

Recurrent Problems

In recurrence, solution to a problem depends on the solution to smaller instance of the same problem.

Tower of Hanoi: We are given a tower of eight disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer the entire tower to one of the other pegs, **moving only one disk at a time and never moving a large one onto a smaller.**

(These Two are Restrictions)

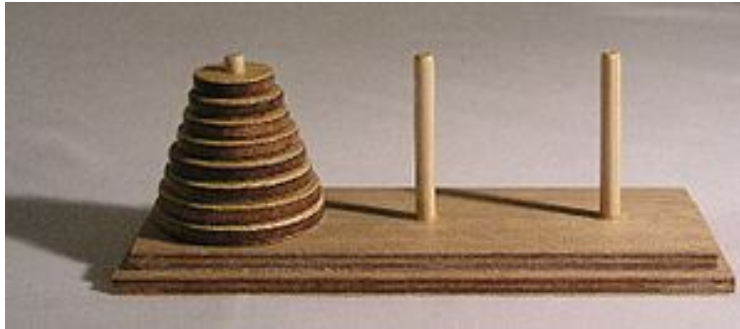


Fig: Tower of Hanoi with 8 disks

Consider small cases:

$T_0 = 0$, [0 disk requires 0 move]

$T_1 = 1$, [1 disk requires only 1 move]

$T_2 = 3$ [*Top disk from L to M; Bottom Disk from L to R; Finally, Top disk from M to R (see the Note below!)*]

$T_3 = 7$, [Transferring small 2 disks to middle peg requires 3 moves. Then moving the largest disk from left peg to right peg requires only 1 move. Finally, moving 2 smaller disks from middle peg to right peg requires another 3 moves. Thus, total $3+1+3=7$ moves require for moving 3 disks from one peg to another]

Similarly, for n disks to transfer from one peg to another we require:

$$T_n = T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1$$

Thus, the recurrence for Tower of Hanoi stands

$$T_0 = 0$$

$$T_n = 2T_{n-1} + 1, \quad \text{for } n > 0$$

We can find out the *closed form* of any recurrence to get quick result from the problem.

$$\begin{aligned} T_n &= 2T_{n-1} + 1 \\ &= 2(2T_{n-2} + 1) + 1 \\ &= 2^2T_{n-2} + 2 + 1 \\ &= 2^2(2T_{n-3} + 1) + 2 + 1 \\ &= 2^3T_{n-3} + 2^2 + 2 + 1 \\ &\vdots \\ &= 2^nT_{n-n} + 2^{(n-1)} + 2^{(n-2)} + \dots + 2^2 + 2 + 1 \\ &= 2^nT_0 + 2^{(n-1)} + 2^{(n-2)} + \dots + 2^2 + 2^1 + 2^0 \end{aligned}$$

$$\begin{aligned}
 T_n &= 2^0 + 2^1 + 2^2 + \cdots + 2^{(n-2)} + 2^{(n-1)} \\
 &= \frac{2^n - 1}{2 - 1} \quad \left[\because a^0 + a^1 + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1} \right] \\
 &= 2^n - 1
 \end{aligned}$$

$T_n = 2^n - 1$ is called the *closed form* solution of the “Tower of Hanoi” problem. *Mathematical induction* is a general way to prove any *closed form* solution. It has three parts.

- Basis – prove the formula for smallest possible value.
- Hypothesis – Consider that, the formula is true for first n values. *(or, first $n-1$ values)*
- Induction – Try to prove the formula for $(n+1)$ -th value. *(or, n -th value, if hypothesis true for $n-1$ values)*

For example, we are going to prove the *closed form* of Tower of Hanoi solution.

Basis: $T_0 = 2^0 - 1 = 1 - 1 = 0$ (OK, because We know T_0 is 0)

Hypothesis: Let, $T_n = 2^n - 1$

Induction: $T_{n+1} = 2T_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$ (Proved)

~~Try to prove $a^0 + a^1 + a^2 + \cdots + a^n = \frac{a^{n+1} - 1}{a - 1}$ using *Mathematical induction* method.~~ Excluded

*** Also, Solve DOUBLE TOH and TRIPLE TOH ***



☺ Good Luck ☺