# Random Variables

Random Variables: A *random variable* does not have a single, fixed value; it can take on a set of possible different values, each with an associated probability. For example, in tossing dice we are often interested in the sum of the two dice and are not really concerned about the actual outcome i.e. we may be interested in knowing that the sum is seven and not be concerned over whether the actual outcome was (1,6) or (2,5) or (3,4) or (4,3) or (5,2) or (6,1). There are two types of random variables; *discrete* and *continuous*.

Example 1: Suppose that our experiment consists of tossing two fair coins. Letting Y denote the number of heads appearing, then Y is a random variable taking on one of the values 0, 1, 2 with respective probabilities

$$P\{Y=0\} = P\{(T,T)\} = \frac{1}{4}$$

$$P\{Y=1\} = P\{(T,H),(H,T)\} = \frac{2}{4}$$

$$P\{Y=2\} = P\{(H,H)\} = \frac{1}{4}$$

Discrete Random Variables: A random variable that can take on at most a countable number of possible values is said to be *discrete*. For a discrete random variable *X*, we define the *probability mass function* *P(a)* of *X* by $P(a) = P\{X=a\}$. The cumulative distribution function *F* can be expressed in terms of *P(a)* by $F(a) = \sum_{all\ x_i \le a} P(x_i)$.

For instance, suppose *X* has a probability mass function given by

$$P(1) = \frac{1}{2}, \quad P(2) = \frac{1}{3}, \quad P(3) = \frac{1}{6}$$

Then, the cumulative distribution function *F* of *X* is given by

$$F(a) = \begin{cases} 0, & a < 1 \\ \frac{1}{2}, & 1 \le a < 2 \\ \frac{5}{6}, & 2 \le a < 3 \\ 1, & 3 \le a \end{cases}$$

See Fig. 2.1 from the Book for Better Understanding!!!

*Bernoullli Random Variable:* Suppose that a trial or an experiment whose outcome can be classified as either a "success" or as a "failure" is performed. If we let *X* equal 1 if the outcome is a success and 0 if it is a failure, then the probability mass function of *X* is given by

$$p(0) = P\{X=0\} = 1-p$$

$$p(1) = P\{X=1\} = p$$

where $p,\ 0 \le p \le 1$, is the probability that the trial is a "success".
A random variable X is said to be a *Bernoulli random variable* if its probability mass function is given by the above equation for some $p \in (0,1)$.

<u>*Binomial Random Variable:*</u> Suppose that $n$ independent trials, each of which results in a "success" with probability $p$ and in a "failure" with probability $1-p$, are to be performed. If $X$ represents the number of successes that occur in the $n$ trials, then $X$ is said to be a *binomial random variable* with parameters $(n, p)$.
The probability mass function of a binomial random variable having parameters $(n, p)$ is given by

$$p(i) = \binom{n}{i} p^i (1-p)^{n-i}, \qquad i = 0, 1, \cdots, n$$

where, $\binom{n}{i} = \dfrac{n!}{(n-i)!\,i!}$

We can check the validity of binomial random variable by following way,

$$\sum_{i=0}^{\infty} p(i) = \sum_{i=0}^{\infty} \binom{n}{i} p^i (1-p)^{n-i} = [p + (1-p)]^n = 1$$

<u>Example 2:</u> Four fair coins are flipped. If the outcomes are assumed independent, what is the probability that two heads and two tails are obtained?

<u>Solution:</u> Let, X = Number of heads (success) that appear, then X is a binomial random variable with parameters, $n = 4$ and $p = 1/2$

$$P\{X = 2\} = \binom{4}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^2 = \frac{3}{8}$$

<u>Example 3:</u> It is known that all items produced by a certain machine will be defective with probability 0.1, independently of each other. What is the probability that in a sample of three items, at most one will be defective?

<u>Solution:</u> Let, X = Number of defective items in the sample, then X is a binomial random variable with parameters, $n = 3$ and $p = 0.1$

$$P\{X = 0\} + P\{X = 1\} = \binom{3}{0}(0.1)^0(0.9)^3 + \binom{3}{1}(0.1)^1(0.9)^2 = 0.972$$

<u>Example 4:</u> Suppose that an airplane engine will fail, when in flight, with probability $1-p$ independently from engine to engine; suppose that the airplane will make a successful flight if at least 50 percent of its engines remain operative. For what value of $p$ is a four-engine plane preferable to a two-engine plane?

Solution: Let, X = Number of engine remain operative, is binomial random variable. Hence the probability that a four-engine plane makes a successful flight is

$$P\{X \geq 2\} = P\{X=2\} + P\{X=3\} + P\{X=4\} = \binom{4}{2} p^2(1-p)^2 + \binom{4}{3} p^3(1-p)^1 + \binom{4}{4} p^4(1-p)^0$$
$$= 6p^2(1-p)^2 + 4p^3(1-p) + p^4$$

Where the corresponding probability for a two-engine plane is

$$P\{X \geq 1\} = P\{X=1\} + P\{X=2\} = \binom{2}{1} p^1(1-p)^1 + \binom{2}{2} p^2(1-p)^0 = 2p(1-p) + p^2$$

Hence the four-engine plane is safer if

$$6p^2(1-p)^2 + 4p^3(1-p) + p^4 \geq 2p(1-p) + p^2$$
$$\Rightarrow 6p(1-p)^2 + 4p^2(1-p) + p^3 \geq 2 - 2p + p$$
$$\Rightarrow 6p(1-2p+p^2) + 4p^2 - 4p^3 + p^3 \geq 2 - p$$
$$\Rightarrow 6p - 12p^2 + 6p^3 + 4p^2 - 3p^3 - 2 + p \geq 0$$
$$\Rightarrow 3p^3 - 8p^2 + 7p - 2 \geq 0$$
$$\Rightarrow 3p^3 - 6p^2 + 3p - 2p^2 + 4p - 2 \geq 0$$
$$\Rightarrow 3p(p^2 - 2p + 1) - 2(p^2 - 2p + 1) \geq 0$$
$$\Rightarrow (3p-2)(p^2 - 2p + 1) \geq 0$$
$$\Rightarrow (3p-2)(p-1)^2 \geq 0$$

$\therefore p \geq 2/3$ . Hence, the four-engine plane is safer when the engine success probability is at least as large as $2/3$, whereas the two engine plane is safer when this probability falls below $2/3$.

*Geometric Random Variable:* Suppose that we toss a coin having a probability *p* of coming up heads, until the first head appears. Letting *N* denote the number of flips required, then assuming that the outcome of successive flips are independent, *N* is said to be a *geometric random variable* with parameter p taking on one of the values 1, 2, 3, … , with respective probabilities

$$P\{N=1\} = P\{(H)\} = p$$
$$P\{N=2\} = P\{(T,H)\} = (1-p)p$$
$$P\{N=3\} = P\{(T,T,H)\} = (1-p)^2 p$$
$$\vdots$$
$$P\{N=n\} = P\{(\underbrace{T,T,\cdots,T}_{n-1},H)\} = (1-p)^{n-1} p$$

As a check, note that

$$P\left\{\bigcup_{n=1}^{\infty}\{N=n\}\right\}=\sum_{n=1}^{\infty}P\{N=n\}=p\sum_{n=1}^{\infty}(1-p)^{n-1}$$

$$=p[1-(1-p)]^{-1}\ ,\ \left[\because 1+x+x^2+\cdots+\infty=(1-x)^{-1}\right]$$

$$=p\cdot\frac{1}{p}=1$$

*Poisson Random Variable:* A random variable X, taking on one of the values 0, 1, 2, … , is said to be a *poison random variable* which parameter $\lambda$, if for some $\lambda>0$,

$$p(i)=P\{X=i\}=e^{-\lambda}\frac{\lambda^i}{i!}\ , \qquad i=0,1,\cdots$$

We can check the validity of poison random variable by following way,

$$\sum_{i=0}^{\infty}p(i)=\sum_{i=0}^{\infty}e^{-\lambda}\frac{\lambda^i}{i!}=e^{-\lambda}(1+\frac{\lambda^1}{1!}+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots)=e^{-\lambda}e^{\lambda}=1$$

Example 5: Suppose that the number of typographical errors on a single page of a book has a poison distribution with $\boxed{\text{parameter } \lambda=1}$. Calculate the probability that there is at least one error on a certain page. **Or, mean = 1**

Solution: Let, X = Number of typographical errors on a single page of a book.

$$P\{X\geq 1\}=1-P\{X=0\}=1-e^{-1}\frac{1^0}{1!}\approx 0.633\ \text{(Ans.)}$$

**For Poisson distribution, Mean = Lambda**
**For Exponential distribution, Mean = 1.0 /Lambda**
**Because, Mean of X = E [X], see Lec 15**

Example 6: If the number of accidents occurring on a highway each day is a poison random variable with $\boxed{\text{parameter } \lambda=3}$ what is the probability that no accident occur today? **Or, mean = 3**

Solution: Let, X = Number of accidents occurring on a highway each day.

$$P\{X=0\}=e^{-3}\frac{3^0}{0!}\approx 0.05\ \text{(Ans.)}$$

Example 7: Consider an experiment that consists of counting the number of $\alpha$ - particles given off in a one-second interval by one gram of radioactive material. If we know from past experience that, on the average, 3.2 such $\alpha$ - particles are given off, what is a good approximation to the probability that no more that 2 $\alpha$ - particles will appear?

Solution: If we think of the gram of radioactive material as consisting of a large number *n* of atoms each of which has probability 3.2/n of disintegrating and sending off an $\alpha$ - particle during the second considered, then we see that to a very close approximation the number of $\alpha$ - particles given off will be a poison random variable with parameter $\lambda=3.2$, Hence the desired probability is

$$P\{X\leq 2\}=\frac{(3.2)^0}{0!}e^{-3.2}+\frac{(3.2)^1}{1!}e^{-3.2}+\frac{(3.2)^2}{2!}e^{-3.2}\approx 0.382\ \text{(Ans.)}$$

☺Good Luck☺