)^1 = 1 \cdot x^1 y^0 + 1 \cdot x^0 y^1 = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1$$

$$(x+y)^2 = 1 \cdot x^2 y^0 + 2 \cdot x^1 y^1 + 1 \cdot x^0 y^2 = \binom{2}{0} x^2 y^0 + \binom{2}{1} x^1 y^1 + \binom{2}{2} x^0 y^2$$

$$(x+y)^3 = 1 \cdot x^3 y^0 + 3 \cdot x^2 y^1 + 3 \cdot x^1 y^2 + 1 \cdot x^0 y^3 = \binom{3}{0} x^3 y^0 + \binom{3}{1} x^2 y^1 + \binom{3}{2} x^1 y^2 + \binom{3}{3} x^0 y^3$$

$$(x+y)^4 = 1 \cdot x^4 y^0 + 4 \cdot x^3 y^1 + 6 \cdot x^2 y^2 + 4 \cdot x^1 y^3 + 1 \cdot x^0 y^4$$

$$= \binom{4}{0} x^4 y^0 + \binom{4}{1} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{3} x^1 y^3 + \binom{4}{4} x^0 y^4$$

In general case the theorem will be,

$$(x+y)^r = \binom{r}{0} x^r y^0 + \binom{r}{1} x^{r-1} y^1 + \cdots + \binom{r}{r-1} x^1 y^{r-1} + \binom{r}{r} x^0 y^r = \sum_k \binom{r}{k} x^{r-k} y^k , \qquad \text{integer } r \ge 0 \text{ or } |x/y| < 1$$

Two special cases of the binomial theorem are worth special attention, even though they are extremely simple. If $x = y = 1$ and $r = n$ is nonnegative, we get

$$2^n = (1+1)^n = \sum_{0 \le k \le n} \binom{n}{k} \cdot 1^{n-k} \cdot 1^k = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} , \qquad \text{integer } n \ge 0$$

This equation tells us that row n of Pascal's triangle sums 2^n.

And when x is -1 instead of +1, we get

$$0^n = (1-1)^n = \sum_{0 \le k \le n} \binom{n}{k} \cdot 1^{n-k} \cdot (-1)^k = \binom{n}{0} - \binom{n}{1} + \cdots + (-1)^{n-1}\binom{n}{n-1} + (-1)^n \binom{n}{n}, \quad \text{integer } n \ge 0$$

For example, $1-4+6-4+1=0$; the elements of row n sum to zero if we give them alternative signs, except in the top row (when n = 0 and $0^0 = 1$).

☺Good Luck☺