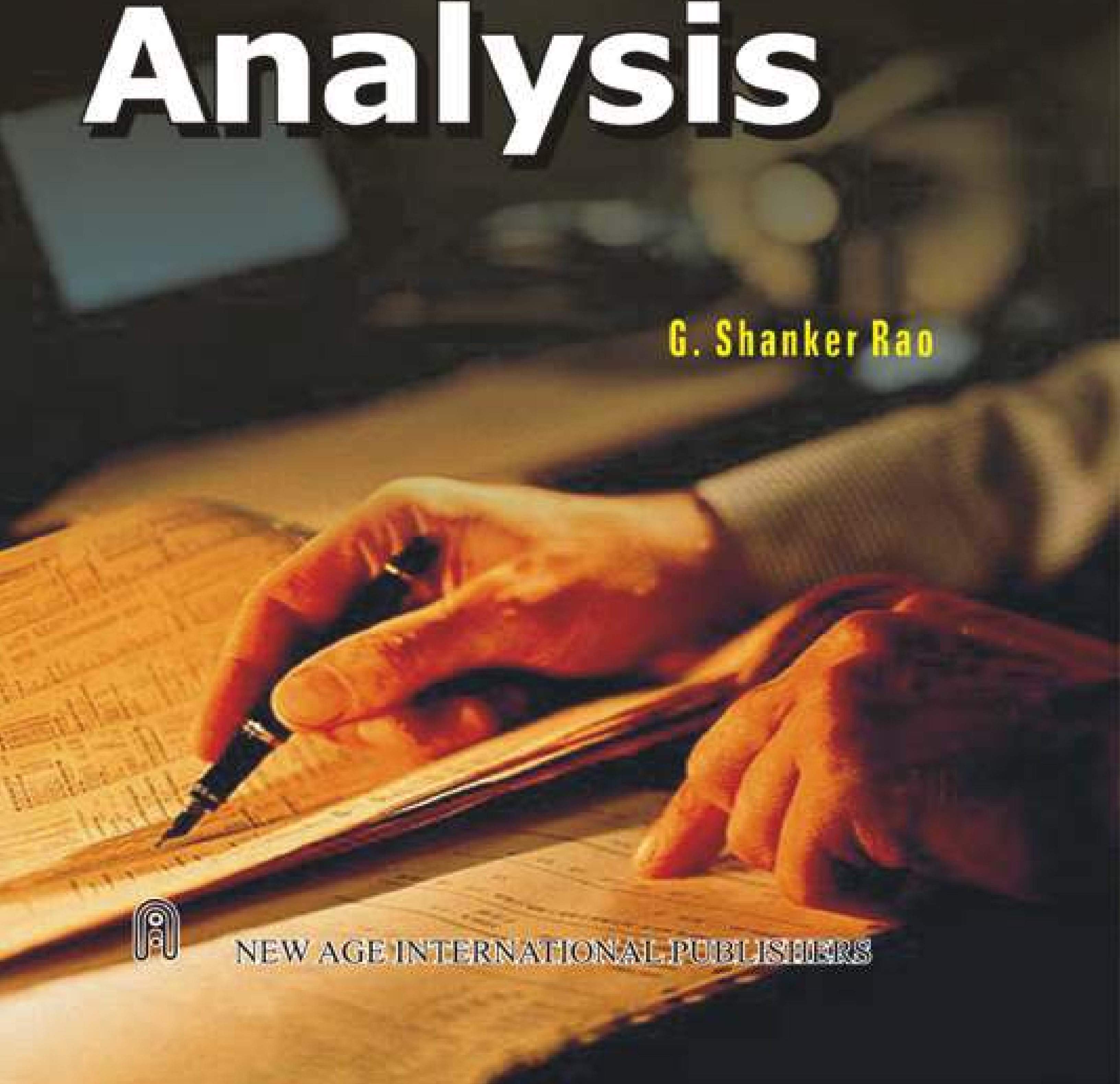
NEW AGE

REVISED THIRD EDITION

Numerical Analysis

G. Shanker Rao



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REVISED THIRD EDITION

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To

My father,
G. Narayana Rao

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PREFACE TO THE THIRD EDITION

This edition is a revision of the 2003 edition of the book. Considerable attention has been given here to improve the second edition. As far as possible efforts were made to keep the book free from typographic and others errors. Many changes have been made in this edition.

A chapter on Regression Analysis has been added in which Scalar diagrams, correlation, linear regression, multiple linear regression, curvilinear regression were briefly discussed. A large number of problems have been added in order to enable students develop better understanding of the theory. Most of these changes were made at the suggestion of individuals who had used my book and who were kind enough to send in their comments. One of the effects of these changes is to place greater emphasis on theory.

I wish to take this opportunity to thank all those who have used my book.

The author would like to express his appreciation to Shri Saumya Gupta, Managing Director, New Age International (P) Ltd., Publishers for the interest and cooperation he has taken in the production of this book.

Finally, I wish to express my sincere thanks to my Publishers, New Age International (P) Ltd., Publishers.

G. SHANKER RAO

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PREFACE TO THE FIRST EDITION

The present book on *Numerical Analysis* is intended to cover the syllabi of different Indian Universities in Mathematics. It meets the continued and persistent demand of the students for a book which could be followed easily.

This book is meant for the students appearing for B.Sc., M.Sc. and B.E. examinations of Indian Universities. The basic aim of this book is to give as far as possible, a systematic and modern presentation of the most important methods and techniques of Numerical Analysis. This book contains large number of solved problems followed by sets of well-graded problems.

I am much indebted to Shri A. Sree Ram Murthy and Shri S. Gangadhar whose inspiration and help had enabled me to write this book.

I am greatly thankful to Shri Govindan, Divisional Manager, New Age International. I am grateful to Smt Supriya Bhale Rao, Publisher, who advised me through all its stages, showing great patience at all times and whose efficient and painstaking help made it possible to bring out this book in a record time of three months.

Nizamabad, 1997

G. SHANKER RAO

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1

ERRORS

1.1 INTRODUCTION

There are two kinds of numbers—exact and approximate numbers.

An approximate number x is a number that differs, but slightly, from an exact number X and is used in place of the latter in calculations.

The numbers 1, 2, 3, …, $\frac{3}{4}, \frac{3}{5}$, …, etc., are all exact, and π, $\sqrt{2}$, e, …, etc., written in this manner are also exact.

1.41 is an approximate value of $\sqrt{2}$, and 1.414 is also an approximate value of $\sqrt{2}$. Similarly 3.14, 3.141, 3.14159, …, etc., are all approximate values of π.

1.2 SIGNIFICANT DIGITS

The digits that are used to express a number are called *significant digits*. Figure is synonymous with digit.

Definition 1 A significant digit of an approximate number is any non-zero digit in its decimal representation, or any zero lying between significant digits, or used as place holder to indicate a retained place.

The digits 1, 2, 3, 4, 5, 6, 7, 8, 9 are significant digits. '0' is also a significant figure except when it is used to fix the decimal point, or to fill the places of unknown or discarded digits.

For example, in the number 0.0005010, the first four '0's' are not significant digits, since they serve only to fix the position of the decimal point and indicate the place values of the other digits. The other two '0's' are significant.

Two notational conventions which make clear how many digits of a given number are significant are given below.

1. The significant figure in a number in positional notation consists of:
   (*a*) All non-zero digits and
   (*b*) Zero digits which

(*i*) lie between significant digits

(*ii*) lie to the right of decimal point, and at the same time to the right of a non-zero digit

(*iii*) are specifically indicated to be significant

2. The significant figure in a number written in scientific notation ($M \times 10^n$) consists of all the digits explicitly in M.

Significant figures are counted from left to right starting with the left most non zero digit.

Example 1.1

Number	*Significant figures*	*No. of Significant figures*
37.89	3, 7, 8, 9	4
5090	5, 0, 9	3
7.00	7, 0, 0	3
0.00082	8, 2	2
0.000620	6, 2, 0	3
5.2×10^4	5, 2	2
3.506×10	3, 5, 0, 6	4
8×10^{-3}	8	1

1.3 ROUNDING OFF NUMBERS

With a computer it is easy to input a vast number of data and perform an immense number of calculations. Sometimes it may be necessary to cut the numbers with large numbers of digits. This process of cutting the numbers is called *rounding off numbers*. In rounding off a number after a computation, the number is chosen which has the required number of significant figures and which is closest to the number to be rounded off. Usually numbers are rounded off according to the following rule.

Rounding-off rule In order to round-off a number to n significant digits drop all the digits to the right of the nth significant digit or replace them by '0's' if the '0's' are needed as place holders, and if this discarded digit is

1. Less than 5, leave the remaining digits unchanged
2. Greater than 5, add 1 to the last retained digit
3. Exactly 5 and there are non-zero digits among those discarded, add unity to the last retained digit

However, if the first discarded digit is exactly 5 and all the other discarded digits are '0's', the last retained digit is left unchanged if even and is increased by unity if odd.

In other words, if the discarded number is less than half a unit in the nth place, the nth digit is unaltered. But if the discarded number is greater than half a unit in the nth place, the nth digit is increased by unity.

And if the discarded number is exactly half a unit in the nth place, the even digit rule is applied.

Example 1.2

Number		*Rounded-off to*	
	Three figures	*Four figures*	*Five figures*
00.522341	00.522	00.5223	00.52234
93.2155	93.2	93.22	93.216
00.66666	00.667	00.6667	00.66667

Example 1.3

Number	*Rounded-off to Four significant figures*
9.6782	9.678
29.1568	29.16
8.24159	3.142
30.0567	30.06

1.4 ERRORS

One of the most important aspects of numerical analysis is the *error analysis*. Errors may occur at any stage of the process of solving a problem.

By the error we mean the difference between the true value and the approximate value.

∴ Error = True value – Approximate value.

1.4.1 Types of Errors

Usually we come across the following types of errors in numerical analysis.

(*i*) *Inherent Errors.* These are the errors involved in the statement of a problem. When a problem is first presented to the numerical analyst it may contain certain data or parameters. If the data or parameters are in some way determined by physical measurement, they will probably differ from the exact values. Errors inherent in the statement of the problem are called *inherent errors*.

(*ii*) *Analytic Errors.* These are the errors introduced due to transforming a physical or mathematical problem into a computational problem. Once a problem has been carefully stated, it is time to begin the analysis of the problem which involves certain simplifying assumptions. The functions involved in mathematical formulas are frequently specified in the form of infinite sequences or series. For example, consider

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

If we compute sin x by the formula

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!},$$

then it leads to an error. Similarly the transformation $e^x - x = 0$ into the equation

$$\left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!}\right) - x = 0,$$

involves an analytical error.

The magnitude of the error in the value of the function due to cutting (truncation) of its series is equal to the sum of all the discarded terms. It may be large and may even exceed the sum of the terms retained, thus making the calculated result meaningless.

(*iii*) *Round-off errors.* When depicting even rational numbers in decimal system or some other positional system, there may be an infinity of digits to the right of the decimal point, and it may not be possible for us to use an infinity of digits, in a computational problem. Therefore it is obvious that we can only use a finite number of digits in our computations. This is the source of the so-called *rounding errors*. Each of the FORTRAN Operations +, –, *, /, **, is subject to possible round-off error.

To denote the cumulative effect of round-off error in the computation of a solution to a given computational problem, we use the computational error and the computational error can be made arbitrarily small by carrying all the calculations to a sufficiently high degree of precision.

Definition 2 By the error of an approximate number we mean the difference between the exact number X, and the given approximate number x.

It is denoted by E (or by Δ)

$$E = \Delta = X - x.$$

Note An exact number may be regarded as an approximate number with error zero.

Definition 3 The absolute error of an approximate number x is the absolute value of the difference between the corresponding exact number X and the number x. It is denoted by E_A. Thus

$$E_A = |X - x|$$

Definition 4 The limiting error of an approximate number denoted by Δx is any number not less than the absolute error of that number.

Note From the definition we have

$$E_A = |X - x| \le \Delta x.$$

Therefore X lies within the range

$$x - \Delta x \le X \le x + \Delta x$$

Thus we can write $X = x \pm \Delta x$

Definition 5 The relative error of an approximate number x is the ratio of the absolute error of the number to the absolute value of the corresponding exact number X, where $(X \neq 0)$. It is denoted by E_R (or by δ)

$$E_R = \delta = \frac{E_A}{|X|}.$$

Definition 6 The limiting relative error δx of a given approximate number x, is any number not less than the relative error of that number.

From the definition it is clear that

$$E_R \le \delta x,$$

i.e.,
$$\frac{E_A}{|X|} \le \delta x,$$

$$\Rightarrow \; E_A \le |X|\, \delta x$$

In practical situations $X \approx x$. Therefore we may use $\Delta x = |x|\, \delta x$.

If Δx denotes the limiting absolute error of x then

$$E_R = \frac{E_A}{X} \le \frac{\Delta x}{x - \Delta x}, \text{ (where } x > 0,\ x > 0 \text{ and } \Delta x < x).$$

Thus we can write $\delta x = \dfrac{\Delta x}{x - \Delta x}$, for the limiting error of the number x.

Definition 7 The percentage error is 100 times the relative error. It is denoted by E_p.

$$\therefore \qquad E_p = E_R \times 100.$$

1.5 RELATIVE ERROR AND THE NUMBER OF CORRECT DIGITS

The relationship between the relative error of an approximate error and the number of correct digits:

Any positive number x can be represented as a terminating or non-terminating decimal as follows:

$$x = \alpha_m 10^m + \alpha_{m-1}10^{m-1} + \ldots + \alpha_{m-n+1}10^{m-n+1} + \ldots \tag{1}$$

where α_i are the digits of the number x, i.e., (α_i = 0, 1, 2, 3, …, 9) and $\alpha_m \ne 0$ (m is an integer).

For example: $1734.58 = 1.10^3 + 7.10^2 + 3.10^1 + 4.10^0 + 5.10^{-1} + 8.10^{-2} + \ldots$

Now we introduce the notation of correct digits of an approximate number.

Definition 8 If the absolute error of an approximate number does not exceed one half unit in the nth place, counting from left to right then we say that the first n significant digits of the approximate number are correct.

If x denotes an approximate number as represented by (1) taking the place of an exact number X, we can write

$$E_A = |X - x| \le \left(\frac{1}{2}\right)10^{m-n+1}$$

then by definition the first digits $\alpha_m, \alpha_{m-1}, \alpha_{m-2}, \ldots, \alpha_{m-n+1}$ of this number are correct.

For example if $X = 73.97$ and the number $x = 74.00$ is an approximation correct to three digits, since

$$\left|X - x\right| = 0.03 < \frac{1}{2}\,(10)^{1-3+1},$$

i.e.,
$$E_A = 0.03 < \frac{1}{2}\left(0.1\right).$$

Note 1. All the indicated significant digits in mathematical tables are correct.

2. Sometimes it may be convenient to say that the number x is the approximation to an exact number X to n correct digits. In the broad sense this means that the absolute error E_A does not exceed one unit in the nth significant digit of the approximate number.

Theorem *If a positive number x has n correct digits in the narrow sense, the relative error E_R of this number does not exceed $\left(\frac{1}{10}\right)^{n-1}$ divided by the first significant digit of the given number or $E_R \le \frac{1}{\alpha_m}\left(\frac{1}{10}\right)^{n-1}$, where α_m is first significant digit of number x.*

Proof Let

$$x = \alpha_m 10^m + \alpha_{m-1}10^{m-1} + \ldots + \alpha_{m-n+1}10^{m-n+1} + \ldots,$$

$(where\ \alpha_m \ge 1)$

denote an approximate value of the exact number X and let it be correct to n digits.

Then by definition we have

$$E_A = \left|X - x\right| \le \frac{1}{2}\left(10\right)^{m-n+1},$$

Therefore
$$X \le x - \frac{1}{2}(10)^{m-n+1}.$$

If x is replaced by a definitely smaller number $\alpha_m 10^m$ we get

$$X \ge \alpha_m 10^m - \frac{1}{2}10^{m-n+1},$$

$$\Rightarrow\ X \ge \frac{1}{2}10^m\left(2\alpha_m - \frac{1}{10^{n-1}}\right),$$

$$\therefore\ X \ge \frac{1}{2}(10)^m\left(2\alpha_m - 1\right).$$

Since

$$2\alpha_m - 1 = \alpha_m + (\alpha_m - 1) \geq \alpha_m$$

we get $\quad X \geq \frac{1}{2} 10^m \alpha_m.$

$$\therefore \; E_R = \frac{E_A}{X} \leq \frac{\frac{1}{2} 10^{m-n+1}}{\frac{1}{2} \alpha_m 10^m}.$$

$$\Rightarrow \; E_R \leq \frac{1}{\alpha_m} \left(\frac{1}{10}\right)^{n-1},$$

proving the theorem.

Corollary 1 The limiting relating error of the number x is $\delta x = \frac{1}{\alpha_m} \left(\frac{1}{10}\right)^{n-1}$, where δ_m is the significant digit of the number x.

Corollary 2 If the number x has more than two correct digits that is $n \geq 2$, then for all practical purpose the formula

$$E_R \; (= \delta R) = \frac{1}{2\alpha_m} \left(\frac{1}{10}\right)^{n-1} \quad \text{holds.}$$

1.5.1 Important Rules

Rule 1 If x is the approximate value of X correctly rounded to m decimal places then

$$|X - x| \leq \frac{1}{2} \times 10^{-m}$$

Rule 2 If x is the approximate value of X, after truncating to k digits, then

$$\left|\frac{X - x}{X}\right| < 10^{-k+1}$$

Rule 3 If x is the approximate value of X, after rounding-off to k digit, then

$$\left|\frac{X - x}{X}\right| < \frac{1}{2} \times 10^{-k+1}$$

Rule 4 If x is the approximate value of X correct to m significant digits, then

$$\left|\frac{X - x}{X}\right| < 10^{-m}$$

Rule 5 If a number is correct to n significant figures, and the first. Significant digit of the number is α_m, then the relative error $E_R < \dfrac{1}{\alpha_m 10^{n-1}}$.

Example 1.4 *How many digits are to be taken in computing $\sqrt{20}$ so that the error does not exceed 0.1%?*

Solution The first digit of $\sqrt{20}$ is 4.

$$\therefore \alpha_m = 4,\ E_R = 0.001$$

$$\therefore \frac{1}{\alpha_m}\left(\frac{1}{10}\right)^{n-1} = \frac{1}{4.10^{4-1}} \leq 0.001$$

$$\Rightarrow 10^{n-1} \geq 250$$

$$\therefore n \geq 4.$$

Example 1.5 *If $X = \dfrac{8}{9}$ and the exact decimal representation of X is 0.888 …, verify rule 1, numerically when X is rounded-off to three decimal digits.*

Solution We have $X = \dfrac{8}{9},\ k = 3$

The decimal representation of X rounded-off to three decimal digits is $x = 0.889$

Then

$$E_A = \left|\frac{8}{9} - 0.889\right| = \left|\frac{8}{9} - \frac{889}{1000}\right|$$

$$= \left|\frac{8000 - 8001}{9 \times 10^3}\right| = \left|\frac{-1}{9 \times 10^3}\right|$$

$$= \frac{1}{9} \times 10^{-3} < \frac{1}{2} \times 10^{-3}$$

$$\therefore E_A < \frac{1}{2} \times 10^{-3}$$

Hence, rule 1 is verified.

1.5.2 Tables for Determining the Limiting Relative Error from the Number of Correct Digits and *vice-versa*

It is easy to compute the limiting relative error of an approximate number when it is written with indicated correct digits. The table given below indicates the relative error as a percentage of the approximate number depending upon the number of correct digits (in the broad sense) and on the first two significant digits of the number, counting from left to right.

Relative error (in %) of numbers correct to *n* digits.

First two significant digits		*n*	
	2	*3*	*4*
10–11	10	1	0.1
12–13	8.3	0.83	0.083
14, …, 16	7.1	0.71	0.071
17, …, 19	5.9	0.59	0.059
20, …, 22	5	0.5	0.05
23, …, 25	4.3	0.43	0.043
26, …, 29	3.8	0.38	0.038
30, …, 34	3.3	0.33	0.033
35, …, 39	2.9	0.29	0.029
40, …, 44	2.5	0.25	0.025
45, …, 49	2.2	0.22	0.022
50, …, 59	2	0.2	0.02
60, …, 69	1.7	0.17	0.017
70, …, 79	1.4	0.14	0.14
80, …, 89	1.2	0.12	0.012
90, …, 99	1.1	0.11	0.011

The table below gives upper bounds for relative errors (in %) that ensure a given approximate value, a certain number of correct digits in the broad sense depending on its first two digits.

Number of correct digits of an approximate number depending on the limiting relative error (in %).

First two significant digits		*n*	
	2	*3*	*4*
10–11	4.2	0.42	0.042
12–13	3.6	0.36	0.036
14, …, 16	2.9	0.29	0.029
17, …, 19	2.5	0.25	0.025
20, …, 22	1.9	0.19	0.019
23, …, 25	1.9	0.19	0.019
26, …, 29	1.7	0.17	0.017
30, …, 34	1.4	0.14	0.014
35, …, 39	1.2	0.12	0.012
40, …, 44	1.1	0.11	0.011
45, …, 49	1	0.1	0.01
50, …, 54	0.9	0.09	0.009

Contd.

First two significant digits		*n*	
	2	*3*	*4*
55, …, 59	0.8	0.08	0.008
60, …, 69	0.7	0.07	0.007
70, …, 79	0.6	0.06	0.006
80, …, 99	0.5	0.05	0.005

1.6 GENERAL ERROR FORMULA

Let u be a function of several independent quantities $x_1, x_2, \ldots, x_n$ which are subject to errors of magnitudes $\Delta x_1, \ldots, \Delta x_n$ respectively. If Δu denotes the error in u then

$$u = f(x_1, x_2, \ldots, x_n)$$

$$u + \Delta u = f(x_1 + \Delta x_1, x_2 + \Delta x_2, \ldots, x_n + \Delta x_n).$$

Using Taylor's theorem for a function of several variables and expanding the right hand side we get

$$u + \Delta u = f(x_1, x_2, \ldots, x_n) + \Delta x_1 \frac{\partial f}{\partial x_1} + \Delta x_2 \frac{\partial f}{\partial x_2} + \ldots + \partial x_n \frac{\partial f}{\partial x_n} +$$

terms involving $(\Delta x_i)^2$, etc.,

$$u + \Delta u = u + \frac{\partial f}{\partial x_1} + \Delta x_1 \frac{\partial f}{\partial x_2} \Delta x_2 + \cdots + \frac{\partial f}{\partial x} \Delta x_n +$$

terms involving $(\Delta x_i)^2$, etc.

The errors $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, are very small quantities. Therefore, neglecting the squares and higher powers of Δx_i, we can write

$$\Delta u \approx \frac{\partial f}{\partial x_1}\Delta x_1 + \frac{\partial f}{\partial x_2}\Delta x_2 + \ldots + \frac{\partial x}{\partial x_n}\Delta x_n. \qquad (1)$$

The relative error in u is

$$E_R = \frac{\Delta u}{u} = \frac{1}{u}\left(\frac{\partial u}{\partial x_1}\Delta x_1 + \frac{\partial u}{\partial x_2}\Delta x_2 + \ldots + \frac{\partial u}{\partial x_n}\Delta x_n\right). \qquad (2)$$

$$\left(\because \frac{\partial f}{\partial x_i} = \frac{\partial u}{\partial x_i}\right)$$

$(i = 1, 2, \ldots, n)$

Formula (2) is called *general error formula*.

1.7 APPLICATION OF ERRORS TO THE FUNDAMENTAL OPERATIONS OF ARITHMETIC

(i) Addition

Let $$u = x_1 + x_2 + \ldots + x_n$$

then $$\Delta u = E_A = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n$$

Thus the absolute error of an algebraic sum of several approximate numbers does not exceed the sum of the absolute errors of the numbers.

Note To add numbers of different absolute accuracies the following rules are useful:

(*a*) Find the numbers with the least number of decimal places and leave them unchanged.

(*b*) Round-off the remaining numbers retaining one or two more decimal places than those with the smallest number of decimals.

(*c*) Add the numbers, taking into account all retained decimals.

(*d*) Round-off the result obtained by reducing it by one decimal.

(ii) Subtraction

Let $$u = x_1 - x_2$$

and $\Delta x_1, \Delta x_2$ denote the errors in x_1 and x_2 respectively then

$$\Delta u = \Delta x_1 - \Delta x_2 .$$

$\Delta x_1, \Delta x_2$ may be positive or negative therefore to obtain the maximum error we take

$$E_R \approx |\Delta x_1| + |\Delta x_2|$$

Note When the numbers are nearly equal most of the significant numbers from the left may disappear which may lead to serious types of errors. Therefore following ways are found useful to lessen the inaccuracy.

1. Each of the numbers may be approximated with sufficient accuracy before subtraction.
2. The given expression may transformed.

(iii) Multiplication

(*a*) A simple formula for the absolute error in a product of two numbers is given below.

Let $X = x_1 x_2$, and E_A denotes the absolute error in the product of the given numbers then

$$E_A = (x_1 + \Delta x_1)(x_2 + \Delta x_2) - x_1 x_2$$

$$= x_1 \Delta x_2 + x_2 \Delta x_1 + \Delta x_1 \cdot \Delta x_2 ,$$

$$\Rightarrow \ E_A = x_1 \Delta x_2 + x_2 \Delta x_1 \text{ (approximately).}$$

and the relative error in X is given by

$$E_R = \frac{E_A}{X} = \frac{\Delta X}{X} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2}$$

(*b*) The relative error in the product of n numbers is given below:

Let $X = x_1, x_2, \ldots, x_n$ and $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, denote the absolute errors in $x_1, x_2, \ldots, x_n$ respectively. Then the relative error in X is given by

$$E_R = \frac{\Delta X}{X} = \frac{\Delta x_1}{x_1} + \frac{\Delta x_2}{x_2} + \ldots + \frac{\Delta x_n}{x_n}$$

(iv) Division

For formula for the absolute error of a quotient can be found as shown under:

Let E_A denote the absolute error in $\dfrac{x_1}{x_2}$ the

$$E_A = \frac{x_1 + \Delta x_1}{x_2 + \Delta x_2} - \frac{x_1}{x_2}$$

$$= \frac{x_1 x_2 + x_2 . \Delta x_1 - x_1 x_2 - x_1 . \Delta x_2}{x_2\,(x_2 + \Delta x_2)}$$

$$= \frac{x_2 \Delta x_1 - x_1 \Delta x_2}{x_2\,(x_2 + \Delta x_2)} = \frac{x_1\left(\dfrac{x_2 \Delta x_1 - x_1 \Delta x_2}{x_1 x_2}\right)}{(x_2 + \Delta x_2)}$$

$$= \frac{x_1\left(\dfrac{\Delta x_1}{x_1} - \dfrac{\Delta x_2}{x_2}\right)}{x_2 + \Delta x_2} = \frac{\dfrac{x_1}{x_2}\left[\dfrac{\Delta x_1}{x_1} - \dfrac{\Delta x_2}{x_2}\right]}{\left(1 + \dfrac{\Delta x_2}{x_2}\right)}$$

$$= \frac{x_1}{x_2}\left[\frac{\Delta x_1}{x_1} - \frac{\Delta x_2}{x_2}\right], \text{ approximately.}$$

Example 1.6 *Round-off 27.8793 correct to four significant figures.*

Solution The number 27.8793 rounded-off to four significant figures is 27.88.

Example 1.7 *Round-off the number 0.00243468 to four significant figures.*

Solution The rounded-off number is 0.002435.

Example 1.8 *Find the sum of the approximate numbers 0.348, 0.1834, 345.4, 235.2, 11.75, 0.0849, 0.0214, 0.000354 each correct to the indicated significant digits.*

Solution 345.4 and 235.4 are numbers with the least accuracy whose absolute error may attain 0.1. Rounding the remaining numbers to 0.01 and adding we get

345.4 + 235.2 + 11.75 + 9.27 + 0.35 + 0.18 + 0.08 + 0.02 + 0.00 = 602.25.

Applying the even-digit rule for rounding the result we get the sum to be equal to 602.2.

∴ The sum of the given numbers = 602.2.

Example 1.9 *Find the number of significant figures in the approximate number 11.2461 given its absolute error as 0.25×10^{-2}.*

Solution Given that absolute error = 0.25×10^{-2} = 0.0025.

∴ The number of significant figure is 4.

Example 1.10 *Find the product 349.1 × 863.4 and state how many figures of the result are trust worthy, assuming that each number is correct to four decimals.*

Solution Let $x_1 = 349.1$. $|\Delta x_1| \le 0.05$

$x_2 = 863.4$, $|\Delta x_2| \le 0.05$

and $u = x_1\, x_2$

then $u = x_1 x_2 = 349.1 \times 863.4 = 301412.94$

now
$$\left|\frac{\Delta u}{u}\right| \le \left|\frac{\Delta x_1}{x_1}\right| + \left|\frac{\Delta x_2}{x_2}\right| \le \left|\frac{0.05}{x_1}\right| + \left|\frac{0.05}{x_2}\right|$$

$$\Rightarrow \qquad \frac{|\Delta u|}{|u|} \le (0.05)\left\{\frac{1}{|x_1|} + \frac{1}{|x_2|}\right\} = (0.05)\left[\frac{|x_1| + |x_2|}{|x_1|\,|x_2|}\right]$$

$$\Rightarrow \qquad |\Delta u| \le (0.05)\,|u|\left[\frac{|x_1| + |x_2|}{|u|}\right] = 0.05\left(|x_1| + |x_2|\right)$$

$$\Rightarrow \qquad |\Delta u| \le (0.05)\lfloor 349.1 + 863.47 \rfloor = 60.6285 \simeq 60.63$$

Therefore; the true value of u lies between

$301412.94 - 60.63$ and $301412.94 + 60.63$

i.e., 301352.31 and 301473.559

i.e., 3014×10^2 and 3015×10^2

We infer that; only the first three figures are reliable.

Example 1.11 *Find the difference* $\sqrt{2.01} - \sqrt{2}$ *to three correct digits.*

Solution We know that $\sqrt{2.01} = 1.41774469 \ldots$ and $\sqrt{2} = 1.41421356 \ldots$

Let X denote the difference

$$\therefore \quad X = \sqrt{2.01} - \sqrt{2}$$
$$= (1.41774469\ \ldots) - (1.41421356\ \ldots)$$
$$= 0.00353$$
$$= 3.53 \times 10^{-3}.$$

Example 1.12 *If* $\Delta x = 0.005$, $\Delta y = 0.001$ *be the absolute errors in* $x = 2.11$ *and* $y = 4.15$, *find the relative error in the computation of* $x + y$.

Solution $x = 2.11,\ y = 4.15$

$\therefore$ $x + y = 2.11 + 4.15 = 6.26,$

and $\Delta x = 0.005,\ \Delta y = 0.001$

$$\Rightarrow \quad \Delta x + \Delta y = 0.005 + 0.001 = 0.006.$$

$\therefore$ The relative error in $(x + y)$ is

$$E_R = \frac{\Delta x + \Delta y}{(x + y)} = \frac{0.006}{6.26}$$
$$= 0.000958.$$

The relative error in $(x + y) = 0.001$ (approximately).

Example 1.13 Given that $u = \dfrac{5xy^2}{z^3}$ Δx, Δy and Δz denote the errors in x, y and z respectively such that $x = y = z = 1$ and $\Delta x = \Delta y = \Delta z = 0.001$, find the relative maximum error in u.

Solution We have

$$\frac{\partial u}{\partial x} = \frac{5y^2}{z^3}, \; \frac{\partial u}{\partial y} = \frac{10xy}{z^3}, \; \frac{\partial u}{\partial z} = \frac{-15xy^2}{z^4}$$

$$\therefore \quad \Delta u = \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \frac{\partial u}{\partial z}\Delta z$$

$$\Rightarrow \quad (\Delta u)_{\max} = \left|\frac{\partial u}{\partial x}\Delta x\right| + \left|\frac{\partial u}{\partial y}\Delta y\right| + \left|\frac{\partial u}{\partial z}\Delta z\right|$$

$$= \left|\frac{5y^2}{z^3}\Delta x\right| + \left|\frac{10xy}{z^3}\Delta y\right| + \left|\frac{-15xy^2}{z^4}\Delta z\right| \tag{1}$$

Substituting the given values in (1) and using the formula to find the relative maximum error we get

$$(E_R)_{\max} = \frac{(\Delta u)_{\max}}{u} = \frac{0.03}{5} = 0.006.$$

Example 1.14 *If $X = 2.536$, find the absolute error and relative error when*

(i) X is rounded and

(ii) X is truncated to two decimal digits.

Solution

(*i*) Here $X = 2.536$

Rounded-off value of X is $x = 2.54$

The Absolute Error in X is

$E_A = |2.536 - 2.54|$

$= |-0.004| = 0.004$

Relative Error $= E_R = \dfrac{0.004}{2.536} = 0.0015772$

$= 1.5772 \times 10^{-3}$.

(*ii*) Truncated Value of X is $x = 2.53$

Absolute Error $E_A = |2.536 - 2.53| = |0.006| = 0.006$

$\therefore$ Relative Error $= E_R = \dfrac{E_A}{X} = \dfrac{0.006}{2.536} = 0.0023659$

$= 2.3659 \times 10^{-3}$.

Example 1.15 *If $\pi = \dfrac{22}{7}$ is approximated as 3.14, find the absolute error, relative error and relative percentage error.*

Solution Absolute Error $= E_A = \left|\dfrac{22}{7} - 3.14\right| = \left|\dfrac{22 - 21.98}{7}\right|$

$$= \left|\frac{0.02}{7}\right| = 0.002857.$$

$$\text{Relative Error } E_R = \left|\frac{0.002857}{22/7}\right| = 0.0009$$

$$\text{Relative Percentage Error } E_P = E_R \times 100 = 0.0009 \times 100$$

$$= 0.09$$

$$\therefore \quad E_P = 0.09\%.$$

Example 1.16 *The number x = 37.46235 is rounded off to four significant figures. Compute the absolute error, relative error and the percentage error.*

Solution We have $X = 37.46235$; $x = 37.46000$

$$\text{Absolute error} = |X - x| = |37.46235 - 37.46000|$$

$$E_A = 0.00235$$

$$E_r = \left|\frac{X - x}{x}\right| = \frac{0.00235}{37.46235} = 6.27 \times 10^{-5}$$

$$E_P = E_r \times 100 = 6.27 \times 10^{-3}$$

Exercise 1.1

1. Round-off the following numbers to two decimal places.

(a) 52.275 (b) 2.375 (c) 2.385 (d) 81.255 (e) 2.375

2. Round-off the following numbers to three decimal places.

(a) 0.4699 (b) 1.0532 (c) 0.0004555 (d) 0.0028561 (**e**) 0.0015

3. Round-off the following numbers to four decimal places.

(a) 0.235082 (b) 0.0022218 (c) 4.50089 (d) 2.36425 (e) 1.3456

4. The following numbers are correct to the last digit, find the sum.

(a) 2.56, 4.5627, 1.253, 1.0534

(b) 0.532, 7.46571, 1.501, 3.62102

(c) 1.3526, 2.00462, 1.532, 28.201, 31.0012

(d) 5.2146, 20.12, 11.2356, 1.8948

5. Find the relative error in computation of

$x - y$ for $x = 12.05$ and $y = 8.02$ having absolute errors $\Delta x = 0.005$ and $\Delta y = 0.001$.

6. Find the relative error in computation of $x - y$ for $x = 9.05$ and $y = 6.56$ having absolute errors $\Delta x = 0.001$ and $\Delta y = 0.003$ respectively.

7. Find the relative error in computation of $x + y$ for $x = 11.75$ and $y = 7.23$ having absolute errors $\Delta x = 0.002$ and $\Delta y = 0.005$.

8. If $y = 4x^6 - 5x$, find the percentage error in y at $x = 1$, if the error is $x = 0.04$.

9. If $\frac{5}{6}$ be represented approximately by 0.8333, find (a) relative error and (b) percentage error.

10. If $f(x) = 4 \cos x - 6x$, find the relative percentage error in $f(x)$ for $x = 0$ if the error in $x = 0.005$.

11. Find the relative percentage error in the approximate representation of $\frac{4}{3}$ by 1.33.

12. Determine the number of correct digits in the number x given its relative error E_R.

(*a*) $x = 386.4$, $E_R = 0.3$

(*b*) $x = 86.34$, $E_R = 0.1$

(*c*) $x = 0.4785$, $E_R = 0.2 \times 10^{-2}$

13. Determine the number of correct digits in the number x, given its absolute error E_A.

(*a*) $x = 0.00985$, $E_A = 0.1 \times 10^{-4}$

(*b*) $x = -33.783$, $E_A = 0.3 \times 10^{-2}$

(*c*) $x = 48.2461$, $E_A = 0.21 \times 10^{-2}$

(*d*) $x = 841.256$, $E_A = 0.1$

(*e*) $x = 0.4942$, $E_A = 0.24 \times 10^{-2}$

14. Evaluate $X = \sqrt{5.01} - \sqrt{5}$ correct to three significant figures.

15. If $\frac{2}{3}$ is approximated to 0.6667. Find

(*a*) absolute error

(*b*) relative error and

(*c*) percentage error

16. Given $X = 66.888$. If x is rounded to 66.89 find the absolute error.

17. If $\frac{1}{3}$ is approximated by 0.333 find

(*a*) absolute error

(*b*) relative error and

(*c*) relative percentage error

18. If $u = \frac{5xy^2}{z^3}$ and error in x, y, z be 0.001, 0.002, and 0.003, compute the relative error in u. Where $x = y = z = 1$.

19. If the true value of a number is 2.546282 and 2.5463 is its approximate value; find the absolute error, relative error and the percentage error in the number.

20. If

$$a = 10.00 \pm 0.05$$
$$b = 0.0356 \pm 0.002$$
$$c = 15300 \pm 100$$
$$d = 62000 \pm 500$$

Find the maximum value of the absolute error in (*i*) $a + b + c + d$

(*ii*) $a + 5c - d$

(*iii*) c^3

21. If $(0.31\,x + 2.73) / (x + 0.35)$

where the coefficients are rounded off find the absolute and relative in y when $x = 0.5 \pm 0.1$.

22. If $u = 4x^2y^3/z^4$ and errors in x, y, z be 0.001 compute the relative maximum error in u when $x = y = z = 1$.

23. If $x = 865\ 250$ is rounded off to four significant figures compute the absolute error, relative error and the percentage error in x.

24. Find the relative error in the function

$$y = k\ x_1^{m_1}\ x_2^{m_2} \ldots x_n^{m_n}$$

25. If $y = 3x\,(x^6 - 2)$ find the percentage error in y at $x = 1$, if the percentage error in x is 5.

26. If $u = 10x^3\ y^2\ z^2$ and errors in x, y, z are 0.03, 0.01, 0.02 respectively at $x = 3$, $y = 1$, $z = 2$. Calculate the asolute error and relative error and percentage erro in u.

27. If the number $X = 3.1416$ is correct to 4 decimal places; then find the error in X.

28. If $u = \dfrac{5xy^2}{z^2}$ and $\Delta x = \Delta y = \Delta z = 0.1$, compute the maximum relative error in u where $x = y = z = 1$.

29. Find the relative error in the evaluation of $x + y$ where $x = 13.24$, $y = 14.32$, $\Delta x = 0.004$ and $\Delta y = 0.002$.

30. If $u = xy + yz + zx$, find the relative percentage error in the evaluation of u for $x = 2.104$, $y = 1.935$, $z = 0.845$, which are the approximate values of the last digit.

31. If $u = 4x^6 + 3x - 9$, find the relative, percentage errors in computing $x = 1.1$ given that error in x is 0.05%.

32. If $a = 5.43$ m and $b = 3.82$ m, where a and b denote the length and breadth of a rectangular plate, measured accurate upto 1 cm, find error in computing its area.

33. Find the percentage error in computing $y = 3x^6 - 6x$ at $x = 1$, given that $\Delta x = 0.05$.

34. Find the percentage error in computing $u = \sqrt{x}$ at $x = 4.44$, when x is corrected to its last digit.

35. Define the terms : (*a*) Absolute error

(*b*) Relative error

(*c*) Percentage error.

36. Explain the rules of round off.

Answers

1. (*a*) 52.28	(*b*) 2.38	(*c*) 2.38	(*d*) 81.26	(*e*) 2.37
2. (*a*) 0.470	(*b*) 1.05	(*c*) 0.000456	(*d*) 0.00286	(*e*) 0.002
3. (*a*) 0.2351	(*b*) 0.002222	(*c*) 4.501	(*d*) 2.364	(*e*) 1.346
4. (*a*) 9.43	(*b*) 13.120	(*c*) 64.091	(*d*) 38.46	
5. 0.00029	**6.** 0.00034	**7.** 0.00037	**8.** 76%	

9. $E_R = 0.00004$, $E_p = 0.004\%$ **10.** 0.75% **11.** $E_p = 0.25\%$

12. (*a*) 1 (*b*) 0 (*c*) 2

13. (*a*) 2 (*b*) 4 (*c*) 4 (*d*) 3 (*e*) 2

14. 2.235×10^{-3} **15.** (*a*) 0.000033 (*b*) 0.00005 (*c*) 0.005 **16.** 0.002

17. (*a*) 0.00033 (*b*) 0.001 (*c*) 0.1% **18.** 0.0141

19. 1.8×10^{-5}, 7.07×10^{-6}, 7.07×10^{-4}%

20. (*i*) 600.05 (*ii*) 1000.05 (*iii*) 5.766×10^{12}

25. 0.075 **26.** 0.036 **27.** $E_A = 50$, $E_P = 6.71 \times 10^{-5}$ $E_P = 6.71 \times 10^{-3}$

28. 0.5 **29.** 0.000217 **30.** 0.062% **31.** 0.55% **32.** $0.0925 \simeq 0.1$ m^2

33. 0% **34.** 0.05%

2

SOLUTION OF ALGEBRAIC AND TRANSCENDENTAL EQUATIONS

2.1 INTRODUCTION

In this chapter we shall discuss some numerical methods for solving algebraic and transcendental equations. The equation $f(x) = 0$ is said to be algebraic if $f(x)$ is purely a polynomial in x. If $f(x)$ contains some other functions, namely, Trigonometric, Logarithmic, Exponential, etc., then the equation $f(x) = 0$ is called a *Transcendental Equation*.

The equations

$$x^3 - 7x + 8 = 0$$

and $$x^4 + 4x^3 + 7x^2 + 6x + 3 = 0$$

are algebraic.

The equations

$$3 \tan 3x = 3x + 1,$$

$$x - 2 \sin x = 0$$

and $$e^x = 4x$$

are transcendental.

Algebraically, the real number α is called the *real root* (or zero of the function $f(x)$) of the equation $f(x) = 0$ if and only if $f(\alpha) = 0$ and geometrically the real root of an equation $f(x) = 0$ is the value of x where the graph of $f(x)$ meets the x-axis in rectangular coordinate system.

We will assume that the equation

$$f(x) = 0 \tag{1}$$

has only isolated roots, that is for each root of the equation there is a neighbourhood which does not contain any other roots of the equation.

Approximately the isolated roots of the equation (1) has two stages.

1. Isolating the roots that is finding the smallest possible interval (a, b) containing one and only one root of the equation (1).
2. Improving the values of the approximate roots to the specified degree of accuracy. Now we state a very useful theorem of mathematical analysis without proof.

Theorem 2.1 If a function $f(x)$ assumes values of opposite sign at the end points of interval (a, b), i.e., $f(a)\, f(b) < 0$ then the interval will contain at least one root of the equation $f(x) = 0$, in other words, there will be at least one number $c \in (a, b)$ such that $f(c) = 0$. Throughout our discussion in this chapter we assume that

1. $f(x)$ is continuous and continuously differentiable up to sufficient number of times.
2. $f(x) = 0$ has no multiple root, that is, if c is a real root $f(x) = 0$ then $f(c) = 0$ and $f'(x) < 0 \quad f'(x) > 0$ in (a, b), (*see* Fig. 2.1).

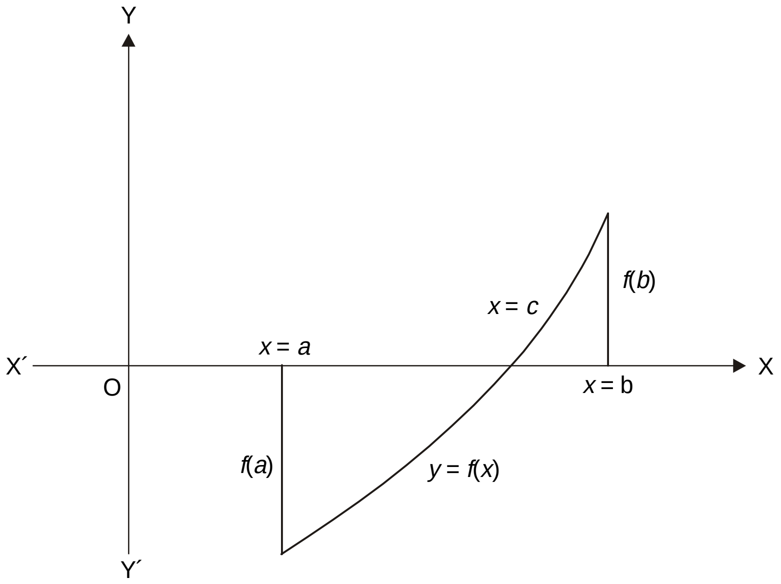


Fig. 2.1

2.2 GRAPHICAL SOLUTION OF EQUATIONS

The real root of the equation

$$f(x) = 0, \qquad \text{refer (1)}$$

can be determined approximately as the abscissas of the points of intersection of the graph of the function $y = f(x)$ with the x-axis. If $f(x)$ is simple, we shall draw the graph of $y = f(x)$ with respect to a rectangular axis X′OX and Y′OY. The points at which the graph meets the x-axis are the location of the roots of (1). If $f(x)$ is not simple we replace equation (1) by an *equivalent equation* say $\phi(x) = \psi(x)$, where the functions $\phi(x)$ and $\psi(x)$ are simpler than $f(x)$. Then we construct the graphs of $y = \phi(x)$ and $y = \psi(x)$. Then the x-coordinate of the point of intersection of the graphs gives the crude approximation of the real roots of the equation (1).

Example 2.1 *Solve the equation $x \log_{10} x = 1$, graphically.*

Solution The given equation

$$x \log_{10} x = 1$$

can be written as

$$\log_{10} x = \frac{1}{x}$$

where $\log_{10}x$ and $\frac{1}{x}$ simpler than $x\log_{10}x$, constructing the curves $y = \log_{10}x$ and $y = \frac{1}{x}$, we get x-coordinate of the point of intersection as 2.5 (*see* Fig. 2.2).

$\therefore$ The approximate value of the root of $x \log_{10}x = 0.1$, is $c = 2.5$.

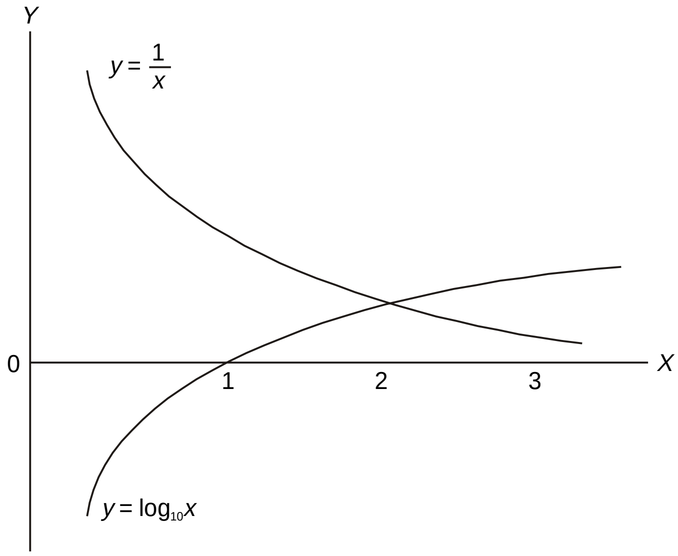


Fig. 2.2

Exercise 2.1

1. Solve $x^2 + x - 1 = 0$ graphically.
2. Solve $-e^{2x} + 2x + 0.1 = 0$ graphically.
3. Solve the cubic equation $x^3 - 1.75x + 0.75 = 0$ graphically.
4. Solve $x^3 + 2x + 7.8 = 0$.
5. Solve graphically the real root of $x^3 - 3.6 \log_{10}x - 2.7 = 0$ correct to two decimal places.
6. Solve graphically the equation $2x^3 - x^2 - 7x + 6 = 0$.
7. Draw the graph of $y = x^3$ and $y + 2x = 20$ and find an approximate solution of the equation $x^3 + 2x - 20 = 0$.
8. Solve $x^3 + 10x - 15 = 0$ graphically.
9. Solve graphically the following equations in the range $(0, \pi/2)$; (*i*) $x = \cos x$ (*ii*) $e^x = 4x$ (*iii*) $x = \tan x$.

Answers

1. 0.6 and –1.6 (approximately)	**2.** 0.3 (approximately)	**3.** –1.5, 0.5 and 1
4. –1.65	**5.** 1.93	**6.** 1, 1.5, –2
7. 2.47	**8.** 1.297	**9.** (*i*) 0.74 (*ii*) 0.36, 2.15 (*iii*) 4.49

2.3 METHOD OF BISECTION

Consider the equation

$$f(x) = 0, \qquad \text{refer (1)}$$

where $f(x)$ is continuous on (a, b) and $f(a)\, f(b) < 0$. In order to find a root of (1) lying in the interval (a, b). We shall determine a very small interval (a_0, b_0) (by graphical method) in which $f(a_0)\, f(b_0) < 0$ and $f'(x)$ maintains the same sign in (a_0, b_0), so that there is only one real root of the equation $f(x) = 0$.

Divide the interval in half and let

$$x_1 = \frac{a_0 + b_0}{2}$$

If $f(x) = 0$ then x_1 is a root of the equation. If $f(x_1) \neq 0$ then either $f(a_0)\, f(x_1) < 0$ of $f(b_0)\, f(x_1) < 0$. If $f(a_0)\, f(x_1) < 0$ then the root of the equation lies in (a_0, x_1) otherwise the root of the equation lies in (x_1, b_0). We rename the interval in which the root lies as (a_1, b_1) so that

$$b_1 - a_1 = \frac{1}{2}(b_0 - a_0),$$

now we take

$$x_2 = \frac{a_1 + b_1}{2}$$

If $f(x_2) = 0$ then x_2 is the root of $f(x) = 0$. If $f(x_2) \neq 0$ and $f(x_2)\, f(a_1) < 0$, then the root lies in (a_1, x_2). In which case we rename the interval as (a_2, b_2), otherwise (x_2, b_1) is renamed as (a_2, b_2) where

$$a_2 - b_2 = \frac{1}{2^2}(b_0 - a_0).$$

Proceeding in this manner, we find

$$x_{n+1} = \frac{a_n + b_n}{2}$$

which gives us the $(n + 1)$th approximation of the root of $f(x) = 0$, and the root lies (a_n, b_n) where

$$b_n - a_n = \frac{1}{2^n}(b_0 - a_0),$$

since the left end points $a_1, a_2, \ldots, a_n, \ldots$ form a monotonic non-decreasing bounded sequence, and the right end points $b_1, b_2, b_2, \ldots, b_n, \ldots$ form a monotonic non-increasing bounded sequence, then there is a common limit

$$c = \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n$$

such that $f(c) = 0$ which means that c is a root of equation (1).

The bisection method is well suited to electronic computers. The method may be conveniently used in rough approximations of the root of the given equation. The bisection method is a simple but slowly convergent method.

Example 2.2 *Solve the equation $x^3 - 9x + 1 = 0$ for the root lying between 2 and 3, correct to three significant figures.*

Solution We have

$$f(x) = x^3 - 9x + 1,$$

$$f(2) = -9,\ f(3) = 1$$

$$\therefore \qquad f(2)\ f(3) < 0.$$

Let $$a_0 = 2,\ b_0 = 3$$

n	a_n	b_n	$x_{n+1} = \left(\dfrac{a_n + b_n}{2}\right)$	$f(x_{n+1})$
0	2	3	2.5	–5.8
1	2.5	3	2.75	–2.9
2	2.75	3	2.88	–1.03
3	2.88	3	2.94	–0.05
4	2.94	3	2.97	0.47
5	2.94	2.97	2.955	0.21
6	2.94	2.955	2.9475	0.08
7	2.94	2.9475	2.9438	0.017
8	2.94	2.9438	2.9419	0.016
check 9	2.9419	2.9438	2.9428	0.003

In the 8th step a_n, b_n and x_{n+1} are equal up to three significant figures. We can take 2.94 as a root up to three significant figures.

$\therefore$ The root of $x^3 - 9x + 1 = 0$ is 2.94.

Example 2.3 *Compute one root of $e^x - 3x = 0$ correct to two decimal places.*

Solution Let

$$f(x) = e^x - 3x$$

$$f(1.5) = -0.02,\ f(1.6) = 0.15$$

$f'(x) = e^x - 3 > 0$ for $x \in [1.5, 1.6]$, only one root of $f(x) = 0$ lies between 1.5 and 1.6, here $a_0 = 1.5$, $b_0 = 1.6$.

n	a_n	b_n	$x_{n+1} = \left(\dfrac{a_n + b_n}{2}\right)$	$f(x_{n+1})$
0	1.5	1.6	1.55	0.06
1	1.5	1.55	1.525	0.02
2	1.5	1.525	1.5125	0.00056
3	1.5	1.5125	1.5062	–0.00904
4	1.5062	1.5125	1.50935	–0.00426
check 5	1.50935	1.5125	1.51092	–0.00184

In the 4th step a_n, b_n and x_{n+1} are equal up to two decimal places. Thus, $x = 1.51$ is the root of $f(x) = 0$, correct up to two decimal places.

Example 2.4 *Find the root of tan x + x = 0 up to two decimal places, which lies between 2 and 2.1.*

Solution Let $$f(x) = \tan x + x$$

Here $$f(2) = -0.18,\ f(2.1) = 0.39$$

Thus, the root lies between 2.0 and 2.1

$$\therefore \quad a_0 = 2,\ b_0 = 2.1$$

n	a_n	b_n	$x_{n+1} = \left(\dfrac{a_n + b_n}{2}\right)$	$f(x_{n+1})$
0	2.0	2.1	2.05	0.12
1	2.0	2.05	2.025	–0.023
2	2.025	2.05	2.0375	0.053
3	2.025	2.0375	2.03125	–0.0152
4	2.025	2.03125	2.02812	–0.0039
5	2.02812	2.03125	2.02968	0.0056
check 6	2.02813	2.02968	2.02890	0.00087

In the 5th step, a_n, b_n and x_{n+1} are equal up to two decimal places.

$\therefore$ $x = 2.03$ is a root of $f(x) = 0$, correct up to two decimal places.

Exercise 2.2

1. Find a root of the equation $x^3 - 4x - 9 = 0$ correct to three decimal places by using bisection method.
2. Find the positive roots of the equation $x^3 - 3x + 1.06 = 0$, by method of bisection, correct to three decimal places.
3. Compute one positive root of $2x - 3 \sin x - 5 = 0$, by bisection method, correct to three significant figures.
4. Compute one root of $x + \log x - 2 = 0$ correct to two decimal places which lies between 1 and 2.
5. Compute one root of $\sin x = 10(x - 1)$ correct to three significant figures.
6. Compute the root of $\log x = \cos x$ correct to two decimal places.
7. Find the interval in which the smallest positive root of the following equation lies. Also find the root correct to two decimal places. Use bisection method

   (a) $\tan x + \tan hx = 0$

   (b) $x^3 - x - 4 = 0$
8. Find the root of the equation $x^3 - x - 11 = 0$, using bisection method correct to three decimal places (which lies between 2 and 3).
9. Find the root of the equation $x^4 - x - 10 = 0$, using bisection method.

10. Solve the equation $x - \exp\left(\frac{1}{x}\right) = 0$, by bisection method

Answers

1. 2.6875	**2.** 0.370	**3.** 2.88
4. 1.56	**5.** 1.09	**6.** 1.30
7. (*a*) (2.3625, 2.36875); 2.37 (*b*) (1.795898, 1.796875), 1.80		**8.** 2.375
9. 1.8556	**10.** 0.567	

2.4 THE ITERATION METHOD

Suppose we have an equation

$$f(x) = 0 \qquad \text{refer (1)}$$

whose roots are to be determined. The equation (1) can be expressed as

$$x = f(x), \qquad (2)$$

putting $x = x_0$ in R.H.S. of (2) we get the first approximation

$$x_1 = \phi(x_o).$$

The successive approximations are then given by

$$\begin{aligned} x_2 &= \phi(x_1) \\ x_3 &= \phi(x_2) \\ x_4 &= \phi(x_3) \\ &\vdots \\ x_n &= \phi(x_{n-1}) \end{aligned}$$

where the sequence of approximations $x_1, x_2, \ldots, x_n$ always converge to the root of $x = \phi(x)$ and it can be shown that if $|\phi'(x)| < 1$ when x is sufficiently close to the exact value c of the root and $x_n \to c$ as $n \to \infty$.

Theorem 2.2 *Let $x = \alpha$ be a root of f(x) = 0, which is equivalent to $x = \phi(x)$ and I be an interval containing α. If $|\phi'(x)| < 1$ for all x in I, then the sequence of approximations $x_0, x_1, \ldots, x_n$ will converge to the root α, provided that the initial approximation x_0 is chosen in I.*

Proof α is a root of $f(x) = 0$

$$\Rightarrow \alpha \text{ is a root of } x = \phi(x)$$

$$\Rightarrow \alpha = \phi(\alpha). \qquad (3)$$

Let x_{n-1} and x_n denote two successive approximations to α, then we have

$$x_n = \phi\left(x_{n-1}\right),$$

$$\therefore\ x_n - \alpha = \phi\left(x_{n-1}\right) - \phi(\alpha).$$

By mean value theorem we have

$$\frac{\phi\left(x_{n-1}\right) - \phi(\alpha)}{x_{n-1} - \alpha} = \phi'(\beta),$$

where $$x_{n-1} < \beta < \alpha$$

$$\Rightarrow\ \phi\left(x_{n-1}\right) - \phi(\alpha) = \left(x_{n-1} - \alpha\right)\phi'(\beta). \tag{4}$$

Let λ be the maximum absolute value of $\phi'(x)$ over I, then from (4) we have

$$\left|x_n - \alpha\right| \le \lambda\left|x_{n-1} - \alpha\right| \tag{5}$$

$$\Rightarrow\ \left|x_{n-1} - \alpha\right| \le \lambda\left|x_{n-2} - \alpha\right|$$

$$\therefore\ \left|x_n - \alpha\right| \le \lambda^2\left|x_{n-2} - \alpha\right| \tag{6}$$

proceeding in this way, we get

$$\left|x_n - \alpha\right| \le \lambda^n\left|x_0 - \alpha\right|. \tag{7}$$

If $\lambda < 1$ over I, then the RHS of (7) becomes small (as n increases) such that

$$\text{Lt}\left|x_n - \alpha\right| = 0$$

$$\text{Lt}\ x_n = \alpha.$$

$\therefore$ The sequence of approximations will converge to the root α if $\lambda < 1$, i.e., $\left|\phi'(x)\right| < 1$.

If $\lambda > 1$, then

$$\left|\phi'(x)\right| > 1$$

$\Rightarrow$ $\left|x_n - \alpha\right|$ will become indefinitely large, as n increases and the sequence approximations does not converge.

Note

1. The smaller the value of $\phi'(x)$ the more rapid will be the convergence.
2. From (1) we have the relation

$$|x_n - \alpha| \le \lambda |x_{n-1} - \alpha|, \ (\lambda \text{ is a constant}).$$

Hence the error at any stage is proportional to the error in the previous stage. Therefore *the iteration method has a linear convergence*.

3. Iteration method is more useful for finding the real roots of an equation which is in the form of an infinite series.

Example 2.5 *Find the root of $x^3 + x - 1 = 0$ by iteration method, given that root lies near 1.*

Solution Given $x = 1$ is the approximate value of the root

$$x^3 + x - 1 = 0$$

can be put in the form
$$x = \frac{1}{1+x^2}$$

such that
$$\phi(x) = \frac{1}{1+x^2} \quad \text{and } x_0 = 1$$

$$\phi'(x) = \frac{-2x}{\left(1+x^2\right)^2}$$

at $x = 1$; we have
$$|\phi'(x)| = |\phi'(1)| = \left|\frac{-2.1}{\left(1+1^2\right)^2}\right| < 1$$

so the iteration method can be applied.

$$\therefore \ x_1 = \phi(x_0) = \frac{1}{1+x_0^2} = \frac{1}{1+1^2} = 0.5$$

$$x_2 = \phi(x_1) = \frac{1}{1+x_1^2} = \frac{1}{1+(0.5)^2} = 0.8,$$

$$x_3 = \phi(x_2) = \frac{1}{1+x_2^2} = \frac{1}{1+(0.8)^2} = 0.61.$$

$\therefore$ The root of the given equation is 0.61 after three iterations.

Note: We can write the equation $x^3 + x - 1 = 0$ in different forms as

$$x = \sqrt{\frac{1}{x} - 1}, \quad \text{and} \quad x = (1-x)^{1/3}.$$

Example 2.6 *Find a real root of $\cos x = 3x - 1$, correct to three decimal places.*

Solution We have

$$f(x) = \cos x - 3x + 1 = 0$$

$$f(0) = \cos 0 - 0 + 1 = 2 > 0$$

$$f\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} - 3\frac{\pi}{2} + 1 = -\frac{3\pi}{2} + 1 < 0$$

$$\Rightarrow \ f(0)\cdot\left(\frac{\pi}{2}\right) < 0$$

$\therefore$ A root of $f(x) = 0$ lies between 0 and $\frac{\pi}{2}$.

The given equation can be written as $x = \frac{1}{3}[1 + \cos x]$.

Here $$\phi(x) = \frac{1}{3}[1 + \cos x]$$

$$\therefore \ \phi'(x) = \frac{-\sin x}{3}$$

$$\Rightarrow \ |\phi'(x)| = \left|\frac{\sin x}{3}\right| < 1 \text{ in } \left(0, \frac{\pi}{2}\right)$$

Iteration method can be applied

Let $x_0 = 0$

be the initial approximation.

$\therefore$ We get $$x_1 = \phi(x_0) = \frac{1}{3}[1 + \cos 0] = 0.66667$$

$$x_2 = \phi(x_1) = \frac{1}{3}[1 + \cos 0.66667] = 0.595295 \approx 0.59530$$

$$x_3 = \phi(x_2) = \frac{1}{3}[1 + \cos 0.59530] = 0.6093267 \approx 0.60933$$

$$x_4 = \phi(x_3) = \frac{1}{3}[1 + \cos 0.60933] = 0.6066772 \approx 0.60668$$

$$x_5 = \phi(x_4) = \frac{1}{3}[1 + \cos 0.60668] = 0.6071818 \approx 0.60718$$

$$x_6 = \phi(x_5) = \frac{1}{3}[1 + \cos 0.60718] = 0.6070867 \approx 0.60709$$

$$x_7 = \phi(x_6) = \frac{1}{3}[1 + \cos 0.60709] = 0.6071039 \approx 0.60710$$

$$x_8 = \phi(x_7) = \frac{1}{3}[1 + \cos 0.60710] = 0.60710$$

The correct root of the equation is 0.607 correct to three decimal places.

Example 2.7 *Find by the iteration method, the root near 3.8, of the equation* $2x - \log_{10} x = 7$ *correct to four decimal places.*

Solution The given equation can be written as

$$x = \frac{1}{2}\left[\log_{10} x + 7\right]$$

clearly $\left|\phi'(x)\right| < 1$ when x is near 3.8

we have

$$x_0 = 3.8$$

$$x_1 = \phi(x_0) = \frac{1}{2}\left[\log_{10} 3.8 + 7\right] = 3.79,$$

$$x_2 = \phi(x_1) = \frac{1}{2}\left[\log_{10} 3.79 + 7\right] = 3.7893,$$

$$x_3 = \phi(x_2) = \frac{1}{2}\left[\log_{10} 3.7893 + 7\right] = 3.7893,$$

$$x_2 = x_3 = 3.7893$$

∴ We can take 3.7893 as the root of the given equation.

Example 2.8 *Find the smallest root of the equation*

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = 0 \qquad \ldots(1)$$

Solution. The given equation can be written as

$$x = 1 + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \ldots = \phi(x) \qquad \text{(say)}$$

omitting x^2 and the other higher powers of x we get

$$x = 1$$

Taking $x_0 = 1$, we obtain

$$x_1 = \phi(x_0) = 1 + \frac{1}{(2!)^2} - \frac{1}{(3!)^2} + \frac{1}{(4!)^2} + \frac{1}{(5!)^2} + \ldots$$

$$= 1.2239$$

$$x_2 = \phi(x_1) = 1 + \frac{(1.2239)^2}{(2!)^2} - \frac{(1.2239)^3}{(3!)^2} + \frac{(1.2239)^4}{(4!)^2} - \frac{(1.2239)^5}{(5!)^2} + \ldots$$

$$= 1.3263$$

$$x_3 = \phi(x_2) = 1 + \frac{(1.3263)^2}{(2!)^2} - \frac{(1.3263)^3}{(3!)^2} + \frac{(1.3263)^4}{(4!)^2} - \frac{(1.3263)^5}{(5!)^2} + \ldots$$

$$= 1.3800$$

$$x_4 = \phi(x_3) = 1 + \frac{(1.3800)^2}{(2!)^2} - \frac{(1.3800)^3}{(3!)^2} + \frac{(1.3800)^4}{(4!)^2} - \frac{(1.3800)^5}{(5!)^2} + \ldots$$

$$= 1.409$$

Similarly, $x_5 = 1.4250$

$x_6 = 1.4340$

$x_7 = 1.4390$

$x_8 = 1.442$

Correct to 2 decimal places, we get

$$x_7 = 1.44 \text{ and } x_8 = 1.44$$

$\therefore$ The root of (1) is 1.44 (approximately).

2.4.1 Aitken's Δ^2 Method

Let $x = \alpha$ be a root of the equation

$$f(x) = 0 \tag{1}$$

and let I be an interval containing the point $x = \alpha$. The equation (1) can be written as

$$x = \phi(x)$$

such that $\phi(x)$ and $\phi'(x)$ are continuous in I and $|\phi'(x)| < 1$ for all x in I. Let x_{i-1}, x_i and x_{i+1} be three successive approximations of the desired root α. Then we know that

$$\alpha - x_i = \lambda(\alpha - x_{i-1})$$

and
$$\alpha - x_{i+1} = \lambda(\alpha - x_i)$$

(λ is a constant such that $|\phi'(x_i)| \le \lambda < 1$ for all i)

dividing we get
$$\frac{\alpha - x_i}{\alpha - x_{i+1}} = \frac{\alpha - x_{i-1}}{\alpha - x_i}$$

$$\Rightarrow \alpha = x_{i+1} - \frac{(x_{i+1} - x_i)^2}{(x_{i+1} - 2x_i - x_{i-1})} \tag{2}$$

Since
$$\Delta x_i = x_{i+1} - x_i$$

and
$$\Delta^2 x_{i-1} = (E-1)^2 x_{i-1}$$

$$= (E^2 - 2E + 1)\, x_{i-1}$$

$$= x_{i+1} - 2x_i + x_{i-1}$$

(2) can be written as

$$\alpha = x_{i+1} - \frac{(\Delta x_i)^2}{\Delta^2 x_{i-1}} \tag{3}$$

formula (3) yields successive approximation to the root α and the method is called Aitken's Δ^2 method.

Note: We know that iteration method is linearly convergent. The slow rate of convergence can be accelerated by using Aitken's method. And for any numerical application the values of the under-mentioned quantities, must be computed.

	Δ	Δ^2
x_{i-1}		
	Δx_{i-1}	
x_i		$\Delta^2 x_{i-1}$
	Δx_i	
x_{i+1}		

Example 2.9 *Find the root of the equation*

$$3x = 1 + \cos x$$

correct to three decimal places.

Solution We have
$$f(x) = \cos x - 3x + 1.$$

$$f(0) = 1 \text{ and } f\left(\frac{\pi}{2}\right) = \cos\frac{\Pi}{2} - 3\left(\frac{\pi}{2}\right) + 1$$

$$= -8.42857$$

$$\therefore\ f(0) > 0 \text{ and } f\left(\frac{\pi}{2}\right) < 0$$

$$\Rightarrow\ f(0)\ f\left(\frac{\pi}{2}\right) < 0, \text{ therefore a root lies between 0 and } \frac{\pi}{2}.$$

The given equation can be written as

$$x = \frac{1}{3}(1 + \cos x) = \phi(x) \qquad \text{(say)}$$

$$\phi'(x) = \frac{-\sin x}{3} \quad \Rightarrow |\phi'(x)| = \left|\frac{-\sin x}{3}\right| < 1 - x \in \left(0, \frac{\pi}{2}\right)$$

Hence, iteration method can be applied.

Let $x_0 = 0$ be the initial approximation of the root

$$\therefore \qquad x_1 = \phi(x_0) = \frac{1}{3}(1 + \cos 0) = 0.6667$$

$$x_2 = \phi(x_1) = \frac{1}{3}(1 + \cos(0.6667)) = 0.5953$$

$$x_3 = \phi(x_2) = \frac{1}{3}(1 + \cos 0.5953) = 0.6093$$

constructing the table, we have

x	Δx	Δ^2
$x_1 = 0.667$	$\underset{\Delta x_1}{-0.0714}$	
$x_2 = 0.5953$		$\underset{\Delta^2 x_1}{0.0854}$
$x_3 = 0.6093$	$\underset{\Delta x_2}{0.014}$	

Hence
$$x_4 = x_3 - \frac{(\Delta x_2)^2}{\Delta^2 x_1} = 0.6093 - \frac{(0.014)^2}{(0.0854)}$$
$$= 0.607$$

$\therefore$ The required root is 0.607.

Exercise 2.3

1. Find the real root of the equation $x^3 + x - 1 = 0$ by the iteration method.

2. Solve the equation $x^3 - 2x^2 - 5 = 0$ by the method of iteration.

3. Find a real root of the equation, $x^3 + x^2 - 100 = 0$ by the method of successive approximations (the iteration method).

4. Find the negative root of the equation $x^3 - 2x + 5 = 0$.

5. Find the real root of the equation $x - \sin x = 0.25$ to three significant digits.

6. Find the root of $x^2 = \sin x$, which lies between 0.5 and 1 correct to four decimals.

7. Find the real root of the equation $x^3 - 5x - 11 = 0$.

8. Find the real root of the equation

$$x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} - \frac{x^{11}}{1320} + \dots + (-1)^{n-1} \frac{x^{2n-1}}{(n-1)!\,(2n-1)} + \dots = 0.4431135$$

9. Find the smallest root of the equation by iteration method

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2} - \frac{x^5}{(5!)^2} + \dots = 0$$

10. Use iteration method to find a root of the equations to four decimal places.

(*i*) $x^3 + x^2 - 1 = 0$

(*ii*) $x = 1/2 + \sin x$ and

(*iii*) $e^x - 3x = 0$, lying between 0 and 1.

(*iv*) $x^3 - 3x + 1 = 0$

(*v*) $3x - \log_{10} x - 16 = 0$

11. Evaluate $\sqrt{30}$ by iteration method.

12. (*a*) Show that the equation $\log_e x = x^2 - 1$, has exactly two real roots $\alpha_1 = 0.45$ and $\alpha_2 = 1$.

(*b*) Determine for which initial approximation x_0, the iteration

$$x_{n+1} = \sqrt{1 + \log_e x_n}$$

converges to α_1 or α_2.

13. Using Aitken's Δ^2 process find the root of

$$x = \frac{1}{2} + \sin x, \quad x_0 = 1$$

14. Apply Aitken's Δ^2 method and show that 1.524 is a root of $2x = \cos x + 3$.

Answers

1. 0.68	**2.** 2.69	**3.** 4.3311
4. –2.09455	**5.** 1.172	**6.** 0.8767
7. 2.95	**8.** 0.47693	**9.** 1.442
10. (*i*) 0.7548 (*ii*) 1.4973 (*iii*) 0.671 (*iv*) 1.532 (*v*) 2.108	**11.** 5.4772	**13.** 1.4973

2.5 NEWTON–RAPHSON METHOD OR NEWTON ITERATION METHOD

This is also an iteration method and is used to find the isolated roots of an equation $f(x) = 0$, when the derivative of $f(x)$ is a simple expression. It is derived as follows:

Let $x = x_0$ be an approximate value of one root of the equation $f(x) = 0$. If $x = x_1$, is the exact root then

$$f(x_1) = 0 \tag{8}$$

where the difference between x_0 and x_1 is very small and if h denotes the difference then

$$x_1 = x_0 + h \tag{9}$$

Substituting in (8) we get

$$f(x_1) = f(x_0 + h) = 0$$

Expanding by Taylor's theorem we get

$$f(x_0) + \frac{h}{1!} f'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots = 0 \tag{10}$$

Since h is small, neglecting all the powers of h above the first from (10) we get

$$f(x_0) + \frac{h}{1!} f'(x_0) = 0, \text{ approximately}$$

$$\Rightarrow \ h = \frac{-f(x_0)}{f'(x_0)}$$

$\therefore$ From (9) we get

$$x_1 = x_0 + h = x_0 - \frac{f(x_0)}{f'(x_0)} \tag{11}$$

The above value of x_1 is a closer approximation to the root of $f(x) = 0$ than x_0. Similarly if x_2 denotes a better approximation, starting with x_1, we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Proceeding in this way we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{12}$$

The above is a general formula, known as Newton–Raphson formula. Geometrically, Newton's method is equivalent to replacing a small arc of the curve $y = f(x)$ by a tangent line drawn to a point of the curve. For definition sake, let us suppose $f''(x) > 0$, for $a \le x \le b$ and $f(b) > 0$ (*see* Fig. 2.3) whose $x_0 = b$, for which $f(x_0)\, f''(x_0) > 0$.

Draw the tangent line to the curve $y = f(x)$ at the point B_0 $[x_0, f(x_0)]$.

Let us take the abscissa of the point of intersection of this tangent with the x-axis, as the first approximation x_1 of the root of c. Again draw a tangent line through B $[x_1, f(x_1)]$, whose abscissa of the intersection point with the x-axis gives us the second approximation x_2 of the root c and so on. The equation of the tangent at the point B_x $[x_n, f(x_n)]$ $[n = 0, 1, 2, \ldots, n]$ is given by

$$y - f(x_n) = f'(x_n)(x - x_n).$$

Putting $y = 0$, $x = x_{n+1}$, we get

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

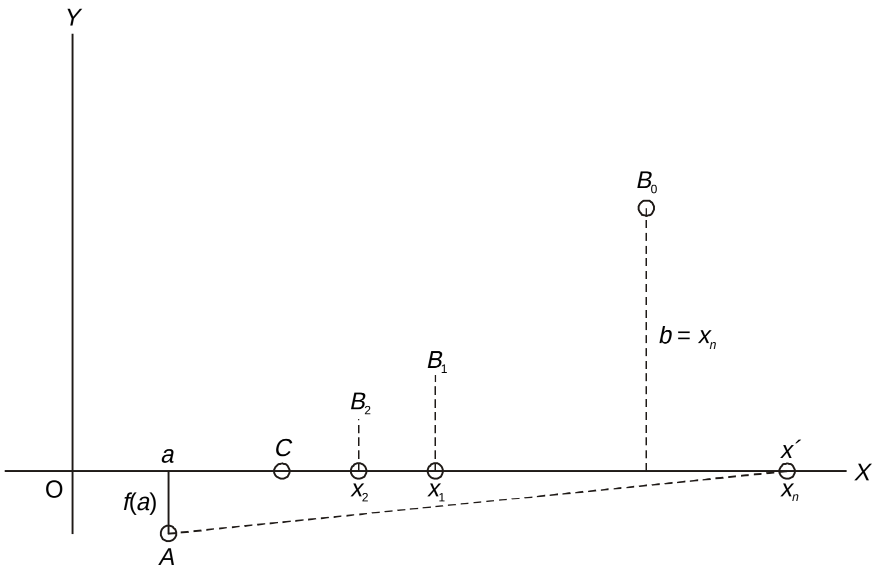


Fig. 2.3

Note :

1. If we put $x_0 = b$, (where $f(x_0)\, f''(x_0) < 0$) then the tangent drawn to the curve $y = f(x)$ at the point $A\,[a, f(a)]$, would give us a point x_1 which lies outside the interval $[a, b]$ from which it is clear that the method is impractical for such a choice. Thus the good choice of initial approximation for which $f(x_0)\, f''(x_0) > 0$ yields better results. Newton's method is applicable to the solution of equation involving algebraic functions as well as transcendental functions. At any stage of the iteration, if

$$\frac{f(x_i)}{f'(x_i)}$$

has n zeros, after decimal point then the result is taken to be correct to $2n$ decimal places.

2. *Criterion for ending the iteration*: The decision of stopping the iteration depends on the accuracy desired by the user. If ε denotes the tolerable error, then the process of iteration should be terminated when

$$|x_{n+1} - x_n| \le \varepsilon.$$

In the case of linearly convergent methods the process of iteration should be terminated when $|f(x_n)| \le \varepsilon$ where ε is tolerable error.

Example 2.10 *Using Newton–Raphson method, find correct to four decimals the root between 0 and 1 of the equation $x^3 - 6x + 4 = 0$*

Solution We have

$$f(x) = x^3 - 6x + 4$$

$$f(0) = 4 \quad \text{and} \quad f(1) = -1$$

$$f(0)\, f(1) = -4 < 0.$$

$\therefore$ A root of $f(x) = 0$ lies between 0 and 1. The value of the root is nearer to 1.

Let $x_0 = 0.7$ be an approximate value of the root

Now

$$f(x) = x^3 - 6x + 4$$

$$\Rightarrow \quad f'(x) = 3x^2 - 6$$

$$\therefore\ f(x_0) = f(0.7) = (0.7)^3 - 6(0.7) + 4 = 0.143$$

$$f'(x_0)\ f'(0.7) = 3(0.7)^2 - 6 = -4.53.$$

Then by Newton's iteration formula, we get

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.7 - \frac{(0.143)}{(-4.53)}$$

$$= 0.7 + 0.0316 = 0.7316$$

$$f(x_1) = (0.7316)^3 - 6 \times (0.7316) + 4 = 0.0019805$$

$$f'(x_1) = 3 \times (0.7316)^2 - 6 = -4.39428.$$

The second approximation of the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.7316 + \frac{0.0019805}{4.39428}$$

$\therefore$ $x_2 = 0.73250699 \approx 0.7321$ (correct to four decimal places).

The root of the equation = 0.7321 (approximately).

Example 2.11 *By applying Newton's method twice, find the real root near 2 of the equation $x^4 - 12x + 7 = 0$.*

Solution Let

$$f(x_1) = x^4 - 12x + 7$$

$$\therefore \quad f'(x) = 4x^3 - 12$$

Here $x_0 = 2$

$$\therefore \quad f(x_0) = f(2) = 2^4 - 12.2 + 7 = -1$$

$$f'(x_0) = f'(2) = 4(2)^3 - 12 = 20$$

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{(-1)}{20} = \frac{41}{20} = 2.05$$

and

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.05 - \frac{(2.05)^4 - 12(2.05) + 7}{4(2.05)^2 - 12} = 2.6706$$

$\therefore$ The root of the equation is 2.6706.

Example 2.12 *Find the Newton's method, the root of the $e^x = 4x$, which is approximately 2, correct to three places of decimals.*

Solution Here

$$f(x) = e^x - 4x$$

$$f(2) = e^2 - 8 = 7.389056 - 8 = -0.610944 = -\text{ ve}$$

$$f(3) = e^3 - 12 = 20.085537 - 12 = 8.085537 = +\text{ ve}$$

$$\therefore \quad f(2)\, f(3) < 0$$

$\therefore$ $f(x) = 0$ has a root between 2 and 3.

Let $x_0 = 2.1$ be the approximate value of the root

$$f(x) = e^x - 4x$$

$$\Rightarrow \quad f'(x) = e^x - 4$$

$$\therefore \quad f(x_0) = e^{2.1} - 4(2.1) = 8.16617 - 8.4 = -0.23383$$

$$f'(x_0) = e^{2.1} - 4 = 4.16617.$$

Let x_1 be the first approximation of the root

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.1 - \frac{(-0.23383)}{4.16617}$$

$$= 2.1 + 0.0561258 = 2.1561 \text{ (approximately).}$$

If x_2 denotes the second approximation, then

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.1561 - \frac{\left[e^{2.561} - 4(2.1561)\right]}{\left[e^{2.561} - 4\right]} = 2.1561 - \frac{0.0129861}{4.6373861}$$

$$= 2.1561 - 0.0028003 = 2.1533 \text{ approximately}$$

$$f(x_2) = f(2.1533) = -0.0013484$$

$$f'(x_2) = f'(2.1533) = 4.6106516.$$

If x_3 denotes the third approximation to the root, then

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.1533 - \frac{(-0.0013484)}{(4.6106516)}$$

$$= 2.1532$$

$\therefore$ The value of the root correct to three decimal places is 2.1532.

Example 2.13 *Find the root of the equation*

$\sin x = 1 + x^3$, between –2 and –1 correct to 3 decimal place by Newton Rappon method.

Solution Given $\sin x = 1 + x^3$

i.e., $x^3 - \sin x + 1 = 0$

Let $f(x) = x^3 - \sin x + 1$

then; we have $f'(x) = 3x^2 - \cos x$

$f(-1) = -1 + 0.8415 + 1 = 0.8415$

and $f(-2) = -8 + 0.9091 + 1 = -6.0907$

$\Rightarrow$ $f(-1)\, f(-2) < 0$

$\therefore$ $f(x) = 0$ has a root between –2 and –1.

Let $x_0 = -1$: be the initial approximation of the root.

The first approximation to the root is given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -1 - \frac{\left[(-1)^3 - \sin(-1) + 1\right]}{3(-1)^2 - \cos(-1)}$$

$$= -1 - \frac{0.8415}{2.4597} = -1.3421$$

The second approximation to the root is

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= -1.3421 - \frac{(-1.3421)^3 - \sin(-1.3421) + 1}{3(-1.3421)^2 - \cos(-1.3421)}$$

$$= -1.2565$$

The third approximation is

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= -1.2565 - \frac{(-1.2565)^3 - \sin(-1.2565) + 1}{3(-1.2565)^2 - \cos(-1.2565)}$$

$$= -1.249 \text{ (correct to three decimal places)}$$

Hence the root is –1.249.

Example 2.14 *Solve $x^4 - 5x^3 + 20x^2 - 40x + 60 = 0$, by Newton–Raphson method given that all the roots of the given equation are complex.*

Solution Let

$$f(x) = x^4 - 5x^3 + 20x^2 - 40x + 60$$

$\therefore$ So that

$$f'(x) = 4x^3 - 15x^2 + 40x - 40$$

The given equation is $f(x) = 0$ (1)

Using Newton–Raphson Method. We obtain

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^4 - 5x_n^3 + 20x_n^2 - 40x_n + 60}{4x_n^3 - 15x_n^2 + 40x_n - 40}$$

$$= \frac{3x_n^4 - 10x_n^3 + 20x_n^2 - 60}{4x_n^3 - 15x_n^2 + 40x_n - 40} \quad (2)$$

Putting $n = 0$ and,

Taking $x_0 = 2(1 + i)$ by trial, we get

$$x_1 = \frac{3x_0^4 - 10x_0^3 + 20x_0^2 - 60}{4x_0^3 - 15x_0^2 + 40x_0 - 40}$$

$$= \frac{3(2+2i)^4 - 10(2+2i)^3 + 20(2+2i)^2 - 60}{4(2+2i)^3 - 15(2+2i)^2 + 40(2+2i) - 40}$$

$$= 1.92(1+i) = 1.92 + 1.92i$$

Similarly,
$$x_2 = \frac{3(1.92 + 1.92i)^4 - 10(1.92 + 1.92i)^3 + 20(1.92 + 1.92i) - 60}{4(1.92 + 1.92i)^3 - 15(1.92 + 1.92i)^2 + 40(1.92 + 1.92i) - 40}$$
$$= 1.915 + 1.908i$$

$\therefore$ $1.915 + 1.908i$ is a root of the given equation

Imaginary roots appear in pairs, therefore $1.915 - 1.908i$ is also a root of the equation.

Since, $f(x) = 0$ is a biquadratic equation, the number of roots of the equation is four. Let us assume that $\alpha + i\beta$ and $\alpha - i\beta$ is the other pair of roots of the given equation.

From (1), we get

$\therefore$ Sum of the roots $= 5$

$$\Rightarrow (1.915 + 1.908i) + (1.915 - 1.908i) + (\alpha + i\beta) + (\alpha - i\beta) = 5$$
$$\Rightarrow 2\alpha + 3.83 = 5$$
$$\Rightarrow \alpha = \frac{5 - 3.83}{3} = 0.585$$

$\therefore$ the product of roots of (1) is 60.

$$\Rightarrow (\alpha + i\beta)(\alpha - i\beta)(1.915 + 1.908i)(1.915 - 1.908i) = 60$$
$$\Rightarrow (\alpha^2 + \beta^2)\left[(1.915)^2 + (1.908i)^2\right] = 60$$
$$\Rightarrow \left((0.585)^2 + \beta^2\right)(7.307689) = 60$$
$$\Rightarrow 0.342225 + \beta^2 = 8.21053$$
$$\Rightarrow \beta^2 = 7.8703$$
$$\beta = \sqrt{7.8703045} = 2.805$$

$\therefore$ The other two roots are $0.585 \pm 2.805i$

Hence the roots of the given equation are $1.915 \pm 1.908i$ and $0.585 \pm 2.805i$.

Example 2.15 *Find the positive root of the equation*

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}e^{0.3x}$$

Solution. Given
$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}e^{0.3x}$$

$$\Rightarrow \quad e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}e^{0.3x} = 0$$

Let
$$f(x) = e^x - 1 - x - \frac{x^2}{2} - \frac{x^3}{6}e^{0.3x}$$

then $$f'(x) = e^x - 1 - x - \frac{x^2}{2}e^{0.3x} - 0.3\,.\,\frac{x^3}{6}e^{0.3x}$$

$$= e^x - 1 - x - e^{0.3x}\,.\,\frac{x^2}{2}\left(1 - \frac{x}{10}\right)$$

We have $f(2) = -0.0404,\ f(3) = 0.5173$

$\therefore$ $f(2)\ f(3) < 0$

Hence: the root of $f(x) = 0$, lies between 2 and 3

let $x_0 = 2.5$ (Initial approximation)

using, the Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)};\quad n = 0, 1, 2, \ldots.$$

we obtain $$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2.5 - \frac{\left[e^{2.5} - 1 - 2.5 - \frac{(2.5)^2}{2} - \frac{(2.5)^3}{6}e^{0.75}\right]}{\left[e^{2.5} - 1 - 2.5 - e^{0.75}\frac{(2.5)^2}{2}\left(1 - \frac{2.5}{10}\right)\right]}$$

$$= 2.461326$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$\Rightarrow$ $$x_2 = 2.461326 - \frac{\left[e^{2.461326} - 1 - 2.461326 - \frac{(2.461326)^2}{2} - \frac{(2.461326)^3}{6}e^{(0.7383978)}\right]}{\left[e^{2.461326} - 1 - 2.461326 - e^{0.7383978}\left(1 - \frac{2.461326}{10}\right)\right]}$$

$\Rightarrow$ $x_2 = 2.379358$

$\therefore$ $$x_3 = x_2 - \frac{f(x_2)}{f'(x_1)}$$

$$= 2.379358 - \frac{\left[e^{2.379358} - 1 - 2.379358 - \frac{(2.379358)^2}{2} - \frac{(2.379358)^3}{6}e^{(0.7138074)}\right]}{\left[e^{2.379358} - 1 - 2.379358 - e^{0.713874}\left(1 - \frac{2.379358}{10}\right)\right]}$$

$\Rightarrow$ $x_3 = 2.363884$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)}$$

$$= 2.363884 - \frac{\left[e^{2.363884} - 1 - 2.363884 - \frac{(2.363884)^2}{2} - \frac{(2.363884)^3}{6}e^{0.7091652}\right]}{\left[e^{2.363884} - 1 - 2.363884 - e^{0.7091652}\left(1 - \frac{2.363884}{10}\right)\right]}$$

$\Rightarrow$ $x_4 = 2.363377$

$\therefore$ $$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)}$$

$$= 2.363377 - \frac{\left[e^{2.363377} - 1 - 2.363377 - \frac{(2.363377)^2}{2} - \frac{(2.363377)^3}{6} e^{0.709131}\right]}{\left[e^{2.363377} - 1 - 2.363377 - e^{0.7090131}\left(1 - \frac{2.363377}{10}\right)\right]}$$

$$= 2.363376$$

Hence, the required root is 2.363376.

Exercise 2.4

1. Apply Newton's method to find the real root of $x^3 + x - 1 = 0$.

2. Find the positive root of the equation $x = 2 \sin x$.

3. The equation $3 \tan 3x = 3x + 1$ is found to have a root near $x = 0.9$, x being in radians.

4. Find the root of $x^3 - 8x - 4 = 0$, which lies between 3 and 4, by Newton–Raphson method, correct to four decimal places.

5. Find a positive root of $x^2 + 2x - 2 = 0$, by Newton–Raphson method, correct to two significant figures.

6. Find a positive root of $x + \log x - 2 = 0$, by Newton–Raphson method, correct to six decimal figures.

7. Find by Newton–Raphson method the real root of $3x - \cos x - 1 = 0$.

8. Find a real root of $x^4 - x - 10 = 0$ by Newton–Raphson method.

9. Find a positive root of $x - e^{-x} = 0$, by Newton–Raphson method.

10. Compute the positive root of $x^3 - x - 0.1 = 0$, by Newton–Raphson method, correct to six decimal figures.

11. Using Newton's method, compute a negative root of the equation $f(x)$ $x^4 - 3x^2 + 75x - 10000 = 0$, correct to five places.

12. Use Newton's method to find the smallest positive root of the equation $\tan x = x$.

13. Apply Newton's method to find a pair of complex roots of the equation $x^4 + x^3 + 5x^2 + 4x + 4 = 0$ starting with $x_0 = i$.

14. Solve $f(z) = z^3 - 3z^2 - z + 9$ using Newton's method (z is a complex variable) starting with $z_0 = 1 + i$.

15. Perform three iterations of Newton–Raphson method for solving

$$1 + z^2 = 0, z_0 = \frac{1+i}{2}$$

Answers

1. 0.68 **2.** 1.895 **3.** 0.8831

4. 3.0514 **5.** 0.73 **6.** 1.55714

7. 0.60710 **8.** 1.8556 **9.** 0.5671

10. 1.046681 **11.** –10.261 **12.** 4.4934 **13.** $-0.573 \pm 0.89i$

14. $\frac{13}{7} \pm \frac{2}{7} i$ **15.** $-0.00172 \pm 0.9973i$

Example 2.16 *Using Newton–Raphson formula, establish the iterative formula* $x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right]$ *to calculate the square root of N.*

Solution Let
$$x = \sqrt{N} \Rightarrow x^2 = N$$
$$\Rightarrow x^2 - N = 0$$

Let
$$f(x) = x^2 - N$$
then
$$f'(x) = 2x$$

By Newton–Raphson rule, if x_n denotes the nth iterate

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{[x_n^2 - N]}{2x_n}$$

$$= \frac{x_n^2 + N}{2x_n} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right], \quad n = 0, 1, 2, 3, \ldots$$

$$\therefore \quad x_{n+1} = \frac{1}{2}\left[x_n + \frac{N}{x_n}\right], \quad n = 0, 1, 2, \ldots$$

Example 2.17 *Find the square root of 8.*

Solution Let $N = 8$

Clearly $2 < \sqrt{8} < 3$ taking $x_0 = 2.5$, we get

$$x_1 = \frac{1}{2}\left[x_0 + \frac{N}{x_0}\right] = \frac{1}{2}\left[2.5 + \frac{8}{2.5}\right] = 2.85$$

$$x_2 = \frac{1}{2}\left[x_1 + \frac{N}{x_1}\right] = \frac{1}{2}\left[2.85 + \frac{8}{2.85}\right] = 2.8285$$

$$x_3 = \frac{1}{2}\left[x_2 + \frac{N}{x_2}\right] = \frac{1}{2}\left[2.8285 + \frac{8}{2.8285}\right] = 2.8284271$$

$$x_4 = \frac{1}{2}\left[x_3 + \frac{N}{x_3}\right] = \frac{1}{2}\left[2.8284271 + \frac{8}{2.8284271}\right] = 2.8284271$$

$$\therefore \quad \sqrt{8} = 2.828427.$$

Example 2.18 *Using Newtons iterative formula establish the iterative formula* $x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]$ *to calculate the cube root of N.*

Solution Let
$$x = \sqrt[3]{N} \Rightarrow x^3 = N$$

$$\Rightarrow \quad x^3 - N = 0$$

we have $$f(x) = x^3 - N$$

such that $$f'(x) = 3x^2$$

By Newton–Raphson rule if x_n denotes the nth iterate then

$$x_{n+1} = x_n - \left[\frac{f(x_n)}{f'(x_n)}\right]$$

$$\Rightarrow \quad x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$= \frac{2x_n^3 + N}{3x_n^2} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right], \quad n = 0, 1, 2, \ldots$$

The iterative formula for the cube root of N is

$$x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right], \quad n = 0, 1, 2, \ldots$$

Example 2.19 *Find the cube root of 12 applying the Newton–Raphson formula twice.*

Solution Clearly $$8 < 12 < 27$$

$$\Rightarrow \quad 8^{1/3} < 12^{1/3} < 27^{1/3}$$

$$\Rightarrow \quad 2 < \sqrt[3]{12} < 3$$

Let $$x_0 = \frac{2+3}{2} = 2.5$$

$\therefore$ we have $N = 12$, $x_0 = 2.5$.

By Newton–Raphson's formula

$$x_1 = \frac{1}{3}\left[2(2.5) + \frac{12}{(2.5)^2}\right] = \frac{1}{3}\left[5 + \frac{12}{6.25}\right] = 2.3066$$

and $$x_2 = \frac{1}{3}\left[2(2.3066) + \frac{12}{(2.3066)^2}\right] = 2.2901$$

$$\therefore \quad \sqrt[3]{12} = 2.2901.$$

Exercise 2.5

1. Find the square root of 5.
2. Compute (*a*) $\sqrt{27}$ (*b*) $\sqrt{12}$.

3. From the equation $x^5 - N = 0$, deduce the Newtonian iterative formula $x_{n+1} = \frac{1}{5}\left[4x_n + \frac{N}{x_n^4}\right]$.
4. Show that the iterative formula for finding the reciprocal of N is $x_{n+1} = x_n\,[2 - Nx_n]$ and hence find the value of $\frac{1}{31}$.
5. Evaluate (*a*) $\sqrt[3]{13}$ (*b*) $\sqrt[7]{125}$.

Answers

1. 2.2361 **2.** (*a*) 5.196154 (*b*) 3.46412 **4.** 0.03226 **5.** (*a*) 2.351 (*b*) 1.993

2.5.1 Convergence of Newton's Method

The Newton–Raphson formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \phi(x_n) \text{ (say)}. \tag{13}$$

The general form of (13) is

$$x = \phi(x), \tag{14}$$

we know that the iteration method given by (14) converges if $|\phi'(x)| < 1$.

Here $$\phi(x) = x - \frac{f(x_n)}{f'(x_n)},$$

$$\therefore\ \phi'(x) = 1 - \left[\frac{[f'(x)]^2 - f(x)\,f''(x)}{[f'(x)]^2}\right] = \frac{f(x)\,f''(x)}{[f'(x)]^2}$$

$$|\phi'(x)| = \left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right|,$$

hence Newton's formula converges if $\left|\frac{f(x)\,f''(x)}{[f'(x)]^2}\right| < 1$

i.e., $$|f(x)\,f''(x)| < [f'(x)]^2. \tag{15}$$

If α denotes the actual root of $f(x) = 0$, then we can select a small interval in which $f(x)$, $f'(x)$, and $f''(x)$ are all continuous and the condition (15) is satisfied. Hence Newton's formula always converges provided the initial approximation x_0 is taken very close to the actual root α.

2.5.2 Rate of Convergence of Newton's Method

Let α denote the exact value of the root of $f(x) = 0$, and let x_n, x_{n+1}, be two successive approximations to the actual root α. If ε_n and ε_{n+1} are the corresponding errors, we have

$$x_n = \alpha + \varepsilon_n \text{ and } x_{n+1} = \alpha + \varepsilon_{n+1}$$

by Newton's iterative formula

$$\alpha + \varepsilon_{n+1} = \alpha + \varepsilon_n - \frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\Rightarrow \varepsilon_{n+1} - \varepsilon_n = -\frac{f(\alpha + \varepsilon_n)}{f'(\alpha + \varepsilon_n)}$$

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n - \frac{f(\alpha) + \varepsilon_n f'(\alpha) + \left(\frac{\varepsilon_n^2}{2}\right) f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots}$$

$$= \varepsilon_n - \frac{\varepsilon_n f'(\alpha) + \frac{\varepsilon_n^2}{2} f''(\alpha) + \ldots}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots} \qquad (\because\ f(\alpha) = 0)$$

$$= \varepsilon_n - \frac{\varepsilon_n \left[f'(\alpha) + \frac{\varepsilon_n}{2} f''(\alpha) + \ldots \right]}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots}$$

$$\approx \frac{1}{2} \left[\frac{\varepsilon_n^2 f''(\alpha)}{f'(\alpha) + \varepsilon_n f''(\alpha) + \ldots} \right]$$

$$\approx \frac{1}{2} \left[\frac{\varepsilon_n^2 f''(\alpha)}{f'(\alpha) \left(1 + \varepsilon_n \frac{f''(\alpha)}{f'(\alpha)}\right) + \ldots} \right]$$

$$\Rightarrow \varepsilon_{n+1} \approx \frac{f''(\alpha)}{2 f'(\alpha)}. \tag{16}$$

From (16) it is clear that the error at each stage is proportional to the sequence of the error in the previous stage. Therefore Newton–Raphson method has a quadratic convergence.

Example 2.20 *Obtain the Newton–Raphson's extended formula*

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2} \frac{\{f(x_0)\}^2 \cdot f''(x_0)}{\{f'(x_0)\}^3}$$

for the root of the equation f(x) = 0.

Solution Expanding $f(x)$ by Taylor's series in the neighbourhood of x_0, we get

$$0 = f(x) = f(x_0) + (x - x_0) f'(x_0),$$

$$\therefore \quad x = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

This is the first approximation to the root of $f(x) = 0$,

$$\therefore \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Again by Taylor's series we have

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0),$$

$$f(x_1) = f(x_0) + (x_1 - x_0) f'(x_0) + \frac{1}{2}(x_1 - x_0)^2 f''(x_0),$$

but $f(x_1) = 0$ as x_1 is an approximation to the root.

$$\therefore \quad f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_0) = 0,$$

or
$$f(x_0) + (x - x_0) f'(x_0) + \frac{1}{2}\frac{\{f(x_0)\}^2 \cdot f''(x_0)}{\{f'(x_0)\}^3} = 0,$$

or
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} - \frac{1}{2}\frac{\{f(x_0)\}^2 \cdot f''(x_0)}{\{f'(x_0)\}^3}.$$

Note: The formula can be used iteratively.

2.6 GENERALISED NEWTONS' METHOD FOR MULTIPLE ROOTS

Let α be a root of the equation

$$f(x) = 0$$

which is repeated p times, then

$$x_{n+1} = x_n - p\,\frac{f(x_n)}{f'(x_n)}$$

The above formula is called the generalised Newton's formula for multiple roots. It reduces to Newton–Raphson formula for $p = 1$.

If α is a root of $f(x) = 0$ with multiplicity p, then it is also a root of $f'(x) = 0$, with multiplicity $p - 1$, of $f''(x) = 0$, with multiplicity $(p - 2)$, of $f'''(x) = 0$, with multiplicity $(p - 3)$ and so on. If the initial approximation x_0 is chosen sufficiently close to the root α then the expressions

$$x_0 - p\,\frac{f(x_0)}{f'(x_0)},\; x_0 - (p-1)\frac{f'(x_0)}{f''(x_0)},\; x_0 - (p-2)\frac{f''(x_0)}{f'''(x_0)},\; \ldots$$

will have the same value.

Example 2.21 *Apply Newton-Raphson method with $x_0 = 0.8$ to the equation*

$$f(x) = x^3 - x^2 - x + 1 = 0$$

and verify that the convergtence is only of first order in each case.

Solution. We have

$$f(x) = x^3 - x^2 - x + 1;$$

$$f'(x) = 3x^2 - 2x - 1$$

and

$$x_0 = 0.8$$

using Newton-Raphson formula : we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0.8 - \frac{(0.8)^3 - (0.8)^2 - 0.8 + 1}{3(0.8)^2 - 2(0.8) - 1} = 0.905882$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 0.905882 - \frac{\left((0.905882)^3 - (0.905882)^2 - 0.905882 + 1\right)}{3(0.905882)^2 - 2(0.905882) - 1}$$

$$= 0.9554132$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 0.954132 - \frac{(0.954132)^3 - (0.954132)^2 - (0.954132) + 1}{3(0.954132)^2 - 2(0.954132) - 1}$$

$$= 0.97738$$

Exact root of the given equation is 1, therefore, we get

$$|\epsilon_0| = |\varepsilon_1 - x_0| = |1 - 0.8| = 0.2 = 0.2 \times 10^0$$

$$|\epsilon_1| = |\varepsilon_1 - x_1| = |1 - 0.9054132| = 0.094118 = 0.94 \times 10^{-1}$$

$$|\epsilon_2| = |\varepsilon_1 - x_2| = |1 - 0.954132| = 0.045868 = 0.46 \times 10^{-1}$$

$$|\epsilon_3| = |\varepsilon_1 - x_3| = |1 - 0.977381| = 0.022662 = 0.22 \times 10^{-1}$$

from the above it is clear that; the error at each stage is of first order.

Hence, verified.

Example 2.22 *Find the double root of the equation*

$$x^3 - x^2 - x + 1 = 0$$

Solution Let

$$f(x) = x^3 - x^2 + x + 1$$

then

$$f'(x) = 3x^2 - 2x + 1$$

and $$f''(x) = 6x - 2$$

The actual root $f(x)$ is $x = 1$

Starting with $x_0 = 0.9$, we get

$$f(x_0) = f(0.9) = (0.9)^3 - (0.9)^2 + (0.9) + 1 = 0.019$$

$$f'(x_0) = f'(0.9) = 3\,(0.9)^2 - 2\,(0.9) + 1 = -\,0.37$$

and $$f''(x_0) = f''(0.9) = 6\,(0.9) - 2 = 5.4 - 2 = 3.4$$

$$x_0 - p\cdot\frac{f(x_0)}{f'(x_0)} = 0.9 - 2\cdot\frac{(0.019)}{(-0.37)} = 0.9 + 0.1027$$

$$= 1.0027$$

$$x_0 - (p-1)\,\frac{f'(x_0)}{f''(x_0)} = 0.9 - (2-1)\cdot\frac{(-0.37)}{3.4} = 0.9 + 0.1088$$

$$= 1.0088.$$

The closeness of these values indicate that there is double root near $x = 1$.

For the next approximation we choose $x_1 = 1.01$

$$\therefore\ x_1 - 2\,\frac{f(x_1)}{f'(x_1)} = 1.01 - 2 \times \frac{0.0002}{0.0403} = 1.0001$$

and $$x_1 - (2-1)\frac{f'(x_1)}{f''(x_1)} = 1.01 - \frac{(0.0403)}{4.06} = 1.0001$$

The values obtained are equal. This shows that there is a double root at $x = 1.0001$. Which is close to the actual root unity.

2.6.1 Newton's Method for System of two non-linear Equations

Now we consider the solution of simultaneous non-linear equations by Newton's method.

Consider the system

$$f(x,\ y) = 0$$
$$f(x,\ y) = 0 \tag{1}$$

involving two non-linear equations.

Let (x_0, y_0) be an initial approximation to the root of the system, and $(x_0 + h,\ y_0 + k)$ be the root of the system given by (1). Then we must have

$$f(x_0 + h,\ y_0 + k) = 0$$

and $$g(x_0 + h,\ y_0 + k) = 0 \tag{2}$$

Let us assume that f and g are differentiable expanding (2) by Taylor's series, we obtain

$$f(x_0 + h,\ y_0 + k) = f_0 + h\frac{\partial f}{\partial x_0} + k\,\frac{\partial f}{\partial y_0} + \ldots = 0$$

and $$g(x_0 + h, y_0 + k) = g_0 + h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} + \ldots = 0 \qquad (3)$$

neglecting, the second and higher order terms and retaining only the linear terms of (3), we obtain

$$h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = -f_0$$

and $$h\frac{\partial g}{\partial x_0} + k\frac{\partial g}{\partial y_0} = -g_0 \qquad (4)$$

where $$f_0 = f(x_0, y_0), \frac{\partial f}{\partial x_0} = \left(\frac{\partial f}{\partial x}\right)_{x = x_0}; \frac{\partial f}{\partial y_0} = \left(\frac{\partial f}{\partial y}\right)_{y = y_0}, \text{ etc.}$$

solving (4) for h and k, the next approximation of the root is given by

$$x_1 = x_0 + h$$

and $$y_1 = y_0 + k$$

The above process is repeated to desired degree of accuracy.

Example 2.23 *Solve*

$$x^2 - y^2 = 4$$

and $$x^2 + y^2 = 16$$

By Newton–Raphson Method.

Solution To obtain the initial approximation we replace the first equation by its asymptote $y = x$, which gives

$$2x^2 = 16 \quad \Rightarrow \quad x = 2\sqrt{2}$$

let $x_0 = 2\sqrt{2}$, $y_0 = 2\sqrt{2}$, and (x_0, y_0) be the initial approximation to the root of the system.

We have $$f = x^2 - y^2 - 4 \Rightarrow f_0 = -4$$

and $$g = x^2 + y^2 - 16 \quad \Rightarrow g_0 = 0$$

differentiating partially, we obtain

$$\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial g}{\partial x} = 2x, \frac{\partial g}{\partial y} = 2y$$

so that $$\frac{\partial f}{\partial x_0} = 2x_0 = 4\sqrt{2}, \frac{\partial f}{\partial y_0} = 2y_0 = 4\sqrt{2}$$

$$\frac{\partial g}{\partial x_0} = 2x_0 = 4\sqrt{2}, \frac{\partial g}{\partial y_0} = -2y_0 = -4\sqrt{2}$$

The system of linear equations can be written as

$$h\frac{\partial f}{\partial x_0} + k\frac{\partial f}{\partial y_0} = -f_0 \Rightarrow h\left(4\sqrt{2}\right) - k\left(4\sqrt{2}\right) = -(-4)$$

$$\Rightarrow h - k = 0.7072$$

and $$h\,\frac{\partial g}{\partial x_0} + k\,\frac{\partial g}{\partial y_0} = -\,g_0 \;\Rightarrow\; h\left(4\sqrt{2}\right) + k\left(4\sqrt{2}\right) = 0$$

$$\Rightarrow\; h + k = 0$$

so that $$h - k = 0.7072 \qquad \text{(i)}$$

$$h + k = 0 \qquad \text{(ii)}$$

solving we get $$h = 0.3536,\; k = -\,0.3536$$

The second approximation to the root is given by

$$x_1 = x_0 + h = 2\sqrt{2} + 0.3536 = 3.1820$$

$$y_1 = y_0 + k = 2\sqrt{2} - 0.3536 = 2.4748$$

The process can be repeated.

Example 2.24 *Solve*

$$f(x, y) = x^2 + y - 20x + 40 = 0$$

$$g(x, y) = x + y^2 - 20y + 20 = 0$$

Solution Let $x_0 = 0$, $y_0 = 0$ be the initial approximation to the root

$$f = x^2 + y - 20x + 40 \;\Rightarrow\; \frac{\partial f}{\partial x} = 2x - 20,\; \frac{\partial f}{\partial y} = 1$$

$$g = x + y^2 - y + 20 \;\Rightarrow\; \frac{\partial g}{\partial x} = 1,\; \frac{\partial g}{\partial y} = 2y - 20$$

and $$f_0 = 40,\; g_0 = 20$$

So that $$\frac{\partial f}{\partial x_0} = -\,20,\; \frac{\partial f}{\partial y_0} = 1$$

$$\frac{\partial g}{\partial x_0} = 1,\; \frac{\partial g}{\partial y_0} = -\,20$$

the linear equations are

$$h\,\frac{\partial f}{\partial x_0} + k\,\frac{\partial f}{\partial y_0} = -\,f_0 \;\Rightarrow\; 20h + k = -\,40 \qquad \text{(i)}$$

$$h\,\frac{\partial g}{\partial x_0} + k\,\frac{\partial g}{\partial y_0} = -\,g_0 \;\Rightarrow\; h - 20k = -\,20 \qquad \text{(ii)}$$

Solving, we get

$$h = 2.055,\; k = 1.103$$

The next approximation is given by

$$x_1 = x_0 + h = 2.055$$

$$y_1 = y_0 + k = 1.103.$$

Exercise 2.6

1. Show the Newton's square root formula has a quadratic convergence.

2. Show that the order of convergence of Newton's inverse formula is two.

3. Show that the modified Newton–Raphson's method $x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n)}$ gives a quadratic convergence when the equation $f(x) = 0$ has a pair of double roots in the neighbourhood of $x = x_n$.

4. (*i*) Show that both of the following two sequences have convergence of the second order with the same limit $\sqrt{a}$

$x_{n+1} = \frac{1}{2}x_n\left(1+\frac{a}{x_n^2}\right)$, and $x_{n+1} = \frac{1}{2}x_n\left(3-\frac{x_n^2}{a}\right)$.

(*ii*) If x_n is suitable close approximation to $\sqrt{a}$, show that error in the first formula for x_{n+1} is about one-third that in the second formula, and deduce that the formula $x_{n+1} = \left(\frac{x_n}{8}\right)\left(6+\frac{3a^2}{x_n^2}-\frac{x_n^2}{a}\right)$ given a sequence with third order convergence.

5. Use Newton-Raphson method to find a solution of the following simultaneous equations

$$x^2 + y - 11 = 0$$
$$x + y^2 - 7 = 0$$

given the approximate values of the roots : $x_0 = 3$, $y_0 = -2$. **Ans.** 3.585, –1.8485

6. Solve $x^2 = 3xy - 7$

$y = 2(x + 1)$ **Ans.** –1.9266, –1.8533

7. Solve $x^2 + y = 5$,

$y^2 + x = 3$ **Ans.** $x = 2$, $y = 1$

$x = -1.683$, $y = 2.164$

8. Solve $x = 2(y + 1)$

$y^2 = 3xy - 7$ **Ans.** –1853, –1.927

9. Solve $x = x^2 + y^2$

$y = x^2 - y^2$

Correct to two decimals, starting with the approximation (0.8, 0.4).

Ans. $x = 0.7974$, $y = 0.4006$

10. Solve $\sin xy + x - y = 0$

$y \cos xy + 1 = 0$

with $x_0 = 1$, $y_0 = 2$, by Newton-Raphson method. **Ans.** $x = 1.0828$, $y = 1.9461$

11. Given $x = x^2 + y^2$, $y = x^2 - y^2$. Solve the equations, by using Newton-Raphson method with the initial approximation as (0.8, 0.4). **Ans.** $x = 0.7719$, $y = 0.4196$

12. Solve $x^2 + y = 11$, $y^2 + x = 7$ with $x_0 = 3.5$, $y_0 = -1.8$. **Ans.** $x = 3.5844$, $y = -1.8481$

2.7 REGULA-FALSI METHOD

Consider the equation $f(x) = 0$ and let a, b be two values of x such that $f(a)$ and $f(b)$ are of opposite signs. Also let $a < b$. The graph of $y = f(x)$will meet the x-axis at the same point between a and b. The equation of the chord joining the two points $[a, f(a)]$ and $[b, f(b)]$ is

$$\frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a} \tag{17}$$

In the small interval (a, b) the graph of the function can be considered as a straight line. So that x-coordinate of the point of intersection of the chord joining $[a, f(a)]$ and $[b, f(b)]$ with the x-axis will give an approximate value of the root. So putting $y = 0$ in (17) we get

$$-\frac{f(a)}{x - x_1} = \frac{f(b) - f(a)}{b - a}$$

or
$$x = a - \frac{f(a)}{f(a) - f(a)}(b - a),$$

or
$$x = \frac{af(b) - b(f(a))}{f(b) - f(a)} = x_0 \quad \text{(say).}$$

If $f(a)$ and $f(x_0)$ are of apposite signs then the root lies between a and x_0 otherwise it lies between x_0 and b.

If the root lies between a and x_0 then the next approximation

$$x_1 = \frac{af(x_0) - x_0 f(a)}{f(x_0) - f(a)}$$

otherwise
$$x_1 = \frac{x_0 f(b) - bf(x_0)}{f(b) - f(x_0)}.$$

The above method is applied repeatedly till the desired accuracy is obtained.

The Geometrical interpretation of the method is as follows In Fig. 2.4, the curve $y = f(x)$ between $A(x = a)$ and $B(x = b)$ cuts OX at Q. The chord AB cuts OX at P. It is clear that $x = OQ$ is the actual value of the root whereas $x = OP = x_0$ is the first approximation to the root $f(x_0)$ and $f(a)$ are of opposite signs. So we apply the false position method to the interval (a, x_0) and get OP_1 the next approximation to the root. The procedure is continued till the root is obtained to the desired degree of accuracy. The points of intersection of the successive chords with x-axis, namely P_1, P_2, … tend to coincide with Q the point where the curve $y = f(x)$ cuts the x-axis and so we get successive approximate values of the root of the equation.

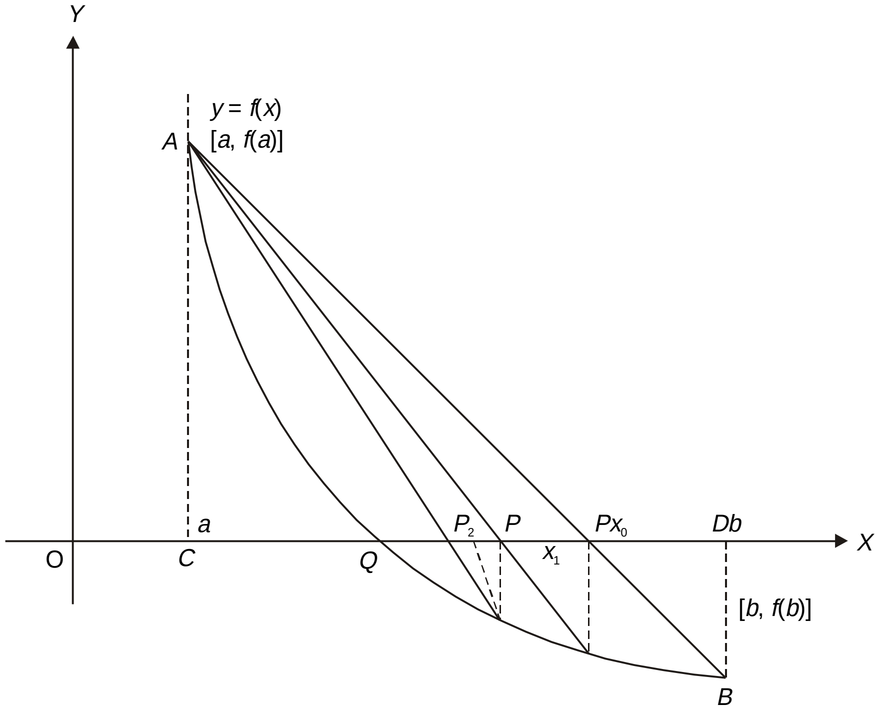


Fig. 2.4

Example 2.25 Find an approximate value of the root of the equation $x^3 + x - 1 = 0$ near $x = 1$, by the method of Falsi using the formula twice.

Solution Here

$$f(x) = x^3 + x - 1$$

$$f(0.5) = -0.375, \quad f(x) = 1$$

Hence the root lies between 0.5 and 1.

We take $a = 0.5$, $b = 1$

$$x_0 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{(0.5)(1) - 1(-0.375)}{1 - (-0.375)} = 0.64.$$

Now

$$f(0.64) = -0.0979 < 0$$

$\therefore$ The root lies between 0.64 and 1,

applying the formula again we get

$$\therefore \ x_1 = \frac{(0.64)(1) - (1)(-0.0979)}{1 - (-0.0979)} = \frac{0.7379}{1.0979} = 0.672.$$

Example 2.26 *Find the real root of the equation $x \log_{10} x - 1.2 = 0$ correct to five decimal places by Regula–Falsi method using the formula four times.*

Solution Here

$$f(x) = x \log_{10} x - 1.2$$

$$f(2) = -0.6, \quad f(3) = 0.23.$$

Thus the root lies between a and 3 and it is nearer to 3.

We take $a = 2$, $b = 3$.

Let x_0 denote the first approximation

$$\therefore \quad x_0 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2(0.23) - 3(-0.6)}{0.23 - (-0.6)} = \frac{2.26}{0.83} = 2.72.$$

Now $\qquad f(2.7) = -0.04$

The root lies between 2.7 and 3.

We note that $\qquad f(2.8) = 0.05$

$$\therefore \quad x_1 = \frac{(2.7)(0.05) - 2.8(-0.04)}{0.05 - (-0.04)} = \frac{0.247}{0.09} = 2.74,$$

since $f(2.74) = -0.0006$

$\therefore$ Root lies between 2.74 and 2.8.

But $\qquad f(2.75) = 0.0081$

$$\therefore \quad x_2 = \frac{(2.84)(0.0081) - (2.75)(-0.0006)}{0.0081 - (-0.0009)} = \frac{0.023844}{0.0087} = 2.7407,$$

since $\qquad f(2.7407) = 0.000045$, the root is < 2.7407,

but $\qquad f(2.7407) = -0.000039$

$$\therefore \quad x_3 = \frac{(2.7406)(0.000045) - (2.7407)(-0.000039)}{0.000045 - (-0.000039)} = \frac{0.0002301}{0.000084}$$

$$= 2.7392$$

is the required value.

Example 2.27 *Solve the equation x tan x = –1, by Regula falsi method starting with 2.5 and 3.0 as the initial approximations to the root.*

Solution. We have

$$f(x) = x \tan x + 1$$

$$f(a) = f(2.5) = 2.5 \tan (2.5) + 1 = -0.8675$$

$$f(b) = f(3) = 3 \tan 3 + 1 = 0.5724$$

By regula falsi method, the first approximation is given by

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$= \frac{3\, f(2.5) - 2.5\, f(3)}{f(2.5) - f(3)} = \frac{3(-0.8675) - 2.5(0.5724)}{-0.8675 - 0.5724}$$

$$= 2.8012$$

Now, $\qquad f(x_1) = f(2.8012) = 2.8012 \tan (2.8012) + 1 = 0.00787$

$$f(2.5) \quad f(2.8012) < 0$$

therefore, the root lies between 2.5 and 2.8012.

The second approximation to the root is given by

$$x_2 = \frac{2.8012\, f(2.5) - 2.5\, f(2.8012)}{f(2.5) - f(2.8012)}$$

$$= \frac{(2.8012)\,(-0.8675) - (2.5)\,(0.00787)}{-0.8675 - 0.00787}$$

$$= 2.7984$$

$$\Rightarrow \quad f(x_2) = f(2.7984) = 2.7984 \tan(2.7984) + 1$$

$$= 0.000039$$

$$\Rightarrow \quad f(2.5)\, f(2.7984) < 0$$

The root lies between 2.5 and 2.79 84

$\therefore$ The third approximation to the root is given by

$$x_3 = \frac{2.7984\; f\,(2.5) - 2.5\; f\,(2.7984)}{f\,(2.5) - f\,(2.7984)}$$

$$= \frac{(2.7984)\,(-0.8675) - (2.5)\,(0.000039)}{-0.8675 - 0.000039}$$

$$= 2.7982$$

The required root is 2.798.

2.8 MULLER'S METHOD

Muller's method is an iterative method. It requires three starting points. $(x_0, f(x_0))$, $(x_1, f(x_1))$ and $(x_2, f(x_2))$. A parabola is constructed that passes through these points then the quadratic formula is used to find a root of the quadratic for the next approximation.

Without loss of generality we assume that x_2 is the best approximation to the root and consider the parabola through the three starting values as shown in Fig. 2.5.

Make the change of variable using the differences

$$t = x - x_2 \tag{18}$$

using the differences $\quad h_0 = x_0 - x_2$ and $h_1 = x_1 - x_2$. (19)

Consider the quadratic polynomial involving t

$$y = at^2 + bt + c \tag{20}$$

each point is used to obtain an equation involving a, b and c

$$\text{at} \quad t = h_0;\ ah_0^2 + bh_0 + c = f_0 \tag{21}$$

$$\text{at} \quad t = h_1;\ ah_1^2 + bh_1 + c = f_1 \tag{22}$$

$$\text{at} \quad t = 0;\ a\,0^2 + b\,0 + c = f_2 \tag{23}$$

from equation (23) we get $c = f_2$. Substituting $c = f_2$ in (21) and (22) and using

$$e_0 = f_0 - c,\ e_1 = f_1 - c,$$

we get $\quad ah_0^2 + bh_0 = f_0 - c = e_0\,,$ (24)

$$ah_1^2 + bh_1 = f_1 - c = e_1 \tag{25}$$

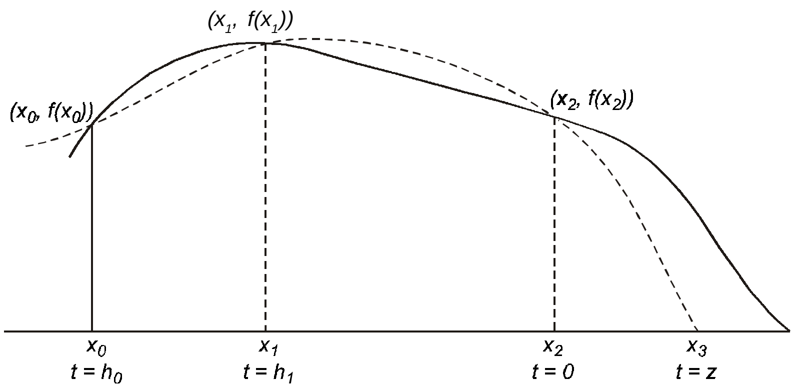


Fig. 2.5

Solving (24) and (25) by using Cramer's rule we get

$$a = \frac{e_0 h_1 - e_1 h_0}{h_0 h_1 [h_0 - h_1]}, \; b = \frac{e_1 h_0^2 - e_0 h_1^2}{h_0 h_1 [h_0 - h_1]}$$

The Quadratic formula is used to find the roots of $t = z_1$, z_2 of (20)

$$z = \frac{-2c}{b \pm \left(b^2 - 4ac\right)^{1/2}} \tag{26}$$

The formula (26) is equivalent to the standard formula for the roots of a quadratic equation since $c = f_2$ is known to give better results.

Note: If $b > 0$, we use the positive sign with the square root and if $b < 0$, we use the negative sign. We choose the root of (26) that has the smallest absolute value. x_3 is given by

$$x_3 = x_2 + z \tag{27}$$

(*see* Fig. 2.5). For the next iteration choose x_0 and x_1 to be the two values selected from (x_0, x_1, x_2) that lie closest to x_3 then replace x_2 with x_3.

Example 2.28 Find the root of the equation

$$f(x) = x^3 - 3x - 5 = 0,$$

which lies between 2 and 3 by using Muller's Method.

Solution We choose $x_0 = 1,\ x_1 = 2,\ x_2 = 3$

$\therefore$ we obtain

$$f_0 = x_0^3 - 3x_0 - 5 = 1 - 3 - 5 = -7$$

$$f_1 = x_1^3 - 3x_1 - 5 = 8 - 6 - 5 = -3$$

and

$$f_2 = x_2^3 - 3x_2 - 5 = 27 - 9 - 5 = 13$$

Let $h_0 = x_0 - x_2 = 1 - 3 = -2$ and $h_1 = x_1 - x_2 = 2 - 3 = -1$

Consider the quadratic polynomial $y = at^2 + bt + c$

$$\Rightarrow \qquad a(-2)^2 + b(-2) + c = -7$$

$$\Rightarrow \qquad 4a - 2b + c = -7 \qquad (i)$$

at $\qquad t = h_1,\ ah_1^2 + bh_1 + c_2 = f_1$

$$\Rightarrow \qquad a(-1)^2 + b(-1) + c = -3$$

$$\Rightarrow \qquad a - b + c = -3 \qquad (ii)$$

at $\qquad t = 0,\ a \times 0 + b \times 0 + c = f_2$

$$\Rightarrow \qquad c = f_2 = 13 \qquad (iii)$$

from (*i*), (*ii*) and (*iii*) we get

$$4a - 2b = -20 \qquad (iv)$$

$$a - b = -16 \qquad (v)$$

Solving (*iv*) and (*v*) we get

$$a = 6 \text{ and } b = 22$$

The quadratic polynomial is

$$y = at^2 + bt + c = 6t^2 + 22b + 13$$

We obtain

$$z = \frac{-2c}{b \pm \sqrt{b^2 - 4ac}} = \frac{-2 \times 13}{22 \pm \sqrt{484 - 312}}$$

$$= \frac{-26}{22 \pm 13.1148}$$

Since $b > 0$, we use positive (+ve) sign for the square root and obtain

$$z = \frac{-26}{35.1148} = -0.7404$$

$\therefore \qquad x_3 = x_2 + z = 3 - 0.7404 = 2.2596$, is the next approximation.

$\therefore$ 2.2596 is the required root of the given equation.

Example 2.29 *Solve* $x^3 - \frac{1}{2} = 0,\ x_0 = 0,\ x_1 = 1,\ x_2 = \frac{1}{2}$ *by using Muller's Method.*

Solution We have $\qquad f(x) = x^3 - \frac{1}{2},\ x_0 = 0,\ x_1 = 1,\ x_2 = \frac{1}{2}$

$$\therefore\ f_0 = 0^3 - \frac{1}{2} = -\frac{1}{2}$$

$$f_1 = 1^3 - \frac{1}{2} = \frac{1}{2}$$

$$f_2 = \left(\frac{1}{2}\right)^3 - \frac{1}{2} = -\frac{3}{8}$$

Also $\qquad h_0 = x_0 - x_2 = 0 - \frac{1}{2} = -\frac{1}{2}$

$$h_1 = x_1 - x_2 = 1 - \frac{1}{2} = \frac{1}{2}$$

Consider the polynomial $at^2 + bt + c$

at $$t = h_0,\quad ah_0^2 + bh_0 + c = f_0$$

$$\Rightarrow a\left(-\frac{1}{2}\right)^2 + b\left(-\frac{1}{2}\right) + c = -\frac{1}{2}$$

$$\Rightarrow \frac{a}{4} - \frac{b}{2} + c = -\frac{1}{2} \tag{i}$$

at $$t = h_1,\quad ah_1^2 + bh_1 + c = f_1$$

$$\Rightarrow a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + c = \frac{1}{2}$$

$$\Rightarrow \frac{a}{4} + \frac{b}{2} + c = \frac{1}{2} \tag{ii}$$

at $$t = 0,\quad a(0)^2 + b(0) + c = f_2$$

$$\Rightarrow c = -\frac{3}{8} \tag{iii}$$

Putting the value of c in (*i*) and (*ii*), we obtain

$$\frac{a}{4} - \frac{b}{2} - \frac{3}{8} = -\frac{1}{2} \Rightarrow 2a - 4b = -1 \tag{iv}$$

$$\frac{a}{4} + \frac{b}{2} - \frac{3}{8} = \frac{1}{2} \Rightarrow 2a + 4b = 7 \tag{v}$$

Solving (*iv*) and (*v*), we get

$$a = 1.5,\ b = 1$$

$$\therefore\ at^2 + bt + c = 1.5 \cdot t^2 + t - \frac{3}{8} = \frac{1}{8}\left[12t^2 + 8t - 3\right]$$

the required quadratic polynomial involving t is taken as $12t^2 + 8t - 3$

We have $a = 12$, $b = 8$, $c = -3$

and $$z = \frac{-2c}{b \pm \sqrt{b^2 + 4ac}} = \frac{-2(-3)}{8 \pm \sqrt{64 + 144}}$$

$$= \frac{6}{8 \pm \sqrt{208}} = \frac{6}{8 \pm 14.22205}$$

Since $b > 0$, we take positive sign with the square root

$$\therefore\ z = \frac{6}{8 + 14.22205} = \frac{6}{22.42205} = 0.26759$$

The root of the equation is

$$\therefore\ x_3 = x_2 + z = 0.5 + 0.26759$$

$$= 0.76759.$$

Exercise 2.7

1. Compute the root of the equation $x^2 + 4x + 4 = 0$, by Regula–Falsi method, correct to three significant figures.
2. Compute the root of the equation $x^3 - 4x - 9 = 0$, by Regula–Falsi method, correct to two decimal places.
3. Compute the root of the equation $x = \log 2(x + 1)$, by Regula–Falsi method, correct to four significant figures.
4. Compute the root of the equation $x^3 - x - 1= 0$ by Regula–Falsi method, correct to four decimal places.
5. Compute the root of the equation $\log x = \cos x$, by Regula–Falsi method, correct to three decimal places.
6. Compute the root of the equation $x \log x = 1$, by Regula–Falsi method, correct to three significant figures.
7. Compute the root of the equation $\sin x + \cos x = 1$, by Regula–Falsi method, correct to four significant figures.
8. Compute the root of the equation, $3x^2 + 5x - 40 = 0$, by Regula–Falsi method, correct to four significant figures.
9. Find the root of

   $f(x) = x - \exp\left(\dfrac{1}{x}\right) = 0$, with a root near $x = 1.5$, using Muller's method.
10. Perform 2 iterations and find the root of

    $\log_{10} x - x + 3 = 0, x_0 = \dfrac{1}{4}, x_1 = \dfrac{1}{2}, x_2 = 1$ by using Muller's Method.
11. Solve $f(x) = 2x^3 - 3x^2 + 2x - 3$ with $x_0 = 0_1$ $x_1 = 1$, $x_2 = 2$, using Muller's method.
12. Find the root of the equation $x^3 - 2x - 5 = 0$, by regula-falsi method. When it is given that the root lies between 2 and 3.
13. Apply Muller's method to find the root of the equation $\cos x = xe^x$ which lies between 0 and 1 taking the initial approximations as -1, 0 and 1.

Answers

1. –1.92	2. 2.71	3. 1.490
4. 1.323	5. 1.303	6. 1.763
7. 1.571	8. 2.138	9. 1.76649
10. 3.20056	11. 1.615.	12. 2.094
13. 0.5177		

3

FINITE DIFFERENCES

3.1 INTRODUCTION

Numerical Analysis is a branch of mathematics which leads to approximate solution by repeated application of four basic operations of Algebra. The knowledge of finite differences is essential for the study of Numerical Analysis. In this section we introduce few basic operators.

3.2 FORWARD DIFFERENCE OPERATOR

Let $y = f(x)$ be any function given by the values $y_0, y_1, y_2, \ldots, y_n$, which it takes for the equidistant values $x_0, x_1, x_2, \ldots, x_n$, of the independent variable x, then $y_1 - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *first differences* of the function y. They are denoted by $\Delta y_0, \Delta y_1, \ldots$, etc.

$\therefore$ We have

$$\begin{aligned}\Delta y_0 &= y_1 - y_0\\ \Delta y_1 &= y_2 - y_1\\ &\ldots\\ \Delta y_n &= y_n - y_{n-1}\end{aligned}$$

The symbol Δ is called the *difference operator*. The differences of the first differences denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots, \Delta^2 y_n$ are called *second differences*, where

$$\begin{aligned}\Delta^2 y_0 &= \Delta[\Delta y_0]\\ &= \Delta[y_1 - y_0]\\ &= \Delta[y_1] - \Delta[y_0]\\ &= \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0)\\ &= y_2 - 2y_1 + y_0\\ &\ldots\\ \Delta^2 y_1 &= \Delta[\Delta y_1]\\ &= y_3 - 2y_2 + y_1\\ &\ldots\end{aligned}$$

Δ^2 is called the *second difference operator*.

Similarly
$$\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$$
$$= y_3 - 3y_2 + 3y_1 - y_0$$
$$\dots$$
$$\Delta^r y_n = \Delta^{r-1} y_{n+1} - \Delta^{r-1} y_n$$
$$= y_{n+r} - \frac{r}{1!} y_{n+r-1} + \frac{r(r-1)}{2!} y_{n+r-2} + \dots + (-1)^r y_n$$
$$\therefore \ \Delta^r y_n = \Delta^{r-1} y_{n+1} - \Delta^{r-1} y_n$$
$$= y_{n+r} - \frac{r}{1!} y_{n+r-1} + \frac{r(n-1)}{2!} y_{n+r-2} + \dots + (-1)^r y_n$$

3.2.1 Difference Table

It is a convenient method for displaying the successive differences of a function. The following table is an example to show how the differences are formed.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
x_0	y_0					
		Δy_0				
x_1	y_1		$\Delta^2 y_0$			
		Δy_1		$\Delta^3 y_0$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The above table is called a *diagonal difference table*. The first term in the table is y_0. It is called the *leading term*.

The differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \dots,$ are called the *leading differences*. The differences $\Delta^n y_n$ with a fixed subscript are called *forward differences*. In forming such a difference table care must be taken to maintain correct sign.

A convenient check may be obtained by noting the sum of the entries in any column equals the differences between the first and the last entries in preceding column.

Another type of difference table called *horizontal difference table* which is more compact and convenient is not discussed here as it is beyond the scope of this book.

3.2.2 Alternative Notation

Let the functions $y = f(x)$ be given at equal spaces of the independent variable x, say at $x = a$, $a + h$, $a + 2h$, …, etc., and the corresponding values of $f(a)$, $f(a + h)$, $f(a + 2h)$, …, etc.

The independent variable x is often called the *argument* and the corresponding value of the dependent variable is of the function at $x = a$, and is denoted by $\Delta f(a)$.

Thus we have $$\Delta f(a) = f(a + h) - f(a),$$

writing the above definition we can write

$$\Delta f(a + h) = f(a + h + h) - f(a + h) = f(a + 2h) - f(a + h)$$

Similarly
$$\begin{aligned}
\Delta^2 f(a) &= \Delta[\Delta f(a)] \\
&= \Delta[f(a + h) - f(a)] \\
&= \Delta f(a + h) - \Delta f(a) \\
&= f(a + 2h) - f(a + h) - [f(a + h) - f(a)] \\
&= f(a + 2h) - 2f(a + h) + f(a),
\end{aligned}$$

Δ^2 is called the *second difference* of $f(x)$ at $x = a$.

Note: The operator Δ is called *forward difference operator* and in general it is defined as

$$\Delta f(x) = f(x + h) - f(x),$$

where h is called the *interval of differencing.* Using the above definition we can write

$$\begin{aligned}
\Delta^2 f(x) &= \Delta[\Delta f(x)], \\
&= \Delta[f(x + h) - f(x)] \\
&= \Delta f(x + h) - f(x) \\
&= f(x + 2h) - f(x + h) - [f(x + h) - f(x)] \\
&= f(x + 2h) - 2f(x + h) - f(x).
\end{aligned}$$

Similarly we can write the other higher order differences as $\Delta^3, \Delta^4, \ldots,$ etc., and $\Delta, \Delta^2, \Delta^3, \ldots, \Delta^n, \ldots,$ etc., are called the *forward differences.*

The difference table called the forward difference table in the new notation is given below.

x	$f(x)$			
		$\Delta f(x)$		
$x+h$	$f(x+h)$		$\Delta^2 f(x)$	
		$\Delta f(x+h)$		$\Delta^3 f(x)$
$x+2h$	$f(x+2h)$		$\Delta^2 f(x+h)$	
		$\Delta f(x+2h)$		
$x+3h$	$f(x+3h)$			

3.2.3 Properties of the Operator Δ

1. *If c is a constant then* $\Delta c = 0$.

Proof Let $f(x) = c$

$\therefore$ $f(x + h) = c$,

(where h is the interval of differencing)

$$\therefore \; \Delta f(x) = f(x+h) - f(x) = c - c = 0$$

$$\Rightarrow \; \Delta c = 0$$

2. Δ *is distributive, i.e.,* $\Delta\left[f(x) \pm g(x)\right] = \Delta f(x) \pm \Delta g(x)$.

Proof $\Delta\left[f(x) + g(x)\right] = \left[f(x+h) + g(x+h)\right] - \left[f(x) + g(x)\right]$

$$= f(x+h) - f(x) + g(x+h) - g(x)$$

$$= \Delta f(x) + \Delta g(x).$$

Similarly we can show that $\Delta\left[f(x) - g(x)\right]$

$$= \Delta f(x) - \Delta g(x)$$

3. *If c is a constant then* $\Delta\left[cf(x)\right] = c\Delta f(x)$.

Proof
$$\Delta\left[cf(x)\right] = cf(x+h) - cf(x)$$

$$= c\left[f(x+h) - f(x)\right] = c\Delta f(x)$$

$$\therefore \; \Delta\left[cf(x)\right] = c\Delta f(x).$$

4. *If m and n are positive integers then* $\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x)$.

Proof $\Delta^m \Delta^n f(x) = (\Delta \times \Delta \times \Delta \ldots m \text{ times})\,(\Delta \times \Delta \ldots n \text{ times})\; f(x)$

$$= (\Delta\,\Delta\,\Delta \ldots (m+n) \text{ times})\; f(x)$$

$$= \Delta^{m+n} f(x).$$

Similarly we can prove the following

5. $$\Delta\left[f_1(x) + f_2(x) + \ldots + f_n(x)\right] = \Delta f_1(x) + \Delta f_2(x) + \ldots + \Delta f_n(x).$$

6. $$\Delta\left[f(x)\; g(x)\right] = f(x)\; \Delta g(x) + g(x)\; \Delta f(x).$$

7. $$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x)\; g(x+h)}.$$

Note:

1. From the properties (2) and (3) it is clear that Δ is a linear operator.
2. If n is a positive integer $\Delta^n.[\Delta^{-n} f(x)] = f(x)] = f(x)$ and in particular when $n = 1$, $\Delta^n.[\Delta^{-1} f(x)] = f(x)] = f(x)$.

Example 3.1 *Find (a) Δe^{ax} (b) $\Delta^2 e^x$ (c) Δ sin x (d) Δ log x (e) Δ tan^{-1} x.*

Solution

(*a*)
$$\Delta e^{ax} = e^{a(x+h)} - e^{ax}$$

$$= e^{ax+ah} - e^{ax} = e^{ax}\left(e^{ah} - 1\right)$$

$$\Delta e^{ax} = e^{ax}\left(e^{ah} - 1\right).$$

(*b*)
$$\Delta^2 e^x = \Delta\left[\Delta e^x\right] = \Delta\left[e^{x+h} - e^x\right]$$

$$= \Delta\left[e^x\left(e^h - 1\right)\right] = \left(e^h - 1\right)\Delta e^x$$

$$= \left(e^h - 1\right)\left(e^{x+h} - e^x\right) = \left(e^h - 1\right)^2 e^x$$

$$\therefore\; \Delta^2 e^x = \left(e^h - 1\right)^2 e^x.$$

(*c*)
$$\Delta \sin x = \sin(x+h) - \sin x$$

$$= 2\cos\left(\frac{x+h+x}{2}\right)\sin\left(\frac{x+h-x}{2}\right)$$

$$= 2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}$$

$$\therefore\; \Delta \sin x = 2\cos\left(x + \frac{h}{2}\right)\sin\frac{h}{2}.$$

(*d*)
$$\Delta \log x = \log(x+h) - \log x$$
$$= \log \frac{x+h}{x} \quad = \log\left[1 + \frac{h}{x}\right]$$
$$\therefore \quad \Delta \log x = \log\left[1 + \frac{h}{x}\right].$$

(e)
$$\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$$
$$= \tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right] \quad = \tan^{-1}\left[\frac{h}{1+hx+x^2}\right].$$

Example 3.2 *Construct a forward difference table for the following data*

x	*0*	*10*	*20*	*30*
y	*0*	*0.174*	*0.347*	*0.518*

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		–0.001	
		0.173		–0.001
20	0.347		–0.002	
		0.171		
30	0.518			

Example 3.3 *Construct a difference table for* $y = f(x) = x^3 + 2x + 1$ *for* $x = 1, 2, 3, 4, 5$.

Solution

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4			
		9		
2	13		12	
		21		6
3	34		18	
		39		6
4	73		24	
		63		
5	136			

Theorem 3.1 *The nth differences of a polynomial of the nth degree are constant when the values of independent variable are at equal intervals.*

Proof Let the polynomial be

$$f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n,$$

where $a_0, a_1, a_2, \dots, a_n$ are constants and $a_0 \neq 0$.

$$\therefore \quad f(x+h) = a_0(x+h)^n + a_1(x+h)^{n-1} + \dots + a_{n-1}(x+h) + a_n,$$

where h is the interval of differencing.

$$\begin{aligned}
\therefore \Delta f(x) &= f(x+h) - f(x) \\
&= a_0(x+h)^n + a_1(x+h)^{n-1} + \dots + a_n(x+h) + a_n - a_0x^n - a_1x^{n-1} - \dots a_{n-1}x - a_n \\
&= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \dots + a_{n-1}[x+h-x] \\
&= a_0[x^n + {}^nc_1x^{n-1}h + {}^nc_2x^{n-2}h^2 + \dots + h^n - x^n] + a_1[x^{n-1} + {}^{n-1}c_1x^{n-2}h + \dots \\
&\quad + h^{n-1} - x^{n-1}] + \dots + a_{n-1}h \\
&= a_0nhx^{n-1} + [a_0{}^nc_2h^2 + a_1h(n-1)]x^{n-2} + \dots + a_{n-1}h \\
&= a_0nhx^{n-1} + b_2x^{n-2} + b_3x^{n-3} + \dots + b_{n-1}x + b_m \qquad (1)
\end{aligned}$$

where $b_2, b_3, \dots, b_m$ are constants.

From (1) it is clear that the first difference of $f(x)$ is a polynomial of $(n-1)$th degree.

Similarly

$$\begin{aligned}
\Delta^2 f(x) &= \Delta[\Delta f(x)] \\
&= \Delta[f(x+h) - f(x)] \\
&= \Delta f(x+h) - \Delta f(x) \\
&= a_0nh\left[(x+h)^{n-1} - x^{n-1}\right] + b_2\left[(x+h)^{n-2} - x^{n-2}\right] + \dots + \\
&\quad b_{n-1}[x+h-x] \\
&= a_0n(n-1)h^2x^{n-2} + c_3a^{n-3} + c_4x^{n-4} + \dots + c_{n-1}x + c_l,
\end{aligned}$$

where $c_3, c_4, \dots, c_{l-1}$ are constants.

Therefore the second differences of $f(x)$ reduces to a polynomial of $(n-2)$th degree. Proceeding as above and differencing for n times we get

$$\Delta^n f(x) = a_0n(n-1)\dots 3 \times 2 \times 1h^x x^{n-n} = a_0n!h^n,$$

which is a constant.

and

$$\begin{aligned}
\Delta^{n+1} f(x) &= \Delta\left[\Delta^n f(x)\right] \\
&= a_nn!h^n - a_0n!h^n = 0
\end{aligned}$$

which completes the proof of the theorem.

Note: The converse of the above theorem is true, i.e., if the nth differences of a tabulated function and the values of the independent variable are equally spaced then the function is a polynomial of degree n.

Example 3.4 *By constructing a difference table and taking the second order differences as constant find the sixth term of the series 8, 12, 19, 29, 42, ….*

Solution Let K be the sixth term of the series. The difference table is

x	y	Δ	Δ^2
1	8		
		4	
2	12		3
		7	
3	19		3
		10	
4	29		3
		13	
5	42		$K - 55$
		$K - 4$	
6	K		

The second differences are constant.

$\therefore \quad K - 55 = 3$

$\Rightarrow \quad K = 58.$

The sixth term of the series is 58.

Example 3.5 *Find (a) Δ^{10} $(1 - ax)(1 - bx^2)(1 - cx^3)(1 - dx^4)$ (b) Δ^3 $(1 - x)(1 - 2x)(1 - 3x)$*

Solution (*a*) Let

$$f(x) = (1 - ax)(a - bx^2)(1 - cx^3)(1 - dx^4)$$

$f(x)$ is a polynomial of degree 10 and the coefficient of x^{10} is *abcd*,

$$\Delta^{10} f(x) = \Delta^{10}\left(abcd\ x^{10}\right)$$

$$= abcd\ \Delta^{10} x^{10}$$

$$= abcd\ 10!.$$

(*b*) Let

$$f(x) = (1 - x)(1 - 2x)(1 - 3x)$$
$$= -6x^3 + 11x^2 - 6x + 1$$

$f(x)$ is a polynomial of degree 3 and the coefficient of x^3 is (-6).

$$\therefore\ \Delta^3 f(x) = (-6)3! = -36.$$

Example 3.6 *Evaluate (a)* $\Delta\left[\dfrac{5x+12}{x^2+5x+6}\right]$ *(b)* $\Delta^n\left(\dfrac{1}{x}\right)$*, taking 1 as the interval of differencing.*

Solution

(*a*) $\Delta\left[\dfrac{5x+12}{x^2+5x+6}\right]$

$$= \Delta\left[\frac{2(x+3)+3(x+2)}{(x+2)(x+3)}\right] = \Delta\left[\frac{2}{x+2}+\frac{3}{x+3}\right]$$

$$= \Delta\left[\frac{2}{x+2}\right] + \Delta\left[\frac{3}{x+3}\right]$$

$$= \left[\frac{2}{x+1+2} - \frac{2}{x+2}\right] + \left[\frac{3}{x+1+3} - \frac{3}{x+3}\right]$$

$$= \frac{-2}{(x+2)(x+3)} - \frac{3}{(x+3)(x+4)}.$$

(*b*) $$\Delta^n\left[\frac{1}{x}\right] = \Delta^{n-1}\Delta\left[\frac{1}{x}\right]$$

Now $$\Delta\left(\frac{1}{x}\right) = \frac{1}{x+1} - \frac{1}{x} = \frac{-1}{x(x+1)},$$

Similarly $$\Delta^2\left(\frac{1}{x}\right) = \frac{(-1)^2}{x(x+1)(x+2)}$$

and so on.

Proceeding as above we get

$$\Delta^n\left[\frac{1}{x}\right] = \frac{(-1)^n}{x(x+1)(x+2)\ldots(x+n)}.$$

Example 3.7 *Show that* $\Delta^n \sin(ax+b) = \left(2\sin\frac{a}{2}\right)^n \sin\left[ax+b+n\left(\frac{a+\pi}{2}\right)\right]$, *1 being the interval of differencing.*

Solution $$\Delta\sin(ax+b) = \sin[a(x+1)+b] - \sin(ax+b)$$

$$= 2\left[\cos\left(ax+b+\frac{a}{2}\right)\right]\sin\left(\frac{a}{2}\right)$$

$$= 2\sin\frac{a}{2}\sin\left(\frac{\pi}{2}+\left(ax+b+\frac{a}{2}\right)\right)$$

$$= 2\sin\frac{a}{2}\sin\left[(ax+b)+\frac{a+\pi}{2}\right],$$

$$\Delta^2\sin(ax+b) = 2\sin\frac{a}{2}\sin\left(a(x+1)+b+\frac{a+\pi}{2}\right) - 2\sin\frac{a}{2}\sin\left((ax+b)+\frac{a+\pi}{2}\right)$$

$$= \left(2\sin\frac{a}{2}\right)2\cos\left(ax+b+\frac{2a+\pi}{2}\right)\sin\frac{a}{2}$$

$$= \left(2\sin\frac{a}{2}\right)^2 \sin\left(\frac{\pi}{2} + ax + b + \frac{2a+\pi}{2}\right)$$

$$\therefore\ \Delta^2 \sin(ax+b) = \left(2\sin\frac{a}{2}\right)^2 \sin\left(ax + b + 2\left(\frac{a+\pi}{2}\right)\right).$$

Proceeding as above and applying the principle of mathematical induction, we get

$$\Delta^n \sin(ax+b) = \left(2\sin\frac{a}{2}\right)^n \sin\left(ax + b + n\left(\frac{a+\pi}{2}\right)\right).$$

3.3 THE OPERATOR *E*

Let $y = f(x)$ be function of x and x, $x + h$, $x + 2h$, $x + 3h$, …, etc., be the consecutive values of x, then the operator E is defined as

$$Ef(x) = f(x + h),$$

E is called *shift operator*. It is also called *displacement operator*.

Note: E is only a symbol but not an algebraic sum.

$E^2f(x)$ means the operator E is applied twice on $f(x)$, i.e.,

$$\begin{aligned} E^2f(x) &= E[Ef(x)] \\ &= Ef(x+h) \\ &= f(x+2h) \\ &\ldots \end{aligned}$$

Similarly $\quad E^nf(x) = f(x + nh)$

and $\quad E^{-n}f(x) = f(x - nh)$.

The operator E has the following properties:

1. $E(f_1(x) + f_2(x) + \ldots + f_n(x)) = Ef_1(x) + Ef_2(x) + \ldots + Ef_n(x)$
2. $E(cf(x)) = cEf(x)$ (where c is constant)
3. $E^m(E^nf(x)) = E^n(E^mf(x)) = E^{m+n}f(x)$ where m, n are positive integers
4. If n is positive integer $E^n[E^{-n}f(x)] = f(x)$

Alternative notation If y_0, y_1, y_2, …, y_n, …, etc., are consecutive values of the function $y = f(x)$ corresponding to equally spaced values x_0, x_1, x_2, …, x_n, etc., of x then in alternative notation

$$\begin{aligned} E\,y_0 &= y_1 \\ E\,y_1 &= y_2 \\ &\ldots \\ E^2y_0 &= y_2 \\ &\ldots \end{aligned}$$

and in general $\quad E^ny_0 = y_n$.

Theorem 3.2 *If n is a positive integer then* $y_n = y_0 + {}^nc_1\ \Delta y_0 + {}^nc_2\ \Delta^2 y_0 + \ldots + \Delta^n y_0$.

Proof From the definition

$$y_1 = Ey_0 = (1+\Delta)y_0 = y_0 + \Delta y_0$$

$$y_2 = E^2 y_0 = (1+\Delta)^2 y_0 = \left(1 + {}^2c_1\,\Delta + \Delta^2\right) y_0$$

$$= y_0 + {}^2c_1\,\Delta y_0 + \Delta^2 y_0$$

$$\cdots$$

Similarly we get
$$y_n = E^n y_0 = (1+\Delta)^n y_0$$

$$= \left(1 + {}^nc_1\,\Delta + \ldots + \Delta^n\right) y_0$$

$$= y_0 + {}^nc_1\,\Delta y_0 + \ldots + \Delta^n y_0,$$

hence proved.

3.3.1 Relation between the Operator E and Δ

From the definition of Δ, we know that

$$\Delta f(x) = f(x+h) - f(x),$$

where h is the interval of differencing. Using the operator E we can write

$$\Delta f(x) = Ef(x) - f(x)$$

$$\Rightarrow \Delta f(x) = (E-1)\, f(x).$$

The above relation can be expressed as an identity

$$\Delta = E - 1$$

i.e.,
$$E = 1 + \Delta.$$

3.3.2 $E\Delta \equiv \Delta E$

Proof
$$E\Delta f(x) = E\left(f(x+h) - f(x)\right)$$

$$= Ef(x+h) - Ef(x)$$

$$= f(x+2h) - f(x+h)$$

$$= \Delta f(x+h)$$

$$= \Delta E f(x)$$

$$\therefore\ E\Delta \equiv \Delta E.$$

Example 3.8 *Prove that* $\Delta \log f(x) = \log\left[1 + \dfrac{\Delta f(x)}{f(x)}\right].$

Solution Let h be the interval of differencing

$$f(x+h) = E\, f(x) \quad = (\Delta + 1)\, f(x) \quad = \Delta\, f(x) + f(x)$$

$$\Rightarrow \frac{f(x+h)}{f(x)} = \frac{\Delta f(x)}{f(x)} + 1,$$

applying logarithms on both sides we get

$$\log\left[\frac{f(x+h)}{f(x)}\right] = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

$$\Rightarrow \log f(x+h) - \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right]$$

$$\Rightarrow \Delta \log f(x) = \log\left[1 + \frac{\Delta f(x)}{f(x)}\right].$$

Example 3.9 *Evaluate* $\left(\frac{\Delta^2}{E}\right)x^3$.

Solution Let h be the interval of differencing

$$\left(\frac{\Delta^2}{E}\right)x^3 = \left(\Delta^2 E^{-1}\right)x^3$$
$$= (E-1)^2 E^{-1} x^3$$
$$= (E^2 - 2E + 1) E^{-1} x^3$$
$$= (E - 2 + E^{-1}) x^3$$
$$= Ex^3 - 2x^3 + E^{-1} x^3$$
$$= (x+h)^3 - 2x^3 + (x-h)^3$$
$$= 6xh.$$

Note If $h = 1$, then $\left(\frac{\Delta^2}{E}\right)x^3 = 6x$.

Example 3.10 *Prove that* $e^x = \frac{\Delta^2}{E}e^x . \frac{Ee^x}{\Delta^2 e^x}$, *the interval of differencing being h.*

Solution We know that

$$E\, f(x) = f(x+h)$$

$$\therefore\; E\, e^x = e^{x+h},$$

again

$$\Delta e^x = e^{x+h} - e^x = e^x (e^h - 1)$$

$$\Rightarrow \Delta^2 e^x = e^x . \left(e^h - 1\right)^2$$

$$\therefore \left(\frac{\Delta^2}{E}\right)e^x = \left(\Delta^2 E^{-1}\right)e^x = \Delta^2 e^{x-h}$$

$$= e^{-h}\left(\Delta^2 e^x\right) = e^{-h} e^x \left(e^h - 1\right)^2$$

$$\therefore \text{ R.H.S.} = e^{-h} e^x (e^h - 1)\, \frac{e^{x+h}}{e^x (e^h - 1)} = e^x.$$

Example 3.11 *Prove that* $f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$.

Solution

$$f(4) - f(3) = \Delta f(3)$$
$$= \Delta\left[f(2) + \Delta f(2)\right] \left(\text{since } f(3) - f(2) = \Delta f(2)\right)$$
$$= \Delta f(2) + \Delta^2 f(2)$$
$$= \Delta f(2) + \Delta^2\left[f(1) + \Delta f(1)\right]$$
$$= \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1)$$
$$\therefore \quad f(4) = f(3) + \Delta f(2) + \Delta^2 f(1) + \Delta^3 f(1).$$

Example 3.12 Given $u_0 = 1$, $u_1 = 11$, $u_2 = 21$, $u_3 = 28$ and $u_4 = 29$, find $\Delta^4 u_0$.

Solution

$$\begin{aligned}
\Delta^4 u_0 &= (E - 1)^4 u_0 \\
&= (E^4 - {}^4c_1 E^3 + {}^4c_2\, E^2 - {}^4c_3\, E + 1)\, u_0 \\
&= E^4 y_0 - 4E^3 u_0 + 6\, E^2\, u_0 - 4Eu_0 + u_0 \\
&= u_4 - 4u_3 + 6u_2 - 4u_1 + u_0 \\
&= 29 - 112 + 126 - 44 + 1 \\
&= 0.
\end{aligned}$$

Example 3.13 Given $u_0 = 3$, $u_1 = 12$, $u_2 = 81$, $u_3 = 200$, $u_4 = 100$, and $u_5 = 8$, find $\Delta^5 u_0$.

Solution

$$\begin{aligned}
\Delta^5 u_0 &= (E - 1)^5 u_0 \\
&= (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)u_0 \\
&= u_5 - 5u_4 + 10u_3 - 10u_2 + 5u_1 - u_0 \\
&= 8 - 500 + 2000 - 810 + 60 - 3 \\
&= 755.
\end{aligned}$$

Example 3.14 Find the first term of the series whose second and subsequent terms are 8, 3, 0, –1, 0, …

Solution Given $f(2) = 8$, $f(3) = 3$, $f(3) = 0$, $f(4) = –1$, $f(5) = 0$, we are to find $f(1)$.

We construct the difference table with the given values.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
2	8				
		–5			
3	3		2		
		–3		0	
4	0		2		0
		–1		0	
5	–1		2		
		1			
6	0				

We have $$\Delta^3 f(x) = \Delta^4 f(x) = \ldots = 0.$$

Using the displacement operator

$$f(1) = E^{-1} f(2) \;\; = (1+\Delta)^{-1} f(2)$$

$$= \left(1 - \Delta + \Delta^2 - \Delta^3 + ...\right) f(2)$$

$$= f(2) - \Delta f(2) + \Delta^2 f(2) - \Delta^3 f(2) + ...$$

$$= 8 - (-5) + 2 \;\; = 15$$

$$\therefore \;\; f(1) = 15.$$

3.4 THE OPERATOR *D*

Dy denotes the differential coefficient of *y* with respect to *x* where $D = \dfrac{d}{dx}$. We have $Dy = \dfrac{dy}{dx}$. The *n*th derivative of *y* with respect to *x* is denoted by $D^n y = \dfrac{d^n y}{dx^n}$.

Relation between the operators Δ, *D* and *E* We know that

$$Df(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x) \;\; \text{etc.}$$

From the definition we have

$$Ef(x) = f(x+h) \qquad \text{(}h\text{ being the interval of differencing)}$$

$$= f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + ...$$

$$= f(x) + \frac{h}{1!} Df(x) + \frac{h^2}{2!} D^2 f(x) + ...$$

(expanding by Taylor's series method)

$$= \left(1 + \frac{h}{1!} D + \frac{h^2}{2!} D^2 + ...\right) f(x)$$

$$= \left(1 + \frac{hD}{1!} + \frac{h^2 D^2}{2!} + ...\right) f(x) = e^{hD} f(x)$$

$$\therefore \;\; Ef(x) = e^{hD} f(x),$$

hence the identity $\qquad E \equiv e^{hD}.$

We have already proved that $E \equiv 1 + \Delta$ and $E \equiv e^{hD}$.

Now consider $E = e^{hD}$. Applying logarithms, we get

$$\Rightarrow \; hD = \log E \quad = \log[1 + \Delta]$$

$$= \Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots$$

$$\Rightarrow \; D = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \ldots\right].$$

3.5 BACKWARD DIFFERENCES

Let $y = f(x)$ be a function given by the values $y_0, y_1, \ldots y_n$ which it takes for the equally spaced values $x_0, x_1, \ldots, x_n$ of the independent variable x. Then $y - y_0, y_2 - y_1, \ldots, y_n - y_{n-1}$ are called the *first backward differences* of $y = f(x)$. They are denoted by $\nabla y_0, \nabla y_1, \ldots, \nabla y_n$, respectively. Thus we have

$$\begin{aligned} y_1 - y_0 &= \nabla y_1 \\ y_2 - y_1 &= \nabla y_2 \\ &\cdots \\ y_n - y_{n-1} &= \nabla y_n, \end{aligned}$$

where ∇ is called the *backward difference operator*.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
x_0	y_0				
		∇y_1			
x_1	y_1		$\nabla^2 y_2$		
		∇y_2		$\nabla^3 y_3$	
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$
		∇y_3		$\nabla^3 y_4$	
x_3	y_3		$\nabla^2 y_4$		
		∇y_4			
x_4	y_4				

Note: In the above table the differences $\nabla^n y$ with a fixed subscript i, lie along the diagonal upward sloping.

Alternative notation Let the function $y = f(x)$ be given at equal spaces of the independent variable x at $x = a, a + h, a + 2h, \ldots$ then we define

$$\nabla f(a) = f(a) - f(a - h)$$

where ∇ is called the *backward difference operator*, h is called the *interval of differencing*.

In general we can define

$$\nabla f(x) = f(x) - f(x-h).$$

We observe that $\nabla f(x+h) = f(x+h) - f(x) = \Delta f(x)$

$$\nabla f(x+2h) = f(x+2h) - f(x+h) = \Delta f(x+h)$$

$$\ldots$$

$$\nabla f(x+nh) = f(x+nh) - f\big(x+(n-1)h\big)$$

$$= \Delta f\big(x+(n-1)h\big).$$

Similarly we get $\nabla^2 f(x+2h) = \nabla\big[\nabla f(x+2h)\big]$

$$= \nabla\big[\Delta f(x+h)\big]$$

$$= \Delta\big[\Delta f(x)\big]$$

$$= \Delta^2 f(x)$$

$$\ldots$$

$$\nabla^n f(x+nh) = \Delta^n f(x).$$

Relation between E and ∇ :

$$\nabla f(x) = f(x) - f(x-h) = f(x) - E^{-1} f(x)$$

$$\Rightarrow \ \nabla = 1 - E^{-1}$$

or

$$\nabla = \frac{E-1}{E}.$$

Example 3.15 *Prove the following (a)* $(1+\Delta)(1-\Delta) = 1$ *(b)* $\Delta\nabla = \Delta - \nabla$ *(c)* $\nabla = E^{-1}\Delta$.

Solution

(*a*) $$(1+\Delta)(1-\nabla) f(x) = E\, E^{-1} f(x)$$

$$= Ef(x-h) = f(x) = 1.\, f(x)$$

$$\therefore\ (1+\Delta)(1-\nabla) \equiv 1.$$

(*b*) $$\nabla\Delta f(x) = (E-1)\big(1-E^{-1}\big) f(x)$$

$$= (E-1)\big[f(x) - f(x-h)\big]$$

$$= Ef(x) - f(x) - Ef(x-h) + f(x-h)$$

$$= f(x+h) - f(x) - f(x) + f(x-h)$$

$$= \left[Ef(x) - f(x)\right] - \left[f(x) - f(x-h)\right]$$

$$= (E-1)\,f(x) - \left(1 - E^{-1}\right) f(x)$$

$$= \left[(E-1) - \left(1 - E^{-1}\right)\right] f(x)$$

$$= (\Delta - \nabla)\, f(x)$$

$$\therefore\ \Delta\nabla f(x) = (\Delta - \nabla) f(x)$$

$$\therefore\ \Delta\nabla = \Delta - \nabla.$$

(*c*) $$\nabla f(x) = \left(1 - E^{-1}\right) f(x) = f(x) - f(x-h)$$

and $$E^{-1}\,\Delta f(x) = E^{-1}\left[f(x+h) - f(x)\right]$$

$$= f(x) - f(x-h)\ \nabla$$

$$\therefore\ \nabla = E^{-1}\,\Delta.$$

3.6 FACTORIAL POLYNOMIAL

A factorial polynomial denoted by x^r is the product of r consecutive factors of which the first factor is x and successive factors are decreased by a constant $h > 0$. Thus

$$x^r = x(x-h)\ (x-2h)\ \ldots\ [x-(r-1)h].$$

When $h = 1$ $$x^r = x(x-1)(x-2)\ldots(x-r+1)$$

and in particular $$x^0 = 1$$

$$x^1 = x$$

$$\Delta x^r = (x+h)^r - x^r$$

$$= (x+h)\,x(x-h)\ldots\left(x+h-(x-1)h\right) - x(x-1)\ldots\ \left(x-(x-1)h\right)$$

$$= rhx^{(r-1)}.$$

In general we can write

$$\Delta^r x^r = r(r-1)\ldots 1 \times h^r \ = h^r r!.$$

Note:

1. $\Delta^{r+1} x^r = 0$
2. If the interval of differencing is unity then the successive differences of x^r, can be obtained by ordinary successive differentiation of x^r.
3. If r is a positive integer then

$$x^{(-r)} = \frac{1}{(x+h)\,(x+2h)\ldots(x+rh)}$$

and if $r = 1$
$$x^{(-r)} = \frac{1}{(x+1)\,(x+2)\ldots(x+r)}.$$

3.6.1 To Express a given Polynomial in Factorial Notation

A polynomial of degree r can be expressed as a fractional polynomial of the same degree.

Let $f(x)$ be a polynomial of degree which is to be expressed in factorial notation and let

$$f(x) = a_0 + a_1 x^1 + a_2 x^2 + \ldots + a_r x^r \tag{2}$$

where a_0, a_1, …, a_r are constants and $a_0 \neq 0$ then

$$\Delta f\ (x) = \Delta \lfloor a_0 + a_1 x^1 + \ldots + a_r x^r \rfloor$$

$$\Rightarrow\ \Delta f(x) = a_1 + 2a_2 x^1 + \ldots + r a_r x^{(r-1)}$$

$$\therefore\ \Delta^2 f(x) = \Delta\left[a_1 + 2a_2 x^1 + \ldots + r a_r x^{(r-1)}\right]$$

$$\Rightarrow\ \Delta^2 f(x) = 2a_2 + 2 \times 3a_3 x^1 + \ldots + r(r-1)\, x^{(r-2)}$$

$$\cdots$$

$$\Delta^r f(x) = a_r r(r-1) \ldots 2 \times 1\ x^{(0)}$$

$$= a_r\ r!.$$

Substituting $x = 0$ in the above we get

$$f(0) = a_0,\ \frac{\Delta f(0)}{1!} = a_1,\ \frac{\Delta^2 f(0)}{2!} = a_2,\ \ldots,\ \frac{\Delta^r f(0)}{r!} = a_r\,.$$

Putting the values of a_0, a_1, a_2, …, a_r in (2) we get

$$f(x) = f(0) + \frac{\Delta f(0)}{1!}\, x^1 + \frac{\Delta^2 f(0)}{2!}\, x^2 + \ldots + \frac{\Delta^r f(0)}{r!}\, x^r\,.$$

Example 3.16 *If m is a positive integer and interval of differencing is 1, prove that*

(a) $\Delta^2 x^{(x)} = m(m-1)\, x^{(m-2)}$ *(b)* $\Delta^2 x^{(-m)} = m(m+1)\, x^{(-m-2)}$

Solution

(a) $x^{(m)} = x(x-1) \ldots \left[x - (m-1)\right]$

$$\Delta x^{(m)} = \left[(x+1)x(x+2)\ldots(x+1-(m-1))\right] - x(x-1)\ldots x-(m-1)$$

$$= mx^{(m-1)}$$

$$\Delta^2 x^{(m)} = \Delta\left[\Delta x^{(m)}\right] = m\Delta x^{(m-1)} = m(m-1)\,x^{m-2}.$$

(*b*) $x^{(-m)} = \dfrac{1}{(x+1)(x+2)\ldots(x+m)}$,

$$\Delta\left[x^{(-m)}\right] = \frac{1}{(x+2)(x+1)\ldots(x+m+1)} - \frac{1}{(x+1)\ldots(x+m)}$$

$$= \frac{1}{(x+2)\ldots(x+m)}\left[\frac{1}{(x+m+1)} - \frac{1}{(x+1)}\right]$$

$$= m\frac{(-1)}{(x+1)(x+2)\ldots(x+m+1)} = (-m)\,x^{(-m-1)}$$

$$\Delta^2\left(x^{(-m)}\right) = (-m)(-m-1)\,x^{(-m-2)} = m(m+1)\,x^{(-m-2)}.$$

3.6.2 Differences of Zero

If n and r are two positive integers and the interval of differencing is 1, then

$$\Delta^n o^r = n^r - {}^nc_1(n-1)^r + {}^nc_2(n-2)^r - \ldots + {}^nc_4(-1)^r.$$

Proof

$$\Delta^n x^r = (E-1)^n x^r$$
$$= [E^n - {}^nc_1E^{n-1} + {}^nc_2E^{n-2} + \ldots + (-1)^n]x^r$$
$$= E^nx^r - {}^nc_1E^{n-1}x^r + {}^nc_2E^{n-2} + \ldots + (-1)\backslash x^r$$
$$= (x+n)^r - {}^nc_1(x+n-1)^r + {}^nc_2(x+n-2)^r + \ldots + {}^nc_1(-1)^{n-1}(x+1)^r + (-1)^nx^r.$$

Substituting $x = 0$, we get

$$\Delta^n o^r = n^r - {}^nc_1\,(n-1)^r + {}^nc_2(n-2)^r + \ldots + {}^nc_n(-1)^r.$$

$$\Delta^3 o^4 = 3^4 - {}^3c_1(3-1)^4 + {}^3c_2(3-2)^4 + {}^3c_3(3-3)^4$$
$$= 81 - {}^3c_1 \times 16 + 3 + 0$$
$$= 36.$$

Note:

1. When $n\,.\,r$, $\Delta^n o^r = 0$
2. $\Delta^n o^n = n!$
3. $\Delta o^r = 1^r = 1$

Example 3.17 *Prove that (a)* $\Delta^2 o^3 = 6$ *(b)* $\Delta^3 o^3 = 6$

Solution

(*a*) $\Delta^2 o^3 = 2^3 - 2.1^3 = 6.$

(*b*) $\Delta^3 o^3 = 3^3 - 3.2^3 + 3.1^3 = 6.$

Example 3.18 *Calculate (a)* $\Delta^3 o^6 = 6$ *(b)* $\Delta^5 o^6$ *(c)* $\Delta^6 o^6$

Solution

(*a*)
$$\Delta^3 o^6 = 3^6 - 3.2^6 + 3.1^6$$
$$= 729 - 192 + 3 = 540.$$

(*b*)
$$\Delta^5 o^6 = 5^6 - 5.4^6 + 10.3^6 - 10.2^6 + 5.1^6$$
$$= 15625 - 20480 + 7290 - 640 + 5 = 1800.$$

(*c*)
$$\Delta^6 o^6 = 6!.$$

Example 3.19 *Express* $f(x) = 3x^3 + x^2 + x + 1$*, in the factorial notation, interval of differencing being unity.*

Solution $f(x)$ is a polynomial of degree 3.

$\therefore$ We can write

$$f(x) = f(0) + \frac{\Delta f(0)}{1!} x^{(1)} + \frac{\Delta^2 f(0)}{2!} x^{(2)} + \frac{\Delta^3 f(0)}{3!} x^{(3)}. \qquad (3)$$

The interval of differencing is unit and finding the values of the function at $x = 0, 1, 2$ and 3, we get

$\therefore$ $f(0) = 1$, $f(1) = 6$, $f(2) = 31$, $f(3) = 94$.

The difference table for the above values is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	1			
		5		
1	6		20	
		25		16
2	31		38	
		63		
3	94			

From the table we have $f(0) = 1, \Delta f(0) = 5, \Delta^2 f(0) = 20, \Delta^3 f(0) = 18.$

Substituting the above values in (3) we get

$$f(x) = 1 + 5x^{(1)} + \frac{20}{2!} x^{(2)} + \frac{18}{3!} x^{(3)},$$

$$\therefore \; f(x) = 3x^{(3)} + 10x^{(2)} + 5x^{(1)} + 1.$$

Alternative method

Let $3x^3 + x^2 + x + 1 = ax^{(3)} + bx^{(2)} + cx^{(1)} + d$

$$= ax(x-1)(x-2) + bx(x-1) + cx + d \qquad (4)$$

Putting $x = 0$ in (4) we get $\quad 1 = d$.

Putting $x = 1$ in (4) we get $\quad c = 5$.

Putting $x = 2$ in (4) we get $\quad 31 = 2b + 11$

$$\Rightarrow \quad b = 10.$$

Comparing the coefficients of x^3 on both sides of (4) we get

$$a = 3$$

$$3x^3 + x^2 + x + 1 = 3x^{(3)} + 10x^{(2)} + 5x^{(1)} + 1.$$

Note:

1. Unless stated the interval of differencing is taken to be unity (i.e., $h = 1$).
2. We can also use another method known as synthetic division to express given polynomial in its factorial notation.

Example 3.20 *Express $3x^3 - 4x^2 + 3x - 11$, in factorial notation.*

Solution Here we apply the method of synthetic division as follows:

Omit the coefficients of x^3, x^2, x the signs of constant terms in $(x - 1)$, $(x - 2)$, $(x - 3)$, are changed so that addition takes the place of subtraction and the remainders are obtained. Thus

1	3	–4	3	–11 = D
	0	3	–1	
2	3	–1	2 = C	
	0	6		
3	–3	5 = B		
	0			
	3 = A			

$$\therefore \quad 3x^3 - 4x^2 + 3x - 11 = 3x^{(3)} + 5x^{(2)} + 2x^{(1)} - 11.$$

Note While applying the methods of synthetic division the coefficients of powers of x should be arranged in descending order, counting zero for the coefficient of missing term.

Example 3.21 *Express $f(x) = x^4 - 5x^3 + 3x + 4$ in terms of factorial polynomials (by using the method of detached coefficients).*

Solution By the method of synthetic division (method of detached coefficients) we get

1	1	–5	0	3	4 = E
	0	1	–4	–4	
2	1	–4	–4	–1 = D	
	0	2	–4		
3	1	–2	–8 = C		
	0	3			
4	1	1 = B			
	0				
	1 = A				

$$\therefore \quad f(x) = x^{(4)} + x^{(3)} - 8x^{(2)} - x^{(1)} + 4.$$

Example 3.22 *Obtain a function whose first difference is $6x^2 + 10x + 11$.*

Solution Expressing the function in factorial notation, we get

$$6x^2 + 10x + 11 = 6x^{(2)} + 16x^{(1)} + 11$$

$$\Delta f(x) = 6x^{(2)} + 16x^{(1)} + 11$$

Integrating we get
$$f(x) = \frac{6x^{(3)}}{3} + \frac{16x^{(2)}}{2} + \frac{11x^{(1)}}{1} + K.$$

$$\therefore \quad f(x) = 2x^{(3)} + 8x^{(2)} + 11x^{(1)} + K,$$

which is the required function.

3.6.3 Recurrence Relation

If n and r are positive integers. Then $\Delta^n o^r = n\left[\Delta^{n-1}o^{r-1} + \Delta^n o^{r-1}\right]$.

Proof

$$\Delta^n o^r = n\left[n^{r-1} - \frac{(n-1)^r}{1!} + \frac{(n-1)(n-2)^r}{2!} + \ldots + (-1)^r\right]$$

$$= n\left[n^{r-1} - \frac{(n-1)(n-1)^{r-1}}{1!} + \frac{(n-1)(n-2)(n-2)^{r-1}}{2!} + \ldots + (-1)^r\right]$$

$$= n\left[(1+n-1)^{r-1} - {}^{(n-1)}c_1(n-1)^{r-1} + {}^{(n-1)}c_2(n-2)^{r-1} \ldots + (-1)^r\right]$$

$$= n\left[(1+n-1) - {}^{(n-1)}c_1(1+n-2)^{r-1} + \ldots + (-1)^r\right]$$

$$= n\left[E^{n-1}1^{r-1} - {}^{(n-1)}c_1 E^{n-2}(1)^{r-1} + {}^{(n-1)}c_2 E^{n-3}(1)^{r-1} \ldots + (-1)^r\right]$$

$$= n[E-1]^{n-1}(1)^{r-1}$$

$$= n\Delta^{n-1}(1)^{r-1}$$

$$= n\Delta^{n-1}E(o)^{r-1} \quad (\text{since } 1 = E(o))$$

$$= n\Delta^{n-1}(1+\Delta)o^{r-1}$$

$$= n\left[\Delta^{n-1}o^{r-1} + \Delta^n o^{r-1}\right]$$

which is the required relation.

Example 3.23 Prove that $\Delta^n o^{n+1} = \dfrac{n(n+1)}{2}\Delta^n o^n$.

Solution We know that $\Delta^n o^m = n\left[\Delta^{n-1}o^{m-1} + \Delta^n o^{m-1}\right]$.

Using this relation, we get

$$\Delta^n o^{n+1} = n\left[\Delta^{n-1} o^n + \Delta^n o^n\right]$$

$$\Delta^{n-1} o^n = (n-1)\left[\Delta^{n-2} o^{n-1} + \Delta^{n-1} o^{n-1}\right]$$

$$\Delta^{n-2} o^{n-1} = (n-2)\left[\Delta^{n-3} o^{n-2} + \Delta^{n-2} o^{n-2}\right]$$

$$\cdots$$

$$\Delta^2 o^3 = 2\left[\Delta o^2 + \Delta^2 o^2\right]$$

$$\Delta o^3 = 1 \times \left[\Delta^0 o^1 + \Delta^1 o^1\right].$$

By back substituting of the above values we have

$$\Delta^n o^{n+1} = n\Delta^n o^n + n(n-1)\Delta^{n-1} o^{n-1} + n(n-1)(n-2)\Delta^{n-2} o^{n-2} +$$

$$\ldots + n(n-1)(n-2)\ldots 2 \times 1\Delta^1 o^1$$

$$= n!\left[n + (n-1) + (n-2) + \ldots + 2 + 1\right]$$

$$= n!\frac{n(n+1)}{2} = \frac{n(n+1)}{2}\Delta^n o^n.$$

3.7 ERROR PROPAGATION IN A DIFFERENCE TABLE

Let $y_0, y_1, y_2, \ldots, y_n$ be the values of the function $y = f(x)$ and the value y_5 be effected with an error $\in$, such that the erroneous value of y_5 is $y_5 + \in$. In this case the error $\in$ effects the successive differences and spreads out facewise as higher orders are formed in the table. The table given below shows us the effect of the error.

y	Δy	$\Delta^2 y$	$\Delta^3 y$
y_0			
	Δy_0		
y_1		$\Delta^2 y_0$	
	Δy_1		$\Delta^3 y_0$
y_2		$\Delta^2 y_0$	
	Δy_2		$\Delta^3 y_1$
y_3		$\Delta^2 y_1$	
	Δy_3		$\Delta^3 y_2 + \in$
y_4		$\Delta^2 y_3 + \in$	
	$\Delta y_4 + \in$		$\Delta^3 y_3 - 3\in$
$y_5 + \in$		$\Delta^2 y_4 - 2\in$	
	$\Delta y_5 - \in$		$\Delta^3 y_4 + 3\in$
y_6		$\Delta^2 y_5 + \in$	
	Δy_6		$\Delta^3 y_5 - \in$
y_7		$\Delta^2 y_6$	
	Δy_7		$\Delta^3 y_6$
y_8		$\Delta^2 y_7$	
	Δy_8		
y_9			

Example 3.24 *The following is a table of values of a polynomial of degree 5. It is given that f(3) is in error. Correct the error.*

x	*0*	*1*	*2*	*3*	*4*	*5*	*6*
y	*1*	*2*	*33*	*254*	*1054*	*3126*	*7777*

Solution It is given that $y = f(x)$ is a polynomial of degree 5.

$\therefore$ $\Delta^5 y$ must be constant, $f(3)$ is in error.

Let $254 + \in$ be the true value, now we form the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
		31		$160 + \in$		
2	33		$190 + \in$		$200 - 4\in$	
		$221 + \in$		$360 - 3\in$		$220 + 10\in$
3	$254 + \in$		$550 - 2\in$		$420 + 6\in$	
		$1771 - \in$		$1780 + 3\in$		$20 - 10\in$
4	1.50		$1330 + \in$		$440 - 4\in$	
		2101		$1220 - \in$		
5	3126		12550			
		4651				
6	7777					

Since the fifth differences of y are constant

$$220 + 10\in = 20 - 10\in$$

$$\Rightarrow \quad 20\in = -220$$

$$\Rightarrow \quad \in = -10$$

$$\therefore \quad f(3) = 254 + \in$$

$$\Rightarrow \quad f(3) = 244.$$

3.8 CENTRAL DIFFERENCES

The operator δ : We now introduce another operator known as the central difference operator to represent the successive differences of a function in a more convenient way.

The central difference operator, denoted by the symbol δ is defined by

$$y_1 - y_0 = \delta y_{1/2}$$

$$y_2 - y_1 = \delta y_{3/2}$$

$$\ldots$$

$$y_n - y_{n-1} = \delta y_{n-1/2}\,.$$

For higher order differences

$$\delta y_{3/2} - \delta y_{1/2} = \delta^2 y_1$$

$$\delta y_2 - \delta y_1 = \delta^2 y_{3/2}$$

$$\ldots$$

$$\delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2} = \delta^n y_r = \left(E^{1/2} - E^{-1/2}\right)^n y_r$$

In its alternative notation

$$\delta\, f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right),$$

where h is the interval of differencing. The central difference table can be formed as follows.

x	y	δ	δ^2	δ^3	δ^4	δ^5	δ^6
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

3.9 MEAN OPERATOR

In addition to the operator Δ, ∇, E and $\delta,$ we define the mean operator (averaging operator) μ as

$$\mu\, f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right) + f\left(x - \frac{1}{2}h\right)\right].$$

Alternative notation If $y = f(x)$ is a functional notation between the variable x and y then it can also denoted by $y = f_x$ or by $y = y_x$.

Let y_x, y_{x+h}, y_{x+2h}, …, etc., denote the values of the dependent variable $y = y_x$, corresponding to the values x, $x + h$, $x + 2h$, …, etc, of the independent variables then the operators $\Delta, \nabla, \delta,$ and μ are defined as

$$\Delta y_x = y_{x+h} - y_x$$

$$\nabla y_x = y_x - y_{x-h}$$

$$\delta y_x = y_{x+\frac{1}{2}h} - y_{x-\frac{1}{2}h},$$

$$\mu = \frac{1}{2}\left(y_{x+\frac{1}{2}h} + y_{x-\frac{1}{2}h}\right),$$

where h is the interval of differencing.

Relation between the operators From the definition we know that

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

(*i*)

$$\delta f(x) = f\left(x + \frac{1}{2}h\right) - f\left(x - \frac{1}{2}h\right)$$

$$= E^{1/2} f(x) - E^{-1/2} f(x)$$

$$= \left(E^{1/2} - E^{-1/2}\right) f(x)$$

$$\therefore \ \delta \equiv \left(E^{1/2} - E^{-1/2}\right)$$

Further

$$\delta f(x) = E^{-1/2}(E - 1) f(x) = E^{-1/2} \Delta f(x)$$

$$\therefore \ \delta = E^{-1/2} \Delta.$$

Note: From the above result we get

$$E^{1/2}\delta = \Delta$$

(*ii*)

$$\mu f(x) = \frac{1}{2}\left[f\left(x + \frac{1}{2}h\right) + f\left(x - \frac{1}{2}h\right)\right]$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] f(x)$$

$$\therefore \ \mu \equiv \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right].$$

(iii) $$E\nabla f(x) = E[f(x) - f(x-h)]$$
$$= Ef(x) - Ef(x-h)$$
$$= f(x+h) - f(x) = \Delta f(x)$$
$$\therefore \; E\nabla \equiv \Delta$$

and $$\nabla Ef(x) = \nabla f(x+h)$$
$$= f(x+h) - f(x) \;\; = \Delta f(x)$$
$$\Rightarrow \; \nabla E = \Delta$$
$$\therefore \; E\nabla = \nabla E.$$

Note: From the above it is clear that operators E and ∇ commute and $\Delta, \nabla, \delta, E$ and μ also commute.

3.10 SEPARATION OF SYMBOLS

The symbolic relation between the operators can be used to prove a number of identities. The method used is known as the *method of separation of symbols*. Few examples based on this method are given below.

Example 3.25 *Use the method of separation of symbols to prove the following identities:*

(a) $u_0 - u_1 + u_2 - ... = \frac{1}{2}u_0 - \frac{1}{4}\Delta u_0 + \frac{1}{8}\Delta^2 u_0 - ...$

(b) $u_0 + \frac{u_1}{1!}x + \frac{u_2}{2!}x^2 + ... = e^x\left[u_0 + \frac{\Delta u_0}{1!} + x^2\frac{\Delta^2 u_0}{2!} + ...\right]$

(c) $(u_1 - u_0) - x(u_2 - u_1) + x^2(u_3 - u_2) - ... = \frac{\Delta u_0}{1+x} - x\frac{\Delta^2 u_0}{(1+x)^2} + x^2\frac{\Delta^2 u_0}{(1+x)^3} ...$

(d) $u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + ... \Delta^{n-1} u_{x-n+1} - n + \Delta^n u_{x-n}$

Solution

(a) $$u_0 - u_1 + u_2 = u_0 - Eu_1 + E^2 u_0 - ...$$
$$= \left(1 - E + E^2 - ...\right)u_0$$
$$= (1+E)^{-1} u_0$$
$$= (1+1+\Delta)^{-1} u_0 \;\; = (2+\Delta)^{-1} u_0$$
$$= 2^{-1}\left(1 + \frac{\Delta}{2}\right)^{-1} u_0$$
$$= \frac{1}{2}\left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{2^2} - \frac{\Delta^3}{2^3} + ...\right]u_0$$

$$= \frac{1}{2}\left[u_0 - \frac{\Delta u_0}{2} + \frac{\Delta^2 u_0}{4} - \frac{\Delta^3 u_0}{8} + \ldots\right]$$

$$\therefore\ u_0 - u_1 + u_2 - u_3 + \ldots = \frac{1}{2}\left[u_0 - \frac{\Delta u_0}{2} + \frac{\Delta^2 u_0}{4} - \frac{\Delta^3 u_0}{8} + \ldots\right]$$

(*b*)
$$u_0 + \frac{u_1 x}{1!} + \frac{u_2 x^2}{2!} + \frac{u_3 x^3}{3!} + \ldots = u_0 + \frac{xEu_0}{1!} + \frac{x^2E^2u_0}{2!} + \frac{x^3E^3u_0}{3!} + \ldots$$

$$= \left[1 - \frac{xE}{1!} + \frac{x^2E^2}{2!} + \frac{x^3E^3}{3!} + \ldots\right]u_0$$

$$= e^{x}Eu_0 \ = e^{x(1+\Delta)}\,u_0 \ = e^{x}\cdot e^{x\Delta}\,u_0\,.$$

$$= e^{x}\left[1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \ldots\right]u_0$$

$$= e^{x}\left[u_0 + \frac{x\Delta u_0}{1!} + \frac{x^2\Delta^2 u_0}{2!} + \ldots\right].$$

(*c*)
$$\text{LHS} = (u - u_0) - x(u_2 - u_0) + x_2\,(u_3 - u_2) - \ldots$$

$$= \Delta u_0 - x\Delta u_1 + x^2\Delta u_2 - x^3\Delta u_3 + \ldots$$

$$= \Delta u_0 - x\Delta Eu_0 + x^2\Delta Eu_0 - x^3\Delta Eu_0 + \ldots = (1 + xE)^{-1}\,\Delta u_0$$

$$\text{RHS} = \frac{\Delta u_0}{1 + x} - x\frac{\Delta^2 u_0}{(1 + x)^2} + x^2\,\frac{\Delta^3 u_0}{(1 + x)^3} - \ldots$$

$$= \frac{1}{x}\left[\frac{x\Delta}{1 + x} - \frac{x^2\Delta^2}{(1 + x)^2} + \frac{x^3\Delta^3}{(1 + x)^3} - \ldots\right]u_0 \ = \frac{1}{x}\left[\frac{\frac{x\Delta}{1 + x}}{1 + \frac{x\Delta}{1 + x}}\right]u_0$$

$$= \left(\frac{\Delta}{1 + x + x\Delta}\right)u_0 = \left(\frac{\Delta}{1 + x(1 + \Delta)}\right)u_0$$

$$= \left(\frac{\Delta}{1 + xE}\right)u_0 = (1 + xE)^{-1}\,\Delta u_0 \ = \text{L.H.S.}$$

Example 3.26 *Use the method of seperation of symbols and prove the following*

(*a*)
$$u_x = u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots + \Delta^n u_{x-n}$$

(*b*)
$$u_x = u_n - (n - x)_{c_1}\,\Delta u_{n-1} + (n - x)_{c_2}\,\Delta^2 u_{n-2} - \ldots$$
$$+ (-1)^{n-x}\,\Delta^{n-x}\,u_{n-(n-x)}$$

Solution. (*a*) Consider R.H.S.

$$u_{x-1} + \Delta u_{x-2} + \Delta^2 u_{x-3} + \ldots + \Delta^n u_{x-n}$$

$$= E^{-1} u_x + \Delta E^{-2} u_{x-2} + \ldots + \Delta^{n-1} E^{-n} u_x + \Delta^n u_{x-n}$$

$$= E^{-1} [1 + \Delta E^{-1} + \Delta^2 E^{-2} + \ldots + \Delta^{n-1} E^{-(n-1)}] u_x + \Delta^n u_{x-n}$$

$$= E^{-1} \left[\frac{1 - \Delta^n E^{-n}}{1 - \Delta E^{-1}}\right] u_x + \Delta^n u_{x-n} = \frac{1}{E}\left[\frac{1 - \Delta^n E^{-n}}{1 - \Delta E^{-1}}\right] u_x + \Delta^n u_{x-n}$$

$$= \left[\frac{1 - \Delta^n E^{-n}}{E - \Delta}\right] u_x + \Delta^n u_{x-n} = \left[\frac{1 - \Delta^n E^{-n}}{1}\right] u_x + \Delta^n u_{x-n}$$

$$(\because E = 1 + \Delta \Rightarrow E - \Delta = 1)$$

$$= u_x - \Delta^n u_{x-n} + \Delta^n u_{x-n} \qquad (\because E^{-n} u_n = u_{n-x})$$

$$= u_x = \text{L.H.S.}$$

(*b*)

$$\text{R.H.S.} = u_n - (n-x)_{c_1} \Delta u_{n-1} + (n-x)_{c_2} \Delta^2 u_{n-2} + \ldots + (-1)^{n-x} \Delta^{n-x} u_{n-(n-x)}$$

$$= (1 - (n-x)_{c_1} \Delta E^{-1} + (n-x)_{c_2} \Delta^2 E^{-2} + \ldots + (-1)^{n-x} \Delta^{n-x} E^{-(n-x)})\, u_n$$

$$= \left[1 - \Delta E^{-1}\right]^{n-x} u_n = \left[1 - \frac{\Delta}{E}\right]^{n-x} u_n$$

$$= \left[\frac{E - \Delta}{E}\right]^{n-x} u_n = \left(\frac{1}{E}\right)^{n-x} u_n = E^{-(n-x)} u_n$$

$$= u_{n-(n-x)} = u_x = \text{L.H.S.}$$

Example 3.27 *If* $u_x = ax^2 + bx + c$, *then show that*

$$u_{2n} - {}^n c_1 2 u_{2n-1} + {}^n c_2 2^2 u_{2n-2} - \ldots + (-2)^n u_n = (-1)^n (c - 2an).$$

Solution Given that $u_x = ax^2 + bx + c$

$$\Rightarrow u_n = an^2 + bn + c.$$

$\therefore$ u_n is a polynomial of degree 2 in n

$$\therefore \Delta^3 u_n = \Delta^4 u_n = \ldots = 0.$$

Let the interval of differencing be equal to 1. Now

$$u_n = an^2 + bn + c$$

$$\Rightarrow \Delta u_n = a(n+1)^2 + b(n+1) + c - an^2 - bn - c = 2an + a + b$$

and

$$\Delta^2 u_n = \Delta[\Delta u_n] = 2a(n+1) + a + b - 2an - a - b = 2a$$

$$\text{LHS} = u_{2n} - {}^nc_1 2u_{2n-1} + {}^nc_2 2^2 u_{2n-2} - \ldots$$

$$= \left[E^n - {}^nc_1 2E^{n-1} + {}^nc_2 2^2 E^{n-2} - \ldots\right]u_n$$

$$= (E-2)^n u_n = (E-1-1)^n u_n$$

$$= (\Delta - 1)^n u_n \; [\because \Delta = \text{E} - 1]$$

$$= (-1)^n (1-\Delta)^n u_n = (-1)^n \left[1 - {}^nc_1 \Delta + {}^nc_2 \Delta^2 + \ldots\right]u_n$$

$$= (-1)^n \left[u_n - n\Delta u_n + \frac{n(n-1)}{2}\Delta^2 u_n + \ldots\right]$$

$$= (-1)^n \left[an^2 + bn + c - n(2an + a + b) + \frac{n^2 - n}{2} 2a\right]$$

$$(\because \; \Delta^3 y_n = \Delta^4 y_n = \ldots = 0)$$

$$= (-1)^n (c - 2an) = \text{R.H.S.}$$

Example 3.28 *Given that* $u_x = e^{ax+b}$, *find* $\Delta^n u_x$.

Solution Let h be the interval of differencing

$$\Delta^n u_x = \Delta^{n-1}\left[\Delta e^{ax+b}\right] = \Delta^{n-1}\left[\Delta e^{ax} e^b\right]$$

$$= \Delta^{n-1} e^b \left[\Delta e^{ax}\right]$$

$$= e^b \Delta^{n-1}\left[e^{ax+h} - e^{ax}\right]$$

$$= e^b \Delta^{n-1} \left(e^{ah} - 1\right) . e^{ax}]$$

$$= e^b \left(e^{ah} - 1\right)\Delta^{n-1} e^{ax}$$

$$= e^b \left(e^{ah} - 1\right)^2 \Delta^{n-2} e^{ax}$$

$$\ldots$$

$$= e^b \left(e^{ah} - 1\right)^n e^{ax}.$$

Example 3.29 *If* Δ, ∇, δ *denote the forward, backward and central difference operators, and E, μ are respectively the shift and averaging operators in the analysis of data with equal spacing. Prove the following.*

(a) $\Delta - \nabla = \delta^2$ (b) $E^{1/2} = \mu + \frac{1}{2}\delta, E^{-1/2} = \mu - \frac{1}{2}\delta$ (c) $\Delta = \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}}$

(d) $\mu^2 = 1 + \frac{1}{4}\delta^2$ (e) $\mu\delta = \frac{1}{2}\Delta E^{-1} + \frac{1}{2}\Delta$

Solution

(*a*) $$\Delta - \nabla = \delta E^{1/2} - \delta E^{-1/2}$$

$$= \delta\left(E^{1/2} - E^{-1/2}\right) = \delta \cdot \delta = \delta^2$$

$$\therefore \ \Delta - \nabla = \delta^2.$$

(*b*) $$\mu + \frac{1}{2}\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left(2E^{1/2}\right) = E^{1/2}$$

$$\mu - \frac{1}{2}\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right) - \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right) = E^{-1/2}.$$

(*c*) $$\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{\delta^2}{4}} = \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \delta\sqrt{1 + \frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}}$$

$$= \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2 + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{4 + E + E^{-1} - 2}{4}}$$

$$= \frac{1}{2}\left(E + E^{-1} - 2\right) + \left(E^{1/2} - E^{-1/2}\right)\sqrt{\frac{\left(E^{1/2} - E^{-1/2}\right)^2}{4}}$$

$$= \frac{1}{2}\left(E + E^{-1} - 2\right) + \frac{\left(E^{1/2} - E^{-1/2}\right)\left(E^{1/2} + E^{-1/2}\right)}{2}$$

$$= \frac{1}{2}\left(E + E^{-1} - 2\right) + \frac{1}{2}\left(E - E^{-1}\right)$$

$$= E - 1 = \Delta.$$

(*d*) $$\mu^2 = \left(\frac{1}{2}\right)^2 \left(E^{1/2} + E^{-1/2}\right)^2$$

$$= \frac{1}{4}\left(E + E^{-1} + 2\right) = \frac{1}{4}\left(\left(E^{1/2} - E^{-1/2}\right)^2 + 4\right)$$

$$= \frac{1}{4}\left(\delta^2 + 4\right) = 1 + \frac{1}{4}\delta^2$$

$$\therefore \ \mu^2 = 1 + \frac{1}{4}\delta^2.$$

(*e*) $$\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$= \frac{1}{2}\left(E - E^{-1}\right) = \frac{1}{2}\left[E - 1 + 1 - e^{-1}\right]$$

$$= \frac{1}{2}(\Delta) + \frac{1}{2}\left(1 - E^{-1}\right) \quad = \frac{1}{2}\Delta + \frac{1}{2}\frac{E-1}{E}$$

$$= \frac{1}{2}\Delta + \frac{1}{2}\Delta E^{-1}.$$

Example 3.30 *If D, E, δ and μ be the operators with usual meaning and if $hD = U$ where h is the interval of differencing. Prove the following relations between the operators:*

(i) $E = e^U$ *(ii)* $\delta = 2\sinh\frac{U}{2}$ *(iii)* $\mu = 2\cosh\frac{U}{2}$ *(iv)* $(E+1)\delta = 2(E-1)\mu$

Solution

(*i*) By definition

$$E = e^{hD}$$

$$\therefore\ E = E^U \quad (\because\ hD = U)$$

(*ii*) Consider R.H.S.

$$2\sinh\frac{U}{2} = 2\left(\frac{e^{\frac{U}{2}} - e^{-\frac{U}{2}}}{2}\right)$$

$$= \left(E^U\right)^{\frac{1}{2}} - \left(E^U\right)^{-\frac{1}{2}}$$

$$= E^{1/2} - E^{-1/2} \quad \left(\because\ E = E^U\right)$$

$$= \delta \ (\text{by definition}) = \text{L.H.S.}$$

(*iii*) Consider R.H.S.

$$\text{R.H.S.} = \cosh\frac{U}{2} = \frac{1}{2}\left(E^{\frac{U}{2}} + E^{-\frac{U}{2}}\right)$$

$$= \frac{\left(E^U\right)^{1/2} + \left(E^U\right)^{-1/2}}{2} = \frac{E^{1/2} + E^{-1/2}}{2}$$

$$= \mu \ (\text{by definition}) = \text{L.H.S.}$$

(*iv*)

$$\text{L.H.S.} = (E+1)\delta$$

$$= (E+1)\left(E^{1/2} - E^{-1/2}\right)$$

$$= \left(E^{1/2}\cdot E^{1/2} + E^{1/2}\cdot E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$= E^{1/2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right)$$

$$= E^{1/2}\left(E^{1/2} - E^{-1/2}\right)\cdot\left(E^{1/2} + E^{-1/2}\right)$$

$$= (E-1)\cdot 2\cdot\left(\frac{E^{1/2}+E^{-1/2}}{2}\right)$$

$$= 2(E-1)\mu \ = \text{R.H.S.}$$

Example 3.31 *If* $\nabla f(x) = f(x) - f(x-1)$ *show that*

$$\nabla^n f(x) = f(x) - {}^n f(x-1) + {}^n c_2\,(x-2) - \ldots + (-1)^n f(x-n)$$

Solution

$$\nabla\, f(x) = f(x) - f(x-1) \quad \text{(given)}$$

$$\Rightarrow\ \nabla\, f(x) = f(x) - E^{-1} f(x)$$

$$\Rightarrow\ \nabla\, f(x) = \left(1 - E^{-1}\right) f(x)$$

$$\therefore\ \nabla^n f(x) = \left(1 - E^{-1}\right) f(x)$$

$$= \left(1 - {}^n c_1\, E^{-1} + {}^n c_2 \left(E^{-1}\right)^2 + \ldots + (-1)^n \left(E^{-1}\right)^n\right) f(x)$$

$$= \left(1 - {}^n c_1\, E^{-1} + {}^n c_2\, E^{-2} + \ldots + (-1)^n E^{-n}\right) f(x)$$

$$= f(x) - {}^n c_1\ f(x-1) + {}^n c_2\ f(x-2) + \ldots + (-1)^n\ f(x-n)$$

3.11 HERCHEL'S THEOREM

Theorem 3.3 *If* $f(x)$ *is a polynomial with constant coefficients and* $E^{nh}\, o^m = n^m\, h^m$, *then*

$$f(e^{ht}) = f(1) + \frac{t}{1!} f(E)o + \frac{t^2}{2!} f(E)o^2 + \frac{t^2}{3!} f(E)o^3 + \ldots$$

Proof Let
$$f(x) = \sum_{n=0}^{\alpha} a_n\, x^n, \tag{5}$$

where *an* (n = 0, 1, 2, …) is real.

We know that
$$e^{(y+nh)t} = 1 + \frac{t(y+nh)}{1!} + \frac{t^2(y+nh)^2}{2!} + \frac{t^3(y+nh)^3}{3!} + \ldots$$

$$\Rightarrow\ e^{(y+nh)t} = 1 + \frac{t}{1!} E^{nh} y + \frac{t^2}{2!} E^{nh} y^2 + \frac{t^3}{3!} E^{nh} y^3 + \ldots \tag{6}$$

Substituting (6) we get

$$e^{(o+nh)t} = e^{nht} = 1 + \frac{t}{1!} E^{nh} o + \frac{t^2}{2!} E^{nh} o^2 + \frac{t^3}{3!} E^{nh} o^3 + \ldots$$

$\therefore$ From (5), we get

$$f\left(e^{ht}\right)=\sum_{n=0}^{\infty}a_n\left(e^{ht}\right)^n=\sum_{n=0}^{\infty}a_n e^{nht}$$

$$=\sum_{n=0}^{\infty}a_n\left[1+\frac{t}{1!}E^{nh}o+\frac{t^2}{2!}E^{nh}o^2+\frac{t^3}{3!}E^{nh}o^3+\ldots\right]$$

$$=\sum_{n=0}^{\infty}a_n+\frac{t}{1!}\sum_{n=0}^{\infty}a_nE^{nh}o+\frac{t^2}{2!}\sum_{n=0}^{\infty}a_nE^{nh}o^2+\frac{t^3}{3!}\sum_{n=0}^{\infty}a_nE^{nh}o^3+\ldots$$

$$=f(1)+\frac{t}{1!}f(E)o+\frac{t^2}{2!}f(E)o^2+\frac{t^3}{3!}f(E)o^3+\ldots$$

Exercise 3.1

1. Show that $\nabla=1-e^{-hD}$

2. Prove the following operator relations

(*a*) $\nabla=1-(1+\Delta)^{-1}$

(*b*) $\delta=\Delta(1+\Delta)^{-1/2}$

(*c*) $\Delta^2=(1+\Delta)\delta^2$

(*d*) $\dfrac{\Delta}{\nabla}-\dfrac{\nabla}{\Delta}=\Delta+\nabla$

(*e*) $\mu^{-1}=1-\frac{1}{8}\delta^2+\frac{3}{128}\delta^2-\frac{5}{1024}\delta^6+\ldots$

(*f*) $\delta=2\sinh\left(\frac{hD}{2}\right)$

(*g*) $\Delta^3y_2=\nabla^3y_5$

(*h*) $\nabla^r f_k=\Delta^r f_{k-r}$

(*i*) $\Delta\left(f_k^2\right)=\left(f_k+f_{k+1}\right)\Delta f_k$

3. Find the values of

(*a*) $\Delta^2 o$ (*b*) $\Delta^5 o^5$ (*c*) $\Delta^6 o^5$

4. Prove the following

(*a*) $u_0+\dfrac{u_1x}{1!}+\dfrac{u_2x^2}{2!}+\dfrac{u_3x^3}{3!}+\ldots=e^x\left[u_0+\dfrac{x\Delta u_0}{1!}+\dfrac{x^2}{2!}\Delta^2u_0+\ldots\right]$

(*b*) $u_n-u_{n+1}+u_{n+2}+u_{n+3}+\ldots$

$$=\frac{1}{2}\left[u_{n-\frac{1}{2}}-\frac{1}{8}\Delta^2u_{n-\frac{3}{2}}+\frac{1\times 3}{2!}\left(\frac{1}{8}\right)^3\Delta^4u_{n-\frac{5}{2}}+\ldots\right]$$

(*c*) $u_0+{}^nc_1u_1x+{}^nc_2u_2x^2+\ldots=(1+x)^n+u_0+$

$${}^nc_1(1+x)^{n-1}x\Delta u_0+{}^nc_2(1+x)^{n-2}x^2\Delta^2u_0+\ldots$$

(*d*) $\Delta^2 u_0 = u_3 - 3u_2 + 3u_1 - u_0$

(*e*) If $u_x = 2^n$ then $\Delta u_n = u_x$

(*f*) $u_{x+h} = u_x + {}^x c_1 \Delta u_{n-1} + {}^{x+1} c_1 \, \Delta^2 u_{n-2} + ...$

5. Show that $y_4 = y_3 + \Delta y_2 + \Delta^2 y_1 + \Delta^3 y_1$

6. Express the following functions in factorial notation

(*a*) $x^4 - 12x^3 + 42x^2 - 30x + 9$

(*b*) $2x^4 - 7x^2 + 5x - 13$

(*c*) $3x^3 - 4x^2 + 3x - 11$

(*d*) $2x^3 - 3x^2 + 3x + 15$

(*e*) $x^4 - 5x^3 + 3x + 4$

(*f*) $x^4 - 2x^2 - x$

7. Obtain the function whose first difference is

(*a*) e^x

(*b*) $2x^3 + 5x^2 - 6x + 13$

(*c*) $x^4 - 5x^3 + 3x + 4$

(*d*) cx

8. If $f(x) = \sin x$, then show that $\Delta^2 f(x) = -\left(2\sin\frac{1}{2}\right)^2 \; Ef(x).$

9. If $f(x) = 2x^3 - x^2 + 3x + 1$ then show that $\Delta^2 f(x) = 12x + 10.$

10. If $f(x) = e^{ax}$, then show that $f(0), \Delta f(0), \Delta^2 f(0)$ are in G.P.

11. Taking 1 as the interval of differencing, prove that $\dfrac{\Delta^2 x^3}{\mathrm{E}x^3} = \dfrac{6}{(x+1)^2}$.

12. Taking 1 as the interval of differencing, prove the following

(*i*) $\Delta \tan ax$

(*ii*) $\Delta^2 ab^c x = a\left(b^c - 1\right)^2 \; bcx$

(*iii*) $\Delta^n ae^x = ae^x \, (e-1)^n$

(*iv*) $\Delta^n \cos(a + bx) = \left(2\sin\frac{h}{2}\right)^n \cos\left(a + bx + \frac{n\,(b+\pi)}{2}\right)$

13. Prove the following

(*i*) $\delta\left[f(x)\, g(x)\right] = \mu \; f(x)\, \delta \, g(x) + \mu \; g(x)\, \delta \; f\;(x)$

(*ii*) $\delta\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\mu \; g(x)\, \delta \; f(x) - \mu \; f(x)\, \delta \, g(x)}{g\left(x - \frac{h}{2}\right) g\left(x + \frac{h}{2}\right)}$

(*iii*) $\mu\left[\dfrac{f(x)}{g(x)}\right] = \dfrac{\mu f(x)\,\mu g(x) - \frac{1}{4}\delta f(x)\,\delta g(x)}{g\left(x - \frac{h}{2}\right) g\left(x + \frac{h}{2}\right)}$

(*iv*) $\mu\left[f(x)\,g(x)\right] = \mu f(x)\,\mu g(x) + \frac{1}{4}\delta f(x)\,\delta g(x)$

14. Prove

$$\Delta\left[f(x-1)\,\Delta g(x-1)\right] = \nabla\left[f(x)\,\Delta g(x)\right] = \Delta\left[f(x-1)\,\nabla g(x)\right].$$

15. Prove

(*i*) $y' = \dfrac{1}{h}\left[\Delta y - \dfrac{\Delta^2 y}{2} + \dfrac{\Delta^3 y}{3} - \dfrac{\Delta^4 y}{4} + \ldots\right]$ and

(*ii*) $y'' = \dfrac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \dfrac{11}{12}\nabla^4 y + \ldots\right]$

where the symbols have their usual meanings.

16. Given $y_0 + y_8 = 1.9243$, $y_1 + y_7 = 1.9540$

$y_2 + y_6 = 1.9823$ and $y_3 + y_5 = 1.9956$

Show that $y_4 = 0.9999557$.

Answers

5. (*a*) $x^{(4)} - 6x^{(3)} + 13x^{(2)} + x^{(1)} + 9$ (*b*) $2x^{(4)} + 12x^{(3)} + 7x^{(2)} - 13$

(*c*) $3x^{(3)} + 5x^{(2)} + 2x^{(1)} - 11$ (*d*) $2x^{(3)} + 3x^{(2)} + 2x^{(1)} + 15$

(*e*) $x^{(4)} + x^{(3)} - 8x^{(2)} - x^{(1)} + 4$ (*f*) $x^{(4)} + 4x^{(3)} + x^{(2)} - 2^1$

6. (*a*) $\dfrac{e^{(n)}}{e-1}$ (*b*) $\dfrac{x^{(4)}}{2} + \dfrac{11}{3}x^{(3)} + \dfrac{1}{2}x^{(2)} + 13x^{(1)} + K$

(*c*) $\dfrac{x^{(5)}}{5} + \dfrac{x^{(4)}}{4} - 8\dfrac{x^{(3)}}{3} - \dfrac{x^{(2)}}{2} + 4x^{(1)} - K$ (*d*) $\dfrac{c^x}{c-1}$

4

INTERPOLATION WITH EQUAL INTERVALS

4.1 INTRODUCTION

The word interpolation denotes the method of computing the value of the function $y = f(x)$ for any given value of the independent variable x when a set of values of $y = f(x)$ for certain values of x are given.

Definition 4.1: Interpoltation is the estimation of a most likely estimate in given conditions. It is the technique of estimating a Past figure (Hiral).

According to Theile: "Interpolation is the art of reading between the lines of a table".

According to W.M. Harper: "Interpolation consists in reading a value which lies between two extreme points".

The study of interpoltation is based on the assumption that there are no sudden jumps in the values of the dependent variable for the period under consideration. It is also assumed that the rate of change of figures from one period to another is uniform.

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$, corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. If the form of the function $y = f(x)$ is known we can very easily calculate the value of y correspondig to any value of x. But in most of the practical problems, the exact form of the function is not known. In such cases the function $f(x)$ is replaced by a simpler function say $\phi(x)$ which has the same values as $f(x)$ for $x_0, x_1, x_2, ..., x_n$. The function $\phi(x)$ is called an *interpolating function*.

4.2 MISSING VALUES

Let a function $y = f(x)$ be given for equally spaced values $x_0, x_1, x_2, ..., x_n$ of the argument and $y_0, y_1, y_2, \ldots, y_n$ denote the corresponding values of the function. If one or more values of $y = f(x)$ are missing we can find the missing values by using the relation between the operators E and Δ.

4.3 NEWTON'S BINOMIAL EXPANSION FORMULA

Let $y_0, y_1, y_2, \ldots, y_n$ denote the values of the function $y = f(x)$ corresponding to the values $x_0, x_0 + h, x_0 + 2h \ldots, x_0 + nh$ of x and let one of the values of y be missing since n values of the functions are known. We have

$$\Delta^n y_0 = 0$$

$$\Rightarrow (E - 1)^n y_0 = 0$$

$$\Rightarrow \left[E^n - {}^nc_1 E^{n-1} + {}^nc_2 E^{n-2} + \ldots + (-1)^n\right] y_0 = 0$$

$$\Rightarrow E^n y_0 - n E^{n-1} y_0 + \frac{n(n-1)}{1 \times 2} E^{n-2} y_0 + \ldots + (-1)^n y_0 = 0$$

$$\Rightarrow y_n - n y_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \ldots + (-1)^n y_0 = 0$$

The above formula is called *Newton's binomial expansion formula* and is useful in finding the missing values without constructing the difference table.

Example 4.1 *Find the missing entry in the following table*

x	0	1	2	3	4
y_x	1	3	9	–	81

Solution Given $y_0 = 1, y_1 = 3, y_2 = 9, \ldots, y_3 = ?, y_4 = 81$ four values of y are given. Let y be polynomial of degree 3

$$\therefore \Delta^4 y_0 = 0$$

$$(E - 1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$\Rightarrow E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\therefore 81 - 4y_3 + 6 \times 9 - 4 \times 3 + 1 = 0$$

$$y_3 = 31.$$

Example 4.2 *Following are the population of a district*

Year (x)	1881	1891	1901	1911	1921	1931
Population (y)	363	391	421	?	467	501

Find the population of the year 1911.

Solution We have

$y_0 = 363,\ y_1 = 391,\ y_2 = 421,\ y_3 = ?,\ y_4 = 467,\ y_5 = 501$

Five values of y are given. Let us assume that y is a polynomial in x of degree 4.

$$\Delta^5 y_0 = 0 \Rightarrow (E - 1)^5 y_0 = 0$$

$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1) y_0 = 0$$

$$\Rightarrow y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$\Rightarrow\ 501 - 5 \times 461 + 10y_3 - 10 \times 421 + 5 \times 391 - 363 = 0$$

$$\Rightarrow\ 10y_3 - 4452 = 0$$

$$\Rightarrow\ y_3 = 445.2$$

The population of the district in 1911 is 445.2 lakh.

Example 4.3 *Interpolate the missing entries*

x	*0*	*1*	*2*	*3*	*4*	*5*
y = f(x)	*0*	–	*8*	*15*	–	*35*

Solution Given $y_0 = 0$, $y_1 = ?$, $y_2 = 8$, $y_3 = 15$, $y_4 = ?$, $y_5 = 35$. Three values are known. Let us assume that $y = f(x)$ is a polynomial of degree 3.

$$\Delta^4 y_0 = 0 \ \Rightarrow\ (E-1)^4 y_0 = 0$$

$$\left(E^4 - 4E^3 + 6E^2 - 4E + 1\right) y_0 = 0$$

$$\therefore\ y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\therefore\ y_4 - 4 \times 15 + 6 \times 8 - 4y_1 - 0 = 0$$

$$\therefore\ y_4 - 4y_1 = 12 \tag{1}$$

and
$$\Delta^5 y_0 = 0 \ \Rightarrow\ (E-1)^5 y_0 = 0$$

$$\Rightarrow\ \left(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1\right) y_0 = 0$$

$$\Rightarrow\ y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$

$$\Rightarrow 35 - 5y_4 + 10 \times 15 - 10 \times 8 + 5y_1 - 0 = 0$$

$$\Rightarrow\ y_4 - y_1 = 21 \tag{2}$$

Solving (1) and (2), we get

$$y_1 = 3,\ y_4 = 24.$$

4.4 NEWTON'S FORWARD INTERPOLATION FORMULA

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the $(n + 1)$ values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values x be equally spaced, i.e.,

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, h$$

where h is the interval of differencing. Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots, x_n$, then, $\phi(x)$ represents the continuous function $y = f(x)$ such that $f(x_r) = \phi(x_r)$ for $r = 0, 1, 2, \ldots, n$ and at all other points $f(x) = \phi(x) + R(x)$ where $R(x)$ is called the *error term* (Remainder term) of the interpolation formula. Ignoring the error term let us assume

$$f(x) \approx \phi(x) \approx a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_o)(x - x_1)\ldots(x - x_{n-1}) \tag{3}$$

the constants $a_0, a_1, a_2, \ldots, a_n$ can be determine as follows.

Putting $x = x_0$ in (3) we get

$$f(x_0) \approx \phi(x_0) = a_0$$

$$\Rightarrow \quad y_0 = a_0$$

putting $x = x_1$ in (3) we get

$$f(x_1) \approx \phi(x_1) = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$

$$\therefore \quad y_1 = y_0 + a_1 h$$

$$\Rightarrow \quad a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}.$$

Putting $x = x_2$ in (3) we get

$$f(x_2) \approx \phi(x_2) = a_0 + a_1(x_2 - x_0)\, a_2(x_2 - x_0)(x_2 - x_1)$$

$$\therefore \quad y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$

$$\Rightarrow \quad y_2 = y_0 + 2(y_1 - y_0) + a_2\left(2h^2\right)$$

$$\Rightarrow \quad a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly by putting $x = x_3$, $x = x_4$ ..., $x = x_n$ in (3) we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3},\ a_4 = \frac{\Delta^4 y_0}{4!h^4},\ \ldots,\ a_n = \frac{\Delta^n y_0}{n!h^n}$$

putting the values of a_0, a_1, …, a_n in (3) we get

$$f(x) \approx \phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) +$$
$$\frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots +$$
$$\frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\left(x - x_{n-1}\right) \tag{4}$$

Writing $u = \dfrac{x - x_0}{h}$, we get $x - x_0 = uh$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= (x - x_0) - (x_1 - x_0) = uh - h = (u - 1)h$$

Similarly
$$x - x_2 = (u - 2)h$$
$$x - x_3 = (u - 3)h$$
$$\cdots$$
$$x - x_{n-1} = (u - n + 1)h$$

Equation (4) can be written as

$$\phi(x) = y_0 + u\frac{\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_o .$$

The above formula is called *Newton's forward interpolation formula*.

Note:

1. Newton forward interpolation formula is used to interpolate the values of y near the beginning of a set of tabular values.
2. y_0 may be taken as any point of the table, but the formula contains only those values of y which come after the value chosen as y_0.

Example 4.4 *Given that*

$\sqrt{12500} = 111.8034, \sqrt{12510} = 111.8481$

$\sqrt{12520} = 111.8928, \sqrt{12530} = 111.9375$

find the value of $\sqrt{12516}$.

Solution The difference table is

x	$y = \sqrt{x}$	Δy	$\Delta^2 y$
12500 x_0	111.8034 y_0		
		0.0447 Δy_0	
12510	111.8481		0 $\Delta^2 y_0$
		0.0447	
12520	111.8928		0
		0.0447	
12530	111.9375		

We have
$$x_0 = 12500,\ h = 10 \text{ and } x = 12516$$
$$u = \frac{x - x_0}{h} = \frac{12516 - 12510}{10} = 1.6$$

from Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$$

$$\Rightarrow \quad f(12516) = 111.8034 + 1.6 \times 0.0447 + 0 + \ldots$$

$$= 111.8034 + 0.07152 \quad = 111.87492$$

$$\therefore \quad \sqrt{12516} = 111.87492.$$

Example 4.5 *Evaluate $y = e^{2x}$ for $x = 0.05$ using the following table*

x	0.00	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.255

Solution The difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.000	1.0000				
		0.2214			
0.10	1.2214		0.0490		
		0.2704		0.0109	
0.20	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.2255				

We have $x_0 = 0.00$, $x = 0.05$, $h = 0.1$.

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} = 0.5$$

Using Newton's forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$f(0.05) = 1.0000 + 0.5 \times 0.2214 + \frac{0.5(0.5-1)}{2}(0.0490) + \frac{0.5(0.5-1)(0.5-2)}{6}(0.0109) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(0.0023)$$

$$= 1.000 + 0.1107 - 0.006125 + 0.000681 - 0.000090 \quad = 1.105166$$

$$\therefore \quad f(0.05) \approx 1.052.$$

Example 4.6 *The values of sin x are given below for different values of x. Find the value of sin 32°*

x	30°	35°	40°	45°	50°
$y = \sin x$	0.5000	0.5736	0.6428	0.7071	0.7660

Solution $x = 32°$ is very near to the starting value $x_0 = 30°$. We compute sin 32° by using Newton's forward interpolation formula.

The difference table is

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30°	0.5000				
		0.0736			
35°	0.5736		–0.0044		
		0.0692		–0.005	
40°	0.6428		–0.0049		0
		0.0643		–0.005	
45°	0.7071		–0.0054		
		0.0589			
50°	0.7660				

$$u = \frac{x - x_0}{h} = \frac{32° - 30°}{5} = 0.4.$$

We have $y_0 = 0.5000, \Delta y_0 = 0.0736, \Delta^2 y_0 = -0.0044, \Delta^3 y_0 = -0.005$

putting these values in Newton's forward interpolation formula we get

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\Rightarrow \quad f(32°) = 0.5000 + 0.4 \times 0.0736 + \frac{(0.4)(0.4-1)}{2}(-0.0044) + \frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.0005)$$

$$f(32°) = 0.5000 + 0.02944 + 0.000528 - 0.00032 = 0.529936 = 0.299.$$

Example 4.7 *In an examination the number of candidates who obtained marks between certain limits were as follows:*

Marks	*30–40*	*40–50*	*50–60*	*60–70*	*70–80*
No. of Students	*31*	*42*	*51*	*35*	*31*

Find the number of candidates whose scores lie between 45 and 50.

Solution Fist of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

The difference table is

x *marks*	y *cumulative frequencies*	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		–25	

60	124		–16		37
		35		12	
70	159		–4		
		31			
80	190				

we have $x_0 = 40,\ x = 45,\ h = 10$

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

and $y_0 = 73,\ \Delta y_0 = 42,\ \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37.$

From Newton’s forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^2 y_0 + \ldots$$

$$\therefore\ f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2} \times 9 + \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24} \times (37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452 = 47.8673$$

$$= 48\ (\text{approximately})$$

$\therefore$ The number of students who obtained marks less than 45 = 48, and the number of students who scored marks between 45 and 50 = 73 – 48 = 25.

Example 4.8 *A second degree polynomial passes through the points (1, –1), (2, –1), (3, 1), (4, 5). Find the polynomial.*

Solution We construct difference table with the given values of x and y

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	–1			
		0		
2	–1		2	
		2		0
3	1		2	
		4		
4	5			

We have $x_0 = 1,\ h = 1,\ y_0 = -1,\ \Delta y_0 = 0,\ \Delta^2 y_0 = 2,$

$$u = \frac{x - x_0}{h} = (x - 1).$$

From Newton's forward interpolation we get

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$$

$$\Rightarrow \quad f(x) = -1 + (x-1)\cdot 0 + \frac{(x-1)(x-1-1)}{2}\cdot 2$$

$$\therefore \quad f(x) = x^2 - 3x + 1.$$

Note: There may be polynomials of higher degree which also fit the data, but Newton's formula gives us the polynomial of least degree which fits the data.

4.5 NEWTON–GREGORY BACKWARD INTERPOLATION FORMULA

Newton's forward interpolation formula cannot be used for interpolating a value of y near the end of a table of values. For this purpose, we use another formula known as Newton–Gregory backward interpolation formula. It can be derived as follows.

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values of x be equally spaced with h as the interval of differencing, i.e.,

Let $$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, n$$

Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots, x_n$, i.e., $\phi(x)$ represents $y = f(x)$ such that $f(x_r) = \phi(x_r)$, $r = 0, 1, 2, \ldots$, we may write $\phi(x)$ as

$$f(x) \approx \phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \ldots + a_n(x - x_n)(x - x_{n-1})\ldots(x - x_1) \tag{5}$$

Putting $x = x_n$ is (5) we get

$$f(x_n) \approx \phi(x_n) = a_0.$$

$$\Rightarrow \quad y_n = a_0.$$

Putting $x = x_{n-1}$ in (5) we get

$$f(x_{n-1}) \approx \phi(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\Rightarrow \quad y_{n-1} = y_n + a_1(-h)$$

$$\Rightarrow \quad a_1 h = y_n - y_{n-1} = \Delta y_n$$

$$\Rightarrow \quad a_1 = \frac{\nabla y_n}{1!\,h}$$

Putting $x = x_{n-2}$, we get

$$f\left(x_{n-2}\right) \approx \phi\left(x_{n-2}\right) = a_0 + a_1\left(x_{n-2} - x_n\right) + \quad a_2\left(x_{n-2} - x_n\right)\left(x_{n-2} - x_{n-1}\right)$$

$$\Rightarrow \; y_{n-2} = y_n + \left(\frac{y_n - y_{n-1}}{h}\right)(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow \; y_{n-2} = y_n - 2y_n + 2y_{n-1} + \left(2h^2\right)a_2$$

$$\Rightarrow \; a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!h^2}$$

similarly putting $x = x_{n-3}, x = x_{n-4}, ..., x = x_{n-5}, ...$ we get

$$a_3 = \frac{\nabla^3 y_n}{3!h^3},\; a_4 = \frac{\nabla^4 \nabla y_n}{4!h^3}, ..., a_n = \frac{\nabla^n y_n}{n!h^n}$$

substituting these values in (5)

$$f(x) \approx \phi(n) = y_n = \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)\left(x - x_{n-1}\right) +$$
$$\frac{\nabla^3 y_n}{3!h^3}(x - x_n)\left(x - x_{n-1}\right)\left(x - x_{n-2}\right) + ... + \frac{\nabla^n y_n}{n!h^n}(x - x_n)\left(x - x_{n-1}\right) ... (x - x_1) \tag{6}$$

writing $u = \dfrac{x - x_n}{h}$ we get $\quad x - x_n = uh$

$$\therefore \; x - x_{n-1} = x - x_n + x_n - x_{n-1} = (uh) + h = (u+1)h$$

$$\Rightarrow \; x - x_{n-2} = (u+2)h, ..., (x - x_1) = (u + x - 1)h$$

$\therefore$ The equation (6) may be written as

$$f(x) \approx \phi(x) = y_n + \frac{u\nabla y_n}{1!} + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{2!}\nabla^3 y_n +$$
$$... + \frac{u(u+1)(u+2)...(u+n-1)}{n!}\nabla^n y_n .$$

The above formula is known as Newton's backward interpolation formula.

Example 4.9 *The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degrees c and P is the percentage of lead in the alloy.*

P	*40*	*50*	*60*	*70*	*80*	*90*
t	*180*	*204*	*226*	*250*	*276*	*304*

Find the melting point of the alloy containing 84 per cent lead.

Solution The value of 84 is near the end of the table, therefore we use the Newton's backward interpolation formula. The difference table is

P	t	∇	∇^2	∇^3	∇^4	∇^5
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have $x_n = 90$, $x = 84$, $h = 10$, $t_n = y_n = 304$, $\nabla t_n = \nabla y_n = 28$, $\nabla^2 y_n = 2$, and

fh = fh

$$\nabla^3 y_n = \nabla^4 y_n = \nabla^5 y_n = 0,$$

$$u = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0.6.$$

From Newton's backward formula

$$f(84) = t_n + u \nabla t_n + \frac{u(u+1)}{2} \nabla^2 t_n + \ldots$$

$$f(84) = 304 - 0.6 \times 28 + \frac{(-0.6)(-0.6+1)}{2} 2$$

$$= 304 - 16.8 - 0.24 = 286.96.$$

Example 4.10 *Calculate the value of* $f(7.5)$ *for the table*

x	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
f(x)	*1*	*8*	*27*	*64*	*125*	*216*	*343*	*512*

Solution 7.5 is near to the end of the table, we use Newton's backward formula to find $f(7.5)$.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	

(*Contd.*)

		61		6		0
5	125		30		0	
		91		6		0
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

We have $x_n = 8,\ x = 7.5,\ h = 1,\ y_n = 512,\ \nabla y_n = 169,\ \nabla^2 y_n = 42,\ \nabla^3 y_n = 6,$

$\nabla^4 y_n = \nabla^5 y_n = \ldots = 0$ $\qquad u = \dfrac{x - x_n}{h} = \dfrac{7.5 - 8}{1} = -\,0.5.$

$\therefore$ we get

$$f(x) = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots$$

$$f(7.5) = 512 + (-0.5)(165) + \frac{(-0.5)(-0.5+1)}{2}(42) + \frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(6)$$

$$= 512 - 84.5 - 5.25 - 0.375$$

$$= 421.87.$$

4.6 ERROR IN THE INTERPOLATION FORMULA

Let the function $f(x)$ be continuous and possess continuous derivatives of all orders with in the interval $[x_0, x_n]$ and let $f(x)$ denote the interpolating polynomial. Define the auxiliary function $F(t)$ as given below

$$F(t) = f(t) - \phi(t) - \{f(x) - \phi(x)\}\,\frac{(t - x_0)(t - x_1)\ldots(t - x_n)}{(x - x_0)(x - x_1)\ldots(x - x_n)}$$

The function $F(t)$ is continuous in $[x_0, x_n]$. $F(t)$ possesses continuous derivatives of all orders in $[x_0, x_n]$ and variables for the values $t = x, x_0, \ldots, x_n$. Therefore $F(t)$ satisfies all the conditions of Rolle's Theorem in each of the subintervals $(x_0, x_1), (x_1, x_2) \ldots (x_{n-1}, x_n)$. Hence $F^{I}(t)$ vanishes at least once in each of the subintervals. Therefore $f'(t)$ vanishes at least $(n + 1)$ times in (x_0, x_n), $f''(t)$ vanishes at least n times in the interval $(x_0, x_n), \ldots, F^{n+1}(t)$ vanishes at least once in (x_0, x_n) say at ξ, where $x_0 < \xi_1 < x_n$.

The expression $(t - x_0)\,(t - x_1) \ldots (t - x_n)$ is a polynomial of degree $(n + 1)$ in t and the coefficient of $t = 1$.

$\therefore$ The $(n + 1)$ the derivative of polynomial is $(n + 1)!$

$$\therefore\ F^{n+1}(\xi) = f^{n+1}(\xi) - \{f(x) - \phi(x)\}\,\frac{(n+1)!}{(x - x_0)(x - x_1)\ldots(x - x_n)} = 0$$

$$\Rightarrow \; f(x) - \phi(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \dots (x - x_n)$$

If $R(x)$ denotes the error in the formula then $R(x) = f(x) - \phi(x)$

$$\therefore \; R(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)(x - x_1) \dots (x - x_n)$$

But $x - x_0 = uh \Rightarrow x - x_1 = (u - 1)h, \dots (x - x_h) = (u - n)h$ where h is the interval of differencing therefore we can write

$$\text{Error } R(x) = \frac{h^{n+1} f^{n+1}(\xi)}{(n+1)!} u(u-1)(u-2) \dots (u-n).$$

Using the relation $D = \frac{1}{h}\Delta$

we get $$D^{n+1} \approx \frac{1}{h^{n+1}} \Delta^{n+1}$$

$$\Rightarrow \; f^{n+1}(\xi) \approx \frac{\Delta^{n+1} f(x_0)}{n+1}$$

The error in the forward interpolation formula is

$$R(x) = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)(u-2) \dots (u-n)$$

Similarly by taking the auxiliary function $F(t)$ in the form

$$F(t) = f(t) - \phi(t) - \{f(x) - \phi(x)\} \frac{(t - x_n)(t - x_{n-1}) \dots (t - x_0)}{(x - x_n)(x - x_{n-1}) \dots (x - x_0)},$$

and proceeding as above we get the error in the Newton backward interpolation formula as

$$R(x) = \frac{\nabla^{n+1} y_n}{(n+1)!} u(u+1) \dots (u+n) \quad \text{where } uh = x - x_n .$$

Example 4.11 *Use Newton's forward interpolation formula and find the value of sin 52° from the following data. Estimate the error.*

x	*45°*	*50°*	*55°*	*60°*
y = sin x	*0.7071*	*0.7660*	*0.8192*	*0.8660*

Solution The difference table is

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		–0.0057	
		0.0532		–0.0007
55°	0.8192		–0.0064	
		0.0468		
60°	0.8660			

$\therefore$ We have $x_0 = 45°$, $x_1 = 52°$, $y_0 = 0.7071$, $\Delta y_0 = 0.0589$, $\Delta^2 y_0 = -0.0057$ and $\Delta^3 y_0 = -0.0007$,

$$u = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4.$$

From Newton's formula

$$y = u_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\therefore \quad f(52) = 0.7071 + 1.4 \times 0.0589 + \frac{(1.4)(1.4-1)}{2} \times (-0.0057) + \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007)$$

$$= 0.7071 + 0.8246 - 0.001596 + 0.0000392 \quad = 0.7880032$$

$\therefore$ $\sin 52° = 0.7880032$

$$\text{Error} = \frac{u(u-1)(u-2)\ldots(u-n)}{(n+1)}\Delta^{n+1} y_0$$

taking $n = 2$ we get

$$\text{Error} = \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 \quad = \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007) = 0.0000392.$$

Exercise 4.1

1. Find the missing figures in the following table

x	0	5	10	15	20	25
y	7	11	–	18	–	32

2. Estimate the production of cotton in the year 1985 from the data given below

Year (x)	1981	1982	1983	1984	1985	1986	1987
Production (y)	17.1	13.0	14.0	9.6	–	12.4	18.2

3. Complete the table

x	2	3	4	5	6	7	8
$f(x)$	0.135	–	0.111	0.100	–	0.082	0.074

4. Find the missing figure in the frequency table

x	15–19	20–24	25–29	30–34	35–39	40–44
f	7	21	35	?	57	58

5. Find the missing term in the following table

x	1	2	3	4	5	6	7
$f(x)$	2	4	8	–	32	64	128

6. Find the missing term

x	1	2	3	4	5
y	7	–	13	21	37

7. Estimate the missing figure in the following table

x	1	2	3	4	5
$f(x)$	2	5	7	–	32

8. Find the missing term in the following data

x	0	1	2	3	4
y	1	3	9	–	81

9. Find $f(1.1)$ from the table

x	1	2	3	4	5
$f(x)$	7	12	29	64	123

10. The following are data from the steam table

Temperature °C	140	150	160	17	180
Pressure kgt/cm^2	3.685	4.84	6.302	8.076	10.225

Using the Newton's formula, find the pressure of the steam for a temperature of 142°C.

11. The area A of circle of diameter d is given for the following values

d	80	85	90	95	100
A	5026	5674	6362	7088	7854

Find approximate values for the areas of circles of diameter 82 and 91 respectively.

12. Compute (1) $f(1.38)$ from the table

x	1.1	1.2	1.3	1.4
$f(x)$	7.831	8.728	9.627	10.744

13. Find the value of y when $x = 0.37$, using the given values

x	0.000	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.2255

14. Find the value of $\log_{10} 2.91$, using table given below

x	2.0	2.2	2.4	2.6	2.8	3.0
$y = \log_{10} x$	0.30103	0.34242	0.38021	0.41497	0.44716	0.47721

15. Find $f(2.8)$ from the following table

x	0	1	2	3
$f(x)$	1	2	11	34

16. Find the polynomial which takes on the following values

x	0	1	2	3	4	5
$f(x)$	41	43	47	53	61	71

17. Find a polynomial y which satisfies the following table

x	0	1	2	3	4	5
y	0	5	34	111	260	505

18. Given the following table find $f(x)$ and hence find $f(4.5)$

x	0	2	4	6	8
$f(x)$	–1	13	43	89	151

19. A second degree polynomial passes through (0,1) (1,3) (2, 7) (3, 13), find the polynomial.

20. Find a cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	0	1	10

21. $u_0 = 560$, $u_1 = 556$, $u_2 = 520$, $u_4 = 385$ show that $u_3 = 465$.

22. In an examination the number of candidates who secured marks between certain limit were as follows:

Marks	0–19	20–39	40–59	60–79	80–99
No. of candidates	41	62	65	50	17

Estimate the number of candidates whose marks are less than 70.

23. Given the following score distribution of statistics

Marks	30–40	40–50	50–60	60–70
No. of students	52	36	21	14

Find

(*i*) the number of students who secured below 35.

(*ii*) the number of students who secured above 65.

(*iii*) the number of students who secured between 35–45.

24. Assuming that the following values of y belong to a polynomial of degree 4, compute the next three values:

x	0	1	2	3	4	5	6	7
y	1	–1	1	–1	1	–	–	–

25. The table gives the distance in nautical miles of the visible horizon for the given heights above the earth's surface:

x = height	100	150	200	250	300	350	400
y = distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

find the values of y when (i) $x = 218$ ft (ii) $x = 410$ ft.

26. Find a cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	2	1	10

Hence or otherwise evaluate $f(4)$.

27. Using Newton's forward formula, find the value of $f(1.6)$, if;

x	1	1.4	1.8	2.2
$f(x)$	3.49	4.82	5.96	6.5

28. Using Newton's Interpolation formulae find the value of y when $x = 1.85$ and $x = 2.4$, if

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
$y = e^x$	5.474	6.050	6.686	7.389	8.166	9.025	9.974

29. From the following table:

x	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$	2.68	3.04	3.38	3.68	3.96	4.21

find $f(0.7)$ approximately.

30. Apply Newton's backward difference formula to the data below, to obtain a polynomial of degree y is x:

x	1	2	3	4	5
y	1	–1	1	–1	1

31. The following data give the melting point of an alloy of lead and zinc, where t is the temperature in °C and p, the percentage of lead in the alloy:

p	60	70	80	90
t	226	250	276	304

Find the melting point of the alloy containing 84 per cent lead, using Newton's interpolation formula.

32. Find a polynomial of degree 4, passing through the points (0, 1) (1, 5) (2, 31), (3, 121), (4, 341), (5, 781)

33. Find the form of the function, given

x	0	1	2	3	4
$f(x)$	3	6	11	18	27

34. Find and correct any error that may be present in the following table:

x	0	1	2	3	4	5	6	7	8	9	10
y	2	5	8	17	38	75	140	233	362	533	752

35. The following table gives the population of Bengal during the period from 1881 to 1931. Estimate the population of Bengal in 1911:

Year	1881	1891	1901	1911	1921	1931
Population (in lakh)	363	391	421	–	467	501

36. Find the index number of exports in 1922, from the table:

Year (x)	1920	1921	1922	1923	1924
Index No. of exports (y)	72	57	–	81	103

37. In the table of values given below the values of y are consecutive terms of a series of which the number 21.6 in the 6th term. Find the first and the tenth terms of the series.

x	3	4	5	6	7	8	9
y	2.7	6.4	12.5	21.6	34.3	51.2	72.9

38. Find the missing figure, in the frequency table:

x	15–19	20–24	25–29	30–34	35–39	40–44
y	7	21	35	–	57	58

39. The table below gives the values of tan x for $0.10 \le x \le 0.30$:

x	0.10	0.15	0.20	0.25	0.30
$y = \tan x$	0.1003	0.1511	0.2027	0.2553	0.3093

find tan (0.12) and tan (0.26).

40. The following data are part of a table for $g(x) = \frac{\sin x}{x^2}$:

x	0.1	0.2	0.3	0.4	0.5
$g(x)$	9.9833	4.9667	3.2836	2.4339	1.9177

Calculate $g(0.25)$ as accurately as possible, by using Newton's forward interpolation formula.

41. Find the value of $e^{1.85}$ give $e^{1.7}$ = 5.4739, $e^{1.8}$ = 6.0496, $e^{1.9}$ = 6.6859, $e^{2.0}$ = 7.3891, $e^{2.1}$ = 8.1662, $e^{2.2}$ = 9.0250, $e^{2.3}$ = 9.9742.

42. Using Newton's formula find sin (θ – 1604) from the table

x	0.160	0.161	0.162
$\sin x$	0.1593182066	0.160305341	0.1612923412

43. Use Newton's forward interpoltation formula and find y at x = 2.5.

x	0	1	2	3	4
y	7	10	13	22	43

44. Applying Newton's interpolation formula, compute $\sqrt{5.5}$ given that $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, $\sqrt{7} = 2.646$, $\sqrt{8} = 2.828$.

45. In the bending of an elastic beam, the normal stress y at a distance x from the middle section is given by the following table

x	0.0	0.25	0.50	0.75	1.0
y	0.46	0.39	0.25	0.12	0.04

find the pressure of the steam for a temperature of 142°C.

46. A rod is rotating in a plane. The following table gives the angle θ (in radians) through which the rod has turned for various values of time t seconds.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3. 20	4.67

obtain the value of θ when t = 0.5.

47. From the table given below compute the value of sin 38°.

$x°$	0	10	20	30°	40°
$\sin x°$	0	0.17365	0.34202	0.50000	0.64279

48. Given the table

x	0	0.1	0.2	0.3	0.4
$y = i^x$	1	1.1052	0.2214	1.3499	1.4918

Find the value of $y = e^x$ when x = 0.38.

49. Apply Newton's backward difference formula, to the data below and find y when $x = 10$.

x	5	6	9	11
y	12	13	14	16

50. Find the expectation of life at age 32 from the following data :

Age	10	15	20	25	30	35
Expectation of life	35.3	32.4	29.2	26.1	23.2	20.5

Answers

1. 17
2. 6.60
3. 48
4. 23.5, 14.25
5. $f(3) = 0.123$, $f(6) = 0.090$
6. 9.5
7. 14
8. 31
9. 7.13
10. 3.899
11. 5281, 6504
12. 10.963
13. 2.0959
14. 0.46389
15. 27.992
16. $x^2 + x + 41$
17. $4x^3 + x$
18. $2x^3 + 3x - 1$
19. $x^2 + x + 1$
20. $x^3 - 2x^2 + 1$
21.
22. 197
23. (*i*) 26 (*ii*) 7 (*iii*) 46
24. 31, 129, 351
25. 15.7 nautical miles, 21.53 nautical miles
26. $2x^3 - 7x^2 + 6x + 1$, 41
27. 5.54
28. 6.36, 11.02
29. 4.43
30. $y = \frac{2}{3}x^4 - 8x^3 + \frac{100}{3}x^2 - 56x + 31$
31. 286.96
32. $x^4 + x^3 + x^2 + x + 1$
33. $x^2 + 2x + 3$
34. The true value of y at $x = 5$ is 77
35. 445.2 lakh
36. 62.8
37. 0.1, 100
38. 47.9
39. 0.1205, 0.2662
40. 3.8647
41. 6.3598
42. 0.1597130489
43. 16.375
44. 2.344
45. 0.308384
46. 0.8
47. 0.61566
48. 1.4623
49. 14.666
50. 22.0948

5

INTERPOLATION WITH UNEQUAL INTERVALS

5.1 INTRODUCTION

The Newton's forward and backward interpolation formulae which were derived in the previous section are applicable only when the values of n are given at equal intervals. In this section we study the problem of interpolation when the values of the independent variable x are given at unequal intervals.

The concept of divided differences: Let the function $y = f(x)$ be given at the point $x_0, x_1, x_2, \ldots, x_n$ (which need not be equally spaced) $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ denote the $(n + 1)$ values the function at the points $x_0, x_2, \ldots, x_n$. Then the first divided differences of $f(x)$ for the arguments x_0, x_1, is defined as

$$\frac{f(x_0) - f(x_1)}{x_0 - x_1}.$$

It is denoted by $f(x_0, x_1)$ or by $\triangle_{x_1} f(x)$ or by $[x_0, x_1]$

$$\therefore \quad f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}.$$

Similarly we can define

$$f(x_1, x_2) = \frac{f(x_1) - f(x_2)}{x_1 - x_2},$$

$$f(x_2, x_3) = \frac{f(x_2) - f(x_3)}{x_2 - x_3},$$

The second divided differences for the arguments $x_0, x_1, x_2, \ldots$ is defined as

$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_1},$$

similarly the third differences for the arguments $x_0, x_1, x_2, x_3 \ldots$ is defined as

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0, x_1, x_2) - f(x_1, x_2, x_3)}{x_0 - x_3}.$$

The first divided differences are called the *divided differences of order one*, the second divided differences are called the *divided differences of order two*, etc.

The divided difference table:

Argument	Entry	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
x	$f(x)$			
x_0	$f(x_0)$			
		$f(x_0, x_1)$		
x_1	$f(x_1)$		$f(x_0, x_1, x_2)$	
		$f(x_1, x_2)$		$f(x_0, x_1, x_2, x_3)$
x_2	$f(x_2)$		$f(x_1, x_2, x_3)$	
		$f(x_2, x_3)$		
x_3	$f(x_3)$			

Example 5.1 *If* $f(x) = \frac{1}{x}$, *then find f(a, b) and f(a, b, c)*

Solution

$$f(x) = \frac{1}{x},$$

$$\Rightarrow \quad f(a, b) = \frac{f(a) - f(b)}{a - b} = \frac{\frac{1}{a} - \frac{1}{b}}{(a - b)} = \frac{b - a}{ab(a - b)} = -\frac{1}{ab}$$

and

$$f(a, b, c) = \frac{f(a, b) - f(b, c)}{a - c}$$

$$= \frac{\frac{-1}{ab} - \left(-\frac{1}{bc}\right)}{a - c} = \frac{1}{b}\left(\frac{-c + a}{ac}\right)\frac{1}{a - c} = \frac{1}{abc}$$

$$\therefore \quad f(a, b) = -\frac{1}{ab}, \; f(a, b, c) = \frac{1}{abc}.$$

Example 5.2 *Prepare the divided difference table for the following data*

x	1	3	4	6	10
$f(x)$	0	18	58	190	920

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	0			

		$\frac{0-18}{1-3}=9$			
3	18		$\frac{9-40}{1-4}=10.33$		
		$\frac{18-58}{3-4}=40$		$\frac{10.33-7}{1-6}=0.6660$	
4	58		$\frac{40-61}{3-6}=7$		$\frac{-6.660-0.4643}{1-60}=0.1248$
		$\frac{58-190}{4-6}=61$		$\frac{7-10.25}{3-10}=0.4643$	
6	190		$\frac{61-182.5}{4-10}=10.25$		
		$\frac{190-920}{6-10}=182.5$			
10	920				

Properties of divided differences: The divided differences are symmetric functions of their arguments:

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{f(x_1) - (x_0)}{x_1 - x_0} = f(x_1, x_0)$$

also
$$f(x_0, x_1) = \frac{f(x_0)}{x_0 - x_1} - \frac{f(x_1)}{x_0 - x_1} = \frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} \tag{1}$$

and
$$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

$$= \frac{1}{x_0 - x_2}\left[\frac{f(x_0)}{x_0 - x_1} + \frac{f(x_1)}{x_1 - x_0} - \frac{f(x_1)}{x_1 - x_2} - \frac{f(x_2)}{x_2 - x_1}\right]$$

$$= \frac{1}{x_0 - x_2}\left[\frac{f(x_0)}{x_0 - x_1} + \frac{(x_1 - x_2) - (x_1 - x_0)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) - \frac{f(x_2)}{x_2 - x_1}\right]$$

$$= \frac{1}{x_0 - x_2}\left[\frac{f(x_0)}{x_0 - x_1} + \frac{(x_0 - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) - \frac{f(x_2)}{x_2 - x_1}\right]$$

$$= \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}. \tag{2}$$

Similarly

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} +$$

$$\frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} + \frac{f(x_3)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}, \quad (3)$$

$$\vdots$$

From (1), (2), (3) it is clear that a divided difference will remain unchanged regardless how much its arguments are interchanged.

By mathematical induction it can be shown that

$$f(x_0, x_1, x_2, ..., x_n) = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} +$$

$$\frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} + \ldots + \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)(x_n - x_{n-1})}$$

which prove that $f(x_0, x_1, \ldots, x_n)$ is a symmetrical function of $x_0, x_1, \ldots, x_n$.

Theorem 5.1 *The divided differences of the product of a constant and a function is equal to the product of the constant and the divided difference of the function is* $\Delta k\, f(x) = k \Delta\, f(x)$, *where* k *is a constant.*

Proof By definition

$$\Delta\, k\, f(x) = \frac{kf(x_0) - kf(x_1)}{x_0 - x_1}$$

$$= k\left[\frac{f(x_0) - f(x_1)}{x_0 - x_1}\right] = k\Delta\; f(x).$$

Theorem 5.2 *The divided difference of the sum (or difference) of two functions is equal to the sum (or difference) of the corresponding separate divided differences.*

Proof Let $f(x) = g(x) + h(x)$, then

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}$$

$$= \frac{[g(x_0) + h(x_0)] - [g(x_1) + h(x_1)]}{x_0 - x_1}$$

$$= \frac{g(x_0) - g(x_1)}{x_0 - x_1} + \frac{h(x_0) - h(x_1)}{x_0 - x_1}$$

$$= g(x_0, x_1) + h(x_0, x_1),$$

similarly we can show that $f(x_0, x_1) = g(x_0, x_1) - h(x_0, x_1)$ where $f(x) = g(x) - h(x)$

Theorem 5.3 *The nth order divided differences of a polynomial of degree in n and x are constants.*

Proof let $f(x) = x^n$, where n is a positive integer. Then

$$f(x_0, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1} = \frac{x_0^n - x_1^n}{x_0 - x_1}$$

$$= x_0^{n-1} + x_1 x_0^{n-2} + \ldots + x_1^{n-1}.$$

$\therefore$ $f(x_0, x_1)$ is a polynomial of degree $(n - 1)$ symmetrical in x_0, x_1 with leading coefficient 1.

$\therefore$ The first order divided difference of $f(x)$ for the arguments x_0, x_1 a polynomial of degree $(n - 1)$.

Now $$f(x_0, x_1, x_2) = \frac{f(x_0, x_1) - f(x_1, x_2)}{x_0 - x_2}$$

$$= \frac{\left(x_0^{n-1} + x_1 x_0^{n-2} + \ldots + x_1^{n-1}\right) - \left(x_1^{n-1} + x_2 x_1^{n-2} + \ldots + x_2^{n-1}\right)}{x_0 - x_2}$$

$$= \frac{x_0^{n-1} - x_2^{n-1}}{x_0 - x_2} + \frac{x_1\left(x_0^{n-2} - x_1^{n-2}\right)}{x_0 - x_2} + \ldots + \frac{x_1^{n-2}(x_0 - x_2)}{x_0 - x_2}$$

$$= \left(x_0^{n-2} + x_2 x_0^{n-1} + \ldots + x_2^{n-2}\right) + \quad x_1\left(x_0^{n-3} + x_2 x_0^{n-4} + \ldots x_2^{n-3}\right) + \ldots + x_1^{n-2}$$

$\therefore$ $f(x_0, x_1, x_2)$ is a polynomial of degree $(n - 2)$ symmetrical in $x_0, x_1, x_2, \ldots, x_n$ with leading coefficient 1.

$\therefore$ The nth divided differences of a polynomial x_n are constant. Similarly when $F(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + a_n$ where $a \neq 0$.

$F(x)$ is a polynomial of degree of n in x.

The nth divided differences of

$$\begin{aligned} F(x) &= a_0 \text{ [}n\text{th divided differences of } x_n] \\ &\quad + [a_1 \text{ (}n\text{th divided differences of } x^{n-1})] \\ &\quad + \ldots + (n\text{th divided differences of } a_n) \\ &= a_0 \times 1 + 0 + 0 \ldots \\ &= a_0, \text{ is a constant.} \end{aligned}$$

Note: The $(n + 1)$th order divided differences will be zero.

5.2 NEWTON'S GENERAL DIVIDED DIFFERENCES FORMULA

Let a function $f(x)$ be given for the $(n + 1)$ values $x_0, x_1, x_2 \ldots, x_n$ as $f(x_0), f(x_1), f(x_2), \ldots, f(x_n)$ where $x_0, x_1, x_2, \ldots, x_n$ are not necessarily equispaced. From the definition of divided difference

$$f(x, x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

$$\Rightarrow \; f(x) = f(x_0) + (x - x_0)\, f(x, x_0) \tag{4}$$

$$f(x, x_0, x_1) = \frac{f(x_1, x_0) - f(x_0, x_1)}{x - x_1}$$

$$\therefore \; f(x, x_0) = f(x_0, x_1) + (x - x_1)\, f(x, x_0, x_1).$$

Substituting in (4) we get

$$f(x) = f(x_0) + f(x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x, x_0, x_1). \tag{5}$$

Proceeding in this way we get

$$f(x) = f(x_0) + (x - x_0)\, f(x, x_0) + (x - x_0)(x - x_1)\, f(x, x_0, x_1) + \ldots + (x - x_0)(x - x_1) .. (x - x_{n-1})\, f(x, x_0, x_1, \ldots, x_{n-1}) + (x - x_0)(x - x_1) \ldots (x - x_n)\, f(x, x_0, x_1, \ldots, x_n) \tag{6}$$

If $f(x)$ is a polynomial of degree n, then the $(n + 1)$th divided differences of $f(x)$ will be zero.

$$\therefore \; f(x, x_0, x_1, \ldots, x_n) = 0.$$

$\therefore$ Equation (6) can be written as

$$f(x) = f(x_0) + (x - x_0)\, f(x_0 - x_1) + \ldots + (x - x_0)(x - x_1) \ldots (x - x_{n-1})\, f(x_0, x_1, \ldots, x_n) \tag{7}$$

$\therefore$ Equation (7) is called *Newton's General divided difference formula*.

Example 5.3 *Use Newton divided difference formula and evaluate f(6), given*

x	*5*	*7*	*11*	*13*	*21*
f(x)	*150*	*392*	*1452*	*2366*	*9702*

Solution

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150				
		121			
7	392		24		
		265		1	
11	1452		32		0
		457		1	
13	2366		46		
		917			
21	9702				

We have $f(x_0) = 150$ $f(x_0, x_1) = 121$, $f(x_0, x_1, x_2) = 24$, $f(x_0, x_1, x_2, x_3) = 1$

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0), (x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_3) f(x_0, x_1, x_2, x_3, x_4) + \ldots$$

$\therefore$ $$f(6) = 150 + (6 - 5)(121) + (6 - 5) j(6 - 7)(24) + (6 - 5)(6 - 7)(6 - 11)\, 1 + 0 + \ldots$$

$\Rightarrow$ $$f(6) = 150 + 121 - 24 + 5$$

$\therefore$ $$f(x) = 252.$$

Example 5.4 *Find the form of the function f(x) under suitable assumption from the following data.*

x	*0*	*1*	*2*	*5*
f(x)	*2*	*3*	*12*	*147*

Solution The divided difference table is given as under:

x	$f(x)$	Δ	Δ^2	Δ^3
0	2			
		1		
1	3		4	
		9		1
2	12		9	
		45		
5	147			

We have $x_0 = 0$, $f(x_0) = 2$ $f(x_0, x_1) = 1$, $f(x_0, x_1, x_2) = 4$, $f(x_0, x_1, x_2, x_3) = 1$.

The Newton's divided difference interpolation formula is

$$f(x) = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3).$$

Substituting we get

$\therefore$ $f(x) = 2 + (x - 0)1 + (x - 0)(x - 1)4 + (x - 0)(x - 1)(x - 2)1$

$f(x) = x^3 + x^2 - x + 2.$

Exercise 5.1

1. If $f(x) = \dfrac{1}{x}$ then show that $f(a, b, c, d) = \dfrac{-1}{abcd}$.

2. If $f(x) = \dfrac{1}{x^2}$ then show that $f(a, b) = -\dfrac{(a+b)}{a^2b^2}$ and $f(a, b, c) = \dfrac{ab + bc + ca}{a^2 b^2 c^2}$.

3. If $f(0) = 8$, $f(1) = 11$, $f(4) = 68$, $f(15) = 123$, then find the form of the function which satisfies the above values.

4. If $u_5 = 150$, $u_7 = 392$, $u_{11} = 1452$, $u_{13} = 2366$, $u_{21} = 9702$, then show that $u_{6.417} = 305.417$.

5. Using the table given below find the value of $f(8)$

x	4	5	7	10	11	12
$f(x)$	48	100	294	900	1210	2028

6. Given the data

x	0	1	2	5
$f(x)$	2	3	12	147

find the cubic function of x.

7. Using the following table, find $f(x)$ as a polynomial in powers of $(x - 6)$.

x	–1	0	2	3	7	10
$f(x)$	–11	1	1	1	141	561

8. Given the following data find $f(x)$ as a polynomial in powers of $(x - 5)$.

x	0	2	3	4	7	9
$f(x)$	4	26	58	112	466	922

9. If $f(x) = \dfrac{1}{x}$ then show that $f(x_0, x_1, x_2, ..., x_n) = \dfrac{(-1)^n}{x_0, x_1, ..., x_n}$.

10. Find the polynomial of the lowest degree which assumes the values 3, 12, 15, –21 when x has the values 3, 2, 1, –1 respectively.

Answers

3. 448 **5.** $x^3 - x^2\ 3x + 8$ **6.** $x^3 + x^2 - x + 2$

7. $x^3 - 9x^2 + 17x + 6$ **8.** $(x - 6)^3 + 13\ (x - 6)^2 + 54\ (x - 6) + 73$

9. $(x - 5)^3 + 17\ (x - 5)^2 + 98\ (x - 5) + 194$

5.3 LAGRANGE'S INTERPOLATION FORMULA

Let $y = f(x)$ be a function which assumes the values $f(x_0)$, $f(x_1)$, $f(x_2)$, …, $f(x_n)$ corresponding to the values $x = x_0, x_1, x_2, \ldots, x_n$, where the values of x are not equispaced. Since $(n + 1)$ values of the function are given corresponding to the $(n + 1)$ values of the independent variable x, we can represent the function $y = f(x)$ be a polynomial in x of degree n.

Let the polynomial be

$$f(x) = a_0 (x - x_1)(x - x_2) \ldots (x - x_n) + a_1 (x - x_0)(x - x_2) \ldots (x - x_n) + a_2 (x - x_0)(x - x_1)(x - x_3) \ldots (x - x_n) + \ldots + a_n (x - x_0)(x - x_1) \ldots (x - x_{n-1}) \quad (8)$$

Each term in equation (8) being a product of n factors in x of degree n, putting $x = x_0$ is (8) we get

$$f(x) = a_0(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)$$

$$\Rightarrow \; a_0 = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}$$

Putting $x = x_2$ in (8) we get

$$f(x_1) = a_1(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)$$

$$\Rightarrow \; a_1 = \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)},$$

similarly putting $x = x_2$, $x = x_3$, $x = x_n$ in (8) we get

$$\Rightarrow \; a_2 = \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)\ldots(x_2 - x_n)},$$

$$\vdots$$

$$\Rightarrow \; a_n = \frac{f(x_n)}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}.$$

Substituting the values of a_0, a_1, …, a_n in (8) we get

$$y = f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}\, f(x_0) +$$

$$\frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}\, f(x_1) + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}\, f(x_n)\cdots \quad (9)$$

The formula given by (9) is called *Lagrange's interpolation formula*. It is simple and easy to remember but the calculations in the formula are more complicated than in Newton's divided difference formula. The application of the formula is not speedy and there is always a chance of committing some error due to the number of positive and negative signs in the numerator and denominator of each term.

Example 5.5 *Using Lagrange's interpolation formula find a polynomial which passes the points (0, –12), (1, 0), (3, 6), (4, 12).*

Solution We have $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $x_3 = 4$, $y_0 = f(x_0) = -12$, $y_1 = f(x_1) = 0$, $y_2 = f(x_2) = 6$, $y_3 = f(x_3) = 12$.

Using Lagrange's interpolation formula we can write

$$f(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_3)}\, f(x_0) + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}\, f(x_1) +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)\dots(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)\dots(x_3-x_2)}f(x_3).$$

$$f(x)=\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}\times(-12)+\frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}\times 0+$$

$$\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}\times(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}\times(12)$$

$$=\frac{(x^3-8x^2+19x-12)}{12}\times 12+\frac{(x^3-5x^2+4x)}{(-6)}\times(6)+\frac{(x^3-4x^2+3x)}{(12)}\times 12$$

$$\therefore\ f(x)=x^3-7x^2+18x-12$$

is the required polynomial.

Example 5.6 *Using Lagrange's interpolation formula, find the value of y corresponding to x = 10 from the following table*

x	*5*	*6*	*9*	*11*
y = f(x)	*12*	*13*	*14*	*16*

Solution We have $x_0 = 5$, $x_1 = 6$, $x_2 = 9$, $x_3 = 11$, $y_0 = 5$, $y_1 = 13$, $y_2 = 14$, $y_3 = 16$.

Using Lagrange's Interpolation formula we can write

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1+$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3.$$

Substituting we get

$$f(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times(13)+$$

$$\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times(16)$$

$$=2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}=\frac{42}{3}.$$

Exercise 5.2

1. Find the polynomial degree three relevant to the following data

x	−1	0	1	2
f(x)	1	1	1	−3

2. Compute $f(0.4)$ using the table

x	0.3	0.5	0.6
$f(x)$	0.61	0.69	0.72

3. Compute sin 39° form the table

$x°$	0	10	20	30	40
sin $x°$	0	1.1736	0.3420	0.5000	0.6428

4. Use Lagrange's interpolation formula and find $f(0.35)$

x	0.0	0.1	0.2	0.3	0.4
$f(x)$	1.0000	1.1052	1.2214	1.3499	1.4918

5. Use Lagrange's interpolation formula to find the value of $f(x)$ for $x = 0$ given the following table

x	–1	–2	2	4
$f(x)$	–1	–9	11	69

6. Use Lagrange's formula and compute

x	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

7. Find by Lagrange's formula the interpolation polynomial which corresponds to the following data

x	–1	0	2	5
$f(x)$	9	5	3	15

8. Find log 5.15 from the table

x	5.1	5.2	5.3	5.4	5.5
$\log_{10} x$	0.7076	0.7160	0.7243	0.7324	0.7404

9. The following table gives the sales of a concern for the five years. Estimate the sales for the year (*a*) 1986 (*b*) 1992

Year	1985	1987	1989	1991	1993
Sales	40	43	48	52	57

(in thousands)

10. Find the polynomial of the least degree which attains the prescribed values at the given points

x	–2	1	2	4
$f(x)$	25	–8	–15	–25

11. Use Lagrange's interpolation formula to find y when $x = 5$ from the following data

x	0	1	3	8
y	1	3	13	123

12. Given

x	0	1	4	5
$f(x)$	4	3	24	39

Find the form of the function $f(x)$; by Lagranges formula.

13. Using Lagrange's interpolation formula find y at $x = 10$, given

x	5	6	9	11
y	12	13	14	16

14. Use Lagrange's formula to find $f(2)$; given

x	0	1	3	4
$y = f(x)$	5	6	50	105

15. Use Lagrange's interpolation formula and find $f(0)$, given the following table

$x - 1$	–2	2	4
$y = f(x) - 1$	–9	11	69

Answers

1. $x^2 - 3x + 5$
2. $-x^3 + x + 1$
3. $x^2 - 10x + 1$
4. 0.65
5. (a) 41.02 (b) 54.46
6. 0.6293
7. 0.7118
8. 1.4191
9. 1.6751
10. 1
11. 38.143
12. $2x^2 - 3x + 4$
13. 14.7
14. 19
15. 1

5.4 INVERSE INTERPOLATION

In interpolation we have discussed various methods of estimating the missing value of the function $y = f(x)$ corresponding to a value x intermediate between two given values. Now we discuss inverse interpolation in which we interpolate the value of argument x corresponding to an intermediate value y of the entry.

Use of Lagrange's interpolation formula for inverse interpolation In Lagrange's Interpolation formula y is expressed as a function of x as given below

$$y = f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)}\, y_0 + \frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)}\, y_1 + \ldots$$

$$+ \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})}\, y_n .$$

By interchanging x and y we can express x as a function of y as follows

$$x = \frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}\,x_0 + \frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}\,x_1 + \ldots$$

$$+ \frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots(y_n-y_{n-1})}\,x_n .$$

The above formula may be used for inverse interpolation.

Example 5.7 *The following table gives the value of the elliptical integral*

$$F(\phi) = \int_0^{\phi} \frac{d\phi}{1-\frac{1}{2}\sin^2\phi}$$

for certain values of ϕ. *Find the values of* ϕ *if* $F(\phi) = 0.3887$

ϕ	*21°*	*23°*	*25°*
F(f)	*0.3706*	*0.4068*	*0.4433*

Solution We have $\phi = 21°, \phi_1 = 23°, \phi_2 = 25°$, $F = 0.3887$, $F_0 = 0.3706$, $F_1 = 0.4068$ and $F_2 = 0.4433$.

Using the inverse interpolation formula we can write

$$\phi = \frac{(F-F_1)(F-F_2)}{(F_0-F_1)(F_0-F_2)}\,\phi_0 + \frac{(F-F_0)(F-F_2)}{(F_1-F_0)(F_1-F_2)}\,\phi_1 + \frac{(F-F_0)(F-F_1)}{(F_2-F_0)(F_2-F_2)}\,\phi_2 ,$$

$$\Rightarrow \phi = \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)} \times 21 + \frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)} \times 23 +$$

$$\frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)} \times 25$$

$$= 7.884 + 17.20 - 3.087 = 21.999\ 22$$

$\therefore\ \phi = 22°$.

Example 5.8 *Find the value of x when y = 0.3 by applying Lagrange's formula inversely*

x	*0.4*	*0.6*	*0.8*
y	*0.3683*	*0.3332*	*0.2897*

Solution From Lagrange's inverse interpolation formula we get

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}\,x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}\,x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}\,x_2 .$$

Substituting $x_0 = 0.4$, $x_1 = 0.6$, $x_2 = 0.8$, $y_0 = 0.3683$, $y_1 = 0.3332$, $y_2 = 0.2899$ in the above formula, we get

$$x = \frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-0.2897)} \times (0.4) + \frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3332-0.2897)} \times (0.6) +$$

$$\frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)} \times (0.8)$$

$= 0.757358.$

Example 5.9 *The following table gives the values of the probability integral* $y = \frac{2}{\sqrt{\pi}}\int_0^n e^{x^2}\,dx$ *corresponding to certain values of x. For what value of x is this integral equal to* $\frac{1}{2}$ *?*

x	0.46	0.47	0.48	0.49
$y = \frac{2}{\sqrt{\pi}}\int_0^n e^{x^2}\,dx$	0.484655	0.4937452	0.5027498	0.5116683

Solution Here $x_0 = 0.46$, $x_1 = 0.47$, $x_2 = 0.48$, $x_3 = 0.48$, $y_0 = 0.484655$, $y_1 = 0.4937452$, $y_2 = 0.5027498$, $y_3 = 0.5116683$ and $y = \frac{1}{2}$.

From Lagrange's inverse interpolation formula

$$x = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1 +$$
$$\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3.$$

$$\therefore x = \frac{(0.5-0.4937452)\,(0.5-0.50274498)\,(0.5-0.5116683)}{(0.4846555-0.4937452)\,(0.484655-0.5027498)\,(0.4846555-0.5116683)} \times 0.46 +$$

$$\frac{(0.5-0.4846555)\,(0.5-0.5\,0.5027498)\,(0.5-0.5116683)}{(0.493752-0.4846555)\,(0.4937452-0.5027498)\,(0.4937452-0.5116683)} \times 0.47 +$$

$$\frac{(0.5-0.486555)\,(0.5-0.4937452)\,(0.5-0.5116683)}{(0.5027498-0.484655)\,(0.5027498-0.4937452)\,(0.5027498-0.5116683)} \times 0.48 +$$

$$\frac{(0.5-0.484655)\,(0.5-0.4937452)\,(0.5-0.5027498)}{(0.5116683-0.484655)\,(0.5116683-0.4937452)\,(0.5-0.5027498)} \times 0.49$$

$= -0.0207787 + 0.157737 + 0.369928 - 0.0299495 = 0.476937.$

Example 5.10 *Show that Lagrange's interpolation formula can be written in the form*

$$f(x) = \sum_{r=0}^{r=n} \frac{\phi(x_r)}{(x-x_r)\,\phi^{\mathrm{I}}(x_r)}$$

where $\phi(x) = (x-x_0)(x-x_1)\ldots(x-x_n)$

and $\phi^{\mathrm{I}}(x_r) = \frac{d}{dx}\left[\phi(x)\right]$ at $x = x_r$

Solution We have

$$\phi(x) = (x - x_0)(x - x_1) \dots (x - x_n)$$

$$\phi'(x) = (x - x_1)(x - x_2) \dots (x - x_n) +$$

$$(x - x_0)(x - x_2) \dots (x - x_n) + \dots +$$

$$(x - x_0)(x - x_1) \dots (x - x_{r-1})(x - x_{r+1}) \dots (x - x_n) +$$

$$\dots + (x - x_0)(x - x_1) \dots (x - x_{n-1})$$

$$\therefore \ \phi'(x_r) = (x_r - x_0)(x_r - x_1) \dots (x_r - x_{r-1})(x_r - x_{r+1}) \dots (x_r - x_n)$$

$$\therefore \ f(x) = \sum_{r=0}^{r=n} \frac{\phi(x)}{(x - x_r)\phi^{\mathrm{I}}(x_r)} f(x_r).$$

Example 5.11 *By means of Lagrange's formula prove that*

(i) $y_1 = y_3 - 0.3\lfloor y_5 - y_{-3} \rfloor + 0.2\lfloor y_{-3} - y_{-5} \rfloor$

(ii) $y_0 = \dfrac{1}{2}(y_1 + y_{-1}) - \dfrac{1}{8}\left[\dfrac{1}{2}(y_3 - y_1) - \dfrac{1}{2}(y_{-1} - y_{-3})\right]$

Solution (*i*) y_{-5}, y_{-3}, y_3 and y_5 are given, therefore the values of the arguments are –5, –3, 3, and 5, y_1 is to be obtained. By Lagrange's formula

$$y_x = \frac{[x-(-3)](x-3)(x-5)}{[-5-(-3)](-5-3)(-5-5)} y_{-5} + \frac{[x-(-5)](x-3)(x-5)}{[-3-(-5)](-3-3)(-3-5)} y_{-3} +$$

$$\frac{[x-(-5)][x-(-3)](x-5)}{[3-(-5)][3-(-3)](3-5)} y_3 + \frac{[x-(-5)][x-(-3)](x-3)}{[5-(-5)][5-(-3)](5-3)} y_5$$

Taking $x = 1$, we get

$$y_1 = \frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)} y_{-5} + \frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)} y_{-3} +$$

$$\frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)} y_3 + \frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)} y_5$$

$$\Rightarrow \ y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)} y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)} y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)} y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)} y_5$$

$$= -\frac{y_{-5}}{5} + \frac{y_{-3}}{2} + y_3 - \frac{3}{10} y_5$$

$$= y_3 - 0.2y_{-5} + 0.5y_{-3} - 0.3y_5$$

$$= y_3 - 0.2y_{-5} + 0.2y_{-3} + 0.3y_{-3} - 0.3y_5$$

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

(*ii*) y_{-3}, y_{-1}, y_1, and y_3 are given, y_0 is to be obtained. By Lagrange's formula

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)} y_{-3} + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)} y_{-1} +$$

$$\frac{(0+3)(0+1)(0-3)}{(1+3)(1+1)(1-3)} y_1 + \frac{(0+3)(0+1)(0-1)}{(3+3)(3+1)(3-1)} y_3$$

$$= -\frac{1}{16} y_{-3} + \frac{9}{16} y_{-1} + \frac{9}{16} y_1 - \frac{1}{16} y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}\left[(y_3 - y_1) - (y_{-1} - y_{-3})\right]$$

$$\therefore \quad y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Example 5.12 *The values of f(x) are given at a, b, and c. Show that the maximum is obtained by*

$$x = \frac{f(a)\cdot(b^2 - c^2) + f(b)\cdot(c^2 - a^2) + f(c)\cdot(a^2 - b^2)}{2\left[f(a)\cdot(b-c) + f(b)\cdot(c-a) + f(c)\cdot(a-b)\right]}$$

Solution By Lagrange's formula $f(x)$ for the arguments a, b, and c is given by

$$f(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$$

$$= \frac{x^2 - (b+c)x + bc}{(a-b)(a-c)} f(a) + \frac{x^2 - (c+a)x + ca}{(b-c)(b-a)} f(a) + \frac{x^2 - (a+b)x + ab}{(c-a)(c-b)} f(c)$$

for maximum or minimum we have $f'(x) = 0$

$$\therefore \quad -\frac{2x - (b+c)}{(a-b)(c-a)} f(a) - \frac{2x - (a+c)}{(a-b)(b-c)} f(b) - \frac{2x - (a+b)}{(b-c)(c-a)} f(c) = 0$$

$$\Rightarrow \quad 2x\left[(b-c) f(a) + (c-a) f(b) + (a-b) f(c)\right] -$$

$$\left[(b^2 - c^2) f(a) + (c^2 - a^2) f(b) + (a^2 - b^2) f(c)\right] = 0$$

$$\therefore \quad x = \frac{(b^2 - c^2) f(a) + (c^2 - a^2) f(b) + (a^2 - b^2) f(c)}{2\left[(b-c) f(a) + (c-a) f(b) + (a-b) f(c)\right]}.$$

Example 5.13 *Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$, find $\log_{10} 656$.*

Solution Here $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 661$ and $f(x) = \log_{10} 656$.

By Lagrange's formula we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 +$$

$$\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\therefore\ \log_{10} 656 = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} \times (2.8156) +$$

$$\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} \times (2.8182) +$$

$$\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} \times (2.8189) +$$

$$\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)} \times (2.8202)$$

$$= \frac{3}{14}(2.8156) + \frac{5}{2}(2.8182) - 2(2.8189) + \frac{2}{7}(2.8202)$$

$$= 0.6033 + 7.045 - 5.6378 + 0.8057$$

$$= 2.8170.$$

Example 5.14 *Write down the Lagrange's polynomial passing through (x_0, f_0), (x_1, f_1) and (x_2, f_2). Hence express $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sum of partial fractions.*

Solution The Lagrange's polynomial through the points (x_0, f_0), (x_1, f_1) and (x_2, f_2) is given by

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} \times f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \times f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \times f_2$$

Consider the numerator $3x^2 + x + 1$.

Let $f(x) = 3x^2 + x + 1$,

tabulating the values of $f(x)$ for $x = 1, 2, 3$ we get

x	1	2	3
$f(x)$	5	15	31

Using Lagrange's formula, we get

$$f(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} \times 5 + \frac{(x-1)(x-3)}{(2-1)(2-3)} \times 15 + \frac{(x-1)(x-2)}{(3-1)(3-2)} \times 31$$

$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

$$\therefore\ \frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{(x-2)} + \frac{31}{2(x-3)}.$$

Exercise 5.3

1. Given u_{-1}, y_0, u_1, and u_2. Using Lagrange's formula show that

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

where $x + y = 1$.

2. If all terms except y_5 of the sequence y_1, y_2, y_3, …, y_9 be given, show that the value of y_5 is

$$\left[\frac{156\left(y_4+y_6\right)-28\left(y_3+y_7\right)+8\left(y_2+y_8\right)-\left(y_1+y_9\right)}{70}\right]$$

3. If y_0, y_1, y_2, …, y_6 are the consecutive terms of a series then prove that

$$y_3 = 0.05(y_0 + y_6) - 0.3(y_1 + y_5) + 0.75(y_2 + y_4).$$

4. Show that the sum of coefficients of y_i's in the Lagrange's interpolation formula is unity.

5. The following values of the function $f(x)$ for values of x are given: $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. Find the values of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

6. The following table is given

x	0	1	2	5
$f(x)$	2	3	12	147

show that the form of $f(x)$ is $x^3 + x^2 - x + 2$.

7. Using Lagrange's formula show that

(*a*) $$\frac{x^3-10x+13}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)}+\frac{3}{(x-2)}-\frac{4}{(x-3)}$$

(*b*) $$\frac{x^2+6x+1}{\left(x^2-1\right)(x-4)(x+6)}=\frac{-2}{25\,(x+1)}-\frac{4}{21\,(x-1)}+\frac{41}{150\,(x-4)}-\frac{1}{350\,(x+6)}$$

8. Express the function $\dfrac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$ as sum of partial functions.

9. Using Lagrange's interpolation formula, express the function $\dfrac{x^2+x-3}{x^3-2x^2-x+2}$ as sum of partial fractions.

Answers

6. $\dfrac{1}{2(x-1)}-\dfrac{1}{(x-2)}-\dfrac{1}{2(x+1)}$

7. $\dfrac{2}{35(x+1)}+\dfrac{4}{15(x-1)}-\dfrac{41}{30(x-4)}+\dfrac{73}{70(x-6)}$

9. $f(6) = 5.66$, maximum at $x = 4.5$.

6

CENTRAL DIFFERENCE INTERPOLATION FORMULAE

6.1 INTRODUCTION

In the preceding sections we have derived and discussed a few interpolation formulae which were suited for interpolation near the beginning and end values of the tabulated data. For interpolation near the middle of a difference table, central difference formulae are preferable. In this section we study some central difference formulae which are used for interpolation near the middle values of the given data.

Let the function $y = y_x = f(x)$

be given for $(2n + 1)$ equispaced values of argument $x_0,\ x_0 \pm h_1,\ x_0 \pm 2h,\ ...,\ x_o, x_h$.

The corresponding values of y be y_r $(r = 0, \pm 1, \pm 2, ..., \pm n)$.

Let $$y = y_0$$

denote the central ordinate corresponding to $x = x_0$. We can form the difference table as given below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
$x_0 - 3h$	y_{-3}						
		Δy_{-3}					
$x_0 - 2h$	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
$x_0 - h$	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
$x_0 + h$	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
		Δy_1		$\Delta^3 y_0$			
$x_0 + 2h$	y_2		$\Delta^2 y_1$				
		Δy_2					
$x_0 + 3h$	y_3						

The above table can also be written in terms of central differences using the Sheppard's operator δ as follows:

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
$x_0 - 3h$	y_{-3}						
		$\delta y_{-5/2}$					
$x_0 - 2h$	y_{-2}		$\delta^2 y_{-2}$				
		$\delta y_{-3/2}$		$\delta^3 y_{-3/2}$			
$x_0 - h$	y_{-1}		$\delta^2 y_{-1}$		$\delta^4 y_{-1}$		
		$\delta y_{-1/2}$		$\delta^3 y_{-1/2}$		$\delta^5 y_{-1/2}$	
x_0	y_0		$\delta^2 y_0$		$\delta^4 y_0$		$\delta^6 y_0$
		$\delta y_{1/2}$		$\delta^3 y_{1/2}$		$\delta^5 y_{1/2}$	
$x_0 + h$	y_1		$\delta^2 y_1$		$\delta^4 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
$x_0 + 2h$	y_2		$\delta^2 y_2$				
		$\delta y_{5/2}$					
$x_0 + 3h$	y_3						

In constructing above table the relation

$$\delta = \Delta E^{-1/2}$$

is used. Both the tables given above are called *central difference tables*. One can very easily observe that the differences given in both the tables are same in corresponding positions.

6.2 GAUSS FORWARD INTERPOLATION FORMULA

The Newton forward interpolation formula is

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{1 \times 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{1 \times 2 \times 3}\Delta^3 y_0 + \ldots, \tag{1}$$

where $u = \dfrac{x - x_0}{h}$ and $x = x_0$ is the origin.

From the central difference table we have

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \ldots$$

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

$$\vdots$$

Substituting the values in (1) we get

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \frac{u(u-1)(u-2)}{3!}\left(\Delta^3 y_{-1} + \Delta^4 y_{-1}\right) +$$

$$\frac{u(u-1)(u-2)(u-3)}{4!}\left(\Delta^4 y_{-1} + \Delta^5 y_{-1}\right) + \ldots$$

The above formula may be written as

$$y_4 = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots \qquad (2)$$

Equation (2) is called *Gauss's forward interpolation formula*.

6.3 GAUSS BACKWARD INTERPOLATION FORMULA

Substituting

$$\Delta y_0 = \Delta y_0 + \Delta^2 y_{-1}$$

$$\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$$

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$$

$$\vdots$$

and

$$\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^2 y_{-2}$$

$$\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$$

In Newton's forward interpolation formula we see

$$y = f(x) = y_0 + \frac{u}{1!}\left(\Delta y_{-1} + \Delta^2 y_{-1}\right) + \frac{u(u-1)}{2!}\left(\Delta^2 y_{-1} + \Delta^3 y_{-1}\right) + \ldots$$

$$\Rightarrow \quad y_4 = y_0 \frac{u}{1!}\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(u+2)\,(u+1)\,u\,(u-1)}{4!}\,\Delta^4 y_{-2} + \ldots$$

This is called *Gauss's Backward Interpolation formula*.

Note: The Gauss's forward interpolation formula employs odd differences above the central line through y_0 and even differences on the central line whereas Gauss's backward formula employs odd differences below the central line through y_0 and even differences on the central line as shown in the table given below.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_{-4}	y_{-4}						
		Δy_{-4}					
x_{-3}	y_{-3}		$\Delta^2 y_{-4}$				
		Δy_{-3}		$\Delta^3 y_{-4}$			
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$		$\Delta^4 y_{-4}$		
		Δy_{-2}		$\Delta^3 y_{-3}$		$\Delta^5 y_{-4}$	
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		$\Delta^6 y_{-4}$
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2					
x_3	y_3		$\Delta^2 y_2$				
		Δy_3					
x_4	y_4						

Gauss's forward formula is used to interpolate the values of the function for the value of u such that $0 < u < 1$, and Gauss's backward formula is used to interpolate line value of the function for a negative value of u which lies between –1 and 0 (i.e., $-1 < u < 0$).

6.4 BESSEL'S FORMULA

Changing the origin in the Gauss's backward interpolation formula from 0 to 1, we have

$$y = y_1 + (u-1)\Delta y_0 + \frac{u(u-1)}{2!}\Delta^3 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_{-1} + \ldots$$

Taking the mean of the above formula and the Gauss's forward interpolation formula, we obtain

$$y_4 = \frac{1}{2}[y_0 + y_1] + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\frac{1}{2}\left[\Delta^2 y_{-1} + \Delta^2 y_0\right] + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} + \ldots$$

This is called *Bessel's formula*.

Note: Bessel's formula involves odd differences below the central line and means of the even differences on and below the line as shown in the table below.

x	y						
x_{-3}	y_{-3}						
		Δy_{-3}					
x_{-2}	y_{-2}		$\Delta^2 y_{-3}$				
		Δy_{-2}		$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}		$\Delta^2 y_{-2}$		$\Delta^4 y_{-3}$		
		Δy_{-1}		$\Delta^3 y_{-2}$		$\Delta^5 y_{-3}$	
x_0	y_0		$\Delta^2 y_{-1}$		$\Delta^4 y_{-2}$		$\Delta^6 y_{-3}$
		Δy_0		$\Delta^3 y_{-1}$		$\Delta^5 y_{-2}$	
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		$\Delta^6 y_{-2}$
		Δy_1		$\Delta^3 y_0$		$\Delta^5 y_{-1}$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$			
x_3	y_3		$\Delta^2 y_2$				
		Δy_3					
x_4	y_4						

The brackets mean that the average of the values has to be taken Bessel's formula is most efficient for $\frac{1}{4} \le u \le \frac{3}{4}$.

6.5 STIRLING'S FORMULA

Gauss's forward interpolation formula is

$$y_u = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} +$$

$$\frac{(u+1)u(u+1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^5 y_{-2} + \ldots \tag{3}$$

Gauss's backward interpolation formula is

$$y_u = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{(u-1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} +$$

$$\frac{(u+2)u(u+1)(u-1)}{4!}\Delta^4 y_{-2} + \ldots \tag{4}$$

Taking the mean of the two Gauss's formulae, we get

$$y_u = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1} +$$

$$\frac{u(u^2-1)}{3!}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{4!}\Delta^4 y_{-2} + \frac{u(u^2-1)(u^2-4)}{5!}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \ldots$$

The above is called Stirling's formula. Stirling's formula gives the most accurate result for $-0.25 \le u \le 0.25$. Therefore, we have to choose x_0 such that u satisfies this inequality.

6.6 LAPLACE–EVERETT FORMULA

Eliminating odd differences in Gauss's forward formula by using the relation

$$\Delta y_0 = y_1 - y_0$$

$$\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$$

$$\Delta^5 y_{-2} = \Delta^4 y_{-1} - \Delta^4 y_{-2} \ldots,$$

we get $y = f(x)y_0 + \dfrac{u}{1!}(y_1 - y_0) + \dfrac{u(u-1)}{2!}\Delta^2 y_{-1} + \dfrac{(u+1)u(u-1)}{3!}(\Delta^2 y_0 - \Delta^2 y_{-1}) +$

$$\frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}(\Delta^4 y_{-1} - \Delta^4 y_{-2}) + \ldots$$

$$= (1-u)y_0 + uy_1 + u(u-1)\left[\frac{1}{1\times 2} - \frac{u+1}{1\times 2\times 3}\right]\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 +$$

$$(u+1)u(u-1)(u-2)\left[\frac{1}{1\times 2\times 3\times 4} - \frac{u+2}{5}\right]\Delta^2 y_{-2} +$$

$$\frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \ldots$$

$$= (1-u)y_0 + \frac{uy_1}{1!} - \frac{u(u-1)(u-2)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 -$$

$$\frac{(u+1)u(u-1)(u-2)(u-3)}{3!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \ldots \tag{5}$$

Writing $v = 1 - u$, i.e., $u = 1 - v$ and changing the terms (5) with a negative sign we get

$$y = vy_0 + \frac{u}{1!}y_1 + \frac{(v+1)v(v-1)}{3!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^2 y_0 +$$

$$\frac{(v+2)(v+1)v(v-1)(v-2)}{5!}\Delta^4 y_{-2} + \frac{(u+2)(u+1)u(u-1)(u-2)}{5!}\Delta^4 y_{-1} + \ldots \tag{6}$$

The above formula may be written

$$y_4 = f(x) = vy_0 + \frac{v\left(v^2 - 1^2\right)}{3!}\Delta^2 y_{-1} + \frac{v(v^2 - 1^2)\,(u^2 - 2^2)}{5!}\Delta^4 y_{-2} + \ldots + uy_1$$

$$+ \frac{u(u^2 - 1^2)}{3!}\Delta^2 y_0 + \frac{u\left(u^2 - 1^2\right)\left(u^2 - 2^2\right)}{5!}\Delta^2 y_{-1} + \ldots \tag{7}$$

Equation (7) is known as *Laplace–Everett's formula*. The formula uses only even differences of the function, hence for a rapidly converging series of differences of the function we will have only a few number of terms in both the u series and v series.

Note: Laplace–Everett's formula can be used in sub-tabulation.

Example 6.1 *Use Gauss forward formula to find y for x = 30 given that*

x	21	25	29	33	37
y	18.4708	17.8144	17.1070	16.3432	15.5154

Solution We construct the difference table by taking as

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
$x_0 - 2h = 21$	18.4708				
		–0.6564			
$x_0 - h = 25$	17.8144		–0.0510		
		–0.7074		–0.0054	
$x_0 = 29$	17.1070		–0.0564		–0.002
		–0.7638		–0.0076	
$x + h + 33$	16.3432		–0.0640		
		–0.8278			
$x_0 + 2h = 37$	15.5154				

Here $h = 4,\ u = \frac{30 - 29}{4} = \frac{1}{4} = 0.25.$

$u = 0.25$ lies between 0 and 1.

$\therefore$ Gauss's forward formula is suitable. Substituting in the Gauss's interpolation formula

$$y = y_0 + \frac{u}{1!}\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots$$

we get

$$y_{0.25} = f(0.25) = 17.1070 + (0.25)(-0.7638) + \frac{(0.25)\,(-0.75)}{2} \times (-0.0564) +$$

$$\frac{(1.25)\,(0.25)\,(-0.75)}{6} \times (-0.0076) + \frac{(1.25)(0.25)(-0.75)(-1.75)}{24}(-0.0022)$$

$$= 16.9216.$$

Example 6.2 *Use Gauss's backward formula and find the sales for the year 1966, given that*

Year	*1931*	*1941*	*1951*	*1961*	*1971*	*1981*
Sales (in lakhs)	*12*	*15*	*20*	*27*	*39*	*52*

Solution We have $h = 10$, we take 1971 as the origin. The central difference table with origin at 1971 is

u	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
–4	12					
		3				
–3	15		2			
		5		0		
–2	20		2		3	
		7		3		–10
–1	27		5		–7	
		12		–4		
0	39		1			
		13				
1	–52					

u at 1996 is $$u = \frac{1966 - 1971}{10} = \frac{-5}{10} = -0.5.$$

Gauss's backward formula is $$y = y_0 + \frac{u}{1!}\Delta y_{-1} + \frac{(u+1)u}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \ldots$$

substituting we get $$y_{-0.5} = 39 + (0.5)(12) + \frac{(0.5)(-0.5)}{2} \times 1 + \frac{0.5 \times (-0.5) \times (-1.5)}{6} \times (-4) + \ldots$$

$$= 39 - 6 - 0.125 - 0.25$$

$$y_{1966} = 32.625.$$

$\therefore$ The sales in the year 1966 is 32.625 lakh of rupees.

Example 6.3 *Apply Gauss's forward formula to find the value of* u_9 *if* $u_0 = 14$, $u_4 = 24$, $u_8 = 32$, $u_{16} = 40$.

Solution Taking origin at 8, we construct the difference table as follows

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–2	14				
		10			
–1	24		–2		
		8		–3	
0	32		–5		10
		3		7	
1	35		2		
		5			
2	40				

We have $$y_0 = 32,\ \Delta y_0 = 3,\ \Delta^2 y_{-1} = -5,\ \Delta^3 y_{-3} = 3,\ \Delta^4 y_{-2} = 7$$

and $$u = \frac{x - x_0}{h} = \frac{9-8}{4} = \frac{1}{4}.$$

From Gauss's forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \frac{(u+1)u(u-1)(u-2)}{4!}\Delta^4 y_{-2} + \ldots$$

$$y_{0.25} = 32 + \frac{1}{4}\times 3 + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{1\times 2}(-5) + \frac{\left(\frac{1}{4}+1\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)}{1\times 2\times 3}(7) + \frac{\left(\frac{1}{4}+1\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-2\right)}{1\times 2\times 3\times 4}\times 10$$

$$= 32 + \frac{3}{4} + \frac{15}{32} + \frac{35}{128} + \frac{175}{1024} = 33.1162$$

$\therefore\ u_9 = 33$ (approximately).

Example 6.4 *Apply Gauss's backward interpolation formula and find the population of a town in 1946, with the help of following data*

Year	*1931*	*1941*	*1951*	*1961*	*1971*
Population (in thousands)	*15*	*20*	*27*	*39*	*52*

Solution We have $h = 10$ taking origin at 1951

$$u = \frac{1946 - 1951}{10} = -0.5.$$

The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
–2	15				
		5			
–1	20		2		
		7		3	
0	27		5		–7
		12		–4	
2	39		1		
		13			
2	52				

Using Gauss's backward formula

$$f(x) = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\Delta^3 y_{-2} + \ldots$$

$$y_{-0.5} = 27 + (-0.5)\times 7 + \frac{(0.5)(-0.5)\times 5}{2} + \frac{(0.5)(-0.5)(-1.5)}{6}\times 3 + \frac{(1.5)(0.5)(-0.5)(-1.5)(-7)}{24} + \ldots$$

$$= 27 - 3.5 - \frac{1.25}{2} + \frac{(0.25)(1.5)}{2} - \frac{(1.5)(0.25)(10.5)}{24}$$

$$= 27 - 3.5 - 0.625 + 0.1875 - 0.1640625$$

$$y_{1946} = 22.898438.$$

$\therefore$ The population of the town in the year 1946 is 22.898 thousand, i.e., 22898.

Example 6.5 *Compute the value of* $\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2}\,dx$ *when x = 0.6538 by using*

(a) Gauss's forward formula

(b) Gauss's backward formula

(c) Stirling's formula

x	*0.62*	*0.63*	*0.64*	*0.65*	*0.66*	*0.67*	*0.68*
y	*0.6194114*	*0.6270463*	*0.634857*	*0.6420292*	*0.6493765*	*0.6566275*	*0.6637820*

Solution The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
0.62	0.6194114						
		0.007649					
0.63	0.6270463		–0.0000955				
		0.0075394		–0.0000004			
0.64	0.6345857		–0.0000959		0.0000001		
		0.0074435		–0.0000003		0.0000001	
0.65	0.6420292		–0.0000962		0.0000002		–0.0000004
		0.0073473		–0.0000001		–0.0000003	
0.66	0.6493765		–0.0000963		–0.0000001		
		0.0072510		–0.0000002			
0.67	0.66275		–0.0000965				
		0.0071545					
0.68	0.6637820						

we have $h = 0.01,\ x = 0.6538,\ x_0 = 0.65$

$$u = \frac{0.6538 - 0.65}{0.01} = 0.38.$$

(*a*) *Using Gauss's forward interpolation formula*

$$y_4 = y_0 + uy_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \ldots$$

$$y_{0.38} = 0.6420292 + 0.3 \times 0.0073473 + \frac{(0.38)(0.38-1)}{2} \times (-0.0000962) + \ldots$$

$$= 0.6448325.$$

(*b*) *Using Gauss's backward interpolation formula*

$$y = y_0 + u\Delta y_{-1} + \frac{u(u+1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-2} + \ldots$$

$$y_{0.38} = 0.6420292 + 0.00282853 - 0.0000252$$

$$= 0.6448325.$$

(*c*) *Using Stirling's formula*

The arithmetic mean of Gauss's forward and Gauss's backward formulae is

$$y_{0.38} = \frac{0.6448325 + 0.6448325}{2}$$

$$\therefore \; y = 0.6448325.$$

Example 6.6 *Apply Bessel's formula to obtain Y_{25} given that $Y_{20} = 2854$, $Y_{24} = 3162$, $Y_{28} = 3544$ and $Y_{32} = 3992$.*

Solution Taking 24 as the origin we get

$$u = \frac{25 - 24}{4} = \frac{1}{4}.$$

The difference table is

X	$u = \dfrac{X-24}{4}$	Y_u	ΔY_u	$\Delta^2 Y_u$	$\Delta^3 Y_u$
20	–1	2854			
			308		
24	0	3162		74	
			382		–8
28	1	3544		66	
			448		
32	2	3992			

The Bessel's formula is given by

$$Y_n = \frac{1}{2}(Y_0 + Y_1) + \left(u - \frac{1}{2}\right)\Delta Y_0 + \frac{u(u-1)}{2!}\frac{(\Delta^2 Y_{-1} + \Delta^2 Y_0)}{2} + \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 Y_{-1} + \ldots$$

$$\Rightarrow \; Y_{0.25} = \frac{1}{2}(3162 + 3544) + \left(\frac{1}{4} - \frac{1}{2}\right)\cdot(382) + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4} - 1\right)}{2!}\frac{(74 + 66)}{2} + \frac{\left(\frac{1}{4} - \frac{1}{2}\right)\frac{1}{4}\left(\frac{1}{4} - 1\right)}{3!}\cdot(-8)$$

$$= 3353 - 95.5 - 6.5625 - 0.0625$$

$$\Rightarrow \; Y_{0.25} = 3250.875,$$

$$\therefore \; y = 3250.875.$$

at $$x = 25$$

$$\therefore \; y_{25} = 3250.875.$$

Example 6.7 *If third difference are constant, prove that*

$$Y_{X-\frac{1}{2}} = \frac{1}{2}(Y_X + Y_{X+1}) - \frac{1}{16}\left(\Delta^2 Y_{X-1} + \Delta^2 Y_X\right).$$

Solution Bessel's formula is

$$Y_X = \frac{Y_0 + Y_1}{2} + \left(X - \frac{1}{2}\right)\Delta Y_0 + \frac{X(X-1)}{2!}\cdot\frac{\Delta^2 Y_{-1} + \Delta^2 Y_0}{2} + \frac{\left(X - \frac{1}{2}\right)X(X-1)}{3!}\Delta^3 Y_{-1} + \ldots \quad (8)$$

Given that the third difference are constant and taking the formula up to third differences and putting $X = \frac{1}{2}$ in (8) we get

$$Y_{\frac{1}{2}} = \frac{Y_0 + Y_1}{2} - \frac{1}{16}\left(\Delta^2 Y_{-1} + \Delta^2 Y_0\right), \quad (9)$$

shifting the origin to X, (9) reduces to

$$Y_{X+\frac{1}{2}} = \frac{1}{2}\left(Y_X + Y_{X+1}\right) - \frac{1}{16}\left(\Delta^2 Y_{X-1} + \Delta^2 Y_X\right).$$

Example 6.8 *Use Stirling's formula to find Y_{28}, given that Y_{20} = 49225, Y_{25} = 48316, Y_{30} = 47236, Y_{35} = 45926, Y_{40} = 44306.*

Solution Taking $X = 30$ as origin and $h = 5$ we get

$$u = \frac{28 - 30}{5} = -0.4.$$

The difference table is

X	$u = \frac{X - 30}{5}$	Y_u	ΔY_u	$\Delta^2 Y_u$	$\Delta^3 Y_u$	$\Delta^3 Y_u$
20	–2	49225				
			–909			
25	–1	48316		–171		
			–1080		–59	
30	0	47236		–230		–21
			–1310		–60	
35	1	45926		–310		
			–1620			
40	2	44306				

The Stirling's formula is

$$Y_u = y_0 + u\frac{\left(\Delta^2 Y_0 + \Delta^2 Y_{-1}\right)}{2} + \frac{u^2\Delta^2 Y_{-1}}{2} + \frac{u\left(u^2 - 1\right)}{6}\frac{\left(\Delta^3 Y_{-1} + \Delta^3 Y_{-2}\right)}{2} + \frac{u^2\left(u^2 - 1\right)}{24}\cdot\Delta^4 Y_{-2} + \ldots,$$

putting u = –0.4 and the values of various differences in the formula we get

$$Y_{-0.4} = 47236 + (-0.4)\frac{(-1310 - 1080)}{2} + \frac{(0.16)}{2}(-230) + \frac{(-0.4)(0.16-1)}{6}\cdot\frac{(-80-59)}{2} + \frac{(0.16)(0.16-1)}{24}\cdot(-21)$$

$$= 47236 + 478 - 18.4 - 3.8920 + 0.1176$$

$$= 47692$$

$$\Rightarrow\ Y_{28} = 47692.$$

Example 6.9 *Use Stirling's formula to compute $u_{12.2}$ from the following table*

x_0	10	11	12	13	14
$10^5 \log x$	23967	28060	31788	35209	38368

Solution The difference table is

x^0	$10^5 u_x$	$10^5 \Delta u_x$	$10^5 \Delta^2 u_x$	$10^5 \Delta^3 u_x$	$10^5 \Delta^4 u_x$
10	23967				
		4093			
11	28060		–365		
		3728		58	
12	31788		–307		–13
		3421		45	
13	35209		–262		
		3159			
14	38368				

We have
$$u = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2,$$
where x_0 = 12 is the origin.

The Stirling's formula is

$$y_u = y_0 + u\frac{(\Delta^2 y_0 + \Delta^2 y_{-1})}{2} + \frac{u^2\Delta^2 y_{-1}}{2} + \frac{u(u^2-1)}{6}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{u^2(u^2-1)}{24}\cdot\Delta^4 y_{-2} + \ldots$$

$$\Rightarrow\ 10^5 u_{12.2} = 31788 + (0.2)\left(\frac{3421 + 3728}{2}\right) + (0.02)(-307) - (0.016)(45+58) - (0.0016)(-13)$$

$$= 31788 + 714.9 - 6.1 - 1.6 + 0.000$$

$$\Rightarrow\ 10^5 u_{12.2} = 32495$$

$$\Rightarrow\quad u_{12.2} = 0.32495.$$

Example 6.10 *Apply Everett's formula to obtain y_{25}, given that y_{20} = 2854, y_{24} = 3162, y_{28} = 3544, y_{32} = 3992.*

Solution Taking origin at x = 24, and h = 4 we get

$$u = \frac{25 - 24}{4} = \frac{1}{4},$$

$$\therefore \; V = 1 - u = 1 - \frac{1}{4} = \frac{3}{4}.$$

The difference table is

x	$u = \dfrac{x-24}{4}$	y_u	Δy_u	$\Delta^2 y_u$	$\Delta^3 y_u$
20	–1	2854			
			308		
24	0	3162		74	
			382		–8
28	1	3544		66	
			448		
32	2	3992			

The Everett's formula is given by

$$y_u = u \cdot y_1 + \frac{u(u^2-1)}{3!}\Delta^2 y_0 + \frac{u(u^2-1)(u^2-4)}{5!}\Delta^4 y_{-1} + \ldots + \; Vy_0 + \frac{V(V^2-1)}{3!}\Delta^2 y_0 + \ldots$$

$$= \frac{1}{4} \cdot (3544) + \frac{\frac{1}{4}\left(\frac{1}{16}-1\right)}{6}(66) + \frac{3}{4} \cdot (3162) + \frac{\frac{3}{4}\left(\frac{9}{16}-1\right)}{6}(74)$$

$$= 886 - 2.5781 + 2371.5 - 4.0469$$

$$= 3254.875$$

$$\therefore \; y_{25} = 3254.875.$$

Exercise 6.1

1. Apply Gauss's forward formula to find the value of $f(x)$ at $x = 3.75$ from the table

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.225	18.644	17.262	16.047

2. Find the value of $f(x)$, by applying Gauss's forward formula from the following data

x	30	35	40	45	50
$f(x)$	3678.2	29995.1	2400.1	1876.2	1416.3

3. Find the value of cos 51° 42′ by Gauss backward formula given that

x	50°	51°	52°	53°	54°
cos x	0.6428	0.6293	0.6157	0.6018	0.5878

4. Given that

x	50°	51°	52°	53°	54°
$f(x)$	1.1918	1.2349	1.2799	1.3270	1.3764

using Gauss's backward formula find the value of tan 51° 42'.

5. $f(x)$ is a polynomial of degree for a and given that $f(4) = 270$, $f(5) = 648$, $\Delta f(5) = 682, \Delta^3 f(4) = 132$ $= 132$ find the value of $f(5.8)$.

6. Use Stirling's formula to find the value of $f(1.22)$ from the table

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8
$f(x)$	0.84147	0.89121	0.93204	0.96356	0.98545	0.99749	0.999570	0.99385	0.9385

7. Apply Stirling's formula to find the value of $f(1.22)$ from the following table which give the values

of $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_0^x e^{-\frac{x^2}{2}} \, dx$ at intervals of $x = 0.5$ from $x = 0$ to 2

x	0	0.5	1.0	1.5	2.0
$f(x)$	0	191	0.341	0.433	0.477

8. Apply Bessel's formula to find the value of $f(27.4)$ from the table

x	25	26	27	28	29	30
$f(x)$	4.000	3.846	3.704	3.571	3.448	3.333

9. Apply Bessel's formula to find the value of $y = f(x)$ at $x = 3.75$ given that

x	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	24.145	22.043	20.2250	18.644	17.262	16.047

10. Find the value of y_{15} using Bessel's formula if $y_{10} = 2854$, $y_{14} = 3162$, $y_{18} = 3544$, $y_{22} = 3992$.

11. Apply Laplace–Everett's formula to find the value of log 23.75, from the table

x	21	22	23	24	25	26
log x	1.3222	1.3424	1.3617	1.3802	1.919	1.4150

12. Find the value of e^{-x} when $x = 1.748$ form the following

x	1.72	1.73	1.74	1.75	1.76	1.77
$f(x) = e^{-x}$	0.1790	0.1773	0.1755	0.1738	0.1720	0.1703

13. Interpolate by means of Gauss's backward formula the population of a town for the year 1974, given that

Year	1939	1949	1959	1969	1979	1989
Population (in thousands)	12	15	20	27	39	52

14. Given that

$$\sqrt{12500} = 111.803399$$

$$\sqrt{12510} = 111.848111$$

$$\sqrt{12520} = 111.892806$$

$$\sqrt{12530} = 111.937483$$

find the value of $\sqrt{12516}$ by means of Gauss's backward formula.

15. $u_{20} = 51203$, $u_{30} = 43931$, $u_{40} = 34563$, $u_{50} = 24348$.Find the value of u_{35}, by using Gauss forward interpolation formula.

16. Use Gauss's backward formula to find the population in the year 1936, given the following table

Year	1901	1911	1921	1931	1941	1951
Population (in thousands)	12	15	20	27	39	52

17. Use (*i*) Stirling's and (*ii*) Bessel's formulae to find the value of sin 25° 40´ 30´´, given that

θ	25°40´0´´	25°40´20´´	25°40´40´´	25°41´0´´	25°41´20´´
$\sin\theta$	0.43313479	0.43322218	0.4330956	0.43339695	0.43348433

18. Compute the value of e^x when $x = 0.644$, by using

(*i*) Bessel's formula

(*ii*) Everett's formula, given that

x	0.61	0.62	0.63	0.64	0.65	0.66	0.67
$y = e^x$	1.840431	1.858928	1.877610	1.896481	1.915541	1.934792	1.954237

19. $f(20) = 51203$, $f(30) = 43931$, $f(40) = 34563$, $f(50) = 24348$ find $f(35)$ using, Bessel's formula.

20. Employ Bessel's formula to find the value of F at $x = 1.95$ given that

x	1.7	1.8	1.9	2.0	2.1	2.2	2.3
F	2.979	3.144	3.283	3.391	3.463	3.997	4.491

21. Given that $f(1) = 1.0000$, $f(1.10) = 1.049$

$f(1.20) = 1.096$, $f(1.30) = 1.40$

Use Everet's formula and find $f(1.15)$.

22. Using Bessel's formula find $f(25)$ given that $f(20) = 24$, $f(24) = 32$, $f(28) = 35$ and $f(32) = 40$.

23. Prove that

(*i*) $$y_{-n} = \sum_{K=0}^{n} (-1)^K \binom{n}{K} \nabla^K y$$

(*ii*) $$\Delta^r f_i = \delta^r f_{i+\frac{r}{2}} = \nabla^r f_{i+r} = (r!)\, h^r\, f(x_i, x_{i+1}, \ldots, x_{i+r})$$

Answers

1. 19.407	**2.** 2290	**3.** 06198
4. 1.2662	**5.** 0.1163	**6.** 0.93910
7. 0.389	**8.** 3.6497	**9.** 19.407
10. 3251	**11.** 3.3756	**12.** 0.1741
13. 32.345 thousands	**14.** 111.874930	**15.** 39431
16. 32.3437	**17.** (*i*) 0.43322218 (*ii*) 0.43326587	**18.** (*i*) 1.904082 (*ii*) 1.904082
19. 39431	**20.** 3.347	**21.** 1.0728
22. 32.95		

7

INVERSE INTERPOLATION

7.1 INTRODUCTION

Inverse interpolation by using Lagrange's interpolation formula was already discussed in the previous chapter in which the roles of x and y were interchanged and x was expressed as a function of y. Now we study two more methods namely

(a) Successive approximations and

(b) Reversion of series.

7.2 METHOD OF SUCCESSIVE APPROXIMATIONS

Let the values of independent variable x be given as $x_0, x_1, x_2, \ldots, x_n$ where x_i are equispaced with h as the interval of differencing (i.e., $x_i = x_0 + ih$, $h = 0, 1, \ldots, n$). If we are required to find x for a given value of y near the beginning of the tabulated value of y, we make use of Newton's forward interpolation formula as follows. From Newton's forward interpolation we have

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0, \ldots \tag{1}$$

where $$u = \frac{x - x_0}{h}.$$

Expression (1) may be written as

$$y - y_0 = u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0.$$

Dividing both sides by Δy_0 we get

$$\frac{y - y_0}{\Delta y_0} = u + \frac{(u-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} + \frac{u(u-1)(u-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} + \ldots + \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\frac{\Delta^n y_0}{\Delta y_0}. \tag{2}$$

Neglecting all the higher order differences other than the first we get

$$u = \frac{y - y_0}{\Delta y_0},$$

we denote it by u_1,

$$\therefore\ u_1 = \frac{y - y_0}{\Delta y_0}.$$

Expression (2) may be written as

$$u_1 = \frac{y - y_0}{\Delta y_0} - \frac{u(u-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u(u-1)(u-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} - \ldots - \frac{u(u-1)(u-2)\ldots(u-n+1)}{n!}\frac{\Delta^n y_0}{\Delta y_0}, \tag{3}$$

and the subsequent approximations u_2, u_3, ..., u_n of u are obtained from (3) as follows

$$u_2 = \frac{y - y_0}{\Delta y_0} - \frac{u_1(u-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \frac{u_1(u_1-1)(u_1-2)}{3!}\frac{\Delta^3 y_0}{\Delta y_0} - \ldots -$$

$$\frac{u_1(u_1-1)(u_1-2)\ldots(u_1-n+1)}{n!}\frac{\Delta^n y_0}{\Delta y_0},$$

$$\ldots$$
$$\ldots$$
$$\ldots$$

$$u_n = \frac{y - y_0}{\Delta y_0} - \frac{(u_{n-1})(u_{n-1}-1)}{2!}\frac{\Delta^2 y_0}{\Delta y_0} - \ldots - \frac{u_{n-1}(u_{n-1}-1)\ldots(u_{n-1}-n+1)}{n!}\frac{\Delta^n y_0}{\Delta y_0}.$$

The process of finding the approximations of u is continued till two successive approximations of u agree with each other to the required accuracy.

If u_n denotes the interpolated value of u, then $x = x_0 + h\, u_n$ gives the required value of x for a given value of y.

Similarly we use same technique with Newton's backward formula and Central difference interpolation formulae and interpolate x for a given y. As an example we consider the Stirling's interpolation formula.

Stirling's interpolation formula is

$$y = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1)}{3!}\left[\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right] + \ldots \tag{4}$$

We construct the difference table, substitute the values of the differences in (4), and write it in the form

$$y = y_0 + ua_1 + u^2 a_2 + u(u^2 - 1)\, a_3 + \ldots \tag{5}$$

where $a_1,\ a_2,\ a_n,\ \ldots$ are constants.

Expression (5) may be written as

$$u\ a_1 = (y - y_0) - u^2\ a_2 - u(u^2 - 1)\ a_3 \ldots$$

which gives
$$u = \frac{y - u_0}{a_1} - \frac{u^2 a_2}{a_1} - \frac{u(u^2 - 1)\ a_3}{a_1} - \ldots$$

To get the first approximation of u, we neglect all differences higher than the first and write

$$u_1 = \frac{y - y_0}{a_1},$$

where u_1 denotes the first approximation of u.

Substituting u_1 in (2) we get u_2, i.e., the second approximation of u. Similarly we can obtain the approximations of $u_3,\ u_4,\ \ldots,\ u_n$ of u.

Example 7.1 *Given table of values of the probability integral* $\frac{2}{\sqrt{\pi}}\int_0^x e^{-x^2} dx$ *corresponding to certain values of x, for what values of x in this integral equal to 0.5?*

x	*0.46*	*0.47*	*0.48*	*0.49*
y = f(x)	*0.4846555*	*0.4937452*	*0.5027498*	*0.5116683*

Solution

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.46	0.4846555			
		0.0090897		
0.47	0.4937452		–0.0000851	
		0.0090046		0.0000345
0.48	0.5027498		0.0001196	
		0.0088850		
0.49	0.5116683			

Taking $x_0 = 0.47$, we get $x_{-1} = 0.46$, $x_1 = 0.48$, $x_2 = 0.49$ and $h = 0.01$. Correspondingly we have $y_0 = 0.4937452$, $y_{-1} = 0.4846555$, $y_1 = 0.5027498$, $y_2 = 0.5116683$.

$\therefore$ Using Gauss's formula we write

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_{-1} + \frac{(u+1)u(u-1)}{3!}\Delta^3 y_{-1} + \ldots$$

$$\frac{y - y_0}{\Delta y_0} = u + \frac{u(u-1)}{2!}\frac{\Delta^2 y_{-1}}{\Delta y_0} + \frac{u(u^2-1)}{3!}\frac{\Delta^3 y_{-1}}{\Delta y_0}. \qquad (6)$$

The first approximation of u be u_1

$$\therefore\ u_1 = \frac{y - y_0}{\Delta y_0} = \frac{0.5 - 0.4937452}{0.0090046} = \frac{0.0062548}{0.0090046}$$

$$\therefore\ u_1 = 0.694623.$$

Expression (6) can be written as

$$u = \frac{y - u_0}{\Delta y_0} - \frac{u(u-1)}{2!}\frac{\Delta^2 y_{-1}}{\Delta y_0} - \frac{u(u^2-1)}{3!}\frac{\Delta^3 y_{-1}}{\Delta y_0}, \tag{7}$$

and for the second approximation of u we have

$$u_2 = \frac{y - u_0}{\Delta y_0} - \frac{u_1(u_1-1)}{2!}\frac{\Delta^2 y_{-1}}{\Delta y_0} - \frac{u_1(u_1^2 -)}{3!}\frac{\Delta^3 y_{-1}}{\Delta y_0}$$

$$\therefore\ u_2 = 0.694623 - \frac{0.694623 \times (-0.305377)}{2} \times \frac{(-0.0000851)}{(0.0090046)} -$$

$$\frac{0.694623 \times \left((0.694623)^2 - 1\right)}{6} \times \frac{(-0.0000345)}{(0.0090046)}$$

$$= 0.694623 - 0.0010024 - 0.0002295 = 0.693391.$$

Similarly

$$u_3 = \frac{y - y_0}{\Delta y_0} - \frac{u_2(u_2-1)}{2!}\frac{\Delta^2 y_{-1}}{\Delta y_0} - \frac{u_2(u_2^2-1)}{3!}\frac{\Delta^3 y_{-1}}{\Delta y_0},$$

$$\therefore\ u_3 = 0.694623 - \frac{0.693391(0.693391-1)}{2}\frac{(-0.0000851)}{(0.0090046)} - \frac{0.693391\left((0.693391)^2 - 1\right)}{6}\frac{(-0.0000345)}{(0.0090046)}$$

$$u_3 = 0.694623 - 0.0010046 - 0.0002299$$

$$\therefore\ u_3 = 0.693389.$$

Taking $u = 0.693389$, we get

$$x = x_0 + hu$$
$$x = 0.47 + 0.01 \times 0.693389$$
$$x = 0.47693389.$$

Example 7.2 *Given f(0) = 16.35, f(5) = 14.88, f(10) = 13.59, f(15) = 12.46 and f(x) = 14.00, find x.*

Solution Let $x_0 = 5$, such that $x_{-1} = 0$, $x_1 = 10$, $x_2 = 15$, then the difference table is

x	y	Δ	Δ^2	Δ^3
–1	$16.35 = y_{-1}$			
		$-1.47 = \Delta y_{-1}$		
0	14.88		$0.18 = \Delta^2 y_{-1}$	
		–1.29		$-0.02 = \Delta^3 y_{-1}$
1	13.59		0.16	
		–1.13		
2	12.46			

$\therefore$ Using Stirling's formula, we write

$$y = f(x) = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!}\Delta^2 y_{-1} + \ldots$$

$$14 = 14.88 + u\left[\frac{-1.29 - 1.47}{2}\right] + \frac{u^2}{2}\ 0.18$$

$$1.38u = 0.88 + 0.09u^2$$

$$u = \frac{0.88}{1.38} + \frac{0.09u^2}{1.38}. \qquad (8)$$

The first approximation u_1 of u is

$$u_1 = \frac{0.88}{1.38} = 0.6232,$$

form (8) second approximation

$$u_2 = \frac{0.88}{1.38} + \frac{0.09u^2}{1.38}u_1^2$$

$$= 0.6232 + \frac{0.09}{1.38}(0.6232)^2$$

$$= 0.6232 + 0.0056 = 0.6288.$$

Taking u = 0 0.6288, we get

$$x = x_0 + hu = 5 + 5(0.6288)$$

$$= 5 + 3.1440 = 8.14440$$

$\therefore$ $x = 8.1440$ at $y = 14.00$.

Note: When the second and higher order differences are very small (i.e., negligible) we form a quadratic equation and solve it for inverse interpolation. This method is clearly explained with an example as follows.

Example 7.3 *Given*

x	*4.80*	*4.81*	*4.82*	*4.83*	*4.84*
y = sinh x	*60.7511*	*61.3617*	*61.9785*	*62.6015*	*63.2307*

in sinh x = 62, find x.

Solution. Taking x_0 = 4.82 we get

h = 1, $x_0 - 2h$ = 4.80, $x_0 - h$ = 4.81, $x_0 + h$ = 4.83, $x_0 + 2h$ = 4.84 and y_0 = 61.9785, y_{-1} = 61.3617, y_{-2} = 60.7511, y_1 = 62.6015 y_2 = 63.2307 and y = 62.

The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4
$x_0 - 2h = 4.80$	$60.7511 = y_{-2}$				
		$0.6106 = \Delta y_{-2}$			
$x_0 - h = 4.81$	$61.3617 = y_{-1}$		0.0062		
		$0.6168 = \Delta y_{-1}$		0	

$x_0 = 4.82$	$61.9785 = y_0$		$0.0062 = \Delta^2 y_{-1}$		0
		$0.6230 = \Delta y_0$		$0 = \Delta^3 y_{-1}$	
$x_0 + h = 4.83$	62.6015		0.0062		
		$0.6292 = \Delta y_1$			
$x_0 - 2h = 4.84$	63.2307				

From the table we have $\Delta y_0 = 0.6230$, $\Delta y_{-1} = 0.6168$, $\Delta^2 y_{-1} = 0.0062$ and using Stirling's formula we may write

$$y = y_0 + u\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{u^2}{2!} u_{-1} + \ldots,$$

$$62 = 61.9785 + u\left[\frac{0.6230 + 0.6168}{2}\right] + \frac{u^2}{2} 0.0062,$$

simplifying we get

$$31u^2 + 6199u - 215 = 0,$$

neglecting the negative roots

$$u = 0.0347$$

$$\therefore \quad x = x_0 + hu$$

$$= 4.82 + 0.0347 \times 0.01$$

$$= 4.82 + 0.000347$$

$$= 4.820347$$

$$\therefore \quad x = 4.820347.$$

7.3 METHOD OF REVERSION SERIES

Let
$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \tag{9}$$

represent any given interpolations formula where $u = \dfrac{x - x_0}{h}$.

The above power series may be reverted and written as

$$x = \frac{y - a_0}{a_1} + c_1\left[\frac{y - a_0}{a_1}\right]^2 + c_2\left[\frac{y - a_0}{3}\right]^3 + \ldots + c_{n-1}\left[\frac{y - a_0}{a_1}\right]^n + \ldots \tag{10}$$

where $c_1, c_2, c_3, \ldots, c_{n-1}, \ldots$ are constants to be determined.

Let $\dfrac{y - a_0}{a_1} = c_1$ then from (10) we get

$$x = u + c_1 u^2 + c_2 u^3 + \ldots + c_{n-1} u^n + \ldots \tag{11}$$

Expression (9) may be written as

$$y - a_0 = a_1x + a_2x^2 + \dots + a_nx^n + \dots \tag{12}$$

Substituting $\quad y - a_0 = a_1c_1$

in (12) using the expression given in (10) for n and then comparing the coefficients of u_2, u_3, u_4, … on both sides we get

$$c_1 = \frac{-a_2}{a_1},$$

$$c_2 = \frac{-a_3}{a_1} + 2\left(\frac{a_2}{a_1}\right)^2,$$

$$c_3 = \frac{-a_4}{a_1} + 5\left(\frac{a_2\,a_3}{a_1}\right) - 5\left(\frac{a_2}{a_1}\right)^3,$$

$$c_4 = \frac{-a_5}{a_1} + 6\left(\frac{a_2\,a_4}{a_1^2}\right) + 3\left(\frac{a_3}{a_1}\right)^2 - \frac{2a_2^2a_3}{a_1^3} + 14\left(\frac{a_2}{a_1}\right)^4,$$

$$c_5 = \frac{-a_6}{a_1} + 7\left(\frac{a_2\,a_5 + a_3\,a_4}{a_1^2}\right) - 28\left(\frac{a_2^2\,a_4 + a_2\,a_3^2}{a_1^3}\right) + 84\frac{a_2^3\,a_3}{a_1^4} - 42\left(\frac{a_2}{a_1}\right)^2 \tag{13}$$

$$\vdots$$

The values of c's are computed by using (5) and then substituted when reverting the series with numerical coefficients. We shall now write Newton's forward interpolation formula, Gauss forward, Gauss backward, Stirling's and Bessel's formula in the form of power series and then write down the values of a_0, a_2, a_3, a_4, … in each case. Since the higher order difference are usually very small, we shall stop computations with forth differences.

(*a*) *Newton's forward interpolation formula*

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$= y_0 + \left(\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4}\right)u + \left(\frac{\Delta^2 y_0}{2} - \frac{\Delta^3 y_0}{3} + \frac{11\Delta^4 y_0}{24}\right)u^2 +$$

$$\left(\frac{\Delta^3 y_0}{6} - \frac{\Delta^4 y_0}{4}\right)u^3 + \frac{\Delta^4 y_0}{24}u^4.$$

Hence $a_0 = y_0$,

$$a_1 = \Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4},$$

$$a_2 = \frac{\Delta^2 y_0}{2} - \frac{\Delta^3 y_0}{2} + \frac{11\Delta^4 y_0}{24},$$

$$a_3 = \frac{\Delta^2 y_0}{6} - \frac{\Delta^4 y_0}{4},$$

$$a_4 = \frac{\Delta^4 y_0}{24}.$$

Using equations of (13) we compute the values of c_1, c_2, c_3, c_4 and the interpolate inversely.

(*b*) *Gauss's forward formula*

$$y = y_0 + u\Delta y_0 + \frac{u^2 - u}{2}\Delta^2 y_{-1} + \frac{u^3 - u}{6}\Delta^3 y_{-1} + \frac{u^2 - 2u^3 - u^2 + 24}{24}\Delta^4 y_{-2} + \ldots$$

$$= y_0 + \left(\Delta y_0 - \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{6}\Delta^3 y_{-1} + \frac{1}{12}\Delta^4 y_{-12}\right)u + \left(\frac{1}{2}\Delta^2 y_{-1} - \frac{1}{24}\Delta^4 y_{-2}\right)u^2 +$$

$$\left(\frac{1}{6}\Delta^3 y_{-1} - \frac{1}{12}\Delta^4 y_{-2}\right)u^3 + \left(\frac{1}{24}\Delta^4 y_{-1}\right)u^4 + \ldots$$

Here $a_0 = y_0$, $\qquad a_1 = \Delta y_0 - \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{6}\Delta^3 y_{-1} + \frac{1}{12}\Delta^4 y_{-2}$,

$$a_2 = \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{24}\Delta^4 y_{-2},$$

$$a_3 = \frac{1}{6}\Delta^3 y_{-1} - \frac{1}{12}\Delta^4 y_{-2},$$

$$a_4 = \frac{1}{24}\Delta^4 y_{-2}.$$

(*c*) *Gauss's backward formula*

$$y = y_0 + u\Delta y_{-1} + \frac{u^2 - u}{6}\Delta^2 y_{-1} + \frac{u^3 - u}{6}\Delta^3 y_{-2} + \frac{u^4 + 2u^3 - u^2 - 24}{24}\Delta^4 y_{-2} + \ldots$$

$$= y_0 + \left(\Delta y_{-1} + \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{6}\Delta^3 y_{-2} - \frac{1}{12}\Delta^4 y_{-2}\right)u + \left(\frac{1}{2}\Delta^2 y_{-1} - \frac{1}{24}\Delta^4 y_{-2}\right)u^2 +$$

$$\left(\frac{1}{6}\Delta^3 y_{-2} + \frac{1}{12}\Delta^4 y_{-2}\right)u^3 + \left(\frac{1}{24}\Delta^4 y_{-2}\right)u^4 + \ldots$$

Here $a_0 = y_0$,

$$a_1 = \Delta y_{-1} + \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{6}\Delta^3 y_{-2} - \frac{1}{12}\Delta^4 y_{-2},$$

$$a_3 = \frac{1}{6}\Delta^3 y_{-2} + \frac{1}{12}\Delta^4 y_{-2},$$

$$a_4 = \frac{1}{24}\Delta^4 y_{-2}.$$

(d) *Stirling's formula*

$$y = y_0 + u\left[\frac{\Delta y_{-1} + \Delta y_0}{2}\right] + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{6}\left[\frac{\Delta^3 y_{-2} + \Delta^3 y_{-3}}{2}\right] + \frac{u^2(u^2-1)}{24}\Delta^4 y_{-2} + \ldots$$

$$y = y_0 + \left(\frac{\Delta y_{-1} + \Delta y_0}{2} - \frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12}\right)u +$$
$$\left(\frac{\Delta^2 y_{-1}}{2} - \frac{\Delta^4 y_{-2}}{24}\right)u^2 + \left(\frac{\Delta^3 y_{-2} + \Delta^3 y_{-1}}{12}\right)u^3 + \frac{\Delta^4 y_{-2}}{24}u^4.$$

Here $a_0 = y_0$,

$$a_1 = \frac{1}{2}[\Delta y_{-1} + \Delta y_0] - \frac{1}{2}[\Delta^3 y_{-2} + \Delta^3 y_{-1}],$$

$$a_2 = \frac{1}{2}\Delta^2 y_{-1} - \frac{1}{24}\Delta^4 y_{-2},$$

$$a_3 = \frac{1}{12}(\Delta^3 y_{-2} + \Delta^3 y_{-1}),$$

$$a_4 = \frac{1}{24}\Delta^4 y_{-2}.$$

(e) *Bessel's formula*

$$y = \frac{y_0 + y_1}{2} + \left(u - \frac{1}{2}\right)\Delta y_0 + \frac{u(u-1)}{2!}\left[\frac{\Delta^2 y_0 + \Delta^2 y_{-1}}{2}\right] + \frac{\left(u-\frac{1}{2}\right)u(u-1)}{3!}\Delta^3 y_{-1} +$$
$$\frac{(u+1)u(u-1)(u-2)}{4!}\frac{(\Delta^4 y_{-1} + \Delta^4 y_{-2})}{2}.$$

Taking $u - \frac{1}{2} = v$ we get

$$y = \frac{y_0 + y_1}{2} + v\Delta y_0 + \frac{\left(v^2 - \frac{1}{4}\right)}{2}\frac{\Delta^2 y_{-1} + \Delta^2 y_0}{2} +$$
$$\frac{v\left(v^2 - \frac{1}{4}\right)}{6}\Delta^2 y_{-1} + \frac{\left(v^2 - \frac{1}{4}\right)\left(v^2 - \frac{9}{4}\right)}{24}\frac{\Delta^4 y_{-2} + \Delta^4 y_{-1}}{2}$$

$$= \frac{y_0 + y_1}{2} - \frac{\Delta^2 y_{-1} + \Delta^2 y_0}{16} + \frac{3}{256}\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right] + \left[\Delta y_0 - \frac{1}{24}\Delta^3 y_{-1}\right]v +$$

$$\left[\frac{1}{4}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) - \frac{5}{96}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right)\right]v^2 + \frac{1}{6}\Delta^3 y_{-1} v^3 + \frac{1}{48}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right)v^4 .$$

Here
$$a_0 = \frac{1}{2}(y_0 + y_1) - \frac{1}{18}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) + \frac{3}{256}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right),$$

$$a_1 = \Delta y_0 - \frac{1}{24}\Delta^3 y_{-1},$$

$$a_2 = \frac{1}{4}\left(\Delta^2 y_{-1} + \Delta^2 y_0\right) - \frac{5}{96}\left(\Delta^4 y_{-2} + \Delta^4 y_{-1}\right),$$

$$a_3 = \frac{1}{6}\Delta^3 y_{-1},$$

$$a_4 = \frac{1}{48}\left[\Delta^4 y_{-2} + \Delta^4 y_{-1}\right].$$

We find the values of c_1, c_2, c_3, c_4 by using equations of (13) and interpolate inversely.

Example 7.4 *If sinh x = 62, find x using the following data*

x	*4.80*	*4.81*	*4.82*	*4.83*	*4.84*
sinh x	*60.7511*	*61.3617*	*61.9785*	*62.6015*	*63.2307*

Solution

x	$y = \sin x$	Δ	Δ^2	Δ^3	Δ^4
$x_0 - 2h = 4.80$	$0.75111 = y_{-2}$				
		$\Delta y_{-2} = 0.6106$			
$x_0 - h = 4.81$	$61.3617 = y_{-1}$		$\Delta^2 y_{-2} = 0.0062$		
		$\Delta y_{-1} = 0.6168$		0	
$x_0 = 4.82$	$61.9785 = y_0$		$\Delta^2 y_{-1} = 0.0062$		0
		$\Delta y_0 = 0.6230$		0	
$x_0 + h = 4.83$	$62.6015 = y_1$		$\Delta^2 y_0 = 0.0062$		
		$\Delta y_1 = 0.6292$			
$x_0 + 2h = 4.84$	$62.2307 = y_2$				

Using Stirling's formula we can write

$$y = 61.9785 + \left[\frac{0.6168 + 0.6230}{2} - \frac{0+0}{2}\right]u + \left[\frac{0.0062}{2} - 0\right]u^2 + \ldots$$

$$\therefore\ y = 61.9785 + 0.6199u + 0.0031u^2 .$$

We have

$\therefore$ $$a_0 = 61.9785,\ a_1 = 0.6199,\ a_2 = 0.0031,\ a_3 = 0,\ a_4 = 0, \text{ and } y = 62,$$

$\Rightarrow$ $$y - a_0 = 62 - 61.9785 = 0.215.$$

$$\therefore\ u = \frac{y - a_0}{a_1} = \frac{0.0215}{0.6199} = 0.034683.$$

$$\frac{a_2}{a_2} = \frac{a_2}{a_1} = \frac{0.0031}{0.6199} = 0.005001,$$

$$\therefore\ c_1 = -0.005001,$$

$$c_2 = 0 + 2(0.00501)^2 = 0.0000502$$

$$c_3 = 3$$

$$\therefore\ u = 0.034683 - 0.005001 + 0.0000502 + (0.034683)3$$

$$u = 0.0347.$$

$$\therefore\ x = x_0 + h = 4.82 + 0.01 + 0.0347$$

$$= 4.8203.$$

Example 7.5 *Find by the method of inverse interpolation the real root of the equation $x^3 + x - 3 = 0$ which lies between 1.2 and 1.3.*

Solution Let

$$f(x) = x^3 + x - 3$$

We construct a difference table for values of x starting from 1 with 0.1 as the step length. The table is given below

x	$f(x)$	$u = \frac{x-1.2}{1}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	–1	–2				
			0.431			
1.1	–0.569	–1		0.066		
			0.497		0.006	
1.2	–0.072	0		0.072		0
			0.569		0.006	
1.3	–0.497	1		0.078		
			0.647			
1.4	1.144	2				

From the table it is clear that $f(x)$ changes its between $x = 1.2$ and $x = 1.3$. Hence the root of $f(x) = 0$ should lie between 1.2 and 1.3 to find the value of x we use Stirling's formula (taking 1.2 as the origin). We have

$$y = y_0 + u\cdot\frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{u^2}{2}\Delta^2 y_{-1} + \frac{u(u^2-1)}{6}\cdot\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \ldots$$

$$0 = -0.072 + u \cdot \frac{0.569 + 0.497}{2} + \frac{u^2}{2}(0.072) + \frac{u(u^2-1)}{2}(0.006)$$

$$\Rightarrow \; 0 = -0.072 + 0.531u + 0.0036u^2 + 0.001u^3$$

$$\Rightarrow \; u = \frac{0.072}{0.531} - \frac{0.0036}{0.531}u^2 - \frac{0.001}{0.531}u^3. \qquad (14)$$

The first approximation u_1 of u is

$$u_1 = \frac{0.072}{0.531} = 0.1353.$$

Putting $u = 0.1353$ in RHS of (14) we get

$$u = 0.1353 - \frac{0.0036}{0.531}(0.1353)^2 - \frac{0.001}{0.531}(0.1353)^2 = 0.134.$$

Taking $u = 0.134$, we get

$$\begin{aligned} x &= x_0 + hu \\ &= 1.2 + (0.1)\ (0.134) \\ &= 1.2 + 0.0134 \\ &= 1.2134 \end{aligned}$$

$\therefore$ The required root is 1.2134.

Exercise 7.1

1. The equation $x^3 - 15x + 4 = 0$ has a root close to 0.3. Obtain this root up to 6 decimal places, using inverse interpolation with Bessel's Interpolation formula.

2. Given $f(0) = 16.35$, $f(5) = 14.88$, $f(10) = 13.5$, $f(15) = 12.46$. Find x for which $f(x) = 14.00$.

3. Given $u_{10} = 544$, $u_{15} = 1227$, $u_{20} = 1775$ find correct to one decimal place, the value of x for which $u_x = 100$.

4. The following values of $f(x)$ are given

x	10	15	20
$f(x)$	1754	2648	3564

Find the value of x for which $f(x) = 3000$, by successive approximation method.

5. Use Lagrange's formula to find the number whose Logarithm is 0.30500 having given

log 1 = 0, log 2 = 0.30103, log 3 = 0.47712 and log 4 = 0.060206

6. Given a table of values of the probability integral $\dfrac{2}{\sqrt{\pi}}\displaystyle\int_0^x e^{-x^2}\,dx$

For what value of x is this integral equal to $\frac{1}{2}$

x	0.45	0.46	0.47	0.48	0.49	0.50
$f(x)$	0.4754818	0.484655	0.4937452	0.5027498	0.4116583	0.5304999

7. For the function $y = x^3$ construct a forward difference table when x takes the values $x = 2, 3, 4, 5$, and find the cube root of 10.

8. Find the value of x for $y = \cos x = 1.285$ by the method of inverse interpolation using difference up to second order only given

x	0.736	0.737	0.738	0.739	0.740	0.741
$y = \cos x$	1.2832974	1.2841023	1.2849085	1.2857159	1.2865247	1.2873348

9. The equation $x^3 - 6x - 11 = 0$ has a root between 3 and 4. Find the root.

10. Find by the method of inverse interpolation the real root of the equation $x^3 + x - 3 = 0$ which lies between 1.2 and 1.3.

Answers

1. 0.267949 **2.** 8.34 **3.** 13.3

4. 16.896 **5.** 2.018 **6.** 0.47693612

7. 2.154 **8.** 0.738110 **9.** 3.091

10. 1.2314

8

NUMERICAL DIFFERENTIATION

8.1 INTRODUCTION

The process of computing the value of the derivative $\frac{dy}{dx}$ for some particular value of x from the given data when the actual form of the function is not known is called *Numerical differentiation*. When the values of the argument are equally spaced and we are to find the derivative for some given x lying near the beginning of the table, we can represent the function by Newton–Gregory forward interpolation formula. When the value of $\frac{dy}{dx}$ is required at a point near the end of the table, we use Newton's backward interpolation formula and we may use suitable Central difference interpolation formula when the derivative is to be found at some point lying near the middle of the tabulated values. If the values of argument x are not equally spaced, we should use Newton's divided difference formula to approximate the function $y = f(x)$.

8.2 DERIVATIVES USING NEWTON'S FORWARD INTERPOLATION FORMULA

Consider Newton's forward interpolation formula

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{1\times 2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 y_0 + \ldots \quad (1)$$

where $$u = \frac{x - x_0}{h} \quad (2)$$

differentiating (1) w.r.t. u we get

$$\frac{dy}{du} = \Delta y_0 + \frac{2u-1}{1\times 2}\Delta^2 y_0 + \frac{3u^2 - 6u + 2}{1\times 2\times 3}\Delta^3 y_0 + \ldots \quad (3)$$

differentiating (2) w.r.t. x we get

$$\frac{du}{dx} = \frac{1}{h}. \tag{4}$$

Now from equations (3) and (4)

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{1\times 2}\Delta^2 y_0 + \frac{3u^2-6u+2}{1\times 2\times 3}\Delta^3 y_0 + \ldots\right] \tag{5}$$

Expression (5) gives the value of $\frac{dy}{dx}$ at any x which is not tabulated. The formula (5) becomes simple for tabulated values of x, in particular when $x = x_0$ and $u = 0$.

Putting $u = 0$ in (5) we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right], \tag{6}$$

differentiating (5) w.r.t. x

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dy}{dx}$$

$$= \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+11}{12}\Delta^4 y_0 + \ldots\right], \tag{7}$$

putting $u = 0$ in (7) we have

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\right],$$

similarly we get

$$\left(\frac{d^3 y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right], \text{ and so on.} \tag{8}$$

Aliter:

We know that $$E = e^{hD}$$

$$\Rightarrow 1 + \Delta = e^{hD}$$

$$\Rightarrow hD = \log(1+\Delta)$$

$$\Rightarrow \; hD = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots$$

$$\Rightarrow \; D = \frac{1}{h}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]$$

$$D^2 = \frac{1}{h^2}\left[\Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \ldots\right]^2$$

$$\Rightarrow \; D^2 = \frac{1}{h^2}\left[\Delta^2 - \Delta^3 + \frac{11}{12}\Delta^4 - \frac{5}{6}\Delta^5 + \ldots\right]$$

$$\vdots$$

Applying the above identities to y_0, we have

$$Dy_0 = \left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$D^2 y_0 = \left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 + \ldots\right]$$

$$\vdots$$

8.3 DERIVATIVES USING NEWTON'S BACKWARD INTERPOLATION FORMULA

Consider the Newton's backward interpolation formula

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots \quad (9)$$

where $$u = \frac{x - x_n}{h} \quad (10)$$

(h being the interval of differencing).

Differentiating (9) w.r.t. u we get

$$\frac{dy}{du} = \nabla y_n + \frac{(2u+1)}{1 \times 2}\nabla^2 y_n + \frac{(3u^2 + 6u + 2)}{1 \times 2 \times 3}\nabla^3 y_n + \ldots \quad (11)$$

and differentiating (10) w.r.t. x we get

$$\frac{du}{dx} = \frac{1}{h}. \quad (12)$$

Now $$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx},$$

using (11) and (12) we can write

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{(2u+1)}{1\times 2}\nabla^2 y_n + \frac{(3u^2+6u+2)}{1\times 2\times 3}\nabla^3 y_n + \ldots\right]. \quad (13)$$

Expression (13) gives us the value of $\frac{dy}{dx}$ at any x which is not tabulated.

At $x = x_n$ we have $$u = \frac{x_n - x_n}{n} = 0.$$

Putting $u = 0$ in (13) we get

$$\left(\frac{dy}{dx}\right)_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right], \quad (14)$$

differentiating (13) w.r.t. x we get

$$\frac{d^2y}{dx^2} = \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx}$$

$$= \frac{1}{h^2}\left[\nabla^2 y_n + (u+1)\nabla^3 y_n + \frac{(6u^2+18u+11)}{12}\nabla^4 y_n + \ldots\right] \quad (15)$$

putting $u = 0$ in (15), we have

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right].$$

In a similar manner we can find the derivatives of higher order at $x = x_n$.

8.4 DERIVATIVES USING STIRLING'S FORMULA

Consider the Stirling's formula

$$y = y_0 + \frac{u}{1!}\frac{(\Delta y_0 + \Delta y_{-1})}{2} + \frac{u_2}{2!}\Delta^2 y_{-1} +$$

$$\frac{u(u^2-1^2)}{3!}\frac{(\nabla^3 y_{-1} + \nabla^3 y_{-2})}{2} + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} +$$

$$\frac{u(u^2-1^2)(u^2-2^2)}{5!}\frac{\left[\Delta^5 y_{-2} + \Delta^5 y_{-3}\right]}{2} + \ldots \quad (16)$$

where $$u = \frac{x - x_0}{h}. \tag{17}$$

Differentiating (16) w.r.t. u we have

$$\frac{dy}{du} = \frac{\Delta y_0 + \Delta y_{-1}}{2} + \frac{2u}{1\times 2}\Delta^2 y_{-1} + \frac{(3u^2 - 1)}{1\times 2\times 3}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$

$$\frac{4u^3 - 2u}{1\times 2\times 3\times 4}\Delta^4 y_{-2} + \frac{(5u^4 - 15u^2 + u)}{1\times 2\times 3\times 4\times 5}\left(\frac{\Delta^5 y_{-2} + \Delta^5 y_{-3}}{2}\right) + \ldots, \tag{18}$$

Differentiating (17) w.r.t. x

$$\frac{dy}{dx} = \frac{1}{h} \tag{19}$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{h}\cdot\frac{dy}{du}$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2}\right] + \frac{2u}{1\times 2}\Delta^2 y_{-1} +$$

$$\frac{(3u^2 - 1)}{1\times 2\times 3}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{(4u^3 - 2u)}{1\times 2\times 3\times 4}\Delta^4 y_{-2} +$$

$$\frac{(5u^4 - 15u^2 + u)}{1\times 2\times 3\times 4\times 5}\frac{(\Delta^5 y_{-2} + \Delta^5 y_{-3})}{2} + \ldots \tag{20}$$

$u = 0$ at $x = x_0$.
Putting $u = 0$ in (20) we get

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{(\Delta^3 y_{-1} + \Delta^3 y_{-2})}{2} + \frac{1}{30}\frac{(\Delta^5 y_{-2} + \Delta^5 y_{-3})}{2} + \ldots\right] \tag{21}$$

Differentiating (20) w.r.t. x and putting $x = x_0$ we get

$$\left(\frac{d^2 y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \ldots\right]. \tag{22}$$

Example 8.1 *From the table of values below compute* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *for* $x = 1$

x	1	2	3	4	5	6
y	1	8	27	64	125	216

Solution The difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
		7			
2	8		12		
		19		6	
3	27		18		0
		37		6	
4	64		24		0
		61		6	
5	125		30		
		91			
6	216				

We have $x_0 = 1$, $h = 1$. $x = 1$ is at the beginning of the table.

$\therefore$ We use Newtons forward formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = \frac{1}{h}\left[7 - \frac{1}{2}12 + \frac{1}{3}6 - 0 + \ldots\right]$$

$$= 7 - 6 + 2 = 3$$

and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \ldots\right]$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1} = \frac{1}{1^2}\left[12 - 6\right] = 6$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1} = 3, \left(\frac{d^2y}{dx^2}\right)_{x=1} = 6.$$

Example 8.2 *From the following table of values of x and y find* $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *for* $x = 1.05$.

x	1.00	1.05	1.10	1.15	1.20	1.25	1.30
y	1.00000	1.02470	1.04881	1.07238	1.09544	1.11803	1.14017

Solution The difference table is as follows

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.00	1.00000					
		0.02470				
1.05	1.02470		–0.00059			
		0.002411		–0.00002		
1.10	1.04881		–0.00054		0.00003	
		0.02357		–0.00001		–0.00006
1.15	1.07238		–0.00051		–0.00003	
		0.02306		–0.00002		
1.20	1.09544		–0.00047			
		0.02259				
1.25	1.11803		–0.00045			
		0.02214				
1.30	1.14017					

Taking $x_0 = 1.05$, $h = 0.05$ we have

$$\Delta y_0 = 0.02411,$$

$$\Delta^2 y_0 = -0.00054,$$

$$\Delta^3 y_0 = 0.00003,$$

$$\Delta^4 y_0 = -0.00001,$$

$$\Delta^5 y_0 = -0.00003,$$

from Newton's formula

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 - \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1.05} = \frac{1}{0.05}\left[0.02411 - \frac{0.00054}{2} + \frac{1}{3}(0.00003)\right] + \frac{1}{0.05}\left[-\frac{1}{4}(0.00001) + \frac{1}{5}(0.00003)\right]$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=1.05} = 0.48763$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

$$\Rightarrow \left(\frac{d^2y}{dx^2}\right)_{x=1.05} = \frac{1}{(1.05)^2}\left[-0.00054 - 0.0003 + \frac{11}{12}(0.00001) - \frac{5}{6}(-0.00003)\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=1.05} = -0.2144.$$

Example 8.3 *A rod is rotating in a plane about one of its ends. If the following table gives the angle θ radians through which the rod has turned for different values of time t seconds, find its angular velocity at t = 7 secs.*

t seconds	*0.0*	*0.2*	*0.4*	*0.6*	*0.8*	*1.0*
θ *radians*	*0.0*	*0.12*	*0.48*	*0.10*	*2.0*	*3.20*

Solution The difference table is given below

t	θ	$\nabla\theta$	$\nabla^2\theta$	$\nabla^3\theta$	$\nabla^4\theta$
0.0	0.0				
		0.12			
0.2	0.12		0.24		
		0.36		0.02	
0.4	0.48		0.26		0
		0.62		0.02	
0.6	1.10		0.28		0
		0.90		0.02	
0.8	2.0		0.30		
		1.20			
1.0	3.20				

Here $$x_n = t_n = 1.0,\ h = 0.2,\ x = t = 0.7$$

$$u = \frac{x - x_n}{h} = \frac{0.7 - 1.0}{0.2} = -1.5.$$

From the Newton's backward interpolation formula, we have

$$\left(\frac{d\theta}{dt}\right)_{n=0.7} = \frac{1}{h}\left[\nabla\theta_0 + \frac{2u+1}{2}\nabla^2\theta_0 + \frac{3u^2 + 6u + 2}{6}\nabla^2\theta_n\right]$$

$$= \frac{1}{0.2}\left[1.20 - 0.30 + \frac{3(-1.5)^2 - 6(-1.5) + 2}{6}(0.02)\right]$$

$$= 5(1.20 - 0.30 - 0.0008) = 4.496 \text{ radian/sec}$$

$$\therefore \quad \frac{d\theta}{dt} = 4.496 \text{ radian/sec}$$

and $$\left(\frac{d^2\theta}{dt^2}\right)_{t=0.7} = \frac{1}{h^2}\left[\nabla^2\theta_0 + (u+1)\nabla^2\theta_0\right]$$

$$= \frac{1}{(0.2)^2}\left[0.30 - 0.5 \times 0.02\right]$$

$$= 25 \times 0.29 = 7.25 \text{ radian/sec}^2.$$

∴ Angular velocity = 4.496 radian/sec and

Angular acceleration = 7.25 radian/sec^2.

Example 8.4 *Find* $\frac{dy}{dx}$ *at x = 0.6 of the function y = f(x), tabulated below*

x	*0.4*	*0.5*	*0.6*	*0.7*	*0.8*
y	*1.5836494*	*1.7974426*	*2.0442376*	*2.3275054*	*2.6510818*

Solution The difference table

x	y	Δy	$\Delta^2 y$	Δ^3	Δ^4
0.4	1.5836494				
		0.2137932			
0.5	1.7974426		0.0330018		
		0.2467950		0.0034710	
0.6	2.0442376		0.0364728		0.0003648
		0.2832678		0.0038358	
0.7	2.3275054		0.0403084		
		0.3235764			
0.8	2.6510818				

substituting these values in the Stirling's formula, i.e., in

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{6}\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2} + \ldots\right]$$

we get
$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=0.6} = \frac{1}{0.1}\left[\frac{1}{2}(0.2832678 + 0.2467950) - \frac{1}{12}(0.0038358 + 0.0034710)\right]$$

$$= 10(0.2650314 - 0.0006089) = 2.644225$$

$$\therefore \left(\frac{dy}{dx}\right)_{x=0.6} = 2.644225.$$

Example 8.5 *From the following table find x correct to two decimal places, for which y is maximum and find this value of y*

x	*1.2*	*1.3*	*1.4*	*1.5*	*1.6*
y	*0.9320*	*0.9636*	*0.9855*	*0.9975*	*0.996*

Solution

The forward difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1.2	0.9320			
		0.0316		
1.3	0.9636		–0.0097	
		0.0219		–0.0002
1.4	0.9855		–0.0099	

		0.0120		–0.0002
1.5	0.9975		–0.0099	
		0.0021		
1.6	0.9996			

we have $x_0 = 1.2$. For maximum value of y we take $\frac{dy}{dx} = 0.$

Differentiating Newton's forward interpolation formula w.r.t. u and neglecting terms of second differences we get

$$0 = 0.0316 + \frac{2u-1}{2}(-0.0097)$$

$$\Rightarrow\ 0 = 0.0712 - (2u-1)(0.0097)$$

$$\Rightarrow\ (2u-1)(0.0097) = 0.0712$$

$$\Rightarrow\ u = 3.8.$$

Substituting in $x = x_0 + uh,$

we get $x = 1.2\ (3.8\ )\ (0.1) = 1.58.$

The value 1.58 is closer to $x = 1.6$, hence we use Newton's backward difference formula

$$f(1.58) = 0.9996 - (0.2)(0.0021) + \frac{(-0.2)(-0.2+1)}{2}(-0.0099)$$

$$= 0.9996 - 0.0004 + 0.0008 \ = 1.000$$

The maximum value occurs at $x = 1.58$ and the maximum value is 1.000.

Example 8.6 *Find the maximum and the minimum values of the function $y = f(x)$ from the following data*

x	*0*	*1*	*2*	*3*	*4*	*5*
f(x)	*0*	*0.25*	*0*	*2.25*	*16.00*	*56.25*

Solution The forward difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
$0 = x_0$	$0 = y_0$					
		$0.25 = \Delta y_0$				
1	0.25		$-0.50 = \Delta^2 y_0$			
		–0.25		$3.00 = \Delta^3 y_0$		
2	0		2.50		$6 = \Delta^4 y_0$	
		2.25		9.00		$0 = \Delta^5 y_0$
3	2.25		11.50		6	
		13.75		15.00		
4	16.00		26.50			
		40.25				
5	56.25					

We have $x_0 = 0$, $h = 1$, differentiating Newton–Gregory forward interpolation formula we get

$$f'(x) = \frac{1}{h}\left[\Delta y_0 + \frac{2u-1}{2!}\Delta^2 y_0 + \frac{3u^2-6u+2}{3!}\Delta^3 y_0 + \frac{4u^3-18u^2+22u-6}{4!}\Delta^4 y_0 + \ldots\right]$$

$$\Rightarrow \quad f'(x) = \frac{dy}{dx} = 0.25 = \frac{2u-1}{2}(-0.50) + \frac{1}{6}\left(3u^2-6u+2\right)(3.00) + \frac{1}{24}\left(4u^3-18u^2+22u-6\right)(6.00)$$

$$\Rightarrow \quad \frac{dy}{dx} = u^3 - 3u^2 + 24.$$

At a maximum point or at a minimum point we have $\frac{dy}{dx} = 0$

$$\therefore \quad u^3 - 3u^2 + 2u = 0$$

$$\Rightarrow \quad u(u-1)(u-2) = 0 \quad \Rightarrow \quad u = 0, u = 1, u = 2,$$

$$\frac{d^2y}{dx^2} = f''(x) = \frac{1}{h^2}\left[\Delta^2 y_0 + (u-1)\Delta^3 y_0 + \frac{6u^2-18u+1}{12}\Delta^4 y_0 + \ldots\right]$$

$$= \frac{1}{1^2}\left[-0.5 + (u-1)(3.00) + \frac{6u^2-18u+11}{12}\times 6\right],$$

clearly $$f''_{(0)} = \left(\frac{d^2y}{dx^2}\right)_{x=0} > 0$$

and $$f''_{(2)} = \left(\frac{d^2y}{dx^2}\right)_{x=2} > 0$$

$\therefore$ $f(x)$ is minimum at $x = 0$ and $x = 2$. The minimum values are $f(0) = 0$, $f(2) = 0$.

Since

$$f''_{(1)} = \left(\frac{d^2y}{dx^2}\right)_{x=1} < 0$$

$f(x)$ has a maximum at $x = 1$. The maximum value is $f(1) = 0.25$.

Exercise 8.1

1. Find the first and second derivatives of the function tabulated below at the point $x = 1.5$

x	1.5	2.0	2.5	3.0	3.5	4.0
$y = f(x)$	3.375	7.0	13.625	240	38.875	59.0

2. Find $f'(1.1)$, $f''(1.1)$ from the following table

x	1.1	1.2	1.3	1.4	1.5
$f(x)$	2.0091	2.0333	2.0692	2.1143	2.1667

3. Find $\frac{dy}{dx}$ at $x = 1$

x	1	2	3	4	5	6
$y = f(x)$	1	8	27	64	125	216

also find $\frac{d^2y}{dx^2}$ at $x = 1$.

4. Find $\frac{dy}{dx}$ at $x = 3.0$ of the function tabulated below

x	3.0	3.2	3.4	3.6	3.8	4.0
y	–14.000	–10.032	–5.296	0.256	6.672	14.000

5. Find $f'(0.4)$ from the following table

x	0.1	0.2	0.3	0.4
$f(x)$	1.10517	1.22140	1.34986	1.49182

6. Find $f'(0.96)$ and $f''(0.96)$ from the following table

x	0.96	0.98	1.00	1.02	1.04
$f(x)$	0.7825	0.7739	0.7651	0.7563	0.7473

7. Compute the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 0$ from the following table

x	0	2	4	6	8	10
$f(x)$	0	12	248	1284	4080	9980

8. The elevations above a datum line of seven points of roads 300 units apart are 135, 149, 157, 183, 201, 205, 193 unit. Find the gradient of the road at the middle point.

9. From the following table, calculate $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.35$

x	1.1	1.2	1.3	1.4	1.5	1.6
y	–1.62628	0.15584	2.45256	5.39168	9.125001	3.83072

10. In a machine a slider moves along a fixed straight rod. Its distance x units along the rod is given below for various values of the time t seconds.

Find (*a*) the velocity of the slider and

(*b*) its acceleration when $t = 0.3$ sec

t (time in sec)	0	0.1	0.2	0.3	0.4	0.5	0.6
x (distance in units)	3.013	3.162	3.207	3.364	3.395	3.381	3.324

11. Using Bessel's formula find $f'(x)$ at $x = 0.04$ from the following table

x	0.01	0.02	0.03	0.04	0.05	0.06
$f(x)$	0.1023	0.1047	0.1071	0.1096	0.1122	0.1148

12. From the following table, find the value of x for which y is minimum and find this value of y

x	0.60	0.65	0.70	0.75
$f(x)$	0.6221	0.6155	0.6138	0.6170

13. From the following data, evaluate $\dfrac{dy}{dx}$ at $x = 0.00$

x	0.00	0.05	0.10	0.15	0.20	0.25
y	0.00000	0.10017	0.20134	0.30452	0.41075	0.52110

14. A rod is rotating in a place the following table gives the angle θ (radians) through with the rod has turned for various values of the time t seconds find (i) the angular velocity of the rod, (ii) its angular acceleration when $t = 0.6$ sec

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.122	0.493	1.123	2.022	3.200	4.666

15. Find the first and second derivatives of at $x = 15$ from the table

x	15	17	19	21	23	25
$\sqrt{x}$	3.873	4.123	4.359	4.583	4.796	5.000

16. From the table below, for what value of x, y is minimum? Also find this value of y?

x	3	4	5	6	7	8
y	0.205	0.240	0.259	0.262	0.250	0.224

17. Find the first and second derivatives of the function $y = f(x)$, tabulated below at the point $x = 1.1$

x	1	1.2	1.4	1.6	1.8	2.0
y	0.00	0.1280	0.5440	1.2960	2.4320	4.0000

18. Find the Force of Mortality $u_x = -\dfrac{1}{l_x}\dfrac{dl_x}{dx}$ at $x = 50$ yrs, using the table below

x	50	51	52	53
l_x	73499	72724	71753	70599

19. Use Stirling's formula to find the first derivative of the function $y = 2e^x - x - 1$, tabulated below at $x = 0.6$

x	0.4	0.5	0.6	0.7	0.8
y	1.5836494	1.7974426	2.0442376	2.3275054	2.6510818

also find the error.

20. Deduce from Bessel's formula the following approximations

(*i*) $\dfrac{d}{dx}(y_x) = \Delta y_{x-\frac{1}{2}} - \dfrac{1}{24}\Delta^3 y_{x-\frac{3}{2}}$

(*ii*) $\dfrac{d^2}{dx^2}(y_x) = \dfrac{1}{2}\left[\Delta^2 y_{x-\frac{3}{2}} + \Delta^2 y_{x-\frac{1}{2}}\right]$

21. A function is according to the table given below

x	0.35	0.40	0.45	0.50	0.55	0.60	0.65
$f(x)$	1.521	1.506	1.488	1.467	1.444	1.418	1.389

Use Stirlings formula and find the value of f'' (0.5)

22. Estimate the annual rate of cloth sales of 1935 from the following data

year	:	1920	1925	1930	1940
Sales of cloth in lakhs of metres	:	250	285	328	444

23. The elevation above a datum line of seven points of a road are given below

x	0	300	600	900	1200	1500	1800
y	135	149	157	183	201	205	193

Find the gradient of the road at the middle point.

Answers

1. 4.75, 9.0
2. 0.1737, 1.4750
3. 3, 6
4. 18
5. 1.4913
6. –0.425, –0.500
7. –2, 0
8. 0.085222
9. 29.32975, 71.33
10. 0.5333 units/sec, –4.56 units/sec^2
11. 0.2563
12. 0.6137
13. 2.0034
14. 3.814 radian/sec, 6.725 radian/sec^2
15. 0.12915, –0.0046
16. x = 5.6875, y = 0.2628
17. 0.630, 6.60
18. 0.099154
19. 2.644225, 0.000013.
20. –0.44
21. 11.55
22. 0.08522

9

NUMERICAL INTEGRATION

9.1 INTRODUCTION

Numerical integration is used to obtain approximate answers for definite integrals that cannot be solved analytically.

Numerical integration is a process of finding the numerical value of a definite integral

$$I = \int_a^b f(x)\,dx,$$

when a function $y = f(x)$ is not known explicitly. But we give only a set of values of the function $y = f(x)$ corresponding to the same values of x.

To evaluate the integral, we fit up a suitable interpolation polynomial to the given set of values of $f(x)$ and then integrate it within the desired limits. Here we integrate an approximate interpolation formula instead of $f(x)$. When this technique is applied on a function of single variable, the process is called *Quadrature*.

Suppose we are required to evaluate the definite integral $I = \int_a^b f(x)\,dx$, firstly we have to approximate $f(x)$ by a polynomial $\phi(x)$ of suitable degree.

Then we integrate $f(x)$ within limits $[a, b]$,

i.e.,
$$\int_a^b f(x)\,dx \approx \int_a^b \phi(x)\,dx,$$

the difference

$$\left[\int_a^b f(x)\,dx - \int_a^b \phi(x)\,dx,\right],$$

is called the *Error of approximation*.

9.2 GENERAL QUADRATURE FORMULA FOR EQUIDISTANT ORDINATES

Consider an integral

$$I = \int_a^b f(x)\,dx \tag{1}$$

Let $f(x)$ take the values $f(x_0) = y_0$, $f(x_0 + h) = y_1$, …, $f(x + nh) = y_n$, when $x = x_0$, $x = x_0 + h$, …, $x = x_0 + nh$ respectively.

To evaluate I, we replace $f(x)$ by a suitable interpolation formula. Let the interval $[a, b]$ be divided into n subintervals with the division points $a = x_0 < x_0 + h < \ldots < x_0 + xh = b$ where the h is the width of each subinterval. Approximating $f(x)$ by Newton's forward interpolation formula we can write the integral (1) as

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+nh} \left(y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots \right) dx, \tag{2}$$

since $$u = \frac{x - x_0}{h},$$

i.e., $$x = x_0 + uh$$

$$\Rightarrow \quad dx = hdu$$

and $$x = x_0$$

$$\Rightarrow \quad u = 0$$

$$x = x_0 + nh$$

$$\Rightarrow \quad u = n.$$

Expression (2) can be written as

$$I = h\int_0^n \left(y_0 + u\Delta y_0 + \frac{u^2 - u}{2}\Delta^2 y_0 + \frac{u^3 - 3u^2 + 2u}{6}\Delta^3 y_0 \right) dx +$$

$$h\int_0^n \left(+ \frac{u^4 - 6u^3 + 11u^2 + 6u}{24}\Delta^4 y_0 + \ldots \right) dx$$

$$\therefore \quad I = h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2} \right)\frac{\Delta^3 y_0}{2} + \right.$$

$$\left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y_0}{6} + \left(\frac{n^5}{5} - \frac{3}{2}n^4 + \frac{11n^3}{3} - 3n^2\right)\frac{\Delta^4 y_0}{24} + \ldots\Bigg]. \qquad (3)$$

The equation (3) is called *General Gauss Legendre Quadrature formula*, for equidistant ordinates from which we can generate any Numerical integration formula by assigning suitable positive integral value to n. Now we deduce four quadrature formulae, namely

(*a*) Trapezoidal rule (*b*) Simpson's one-third rule (*c*) Simpson's three-eighths rule and (*d*) Weddle's rule from the general quadrature formula (3).

9.3 TRAPEZOIDAL RULE

Substituting $n = 1$ in the relation (3) and neglecting all differences greater than the first we get

$$I_1 = \int_{x_0}^{x_0+h} f(x)dx = h\left[y_0 + \frac{1}{2}\Delta y_0\right]$$

$$= \frac{h}{2}(2y_0 + y_1 - y_0) = \frac{h}{2}(y_0 + y_1),$$

for the first subinterval $[x_0, x_0 + h]$,

similarly, we get

$$l_2 = \int_{x_0+h}^{x_0+2h} f(x)dx = \frac{h}{2}(y_1 + y_2),$$

$$l_3 = \int_{x_0+2h}^{x_0+3h} f(x)dx = \frac{h}{2}(y_2 + y_3),$$

$$\ldots$$
$$\ldots$$
$$\ldots$$

$$I_n = \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx = \frac{h}{2}(y_{n-1} + y_n),$$

for the other integrals.

Adding $I_1, I_2, \ldots, I_n$

we get $\quad I_1 + I_2 + \ldots + I_n$

$$= \int_{x_0}^{x_0+h} f(x)dx + \int_{x_0+h}^{x_0+2h} f(x)dx + \int_{x_0+2h}^{x_0+3h} f(x)dx + \ldots + \int_{x_0+(n-1)h}^{x_0+nh} f(x)dx$$

$$= \frac{h}{2}\left[y_0 + y_1\right] + \frac{h}{2}\left[y_1 + y_2\right] + \frac{h}{2}\left[y_2 + y_3\right] + \ldots + \frac{h}{2}\left[y_{n-1} + y_n\right],$$

$$\Rightarrow \quad \int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right],$$

$$I = \int_a^b f(x)\,dx = \frac{h}{2}\left[\left(y_0 + y_n\right) + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right]. \tag{4}$$

The formula (4) is called Trapezoidal rule for numerical integration. The error committed in this formula is given by

$$E \approx -\frac{h^3}{12} f''(\xi) = \frac{-(b-a)^3}{12n^2} f''(\xi),$$

where $a = x_0 < \xi < x_n = b.$

Note: Trapezoidal rule can be applied to any number of subintervals odd or even.

9.4 SIMPSON'S ONE-THIRD RULE

Substituting $n = 2$ in the General quadrature formula given by (3) and neglecting the third and other higher order differences

we get

$$I_1 = \int_{x_0}^{x_0+2h} f(x)\,dx = h\left[2y_0 + 2\Delta y_0 + \left(\frac{8}{3} - 2\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2\left(y_1 - y_0\right) + \frac{1}{3}\left(y_2 - 2y_1 + y_0\right)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right]$$

$$\therefore \quad I_1 = \frac{h}{3}\left[y_0 + 4y_1 + y_2\right],$$

Similarly $$I_2 = \int_{x_{0+2h}}^{x_0+2h} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right],$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$I_{n/2} = \int_{x_{0+(n-2)h}}^{x_0+nh} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right].$$

Adding $I_1, I_2, I_{n/2}$ we get

$$I_1 + I_2 + \ldots + I_{n/2} = \int_{x_0}^{x_0+2h} f(x)\,dx + \int_{x_0+2h}^{x_0+2h} f(x)\,dx + \ldots + \int_{x_{0+(n-2)h}}^{x_0+nh} f(x)\,dx$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] + \frac{h}{3}\left[y_2 + 4y_3 + y_4\right] + \ldots + \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right],$$

$$= \frac{h}{3}\ (y_0 + y_n) + 4(y_1 + y_3 + y_5 + \ldots + y_{n-1})\ + 2\ (y_2 + y_4 + \ldots + y_{n-2})\rfloor$$

$$= \frac{h}{3}\left[\,y_0 + y_n + 4\right] \times \text{(sum of odd ordinates)} + 2 \times \text{(sum of even ordinates)}$$

$$= \frac{h}{3}\left[(y_0 + y_n) + 4 \times (\text{sum of the odd ordinates})\right] +$$

$$= \frac{h}{3}\left[2 \times (\text{sum of the even ordinates})\right]$$

The above rule is known as Simpson's one-third rule. The error committed in Simpson's one-third rule is given by

$$E \approx \frac{-nh^5}{180} f^{\text{iv}}(\xi) = -\frac{(b-a)^5}{2880n^4} f^{\text{iv}}(\xi)$$

where $a = x_0 < \xi < x_n = b$ (for n subintervals of lengths h).

Note:

1. The above formula may written as

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx$$

2. Simpson's one-third rule can be applied only when the given interval $[a, b]$ is subdivided into even number of subintervals each of width h and within any two consecutive subintervals the interpolating polynomial $\phi(x)$ is of degree 2.

9.5 SIMPSON'S THREE-EIGHTH'S RULE

We assume that within any three consecutive subintervals of width h, the interpolating polynomial $\phi(x)$ approximating $f(x)$ is of degree 3. Hence substituting $n = 3$, *i.e.*, the General quadrature formula and neglecting all the differences above Δ^3, we get

$$I_1 = \int_{x_0}^{x_0+3h} f(x)\,dx = h\left[3y_0 + \frac{9}{2}\Delta y_0 + \left(9 - \frac{9}{2}\right)\Delta^2 y_0 + \left(\frac{81}{4} - 27 + 9\right)\frac{\Delta^3 y_0}{6}\right]$$

$$= h\left[3y_0 + 9(y_1 - y_0) + \frac{9}{4}(y_2 - 2y_1 + y_0) + \frac{3}{8}(y_3 - 3y_2 + 3y_1 + y_0)\right]$$

$$= \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3],$$

Similarly
$$I_2 = \int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6],$$

$$I_3 = \int_{x_0+6h}^{x_0+9h} f(x)\,dx = \frac{3h}{8}[y_6 + 3y_7 + 3y_8 + y_9],$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$I_{n/3} = \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx = \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right].$$

Adding $I_1, I_2, ..., I_{n/3}$ we get

$$I_1 + I_2 + ... + I_{n/3} = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + ... + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx,$$

$$\Rightarrow \; I = \frac{3h}{8}[y_0 + 3y_1 + 3y_2 + y_3] + \frac{3h}{8}[y_3 + 3y_4 + 3y_5 + y_6] +$$

$$... + \frac{3h}{8}\left[y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right],$$

$$\therefore \; I = \frac{3h}{8}\Big[(y_0 + y_n) + 3\big(y_1 + y_2 + y_4 + y_5 + ... + y_{n-1}\big) +$$

$$2\big(y_3 + y_6 + ... + y_{n-3}\big)\Big].$$

Note:

1. Simpson's three-eighths rule can be applied when the range $[a, b]$ is divided into a number of subintervals, which must be a multiple of 3.

2. The error in Simpson's three-eighths rule

$$E \approx \frac{-nh^5}{80} f^{iv}(\xi),$$

where x_0 , ξ , x_n (for n subintervals of length h).

9.6 WEDDLE'S RULE

Here we assume that within any six consecutive subintervals of width h each, the interpolating polynomial approximating $f(x)$ will be of degree 6. Substituting $n = 6$ in the General quadrature formula given by expression (3) and neglecting all differences above Δ^6, we get

$$I_1 = \int_{x_0}^{x_0+6h} f(x)dx = h\left[6y_0 + 18\Delta y_0 + 27\Delta^2 y_0 + 24\Delta^3 y_0 + \frac{123}{10}\Delta^4 y_0 + \frac{33}{10}\Delta^5 y_0 + \frac{41}{140}\Delta^6 y_0\right].$$

Since $$\frac{3}{10} - \frac{41}{140} = \frac{1}{140},$$

we take the coefficient of $\Delta^6 y_0$ as $\frac{3}{10}$, so that the error committed is $\frac{1}{140}$ and we write

$$I_1 = \int_{x_0}^{x_0+6h} f(x)dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6],$$

Similarly $$I_2 = \int_{x_0+6h}^{x_0+12h} f(x)dx = \frac{3h}{10}[y_6 + 5y_1 + y_8 + 6y_9 + y_{10} + 5y_{11} + y_{12}],$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$I_{n/6} = \int_{x_0+(n-6)h}^{x_0+nh} f(x)dx = \frac{3h}{10}[y_{n-6} + 5y_{n-1} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n].$$

Adding $I_1, I_2, I_{n/6}$, we get

$$I = \int_{x_0}^{x_0+nh} f(x)dx$$

$$= \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + 2y_6 + 5y_7 + y_8 + 6y_9 + y_{10} + 6y_{11} + 2y_{12} + ... + 2y_{n-6} + 5y_{n-5} + y_{n-4} + 6y_{n-3} + y_{n-2} + 5y_{n-1} + y_n]$$

$$= \frac{3h}{10}\Big[\big(y_0 + y_n\big) + \big(y_2 + y_4 + y_8 + y_{10} + y_{14} + y_{16} + \ldots +$$

$$y_{n-4} + y_{n-2}\big) + 5\big(y_1 + y_5 + y_7 + y_{11} + \ldots + y_{n-5} + y_{n-1}\big) +$$

$$6\big(y_3 + y_9 + y_{15} + \ldots + y_{n-3}\big) + 2\big(y_6 + y_{12} + \ldots + y_{n-6}\big)\Big].$$

Note:

1. Weddle's rule requires at least seven consecutive equispaced ordinates with in the given interval (a, b).
2. It is more accurate than the Trapezoidal and Simpson's rules.
3. If $f(x)$ is a polynomial of degree 5 or lower, Weddle's rule gives an exact result.

Example 9.1 *Calculate the value* $\int_0^x \frac{x}{1+x}dx$ *correct up to three significant figures taking six intervals by Trapezoidal rule.*

Solution Here we have

$$f(x) = \frac{x}{1+x},$$

$$a = 0,\ b = 1 \text{ and } n = 6,$$

$$\therefore\ h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}.$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6 = 1
$y = f(x)$	0.00000	0.14286	0.25000	0.33333	0.40000	0.45454	0.50000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Trapezoidal rule can be written as

$$I = \frac{h}{2}\big[\big(y_0 + y_6\big) + 2\big(y_1 + y_2 + y_3 + y_4 + y_5\big)\big]$$

$$= \frac{1}{12}\big[(0.00000 + 0.50000 + 2\big(0.14286 + 0.25000 + 0.3333 + 0.40000 + 0.45454\big)\big]$$

$$= 0.30512.$$

$\therefore\ I = 0.0305,$ correct to three significant figures.

Example 9.2 *Find the value of* $\int_0^1 \frac{dx}{1+x^2}$, *taking 5 subinterval by Trapezoidal rule, correct to five significant figures. Also compare it with its exact value.*

Solution Here

$$f(x) = \frac{1}{1+x^2},$$

$$a = 0,\ b = 1 \text{ and } n = 5,$$

$$\therefore\ h = \frac{1-0}{5} = \frac{1}{5} + 0.2.$$

x	0.0	0.2	0.4	0.6	0.8	1
$y = f(x)$	1.000000	0.961538	0.832069	0.735294	0.609756	0.500000
	y_0	y_1	y_2	y_3	y_4	y_5

Using trapezoidal rule we get

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[(1.000000 + 0.500000) + \ 2(0.961538 + 0.862069 + 0.735294 + 0.609756)\right]$$

$$= 0.783714,$$

$\therefore\ I = 0.78373,$ correct to five significant figures.

The exact value

$$= \int_0^1 \frac{1}{1+x^2}\,dx = \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0 \ = \frac{\pi}{4} = 0.7853981$$

$$\int_0^1 \frac{1}{1+x^2}\,dx = 0.78540,$$

correct to five significant figures.

$\therefore$ The error is $= 0.78540 - 0.78373 = 0.00167$

$\therefore$ Absolute error $= 0.00167.$

Example 9.3 *Find the value of* $\int_1^5 \log_{10} x\,dx$, *taking 8 subintervals correct to four decimal places by Trapezoidal rule.*

Solution Here

$$f(x) = \log_{10} x,$$

$$a = 1,\ b = 5 \text{ and } n = 8,$$

$$\therefore\ h = \frac{b-a}{n} = \frac{5-1}{8} = 0.5.$$

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0.00000	0.17609	0.30103	0.39794	0.47712	0.54407	0.60206	0.65321	0.69897
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Trapezoidal rule we can write

$$I = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.5}{2}\left[(0.00000 + 0.69897) + 2(0.17609 + 0.30103 + 0.39794)\right] +$$

$$\frac{0.5}{2}\left[2(0.47712 + 0.54407 + 0.60206 + 0.65321)\right]$$

$$= 1.7505025$$

$$\therefore\ I = \int_1^5 \log_{10} x dx = 1.75050.$$

Example 9.4 *Find the value* $\int_0^{0.6} e^x \, dx$, *taking n = 6, correct to five significant figures by Simpson's one-third rule.*

Solution We have

$$f(x) = e^x,$$

$$a = 0,\ b = 0.6,\ n = 6.$$

$$\therefore\ h = \frac{b-a}{n} = \frac{0.6 - 0}{6} = 0.1.$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$y = f(x)$	1.0000	1.10517	1.22140	1.34986	1.49182	1.64872	1.82212
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Simpson's rule is

$$I = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.1}{3}\left[(1.00000 + 1.82212) + 4(1.10517 + 1.34986 + 1.64872) + 2(1.22140 + 1.49182)\right]$$

$$= \frac{0.1}{3}\left[(2.82212) + 4(4.10375) + 2(2.71322)\right]$$

$$= 0.8221186 \approx 0.82212$$

$$\therefore\ I = 0.82212.$$

Example 9.5 *The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in km/hour.*

t (minutes)	*2*	*4*	*6*	*8*	*10*	*12*	*14*	*16*	*18*	*20*
v (km/hr)	*16*	*28.8*	*40*	*46.4*	*51.2*	*32.0*	*17.6*	*8*	*3.2*	*0*

Estimate approximately the total distance run in 20 minutes.

Solution
$$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v.dt$$

$$\Rightarrow \quad \int ds = \int v.dt$$

$$s = \int_0^{20} v.dt.$$

The train starts from rest, $\therefore$ the velocity $v = 0$ when $t = 0$.

The given table of velocities can be written

t	0	2	4	6	8	10	12	14	16	18	20
v	0	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$h = \frac{2}{60}\text{hrs} = \frac{1}{30}\text{hrs}.$$

The Simpson's rule is

$$s = \int_0^{20} v.dt = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{1}{30 \times 3}\left[(0 + 0) + 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8)\right]$$

$$= \frac{1}{90}\left[0 + 4 \times 128 + 2 \times 115.2\right] = 8.25\,\text{km}.$$

$\therefore$ The distance run by the train in 20 minutes = 8.25 km.

Example 9.6 *A tank in discharging water through an orifice at a depth of x meter below the surface of the water whose area is Am^2. The following are the values of x for the corresponding values of A.*

A	1.257	1.39	1.52	1.65	1.809	1.962	2.123	2.295	2.462	2.650	2.827
x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00

Using the formula (0.018) $T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}}dx$, *calculate T the time in seconds for the level of the water to drop from 3.0 m to 15 m above the orifice.*

Solution We have $h = 0.15$,

The table of values of x and the corresponding values of $\frac{A}{\sqrt{x}}$ is

x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00
$y = \frac{A}{\sqrt{x}}$	1.025	1.081	1.132	1.182	1.249	1.308	1.375	1.438	1.498	1.571	1.632

Using Simpson's rule, we get

$$\int_{1.5}^{3.0} \frac{A}{\sqrt{x}} dx = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{0.15}{3}\left[(1.025 + 1.632) + 4(1.081 + 1.182 + 1.308 + 1.438 + 1.571)\right] +$$

$$\frac{0.15}{3}\left[2(1.132 + 1.249 + 1.375 + 1.498)\right]$$

$$= 1.9743$$

$$\therefore \int_{1.5}^{3} \frac{A}{\sqrt{x}} dx = 1.9743.$$

Using the formula $\qquad (0.018)T = \int_{1.5}^{3} \frac{A}{\sqrt{x}} dx,$

we get $\qquad (0.018)T = 1.9743$

$$\Rightarrow T = \frac{1.9743}{0.018} = 110 \text{ sec (approximately)}$$

$$\therefore T = 110 \text{ sec.}$$

Example 9.7 *Evaluate* $\int_0^1 \frac{1}{1+x^2} dx$, *by taking seven ordinates.*

Solution We have

$$n + 1 = 7 \Rightarrow n = 6$$

The points of division are

$$0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, 1.$$

x	0	1/6	2/6	3/6	4/6	5/6	1
$y = \frac{1}{1+x^2}$	1.0000000	0.9729730	0.9000000	0.8000000	0.6923077	0.5901639	0.5000000

Here $h = \frac{1}{6}$, the Simpson's three-eighths rule is

$$I = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$$

$$= \frac{3}{6 \times 8}\left[(1 + 0.5000000) + 3(0.9729730 + 0.9000000)\right] +$$

$$\frac{3}{6 \times 8}\left[3(0.6923077 + 0.5901639) + 2(0.8000000)\right]$$

$$= \frac{1}{16}\left[1.5000000 + 9.4663338 + 1.6000000\right]$$

$$= 0.7853959.$$

Example 9.8 *Calculate* $\int_0^{\pi/2} e^{\sin x} dx$, *correct to four decimal places.*

Solution We divide the range in three equal points with the division points

$$x_0 = 0,\ x_1 = \frac{\pi}{6},\ x_2 = \frac{\pi}{3},\ x_3 = \frac{\pi}{2}$$

where $$h = \frac{\pi}{6}.$$

The table of values of the function is

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y = e^{\sin x}$	1	1.64872	2.36320	2.71828
	y_0	y_1	y_2	y_3

By Simpson's three-eighths rule we get

$$I = \int_0^{\pi/2} e^{\sin x} dx = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

$$= \frac{3}{8}\frac{\pi}{6}\left[(1 + 2.71828) + 3(1.64872 + 2.36320)\right]$$

$$= \frac{\pi}{16}\left[(3.71828 + 12.03576)\right] = 0.091111$$

$$I = \int_0^{\pi/2} e^{\sin x} dx = 0.091111.$$

Example 9.9 *Compute the integral* $\int_0^{\pi/2} \sqrt{1-0.162\sin^2\phi}\, d\phi$ *by Weddle's rule.*

Solution Here we have

$$y = f(\phi) = \sqrt{1 - 0.162\sin^2\phi},$$

$$a = 0,\ b = \frac{\pi}{2},$$

taking $n = 12$ we get

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2} - 0}{12} = \frac{\pi}{24}.$$

f	$y = f(f)$		f	$y = f(f)$	
0	1.000000	y_0	$\frac{6\pi}{24}$	0.958645	y_6
$\frac{\pi}{24}$	0.998619	y_1	$\frac{7\pi}{24}$	0.947647	y_7
$\frac{2\pi}{24}$	0.994559	y_2	$\frac{8\pi}{24}$	0.937283	y_8
$\frac{3\pi}{24}$	0.988067	y_3	$\frac{9\pi}{24}$	0.928291	y_9
$\frac{4\pi}{24}$	0.979541	y_4	$\frac{10\pi}{24}$	0.921332	y_{10}
$\frac{5\pi}{24}$	0.969518	y_5	$\frac{11\pi}{24}$	0.916930	y_{11}
			$\frac{12\pi}{24} = \frac{\pi}{2}$	0.915423	y_{12}

By Weddle's rule we have

$$I = \int_0^{\pi/2} \sqrt{1 - 0.162\sin^2 f}\, df$$

$$= \frac{3h}{10}\left[(y_0 + y_{12}) + 5(y_1 + y_5 + y_7 + y_{11})\right] + \frac{3h}{10}\left[(y_2 + y_4 + y_8 + y_{10}) + 6(y_3 + y_9) + 2y_6\right]$$

$$= \frac{3\pi}{240}\left[(11.000000 + 0.915423) + 5(0.998619 + 0.969518 + 0.947647 + 0.916930)\right] +$$

$$\frac{3\pi}{240}\left[(0.994559 + 0.979541 + 0.937283 + 0.9213322)\right] + \frac{3\pi}{240}\left[6(0.988067 + 0.928291) + 2(0.958645)\right]$$

$$\therefore\ I = 1.505103504.$$

Example 9.10 *Find the value of* $\int_4^{5.2} log_e x dx$ *by Weddle's rule.*

Solution Here $f(x) = \log_e n$, $a = x_0 = 4$, $b = x_n = 5.2$ taking $n = 6$ (a multiple of six) we have

$$h = \frac{5.2 - 4}{6} = 0.2,$$

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$y = f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6457

Weddle's rule is

$$I = \int_4^{5.2} \log_e x\,dx = \frac{3h}{10}[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3 \times (0.2)}{10}[1.3863 + 7.1755 + 1.4816 + 9.1566 + 1.5686 + 8.0470 + 1.6487]$$

$$= 0.06[30.4643]$$

$$= 1.827858$$

$$\int_4^{5.2} \log_e x = 1.827858.$$

Exercise 9.1

1. Evaluate $\int_0^1 x^3\,dx$ by Trapezoidal rule.

2. Evaluate $\int_0^1 (4x - 3x^2)\,dx$ taking 10 intervals by Trapezoidal rule.

3. Given that $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$, find an approximation value of $\int_0^4 e^x$ by Trapezoidal rule.

4. Evaluate $\int_0^1 \sqrt{1 - x^3}\,dx$ by (*i*) Simpson's rule and (*ii*) Trapezoidal rule, taking six interval correct to two decimal places.

5. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$ taking $x = 6$, correct to four significant figures by (*i*) Simpson's one-third rule and (*ii*) Trapezoidal rule.

6. Evaluate $\int_0^2 \frac{dx}{x}$ taking 4 subintervals, correct to five decimal places (i) Simpson's one-third rule (ii) Trapezoidal rule.

7. Compute by Simpson's one-third rule, the integral $\int_0^1 x^2 (1 - x)\, dx$ correct to three places of decimal, taking step length equal to 0.1.

8. Evaluate $\int_0^1 \sin x^2 dx$ by (*i*) Trapezoidal rule and (*ii*) Simpson's one-third rule, correct to four decimals taking x = 10.

9. Calculate approximate value of $\int_{-3}^{3} \sin x^4\, dx$ by using (*i*) Trapezoidal rule and (*ii*) Simpson's rule, taking n = 6.

10. Find the value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$ by (*i*) Trapezoidal rule and (*ii*) Simpson's one-third rule taking x = 6.

11. Compute $\int_1^{15} e^x\, dx$ by (*i*) Trapezoidal rule and (*ii*) Simpson's one-third rule taking x = 10.

12. Evaluate $\int_0^{0.5} \frac{x}{\cos x} dx$ taking n = 10, by (*i*) Trapezoidal rule and (*ii*) Simpson's one-third rule.

13. Evaluate $\int_0^{0.4} \cos x\, dx$ taking four equal intervals by (*i*) Trapezoidal rule and (*ii*) Simpson's one-third rule.

14. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$ by Weddle's rule taking n = 6.

15. Evaluate $\int_0^1 \frac{x^2 + 2}{x^2 + 1}\, dx$ by Weddle's rule, correct to four decimals taking $n = 12$.

16. Evaluate $\int_0^2 \frac{1}{1 + x^2}\, dx$ by using Weddle's rule taking twelve intervals.

17. Evaluate $\int_{0.4}^{16} \frac{x}{\sin hx}\, dx$ taking thirteen ordinates by Weddle's rule correct to five decimals.

18. Using Simpson's rule evaluate $\int_{0}^{\frac{\pi}{2}} \sqrt{2+\sin x}\, dx$ with seven ordinates.

19. Using Simpson's rule evaluate $\int_{1}^{2} \sqrt{x - 1/x}\, dx$ with five ordinates.

20. Using Simpson's rule evaluate $\int_{2}^{6} \frac{1}{\log_e x}\, dx$ taking $n = 4$.

21. A river is 80 unit wide. The depth at a distance x unit from one bank d is given by the following table

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

find the area of cross-section of the river.

22. Find the approximate value of $\int_{0}^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ using Simpson's rule with six intervals.

23. Evaluate $\int_{0.5}^{0.7} x^{\frac{1}{2}} e^{-x}\, dx$ approximately by using a suitable formula for at least 5 points.

24. Evaluate $\int_{0}^{1} \sqrt{\sin x + \cos x}\, dx$, correct to two decimal places using seven ordinates.

25. Use Simpson's three-eighths rule to obtain an approximate value of $\int_{0}^{0.3} (1 - 8x^3)^{\frac{1}{2}}\, dx$.

26. Find the value of $\int_{0}^{1/2} \frac{dx}{\sqrt{1-x^2}}$, using Weddle's rule.

27. Prove that

$$\int_{-1}^{1} f(x)\, dx = \frac{1}{12} \left| 13 f(1) - f(3) - f(-3) \right|$$

28. If $u_x = a + bx + cx^2$, prove that

$$\int_1^3 u_x \, dx = 2u_2 + \frac{1}{12}(u_0 - 2u_2 + u_4)$$

and hence approximate the value for

$$\int_{-1/2}^{1/2} e^{\frac{-x^2}{10}} \, dx$$

29. If $f(x)$ is a polynomial in x of degree 2 and $u_{-1} = \int_{-3}^{-1} f(x)\, dx,\ u_0 = \int_{-1}^{1} f(x)\, dx,\ u_1 = \int_{1}^{3} f(x)\, dx$ then show that $f(0) = \frac{1}{2}\left[u_0 - \frac{\Delta^2 u_{-1}}{2u}\right]^{-1}$.

Answers

1.	0.260	**2.**	0.995	**3.**	58.00
4.	0.83	**5.**	1.187, 1.170	**6.**	0.69326, 0.69702
7.	0.083	**8.**	0.3112, 0.3103	**9.**	115, 98
10.	1.170, 1.187	**11.**	1.764, 1.763	**12.**	0.133494, 0.133400
13.	0.3891, 0.3894	**14.**	1.18916	**15.**	1.7854
16.	1.1071	**17.**	1.1020	**18.**	2.545
19.	1.007	**20.**	3.1832	**21.**	710 Sq units
22.	1.1872	**23.**	0.08409	**24.**	1.14
25.	0.2899	**26.**	0.52359895		

9.7 NEWTON–COTES FORMULA

Consider the Lagrange's interpolation formula

$$f(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_n)}{(x_0 - x_1)(x_0 - x_2)\ldots(x_0 - x_n)} f(x_0) +$$

$$\frac{(x - x_0)(x - x_2)\ldots(x - x_n)}{(x_1 - x_0)(x_1 - x_2)\ldots(x_1 - x_n)} f(x_1) + \ldots + \frac{(x - x_0)(x - x_1)\ldots(x - x_{n-1})}{(x_n - x_0)(x_n - x_1)\ldots(x_n - x_{n-1})} f(x_n),$$

Integrating between the limits x_0 and $x_0 + nh$ we get

$$\int_{x_0}^{x_0+h} f(x)dx = H_0 f(x_0) + H_1 f(x_1) + \ldots + H_r f(x_r) + \ldots + H_n f(x_n). \tag{5}$$

Expression (5) is known as Newton–Cotes formula. Taking

$$x_{r+1} - x_r = h$$

for all r such that $\quad x_r = x_0 + rh$

and substituting $\quad u = \dfrac{x - x_0}{h}$

we get $\quad hdu = dx$

and
$$H_r = h\int_0^n \frac{uh[(u-1)h]\ldots[(u-r+1)h][(u-r-1)h]\ldots(u-n)h}{(rh)[(r-1)h][(r-2)h]\ldots(1h)(-1h)\ldots[-(n-r)h]}du$$

$$= h\int_0^n h^n \frac{u(u-1)(u-2)\ldots(u-r+1)(u-r-1)\ldots(u-n)}{h^n\, r!(-1)^{n-r}\, h^{n-r}\,(n-r)!}du$$

$$= \frac{(-1)^{n-r}}{r!(n-r)!}h\int_0^n \frac{u(u-1)(u-2)\ldots(u-n)}{(u-r)}du, \tag{6}$$

expression (6) gives the values of H_r.

To obtain the error in the Newton–Cotes formula. We integrate the error term of the Lagrange's interpolation formula over the range x_0 to $x_n = x_0 + nh$.

The error term in Lagrange's formula is

$$\text{Error} = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n),$$

where $x_0 < \xi < x_n$.

$\therefore$ The error in the Newton–Cotes formula is

$$E = \int_{x_0}^{x_0+nh} \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\ldots(x-x_n)dx$$

$$= h\int_0^n h^{n+1}\frac{f^{(n+1)}(\xi)}{(n+1)!}\,u(u-1)(u-2)\ldots(u-n)du$$

where $x_0 < \xi < x_n$.

Since $\quad \dfrac{\Delta^{n+1}y_0}{h^{n+1}} \approx f^{(n+1)}(\xi),$

we can write
$$E = h\int_0^n \frac{\Delta^{n+1}y_0}{(n+1)}u(u-1)(u-2)\ldots(u-n)du.$$

Note: Replacing H_r by $nh\ {}^nc_r$ in the Newton–Cotes formula we get

$$I = \int_{x_0}^{x_0+nh} f(x)\,dx = nh \sum_{r=0}^{n} f(x_r)\,{}^nc_r$$

$$= (x_n - x_0) \sum_{0}^{n} f(x_r)\,{}^nc_r.$$

The numbers ${}^nc_r$, $0 \le r \le n$ are called Cotes numbers.

Corollary *The coefficients of the Newton–Cotes formula are symmetric from both the ends.*

Proof Putting $n - v = u$, in (6) we get, $u - n = -v$

$$du = -dv$$

and

$$H_r = -h\int_n^0 (-1)^{n-r} \frac{(n-v)(n-v+1)\ldots(-v)}{(n-v+1)}\,du$$

$$= h\int_0^n \frac{(-1)^{n-r}}{r!(n-r)!} \frac{(-v)(-v+1)\ldots(-v+n)}{-(v-n+1)}\,dv$$

$$= h\int_0^n \frac{(-1)^{n-r}}{r!(n-r)!}(-1)^{n+1} \frac{(v)(v-1)\ldots(v-n)}{(-1)(v-n+1)}\,dv$$

$$= h\int_0^n \frac{(-1)^{2n-r}}{r!(n-r)!} \frac{(v)(v-1)\ldots(v-n)}{(-1)(v-n+1)}\,dv$$

$$= h\int_0^n \frac{(-1)^{2n}(-1)^r}{r!(n-r)!} \frac{(v)(v-1)\ldots(v-n)}{(-1)(v-n+1)}\,dv$$

$$= h\int_0^n \frac{(-1)^r}{r!(n-r)!} \frac{(v)(v-1)\ldots(v-n)}{(-1)(v-n+1)}\,dv$$

$$= \frac{(-1)^r}{r!(n-r)!}\, h\int_0^n \frac{(v)(v-1)\ldots(v-n)}{(-1)(v-n+1)}\,dv = H_{n-r}$$

$$\therefore\ H_r = H_{n-r} \text{ proved}.$$

9.8 DERIVATION OF TRAPEZOIDAL RULE, AND SIMPSON'S RULE FORM NEWTON–COTES FORMULA

1. Trapezoidal rule

Putting $n = 1$ in (5), we get

$$I = \int_{x_0}^{x_0+h} f(x)\,dx = H_0 f(x_0) + H_1 f(x_1)$$

where $$n_1 = n_0 + h$$

and $$H_0 = \frac{(-1)^{1-0}}{0!\,1!} h \int_0^1 (u-1)\,du = (-1)h\left[\frac{u^2}{2} - u\right] = \frac{h}{2}$$

$$H_1 = h\int_0^1 u\,du = \frac{h}{2},$$

$$I = \int_{x_0}^{x_0+h} f(x)\,dx = \frac{h}{2}\left[f(x_0) + f(x_1)\right].$$

2. Simpson's one-third rule

Putting $n = 2$ in (5) we get

$$I = \int_{x_0}^{x_0+2h} f(x)\,dx = H_0 f(x_0) + H_1 f(x_1) + H_2 f(x_2)$$

$$H_0 = \frac{(-1)^{2-0}}{2!\,0!} h \int_0^2 \frac{u(u-1)(u-2)}{u}\,du$$

$$= \frac{1}{2} h \int (u^2 - 3u + 2)\,du = \frac{h}{2}\left[\frac{u^3}{3} - \frac{3u^2}{2} + 2u\right]_0^2$$

$$= \frac{h}{2}\left[\frac{8}{3} - 6 + 4\right] = \frac{h}{3},$$

$$H_1 = \frac{(-1)^{2-1}}{1!\,1!} h \int_0^2 u(u-2)\,du = -h\left[\frac{u^3}{3} - u^2\right]_0^2$$

$$= -h\left[\frac{8}{3} - 4\right] = \frac{4h}{3},$$

since $$H_r = H_{n-r}$$

we get $$H_2 = H_0 = \frac{h}{3}$$

$$I = \int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{3} f(x_0) + \frac{4h}{3} f(x_1) + \frac{h}{3} f(x_2)$$

$$\Rightarrow \int_{x_0}^{x_0+2h} f(x)dx = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right].$$

Note: Similarly by putting $n = 3$, and $n = 6$ in respectively (5) we can derive Simpson's three-eighths rule and Weddle's rule.

Example 9.11 *If y_x is a polynomial in x of the third degree, find an expression for $\int_0^2 y_x dx$ in terms of y_0, y_1, y_2 and y_3. Use this results to show that:*

$$\int_1^2 y_x dx = \frac{1}{24}\left[-y_0 + 13y_1 + 13y_2 - y_3\right].$$

Solution We have

$$y_x = \frac{(x-1)(x-2)(x-3)}{(-1)(-2)(-3)} y_0 + \frac{x(x-2)(x-3)}{(1)(-2)(-3)} y_1 + \frac{x(x-1)(x-3)}{(2)(1)(-1)} y_2 + \frac{x(x-1)(x-2)}{(3)(2)(1)} y_3$$

$$= \frac{x^3 - 6x^2 + 11x - 6}{-6} y_0 + \frac{x^3 - 5x^2 + 6x}{2} y_1 + \frac{x^3 - 4x^2 + 3x}{-2} y_2 + \frac{x^3 - 3x^2 + 2x}{6} y_3 .$$

$\therefore$ We get

$$\int_0^1 y_x\, dx = \left[-\frac{1}{6}\left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x\right) y_0 + \frac{1}{2}\left(\frac{x^4}{4} - \frac{5}{3}x^3 + 3x^2\right) y_1\right]_0^1 +$$

$$\left[-\frac{1}{2}\left(\frac{x^4}{4} - \frac{4}{3}x^3 + \frac{3x^2}{2}\right) y_2 + \frac{1}{6}\left(\frac{x^4}{4} - x^3 + x^2\right) y_0\right]_0^1$$

$$= \left(-\frac{1}{6}\right)\left(-\frac{9}{4}\right) y_0 + \left(\frac{1}{2}\right)\left(\frac{19}{12}\right) y_1 - \left(\frac{1}{2}\right)\left(\frac{5}{12}\right) y_2 + \frac{1}{6}\left(\frac{1}{4}\right) y_3 ,$$

similarly

$$\int_0^2 y_x\, dx = \left(-\frac{1}{6}\right)(-2)\, y_0 + \left(\frac{1}{2}\right)\left(\frac{8}{3}\right) y_1 \cdot \left(\frac{1}{2}\right)\left(\frac{8}{3}\right) y_{-2} + \left(\frac{1}{6}\right)(0)\, y_3$$

subtracting we get

$$\int_1^2 y_x\, dx = \frac{1}{24}\left[-y_0 + 13y_1 + 13y_2 - y_3\right].$$

Example 9.12 *Show that*

$$\int_0^1 y_x dx = \frac{1}{12}(5u_1 + 8u_0 - u_{-1})$$

Solution We have

x	-1	0	1
u_x	u_{-1}	u_0	u_1

Using Lagrange's formula, we get

$$u_x = \frac{(x-0)(x-1)}{(-1-0)(-1-1)}u_{-1} + \frac{(x+1)(x-1)}{(0+1)(0-1)}u_0 + \frac{(x+1)(x-0)}{(1+1)(1-0)}u_1$$

$$= \frac{x^2 - x}{2}u_{-1} - (x^2 - 1)u_0 + (x^2 + x)u_1$$

$$\int_0^1 u_x\, dx = \frac{1}{2}u_{-1}\int_0^1 (x^2 - x)dx - u_0\int_0^1 (x^2 - 1)dx + \frac{1}{2}u_1\int_0^1 (x^2 + x)dx$$

$$= -\frac{1}{12}u_{-1} + \frac{2}{3}u_0 + \frac{5}{12}u_1$$

$$= \frac{1}{12}(5u_1 + 8u_0 - u_{-1}).$$

9.9 BOOLE'S RULE

Retaining differences up to those of the fourth order in the general formula and integrating between x_0 and x_4 we get

$$I_1 = \int_{x_0}^{x_4} f(x)dx = \int_{x_0}^{x_0+uh} f(x)dx$$

$$= 4h\left[u_0 + 2\Delta y_0 + \frac{5}{3}\Delta^2 y_0 + \frac{2}{3}\Delta^3 y_0 + \frac{7}{90}\Delta^4 y_0\right]$$

$$= 4h\left[y_0 + 2(y_1 - y_0) + \frac{5}{3}(y_2 - 2y_1 + y_0) + \right.$$

$$\left.\frac{2}{3}(y_3 - 3y_2 + 3y_1 - y_0) + \frac{7}{90}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0)\right]$$

$$= \frac{2h}{45}[7y_0 + 32y_1 + 12y_2 + 32y_3 + 7y_4]$$

Similarly $$I_2 = \int_{x_0+4h}^{x_0+8h} f(x)\,dx = \frac{2h}{45}\left[7y_4 + 32y_5 + 12y_6 + 32y_7 + 7y_8\right]$$

$$\cdots$$
$$\cdots$$
$$\cdots$$

$$I_{n/4} = \int_{x_0+(n-4)h}^{x_0+nh} f(x)\,dx = \frac{2h}{45}\left[7y_{n-4} + 32y_{n-3} + 12y_{n-2} + 32y_{n-1} + 7y_n\right]$$

Adding $I_1, I_2, ..., I_{n/4}$ we get

$$I_1 + I_2 + I_3 + ... + I_{n/4} = I = \int_{x_0}^{x_0+nh} f(x)\,dx = \int_a^b f(x)\,dx$$

$$= \frac{2h}{45}\left[7y_0 + 32y_1 + 12y_2 + 32y_3 + 14y_4 + 32y_5 + 12y_6 + 32y_7 + 14y_8 + ... + 14y_{n-4} + 32y_{n-3}\right.$$
$$\left. 12y_{n-2} + 32y_{n-1} + 7y_n\right]$$

The above formula is known as *Boole's rule*. The leading term in the error of the formula is

$$\frac{-8h^7}{945} f^{VI}(\xi).$$

Example 9.13 *Evaluate the integral of f(x) = 1 + e^{-x} sin 4x over the interval [0, 1] using exactly five functional evaluations.*

Solution Taking $h = \frac{1}{4}$ and applying Boole's rule we get

$$\int_0^1 f(x)\,dx = \frac{1}{4} \times \frac{2}{45}\left[7f(0) + 32f\left(\frac{1}{4}\right) + 12f\left(\frac{1}{2}\right) + 32f\left(\frac{3}{4}\right) + 7f(1)\right]$$

$$= \frac{1}{90}\left[7 \times 1.0000 + 32 \times 1.65534 + 12 \times 1.55152\right] + \frac{1}{90}\left[32 \times 1.06666 + 7 \times 0.72159\right]$$

$$= 1.30859.$$

9.10 ROMBERG INTERGRATION

We modify the Trapezoidal rule to find a better approximation to the value of an integral. We know that the truncation error in the trapezoidal rule is nearly proportional to h^2 an interval of size h. The error in the Trapezoidal rule

$$E = -\frac{(b-a)}{12} y''(\xi) h^2$$

where $(a < \xi < b)$.

If we put $$c = -\frac{(b-a)}{12} y''(\xi)$$

then the error in the Trapezoidal rule = ch^2.

If $y''(\xi)$, the second derivative, is reasonably constant c may be taken to be constant. Consider the valuation of the integral

$$I = \int_a^b y\,dx$$

by the Trapezoidal rule with two different intervals say h_1, h_2. Let I_1 and I_2 denote the approximate values with the corresponding errors E_1 and E_2 respectively.

Then $$I = I_1 + ch_1^2$$

and $$I = I_2 + ch_2^2$$

$\therefore$ We get $$I_1 + ch_1^2 = I_2 + ch_2^2$$

or $$c = \frac{I_1 - I_2}{h_2^2 - h_1^2}.$$

$$\therefore\; I = I_1 + \left(\frac{I_1 - I_2}{h_2^2 - h_1^2}\right)h_1^2$$

$$\Rightarrow\; I = \frac{I_1 h_2^2 - I_2 h_1^2}{h_2^2 - h_1^2}. \tag{7}$$

This will be a better approximation to I than I_1 or I_2. The above method is called Richardson's method.

If we take $h = h_1$ and $h_2 = \frac{1}{2}h$ in (7)

we get $$I = \frac{I_1 \dfrac{h^2}{4} - I_2 h^2}{\dfrac{h^2}{4} - h^2} = \frac{\dfrac{1}{4}I_1 - I_2}{\dfrac{-3}{4}} = \frac{4I_2 - I_1}{3}$$

$$\therefore\; I = I_2 + \frac{I_2 - I_1}{3}. \tag{8}$$

If we apply the Trapezoidal rule several times successively halving h, every time the error is reduced by a factor $\frac{1}{4}$. Let $A_1, A_2, A_3, \ldots$ denote the results. Let the formula (8) be applied to each pair of A_is' and denote the results by $B_1, B_2, B_3, \ldots$, etc.

Applying formula (8) to each pair of B_is' we get next results $C_1, C_2, \ldots$ in this process the following array of results is obtained.

$$\begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & \dots \\ & B_1 & B_2 & B_3 & \dots \\ & & C_1 & C_2 & \dots \end{array}$$

The above computation is continued with two successive values are very close to each other. This refinement of Richardson's method is known as Romberg integration.

The values of the integral, in Romberg integration can be tabulated as follows.

$$\begin{array}{llll} I(h) & & & \\ & I(h,\ h/2) & & \\ I(h/2) & & I(h,\ h/2,\ h/4) & \\ & I(h/2,\ h/4) & & I(h,\ h/2,\ h/4,\ h/8) \\ I(h/4) & & I(h/2,\ h/4,\ h/8) & \\ & I(h/4,\ h/8) & & \\ I(h/8) & & & \end{array}$$

where
$$I(h, h/2) = \frac{1}{3}\left[4\,I(h/2) - I(h)\right]$$

$$I(h/2, h/4) = \frac{1}{3}\left[4\,I(h/4) - I(h/2)\right]$$

$$\dots$$

$$I(h, h/2, h/4) = \frac{1}{3}\left[4\,I(h/2, h/4) - I(h, h/2)\right]$$

$$I(h/2, h/4, h/8) = \frac{1}{3}\left[4\,I(h/4, h/8) - I(h/2, h/4)\right]$$

$$I(h, h/2, h/4, h/8) = \frac{1}{3}\left[4\,I(h/2, h/4, h/8) - I(h, h/2, h/4)\right]$$

Example 9.14 *Using Romberg's method compute* $I = \int_0^{1.2} \frac{1}{1+x}\,dx$ *correct to 4 decimal places.*

Solution Here
$$f(x) = \frac{1}{1+x}$$

We can take $h = 0.6,\ 0.3,\ 0.15$

i.e.,
$$h = 0.6,\ \frac{h}{2} = 0.3,\ \frac{h}{4} = 0.15$$

x	0	0.15	0.30	0.40	0.60	0.75	0.90	1.05	1.20
$f(x)$	1	0.8695	0.7692	0.6896	0.6250	0.5714	0.5263	0.48780	0.4545

Using Trapezoidal rule with $h = 0.6$ we get

$$I(h) = I(0.6) = I_1 = \frac{0.6}{2}\left[(1 + 0.4545) + 2 \times 0.6256\right] = 0.8113,$$

with $h = \dfrac{0.6}{2} = 0.3$ we get

$$I(h/2) = I(0.3) = I_2$$

$$= \frac{0.3}{2}\left[(1 + 0.4545) + 2 \times (0.7692 + 0.625 + 0.5263)\right]$$

$$= 0.7943,$$

with $h = \dfrac{0.6}{4} = 0.15$ we get

$$I(h/4) = I(0.15) = I_3$$

$$= \frac{0.15}{2}\left[(1 + 0.4545) + 2 \times (0.8695 + 0.7692 + 0.6896)\right] + \frac{0.15}{2}\left[2 \times (0.6250 + 0.5714 + 0.5263 + 0.4878)\right]$$

$$= 0.7899.$$

Now $$I(h, h/2) = I(0.6, 0.3)$$

$$\therefore\ I(0.6, 0.3) = \frac{1}{3}\left[4 \times I(0.3) - I(0.6)\right]$$

$$= \frac{1}{3}\left[4 \times 0.7943 - 0.8113\right] = 0.7886,$$

Similarly $$I(h/2, h/4) = I(0.3, 0.15)$$

$$\therefore\ I(0.3, 0.15) = \frac{1}{3}\left[4 \times I(0.15) - I(0.3)\right]$$

$$= \frac{1}{3}\left[4 \times 0.7899 - 0.7943\right] = 0.7884.$$

$\therefore$ We get $$I(h, h/2, h/4) = I(0.6, 0.3, 0.15)$$

$$\therefore\ I(0.6, 0.3, 0.15) = \frac{1}{3}\left[4 \times I(0.15, 0.3) - I(0.3, 0.6)\right]$$

$$= \frac{1}{3}\left[4 \times 0.7884 - 0.7886\right] = 0.7883$$

The table of these values is

0.8113		
	0.7886	
0.7948		0.7883
	0.7884	
0.7899		

$$\therefore\ I = \int_0^{1.2} \frac{1}{1+x}\,dx = 0.7883$$

Exercise 9.2

1. If H_0, H_1, H_2, …, H_n are Cotes coefficients, show that
   (a) $H_0 + h_1 + H_2 + \ldots + H_n = nh$
   (b) $H_r = H_{n-r}$

2. Using Cotes formula, show that $\int_{x_0}^{x_2} f(x)\,dx = (x_2 - x_0)\left(\frac{1}{6}y_0 + \frac{4}{6}y_1 + \frac{1}{6}y_2\right)$ and also show that

   $C_0^2 = \frac{1}{6}, C_1^2 = \frac{1}{2}, C_2^2 = \frac{1}{6}$ where C_0^2, C_1^2, C_2^2 are Cotes numbers.

3. Using Romberg's method prove that $\int_0^1 \frac{1}{1+x}\,dx = 0.6931.$

4. Apply Romberg's method to show that $\int_0^{\frac{\pi}{2}} \sin x\,dx = 1.$

5. Apply Romberg's method to evaluate $\int_4^{5.2} \log xk\,dx$ given that

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$\log_e x$	1.3863	1.4351	1.4816	1.526	1.5686	1.6094	1.6486

6. Use Romberg's method and show that $\int_0^1 \frac{dx}{1+x^2} = 0.7855.$

9.11 DOUBLE INTEGRATION

In this section we obtain double integration formulae by shifting the single integration formulae.

Trapezoidal rule Consider the integral of the form

$$I = \int_c^d \left(\int_a^b f(x, y)\,dx\right) dy, \tag{9}$$

over the rectangles $x = a$, $x = b$, and $y = c$, $y = d$.

Evaluating the inner integral by the Trapezoidal rule we get

$$I = \frac{b-a}{2}\int_c^d [f(a, y) + f(b, y)]\,dy, \tag{10}$$

applying Trapezoidal rule again to evaluate the integral on the right hand side of (10) we get

$$I = \frac{(b-a)(d-c)}{4}\left[f(a, c) + f(a, d) + f(b, c) + f(b, d)\right]. \tag{11}$$

If discrete value are given we can use the composite Trapezoidal rule by dividing the interval $[a, b]$ into n equal subintervals each of length h and the interval $[c, d]$ into m equal subintervals each of length k.

We have

$$x_i = x_0 + ih,\ x_0 = a,\ x_n = b$$
$$y_j = y_0 + jk,\ y_0 = c,\ y_m = d$$

applying composite Trapezoidal rule in both the directions we get

$$\begin{aligned} I = \frac{hk}{4}\Big\{ & f(x_0, y_0) + 2\big(f(x_0, y_1) + f(x_0, y_2) + \ldots + f(x_0, y_{m-1})\big) + \\ & 2\sum_{i=1}^{n-1}\Big[f(x_i, y_0) + 2\big(f(x_i, y_1) + f(x_i, y_2) + \ldots + \\ & f(x_i, y_{m-1})\big) + f(x_i, y_m)\Big] + f(x_n, y_0) + \\ & 2\big(f(x_n, y_1) + f(x_n, y_2) + \ldots + f(x_n, y_{m-1})\big) + f(x_n, y_m)\Big\}. \end{aligned} \tag{12}$$

Simpson's method Taking $h = \dfrac{b-a}{2},\ k = \dfrac{d-c}{2}$ and applying Simpson's rule to evaluate (9) we get

$$\begin{aligned} I = \frac{hk}{9}\Big\{ & f(a, c) + f(a, d) + f(b, c) + f(b, d) + \\ & 4\big[f(a, c+k),\ f(a+h, c) + f(a+h, d) + f(b, c+k)\big] + \\ & 16 f(a+h, c+k)\Big\}. \end{aligned} \tag{13}$$

Example 9.15 *Evaluate the integral* $I = \int_1^2\int_1^2 \dfrac{dx\,dy}{x+y}$, *using Trapezoidal rule with $h = k = 0.5$.*

Solution Using Trapezoidal rule, we get

$$I = \int_1^2\int_1^2 \frac{dx\,dy}{x+y}$$

$$= \frac{1}{16}\left[f(1, 1) + f(2, 1) + f(1, 2) + f(2, 2)\right] + \frac{1}{16}\left[2\left[f\left(\frac{3}{2}, 1\right) + f\left(1, \frac{3}{2}\right) + f\left(2, \frac{3}{2}\right) + f\left(\frac{3}{2}, 2\right)\right] + 4f\left(\frac{3}{2}, \frac{3}{2}\right)\right]$$

$$= \frac{1}{16}\left[0.5 + \frac{1}{3} + \frac{1}{3} + 0.25 + 2\left[0.4 + 0.4 + \frac{2}{7} + \frac{2}{7}\right] + \frac{4}{3}\right]$$

$$= 0.343304.$$

Example 9.16 *Using the table of values given below evaluate the integral of f(x, y) =* e^y *sin x over the interval* $0 \le x \le 0.2, 0 \le y < 0.2$

(a) by the Trapezoidal rule with h = k = 0.2 and

(b) by Simpson's one-third rule with h = k = 0.1

y *x*	*0.0*	*0.1*	*0.2*
0.0	*0.0*	*0.998*	*0.1987*
0.1	*0.0*	*0.1103*	*0.2196*
0.2	*0.0*	*0.1219*	*0.2427*

Solution

(*a*) Applying Trapezoidal rule we get

$$I = \frac{(0.2)^2}{4}[0 + 0 + 0.1987 = 0.2427] = 0.004414.$$

(*b*) By Simpson's rule we get

$$I = \frac{(0.1)^2}{9}[1.0 + 4(0.998) + 1(0.1987) + 4.0 + 16(0.1103)] + \frac{(0.1)^2}{9}[4(0.2196) + 1.0 + 4(0.1219) + 1(0.2427)]$$

$= 0.004413.$

Example 9.17 *Evaluate* $\int_0^{0.5}\int_0^{0.5} \frac{xy}{1+xy}\,dx\,dy$ *using Simpson's rule for double integrals with both step sizes equal to 0.25.*

Solution Taking $n = k = 0.25,$

we have

$$x_0 = 0,\ x_1 = 0.25,\ x_2 = 0.5$$

$$y_0 = 0,\ y_1 = 0.25,\ y_2 = 0.5$$

$$f(0, 0) = 0,\ \ f(0, 0.25) = 0,\ \ f(0, 0.5) = 0$$

$$f(0.25, 0) = 0,\ f(0.25, 0.25) = 0.05878525,\ f(0.25, 0.5) = 0.110822$$

$$f(0.5, 0) = 0,\ f(0.5, 0.25) = 0.110822,\ f(0.5, 0.5) = 0.197923$$

Applying, Trapezoidal rule we get

$$I = \frac{1}{9.44}\left[f(0,0) + 4\,f\left(\frac{1}{4},0\right) + f\left(\frac{1}{2},0\right)\right] + 4\left\{f\left(0,\frac{1}{4}\right) + 4\,f\left(\frac{1}{4},\frac{1}{4}\right) + f\left(\frac{1}{2},\frac{1}{4}\right)\right\}$$

$$+ f\left(0,\frac{1}{2}\right) + 4\,f\left(\frac{1}{4},\frac{1}{2}\right) + f\left(\frac{1}{2},\frac{1}{2}\right)$$

$$= \frac{1}{144}[0 + 0 + 0 + 4(0 + 0.235141 + 0.110822) + 0 + 0.443288 + 0.197923]$$

$= 0.014063$

9.12 EULER-MACLAURIN SUMMATION FORMULA

Consider the function $F(x)$, such that

$$\Delta F(x) = f(x) \qquad \text{...(1)}$$

let x_0, x_1, x_2, ..., x_n, be equi-spaced values of x, with difference.

From (1), we get $\quad \Delta F(x) = f(x_0)$

$\Rightarrow \quad F(x_1) - F(x_0) = f(x_0)$

Similarly, $\quad F(x_2) - F(x_1) = f(x_1)$

$$\cdots$$

$$F(x_n) - F(x_{n-1}) = f(x_{n-1})$$

Adding, these we get

$$F(x_n) - F(x_0) = \sum_{i=0}^{n-1} f(x_i) \qquad \text{...(2)}$$

from (1) we have

$$\begin{aligned}
F(x) &= \Delta^{-1} f(x) \\
&= (E - 1)^{-1} f(x) \\
&= (e^{hD} - 1)^{-1} f(x) \\
&= \left[\left(1 + hD + \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \ldots\right) - 1\right]^{-1} f(x) \\
&= \left[(hD) + \frac{(hD)^2}{2!} + \frac{(hD)^3}{3!} + \ldots\right]^{-1} f(x) \\
&= (hD)^{-1}\left[1 + \frac{hD}{2!} + \frac{h^2D^2}{3!} + \ldots\right]^{-1} f(x) \\
&= \frac{1}{h} D^{-1}\left[1 - \frac{hD}{2} + \frac{h^2D^2}{12} - \frac{h^4D^4}{720} + \ldots\right] f(x) \\
&= \frac{1}{h} D^{-1}\left[1 - \frac{hD}{2} + \frac{h^2D^2}{12} - \frac{h^4D^4}{720} + \ldots\right] f(x) \\
&= \frac{1}{h}\int f(x)\,dx - \frac{1}{2} f(x) + \frac{h}{12} f'(x) - \frac{h^3}{720} f'''(x) \\
&= \frac{1}{h}\int f(x)\,dx - \frac{1}{2} f(x) + \frac{h}{12} f'(x) - \frac{h^3}{720} f'''(x) + \ldots
\end{aligned}$$

$$\text{...(3)}$$

Putting $x = x_n$, and $x = x_0$ in (3) and then subtracting we get

$$F(x_n) - F(x_0) = \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx - \frac{1}{2}\left[f(x_n) - f(x_0)\right]$$
$$+\frac{h}{12}\left[f'(x_n) - f'(x_0)\right] - \frac{h^3}{720}\left[f'''(x_n) - f'''(x_0)\right] + \ldots \qquad \ldots(4)$$

$$\Rightarrow \qquad \sum_{i=0}^{n-1} f(x_i) = \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx - \frac{1}{2}\left[f(x_n) - f(x_0)\right] +$$
$$\frac{h}{12}\left[f'(x_n) - f'(x_0)\right] - \frac{h^3}{720}\left[f'''(x_n) - f'''(x_0)\right] + \ldots$$

$$\Rightarrow \qquad \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx = \sum_{i=0}^{n-1} f(x_i) + \frac{1}{2}\left[f(x_n) - f(x_0)\right]$$
$$-\frac{h}{12}\left[f'(x_n) - f'(x_0)\right] + \frac{h^3}{720}\left[f'''(x_n) - f'''(x_0)\right]\ldots$$

Hence, we obtain

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_{0+nh}} y\,dx$$
$$= \frac{h}{2}\left[y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n\right]$$
$$-\frac{h^2}{12}(y'_n - y'_0) + \frac{h^4}{720}(y'''_n - y'''_0) + \ldots \qquad \ldots(5)$$

(5) is called the Euler-Maclaurin formula.

Example 9.18 *Find the value of $\log_e 2$ from $\int_0^1 \frac{1}{1+x}dx$, using Euler-Maclaurin formula.*

Solution Taking $y = \frac{1}{1+x}$, and $n = 10$

We have $x_0 = 0,\ x_n = 1,\ h = 0.1$

$$y' = \frac{-1}{(1+x)^2},\quad y'' = \frac{2}{(1+x)^3},\quad y''' = \frac{-6}{(1+x)^4}$$

$$y_0 = \frac{1}{1+x_0} = \frac{1}{1+0} = 1,\quad y_1 = \frac{1}{1+x_1} = \frac{1}{1+0.1} = \frac{1}{1.1},\ \ldots,\ x_n = \frac{1}{1+1} = \frac{1}{2}$$

from Euler-Maclaurin Summation formula.

$$\int_0^1 \frac{1}{1+x}\,dx = \frac{0.1}{2}\left[1+\left(\frac{2}{1.1}+\frac{2}{1.2}+\ldots+\frac{2}{1.9}\right)+\frac{1}{2}\right]$$

$$-\frac{(0.1)^2}{12}\left[\frac{-1}{2^2}-\frac{(-1)^2}{1^2}\right]+\frac{(0.1)^4}{720}\left(\frac{-6}{2^4}-\frac{(-8)}{1^4}\right)$$

$$= 0.693773 - 0.000625 + 0.000001$$

$$= 0.693149 \qquad \ldots(1)$$

Now $$\int_0^1 \frac{1}{1+x}\,dx = \log_e |1+x|_0^1 = \log_{e^2} \qquad \ldots(2)$$

Hence from (2) we get

$$\log_e 2 \simeq 0.693149$$

Example 9.19 *Use the Euler-Maclaurin expansion to prove*

$$\sum_{x=1}^{n} x^2 = \frac{n(n+1)(2n+1)}{6}$$

Solution We have

$$y = f(x) = x^2$$
$$y' = f'(x) = 2x$$
$$y'' = 2, \quad y''' = 0, \ldots$$

Taking $h = 1$, we get $x_0 = 1$, $x_n = n$, $y_0 = 1$, $y_n = n^2$.

From Euler-Maclaurin formula we have

$$y_0 + y_1 + \ldots y_n = \sum_{x=1}^{n} x^2$$

$$= \frac{1}{h}\int_{x_0}^{x_n} f(x)\,dx + \frac{1}{2}(y_n + y_0) + \frac{1}{12}(y'_n - y'_n) + \ldots$$

$$= \int_1^n x^2 dx + \frac{1}{2}(n^2+1) + \frac{1}{12}(2n-2)$$

$$= \frac{1}{3}(n^2-1) + \frac{1}{2}(n^2+1) + \frac{1}{6}(n-1)$$

$$= \frac{1}{6}\left(2n^3 - 2 + 3n^2 + 3 + n - 1\right)$$

$$= \frac{2n^3+3n^2+n}{6} = \frac{1}{6}n(n+1)(2n+1)$$

Hence Proved.

Exercise 9.3

1. Evaluate $\int_1^2\int_1^2 \frac{dx\,dy}{x+y}$ using the Trapezoidal rule with $h = k = 0.25$.

2. Evaluate the double integral $\int_0^1 \left(\int_0^2 \frac{2xy}{\left(1+x^2\right)\left(1+y^2\right)}\, dy \right) dx$ using (i) the Trapezoidal rule with $h = k = 0.25$ (ii) the Simpson's rule $h = k = 0.25$.

3. Evaluate the double integral $\int_1^5 \left(\int_1^5 \frac{dx}{\left(x^2+y^2\right)^{\frac{1}{2}}} \right) dy$ using the Trapezoidal rule with two and four subintervals.

4. Using the table of values given below evaluate the integral of $f(x, y) = e^y \sin x$ over the interval $0 \le x \le 0.2$, $0 \le y \le 0.2$

(a) by the Trapezoidal rule with $h = k = 0.1$

(b) by Simpson's one-third rule with $h = k = 0.1$

yx	0.0	0.1	0.2
0	0.0	0.0998	0.1987
0.1	0.0	0.1103	0.2196
0.2	0.0	0.1219	0.2427

5. Integrate the following functions over the given domains by the Trapezoidal formula, using the indicated spacing.

(a) $f\left(x,y\right) = \sqrt{1-xy}$; $0 \le x \le 1, 0 \le y \le 1$, with $h = k = 0.5$

(b) $f\left(x, y\right) = \sin x \cos y$; $0 \le x \le \frac{\pi}{2}$ with $h = k = \frac{\pi}{4}$

6. Find the value of the double integral $I = \int_2^{3.2}\int_1^{3.6} \frac{1}{x+y}\, dy\, dx.$

7. Use Euler-Macleurin formula to prove that

$$\sum_{x=1}^{n} x^3 = \frac{n^2(n+1)^2}{4}$$

8. Use Euler-Maclaurin formula to show that

$$\frac{1}{51^2} + \frac{1}{53^2} + ... + \frac{1}{99^2} = 0.00499$$

Answers

1. 0.340668
2. (i) 0.31233 (ii) 0.31772
3. $n = 2$, $I = 4.134$; $n = 4$, $I = 3.997$
4. 0.004413
5. (a) 0.8308 (b) 0.8988
6. 0.48997

10

NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

10.1 INTRODUCTION

The most general form of an ordinary differential equation of nth order is given by

$$\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \cdots, \frac{d^ny}{dx^n}\right) = 0 \tag{1}$$

A general solution of an ordinary differential equation such as (1) is a relation between y, x and n arbitrary constants which satisfies the equation and it is of the form

$$f\left(x, y, c_1, c_2, ..., c_n\right) = 0. \tag{2}$$

If particular values are given to the constants c_1, c_2, …, c_n, then the resulting solution is called a *Particular solution*. To obtain a particular solution from the general solution given by (2), we must be given n conditions so that the constants can be determined. If all the n conditions are specified at the same value of x, then the problem is termed as *initial value problem*. Though there are many analytical methods for finding the solution of the equation form given by (1), there exist large number of ordinary differential equations, whose solution cannot be obtained by the known analytical methods. In such cases we use numerical methods to get an approximate solution of a given differential equation under the prescribed initial condition.

In this chapter we restrict ourselves and develop the numerical methods for findings a solution of an ordinary differential equation of first order and first degree which is of the form

$$\frac{dy}{dx} = f(x, y),$$

with the initial condition $y(x_0) = y_0$, which is called *initial value problem*.

The general solutions of equation (3) will be obtained in two forms: (1) the values of y as a power series in independent variable x and (2) as a set of tabulated values of x and y.

We shall now develop numerical methods for solution of the initial value problem of the form given by (3). We partition the interval [a, b] on which the solution is derived in finite number of sub-intervals by the points

$$a = x_0 < x_1, < x_2, \cdots < x_n = b.$$

The points are called *Mesh Points*. We assume that the points are spaced uniformly with the relation

$$x_n = x_0 + nh.$$

The existence of uniqueness of the solution of an initial value problem in $[x_0, b]$ depends on the theorem due to Lipschitz, which states that:

(1) If $f(x, y)$ is a real function defined and continuous in $[x_0, b]$, $y \in (-\infty, +\infty)$, where x_0, and b are finite.

(2) There exists a constant $L > 0$ called *Lipschitz's constant* such that for any two values $y = y_1$ and $y = y_2$

$$|f(x, y_1) - f(x, y_2)| < L\,|L_1 - L_2|$$

where $x \in [x_0, b]$, then for any $y(x_0) = y_0$, the initial value problem (3), has unique solution for $x \in [x_0, b]$.

Now we shall discuss the Taylor's series method and the Euler's method.

10.2 TAYLOR'S SERIES METHOD

Let $y = f(x)$, be a solution of the equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{refer (3)}$$

with $y(x_0) = y_0$.

Expanding it by Taylor's series about the point x_0, we get

$$f(x) = f(x_0) + \frac{(x - x_0)}{1!} f'(x_0) + \frac{(x - x_0)^2}{2!} f'(x_0) + \cdots,$$

this may be written as

$$y = f(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots$$

Putting $x = x_1 = x_0 + h$, we get

$$f(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h_3}{3!} y_0''' + \cdots \tag{4}$$

Similarly we obtain

$$y_{n+1} = y_n + \frac{h}{1!} y_n' + \frac{h^2}{2!} y_n'' + \frac{h}{3!} y_n''' + \cdots \tag{5}$$

Equation (5) may be written as

$$y_{n+1} = y_n + \frac{h}{1!} y'_n + \frac{h^2}{2!} y''_n + O\left(h^3\right), \tag{6}$$

where $O(h^3)$ means that all the succeeding terms containing the third and higher powers of h. If the terms containing the third and higher powers of h are neglected then the local truncation error in the solution is kh^3 where k is a constant. For a better approximation terms containing higher powers of h are considered.

Note: Taylor's series method is applicable only when the various derivatives of $f(x, y)$ exist and the value of $(x - x_0)$ in the expansion of $y = f(x)$ near x_0 must be very small so that the series converge.

Example 10.1 *Solve* $\frac{dy}{dx} = x + y, y(1) = 0$, *numerically up to* $x = 1.2$, *with* $h = 0.1$.

Solution We have $x_0 = 1$, $y_0 = 1$ and

$$\frac{dy}{dx} = y' = x + y \Rightarrow y'_0 = 1 + 0 + 1,$$

$$\frac{d^2y}{dx^2} = y'' = 1 + y' \Rightarrow y''_0 = 1 + 1 = 2,$$

$$\frac{d^3y}{dx^3} = y''' = y'' \Rightarrow y'''_0 = 2,$$

$$\frac{d^4y}{dx^4} = y'^v = y''' \Rightarrow y_0'^v = 2,$$

$$\frac{d^5y}{dx^5} = y^v = y'^v \Rightarrow y^v{}_0 = 2,$$

$$\vdots$$

Substituting the above values in

$$y_1 = y_0 + \frac{h}{1!} y'_0 + \frac{h^2}{2!} y''_0 + \frac{h^3}{3!} y'''_0 + \frac{h^4}{4!} y_0'^v + \frac{h^5}{5!} y_0{}^v + \cdots$$

we get

$$y_1 = 0 + (0.1) + \frac{(0.1)^2}{2} 2 + \frac{(0.1)^3}{6} 2 + \frac{(0.1)^4}{24} 2 + \frac{(0.1)^5}{120} 2 + \cdots$$

$$\Rightarrow \quad y_1 = 0.11033847$$

$$\therefore \quad y_1 = y(0.1) \approx 0.110.$$

Now $\quad x_1 = x_0 + h = 1 + 0.1 = 1.1,$

we have $\quad y'_1 = x_1 + y_1 = 1.1 + 0.110 = 1.21,$

$$y''_1 = 1 + y'_1 = 1 + 1.21 = 2.21,$$

$$y'''_1 = y''_1 = 2.21,$$

$$y_1'^v = 2.21,$$

$$y_1^{\,v} = 2.21,$$

$$\vdots$$

Substituting the above values in (1), we get

$$y_2 = 0.110 + (0.1)(1.21) + \frac{(0.1)^2}{2}(2.21) + \frac{(0.1)^3}{6}(2.21) +$$

$$\frac{(0.1)^4}{24}(2.21) + \frac{(0.5)^5}{120}(2.21),$$

$$\therefore \; y_2 = 0.24205$$

$$\therefore \; y(0.2) = 0.242$$

Example 10.2 *Given* $\frac{dy}{dx} = 1 + xy$ *with the initial condition that* $y = 1, x = 0$. *Compute* $y\,(0.1)$ *correct to four places of decimal by using Taylor's series method.*

Solution Given $$\frac{dy}{dx} = 1 + xy \text{ and } y(0) = 1,$$

$$\therefore \; y_1(0) = 1 + 0 \times 1 = 1.$$

Differentiating the given equation w.r.t. x, we get

$$\frac{d^2y}{dx^2} = y + x\,\frac{dy}{dx},$$

$$y_0'' = 1 + 0 \times 1 = 1 + 0 = 1$$

similarly $$\frac{d^3y}{dx^3} = x\,\frac{d^2y}{dx^2} + 2\,\frac{dy}{dx},$$

$$\Rightarrow \; y_0''' = 2,$$

and $$\frac{d^4y}{dx^4} = x\,\frac{d^3y}{dx^3} + 3\,\frac{d^2y}{dx^2},$$

$$\Rightarrow \; y_0'^{\,v} = 3,$$

from Taylor's series method, we have

$$y_1 = 1 + hy_0' + \frac{h^2}{2}\,y_0'' + \frac{h^3}{3}\,y_0''' + \frac{h^4}{24}\,y_0'^{\,v} + \cdots$$

$$\therefore \; y(0.1) = 1 + (0.1)(1) + \frac{(0.1)^2}{2}\,1 + \frac{(0.1)^3}{6}\,2 + \frac{(0.1)^4}{24}\,3 + \cdots$$

$$= 1.1053425$$

$$\therefore \; y(0.1) = 1.1053$$

correct to four decimal places.

Example 10.3 *Apply the Taylor's series method to find the value of y (1.1) and y (1.2) correct to three decimal places given that* $\frac{dy}{dx} = xy^{\frac{1}{3}}$, *y(1) = y(1) = 1 taking the first three terms of the Taylor's series expansion.*

Solution Given
$$\frac{dy}{dx} = xy^{\frac{1}{3}},\ y_0 = 1,\ x_0 = 1,\ h = 0.1$$

$$y_0' = x_0\, y_0^{\frac{1}{3}} = 1.1^{\frac{1}{3}} = 1,$$

differentiating the given equation w.r.t. x we get

$$\frac{d^2y}{dx^2} = \frac{1}{3}\, xy^{\frac{-2}{3}}\ \frac{dy}{dx} + y^{\frac{1}{3}}$$

$$= \frac{1}{3}\, xy^{\frac{-2}{3}} \left(xy^{\frac{1}{3}}\right) + y^{\frac{1}{3}} = \frac{1}{3}\, x^2 y^{-\frac{1}{3}} + y^{\frac{1}{3}}$$

$$\Rightarrow\ y_0'' = \frac{1}{3}\, 1.1 + 1 = \frac{4}{3}.$$

Taking the first three terms of the Taylor's formula we get

$$y_1 = y_0 = hy_0' + \frac{h^2}{2}\, y'' \tag{7}$$

substituting the values in (7)

$$y_1 = y(1.1) = 1 + (0.1) \times 1 + \frac{(0.1^2)}{2} \times \frac{4}{3} = 1.1066$$

$$\therefore\ y_1(1.1) = 1.1066,$$

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$y_1' = \left(x_1 \times y_1^{\frac{1}{3}}\right) = (1.1)(1.1066)^{\frac{1}{3}} = 1.138,$$

$$y_1'' = \frac{1}{3} x_1^2\, y_1^{\frac{1}{3}} + y_1^{\frac{1}{3}}$$

$$= \frac{1}{3}(1.1)^2 (1.1066)^{\frac{-1}{3}} + (1.1066)^{\frac{1}{3}}$$

$$= 1.4249.$$

Substituting in
$$y_2 = y_1 + hy_1' + \frac{h^2}{2}\, y_1'',$$

we get
$$y_2 = y(1.2) = 1.1066 + 0.1 \times 1.138 + \frac{(0.1)^2}{2} \times 1.4249$$

$$= 1.2275245 \approx 1.228$$

$$\therefore \; y_2 = y(1.2) = 1.228.$$

10.3 EULER'S METHOD

Consider the first order and first degree differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{refer (3)}$$

with the condition that $y(x_0) = y_0$. Suppose we want to find the approximate value of y say y_n when $x = x_n$. We divide the interval $[x_0, x_n]$ into n–subintervals of equal length say h, with the division point $x_0, x_1, \ldots, x_n$, where $x_r = x_0 + rh$, $(r - 1, 2, \ldots, n)$.

Let us assume that

$$f(x, y) \approx f(x_{r-1}, y_{r-1})$$

in $[x_{r-1}, x_r]$. Integrating (3) in $[x_{r-1}, x_r]$, we get

$$\int_{x_{r-1}}^{x_r} dy = \int_{x_{r-1}}^{x_r} f(x, y)\,dx$$

$$\Rightarrow \; [y_r - y_{r-1}] = \int_{x_{r-1}}^{x_r} f(x, y)\,dx$$

$$\Rightarrow \; y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1}) \int_{x_{r-1}}^{x_r} dx$$

$$\Rightarrow \; y_r \approx y_{r-1} + f(x_{r-1}, y_{r-1})(x_r - x_{r-1})$$

$$\therefore \; y_r \approx y_{r-1} + hf(x_{r-1}, y_{r-1}). \qquad (8)$$

Equation (8) is called *Euler's iteration formula*.

Taking $r = 1, 2, \ldots, n$ in (8), we get the successive approximately of y as follows

$$y_1 = y(x_1) = y_0 + hf(x_0, y_0)$$

$$y_2 = y(x_2) = y_1 + hf(x_1, y_1)$$

$$\vdots$$

$$y_n = y(x_n) = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

Note: Euler's method has limited usage because of the large error that is accumulated as the process proceeds. The process is very slow and to obtain reasonable accuracy with Euler's method we have to take a smaller value of h. Further, the method should not be used for a larger range of x as the values found by this method go on becoming farther and farther away from the true values. To avoid this difficulty one can choose Euler's modified method to solve the equation (3).

10.4 MODIFIED EULER'S METHOD

From Euler's iteration formula we h known that

$$y_r \approx y_{r-1} + hf\left(x_{r-1}, y_{r-1}\right) \tag{9}$$

Let $y(x_r) = y_r$ denote the initial value using (9) an approximate value of y_r^0 can be calculated as

$$y_r^{(0)} = y_{r-1} + \int_{x_{r-1}}^{x_r} f(x, y)\,dx$$

$$\Rightarrow \; y_r^{(0)} \approx y_{r-1} + hf\left(x_{r-1}, y_{r-1}\right) \tag{10}$$

replacing $f(x, y)$ by $f(x_{r-1}, y_{r-1})$ in $x_{r-1} \le x < x_r$ using Trapezoidal rule in $[x_{r-1}, r_x]$, we can write

$$y_r^{(0)} = y_{r-1} + \frac{h}{2}\left[f\left(x_{r-1}, y_{r-1}\right) + f\left(x_r, y_r\right)\right].$$

Replacing $f(x_r, y_r)$ by its approximate value $f\left(x_r, y_r^{(0)}\right)$ at the end point of the interval $[x_{r-1}, x_r]$, we get

$$y_r^{(1)} = y_{r-1} + \frac{h}{2}\left[f\left(x_{r-1}, y_{r-1}\right) + f\left(x_r, y_r^{(0)}\right)\right],$$

where $y_r^{(1)}$ is the first approximation to $y_r = y(x_r)$ proceeding as above we get the iteration formula

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f\left(x_{r-1}, y_{r-1}\right) + f\left(x_r, y_r^{(n-1)}\right)\right], \tag{11}$$

where y_r^n denoted the nth approximation to y_r

$\therefore$ we have

$$y_r \approx y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f\left(x_{r-1}, y_{r-1}\right) + f\left(x_r, y_r^{(n-1)}\right)\right].$$

Example 10.4 *Solve the equation* $\frac{dy}{dx} = 1 - y$, *with the initial condition $x = 0$, $y = 0$, using Euler's algorithm and tabulate the solutions at $x = 0.1, 0.2, 0.3$.*

Solution Given $\frac{dy}{dx} = 1 - y$, with the initial condition $x = 0$, $y = 0$

$$\therefore \quad f(x, y) = 1 - y$$

we have

$$h = 0.1$$

$$\therefore \quad x_0 = 0, \; y_0 = 0$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = 0.2, \; x_3 = 0.3.$$

Taking $n = 0$ in

$$y_{n+1} = y_n + hf(x_n, y_n)$$

we get

$$y_1 = y_0 + hf(x_0, y_0) \;\; = 0 + (0.1)(1 - 0) = 0.1$$

$$\therefore y_1 = 0.1, \quad i.e., \quad y(0.1) = 0.1,$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 0.1 + (0.1)(1 - y_1)$$

$$= 0.1 + (0.1)(1 - 0.1) \;\; = 0.19$$

$$\therefore \quad y_2 = y(0.2) = 0.19,$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$\therefore \quad y_3 = 0.19 + (0.1)(1 - y_2) \;\; = 0.19 + (0.1)(1 - 0.19)$$

$$= 0.19 + (0.1)(0.81) \;\; = 0.271$$

$$\therefore y_3 = y(0.3) = 0.271.$$

x	*Solution by Euler's method*
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Example 10.5 *Given* $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, *compute y(0.2) by Euler's method taking h = 0.01.*

Solution Given

$$\frac{dy}{dx} = x^3 + y,$$

with the initial condition $y(0) = 1$.

$\therefore$ We have $$f(x, y) = x^3 + y$$

$$x_0 = 0,\ y_0 = 1,\ h = 0.01$$

$$x_1 = x_0 + h = 0 + 0.01 = 0.01,$$

$$x_2 + x_0 + 2h = 0 + 2(0.01) = 0.02.$$

Applying Euler's formula we get

$$y_1 = y_0 + hf(x_0, y_0)$$

$$\therefore\ y_1 = 1 + (0.01)(x_0^3 + y_0)$$

$$= 1 + (0.01)(0^3 + 1) = 1.01$$

$$\therefore\ y_1 = y(0.01) = 1.01,$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.01 + (0.01)\left[x_1^3 + y_1\right]$$

$$= 1.01 + (0.01)\left[(0.01)^3 + 1.01\right] = 1.0201$$

$$\therefore\ y_2 = y(0.02) = 1.0201.$$

Example 10.6 *Solve by Euler's method the following differential equation x = 0.1 correct to four decimal places* $\frac{dy}{dx} = \frac{y - x}{y + x}$ *with the initial condition y(0) = 1.*

Solution Here $$\frac{dy}{dx} = \frac{y - x}{y + x}$$

$$\Rightarrow\ f(x, y) = \frac{y - x}{y + x},$$

the initial condition is $y(0) = 1$.

Taking $h = 0.02$, we get

$$x_1 = 0.02,$$
$$x_2 = 0.04,$$
$$x_3 = 0.06,$$
$$x_4 = 0.08,$$
$$x_5 = 0.1.$$

Using Euler's formula we get

$$y_1 = y(0.02) = y_0 + hf(x_0, y_0)$$

$$= y_0 + h\left(\frac{y_0 - x_0}{y_0 + x_0}\right) = 1 + (0.02)\left(\frac{1-0}{1+0}\right) = 1.0200$$

$$\therefore \quad y(0.02) = 1.0200,$$

$$y_2 = y(0.04) = y_1 + hf(x_1, y_1) = y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.0200 + (0.02)\left(\frac{1.0200 - 0.02}{1.0200 + 0.02}\right) = 1.0392$$

$$y_2 = y(0.04) = 1.0392,$$

$$y_3 = y(0.06) = y_2 + h\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$$

$$= 1.0392 + (0.02)\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$\therefore \quad y_3 = y(0.06) = 1.0577,$$

$$y_4 = y(0.08) = y_3 + hf(x_3, y_3) = y_3 + h\left(\frac{y_3 - x_3}{y_3 + x_3}\right)$$

$$= 1.0577 + (0.02)\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right] = 1.0756$$

$$\therefore y_4 = y(0.08) = 1.0756,$$

$$y_5 = y(0.1) = y_4 + hf(x_4, y_4)$$

$$= y_4 + h\left(\frac{y_4 - x_4}{y_4 + x_4}\right)$$

$$= 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right] = 1.0928$$

$$\therefore \quad y(0.1) = 1.0928.$$

Example 10.7 *Solve the Euler's modified method the following differential equation for $x = 0.02$ by taking $h = 0.01$ given $\frac{dy}{dx} = x^2 + y, y = 1,$ when $x = 0.$*

Solution Here we have

$$f(x, y) = x^2 + y$$

$$h = 0.01,\ x_0 = 0,\ y_0 = y(0) = 1$$

$$x_1 = 0.01,\ x_2 = 0.02$$

we get

$$\therefore\ y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.01)\left(x_0^2 + y_0\right) = 1 + (0.01)\left(0^2 + 1\right) = 1.01$$

$$\therefore\ y_1^{(0)} = 1.01.$$

Applying Euler's modified formula we get

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.01\right] = 1.01005$$

$$\therefore\ y_1^{(1)} = 1.01005,$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.1005\right] = 1.01005$$

$$y_1^{(2)} = 1.01005,$$

$$\therefore\ y_1^{(1)} = y_1^{(2)} = 1.01005,$$

$$\therefore\ y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$= 1.01005 + (0.01)\left(x_1^2 + y_1\right)$$

$$= 1.01005 + (0.01)\left((0.01)^2 + 1.01005\right) = 1.02015,$$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left| f(x_1, y_1) + f(x_2, y_2^{(0)})\right|$$

$$= 1.01005 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.02015)\right]$$

$$= 1.020204,$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 1.01 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.020204)\right]$$

$$\therefore\ y_2 = 1.020204$$

$$\therefore\ y_2 = y(0.02) = 1.020204.$$

Exercise 10.1

1. Given $\frac{dy}{dx} = \frac{1}{x^2 + y}$, $y(4) = 4$ find $y(4.2)$ by Taylor's series method, taking $h = 0.1$.

2. Given that $\frac{dy}{dx} = x + y^2$, $y(0) = 1$ find $y(0.2)$.

3. Solve $\frac{dy}{dx} = 3x + y^2$, $y = 1$, when $x = 0$, numerically up to $x = 0.1$ by Taylor's series method.

4. Apply Taylor's algorithm to $y^1 = x^2 + y^2$, $y(0) = 1$. Take $h = 0.5$ and determine approximations to $y(0.5)$. Carry the calculations up to 3 decimals.

5. Find $y(1)$ by Euler's method from the differential equation $\frac{dy}{dx} = -\frac{-y}{1+x}$, when $y(0.3) = 2$. Convert up to four decimal places taking step length $h = 0.1$.

6. Find $y(4.4)$, by Euler's modified method taking $h = 0.2$ from the differential equation $\frac{dy}{dx} = \frac{2 - y^2}{5x}$, $y = 1$ when $x = 4$.

7. Given $\frac{dy}{dx} = x^2 + y$, with $y(0) = 1$, evaluate $y(0.02)$, $y(0.04)$ by Euler's method.

8. Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ by Euler's method up to two decimal places.

9. Given $\frac{dy}{dx} = -\frac{y - x}{1+x}$, with boundary condition $y(0) = 1$, find approximately y for $x = 0.1$, by Euler's method (five steps).

10. Use modified Euler's method with one step to find the value of y at $x = 0.1$ to five significant figures, where $\frac{dy}{dx} = x^2 + y$, $y = 0.94$, when $x = 0$.

11. Solve $y' = x - y^2$, by Euler's method for $x = 0.2$ to 0.6 with $h = 0.2$ initially $x = 0$, $y = 1$.

12. Solve the differential equation

$$\frac{dy}{dx} = 2y + 3e^x$$

with $x_0 = 0$, $y_0 = 0$, using Taylor's series method to obtain and check the value of y for $x = 0.1$, 0.2.

13. Solve $y^1 = y \sin x + \cos x$, subject to $x = 0$, $y = 0$ by Taylor's series method.

14. Using Euler's modified method, solve numerically the equation

$$\frac{dy}{dx} = x + \left|\sqrt{y}\right|$$

with boundary condition $y = 1$ at $x = 0$ for the range $0 \le x \le 0.4$ in steps of 0.2.

15. Solve $\frac{dy}{dx} = 1 - y$ with $y = 0$ when $x = 0$ in the range $0 \le x \le 0.2$ by taking $h = 0.1$ (apply Euler's modified formula).

Answers

1. 4.0098 **2.** 1.2375 **3.** 1.12725

4. 1.052 **5.** 1.2632 **6.** 1.01871

7. 1.0202, 1.0408, 1.0619 **8.** 2.42, 2.89 **9.** 1.0928

10. 1.039474 **11.** $y(0.2) = 0.8512$, $y(0.4) = 0.7798$, $y(0.6) = 0.7260$

12.

x	*Calculated values of y*	*Exact values*
0.1	0.3488	0.3486
0.2	0.8112	0.8112

13. $y = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots$

14.

x	y
0.0	1
0.2	1.2309
0.4	1.5253

15.

x	y
0.0	0.0000
0.1	0.09524
0.2	0.1814076

10.5 PREDICTOR–CORRECTOR METHODS

Predictor–corrector formulae are easily derived but require the previous evaluation of y and $y_1 = f(x, y)$ at a certain number of evenly spaced pivotal point (discrete points of x_i of x-axis) in the neighbourhood of x_0.

In general the Predictor–corrector methods are the methods which require the values of y at $x_n, x_{n-1}, x_{n-2}, \ldots$ for computing the value of y at x_{n+1}. A Predictor formula is used to predict the value of y_{n+1}. Now we discuss Milne's method and Adams–Bashforth–Moulton methods which are known as Predictor–corrector methods.

10.6 MILNE'S METHOD

This method is a simple and reasonable accurate method of solving the ordinary first order differential equation numerically. To solve the differential equation

$$\frac{dy}{dx} = y' = f(x, y),$$

by this method we first obtain the approximate value of y_{n+1} by predictor formula and then improve the value of y_{n+1} by means of a corrector formula. Both these formulas can be derived from the Newton forward interpolation formula as follows:

From Newton's formula, we have

$$f(x) = f(x_0 + uh) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{1\times 2}\Delta^2 f(x) + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 f(x_0) + \cdots \tag{14}$$

where $u = \dfrac{x - x_0}{h}$, or $x = x_0 + uh$.

Putting $y' = f(x)$ and $y_0' = f(x_0)$ in the above formula we get

$$y' = y_0' + u\Delta y_0' + \frac{u(u-1)}{1\times 2}\Delta^2 y_0' + \frac{u(u-1)(u-2)}{1\times 2\times 3}\Delta^3 y_0' + \frac{u(u-1)(u-2)(u-3)}{1\times 2\times 3\times 4}\Delta^4 y_0^1 + \ldots \tag{15}$$

Integrating (15) from $x_0 + x_0 + 4h$, *i.e.*, from $u = 0$ to $u = 4$, we get

$$\int_{x_0}^{x_0+4h} y'dx = h\int_0^4 \left[y_0' + u\Delta y_0' + \frac{u(u-1)}{2}\Delta^2 y_0' + \right.$$

$$\left. \frac{u(u-1)(u-2)}{6}\Delta^3 y_0' + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0' + \ldots \right] du$$

$(\because \quad hdu = dx)$ which gives

$$y_{x_0+4h} - y_{x_0} = h\left[4y_0' + \Delta y_0' + \frac{20}{3}\Delta^2 y_0' + \frac{8}{3}\Delta^3 y_0' + \frac{28}{90}\Delta^4 y_0'\right]$$

[considering up to fourth differences only].

Using $\Delta = E - 1$

$$y_4 - y_0 = h\left[4y_0' + 8(E-1)y_0' + \frac{20}{3}(E-1)^2 y_0' + \frac{8}{3}(E-1)^3 y_0' + \frac{14}{45}\Delta^4 y_0'\right]$$

$$\Rightarrow \; y_4 - y_0 = \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] + \frac{14}{45}\Delta^4 y_0'. \tag{16}$$

This is known as Milne's predictor formula. The corrector formula is obtained by integrating (15) from x_0 to $x_0 + 2h$, *i.e.*, from $u = 0$ to $u = z$

$$\int_{x_0}^{x_0+2h} y\,dx = h\int_0^2 \left(y_0^I + u\,\Delta y_0^I + \frac{u(u-1)}{2}\Delta^2 y_0^I + \ldots \right) du$$

$$y_2 - y_0 = h\left[2y_0' + 2\Delta y_0' + \frac{1}{3}\Delta^2 y_0' - \frac{1}{90}\Delta^4 y_0'\right]$$

using $\Delta = E - 1$, and simplifying we get

$$y_2 = y_0 + \frac{h}{3}\left[y_0' + 4y_1' + y_2'\right] - \frac{h}{90}\Delta^4 y_0' \tag{17}$$

Expression (17) is called Milne's corrector formula.

The general forms of Equations (16) and (17) are

$$y_{n+1} = y_{n-3} + \frac{4h}{3}\left[2y_{n-2}' + y_{n-1}' + 2y_n'\right], \tag{18}$$

and
$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + y_{n+1}'\right], \tag{19}$$

i.e.,
$$\bar{y}_{n+1} - y_{n-3} = \frac{4h}{3}\left[2y_{n-2}' + y_{n-1}' + 2y_n'\right], \tag{20}$$

and
$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + \bar{y}_{n+1}'\right]. \tag{21}$$

In terms of f the Predictor formula is

$$\bar{y}_{n+1} = y_{n-3} + \frac{4h}{3}\left[2f_{n-2} - f_{n-1} + 2f_n\right], \tag{22}$$

the corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[f_{n-1} + 4f_n + \bar{f}_{n+1}\right].$$

Note: In deriving the formula we have considered the differences up to third order because we fit up with a polynomial of degree four. To solve the first order differential equation, we first find the three consecutive values of y and x in addition to the initial values and then we find the next value of y by (18) or (22). The value of y thus obtained is then substituted in $y' = f(x, y)$ to get y'. The value is then substituted in (19), to get the corrected value of the new y. If the corrected value of y agrees closely with the predicted value then we proceed to the next interval and if the corrected value of y differs with the predicted value, we then compute the value of

$$\delta = \frac{1}{29}(y_{n+1} - \bar{y}_{n+1}).$$

If the value of δ is very small we proceed to the next interval otherwise value of δ is made small.

Example 10.8 *Given $\frac{dy}{dx} = \frac{1}{2}(1 + x^2)y^2$ and y(0) = 1, y(0.1) = 1.06, y(0.2) = 1.12, y(0.3) = 1.21. Evaluate y(0.4) by Milne's Predictor–Corrector method.*

Solution Milne's predictor formula is

$$\bar{y}_{n+1} = y_{n-3} + \frac{4h}{3}(2y_{n-2}' - y_{n-1}' + 2y_n').$$

Putting $n = 3$ in the above formula we get

$$\bar{y}_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]. \tag{23}$$

We have $y_0 = 1,\ y_1 = 1.06,\ y_2 = 1.12,\ y_3 = 1.21$ and $h = 0.1.$

The given differential equation is

$$y' = \frac{1}{2}\left(1 + x^2\right)y^2$$

$$y_1' = \frac{1}{2}\left[\left(1 + x_1^2\right)y_1^2\right] = \frac{1}{2}\left[\left(1 + (0.1)^2\right)\right]\cdot(1.06)^2$$

$$= 0.505 \times (1.06)^2 = 0.5674,$$

$$y_2' = \frac{1}{2}\left[\left(1 + x_2^2\right)y_2^2\right] = \frac{1}{2}\left[\left(1 + (0.2)^2\right)\right]\cdot(1.12)^2$$

$$= 0.52 \times (1.12)^2 = 0.6522,$$

$$y_3' = \frac{1}{2}\left[\left(1 + x_2^3\right)y_3^2\right] = \frac{1}{2}\left[\left(1 + (0.3)^2\right)\right]\cdot(1.21)^2$$

$$= 0.545 \times (1.21)^2 = 0.7980.$$

Substituting these values in (23) that is in the predictors formula, we get

$$\bar{y}_4 = 1 + \frac{4 \times (0.1)}{3}\left[2 \times 0.5674 - 0.6522 + 2 \times 0.7980\right]$$

$$= 1.27715 = 1.2772 \tag{24}$$

(correct to 4 decimal places).

$\therefore$ We get
$$y_4' = \frac{1}{2}\left[\left(1 + x_4^2\right)\bar{y}_4^2\right]$$

$$= \frac{1}{2}\left[\left(1 + (0.4)^2\right)\right]\cdot(1.2772)^2 = 0.9458.$$

Milne’s corrector formula is

$$y_{n+1} = y_{n-1} + \frac{h}{3}\left[y_{n-1}' + 4y_n' + \bar{y}_{n+1}'\right], \tag{25}$$

putting $n = 3$ in (25) we get

$$y_4 = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + \bar{y}_4'\right]$$

$$= 1.12 + \frac{0.1}{3}\left[0.6522 + 4 \times 0.798 + 0.9458\right]$$

$$= 1.2797$$

(correct to 4 decimal places)

$$\therefore\ y(0.4) = 1.2797.$$

Example 10.9 *Tabulate by Milne's method the numerical solution of* $\frac{dy}{dx} = x + y$ *with initial conditions* $x_0 = 0, y_0 = 1$, *from* $x = 0.20$ *to* $x = 0.30$ *with* $y_1 = 1.1026$, $y_2 = 1.2104$, $y_3 = 1.3237$.

Solution Here $y_1' = x + y$

$$y'' = 1 + y', \; y''' = y'', \; y''' = y'^{v}, \; y^{v} = y'^{v}, \ldots$$

Hence $$y_0' = x_0 + y_0 = 0 + 1 = 1$$

$$y_0'' = 1 + y_0' = 1 + 1 = 2$$

$$y_0''' = y_0'' = 2$$

$$y_0'^{v} = 2, \; y_0^{v} = 2$$

Now taking $h = 0.05$, we get $x_4 = 0.20$, $x_5 = 0.25$, $x_6 = 0.30$

and $y_1 = 1.1026$, $y_2 = 1.2104$, $y_3 = 1.3237$.

Using Milne's predictor formula we get

$$\bar{y}_4 = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 1 + \frac{4(0.05)}{3}\left[2.2052 - 1.2104 + 2.6474\right] = 1.2428$$

$$\bar{y}_4' = x_4 + \bar{y}_4 = 0.2 + 1.2428 = 1.4428$$

using corrector formula we get

$$y_4 = y_2 + \frac{h}{2}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 1.1104 + \frac{(0.05)}{3}\left[1.2104 + 5.2948 + 1.4428\right] = 1.2428$$

which is the same as the predicted value,

$$\therefore \; y_4 = y_{0.20} = 1.2428$$

and $$y_4' = 1.4428.$$

Again putting $n = 4$, $h = 0.05$ we get

$$\bar{y}_5 = y_1 + \frac{4h}{3}\left[2y_2' - y_3' + 2y_4'\right]$$

$$= 1.0526 + \frac{4(0.05)}{3}\left[2.4208 - 1.3237 + 2.8856\right]$$

$$= 1.3181$$

$$\bar{y}_5' = x_5 + \bar{y}_5 = 0.25 + 1.3181 = 1.5681.$$

Using Milne's corrector formula we get

$$y_5 = y_3 + \frac{h}{3}\left[y_3' + 4y_4' + \bar{y}_5'\right]$$

$$\therefore \; y_5 = 1.1737 + \frac{(0.05)}{3}\left[1.3237 + 5.7712 + 1.5681\right] = 1.3181$$

which is same as the predicted value

$$\therefore \; y_5 = y_{0.25} = 13181 \text{ and } y_5' = 1.5681.$$

Again putting $n = 5$, $h = 0.05$ and using Milne's predictor formula we get

$$\bar{y}_6 = y_2 + \frac{4h}{3}\left[2y_3' - y_4' + 2y_5'\right]$$

$$= 1.1104 + \frac{4 \times (0.05)}{3}\left[2.6474 - 1.4428 + 3.1362\right]$$

$$y_6' = 1.3997$$

$$y_6' = 0.3 + 1.39972 = 1.6997$$

which is corrected by

$$y_6 = y_4 + \frac{h}{3}\left[y_4' + 4y_5' + y_6'\right]$$

$$= 1.2428 + \frac{(0.05)}{3}\left[1.4428 + 6.2724 + 1.6997\right]$$

$$= 1.3997$$

which is same as the predicted value

$$\therefore \; y_6 = y_{0.30} = 1.3997$$

and $$y_6' = 1.6997.$$

x	y	y'
0.20	1.2428	1.4428
0.25	1.3181	1.5681
0.30	1.3997	1.6997

Example 10.10 *Part of a numerical solution of difference equation*

$$\frac{dy}{dx} = 0.2x = 0.1y$$

is shown in the following table.

x	*0.00*	*0.05*	*0.10*	*0.15*
y	*2.0000*	*2.0103*	*2.0211*	*2.0323*

use Milne's method to find the next entry in the table.

Solution We have $x_0 = 0.00$, $x_1 = 0.05$, $x_2 = 0.10$, $x_3 = 0.15$, $x_4 = 0.20$, and $u = 0.05$

The corresponding values of y are

$$y_0 = 2, \; y_1 = 2.0103, \; y_2 = 2.0211, \; y_3 = 2.0323 \text{ and } y_4 = ?$$

$$y_1^1 = 0.2x_1 + 0.1y_1 = 0.2 \times 0.05 + 0.1 \times 2.0103$$

$$= 0.21103$$

$$y_2^1 = 0.2x_2 + 0.1y_2 = 0.2 \times 0.10 + 0.1 \times 2.0211$$
$$= 0.22211$$
$$y_3^1 = 0.2x_3 + 0.1y_3 = 0.2 \times 0.15 + 0.1 \times 2.0323$$
$$= 0.23323$$

Using Milne's Predictors' formula,

$$\bar{y}_4 = y_0 + \frac{4h}{4}\left[2y'_1 - y'_2 - 2y'_3\right]$$
$$= 2.0 + \frac{4 \times 0.05}{3}\left[2 \times 0.21103 - 0.22211 + 2 \times 0.23323\right]$$
$$= 2.0 + \frac{0.2}{3}\left[0.42206 - 0.22211 + 0.46646\right]$$
$$= 2.044427$$
$$y_4^1 = 0.2x_4 + 0.1\bar{y}_4$$
$$= 0.2 \times 0.2 + 0.1 \times 2.044427$$
$$= 0.2444427$$

By the corrector formula, we have

$$y_4 = y_2 + \frac{h}{3}\left[y'_2 + 4y'_3 + y'_4\right]$$
$$= 2.0211 + \frac{0.05}{3}\left[0.22211 + 4 \times 0.23323 + 0.2444427\right]$$
$$= 2.0211 + 0.0233245$$
$$= 2.0444245$$

$\therefore$ the next entry in the table is 2.0444.

10.7 ADAMS–BASHFORTH–MOULTON METHOD

We give below another Predictor–corrector method known as the Adams–Bashforth–Moulton method. This method is a multistep method based on the fundamental theorem of calculus.

$$y(x_{k+1}) = y(x_k) + \int_{x_k}^{x_{k+1}} f(x, y(x))dx$$

where $y^1 = f(x, y)$ is given with boundary condition $y = y_0$ at $x = x_0$. The predictor uses the Lagrange's polynomial approximation for $f(x, y(x))$ based on the points (x_{k-3}, y'_{k-3}), (x_{k-2}, y'_{k-2}), (x_{k-1}, y'_{k-1}) and (x_k, y'_k). It is integrated over the interval $[x_k, x_{k+1}]$. This produces the predictor known as Adams–Bashforth predictor.

$$\bar{y}_{k+1} = y_k + \frac{h}{24}\left[55y'_k - 59y'_{k-1} + 37y'_{k-2}\ 9y'_{k-3}\right]$$

The corrector can be developed by using y_{k+1}. A second Lagrange's polynomial for $f(x, y(x))$ is constructed which is based on the points $\left(x_{k-2}, y'_{k-2}\right), \left(x_{k-1}, y'_{k-1}\right)$ and new point $\left(x_{x+k}, y'_{k+1}\right) = \left(x_{k+1}, f\left(x_{k+1}\right) \bar{y}_{k+1}\right)$.

It is integrated over $\left[x_k, x_{k-1}\right]$ to produce the Adams–Moulton corrector

$$y_{k+1} = y_k + \frac{h}{24}\left(y'_{k+2} - 5y'_{k-1} + 19y'_k + 9y'_{k+1}\right).$$

Example 10.11 *Obtain the solution of the initial value problem* $\frac{dy}{dx} = x^2 + x^2 y, y(1) = 1$ *at* $x = 1(0.1)\ 1.3$, *by any numerical method you know and at* $x = 1.4$ *by Adams–Bashforth method.*

Solution The given differential equation is

$$\frac{dy}{dx} = x^2 + x^2 y = x^2(1 + y)$$

$$y' = x^2\ (1 + y)$$

and we have $x_0 = 1\ y_0 = 1$.

Computing the values of $y(1.1)$, $y(1.2)$, $y(1.3)$ by Taylor's algorithm we get

$$y(1.1) = y_1 = 1.233,\ y_2 = 1.548488,\ y_3 = 1.9789$$

and $$y'_0 = 2y'_1 = 2.702,\ y'_2 = 3.669,\ y'_3 = 5.035.$$

Using Adams–Bashforth predictor formula we get

$$\bar{y}_4 = y_3 + \frac{h}{24}\left[55y'_3 - 59y'_2 + 37y'_1 - 9y'_0\right]$$

$$= 1.9789 + \frac{(0.1)}{24}\left[55 \times 5.035 - 59 \times 3.669 - 37 \times 2.702 - 9 \times 2\right]$$

$$= 2.5762$$

$$\bar{y}'_4 = x_4^2\left[1 + \bar{y}_4\right]\left[x_4^2\right]\left[1 + \bar{y}_4\right] = (1.4)^2 \times (3.5726) = 7.004.$$

By Adam–Moulton corrector formula

$$y_4 = y_3 + \frac{h}{24}\left[9y'_4 + 19y'_3 - 5y'_2 + y'_1\right]$$

$$= 1.9789 + \frac{(0.1)}{24}\left[9 \times 7.004 + 19 \times 5.035 - 5 \times 3.669 + 2.702\right]$$

$$= 1.9789 + \frac{(0.1)}{24}(63.036 + 95.665 - 18.345 + 2.702)$$

$$= 1.9879 + 0.5962\ = 2.5751$$

$$\therefore\ y(1.4) = 2.5751.$$

Exercise 10.2

1. Solve numerically the equation $y' = x + y$ with the initial conditions $x_0 = 0$, $y_0 = 1$ by Milne's method from (1) $x = 0$, $x = 0.4$.

2. Solve the differential equation $y' = x^3 - y^2 - 2$ using Milne's method for $x = 0.3$ (0.1) (0.6). Initial value $x = 0$, $y = 1$. The values of y for $x = -0.1$, 0.1, and 0.2 are two be computed by series expansion.

3. Solve the differential equation $y' = x^2 + y^2 - 2$ using Milne's predictor-corrector method for $x = 0.3$ given the initial value $x = 0$, $y = 1$. The values of y for $x = -0.1$, 0.1 and 0.2 should be computed by Taylor's expansion.

4. Use Milne's method to solve $\frac{dy}{dx} = y + x$, with initial condition $y_{(0)} = 1$, from $x = 0.20$ to $x = 0.30$.

5. Solve numerically at $x = 0.4$, 0.5 by Milne's predictor-corrector method given their values at the four points $x = 0$, 0.1, 0.2, 0.3, $\frac{dy}{dx} = 2e^x - y$, given $y_0 = 2$, $y_1 = 2.010$, $y_2 = 2.040$, $y_3 = 2.09$.

6. Using the Adams-Bashforth-Moulton predictor-corrector formulas, evaluate $y(1.4)$, if y satisfies $\frac{dy}{dx} + \frac{y}{x} = \frac{1}{x^2}$ and $y_{(1)} = 1$, $y_{(1.1)} = 0.996$, $y_{(1.2)} = 0.986$, $y_{(1.3)} = 0.972$.

7. Find $y(2)$ if $y(x)$ is the solution of

$$\frac{dy}{dx} = \frac{1}{2}(x + y)$$

assuming $y(0) = 2$, $y(0.5) = 2.636$.

$y(1.0) = 3.595$ and $y(1.5) = 4.968$.

8. Given $y^1 = 2 - xy^2$ and $y(0) = 10$. Show by Milne's method, that $y(1) = 1.6505$ taking $h = 0.2$.

9. Solve $y^1 = -y$ with $y(0) = 1$ by using Milne's method from $x = 0.5$ to $x = 0.8$ with $h = 0.1$

10. Solve the initial value problem $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ to find $y(0.4)$ by Adam's method. With $y(0.1) = 0.9117$, $y(0.2) = 0.8494$, $y(0.3) = 0.8061$.

11. Using Adams–Bashforth formula, determine $y(0.4)$ given the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$, and the data

x	0	0.1	0.2	0.3
y	1	1.0025	1.0101	1.0228

12. Using Adams-Bashforth formula, determine y (0.4) given the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$ and the data

x	0	0.1	0.2	0.3
y	1	1.0025	1.0101	1.0228

13. Given the differential equation $\frac{dy}{dx} = x^2 y + x^2$ and the data

x	1	1.1	1.2	1.3
y	1	1.233	1.548488	1.978921

determine $y(1.1)$ by Adams-Bashforth formula.

14. Using Adams-Bashforth method, obtain the solution of $\frac{dy}{dx} = x - y^2$ at $x = 0.8$, given

x	9	0.2	0.4	0.6
y	0	0.0200	0.0795	0.1762

Answers

1. $y_4' = 1.583627, = 1.5703$

2.

x	0.3	0.4	0.5	0.6
y	0.061493	0.45625	0.29078	0.12566

3. $y_{(0.3)} = 0.6148$ **4.** $y_{0.2} = 12428$, $y_{0.3} = 1.3997$ **5.** $y_4 = 2.162$, $y_5 = 2.256$

6. $y_{(1.4)} = 0949$ **7.** 6.8733

9. $y(0.5) = 0.6065$, $y(0.6) = 0.5490$, $y(0.7) = 0.4965$, $y(0.8) = 4495$

10. $y(0.4) = 0.7785$ **11.** 1.1749

10.8 RUNGE-KUTTA METHOD

The method is very simple. It is named after two German mathematicians Carl Runge (1856–1927) and Wilhelm Kutta (1867–1944). It was developed to avoid the computation of higher order derivations which the Taylor's method may involve. In the place of these derivatives extra values of the given function $f(x, y)$ are used.

The Runge-Kutta formulas for several types of differential equations are given below.

Fourthorder Runge-Kutta Method:

Let

$$\frac{dy}{dx} = f(x, y)$$

represent any first order differential equation and let h denote the step length. If x_0, y_0 denote the initial values, then the first increment Δy in y is computed from the formulae

$$k_1 = h\ f(x_0, y_0),$$

$$k_2 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right),$$

$$k_3 = h\ f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_3}{2}\right),$$

$$k_4 = h\ f(x_0 + h, y_0 + k_3),$$

and
$$\Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Thus we can write

$$x_1 = x_0 + h, y_1 = y_0 + \Delta y.$$

Similarly the increments for the other intervals are computed. It will be noted that if $f(x, y)$ is independent of y then the method reduces to Simpson's formula. Though approximately the same as Taylor's polynomial of degree four. Runge-Kutta formulae do not require prior calculations of higher derivatives of $y(x)$, as the Taylor's method does. These formulae involve computation of $f(x, y)$ of various position. This method known as Runge-Kutta fourth order method is very popular and extensively used but the errors in method are not easy to watch. The error in the Runge-Kutta method is of the order h^5. Runge-Kutta methods agree with Taylor's series solution up to the term h^m where m differs from the method and is called the order of that method.

First order Runge-Kutta method:

Consider the first order equation

$$\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \cdots \tag{1}$$

We have seen that Euler's method gives

$$y_1 = y_0 + h\ f(x_0, y_0) = y_0 + hy_0' \cdots \tag{2}$$

Expanding by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2} y_0'' + \cdots \tag{3}$$

Comparing (2) and (3), it follows that, Euler's method agrees with Taylor's series solution up to the term in h.

Hence Euler's method is the Runge-Kutta method of the first order.

Second order Runge-Kutta method:

The modified Euler's method gives

$$y_1 = y_0 + \frac{h}{2}\left[f\ (x_0, y_0) + f(x_0 + h, y_1)\right] \tag{4}$$

Taking $f_0 = f(x_0, y_0)$ and substituting $y_1 = y_0 + h\ f_0$ the RHS of (4), we obtain

$$y_1 = y(x_0 + h) = y_0 + \frac{h}{2}\left[f_0 + f(x_0 + h, y_0 + h\ f_0)\right] \tag{5}$$

Expanding LHS by Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \tag{6}$$

Expanding $f(x_0 + h, y_0 + h f_0)$ by Taylor's series for a function of the variables, we obtain

$$f(x_0 + h, y_0 + h f_0) = f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + h f_0\left(\frac{\partial f}{\partial y}\right)_0$$

+ terms containing second and other higher powers of h.

$$f_0 + h = \left(\frac{\partial f}{\partial x}\right)_0 + h f_0\left(\frac{\partial f}{\partial y}\right)_0 + O\left(h^2\right)$$

$\therefore$ (5) can be written as

$$y_1 = y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left[\left(\frac{\partial f}{\partial x}\right)_0 + f_0\left(\frac{\partial f}{\partial y}\right)_0\right] + O\left(h^3\right)\right]$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O\left(h^3\right)$$

$$\left[\because \frac{df}{dx} = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} \text{ where } f = f(x, y)\right]$$

$$= y_0 + hy_0' + \frac{h^2}{2!} y_0'' + O\left(h^3\right). \tag{7}$$

Comparing (6) and (7), it follows that the modified Euler's method agrees with Taylor's series solution up to the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of second order.

The second order Runge-Kutta formula is as follows:

$$k_1 = h f\left(x_n, y_n\right)$$

$$k_2 = h f\left(x_n + h, y_n + k_1\right)$$

$$y_{n+1} = y_n + \Delta y_n$$

where $$\Delta y_n = \frac{1}{2}\left(k_1 + k_2\right)$$

which gives $$k_1 = h f\left(x_0, y_0\right)$$

$$k_2 = h f\left(x_0 + h, y_0 + k_1\right)$$

and $$y_1 = y_0 + \frac{1}{2}\left(k_1 + k_2\right) = y_0 + \Delta y_0 .$$

Third order Runge-Kutta method

This method agrees with Taylor's series solution up to the term in h^3. The formula is as follows:

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where
$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = h\,f(x_0 + h, y_0 + 2k_2 - k_1)$$

The general formula is

$$y_{n+1} = y_n + \Delta y$$

where
$$k_1 = h\,f(x_n + y_n)$$

$$k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$$

$$k_3 = hf\ (x_n + h, y_n + 2k_2 - k_1)$$

and
$$\Delta y = \frac{1}{6}(k_1 + 4k_2 + k_3)$$

Runge-Kutta methods are one-step methods and are widely used. Fourth order R-K method is most commonly used and is known as Runge-Kutta method only. We can increase the accuracy of Runge-Kutta method by taking higher order terms.

Example 10.12 *Use Runge–Kutta method to approximate y when x = 0.1, given that y = 1, when*

$$x = 0 \ \ and \ \ \frac{dy}{dx} = x + y.$$

Solution We have

$$x_0 = 0,\ y_0 = 1$$

$$f(x, y) = x + y, \ \text{and} \ \ h = 0.1.$$

$$\therefore \ \ f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1,$$

we get
$$k_1 = h\,f(x_0, y_0) \ = 0.1 \times 1 = 0.1,$$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) \ = (0.1)(f\ (0 + 0.05), 1 + 0.05)$$

$$= (0.1)\,f(0.05, 1.05) \ = (0.1)(0.05 + 1.05) = 0.11,$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)\big(f\,(0 + 0.05), 1 + 0.055\big) \;= (0.1)(0.05 + 1.055)$$

$$= (0.1)(1.105) \;= 0.1105,$$

$$k_4 = f\,(x_0 + h,\, y_0 + k_3)$$

$$= (0.1)\,f\,(0 + 0.01, 1 + 0.1105) \;= (0.1)\,f\,(0.1, 1.1105)$$

$$= (0.1)(1.2105) \;= 0.12105,$$

$$\therefore\;\; \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1 + 0.22 + 0.2210 + 0.12105) \;= 0.11034.$$

We get $\quad x_1 = x_0 + h = 0 + 0.1 = 0.1$

$$y_1 = y_0 + \Delta y = 1 + 0.11034 \;= 1.11034.$$

Example 10.13 *Using Runge–Kutta method, find an approximate value of y for x = 0.2, if* $\dfrac{dy}{dx} = x + y^2$, *gives that* $y = 1$ *when* $x = 0$.

Solution Taking step-length $h = 0.1$, we have

$$x_0 = 0,\; y_0 = 1,\; \frac{dy}{dx} = f\,(x, y) = x + y^2.$$

Now $\quad k_1 = h\,f\,(x_0, y_0) = (0.1)(0 + 1) - 0.1,$

$$k_2 = h\,f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_1}{2}\right) \;= (0.1)(0.05 + 1.1025)$$

$$= (0.1)(1.1525) \;= 0.11525,$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2},\, y_0 + \frac{k_2}{2}\right) \;= (0.1)(0.05 + 1.1185)$$

$$= (0.1)(1.1685) \;= 0.11685,$$

$$k_4 = h\,f\,(x_0 + h,\, y_0 + k_3)$$

$$= (0.1)(0.01 + 1.2474) \;= (0.1)(1.3474) \;= 0.13474,$$

$$\therefore\;\; \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\left(0.1 + 2(0.11525) + 2(0.11685) + 0.13474\right)$$

$$= \frac{1}{6}(0.6991) = 0.1165.$$

We get $y_1 = y_0 + \Delta y = 1 + 0.1165$

$$\therefore \; y(0.1) = 1.1165.$$

For the second step, we have

$$x_0 = 0.1, \; y_0 = 1.1165,$$

$$k_1 = (0.1)(0.1 + 1.2466) = 0.1347,$$

$$k_2 = (0.1)(0.15 + 1.4014) = (0.1)(1.5514) \quad = 0.1551,$$

$$k_3 = (0.1)(0.15 + 1.4259) = (0.1)(1.5759) \quad = 0.1576,$$

$$k_4 = (0.1)(0.2 + 1.6233) = (0.1)(1.8233) \quad = 0.1823,$$

$$\Delta y = \frac{1}{6}(0.9424) = 0.1571,$$

$$\therefore \; y(0.2) = 1.1165 + 0.1571 = 1.2736$$

$$\therefore \; y(0.1) = 1.1165 \text{ and } y(0.2) = 1.2736.$$

Example 10.14 *Using Runge-Kutta method of order 4, find y for x = 0.1, 0.2, 0.3, given that* $\frac{dy}{dx} = xy + y^2, y(0) = 1.$. *Continue the solution at x = 0.4 using Milne's method.*

Solution. We have $f(x, y) = xy + y^2$

$x_0 = 0, \; y_0 = 1$

$x_1 = 0.1, \; x_2 = 0.2, \; x_3 = 0.3, \; x_4 = 0.4$, and $h = 0.1$

To find $y_1 = y\,(0.1)$:

$$k_1 = hf\,(x_0, y_0) = (0.1)\,(0 \times 1 + 1^2) = 0.1000$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\, f\,(0.05,\; 1.05)$$

$$= 0.1155$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\; f\,(0.05, 1.0577)$$

$$= 0.1172$$

$$k_4 = hf\,(x_0 + h, \; y_0 + k_3) = (0.1)\; f(0.1,\; 1.1172)$$

$$= 0.13598$$

and $$k = \frac{1}{6}\left(k_1 + k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}(0.1000 + 2 \times 0.1155 + 2 \times 0.1172 + 0.13598)$$

$$= 0.11687$$

$\therefore$ $\quad y_1 = y(0.1) = y_0 + k = 1 + 0.11687 = 1.11687 \simeq 1.1169$

$\Rightarrow$ $\quad y_1 = 1.1169$

To find $y_2 = y(0.2)$

Here we have

$$k_1 = hf(x_1, y_1) = (0.1)\ f(1,\ 1.1169) = 0.1359$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)\ f(0.15, 1.1848)$$

$$= 0.1581$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1)\ f(0.15, 1.0959)$$

$$= 0.1609$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\ f(0.2, 1.2778)$$

$$= 0.1888$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1359 + 2 \times 0.1581 + 2 \times 0.1609 + 0.1888)$$

$$= 0.1605$$

$$y_2 = y(0.2) = y_1 + k = 1.1169 + 0.1605 = 1.2774$$

To find $y_3 = y(0.3)$

Here we have

$$k_1 = hf(x_2, y_2) = (0.1)\ f(0.2,\ 1.2774)$$

$$= 0.1887$$

$$k_2 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1}{2}\right) = (0.1)\ f(0.25, 1.3716)$$

$$= 0.2224$$

$$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.1)\ f(0.25, 1.3885)$$

$$= 0.2275$$

$$k_4 = hf(x_2 + h, y_2 + k_3) = (0.1)\ f(0.3, 1.5048)$$

$$= 0.2716$$

$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.1887 + 2 \times 0.2224 + 2 \times 0.2275 + 0.2716)$$

$$= 0.2267$$

$\therefore \quad y_3 = y(0.3) = y^2 + k = 1.2774 + 0.2267$

$= 1.5041$

we have $x_0 = 0.0,\ x_1 = 0.1,\ x_2 = 0.2,\ x_3 = 0.3,\ x_4 = 0.4$

$y_0 = 1,\ y_1 = 1.1169,\ y_2 = 1.2774,\ y_3 = 1.5041$

$y'_0 = 1.000,\ y'_1 = 1.3591,\ y'_2 = 1.8869,\ y'_3 = 2.7132$

Using Milne's predictor:

$$\bar{y}_4 = y_0 + \frac{4h}{3}[2y'_1 - y'_2 - 2y'_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2 \times 1.3591 - 1.8869 + 2 \times 2.7132]$$

$$= 1.8344$$

$\Rightarrow \quad y'_4 = 4.0988$

and the corrector is
$$y_4 = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

$$= 12773 + \frac{0.1}{3}[1.8869 + 4 \times 2.7132 + 4.0988] = 1.8366$$

$\therefore \quad y(0.4) = 1.8366$

Exercise 10.3

1. Solve the equation $\frac{dy}{dx} = x - y^2,\ y(0) = 1$ for $x = 0.2$ and $x = 0.4$ to 3 decimal places by Runge-Kutta fourth order method.

2. Use the Runge-Kutta method to approximate y at $x = 0.1$ and $x = 0.2$ for the equation $\frac{dy}{dx} = x + y,\ y(0) = 1.$

3. For the equation $\frac{dy}{dx} = 3x + \frac{y}{2},\ y(0) = 1.$ Find y at the following points with the given step-length.

4. Use Runge-Kutta method to solve $y' = xy$ for $x = 1.4$, initially $x = 1$, $y = 2$ (by taking step-length $h = 0.2$).

5. $\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$, use Runge–Kutta method to find y at $x = 0.1, 0.2, 0.3$ and 0.4, given that $y = 1$ when $x = 0$.

6. Use Runge-Kutta method to obtain y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and y satisfies the equation $\frac{dy}{dx} = 3x + y^2.$

7. Solve the differential equation $\frac{dy}{dx} = \frac{1}{x + y}$ for $x = 2.0$ by using Runge-Kutta method. Initial values $x = 0$, $y = 1$, interval length $h = 0.5$.

8. Use Runge-Kutta method to calculate the value of y at $x = 0.1$, to five decimal places after a single step of 0.1, if

$$\frac{dy}{dx} = 0.31 + 0.25y + 0.3x^2$$

and $y = 0.72$ when $x = 0$

9. Using Runge-Kutta method of fourth order solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{y^2 + x^2}$, with $y(0) = 1$ at $x = 0.2, 0.4$.

10. Using Runge-Kutta method of order, 4 compute $y(0.2)$ and $y(0.4)$ from $10\dfrac{dy}{dx} = x^2 + y^2,\ y(0) = 1,$ taking $x = 0.1$.

11. Find by Runge-Kutta method an approximate value of y for $x = 0.8$, given that $y = 0.41$ when $x = 0.4$ and

$$\frac{dy}{dx} = \sqrt{x + y}.$$

12. The unique solution of the problem

$$\frac{dy}{dx} = -xy,\ y(0) = 1$$

is $y = e^{-x^2/3}$, find approximate value of $y(0.2)$ using one application of R-K method.

Answers

1. 0851, 0.780

2. 1.1103, 1.2428

3.

x	h	y
0.1	0.1	1.0665242
0.2	0.2	1.1672208
0.4	0.4	1.4782

4. 2.99485866

5. $y(0.1) = 1.0874$, $y(0.2) = 1.1557$, $y(0.3) = 1.2104$, $y(0.4) = 1.2544$

6. $y(1.1) = 1.7271$

7.

x	0.5	1.0	1.5	2.0
y	1.3571	1.5837	1.7555	1.8957

8. 0.76972

9. 1.196, 1.3752

10. 1.0207, 1.038

11. 1.1678

12. 0.9802

10.9 PICARD'S METHOD OF SUCCESSIVE APPROXIMATION

Consider the initial value problem $\frac{dy}{dx} = f(x, y)$, with the initial condition $y(x_0) = y$. Integrating the differential equation between x_0 and x, we can write

$$\int_{x_0}^{x} dy = \int_{x_0}^{x} f(x, y)\,dx$$

$$\Rightarrow\ y - y_0 = \int_{x_0}^{x} f(x, y)\,dx$$

$$\Rightarrow\ y = y_0 + \int_{x_0}^{x} f(x, y)\,dx, \tag{26}$$

satisfying the initial condition $y(x_0) = y_0$. Equation (26) is known as an *integral equation*, because the dependent variable y in the function $f(x, y)$ on the right hand side occurs under the sign of integration.

The first approximation y_1 of y is obtained by replacing y by y_0 in $f(x, y_0)$ in equation (26). This gives

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx. \tag{27}$$

The value of y_1 obtained from equation (27) is substituted for y in the integral equation (26) to get second approximation y_2, such that

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx. \tag{28}$$

The successive approximation of y may be written as

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$\vdots$$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx.$$

The process of iteration is stopped when the values of y_{n-1} and y_n are approximately the same.

Example 10.14 Solve $\frac{dy}{dx} = 1 + y^2$, $y(0) = 0$ by Picard's method.

Solution Here we have

$$x_0 = 0,\ y_0 = 0,\ f(x, y) = 1 + y^2.$$

By Picard's iterative formula,

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx,$$

$$\Rightarrow \quad y_n = \int_0^x \left[1 + y_{n-1}^2\right] dx,$$

$$\Rightarrow \quad y_n = x + \int_0^x y_{n-1}^2\, dx.$$

Now taking $n = 1, 2, 3 \ldots$, we get the following successive approximations to y

$$y_1 = x + \int_0^x y_0^2\, dx = x + \int_0^x (0)\, dx = x,$$

$$y_2 = x + \int_0^x y_1^2\, dx = x + \int_0^x x^2\, dx = x + \frac{x^3}{3},$$

$$y_3 = x + \int_0^x y_2^2\, dx = x + \int_0^x \left(x + \frac{x^3}{3}\right)^2 dx,$$

$$y_3 = x + \int_0^x \left(x^2 + \frac{2x^4}{3} + \frac{x^6}{9}\right) dx$$

$$\Rightarrow \quad y_3 = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{x^7}{63}.$$

The solution can successively be improved further.

Example 10.15 *Use Picard's method to approximate y when x = 0.1, x = 0.2, for* $\frac{dy}{dx} = x + y^2$, *where y = 0, when x = 0.*

Solution The first approximation be y_1

$$y_1 = 0 + \int_0^x (x + 0)\, dx = \frac{x^2}{2},$$

the second approximation be y_2. Then

$$y_2 = 0 + \int_0^x \left(x + \frac{x^4}{4}\right) dx = \frac{x^2}{2} + \frac{1}{20}x^5,$$

the third approximation be y_3. Then

$$y_3 = 0 + \int_0^x \left[x + \left(\frac{x^2}{2} + \frac{1}{20}x^5\right)^2\right] dx$$

$$= \int_0^x \left(x + \frac{x^4}{4} + \frac{1}{400} x^{10} + \frac{1}{20} x^7 \right) dx$$

$$= \frac{x^2}{2} + \frac{1}{20} x^5 + \frac{1}{4400} x^{11} + \frac{1}{1600} x^8$$

$$= \frac{1}{2} x^2 + \frac{1}{20} x^5 + \frac{1}{160} x^8 + \frac{1}{4400} x^{11}.$$

For $x = 0.1$

$$y_2 = \frac{0.01}{2} + \frac{0.00001}{20} = 0.005 + 0.0000005 = 0.00500,$$

$$y_3 = \frac{0.01}{2} + \frac{0.00001}{20} + \frac{0.00000001}{160} + \cdots = 0.00500.$$

There is no difference between y_2 and y_3 (up to 5 decimal places)

$$\therefore \quad y = 0.00500 \text{ for } x = 0.1.$$

For $x = 0.2$, we may take $x = 0.1$, $y_1 = 0.005$ as the initial values, we may write the first approximation y_1 as

$$y_1 = 0.005 + \int_{0.1}^x (x + 0.000025)\, dx$$

$$= 0.005 \left[\frac{x^2}{2} + 0.000025 \right]_{0.1}^x$$

$$= 0.005 - 0.005 - 0.0000025 + \frac{1}{2} x^2 + \frac{25}{10^6} x.$$

For $x = 0.2$

$$y_1 = \frac{0.04}{2} + \frac{25}{10^6} \times 0.2 = 0.02 + 0.000005 \approx 0.0200,$$

$$y_2 = 0.005 + \int_{0.1}^x (x + 0.0004)\, dx$$

$$= 0.005 + \left[\frac{x^2}{2} + 0.0004x \right]_{0.1}^x = \frac{x^2}{2} + \frac{4}{10^4} x - \frac{4}{10^5}.$$

$\therefore$ For $x = 0.2$

$$y_2 = 0.02 + 0.00008 - 0.00004 = 0.02004.$$

y_1 and y_2 are approximately the same up to the 4 decimal places

x	y	$\frac{dy}{dx}$
0.0	0	0
0.1	0.0050	0.100025
0.2	0.0200	0.200400

Example 10.16 *Given the differential equation* $\frac{dy}{dx} = x - y$, *with the condition y = 1 when x = 0, use Picard's method to obtain y for x = 0.2 correct to five decimal places.*

Solution Here $f(x, y) = x - y$, $x_0 = 0$, $y_0 = 1$. The successive approximations are given by

$$y_1 = 1 + \int_0^x (x - 1)\, dx = 1 - x + \frac{x^2}{2},$$

$$y_2 = 1 + \int_0^x \left(x - 1 + x - \frac{x^2}{2} \right) dx = 1 - x + x^2 - \frac{x^3}{6},$$

$$y_3 = 1 + \int_0^x \left(x - 1 + x - x^2 + \frac{x^3}{6} \right) dx = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{24},$$

$$y_4 = 1 + x^2 - x - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{120},$$

$$y^5 = 1 - x + x^2 - \frac{x^3}{3} + \frac{x^4}{12} - \frac{x^5}{60} + \frac{x^6}{720}.$$

When $x = 0.2$, the successive approximation of y are given by $y_0 = 1$, $y_1 = 0.82$, $y_2 = 0.83867$, $y_3 = 0.43740$, $y_4 = 0.83746$, $y_5 = 0.83746$

$\therefore$ The value of y is given by

$$y = 0.83746.$$

Example 10.17 *Solve* $\frac{dy}{dx} = y$, *y(0) = 1 by Picard's method and compare the solution with the exact solution.*

Solution *We have* $f(x, y) = y$, $x_0 = 0$, $y_0 = 1$, *Picard's formula takes the form*

$$y_n = 1 + \int_0^x y_{n-1}\, dx,\ n = 1, 2, 3, \ldots$$

Therefore taking $n = 1, 2, 3, \ldots$, we get

$$y_1 = 1 + \int_0^x y_0 dx = 1 + \int_0^x (1)\, dx = 1 + x,$$

$$y_2 = 1 + \int_0^x y_1 dx = 1 + \int_0^x (1 + x)\, dx,$$

$$\Rightarrow\ y_2 = 1 + \left[x + \frac{x^2}{2} \right]_0^x = 1 + x + \left(\frac{x^2}{2} \right),$$

$$y_3 = 1 + \int_0^x \left[1 + x + \left(\frac{x^2}{2} \right) \right] dx = 1 + x + \left(\frac{x^2}{2} \right) + \frac{x^2}{3.2},$$

$$y_4 = 1 + \int_0^x \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3!} \right) dx,$$

$$\Rightarrow \quad y_4 = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{29}$$

Analytical Solution The given differential equation is $\frac{dy}{dx} = y$, by separating the variables, we have $\frac{dy}{y} = dx$, on integrating, we get log $y = x + k$,

or $$y = e^{x+k} = ce^x \tag{30}$$

where $c = e^k$ is an arbitrary constant. Substituting the initial values $x = 0$, $y = 1$ in (2) we get $c = 1$.

The exact solution is $y = e^x$ which has the expansion

$$y = e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \tag{31}$$

The Picard solution is given by (29) is the same as the first five terms of (31).

Exercise 10.4

1. Solve $\frac{dy}{dx} = 1 + xy$, given that the integral curve passes through the point (0,1) tabulate the values of y is 0(0.1) 0.5.
2. Solve $\frac{dy}{dx} = x + y^2$, $x = 0$, $y = 1$.
3. Use Picard's method, to obtain the second approximation to the solution of $\frac{d^2y}{dx^2} - x^3\frac{dy}{dx} + x^3y = 0$ given that $y = 1$, $\frac{dy}{dx} = \frac{1}{2}$ at $x = 0$.
4. Solve $\frac{dy}{dx} = x + y$, with the initial conditions $x_0 = 0$, $y_0 = 1$.
5. Find the value of y for $x = 0.1$ by Picard's Method given that
$$\frac{dy}{dx} = \frac{y - x}{y + x},\ y(0) = 1.$$
6. Obtain Picard's Second approximate solution of the initial value problem
$$y' = \frac{x^2}{y^2 + 1};\ y(0) = 0$$

Answers

1. $y_3 = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \dfrac{x^4}{8} + \dfrac{x^5}{15} + \dfrac{x^6}{48} + \cdots$

2. $y_2 = 1 + x + \dfrac{3x^2}{2} + \dfrac{2x^3}{3} + \dfrac{x^4}{4} + \dfrac{x^5}{20} + .$

3. $y_2 = 1 + \dfrac{x}{2} + \dfrac{3x^5}{40} + \cdots$

4. For $x = 0.1$, $y = 1.1103$, for $x = 0.2$, $y = 1.2427$.

5. 0.9828 **6.** $y = \dfrac{1}{3}x^3 - \dfrac{1}{8\pi}x^9 + \ldots$

11

SOLUTION OF LINEAR EQUATIONS

11.1 MATRIX INVERSION METHOD

System of linear equations arise frequently and if n equations in n unknowns are given, we write

$$\left.\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots a_{1n} x_n &= b_1 \\ a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\ a_{31} x_1 + a_{32} x_2 + \dots + a_{3n} x_n &= b_3 \\ &\vdots \\ a_{n1} x_1 + a_{n2} x_2 + \dots + a_{nn} x_n &= b_n \end{aligned}\right\} \tag{1}$$

The set of numbers $x_1, x_2, x_3, \dots x_n$ which reduces (1) to an identity is called the *solution set of the system*. If we denote the matrix of coefficients by

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

the column of its constant terms by

$$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix},$$

and the column of the unknowns by

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then the system (1) can be compactly written in the form of a matrix equation

$$AX = B \tag{2}$$

If the matrix A is non-singular, that is if

$$\det A = \Delta = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \neq 0,$$

then (1) has a unique solution.

If $\det A \neq 0$, there is an inverse matrix A^{-1}. Pre multiplying both sides of (2) by the matrix A^{-1}, we obtain

$$A^{-1}AX = A^{-1}B$$

or

$$X = A^{-1}B. \tag{3}$$

Formula (3) yields a solution of (2) and the solution is unique.

Example 11.1 *Solve the system of equations by matrix inversion method.*

$$x_1 + x_2 + x_3 = 1$$
$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + 3x_2 + 4x_3 = 6,$$

Solution The given equation can be put in the form

$$AX = B$$

where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}, \ X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix}.$$

$$\det A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} = -1$$

$$\Rightarrow \ \det A = -1 \neq 0$$

$$\therefore \ A^{-1} \text{ exists.}$$

$$\therefore \ A^{-1} = \frac{1}{\det A} \operatorname{Adj} A$$

$$= \frac{1}{-1}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}.$$

From (3)

$$X = A^{-1}B$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x_1 = 1,\ x_2 = -5,\ x_3 = 5.$$

Exercise 11.1

Solve the following system of equations by matrix inversion method

1. $3x_1 - x_2 = 5$

$-2x_1 + x_2 + x_3 = 0$

$2x_1 - x_2 = 4x_3 = 15$

2. $2x - 3y - 5z = 11$

$5x + 2y - 7z = -12$

$-4x + 3y + z = 5$

3. $x + y + z = 7$

$x + 2y + 3z = 16$

$x + 3y + 4z = 22$

4. $7x_1 + 7x_2 - 7x_3 = 2$

$-x_1 + 11x_2 + 7x_3 = 1$

$11x_1 + 5x_2 + 7x_3 = 0$

5. $x_1 + x_2 + x_3 = 6$

$x_1 + 2x_2 + 3x_3 = 14$

$-x_1 + x_2 - x_3 = -2$

Answers

1. $x_1 = 2,\ x_2 = 1,\ x_3 = 3$ **2.** $x = 1,\ y = 2,\ z = 3$ **3.** $x = 1,\ y = 3,\ z = 3$

4. $x_1 = 0,\ x_2 = 16,\ x_3 = -105/882$ **5.** $x_1 = 1,\ x_2 = 2,\ x_3 = 3$

11.2 GAUSS–ELIMINATION METHOD

The system equations given by (1) may also be written the tabular form as

x_1	x_2	x_3	$\cdots$	x_n	b_i
a_{11}	a_{12}	a_{13}	$\cdots$	a_{1n}	b_1
a_{21}	a_{22}	a_{23}	$\cdots$	a_{2n}	b_2
a_{31}	a_{32}	a_{33}	$\cdots$	a_{3n}	b_3
$\vdots$					
a_{n1}	a_{n2}	a_{n3}	$\cdots$	a_{nn}	b_n

The above system is first reduced to triangular form by eliminating one of the unknowns at a time. The unknown x_1 is first eliminated from (n–1) equations by dividing the first equation by a_{11} and by subtracting this equation multiplied by $a_{ii}(i = 2, 3, .., n)$ from the remaining (n–1) equations. The unknown x_2 is then eliminated from the (n–2) equations of (n–1) equations not containing x_1. The Process of elimination is continued until appears only in the last equation as shown in the table below.

x_1	x_2	$x_3 \cdots$	x_n	b_i
1	c_{12}	$c_{13} \cdots$	c_{1n}	d_1
0	1	$c_{22} \cdots$	cz_n	d_2
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	1	d_n

The unknowns $x_1, x_2, \ldots, x_n$ are then evaluated by backward substitutions.

The value of x_n is obtained from the nth equation the value of x_2 is then substituted in (n–1)th equation which gives x_{n-1}.

The values of x_n and x_{n-1} are then substituted in (n–2)nd equation to get the value of x_{n-2}, etc.

The computations are checked at each row by means of an additional column. The additional column of such checks is denoted by s. The Process described above is called Gauss–Elimination.

If $a_{ii} \neq 0$, the ith row cannot be used to eliminate the element in ith column and row r must be changed with some row below the diagonal to obtained a zero element which is used eliminate x_2.

The example given below is an illustrations of the method.

Example 11.2 *Solve by Gauss–Elimination method*

$$2x + 2y + 4z = 18$$
$$x + 3y + 2z = 13$$
$$3x + y + 3z = 14$$

Row No.	*x*	*y*	*z*	*d*	*s*	*Explanation*
I	2	2	4	18	26	
II	1	3	2	13	19	
III	3	1	1	14	21	
IV	1	1	2	9	13	I/a_{11} = I/2
V	0	2	0	4	6	II – 1 × (III)
VI	0	–2	–3	–13	–18	III –3 × (IV)
VII	–	1	0	2	3	V/a_{22} = V/2
VIII	–	0	–3	–9	–12	VI + 2 × VII
IX	–	–	1	3	4	VIII/a_{33} = VIII/(–3)

From row IX, we get

$$z = 3,$$

From VII, we have

$$y + 0 = 2$$

$\Rightarrow$ $$y = 2$$

From IV,

$$x + y + 2z = 9$$

$\Rightarrow$ $$x = 9 - y - 2z = 9 - 2 - 2 \times 3$$

$\therefore$ $$x = 1,\ y = 2,\ z = 3.$$

Exercise 11.2

Solve the following by Gauss–Elimination method

1. $x_1 + 2x_2 + 3x_3 = 7$

$2x_1 + 7x_2 + 15x_3 = 26$

$3x_1 + 15x_2 + 41x_3 = 26$

2. $2x_1 + 6x_2 - x_3 = 23$

$4x_1 - x_2 + 3x_3 = 9$

$3x_1 + x_2 + 2x_3 = 13$

3. $2x + 2y + 4z = 14$

$3x - y + 2z = 13$

$5x + 2y - 2z = 2$

4. $4x - y + 2z = 15$

$-x + 2y + 3z = 5$

$5x - 7y + 9z = 8$

5. $2x + y + 4z = 12$

$8x - 3y + 2z = 20$

$4x + 11y - z = 33$

Answers

1. $x_1 = 2,\ x_2 = 1,\ x_3 = 1$ **2.** $x_1 = 1,\ x_2 = 4,\ x_3 = 3$ **3.** $x = 2,\ y = -1,\ z = 3$

4. $x = 4,\ y = 3,\ z = 1$ **5.** $x = 3,\ y = 2,\ z = 1$

11.3 ITERATION METHODS

The Matrix inversion method and Gauss elimination methods are called *Direct methods* (or exact methods) they are based on the elimination of variables in order to reduce the given system of equations to a triangular form. When a linear system has a large number of unknowns, the Gaussian scheme becomes very unwieldily. Under such conditions it is more convenient to use indirect or iterative methods. The iterative methods are not applicable to all systems of equations. In order that the iteration may succeed, each equation of the system must contain one large coefficient and the large coefficient must be attached to a different unknown in that equation. Successful use of the iteration process requires that the moduli of the diagonal coefficients of the given systems be large in comparison with the moduli of the non-diagonal coefficients. We shall discuss two particular methods of iteration.

11.3.1 Jacobi's Method

Consider the system of equations

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 &= b_2 \\ a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 &= b_3, \end{aligned} \tag{4}$$

assume that the diagonal coefficients, a_{11}, a_{22} and a_{33} are large, compared to other coefficients solving for x_1, x_2 and x_3 respectively. We get

$$\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right], \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right], \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_3x_1 - a_{32}x_2\right], \end{aligned} \tag{5}$$

let $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$, denote the initial estimates for the values of the unknowns x_1, x_2, x_3. Substituting these values in the right sides of (5) we get the first iterative values x_1, x_2, x_3 as follows:

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right],$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right],$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right].$$

Substituting the values $x_1^{(1)}, x_2^{(1)}, x_3^{(1)}$ the right sides of (5), we get

$$x_1^{(2)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(1)} - a_{13}x_3^{(1)}\right],$$

$$x_2^{(2)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(1)}\right],$$

$$x_3^{(2)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right].$$

If $x_1^{(n)}, x_2^{(n)}, x_3^{(n)}$ denote the *n*th iterates then

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)}\right],$$

$$x_2^{(n+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(n)} - a_{23}x_3^{(n)}\right],$$

$$x_3^{(n+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(n)} - a_{32}x_2^{(n)}\right].$$

The process is continued till convergence is secured.

Note: In the absence of any better estimates, the initial estimates for the values of x_1, x_2, x_3, are taken as $x_1^{(0)} = 0$, $x_2^{(0)} = 0$, $x_3^{(0)} = 0$.

Example 11.3 *Solve by Gauss–Jacobi's method*

$$5x + 2y + z = 12,$$
$$x + 4y + 2z = 15,$$
$$x + 2y + 5z = 20.$$

Solution The above system is diagonally dominant, *i.e.*, in each equation the absolute value of the largest coefficient is greater than the sum of the remaining coefficients. The given equations can be written as

$$x = \frac{1}{5}\left[12 - 2y - 3\right],$$

$$y = \frac{1}{4}\left[15 - x - 2z\right],$$

$$z = \frac{1}{5}\left[20 - x - 2y\right].$$

We start the iteration by putting $x = 0$, $y = 0$, $z = 0$

$\therefore$ For the first iteration we get

$$x^{(1)} = \frac{1}{5}\left[12 - 0 - 0\right] = 2.40,$$

$$y^{(1)} = \frac{1}{4}\left[15 - 0 - 0\right] = 3.75,$$

$$z^{(1)} = \frac{1}{5}\left[20 - 0 - 0\right] = 4.00,$$

putting the values $y^{(1)}$, $z^{(1)}$ in the right side of (1), we get

$$x^{(2)} = \frac{1}{5}[12 - 2(3.75) - 4.00] = 0.10,$$

similarly putting the values of $z^{(1)}$ and $x^{(1)}$ in (2) we get

$$y^{(2)} = \frac{1}{4}[15 - 2.40 - 2.40] = 1.15,$$

and putting the values of $x^{(1)}$ and $y^{(1)}$ in the right side of (3) we get

$$z^{(2)} = \frac{1}{5}[20 - 2.40 - 3.75] = 2.02.$$

The iteration process is continued and the results are tabulated as follows

Iterations	1	2	3	4	5	6	7	8
x	2.40	0.10	1.54	0.61	1.41	0.80	1.08	1.084
y	3.75	1.15	1.72	1.17	2.29	1.69	1.95	1.95
z	4.00	2.02	3.57	2.60	3.41	3.20	3.16	3.164

The values of x, y, z, at the end of the 8th iteration are $x = 1.084$, $y = 1.95$, and $z = 3.164$.

Exercise 11.3

Solve by Gauss–Jacobi's method of iteration:

1. $27x_1 + 6x_2 - x_3 = 85$
$6x_1 + 5x_2 + 2x_3 = 72$
$x_1 + x_2 + 54x_3 = 110$

2. $x + 10y + 3 = 6$
$10x + y + z = 6$
$x + y + 10z = 6$

3. $13x_1 + 5x_2 - 3x_3 + x_4 = 18$
$2x_1 + 12x_2 + x_3 - 4x_4 = 13$
$3x_1 - 4x_2 + 10x_3 + x_4 = 29$
$2x_1 + x_2 - 3x_3 + 9x_4 = 31$

4. $4x_1 + 0.24x_2 - 0.08x_3 = 8$
$0.09x_1 + 3x_2 - 0.15x_3 = 9$
$0.04x_1 - 0.08x_2 + 4x_3 = 20$

Answers

1. $x_1 = 2.4255$, $x_2 = 3.5730$, $x_3 = 1.9260$

2. $x = 0.5$, $y = 0.5$, $z = 0.5$

3. $x_1 = 1$, $x_2 = 2$, $x_3 = 3$, $x_4 = 4$

4. $x_1 = 1.90923$, $x_2 = 3.19495$, $x_3 = 5.04485$

11.3.2 Gauss–Seidel Method

Consider the system of equations

$$\begin{aligned} a_{11}\, x_1 + a_{12}\, x_2 + a_{13}\, x_3 &= b_1 \\ a_{21}\, x_1 + a_{22}\, x_2 + a_{23}\, x_3 &= b_2 \\ a_{31}\, x_1 + a_{32}\, x_2 + a_{33}\, x_3 &= b_3 \end{aligned} \qquad \text{refer (4)}$$

Suppose in the above system, the coefficients of the diagonal terms are large in each equation compared to other coefficients, solving for x_1, x_2, x_3 respectively

$$\left.\begin{aligned} x_1 &= \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right], \\ x_2 &= \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right], \\ x_3 &= \frac{1}{a_{33}}\left[b_3 - a_3 x_1 - a_{32}x_2\right]. \end{aligned}\right\} \qquad (6)$$

Let $x_1^{(0)}, x_2^{(0)}, x_3^{(0)}$, denote the initial approximations of x_1, x_2, x_3 respectively. Substituting $x_2^{(0)}$, and $x_3^{(0)}$, in the first equation of (6) we get

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right].$$

Then we substitute $x_1^{(1)}$, and x_1 and $x_3^{(0)}$ for x_3 in second equation of (6) which gives

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right].$$

We substitute $x_1^{(1)}$ for x, and $x_2^{(1)}$ for x_2 in the third equation of (6) which gives

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right].$$

In the above process, we observe that the new value when found is immediately used in the following equations:

$$x_1^{(n+1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(n)} - a_{13}x_3^{(n)}\right],$$

$$x_2^{(n+1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(n+1)} - a_{23}x_3^{(n)}\right],$$

$$x_3^{(n+1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(n+1)} - a_{32}x_2^{(n+1)}\right].$$

The above process is continued till convergency is secured.

Note: Gauss–Seidel method is a modification of Gauss–Jacobi method. The convergence is Gauss–Seidel method is more rapid than in Gauss–Jacobi Method.

Example 11.4 *Solve by Gauss–Seidel method of iteration the equations*

$$10x_1 + x_2 + x_3 = 12$$
$$2x_1 + 10x_2 + x_3 = 13$$
$$2x_1 + 2x_2 + 10x_3 = 14$$

Solution From the given equations we have

$$x_1 = \frac{1}{10}\left[12 - x_1 - x_2\right], \tag{7}$$

$$x_2 = \frac{1}{10}\left[13 - 2x_1 - x_3\right], \tag{8}$$

$$x_3 = \frac{1}{10}\left[14 - 2x_1 - 2x_2\right]. \tag{9}$$

Putting $x = 0$ in right side of (7) we get

$$x_1^{(1)} = \frac{12}{10}\ 1.2,$$

putting $x_1 = x_1^{(1)} = 1.2,\ x_3 = 0$ in (8), we get

$$x_2^{(1)} = \frac{1}{10}\left[13 - 2.4 - 0\right] = \frac{10.6}{10} = 1.06,$$

putting $x_1 = x_1^{(1)} = 1.2$ and $x_2 = x_2^{(1)} = 1.6$ in (9), we get

$$x_3^{(1)} = \frac{1}{10}\left[14 - 2.4 - 2.12\right] = 0.948,$$

For second iteration we have

$$x_1^{(2)} = \frac{1}{10}\left[12 - 1.06 - 0.948\right] = 0.9992,$$

$$x_2^{(2)} = \frac{1}{10}\left[13 - 2(0.9992) - 0.948\right] = 1.00536,$$

$$x_3^{(2)} = \frac{1}{10}\left[14 - 2(0.9992) - 2(1.00536)\right] = 0.999098.$$

Thus the iteration process is continued. The results are tabulated as follows correcting to four decimal places

i	$x_1^{(i)}$	$x_2^{(i)}$	$x_3^{(i)}$
0	1.2000	0.000	0.0000
1	1.2000	1.0600	0.9480
2	0.9992	1.0054	0.9991
3	0.9996	1.001	1.001
4	1.0000	1.0000	1.00
5	1.000	1.000	1.000

$\therefore$ The exact values of the roots are

$$x_1 = 1, x_2 = 1, x_3 = 1.$$

Exercise 11.4

Solve by Gauss–Seidel method, the equations:

1. $27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$

2. $10x_1 - x_2 - x_3 = 13$
$x_1 + 10x_2 + x_3 = 36$
$-x_1 - x_2 + 10x_3 = 35$

3. $x_1 + 10x_2 + x_3 = 6$
$10x_1 + x_2 + x_3 = 6$
$x_1 + x_2 + 10x_3 = 6$

4. $13x_1 + 5x_2 - 3x_3 + x_4 = 18$
$2x_1 + 12x_2 + x_3 - 4x_4 = 13$
$3x_1 - 4x_2 + 10x_3 - x_4 = 29$
$2x_1 + x_2 - 3x_3 + 9x_4 = 31$

5. $5x + 2y + z = 12$
$x + 4y + 2z = 15$
$x + 2y + 5z = 20$

Answers

1. $x = 2.4255, y = 3.5730, z = 1.9260$ **2.** $x_1 = 2, x_2 = 3, x_3 = 4$ **3.** $x_1 = 0.5, x_2 = 0.5, x_3 = 0.5$

4. $x_1 = 1, x_2 = 2, x_3 = 3, x_4 = 4$ **5.** $x = 0.996, y = 2, z = 3.$

11.4 CROUT'S TRIANGULARISATION METHOD (METHOD OF FACTORISATION)

Basic Definitions

Consider the square matrix

$$A = [a_{ij}]_{n \times n}$$

(*i*) If $a_{ij} = 0$ for $i > j$. Then A is called an upper triangular matrix.

(*ii*) If $a_{ij} = 0$ for $i < j$. Then the matrix A is called a lower triangular matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} \text{ is an upper triangular matrix}$$

and $$A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ is a lower triangular matrix}$$

from the above; it is clear that; A is an upper triangular matrix if all the elements below the main diagonal are zero, and A is lower triangular if A has all elements above the principal diagonal as zero.

Triangularization Method

Consider the system

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = b_1$$
$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$$
$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = b_3$$

The above system can be written as

$$AX = B \quad ... \tag{1}$$

where $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Let $$A = LU ... \tag{2}$$

where $$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Hence the equation (1) becomes

$$LUX = B ... \tag{3}$$

If we write $$UX = V ... \tag{4}$$

Equation (3) becomes

$$LV = B ... \tag{5}$$

Which is equivalent to the system

$$v_1 = b_1$$
$$l_{21} v_1 + v_2 = b_2$$
$$l_{31} v_1 + l_{32} v_2 + v_3 = b_3$$

the above system can be solved to know the values of v_1, v_2 and v_3 which give us the matrix V. When V is known the system.

$$UX = V, \text{ becomes}$$

$$u_{11} x_1 + u_{12} x_2 + u_{13} x_3 = v_1$$
$$u_{22} x_2 + u_{23} x_3 = v_2$$
$$u_{33} x_3 = v_3$$

which can be solved for x_3, x_2 and x_1 by the backward substitution

To compute the Matrices L and U, we write (2) as

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

multiplying the matrices on the left and equating the corresponding elements of both sides, we obtain

$$u_{11} = a_{11}, \; u_{12} = a_{12}, \; u_{13} = a_{13} \qquad (i)$$

$$\left.\begin{aligned} l_{21} u_{11} &= a_{21} \;\Rightarrow\; l_{21} = \frac{a_{21}}{a_{11}} \\ l_{31} u_{11} &= a_{31} \;\Rightarrow\; l_{31} = \frac{a_{31}}{a_{11}} \end{aligned}\right\} \qquad (ii)$$

$$\left.\begin{aligned} l_{21} u_{12} + u_{22} &= a_{22} \;\Rightarrow\; u_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12} \\ l_{21} u_{13} + u_{23} &= a_{23} \;\Rightarrow\; u_{23} = a_{23} - \frac{a_{21}}{a_{11}} a_{13} \end{aligned}\right\} \qquad (iii)$$

$$l_{31} u_{12} + l_{32} u_{22} = a_{32} \;\Rightarrow\; l_{32} = \frac{1}{u_{22}} \left[a_{32} - \frac{a_{31}}{a_{11}} a_{12} \right] \ldots \qquad (iv)$$

and $$l_{31} u_{13} + l_{32} u_{23} + u_{33} = a_{33} \qquad (v)$$

The value of u_{33} can be computed from (v)

To evaluate the elements of L and U, we first find the first row of U and the first column of L: then we determine the second row of U and the second column of L: and finally, we compute the third row of U. The procedure can be generalised.

Example 11.5 *Apply Triangularization (factorization) method to solve the equation*

$$2x + 3y + z = 9$$
$$x + 2y + 3z = 6$$
$$3x + y + 2z = 8$$

Solution We have
$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; B = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Let
$$\begin{bmatrix} l & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$$

multiplying and equating we get

$$u_{11} = 2, \; u_{12} = 3, \; u_{13} = 1 \ldots \qquad (i)$$

$$\left.\begin{aligned} l_{21}u_{11} &= 1 \Rightarrow 2l_{21} = 1 \Rightarrow l_{21} = \frac{1}{2} \\ l_{31}u_{11} &= 3 \Rightarrow 2l_{31} = 3 \Rightarrow l_{31} = \frac{3}{2} \end{aligned}\right\} \ldots \quad (ii)$$

$$\left.\begin{aligned} l_{21}u_{12} + u_{22} &= 2 \Rightarrow \frac{1}{2}\cdot 3 + u_{22} = 2 \Rightarrow u_{22} = \frac{1}{2} \\ l_{21}u_{13} + u_{23} &= 3 \Rightarrow \frac{1}{2} + u_{23} = 3 \Rightarrow u_{23} = \frac{5}{2} \end{aligned}\right\} \ldots \quad (iii)$$

$$l_{31}u_{12} + l_{32}\,u_{22} = 1 \Rightarrow \frac{3}{2}\cdot 3 + l_{32}\left(\frac{1}{2}\right) = 1 \Rightarrow l_{32} = -7 \quad (iv)$$

Finally, $l_{31}\,u_{13} + l_{32}\,u_{23} + u_{33} = 2$

$$\Rightarrow \frac{3}{2}\cdot 1 + (-7)\cdot\frac{5}{2} + u_{33} = 2 \Rightarrow u_{33} = 18 \quad (v)$$

Thus, we get

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}$$

and the given system can be written as

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

Writing; $LV = B$ we get

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$$

which gives

$$v_1 = 9$$

$$\frac{v_1}{2} + v_2 + 6 \text{ or } v_2 = \frac{3}{2}$$

and $$\frac{3}{2}v_1 - 7v_2 + v_3 = 8 \Rightarrow v_3 = 5$$

∴ The solution to the original system; is given by; $UX = V$; *i.e.,*

$$\begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ \frac{3}{2} \\ 8 \end{bmatrix}$$

i.e., $$2x + 3y + z = 9$$

$$\frac{y}{2} + \frac{5z}{2} = \frac{3}{2}$$

$$18z = 5$$

by back substituting, we have

$$x = \frac{35}{18}, y = \frac{29}{18}, z = \frac{5}{18}.$$

Crout's Method

Consider the system

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + a_{13} x_3 &= b_1 \\ a_{21} x_1 + a_{22} x_2 + a_{23} x_3 &= b_2 \\ a_{31} x_1 + a_{32} x_2 + a_{33} x_3 &= b_3 \end{aligned} \quad \text{...(1)}$$

The above system can be written as

$$AX = B \quad \text{...(2)}$$

Let $$A = LU \quad \text{...(3)}$$

where $$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

L is a lower triangular matrix and u is an upper triangular matrix with diagonal elements unity.

$$A = LU \Rightarrow A^{-1} = U^{-1}L^{-1} \quad \text{...(4)}$$

Now $$A = LU \Rightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

Equating the corresponding elements, we get

$$l_{11} = a_{11} \qquad l_{21} = a_{21} \qquad l_{31} = a_{31} \qquad \ldots(i)$$

$$l_{11}u_{12} = a_{12} \qquad l_{11}u_{13} = a_{13} \qquad \ldots(ii)$$

$$l_{21}\ u_{12} + l_{22} = a_{22} \qquad l_{31}\ u_{12} + l_{32} = a_{32} \qquad \ldots(iii)$$

$$l_{21}\ u_{13} + l_{22}\ u_{23} = a_{23} \qquad \ldots(iv)$$

and $$l_{31}\ u_{13} + l_{32}\ u_{23} + l_{33} = a_{33} \qquad \ldots(v)$$

from (ii) we get $$u_{12} = a_{12}/l_{11} \qquad \text{(using } (i))$$

$$= a_{12}/a_{11}$$

from (iii) we get $$l_{22} = a_{22} - l_{21}\ u_{12} \qquad \ldots(vi)$$

$$l_{32} = a_{32} - l_{31}\ u_{12} \qquad \ldots(vii)$$

(iv) gives $$u_{23} = (a_{23} - l_{21}\ u_{23})/l_{22} \qquad \ldots(viii)$$

from the relation (v) we get

$$l_{33} = a_{33} - l_{31}\ u_{13} - l_{32}\ u_{23} \qquad \ldots(ix)$$

Thus, we have determined all the elements of L and U.

From (2) and (3) we have

$$LUX = B \qquad \ldots(5)$$

Let $$UX = V \quad \text{where} \quad V = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

From (5) we have $LV = B$, which on forward substitution gives V.

From $UX = V$, we find X (by backward substitution)

Note: Using (4) we can also find the in case of A.

Example 1. *Solve*
$$2x + y = 7$$
$$x + 2y = 5$$

Solution. The given system can be written as $AX = B$ ...(1)

where $$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \end{bmatrix},\ B = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

Let $A = LU$ where $$L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix},\ U = \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$\therefore \qquad A = LU \Rightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix} \begin{bmatrix} 1 & u_{12} \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} \\ l_{21} & l_{21}u_{12} + l_{22} \end{bmatrix}$$

Equating the corresponding elements we get

$$l_{11} = 2, \qquad l_{11}\, u_{12} = 1 \quad \Rightarrow \quad u_{12} = 1/2$$

$$l_{21} = 1, \qquad l_{21}\, u_{12} + l_{22} = 2$$

$$\Rightarrow \qquad 1.u_{12} + l_{22} = 2$$

$$\Rightarrow \qquad \frac{1}{2} + l_{22} = 2 \quad \Rightarrow \quad l_{22} = \frac{3}{2}$$

$$\therefore \qquad L = \begin{bmatrix} 2 & 0 \\ 1 & \frac{3}{2} \end{bmatrix}, \; U = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}$$

$\therefore$ From (1) we get $\qquad LUX = B \qquad$ ...(2)

Let $\qquad UX = V$, than from we have

$$LV = B$$

$$\Rightarrow \qquad \begin{bmatrix} 2 & 0 \\ 1 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\Rightarrow \qquad 2v_1 = 7, \quad v_1 + \frac{3}{2}v_2 = 5$$

$$v_1 = \frac{7}{2} \text{ and } \frac{7}{2} + \frac{3}{2}v_2 = 5 \Rightarrow u_2 = 1$$

$$\therefore \qquad v_1 = \frac{7}{2}, \; v_2 = 1 \quad \Rightarrow \quad U = \begin{bmatrix} \frac{7}{2} \\ 1 \end{bmatrix}$$

Now $\qquad UX = V \quad \Rightarrow \quad \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7/2 \\ 1 \end{bmatrix}$

$$\Rightarrow \qquad x + \frac{1}{2}y = \frac{7}{2}, \; y = 1$$

$$\Rightarrow \qquad x + \frac{1}{2} = \frac{7}{2}$$

$$\Rightarrow \qquad x = \frac{7}{2} - \frac{1}{2} = \frac{6}{2} = 3$$

$\therefore$ $x = 3$, $y = 1$ is the required solution.

Example 2. *Solve the equation by Crout's method.*

$$x + y + z = 9$$

$$2x - 3y + 4z = 13$$
$$3x + 4y + 5z = 40$$

Solution. We have $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$

The given system of equation is $AX = B$ ...(1)

Let $A = LU$ ...(2)

where $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$; $U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$

$$A = LU$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{22}u_{12} + l_{22} & l_{21}u_{13} + l_{22}u_{23} \\ l_{31} & l_{31}u_{12} + l_{32} & l_{33}u_{13} + l_{32}u_{23} + l_{33} \end{bmatrix}$$

Equating the corresponding elements, we have

$l_{11} = 1$, $\quad l_{11}\, u_{12} = 1 \Rightarrow 1.u_{12} = 1 \Rightarrow u_{12} = 1$; $\quad l_{11}u_{13} = 1 \Rightarrow u_{13} = 1$

$l_{21} = 2$, $\quad l_{21}\, u_{12} + l_{22} = -3 \Rightarrow 2 + l_{22} = -3$; $\quad l_{21}u_{13} + l_{22}u_{23} = 4$

$\Rightarrow l_{22} = -5$; $\quad \Rightarrow 2 - 5u_{23} = 4$

$\Rightarrow u_{23} = -\dfrac{2}{5}.$

$l_{31} = 3$, $\quad l_{31}u_{12} + l_{32} = 4$; $\quad l_{31}\, u_{13} + l_{32}\, u_{23} + l_{33} = 5$

$\Rightarrow \quad 3 + l_{32} = 4$; $\quad \Rightarrow 3 - \dfrac{2}{5} + l_{33} = 5$

$\Rightarrow \quad l_{32} = 1$; $\quad \Rightarrow l_{33} = \dfrac{12}{5}$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 3 & 1 & \frac{12}{5} \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 1 \end{bmatrix}$$

Substituting $A = LU$ in (1) we get

$LUX = B$ ...(3)

Let $UX = V$ where $V = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$

$$\therefore \quad LV = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & -5 & 0 \\ 3 & 1 & \frac{12}{5} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} v_1 \\ 2v_1 - 5v_2 \\ 3v_1 + v_2 + \frac{12}{5}v_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$$

$$\therefore \; v_1 = 9, \quad 2v_1 - 5v_2 = 13 \;;\; 3v_1 + v_2 + \frac{12}{5}v_3 = 40$$

$$\Rightarrow \quad 18 - 5v_2 = 13 \;;\; 27 + 1 + \frac{12}{5}v_3 = 40$$

$$\Rightarrow \quad v_2 = 1 \;;\; v_3 = 5$$

$$\therefore \quad V = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix}$$

Now $\quad UX = V$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & \frac{-2}{5} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 1 \\ 5 \end{bmatrix}$$

$$\Rightarrow \quad x + y + z = 9$$

$$y - \frac{2}{5}z = 1$$

$$z = 5$$

By back substitution, we get

$$y - \frac{2}{5}z = 1 \Rightarrow y - \frac{2}{5}.5 = 1 \Rightarrow y - 2 = 1 \Rightarrow y = 3$$

and $\quad x + y + z = 9 \Rightarrow x + 3 + 5 = 9 \Rightarrow x = 1$

$\therefore$ The required solution is $x = 1$, $y = 2$, $z = 3$

Exercise 11.5

Applying (*a*) Crouts' method (*b*) triangularization method solve the equations

1. $3x + 2y + 7z = 4$

$2x + 3y + z = 5$

$3x + 4y + z = 7$

2. $10x + y + z = 12$
$2x + 10y + z = 13$
$2x + 2y + 10z = 14$

3. $5x + 2y + z = -12$
$-x + 4y + 2z = 20$
$2x - 3y + 10z = 3$

4. $2x - 6y + 8z = 24$
$5x + 4y - 3z = 2$
$3x + y + 2z = 16$

5. $10x_1 + 7x_2 + 8x_3 + 7x_4 = 32$
$7x_1 + 5x_2 + 6x_3 + 5x_4 = 23$
$8x_1 + 6x_2 + 10x_3 + 9x_4 = 33$
$7x_1 + 5x_2 + 9x_3 + 10x_4 = 31$

6. $2x_1 - x_2 + x_3 = -1,\ 2x_2 - x_3 + x_4 = 1$
$x_1 + 2x_3 - x_4 = -1,\ x_1 + x_2 + 2x_4 = 3$

Answers

1. $x = 7/8,\ y = 9/8,\ z = -1/8$ **2.** $x = y = z = 1$ **3.** $z = 2,\ y = 3,\ x = -4$
4. $x = 1,\ y = 3,\ z = 5$ **5.** $x_1 = x_2 = x_3 = x_4 = 1$ **6.** $x_1 = -1,\ x_2 = 0,\ x_3 = 1,\ x_4 = 2.$

12

CURVE FITTING

12.1 INTRODUCTION

In this chapter we are concerned with the problem of fitting an equation or a curve to data involving paired values. An approximate non-mathematical relationship between the two variables, can be established by a diagram called *scatter diagram*. The exact mathematical relationship between the two variables is given by simple algebraic expression called *curve fitting*. Though there are infinite variety of curves in mathematics, the curves used for the purpose of curve fitting are relatively limited in type. The straight line is the simplest and one of the most important curves used.

12.2 THE STRAIGHT LINE

The equation

$$y = a + bx \tag{1}$$

is an equation of the first degree in x and y. It represents a straight line.

The difference

$$y_i - (a + bx_i) \tag{2}$$

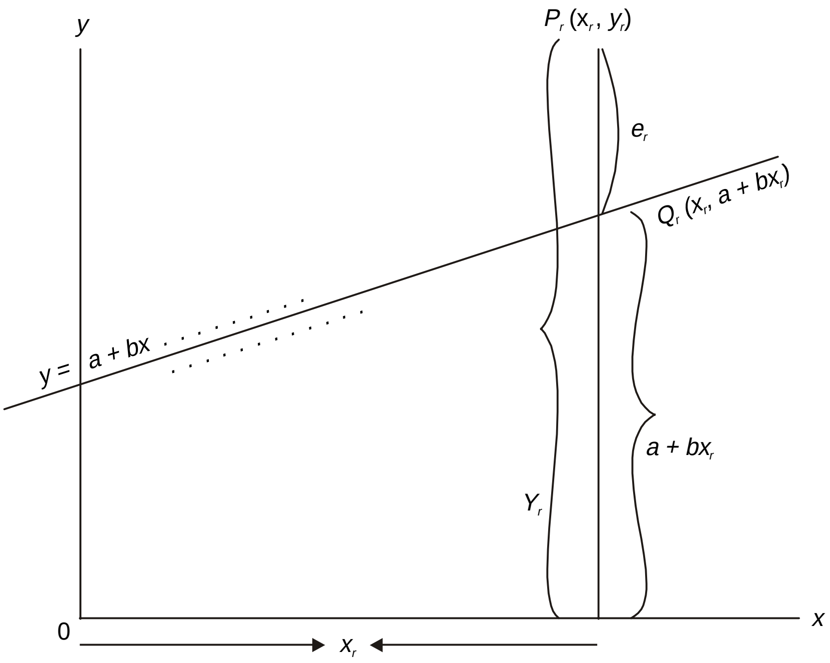
is zero if and only if the point (x_i, y_i) lies on the line given by (1).

12.3 FITTING A STRAIGHT LINE

Usually fitting a straight line means finding the values of the parameters a and b of the straight line given by (1), as well as actually constructing the line itself. The Graphic methods and the Method of least squares are two useful methods for fitting a straight line.

12.3.1 The Graphical Method

This method can be used whenever the given formula can be plotted as a straight line either directly or after suitable transformation. The straight line drawn after a careful visual estimate of its position has been made with the aid of a ruler. The co-ordinates of any two points on the line, not too near together are then measures and substituted in equation (1). The resulting equations in a and b are then solved for their parameters. Graphical method, whatever its theoretical attractions, suffers from the disadvantage that it is difficult to apply in practice except for the straight line. This method will give fairly good results when finely divided co-ordinate paper is used, but in general it is not recommended.



An alternative method, which is in almost universal use at present time, is known as the method of *Least Squares* and we proceed to discuss it at length.

12.3.2 The Method of Least Squares

This method of curve fitting was suggested early in the nineteenth century by the French mathematician Adrien Legendre. The method of least squares assumes that the best fitting line in the curve for which the sum of the squares of the vertical distances of the points (x_r, y_r) from the line is minimum. This method is more accurate than the graphical method.

Let $$y = a + bx \qquad \text{refer (1)}$$

be the equation of the line. The ordinate of any point Q_r on the line vertically above or below a given point P_r, can be found by substituting the abscissa x_r, in the right-hand side of (1). The two co-ordinates of Q_r, will be $(x_r, a + bx_r)$ (*see* in the above figure). The vertical distance e_r from the line of any point, P_r with co-ordinates (x_r, y_r), will therefore be given by the equation

$$e_r = y_r - (a + x_r). \qquad (3)$$

We may say that e_r represents the difference between the actual ordinate y_r, of a point and its theoretical ordinate $a + bx_r$.

Let $$q = \sum_{r=1}^{n} e_r^2 = \sum_{r=1}^{n} \left(y_r - \left(a + bx_r\right)\right)^2. \qquad (4)$$

The best fitting line is that line for which the sum of the squares, $q = \sum e_r^2$ is a minimum. We find the values of a and b which make q minimum as follows:

Differentiating q partially with respect to a and b and equally these partially to zero, we obtain

$$\therefore \ \frac{\partial q}{\partial a} = \sum_{r=1}^{n} (-2)\left[y_r - (a + bx_r)\right] = 0,$$

and
$$\frac{\partial q}{\partial b} = \sum_{r=1}^{n} (-2)\, x_r \left[y_r - (a + bx_r)\right] = 0,$$

which yield the so-called *system of normal equations,*

$$\sum_{r=1}^{n} y_r = an + b \sum_{r=1}^{n} x_r \,, \tag{5}$$

and
$$\sum_{r=1}^{n} x_r y_r = a \sum_{r=1}^{n} x_r + b \sum_{r=1}^{n} x_r^2 \,. \tag{6}$$

Solving this system of equations we get

$$a = \frac{\sum_{1}^{n} x_r^2 \sum_{1}^{n} y_r - \sum_{1}^{n} x_r \sum x_r y_r}{n \sum_{1}^{n} x_r^2 - \left(\sum_{1}^{n} x_r\right)^2}, \tag{7}$$

$$b = \frac{n\left(\sum_{r=1}^{n} x_r y_r\right) - \left(\sum_{r=1}^{n} x_r\right)\left(\sum_{r=1}^{n} x_r^2\right)}{n\left(\sum_{r=1}^{n} x_r^2\right) - \left(\sum_{r=1}^{n} x_r\right)}, \tag{8}$$

where n is the number of points (x_r, y_r).

To simplify the formulae, we let

$$t_r = \frac{x_r - \bar{x}}{h}, \tag{9}$$

where h is the interval between successive values of x assumed to be equally spaced. Shifting the origin to the mean $\bar{x}$ and compressing the intervals to unity we get the transformed equation of (2) referred to the new (t, y) axes as

$$y = \bar{a} + \bar{b}\, t\,, \tag{10}$$

where $\bar{a}$ and $\bar{b}$ are the parameters in the new equation. In terms of the new co-ordinate formulae (7) and (8) become

$$\bar{a} = \frac{\sum_1^n t_r^2 \sum_1^n y_r - \sum_1^n t_r \sum t_r y_r}{n\sum t_r^2 - \left(\sum t_r\right)^2}, \tag{11}$$

$$\bar{b} = \frac{n\sum_1^n t_r y_r - \sum_1^n t_r \sum y_r}{n\sum_1^n t_r^2 - \left(\sum_1^n t_r\right)^2}. \tag{12}$$

Transformation (9) replace the x_rs' by unit deviations t_r from the mean $\bar{x}$. Since

$$\sum_1^n t_r = 0,$$

the formulae (11) and (12) reduce to

$$\bar{a} = \frac{\sum_1^n t_r^2 \sum_1^n y_r - 0}{n\sum t_r^2 - 0},$$

and

$$\bar{b} = \frac{n\sum_1^n t_r\, y_r - 0}{n\sum_1^n t_r^2 - 0},$$

which further simplify to the form

$$\bar{a} = \frac{1}{n}\sum_1^n y_r = \bar{y}, \tag{13}$$

$$\bar{b} = \frac{\sum_1^n t_r\, y_r}{\sum_1^n t_r^2}. \tag{14}$$

12.4 FITTING A PARABOLA

When a set of points exhibits a parabolic trend, the fitting of a quadratic function

$$y = a + bx + cx^2,$$

to the data may be carried out by the method of least squares, which leads to the three equations

$$na + b\sum_{1}^{n} x_r + c\sum x_r^2 = \sum_{1}^{n} y_r,$$

$$a\sum_{1}^{n} x_r + b\sum_{1}^{n} x_r^2 + c\sum_{1}^{n} x_r^3 = \sum_{1}^{n} x_r\, y_r,$$

$$a\sum_{1}^{n} x_r^2 + b\sum_{1}^{n} x_r^3 + c\sum_{1}^{n} x_r^4 = \sum_{1}^{n} x_r^2\, y_r.$$

The above equations can be solved for a, b, c.

12.5 EXPONENTIAL FUNCTION $y = ae^{bx}$

Transforming the exponential equation $y = ae^{bx}$ by taking logarithms on both sides we get

$$\log y = \log a + bx \log e.$$

If we replace $\log y$ by y and the constants $\log a$ and $b \log e$ by a' and b' respectively, we obtain

$$Y = a' + b'x,$$

which defines a straight line.

Example 12.1 *Find the least square line y = a + bx for the data points (–1, 10), (0,9), (1,7), (2,5), (3,4), (4,3), (5,0) and (6, –1).*

Solution Here

x_r	y_r	x_r^2	$x_r y_r$
–1	10	1	–10
0	9	0	0
1	7	1	7
2	5	4	10
3	4	9	12
4	3	9	12
5	0	25	0
6	–1	36	–6
20	37	92	25

From the table we have

$$n = 8, \ \sum x_r = 20, \ \sum y_r = 37, \ \sum x_r 2 = 92, \text{ and } \sum x_r y_r = 25.$$

The normal equations are

$$na + b\sum x_r = \sum y_r ,$$

$$a\sum x_r + b\sum x_r^2 = \sum x_r y_r .$$

Putting the values in normal equations

$$8a + 20b = 37,$$

and $$20a = 97b = 25,$$

and solving these equations we get

$$a = 8.6428571,$$
$$b = -1.6071429.$$

Therefore the least square line is

$$y = 8.6428571 + (-1)\ 1.6071429x,$$

i.e., $$y = -1.6071429x + 8.6428571.$$

Example 12.2 *Find the least square line y =a + bx for the data*

x_r	-2	-1	0	1	2
y_r	1	2	3	3	4

Solution

x_r	y_r	x_r^2	$x_r y_r$
–2	1	4	–2
–1	2	1	–2
0	3	0	0
1	3	1	3
2	4	4	8
0	13	10	7

∴ In this case $n = 5,\ \sum x_r = 0,\ \sum y_r = 13,\ \sum x_{r2} = 10,$ and $\sum x_r y_r = 7.$

The normal equations are

$$na + b\sum x_r = \sum y_r ,$$

$$a\sum x_r + b\sum x_r^2 = \sum x_r y_r .$$

Putting the values of $n, \sum x_r, \sum y_r, \sum x_r^2$ and $\sum x_r y_r$ in the above equation we get $5a = 13,\ 10b = 7.$

Solving

$$a = 2.6,\ b = 0.7.$$

∴ The required line of fit is

$$y = 2.6 + (0.7)x.$$

Example 12.3 *Find a formula for the line of the form y =a + bx +cx² which will fit the following data*

x	*0*	*0.1*	*0.2*	*0.3*	*0.4*	*0.5*	*0.6*	*0.7*	*0.8*	*0.9*
y	*3.1950*	*3.2299*	*3.2532*	*3.2611*	*3.2516*	*3.2282*	*3.1807*	*3.1266*	*3.0594*	*2.9759*

Solution The normal equations are

$$na + b\sum x_r + x\sum x_r^2 = \sum y_r ,$$

$$a\sum x_r + b\sum x_r^2 = c\sum y_r^3 = \sum x_r y_r ,$$

$$a\sum x_r^2 + b\sum x_r^3 + c\sum x_r^4 = \sum y_r^2 y_r .$$

Substituting the values of $\sum x_r, \sum x_r^2, \sum x_r^3, \sum x_r y_r, \sum x_r^2 y_r, \sum y_r,$ and n, we get

$$10a + 4.5b + 2.85c = 31.7616,$$
$$4.5a + 2.85b + 2.025c = 14.0896,$$
$$2.85a + 2.025b + 1.5333c = 8.82881,$$

and solving these equations we obtain

$$a = 3.1951,\ b = 0.44254,\ c = -\ 0.76531.$$

∴ The required equation is

$$y = 3.1951 + 0.44254x - 0.76531x^2.$$

Example 12.4 *Fit a second degree parabola to the following data*

x	*0*	*1*	*2*	*3*	*4*
y	*0*	*1.8*	*1.3*	*2.5*	*6.3*

Solution The values of x are, 0, 1, 2, 3, 4

The number of values is odd. Shifting the origin to the middle value 2 of x, and making the substitution

$$u = x - 2,\ v = y,$$

the curve of fit as

$$v = a + bu + cu^2,$$

we obtain

x	v	u	v	uv	u^2	u^2v	u^3	u^4
0	1	–1	1	–2	4	4	–8	16
1	1.8	–2	1.8	1.6	1	1.8	–1	1
2	1.2	0	1.3	0	0	0	0	0
3	2.5	1	2.5	2.5	1	2.5	1	1
4	6.3	2	6.3	12.6	4	25.2	8	16
–	–	0	12.9	11.3	10	33.5	0	34

The normal equations are

$$\sum v = na + b\sum u + c\sum u^2 ,$$

$$\sum uv = a\sum u + b\sum u^2 + c\sum u^3 ,$$

$$\sum u^2 v = a\sum u^2 + b\sum u^3 + c\sum u^4 .$$

Putting the values of $n, \sum u, \sum v, \sum uv, \ldots,$ etc.,

$$12.9 = 5a + 10c,$$
$$11.3 = 10b,$$
$$33.5 = 10a + 34c,$$

solving these normal equations we get

$$a = 1.48,\ b = 1.13,\ c = 0.550.$$

∴ The required equation is

$$v = 1.48 + 1.13u + 1.55u^2,$$

substituting $u = x - 2$, $v = y$ in the above equation we get,

$$y = 1.42 + 1.13(x - 2) + (0.55)\ (x - 2)^2.$$

∴ The required curve of fit is the parabola

$$y = 1.42 - 1.07x + 0.55x^2.$$

Example 12.5 *Obtain normal equations for fitting a curve of the form*

$$y = ax + \frac{b}{x}$$

for a point (x_r, y_r), $r = 1, 2, ..., x$

Solution The curve $$y = ax + \frac{b}{x} \quad ...(1)$$

Passes through the points $(x_r,\ y_r)$ $r = 1, 2, ..., n$

Therefore, we have

$$y = ax_1 + \frac{b}{x_1}$$
$$y_2 = ax_2 + \frac{b}{x_2} \quad ...(2)$$
$$\vdots$$
$$y_n = ax_n + \frac{b}{x_n}$$

putting $$\frac{1}{x} = z \text{ in (1) we get}$$

$$y = ax + bz \quad ...(3)$$

$$\Rightarrow \quad S = \Sigma(y_r - ax_r - bz_r)^2 \quad ...(4)$$

differentiating (4); partially with respect a and b we get

$$\frac{\partial s}{\partial a} = -2\ \Sigma x_r\,(y_r - ax_r - bz_r) = -2\ \Sigma\ (x_r y_r - ax_r^2 - bx_r z_r)$$

and $$\frac{\partial s}{\partial a} = -2\ \Sigma z_r\ (y_r - ax_r - bz_r) = -2\Sigma(y_r z_r - ax_r z_r - bz_r^2)$$

for S to be minimum : we have

$$\frac{\partial s}{\partial a} = 0 \quad \text{and} \quad \frac{\partial s}{\partial b} = 0$$

$$\frac{\partial s}{\partial a} = 0 \Rightarrow -2\,\Sigma(x_r,\, y_r - ax_r^2 - bx_r z_r) = 0$$

$$\Rightarrow \Sigma x_r y_r = a\,\Sigma x_r^2 + b\Sigma x_r\, z_r \qquad ...(5)$$

$$\frac{\partial s}{\partial b} = 0 \Rightarrow -2\,\Sigma(y_r\, z_r - ax_r z_r - bz_r^2) = 0$$

$$\Rightarrow \Sigma y_r z_r = a\,\Sigma x_r z_r + b\Sigma z_r^2 \qquad ...(6)$$

but $z = \frac{1}{x}$

$\therefore$ from (5) and (6) obtain the required normal equations can be written as

$$\Sigma x_r\, y_r = a\,\Sigma x_r^2 + b\Sigma x_r \frac{1}{x_r}$$

i.e.,
$$nb + a\Sigma x_r^2 = \Sigma x_r y_r \qquad ...(7)$$

and
$$\Sigma y_r\, z_r = a\Sigma x_r\, z_r + b\Sigma y_r^2$$

$\Rightarrow$
$$na + b\Sigma\frac{1}{x_r^2} = \Sigma\frac{y_r}{x_r} \qquad ...(8)$$

Example 12.6 *Given the following data*

v (ft/min)	:	*350*	*400*	*500*	*600*
t (min)	:	*61*	*26*	*7*	*2.6*

If v and t an connected by the relation $v = at^b$, find the best possible values of a and b.

Solution
$$v = at^b$$

$\Rightarrow$
$$\log_{10} v = \log_{10} a + b\log_{10} t \qquad ...(1)$$

Substituting $x = \log_{10} t$, $y = \log_{10} v$, $a' = \log_{10} a$ in (1) we get

$$y = a' + bx$$

The normal equations can be wirtten as

$$4a' + b\Sigma x_r = \Sigma y_r \qquad ...(2)$$

$$a'\Sigma x_r + b\Sigma x_r^2 = \Sigma x_r y_r \qquad ...(3)$$

v	k	x	y	xy	x^2
350	61	1.7858	2.5441	4.542	3.187
400	26	1.4150	2.6021	3.682	2.002
500	7	0.8451	2.6990	2.281	0.714
600	2.6	0.4150	2.7782	1.153	0.172
		4.4604	10.6234	11.658	6.075

Substituting the above values in (2) and (3) we get

$$4a' + 4.4604b = 10.623 \qquad ...(4)$$

$$4.4604a' + 6.075b = 11.658 \qquad ...(5)$$

Solving (4) and (5) we get

$$a' = 2.845, \; b = -0.1697$$

$\therefore$ a = antilog a' = antilog 2.845 = 699.8

Example 12.7 *Using the method of least squares bit a curve of the form $y = ab^x$ to the following data*

x :	*2*	*3*	*4*	*5*	*6*
y :	*8.3*	*15.4*	*33.1*	*65.2*	*127.4*

Solution. Here we have $n = 5$ (number of observation)

Consider $y = ab^x$ ...(1)

Applying logarithms on both sides we get

$$\log_{18} y = \log_{10} a + x \log_{10} b \quad ...(2)$$

taking $\log_{10} y = Y$, the equaiton (2) can be written as

$$Y = a' + b'_x \quad ...(3)$$

where $a' = \log_{10} a$, $b' = \log_{10} b$

Equation (3) is linear in X and Y, hence the normal equations are

$$na' + b'\Sigma x = \Sigma Y$$

$$a'\Sigma x + b'\Sigma x^2 = \Sigma XY$$

$\therefore$ we have

x	y	$Y = log_{10} y$	XY	x^2
2	8.3	0.9191	1.8382	4
3	15.4	1.1875	3.5625	9
4	33.1	1.5198	6.0792	16
5	65.2	1.1842	9.0710	25
6	127.4	2.1052	12.6312	36
Total 20	–	7.5458	33.1821	90

The normal equations are

$$5a' + 20b' = 7.5458 \quad ...(4)$$

$$20a' + 90b' = 33.1821 \quad ...(5)$$

Solving (4) and (5) we get

$$a' = 0.3096, \quad b' = 0.2995$$

Now $a' = 0.3096$

$\Rightarrow$ $\log_{10} a = 0.3096$

$\Rightarrow$ $a = 2.0399$

$b' = 0.2999$

$\Rightarrow$ $\log_{10} b = 0.2999$

$\Rightarrow$ $b = 1.9948$

$\therefore$ The required least square curve is

$$y = 2.0399 \; (1.9948)^x$$

Example 12.8 *Fit a least square curve of the form $y = ae^{bx}$ $(a > 0)$ to the data given below*

x :	1	2	3	4
y :	1.65	2.70	4.50	7.35

Solution. Consider $y = ae^{bx}$

Applying logarithms (with base 10) on both sides, we get

$$\log_{10} y = \log_{10} a + bx \log_{10} e \quad ...(1)$$

taking $\log_{10} y = Y$, the equaiton (1) can be written as

$$Y = a' + b'x \quad ...(2)$$

where $a' = \log_{10} a$, $b' = \log_{10} e$

(2) is a linear equation in x and Y, the normal equations are

$$na' + b'\Sigma x = \Sigma Y$$

$$a'\Sigma x + b'\Sigma x^2 = \Sigma XY$$

x	y	$Y = \log_{10} y$	xY	x^2
1	1.65	0.2175	0.2175	1
2	2.70	0.4314	0.8628	4
3	4.50	0.6532	1.9596	9
4	7.35	0.8663	3.4652	16
Total 10	–	2.1684	6.5051	30

The normal equation can be written as

$$4a' + 10b' = 2.1684 \quad ...(3)$$

$$10a' + 30b' = 6.5051 \quad ...(4)$$

Solving the euqaitons (3) and (4), we get

$$a' = 0.0001, \quad b' = 0.2168$$

Now $a' = 0.0001 \Rightarrow \log_{10} a = 0.0001 \Rightarrow a = 1.0002$

$b' = 0.2168 \Rightarrow b \log_{10} e = 0.2168$

$$\Rightarrow \quad b = \frac{0.2168}{\log_{10} e} = \frac{0.2169}{0.4343}$$

$$\Rightarrow \quad b = 0.4992$$

$\therefore$ The required curve is $y = (1.0002).e^{0.4992x}$.

Exercise 12.1

1. Find the least square line $y = a + bx$ for the data

x_r	–4	–2	0	2	4
y_r	1.2	2.8	6.2	7.8	13.2

2. Fit a straight line to the following data regarding x as the independent variable

x	0	1	2	3	4
y	1	1.8	3.3	4.5	6.3

3. Find the least square line $y = a + bx$

x	–2	0	2	4	6
y	1	3	6	8	13

4. Find the least squares parabolic fit $y = a + bx + cx^2$

x	–3	–1	1	3
y	15	5	1	5

5. Find the least squares parabola for the points (–3, 3), (0, 1), (2, 1), (4, 3).

6. The profit of a certain company in the xth year of its life are given by

x	1	2	3	4
y	1250	1400	1950	2300

Taking $u = x - 3$ and $V = \dfrac{y-1650}{50}$, show that the parabola of the second degree of v on u is $v + 0.086 = 5.30u + 0.643u^2$ and deduce that the parabola of the second degree of y on x is $y = 114 + 72x + 32.15x^2$.

7. Find the least square line $y = a_0 + a_1x$ for the data

x	1	2	3	4
y	0	1	1	2

8. Find the least square fit straight line (of the form $y = a_0 + a_1x$) for the data of fertilize application and yield of a plant

Fertilizer applied (gm/week/plant)	0	10	20	30	40	50
Yield (kg)	0.8	0.8	1.3	1.6	1.7	1.8

9. Find the normal equations that arise from filling by the least squares method, an equation of the form $y = a_0a_1\sin x$, to the set of points (0, 0), (π/6, 1), (π/2, 3), and (5π/6, 2) solve for a_0 and a_1.

10. The following data relates the percentage of alloying element to the compressive strength of an alloy:

% alloying element	10	15	20	25	30
Compressive strength	27.066	29.57	31.166	31.366	31.0

11. Find the least squares parabolic fit $y = ax^2 + bx + c$, for the following data

x	–3	–1	1	3
y	15	5	1	5

12. Find the polynomial of degree two that best fits the following data in least square sense

x	–2	–1	0	1	2
y	–3.150	–1.390	0.620	2.880	5.378

13. If P is the Pull required to lift a load W by means of a pulley block, find a liner law of the form $P = mW + c$, connecting P and W, using the data

P	12	15	21	25
W	50	70	100	120

where P and W are taken in kg-wt. Compute P when $W = 150$ kg.

14. In some determination of the volume v of carbondioxide dissolved in a given volume of water at different temperatures θ, the following pairs of values were obtained

θ	0	5	10	15
v	1.80	1.45	1.80	1.00

obtain by the method of least squares, a relation of the form $v = a + b\theta$. Which best fits to these observations.

15. The observations from an experiment are as given below

y	2	10	26	61
x	600	500	400	350

It is known that a relation of type $y = ae^{bx}$ exists.

Find the best possible values of a and b.

Hint : The normal equations are

$a' + b\Sigma x_r = \Sigma \log y_r$

$a'\Sigma x_r + b\Sigma x_r^2 = \Sigma x_r \log y_r$

where $a' = \log a$

i.e., $4a' + 1850b = 4.501333$

$1850\,a' + 892500\,b = 1871.473$

Solving we get : $a' = 3.764167 \Rightarrow a = 43.1277$

$b = -\,0.0057056$

and the curve is $y = 43.12777\ e^{-0.0057056x}$

16. Fit a least square geometric curve $y = ax^b$ to the following data.

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

17. Using the method of least squares, fit a relation of the form $y = abx$ to the following data

x	2	3	4	5	6
y	144	172.8	207.4	248.8	298.5

18. Using the method of least squares fit a curve of the form $y = ab^x$ to the following data

x	1	2	3	4
y	4	11	35	100

19. Fit a least square curve of the form $y = ax^b$ for the following data where a and b are constants

x	61	26	7	2.6
y	350	400	500	600

20. The pressure and volume of a gas are related by the equation $PV^{\lambda} = k$ (λ an k are constants). Fit this equation for the data given below:

p	0.5	1.0	1.5	2.0	2.5	3.0
v	1.62	1.00	0.75	0.62	0.52	.046

Answers

1. $y = 6.24 + 1.45x$ **2.** $y = 0.72 + 1.33x$ **3.** $y = 3.3 + 1.45x$

4. $y = 2.125 - 1.70x + 0.875x^2$ **5.** $y = 0.850519 - 0.192495x + 0.178462x^2$

7. $y = -\frac{1}{2} + \frac{3}{5}x$ **8.** $0.7575 + 0.0229x$ **9.** $a_0 = 0, a_1 = 3$

10. $y = 18.988 + 1.0125x - 0.02048x^2$ **11.** $y = \frac{7}{8}x^2 - \frac{17}{10}x + \frac{17}{8}$

12. $y = 0.621 + 2.1326x + 0.1233x^2$ **13.** $2.2759 + 0.1879\ W$, 30.4635 kg

14. $v = 1.758 - 0.053\,\theta$. **16.** $y = 0.5012\ x^{1.9977}$

17. $y = 9986\ x^{1.2}$ **18.** $y = (1.3268)\ .\ (2.9485)^x$

19. $y = (701.94)\ x^{-0.1708}$ **20.** $pv^{1.4225} = 0.997$

13

EIGEN VALUES AND EIGEN VECTORS OF A MATRIX

13.1 Let $A = [a_{ij}]$ be a square matrix of dimension $n \times n$. The equation

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} = 0, \tag{1}$$

where λ is a parameter called the *characteristic equation* of A. It is of degree n. The roots of this characteristic equation are called the *characteristic roots* or the *Eigen values* of the matrix A. A square matrix of order n has always n eigen values.

When the determinant in (1) is expanded it becomes a polynomial of degree n, which is called *characteristic polynomial*.

From (1) we get

$$\lambda^n - (a_{11} + a_{22} + ... + a_{nn})\lambda^{n-1} + ... + (-1)^n \det A = 0$$

$$\therefore\ \lambda_1 + \lambda_2 + ... + \lambda_n = a_{11} + a_{22} + ... + a_{nn} = \text{Trace } A$$

$$\lambda_1 \lambda_2 ... \lambda_n = \det A = |A|.$$

Consider the homogeneous equations

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= \lambda x_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= \lambda x_2 \\ &\vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= \lambda x_n \end{aligned}\right\} \tag{2}$$

where λ is an undetermined parameter. The n values of λ for which non-zero roots of the homogeneous equations (2) exist, are called the *Eigen values* or *Characteristic values* of the parameter λ. The non-zero column vector x satisfying $(A— \lambda I)\ X = 0$, is called *Eigen vector* of A. Corresponding to each of the eigen value there is an eigen vector of A.

Consider the projection transformation in two dimensional space OX_1X_2 defined by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Here the eigen vectors are:

(i) The non-zero vectors directed along the X_1-axis with eigen value $\lambda_1 = 1$ and

(ii) The non-zero vectors directed along the X_2-axis with eigen value $\lambda_2 = 1$ (*see* Fig. 1)

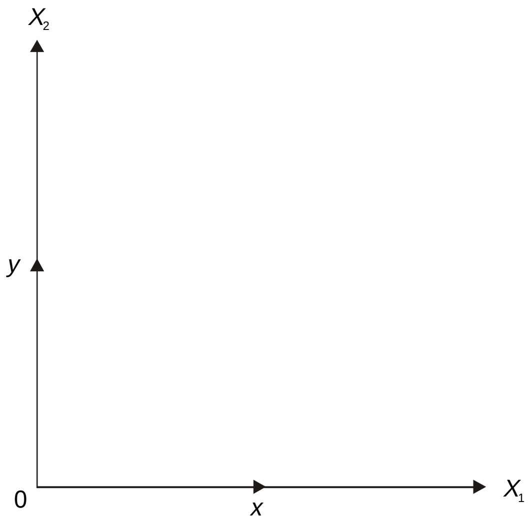


Fig. 1

We state below some important properties of eigen values and eigen vectors of a matrix:

1. For each distinct eigen value λ, there exists at least one eigen vector corresponding to λ.
2. If A is a square matrix and $\lambda_1, \lambda_2, ..., \lambda_n$ are distinct eigen values of A with associated eigen vectors $v_1, v_2, \ldots, v_n$ respectively, then $\{v_1, v_2, \ldots, v_n\}$ is a set of linearly independent vectors.
3. If B is a non-singular matrix then A and $B^{-1} AB$ have same eigen values.
4. The eigen values of a Hermitian matrix are real.
5. The eigen values and eigen vectors of a real symmetric matrix are real.
6. The number of linearly independent eigen vectors corresponding to one and the same root of the characteristic equation does not exceed the multiplicity of that root.

Note: In this chapter, we write

$$|A = \lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22}-\lambda & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn}-\lambda \end{vmatrix} = p_0\lambda^n + p_1\lambda^{n-1} + \ldots + p_n \text{ (say)}$$

Example 13.1 *Find the characteristic values (Eigen values) of the matrix*

$$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$$

Solution The characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e.,
$$\begin{vmatrix} 3-\lambda & -1 \\ x & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (-\lambda)\ (3 - \lambda) + 2 = 0$$

$$\lambda^2 - 3 + 2 = 0$$

$$\Rightarrow \quad (\lambda - 1)\ (\lambda - 2) = 0 \qquad ...(1)$$

The roots of are $\lambda = 1, \lambda = 2$

$\therefore$ The characteristic roots of values of A are 1, 2.

Example 13.2 *Find the characteristic values and the corresponding characteristic vectors of*

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Solution The characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e.,
$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow (3 - \lambda)^2 - 1 = 0$$

$$\Rightarrow \quad \lambda^2 - 6\lambda + 9 - 1 = 0 \Rightarrow \lambda^2 - 6\lambda + 8 = 0$$

$$\Rightarrow \quad (\lambda - 2)\ (\lambda - 4) = 0$$

$\therefore$ $\lambda = 2, \lambda = 4$ are the characteristic values of A.

Let
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

denote the characteristic vector corresponding to the value $\lambda = 2$; then

$$(A - \lambda I)x = 0$$

$$\Rightarrow \quad \begin{bmatrix} 3-2 & 1 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x_1 + x_2 = 0, \quad x_1 + x_2 = 0$$

i.e.,
$$x_1 + x_2 = 0$$

$$\Rightarrow \quad x_1 = -x_2$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{-1} = k \text{ (say)}$$

$\therefore$ The characteristic vector corresponding to the value $\lambda = z$ is

$$X_1 = \begin{bmatrix} k \\ -k \end{bmatrix}$$

or
$$X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

when $\lambda = 4$: we have

$$(A - \lambda I)\, X = 0 \Rightarrow \begin{bmatrix} 3-4 & 1 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad -x_1 + x_2 = 0, \quad x_1 - x_2 = 0$$

$$\Rightarrow \quad x_1 = x_2$$

$$\Rightarrow \quad \frac{x_1}{1} = \frac{x_2}{1} = k' \text{ (say)}$$

$\therefore$ The characteristic vector corresponding to the value

$$\lambda = 4 \text{ is } X_2 = \begin{bmatrix} k' \\ k' \end{bmatrix} \text{ or } X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The characteristic roots of A are $\lambda = 2$, $\lambda = 4$ and the characteristic vectors of A are $X_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Note: The characteristic vectors may be normalized and expressed as

$$X_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad X_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

Example 13.3 *Obtain the eigen values and eigen vectors of the symmetric matrix*

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution The characteristic equation

$$|A - \lambda I| = 0 \text{ gives } \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0,$$

$$\Rightarrow \quad (\lambda + 1)^2 (\lambda - 2) = 0,$$

hence eigen values are $\lambda = -1, -1, 2$.

Case 1 When $\lambda = 2$

The corresponding eigen vector is given by

$$(A - 2I)X = 0$$

$$\Rightarrow \quad -2x + y + z = 0$$

$$x - 2y + z = 0$$

$$x + y - 2z = 0$$

solving we get $x = y = z = k$ (say), thus

$$X_1 = [k \; k \; k]^T.$$

Case 2 When $\lambda = -1, -1$

The corresponding eigen vector is given by

$$(A - (-1)I)\, X = 0, \text{ i.e., } (A + I)\, X = 0,$$

$$\Rightarrow \qquad x + y + z = 0$$

(all three equations are equivalent).

The trial solutions are

$$[1\ 0\ -1]^T \text{ and } [1 + k\ -1\ -k]^T$$

since these are orthogonal, we have

$$1 + k + k = 0 \quad \Rightarrow \quad k = -1/2$$

hence
$$X_2 = \left[\frac{1}{\sqrt{2}}\ 0\ \frac{-1}{\sqrt{2}}\right]^T, \ X_3 = \left[\frac{1}{\sqrt{6}}\ \frac{-2}{\sqrt{6}}\ \frac{1}{\sqrt{6}}\right]^T.$$

Example 13.4 *Determine the eigen values and the corresponding eigen vectors of the following system*

$$10x_1 + 2x_2 + x_3 = \lambda x_1$$

$$2x_1 + 10x_2 + x_3 = \lambda x_2$$

$$2x_1 + x_2 + 10x_3 = \lambda x_3$$

Solution We have

$$A = \begin{bmatrix} 10 & 2 & 1 \\ 2 & 10 & 1 \\ 2 & 1 & 10 \end{bmatrix}$$

The characteristic equation

$$|A - \lambda I| = \begin{vmatrix} 10-\lambda & 1 & 1 \\ 2 & 10-\lambda & 1 \\ 2 & 1 & 10-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \ (10-\lambda)^3 - 7(10-\lambda) + 6 = 0$$

$$\Rightarrow \ \lambda = 13, \lambda = 9, \lambda = 8$$

i.e.,
$$\lambda_1 = 13, \lambda_2 = 9, \lambda_3 = 8.$$

Case 1 When $\lambda_1 = 13$

$\therefore$ We get

$$-3x_1 + 2x_2 + x_3 = 0,$$
$$2x_1 - 3x_2 + x_3 = 0,$$
$$2x_1 + x_2 + 3x_3 = 0,$$

solving the first two equations with $x_3 = 1$, we get

$$x_1 = 1, \ x_2 = 1$$

these values satisfies all the three equations. The first eigen vector is $[1\ 1\ 1]^T$.

Case 2 When $\lambda_2 = 9$

$$\Rightarrow \quad (A - 9I)X = 0$$

$$\Rightarrow \quad \begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \\ 2x_1 + x_2 + x_3 &= 0 \end{aligned}$$

with $x_3 = 1$, the second eigen vector becomes

$$\left[-\frac{1}{3}\ \frac{1}{3} 1\right]^T.$$

Case 3 When $\lambda_3 = 8$

The corresponding eigen vector is given by

$$(A - 8I)X = 0$$

$$\Rightarrow \quad \begin{aligned} 2x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 + 2x_3 &= 0 \end{aligned}$$

with $x_3 = 1$, the third eigen vector becomes

$$\left[-\frac{3}{2}\ 1\ 1\right]^T.$$

Example 13.5 *Find the Eigen values and eigen vectors of the matrix*

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Solution The characteristic equation of the matrix is

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 2 & 2-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (\lambda - 1)^2 (4 - \lambda) = 0$$

we get $\lambda_1 = \lambda_2 = 1, \lambda_3 = 4$

Case 1 $\lambda_1 = 1$, we get

$$(A - \lambda I)X = 0 \Rightarrow \begin{vmatrix} 2-1 & 1 & 1 \\ 2 & 2-1 & 1 \\ 2 & 1 & 2-1 \end{vmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow \left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \\ x_1 + x_2 + x_3 &= 0 \end{aligned}\right\} \tag{3}$$

The rank of the system (3) is one, therefore two of the equations are consequences of the third. It suffices to solve the equation

$$x_1 + x_2 + x_3 = 0.$$

Putting

$$x_1 = k_1,\ x_2 = k_2,$$

we get $$x_3 = -[k_1 + k_2],$$

where k_1 and k_2 are arbitrary scalars not simultaneously zero.

In particular choosing $k_1 = 1$, $k_2 = 0$, and then $k_1 = 0$, $k_2 = 1$, we get the solution consisting of two linearly independent eigen vectors of matrix A

$$X_1 = [1\ 0\ -1]^T,\ X_2 = [0\ 1\ -1]^T.$$

All the other eigen vectors of A that correspond to the eigen value $\lambda_1 = 1$ are linear combinations of these basis vectors and fill the plane spannel by the vectors X_1 and X_2.

Case 2 Now $\lambda_3 = 4$, gives

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0,$$

or

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \\ x_1 + x_2 - 2x_3 &= 0 \end{aligned}$$

The rank of the above system is 2. The third equation of the system is a consequence of the first two equations.

$\therefore$ Solving first two equations

$$\begin{aligned} -2x_1 + x_2 + x_3 &= 0 \\ x_1 - 2x_2 + x_3 &= 0 \end{aligned}$$

we get $$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

i.e., $$x_1 = x_2 = x_3 = k$$

where k is a constant different from zero, putting k =1, we get the simplest solution that effects the eigen vector of A.

$$\therefore\ \lambda_3 = [1\,1\,1]^T.$$

Example 13.6 *If $a + b + c = 0$; find the characteristic roots of the matrix*

$$A = \begin{bmatrix} a & c & b \\ c & b & a \\ b & a & c \end{bmatrix}$$

Solution The characteristic matrix of A is

$$\mathrm{A} - \lambda \mathrm{I} = \begin{bmatrix} a-\lambda & c & b \\ c & b-\lambda & a \\ b & a & c-\lambda \end{bmatrix}$$

applying $c_1 \to c_1 + c_2 + c_3$

$$\sim \begin{bmatrix} -\lambda & c & b \\ -\lambda & b-\lambda & a \\ -\lambda & a & c-\lambda \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} -\lambda & c & b \\ 0 & b-\lambda-c & a-b \\ 0 & a-c & c-b-\lambda \end{bmatrix}$$

$\therefore$ The characteristic Equation of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & c & b \\ 0 & b-\lambda-c & a-b \\ 0 & a-c & c-b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-\lambda)\left|(b-c-\lambda)(c-b-\lambda) - (a-b)(a-c)\right| = 0$$

$$\Rightarrow \lambda \left| -[(b-c) - \lambda] (b-c+\lambda) - a^2 + ac + ab - bc \right| = 0$$

$$\Rightarrow \lambda \left| -\left[(b-c)^2 - \lambda^2\right] - a^2 + ac + ab - bc \right| = 0$$

$$\Rightarrow \lambda \left| \lambda^2 - b^2 - c^2 + 2bc - a^2 + ac + ab - bc \right| = 0$$

$$\Rightarrow \lambda \left| \lambda^2 - (a^2 + b^2 + c^2) + ab + bc + ca \right| = 0 \qquad ...(1)$$

but $\qquad a + b + c = 0 \Rightarrow (a+b+c)^2 = 0$

$$\Rightarrow \quad a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$$

$$\Rightarrow \quad (ab + bc + ca) = -\frac{(a^2 + b^2 + c^2)}{2} \qquad ...(2)$$

from (1) and (2), we get

$$\lambda\,(\lambda^2 - (a^2 + b^2 + c^2) - \frac{(a^2 + b^2 + c^2)}{2} = 0$$

$$\Rightarrow \quad \lambda\left(\lambda^2 - \frac{3(a^2 + b^2 + c^2)}{2}\right) = 0$$

$\therefore$ The characteristic roots of A are

$$\lambda = 0, \quad \lambda = \pm\sqrt{\frac{3\,(a^2 + b^2 + c^2)}{2}}$$

13.2 METHOD FOR THE LARGEST EIGEN VALUE (POWER METHOD)

Let A be a given matrix whose (largest) eigen value is to be determined and X_0 be an arbitrary vector. We use X_0 as the initial approximation to an eigen value of the matrix A.

Suppose we "normalize" the vector X_0; by requiring that one component say the last by unity. Compute the sequence

$$\begin{aligned} AX_0 &= \lambda_1 x_1 \\ AX_1 &= \lambda_2 x_2 \\ AX_2 &= \lambda_3 x_3 \\ \cdots & \quad \cdots \\ AX_{i-1} &= \lambda_i\, x_i \\ \cdots & \quad \cdots \end{aligned}$$

In this sequence; all the vectors X_1, X_2, ... are to be normalized in whatever manner was chosen originally. The iterative procedure converges, and we get a relation of this form

$$AX = \lambda X$$

where $$X = \lim x_i,\ \lambda = \lim \lambda_i$$

If the eigen value of A is real and unrepeated the above process will converge to give the largest eigen value of the matrix A.

Example 1 *Find the largest eigen value for the matrix*

$$\begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

also find the eigen vector corresponding to the largest eigen vector.

Solution

Let $$A = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

and $$X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ (Initial approximation of the eigen vector)}$$

$$AX_0 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 13 \\ 9 \\ 3 \end{bmatrix}$$

$$= 13 \begin{bmatrix} 1.0 \\ 0.62930 \\ 0.23076 \end{bmatrix} = 13X_1 \text{ where}$$

$$X_1 = \begin{bmatrix} 1.0 \\ 0.62930 \\ 0.23076 \end{bmatrix}$$

$$AX_1 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.62930 \\ 0.23076 \end{bmatrix}$$

$$= \begin{bmatrix} 12.5384 \\ 6.07692 \\ 1.30796 \end{bmatrix} = 12.5384 \begin{bmatrix} 1.0 \\ 0.48466 \\ 0.10429 \end{bmatrix}$$

$$= 12.5384\ x_2 \text{ where } x_2 = \begin{bmatrix} 1.0 \\ 0.48466 \\ 0.10429 \end{bmatrix}$$

$$AX_2 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.48466 \\ 0.10429 \end{bmatrix}$$

$$= \begin{bmatrix} 11.83436 \\ 5.28220 \\ 0.55828 \end{bmatrix} = 11.834836 \begin{bmatrix} 1.0 \\ 0.44634 \\ 0.04717 \end{bmatrix}$$

$$= 11.834836\ x_3 \text{ where } x_3 = \begin{bmatrix} 1.0 \\ 0.44634 \\ 0.04717 \end{bmatrix}$$

$$AX_3 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.44634 \\ 0.04717 \end{bmatrix}$$

$$= \begin{bmatrix} 11.73821 \\ 5.03421 \\ 0.386210 \end{bmatrix} = 11.73821 \begin{bmatrix} 1.0 \\ 0.42887 \\ 0.032902 \end{bmatrix}$$

$$= 11.73821\ x_4$$

where $$X_4 = \begin{bmatrix} 1.0 \\ 0.42887 \\ 0.032902 \end{bmatrix}$$

$$AX_4 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.42887 \\ 0.032902 \end{bmatrix}$$

$$= \begin{bmatrix} 11.6826 \\ 4.95645 \\ 0.319524 \end{bmatrix} = 11.6826 \begin{bmatrix} 1.0 \\ 0.424259 \\ 0.027350 \end{bmatrix}$$

$$= 11.6826\ x_5 \text{ where } x_5 = \begin{bmatrix} 1.0 \\ 0.424259 \\ 0.027350 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.424259 \\ 0.027350 \end{bmatrix} = \begin{bmatrix} 11.66969 \\ 4.93057 \\ 0.300129 \end{bmatrix}$$

$$= 11.66969 \begin{bmatrix} 1.0 \\ 0.422510 \\ 0.025718 \end{bmatrix}$$

$$= 11.66969\ x_6 \text{ where } x_6 = \begin{bmatrix} 1.0 \\ 0.422510 \\ 0.025718 \end{bmatrix}$$

$$AX_6 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.422510 \\ 0.025718 \end{bmatrix}$$

$$= \begin{bmatrix} 11.6643 \\ 4.92217 \\ 0.29325 \end{bmatrix} = 11.6643 \begin{bmatrix} 1.0 \\ 0.42198 \\ 0.02514 \end{bmatrix}$$

$$= 11.6643\ x_7 \text{ where } x_7 = \begin{bmatrix} 1.0 \\ 0.42198 \\ 0.02514 \end{bmatrix}$$

$$AX_7 = \begin{bmatrix} 10 & 4 & -1 \\ 4 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.42198 \\ 0.02514 \end{bmatrix}$$

$$= \begin{bmatrix} 11.6628 \\ 4.91939 \\ 0.291099 \end{bmatrix} = 11.6628 \begin{bmatrix} 1.0 \\ 0.42180 \\ 0.02495 \end{bmatrix}$$

$$= 11.6628\ x_7 \text{ where } X_8 = \begin{bmatrix} 1.0 \\ 0.42180 \\ 0.02495 \end{bmatrix}$$

$\therefore$ The largest eign value is 11.6628 and corresponding eigen vector is $\begin{bmatrix} 1.0 \\ 0.42180 \\ 0.02495 \end{bmatrix}$.

Example 2 *Find the largest eigen value and the corresponding eigen vector of the matrix.*

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \text{ taking } x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Solution We have

$$AX_0 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 . \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= 1.X_1, \quad \text{where } X_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$

$$AX_1 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = 7X_2$$

where

$$X_2 = \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix}$$

Then

$$AX_2 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.4 \\ 1.4 \\ 0 \end{bmatrix} = 3.4 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$$= 3.4X_3, \quad \text{where } X_3 = \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix}$$

$\therefore$

$$AX_3 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12X_4$$

where

$$X_4 = \begin{bmatrix} 1 \\ 0.49 \\ 0 \end{bmatrix}$$

Now

$$AX_4 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.49 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.94 \\ 1.98 \\ 0 \end{bmatrix}$$

$$= 3.94 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = 3.94X_5$$

where $$X_5 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$AX_5 = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$$= 4X_6, \text{ where } X_6 = \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$$

$\therefore$ The largest eigen value is 4 and the corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$.

13.3 CAYLEY-HAMILTON THEOREM

Theorem: Every square matrix A satisfies its own characteristic equation.

Proof: Let $$A = [a_{ij}]_{n\times n}$$

Then the characteristic matrix of A is $A - \lambda I$ and the cofactors of $|A - \lambda I|$ are of degree at most degree n–1. Therefore the highest power of λ in the polynomial of Adj $(A - \lambda I)$ is $n - 1$.

We can write

$$\text{Adj } (A - \lambda I) = B_0\lambda^{n-1} + B_1 \lambda^{n-2} + \ldots + B_{n-1} \qquad \text{...(1)}$$

where $B_0, B_1, \ldots, B_{n-1}$ are matrices of order n and whose elements are polynomials in the elements of A

$\therefore$ we have

$$(A - \lambda I) \text{ Adj} (A - \lambda I) = |A - \lambda I|\, I \qquad \text{...(2)}$$

or $$(A - \lambda I) \text{ Adj} (A - \lambda I) = \left[P_0\lambda^n + P_1\lambda^{n-1} + P_2\lambda^{n-2} + \ldots + P_n\right] I \quad ; \quad \text{say} \qquad \text{...(3)}$$

$$(\text{where } |A - \lambda I| = P_0\lambda^n + P_1\lambda^{n-1} + \ldots + P_n)$$

Using (1), we can write

$$(A - \lambda I)\,(B_0\lambda^{n-1} + B_1\lambda^{n-2} + \ldots + B_{n-1}) = (P_0\lambda^n + P_1\lambda^{n-1} + \ldots + P_n)\, I \qquad \text{...(4)}$$

(4) is an identity in λ : therefore equating the coefficients of like powers of λ from both sides we obtain

$$\begin{aligned} -B_0 &= P_0 I \\ AB_0 - B_1 &= P_1 I \\ AB_1 - B_2 &= P_2 I \\ &\;\;\vdots \\ AB_{r-1} - B_r &= P_r I \\ &\;\;\vdots \\ AB_{n-1} &= P_n I \end{aligned} \qquad \text{....(5)}$$

By pre-multiplying these equations with A^n, A^{n-1}, ..., A, I respectively and adding we have

$$0 = P_0A^n + P_1A^{n-1} + ... + P_nI \quad ...(6)$$

The matrix A satisfies its own characteristic equation.

Hence Proved.

Computation of the inverse of a non-singular Matrix

Cayley-Hamilton theorem can be used to compute the inverse of a non-singular matrix;

Let A be non-singular matrix of order. Then by Cayley-Hamilton theorem; we have

$$P_0A^n + P_1A^{n-1} + P_2A^{n-2} + ... + P_{n-1} A + P_nI = 0 \quad ...(i)$$

on multiplying (*i*) by A^{-1}; we obtain

$$P_0A^{n-1} + P_1A^{n-2} + P_2A^{n-3} + ... + P_{n-1} A + P_nI = 0.A^{-1} \quad ...(ii)$$

or $$P_0A^{n-1} + P_1A^{n-2} + ... + P_n A^{-1} = 0$$

i.e., $$P_nA^{-1} = -(P_0A^{n-1} + P_1A^{n-2} + ... + P_{n-1} I)$$

$$\Rightarrow \quad A^{-1} = -\frac{1}{P_n}(P_0A^{n-1} + P_1A^{n-2} + ... + P_{n-1}I) \quad ...(iii)$$

Thus; inverse of A can be evaluated by putting the values of A^{n-1}; A^{n-2}, ... in (iii)

Remark : We can also apply cayley-Hamilton theorem to find A^{-2}, A^{-3},...

Example 13.7 *Verify cayley-Hamilton theorem for the matrix*

$$\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$$

Also; find the inverse of the matrix A.

Solution Let $$A = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$$

then the characteristic equation of A is

$$|A - \lambda I| = 0$$

i.e., $$\begin{vmatrix} 5-\lambda & 6 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

i.e., $$(5-\lambda)(2-\lambda) - 6 = 0$$

$$\Rightarrow \quad \lambda^2 - 7\lambda + 4 = 0 \quad ...(1)$$

we have to show that A satisfies (1)

now $$A^2 = \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 31 & 42 \\ 7 & 10 \end{bmatrix}$$

$\therefore$ $A^2 - 7A + 4I$

$$= \begin{bmatrix} 31 & 42 \\ 7 & 10 \end{bmatrix} - 7\begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 31-35+4 & 42-42+0 \\ 7-7+0 & 10-14+4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore \quad A^2 - 7A + 4I = 0 \qquad$ ...(2)

Hence, the matrix A satisfies its own characteristic equation.

Cayley-Hamilton theorem is verified

from (2), we have

$$A^2 - 7A + 4I = 0$$

$$\Rightarrow \quad A^{-1}(A^2 - 7A + 4I) = 0$$

$$\Rightarrow \quad A - 7I + 4IA^{-1} = 0$$

$$\Rightarrow \quad 4A^{-1} = 7I - A = 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 1 & 2 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 2 & -6 \\ -1 & 5 \end{bmatrix}$$

Example 13.8 *Show that the matrix*

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Satisfies its own characteristic equation and find A^{-1}.

Solution The characteristic equation of A is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (1-\lambda)\begin{vmatrix} -1 & -\lambda & 0 \\ 0 & -1 & -\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 0 \\ 0 & -1-\lambda \end{vmatrix} + 0 = 0$$

$$\Rightarrow \quad (1-\lambda)(1+\lambda)^2 + 4(1+\lambda) = 0$$

$$\Rightarrow \quad -\lambda^3 - \lambda^2 + 5\lambda + 5 = 0$$

$$\Rightarrow \quad x^3 + \lambda^2 - 5\lambda - 5 = 0 \qquad \text{...(1)}$$

We have to show that A satisfies the equation (1); *i.e.,* $A^3 + A^2 - 5A - 5I = 0$

now $\quad A^2 = A \,.\, A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A^3 = A^2 . A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore\ A^3 + A^2 - 5A - 5I$

$$= \begin{bmatrix} 5 & 10 & 0 \\ 10 & -5 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5+5-5-5 & 10+0-10+0 & 0+0+0+0 \\ 10+0-10+0 & -5+5+5-5 & 0+0+0+0 \\ 0+0+0+0 & 0+0+0+0 & -1+1+5-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ A satisfies its own characteristic equation

i.e., $\quad A^3 + A^2 - 5A - 5I = 0$

Consider $\quad A^3 + A^2 - 5A - 5I = 0$

$\Rightarrow \quad A^{-1}(A^3 + A^2 - 5A - 5I) = 0$

$\Rightarrow \quad A^2 + A - 5I - 5A^{-1} = 0$

$\Rightarrow \quad 5A^{-1} = 5 - A - A^2$

$$\therefore\ 5A^{-1} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Exercise 13.1

1. Determine the eigen values and the corresponding eigen vectors for the matrices

$$(a)\begin{bmatrix}2 & \sqrt{2}\\ \sqrt{2} & 1\end{bmatrix}\qquad (b)\begin{bmatrix}1 & 4\\ 3 & 2\end{bmatrix}\qquad (c)\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$$

2. Find the characteristic roots and the corresponding characteristic vectors of the matrices

$$(a)\begin{bmatrix}8 & -6 & 2\\ -6 & 7 & -4\\ 2 & -4 & 3\end{bmatrix}\qquad (b)\begin{bmatrix}1 & 2 & 3\\ 0 & -4 & 2\\ 0 & 0 & 7\end{bmatrix}\qquad (c)\begin{bmatrix}3 & -1 & 3\\ -1 & 2 & -1\\ 0 & -1 & 3\end{bmatrix}$$

3. Find the largest eigen values and the corresponding eigen vector of the matrix

$$A=\begin{bmatrix}-15 & 4 & 3\\ 10 & -12 & 6\\ 20 & -4 & 2\end{bmatrix}$$

4. Determine the largest eigen value and the corresponding eigen vector of the matrix

$$A=\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 2\end{bmatrix}$$

5. Find the latent roots and the characteristic vectors of the matrices

$$(i)\begin{bmatrix}2 & 0 & 1\\ 0 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}\quad ii)\ A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}\quad (iii)\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 3\\ 0 & 0 & 2\end{bmatrix}$$

6. Find the latent roots and latest vectors of the matrix

$$A=\begin{bmatrix}a & h & g\\ o & b & o\\ o & o & c\end{bmatrix}$$

7. Verify Cayley-Hamilton's theorem for the matrix

$$A=\begin{bmatrix}0 & 0 & 1\\ 3 & 1 & 0\\ -2 & 1 & 4\end{bmatrix}$$

8. Show that the matrix A, satisfies the matrix equation $A^2-4A-5I=0$

$$\text{where }A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$$

9. Show that the matrix $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$ satisfies Cayley-Hamilton's theorem

10. Verify that the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ satisifies Cayley-Hamilton theorem.

Answers

1. (*a*) $\lambda_1 = 0, \lambda_2 = 3, \left[\frac{-1}{\sqrt{3}}\ \frac{\sqrt{2}}{3}\right]^T$ and $\left[\frac{\sqrt{2}}{\sqrt{3}}\ \frac{1}{\sqrt{3}}\right]^T$

(*b*) $\lambda_1 = -2, \lambda_2 = 5, \left[\frac{-4}{3}\ 1\right]^T$ and $[1\ 1]^T$

(*c*) $5.38, \begin{bmatrix} 0.46 \\ 1 \end{bmatrix}$

2. (*a*) $\lambda_1 = 0, \lambda_2 = 3, \lambda_3 = 15\left[\frac{1}{2}\ 1\ 1\right]^T \left[-1\ \frac{-1}{2}\ 1\right]^T, \left[2 - 2\ 1\right]^T$

(*b*) $[1\ -4\ 7]^T$

(*c*) $\lambda_1 = 1, \lambda_2 = 3, \lambda_3 = 4, [1\ 2\ 1]^T, [1\ 0\ -1]^T, [1 -1\ 1]^T$

3. $[1\ -0.5\ -1]^T$

4. $3.41[0.74\ -1\ 0.67]^T$

5. (*i*) 1, 2, 3, $[1\ 0\ -1]^T$, $[0\ 1\ 0]^T$, $[1\ 0\ 1]^T$

(*ii*) 8, 2, 2, $[2\ -1\ 1]^T$, $[1\ 0\ -2]^T$, $[1\ 2\ 0]^T$.

(*iii*) $[1\ 0\ 0]^T$, $[2\ 10]^T$

6. $[k,\ 0\ 0]^T$, $[k_2 h\ \ k_2(b - a)0]^T$

$[k_3\ g\ 0\ \ k_3(c - a)]^T$

14

REGRESSION ANALYSIS

14.1 REGRESSION ANALYSIS

In this chapter we discuss regression which measures the nature and extent of correlation. Regression methods are meant to determine the best functional relationship between a dependent variable y with one or more related variable (or variables) x. The functional relationship of a dependent variable with one or more independent variables is called a regression equation.

14.2 CORRELATION

Correlation is a statistical measure for finding out the degree of association between two variables with the help of correlation we study the relationship between variables.

Definition 14.1. If two or more quantities varies in other sympathy so that movements in the one tends to be accompanied by corresponding movements in the other, then they are said to be correlated.

Types of correlation: Correlation may be

(*i*) Positive or negative

(*ii*) Simple or partial or multiple

(*iii*) Linear or non-linear.

14.3 COEFFICIENT OF CORRELATION (*r*)

Coefficient of correlation is a measure of degree or extent of linear relationship between two variables x and y. It is denoted by r.

14.4 SCATTER DIAGRAM

When the pair of values (x_1, y_1), (x_2, y_2), (x_3, y_3),... (x_n, y_n) are plotted on a graph paper, the points show the pattern in which they lie, such a diagram is called a scatter diagram.

Consider the points (x_1, y_1), (x_2, y_2),.... (x_n, y_n). In scatter diagram the variable x is shown along the x-axis (horizontal axis) and the variable y is shown along the y-axis (vertical axis) and all the pairs of values of x and y are shown by points (or dots) on the graph paper. The scatter diagram of these points reveals the nature and strength of correlation between these variable x and y. We observe the following.

If the points plotted lie on a straight line rising from lower left to upper right, then there is a perfect positive correlation between the variables x and y (Fig. 1(*a*)). If all the points do not lie on a straight line, but their tendency is to rise from lower left to upper right then there is a positive correlation between the variable x and y (Fig. 1(*b*)). In these cases the two variables x and y are in the same direction and the association between the variables is direct.

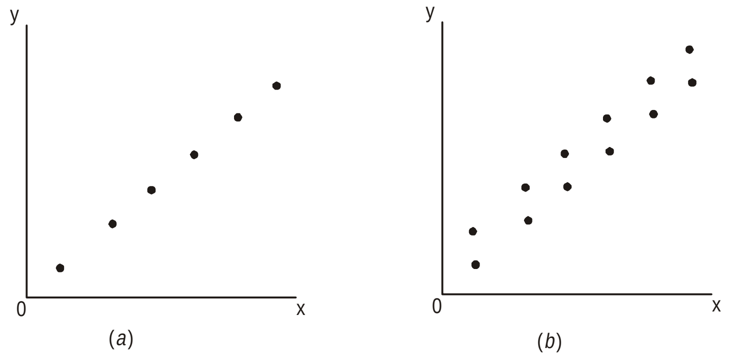


Fig. 1. (*a*) Perfect positive correlation (r = 1) (*b*) Positive correlation.

If the movements of the variables x and y are opposite in direction and the scatter diagram is a straight line, the correlation is said to be negative *i.e.*, association between the variables is said to be indirect.

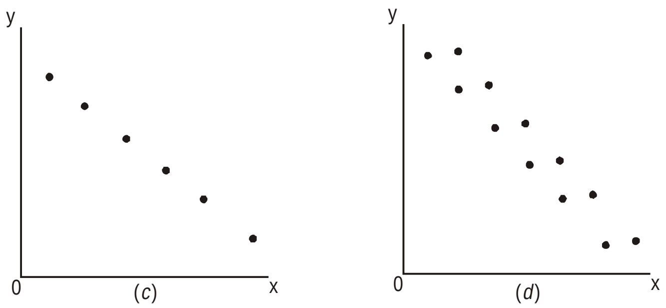
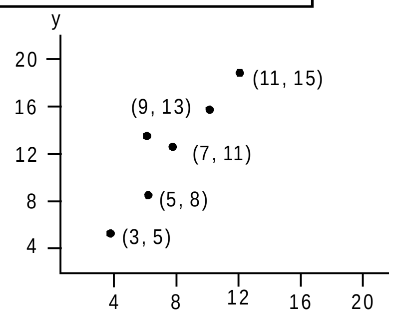


Fig. 1. (*c*) Perfect negative correlation (*d*) Negative correlation.

Example. *Draw a Scatter diagram for the following data*

x	*3*	*5*	*7*	*9*	*11*	*13*	*15*
y	*5*	*8*	*11*	*13*	*15*	*17*	*19*



14.5 CALCULATION OF *r* (CORRELATION COEFFICIENT) (KARL PEARSON'S FORMULA)

If (x_1, y_1), (x_2, y_2), ... (x_n, y_n) be n paired observation; then

$$r = \frac{\Sigma[(x_i - \bar{x})(y_i - \bar{y})]}{n\sigma_x \sigma_y}$$

or simply

$$r = \frac{\Sigma[(x_i - \bar{x})(y_i - \bar{y})]}{\sqrt{\Sigma(x - \bar{x})^2}\sqrt{\Sigma(y - \bar{y})^2}}$$

where

σ_x = standard deviation of $x_1, x_2, ..., x_n$

σ_y = standard deviation of $y_1, y_2, ..., y_n$

$$\bar{x} = \frac{\Sigma x_i}{n}, \ \bar{y} = \frac{\Sigma y_i}{n}$$

and

$$\sigma_x = \sqrt{\frac{\Sigma x_i^2}{n}}, \quad \sigma_i = \sqrt{\frac{\Sigma y_i^2}{n}}$$

If $x_i = x_i - \bar{x}$ and $y_i = y_i - \bar{y}$ then

$$r = \frac{\Sigma\, x_i\, y_i}{n\, \sigma_x \sigma_y}$$

$$= \frac{\Sigma\, x_i\, y_i}{\sqrt{\Sigma\, x_i^2}\sqrt{\Sigma\, y_i^2}}$$

If A and B denote the assumed means then

$$r = \frac{\Sigma x_i y_i - \dfrac{(\Sigma x_i)(\Sigma y_i)}{n}}{\sqrt{\Sigma x_i^2 - \dfrac{(\Sigma x_i)^2}{n}}\sqrt{\Sigma y_i^2 - \dfrac{(\Sigma y_i)^2}{n}}}$$

Karl Pearson's formula; is a direct method of computing r. It can be proved mathematically that $-1 \le r \le 1$. The Karl Pearson's coefficient of correlation r; is also denoted by P (rho) and is also called Karl Pearsons moment correlation coefficient.

14.6 REGRESSION

Correlation methods are used to know, how two or more variables are interrelated. Correlation; cannot be used to estimate or predict the most likely values of one variable for specified values of the other variable. The terms 'Regression' was coined by Sir Francis Galton (while studying the linear relation between two variables).

Definition: 14.2. Regression is the measure of the average relationship between two or more variables in terms of the original units of data.

14.7 REGRESSION EQUATION

The functional relationship of a dependent variable with one or more independent variables is called a regression equation: It is also called prediction equation (or estimating equation).

14.8 CURVE OF REGRESSION

The graph of the regression equation is called the curve of regression: If the curve is a straight line; then it is called the line of regression.

14.9 TYPES OF REGRESSION

If there are only two variables under consideration; then the regression is called simply regression

Example. *(i) Study of regression between heights and age for a group of persons (ii) The study of regression between 'income' and expenditure for a group of persons.*

In this case the relationship is linear.

If there are more than two variables under considerations then the regression is called multiple regression.

If there are more than two variables under considerations and relation between only two variables is established, after excluding the effect of the remaining variables, then the regression is called partial regression.

If the relationship between x and y is nonlinear, then the regression is curvilinear regression. In some cases polynomials are selected to predict or estimate; which is called polynomial regression.

14.10 REGRESSION EQUATIONS (LINEAR FIT)

14.10.1 Linear Regression Equation of y on x

In linear regression if we fit a straight line of the form $y = a + bx$ to the given data by the method of least squares, we obtain the regression of y on x:

Let $(x_1\ y_1), (x_2, y_2), \ldots, (x_n, y_n)$ denote n pairs of observations and let the corresponding straight line to be fitted, to these data points be

$$y = a + bx \qquad \ldots(1)$$

applying the method of least squares, we get the following normal equations:

$$na + b\ \Sigma x_i = \Sigma y_i \qquad \ldots(2)$$

$$a\ \Sigma x_i + b\ \Sigma x_i^2 = \Sigma x_i y_i \qquad \ldots(3)$$

dividing equation (2) by n, we get

$$a + b\frac{\Sigma x_i}{n} = \frac{\Sigma y_i}{n}$$

or $$a + b.\bar{x} = \bar{y} \qquad ...(4)$$

Subtracting (4) from (1), we obtain

$$y - \bar{y} = b(x - \bar{x}) \qquad ...(5)$$

multiplying (2) by Σx_i and (3) by n and then subtracting, we get

$$\Sigma x_i y_i - n\,\Sigma x_i y_i = b(\Sigma x_i)^2 - nb\Sigma x_i^2$$

or $$b[n\Sigma x_i^2 - (\Sigma x_i)^2] = n\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)$$

or $$b = \frac{n\,\Sigma x_i y_i - \Sigma x_i \Sigma y_i}{n\Sigma x_i^2 - (\Sigma x_i)^2}$$

or $$b = \frac{\dfrac{\Sigma x_i y_i}{n} - \dfrac{\Sigma x_i}{n}.\dfrac{\Sigma y_i}{n}}{\dfrac{\Sigma x_i^2}{n} - \dfrac{(\Sigma x_i)^2}{n}} \qquad ...(6)$$

replacing b by b_{yx} in (5) we get

$$y - \bar{y} = b_{yx}(x - \bar{x}) \qquad ...(7)$$

where $$b_{yx} = \frac{\dfrac{\Sigma x_i y_i}{n} - \dfrac{\Sigma x_i}{n}.\dfrac{\Sigma y_i}{n}}{\dfrac{\Sigma x_i^2}{n} - \dfrac{(\Sigma x_i)^2}{n}}$$

Equation (7) is called regression equation of y on x, and is used to estimate the values of y for given values of n.

b_{yx} is also given by

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

and it is called the regression coefficient of y or x.

14.10.2 Regression equation of *x* on *y*

It is the best equation of best filted straight line of the type

$$x = a' + b'y$$

to the given data.

Applying the principle of least squares, we get the following two normal equations.

$$na' + b'\Sigma y = \Sigma x \qquad ...(i)$$

$$a'\Sigma y + b'\Sigma y^2 = \Sigma xy \qquad ...(ii)$$

Solving (i) and (ii) for a' and b' and proceeding as before, we obtain the regression equation of x on y as follows

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

where $$b_{xy} = \frac{\dfrac{\Sigma x_i y_i}{n} - \dfrac{\Sigma x_i}{n} \cdot \dfrac{\Sigma y_i}{n}}{\dfrac{\Sigma y_i^2}{n} - \dfrac{(\Sigma y)^2}{n}} \qquad \ldots(iii)$$

b_{xy} is also given by $$b_{yx} = r\frac{\sigma_x}{\sigma_y}$$

and is called the regression coefficient of x on y.

Since $$b_{yx} = r\frac{\sigma_y}{\sigma_x} \text{ and } b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

we have $$b_{yx} \cdot b_{xy} = r^2$$

Note.

(1) If $x_i = x_i - \bar{x}$, $y_i = y_i - \bar{y}$; then

$$b_{yx} = \frac{\dfrac{\Sigma x_i y_i}{n}}{\dfrac{\Sigma x_i^2}{n}} = \frac{\Sigma x_i y_i}{\Sigma x_i^2} \qquad (\because \Sigma x_i = 0, \Sigma y_i = 0)$$

Similarly $$b_{xy} = \frac{\Sigma x_i y_i}{\Sigma y_i^2}$$

The two regression equation lines

$$y - \bar{y} = b_{yx}(x - \bar{x}) \qquad \ldots(iv)$$

and $$x - \bar{x} = b_{xy}(y - \bar{y}) \qquad \ldots(v)$$

are identical if $b_{yx} \times b_{xy} = 1$

or $$b_{yx} = \frac{1}{b_{xy}} \text{ or } r^2 = 1$$

i.e. the lines (*iv*) and (*v*) are identical if

$$r^2 = 1$$

i.e. $$r = \pm 1$$

(2) The two regression lines always intersect at $(\bar{x}, \bar{y})$

14.11 ANGLE BETWEEN TWO LINES OF REGRESSION

Consider the regression lines

$$y - \bar{y} = b_{yx}\,(x - \bar{x}) \qquad ...(1)$$

and

$$(x - \bar{x}) = b_{xy}\,(y - \bar{y}) \qquad ...(2)$$

equation (2) can be written as

$$y - \bar{y} = \frac{1}{b_{xy}}\,(x - \bar{x}) \qquad ...(3)$$

Let θ be the angle between the regression lines, then, the slopes of the lines (1) and (3) are

$$m_1 = b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

and

$$m_2 = \frac{1}{b_{xy}} = \frac{\sigma_y}{r\sigma_x}$$

we have

$$\tan\theta = \pm\frac{m_2 - m_1}{1 + m_2 m_1} = \pm\frac{\dfrac{\sigma_y}{r\sigma_x} - \dfrac{r\sigma_y}{\sigma_x}}{1 + \dfrac{\sigma_y}{r\sigma_x}\,\dfrac{r\sigma_x}{\sigma_y}} = \pm\frac{\dfrac{\sigma_y}{\sigma_x}\left(\dfrac{1}{r} - r\right)}{1 + \dfrac{\sigma_y^2}{\sigma_x^2}}$$

$$= \pm\left(\frac{1-r^2}{r}\right)\frac{\sigma_y}{\sigma_x}\cdot\frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2} = \pm\left[\left(\frac{1-r^2}{r}\right)\frac{\sigma_x\,\sigma_y}{\sigma_x^2 + \sigma_y^2}\right]$$

Since $r^2 \le 1$ and σ_x, σ_y are positive, the positive sign gives the acute angle between the lines and the negative sign gives the obtuse angle between the lines.

If θ_1 denotes acute angle and θ_2 denotes the obtuse angle between the regression lines, then

$$\theta_1 = \tan^{-1}\left[\frac{1-r^2}{r}\cdot\frac{\sigma_x\,\sigma_y}{\sigma_x^2 + \sigma_y^2}\right]$$

and

$$\theta_2 = \tan^{-1}\left[\frac{r^2-1}{r}\cdot\frac{\sigma_x\,\sigma_y}{\sigma_x^2 + \sigma_y^2}\right]$$

If $r = 0$; then $\tan\theta = \infty$ and $\theta = \dfrac{\pi}{2}$ $\therefore$ in this case x and y are uncorrelated and lines of regression are perpendicular to each other.

If $r = \pm 1$, then $\tan\theta = 0$, and $\theta = 0$ or π; in this case there is a perfect correlation (positive or negative) between x and y. The two lines of regression coincide, but are not parallel since the lines pass through the point $(\bar{x}, \bar{y})$.

Note: If $r = 0$, then from the equation of lines of regression we have

$$(y, \bar{y}) = r\frac{\sigma_y}{\sigma_x}(x - \bar{x}) = 0$$

i.e.,
$$(y, \bar{y}) = 0 \text{ or } y = \bar{y}$$

and
$$x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y}) = 0$$

i.e.,
$$x - \bar{x} = 0 \text{ or } x = \bar{x}$$

$\therefore$ when $r = 0$, the equations of lines of regression are $x = \bar{x}$ and $y = \bar{y}$ which are the equations of the lines parallel to the axis.

14.12 SOLVED EXAMPLES

Example. 14.1 *For the following data, find the regression line (by applying the method of least squares)*

x	5	10	15	20	25
y	20	40	30	60	50

Solution: We have

x	y	x^2	y^2	xy
5	20	25	400	100
10	40	100	1600	400
15	30	225	900	450
20	60	400	3600	1200
25	50	625	2500	1250
75	200	1375	9000	3400

$\therefore\ \Sigma x_i = 75,\ \Sigma y_i = 200,\ \Sigma x_i^2 = 1375,\ \Sigma y_i^2 = 9000,\ \Sigma x_i y_i = 3400$

Regression of y on x:

The normal equations are

$$na + b\Sigma x_i = \Sigma y_i$$

$$\Sigma x_i y_i + b\Sigma x_i^2 = \Sigma x_i y_i$$

i.e.,
$$5a + 75b = 200 \quad ...(1)$$
$$75a + 1375b = 3400 \quad ...(2)$$

Solving these equations we get

$a = 16$, and $b = 1.6$

$\therefore$ The regression equation of y or x is

$$y = 16 + 1.6x$$

Regression of equation x on y:

The normal equations are

$$na + b\Sigma y_i = \Sigma x_i$$

$$a\Sigma y_i + b\Sigma y_i^2 = \Sigma x_i y_i$$

i.e., $5a + 200b = 75$ ...(3)

$200a + 9000b = 3400$ ...(4)

Solving equations (3) and (4) we get

$$a = -1 \text{ and; } b = 0.4$$

$\therefore$ The regression equation of x on y is

$$x = -1 + 0.4y$$

$\therefore$ The regression equations are

$$y = 16 + 1.6x$$

and $$x = -1 + 0.4y$$

Example 14.2 *For the following data find the regression line of y on x*

x	*1*	*2*	*3*	*4*	*5*	*8*	*10*
y	*9*	*8*	*10*	*12*	*14*	*16*	*15*

Solution. We have

x_i	y_i	$x_i y_i$	x_i^2
1	9	9	1
2	8	16	4
3	10	30	9
4	12	48	16
5	14	70	25
8	16	128	64
10	15	150	100

Total : $\Sigma x_i = 33$ $\Sigma y_i = 84$ $\Sigma x_i y_i = 451$ $\Sigma x_i^2 = 219$

$$n = 7$$

$$\therefore \quad \bar{x} = \frac{\Sigma x_i}{n} = \frac{33}{7} = 4.714$$

$$\bar{y} = \frac{\Sigma y_i}{n} = \frac{84}{7} = 12$$

and $$b_{yx} = \frac{n\Sigma x_i y_i - (\Sigma x_i)(\Sigma y_i)}{n\Sigma x_i^2 - (\Sigma x_i)^2} = \frac{7.(451) - (33)(84)}{7(2119) - (33)^2}$$

$$= 0.867$$

The regression equation of y on x is:

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

i.e., $$y - 12 = 0.867(x - 4.714)$$

or $$y = 0.867x + 7.9129$$

Example. 14.3 *From the following data, fit two regression equations by find actual means (of x and y.) i.e. by actual means method.*

x	*1*	*2*	*3*	*4*	*5*	*6*	*7*
y	*2*	*4*	*7*	*6*	*5*	*6*	*5*

Solution. We change the origin and find the regression equations as follows:

we have
$$\bar{x} = \frac{\Sigma x_i}{n} = \frac{1+2+3+4+5+6+7}{7} = \frac{28}{7} = 4$$

$$\bar{y} = \frac{\Sigma y_i}{n} = \frac{2+4+7+6+5+6+5}{7} = \frac{35}{7} = 5$$

x	y	$X = x - \bar{x}$	$Y = y - \bar{y}$	X^2	Y^2	XY
1	2	–3	–3	9	9	9
2	4	–2	–1	4	1	2
3	7	–1	2	1	4	–2
4	6	0	1	0	1	0
5	5	1	0	1	0	0
6	6	2	1	4	1	2
7	5	3	0	9	0	0
Totals 28	35	0	0	28	16	11

∴ we have $\Sigma x_i = 28$, $\Sigma y_i = 35$, $\Sigma X_i = 0$, $\Sigma Y_i = 0$; $\Sigma X_i^2 = 28$, $\Sigma Y_i^2 = 16$, $\Sigma X_i Y_i = 11$

$$b_{yx} = \frac{\Sigma X_i Y_i}{\Sigma X_i^2} = \frac{11}{28} = 0.3928 = 0.393 \text{ (approximately)}$$

and
$$b_{xy} = \frac{\Sigma X_i Y_i}{\Sigma X_i^2} = \frac{11}{28} = 0.6875 = 0.688 \text{ (approximately)}$$

∴ The regression equation of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

i.e.
$$y - 5 = 0.393\ (x - 4)$$

or
$$y = 0.393x + 3.428$$

and the regression equation of x on y is

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

i.e.
$$x - 4 = 0.688\ (y - 5)$$

or
$$x = 0.688\ y + 0.56$$

∴ The required regression equations are

$$y = 0.393x + 3.428$$

and
$$x = 0.688y + 0.56$$

Example 14.4. *From the following results obtain the two regression equations and estimate the yield of crops when the rainfall is 29 cms. and the rainfall when the yield is 600 Kg.*

Mean	*y (yield in kgs.)*	*(Rainfall in cms.)*
	508.4	*26.7*
S.D.	*36.8*	*4.6*

Coefficient of correlation between yield and rain fall in 0.52.

Solution. We have

$$\bar{x} = 26.7, \qquad \bar{y} = 508.4$$

$$\sigma_x = 4.6, \qquad \sigma_y = 36.8$$

and $$r = 0.52$$

$\therefore$ $$b_{yx} = r\frac{\sigma_y}{\sigma_x} = (0.52)\,\frac{36.8}{4.6} = 4.16$$

and $$b_{xy} = r\frac{\sigma_x}{\sigma_y} = (0.52)\,\frac{4.6}{36.8} = 0.065$$

Regression equation of y on x

$$y - \bar{y} = b_{yx}(x - \bar{x})$$

i.e., $$y - 508.4 = 4.16\ (x - 26.7)$$

or $$y = 397.328 + 4.16x$$

when $x = 29$, we have $$y = 397.328 + 4.16\ (29) = 517.968 \text{ kgs.}$$

Regression equation of x on y

$$x - \bar{x} = b_{xy}(y - \bar{y})$$

or $$x - 26.7 = -\ 0.065\ (y - 508.4)$$

or $$x = -\ 6.346 + 0.065x$$

when $y = 600$ kg, $$x = -\ 6.346 - 0.065 \times 600$$

$$= 32.654 \text{ cms}$$

$\therefore$ The regression equations are:

$$y = 397.328 + 4.16\ x$$

and $$x = -\ 6.346 + 0.065\ y$$

When the rain fall is 29 cms the yield of crops is 517.968 kg and when the yield is 600 kg the temperature is 32.654 cms.

Example 14.5. *Find the most likely price of a commodity in Bombay corresponding to the price of Rs.70. at Calcutta from the following*

	Calcutta	*Bombay*
Average price	*65*	*67*
Standard deviation	*2.5*	*3.5*

Correlation coefficient between the price of commodity in the two cities is 0.8.

Solution. We have $$\bar{x} = 65, \qquad \bar{y} = 67$$

$$\sigma_x = 2.5, \qquad \sigma_y = 3.5 \text{ and } r = 0.8$$

$\therefore$ $$y - \bar{y} = r\,\frac{\sigma_y}{\sigma_x}\,(x - \bar{x})$$

$\Rightarrow$ $$y - 67 = (0.8)\,.\left(\frac{3.5}{2.5}\right)(x - 65)$$

$$\Rightarrow \quad y = 67 + 1.12x - 72.8$$

$$\Rightarrow \quad y = -1.12x$$

$$\Rightarrow \quad y = -5.8 + 1.12x$$

When $x = 70;$

$$y = -5.8 + 1.12 \times 70 = -5.8 + 78.4$$

$$\Rightarrow \quad y = 72.60$$

$\therefore$ The price of the commodity in Bombay corresponding to Rs. 70 at Calcutta is 72.60.

Example 14.6. *The regression equation calculated from a given set of observation*

$$x = -0.4y + 6.4$$

and $y = -0.6x + 4.6$

Calculate $\bar{x}, \bar{y}$ *and* r_{xy}.

Solution. We have $x = -0.4y + 6.4$ ...(1)

and $y = -0.6x + 4.6$ ...(2)

From (2), we have $y = -0.6(-0.4y + 6.4) + 4.6$ (using (1))

$$\Rightarrow \quad y = 0.24y - 3.84 + 4.6$$

$$\Rightarrow \quad 0.76y = 0.76$$

$$\Rightarrow \quad y = 1$$

From (1) we have $y = -0.4 \times 1 + 6.4 = 6.0$

but $(\bar{x}, \bar{y})$ in the point of intersection of (1) and (2)

Hence $(\bar{x}, \bar{y}) = (1, 6)$

$$\therefore \quad \bar{x} = 1,$$

$$\bar{y} = 6$$

Clearly, equation (1) in the regression equation x or y and equation (2) is the regression equation y on x.

$\therefore$ We have $b_{xy} = -0.4$ and $b_{yx} = -0.6$

and $r^2 = (-0.4)(-0.6) = 0.24$

$$r = r_{xy} = \pm\sqrt{0.24}$$

Since b_{xy} and b_{yx} are both negative $r = r_{xy}$ is negative

$$r_{xy} = -\sqrt{0.24}$$

Example 14.7. *Show that the coefficient of correlation is the Geometric mean (G.M.) of the coefficients of regression.*

Solution. The coefficients of regression are $r\dfrac{\sigma_x}{\sigma_y}$ and $r\dfrac{\sigma_y}{\sigma_x}$

$\therefore$ Geometric mean of the regression coefficients is

$$\sqrt{r\frac{\sigma_x}{\sigma_y} \cdot r\frac{\sigma_y}{\sigma_x}} = \sqrt{r^2} = r = \text{coefficient of correlation.}$$

Example 14.8. *In a partially destroyed laboratory record of an analysis of correlation data; the following results only are legible:*

$$\textit{variance of } x = 9$$

Regression equations: $8x - 10y + 66 = 0,\ 40x - 18y = 214$

What were (a) *the mean values of x and y*

(b) *the standard deviation of y*

(c) *the coefficient of correlation between x and y.*

Solution. Variance of $x = 9$

i.e., $\sigma_x^2 = 9 \quad \Rightarrow \quad \sigma_x = 3$

Solving the regression equations

$$8x - 10y + 66 = 0 \qquad \text{...(1)}$$

$$40x - 18y = 214 \qquad \text{...(2)}$$

We obtain $x = 13,\ y = 17.$

Since the point of intersection of the regression lines is $(\bar{x}, \bar{y})$ we have

$$(\bar{x}, \bar{y}) = (x, y) = (13, 17)$$

$$\therefore \quad \bar{x} = 13, \quad \bar{y} = 17$$

The regression (1) and (2) can written as

$$y = 0.8x + 6.6 \qquad \text{...(3)}$$

and $$x = 0.45y + 5.35 \qquad \text{...(4)}$$

$\therefore$ The regression coefficient of y on x is

$$r\frac{\sigma_y}{\sigma_x} = 0.8 \qquad \text{...(5)}$$

and the regression coefficient of x on y is

$$r\frac{\sigma_x}{\sigma_y} = 0.45 \qquad \text{...(6)}$$

Multiplying (5) and (6), we get

$$r^2 = 0.45 \times 0.8$$

$$\Rightarrow \quad r^2 = 0.36$$

$$\Rightarrow \quad r = 0.6$$

$\therefore$ Putting the values of r and σ_x in (5), we get the value of σ_y as follows:

$$r\frac{\sigma_y}{\sigma_x} = 0.8$$

$$\Rightarrow \quad (0.6)\ \frac{\sigma_y}{3} = 0.8$$

$$\Rightarrow \quad \sigma_y = \frac{0.8}{0.2} = 4$$

Example 14.9. *If one of the regression coefficients is greater than unity. Show that the other regression coefficient is less than unity.*

Solution. Let one of regression coefficient; say $b_{yx} > 1$

Then $$b_{yx} > 1 \quad \Rightarrow \quad \frac{1}{b_{yx}} < 1$$

Since, $$b_{yx}\, b_{xy} = r^2 \leq 1$$

We have $$b_{xy} \leq \frac{1}{b_{yx}}$$

$$\Rightarrow \qquad b_{xy} < 1 \qquad \left(\because \frac{1}{b_{yx}} < 1\right)$$

Example 14.10. *Show that the arithmetic mean of the regression coefficients is greater than the correlation coefficient.*

Solution.

We have to show that $\dfrac{b_{yx} + b_{xy}}{2} > r$

Consider $(\sigma_y - \sigma_x)^2$

Clearly $$(\sigma_y - \sigma_x)^2 > 0$$

($\therefore$ Since square of two real qualities is always > 0)

$$\Rightarrow \qquad \sigma_y^2 + \sigma_x^2 - 2\sigma_y\sigma_x > 0$$

$$\Rightarrow \qquad \frac{\sigma_y^2}{\sigma_y\sigma_x} + \frac{\sigma_x^2}{\sigma_y\sigma_x} > 2$$

$$\Rightarrow \qquad \frac{\sigma_y}{\sigma_x} + \frac{\sigma_x}{\sigma_y} > 2$$

$$\Rightarrow \qquad r\frac{\sigma_y}{\sigma_x} + r\frac{\sigma_x}{\sigma_y} > 2r$$

$$\Rightarrow \qquad b_{yx} + b_{xy} > 2r$$

$$\Rightarrow \qquad \frac{b_{yx} + b_{xy}}{2} > r$$

Hence proved.

Example 14.11. *Given that $x = 4y + 5$ and $y = kx + 4$ are two lines of regression. Show that $0 \leq k \leq \frac{1}{4}$. If $k = \frac{1}{8}$ find the means of the variables, ratio of their variables.*

Solution.

$$x = 4y + 5 \Rightarrow b_{xy} = 4$$

$$y = kx + 4 \Rightarrow b_{yx} = k$$

$\therefore$ $$r^2 = b_{xy} \cdot b_{yx} = 4.k$$

but $-1 \le r \le 1$

$\Rightarrow$ $$0 \le r^2 \le 1$$

$\Rightarrow$ $$0 \le 4k \le 1$$

$\Rightarrow$ $$0 \le k \le \frac{1}{4}$$

When $k = \frac{1}{8}$; $$r^2 = 4 \cdot \frac{1}{8} = \frac{1}{2} \Rightarrow r = 0.7071$$

$\therefore$ $$y = kx + 4$$

$\Rightarrow$ $$y = \frac{1}{8}x + 4$$

$\Rightarrow$ $$8y = x + 32$$

$\Rightarrow$ $$8y = 4y + 5 + 32$$

$\Rightarrow$ $$4y = 37 \Rightarrow y = 9.25$$

Now $$x = 4y + 5 \Rightarrow x = 4(9.25) + 5$$

$\Rightarrow$ $$x = 42$$

$\therefore$ $$(x, y) = (\bar{x}, \bar{y}) \quad (\because \text{ the point of intersection is } (\bar{x}, \bar{y}))$$

$$= (42, 9.25)$$

i.e., $$\bar{x} = 42, \quad \bar{y} = 9.25$$

$\therefore$ $$\frac{b_{xy}}{b_{yx}} = \frac{r\frac{\sigma_x}{\sigma_y}}{r\frac{\sigma_y}{\sigma_x}} = \frac{\sigma_x . \sigma_x}{\sigma_y . \sigma_y} = \frac{4}{\left(\frac{1}{8}\right)} = 32$$

i.e., $$\frac{\sigma_x^2}{\sigma_y^2} = 32$$

$\therefore$ The ratio of the variances is 32 : 1.

14.13 MULTILINEAR LINEAR REGRESSION

In some cases, the value of a variate may not depend only on a single variable. It may happen that these are several variable; which when taken jointly, will serve as a satisfactory basis for estimating the desired variable. If $x_1, x_2, ..., x_k$ represent the independent variables, y' is the variable which is to be predicted, and represents the regression equation,

$$y' = a_0 + a_1x_1 + a_2x_2 + \ldots + a_kx_k$$

the unknown coefficients $a_0, a_1, \ldots, a_k$ will be estimateds by the method of least squares. To obtain the values of the variables; we have n sets of values of $(k + 1)$ variables. Geometrically, the problem is one of finding the equation of the plane which best fits in the sense of least squares a set of n points in $(k + 1)$ dimension. The normal equations are:

$$na_0 + a_1\Sigma x_1 + a_2\Sigma x_2 + \ldots + a_k\Sigma x_k = \Sigma y$$

$$a_0\Sigma x_1 + a_1\Sigma x_1^2 + a_2\Sigma x_1x_2 + \ldots + a_k\ \Sigma x_1x_k = \Sigma x_1y$$

$$\vdots \qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots$$

$$a_0\Sigma x_k + a_1\Sigma x_1x_k + \ldots + a_k\ \Sigma x_k^2 = \Sigma x_ky$$

If there are two independent variables say x_1 and x_2, the normal equations are

$$na_0 + a_1\Sigma x_1 + a_2\Sigma x_2 = \Sigma y$$

$$a_0\Sigma x_1 + a_1\Sigma x_1^2 + a_2\Sigma x_1x_2 = \Sigma x_1y$$

$$a_0\Sigma x_2 + a_1\Sigma x_1x_2 + a_2\Sigma x_2^2 = \Sigma x_2y$$

and the regression equation is

$$\Sigma y = a_0y + a_1\ \Sigma x_1y + a_2\ \Sigma xy_2$$

$$y = a_0 + a_1x_1 + a_2x_2$$

Example. 1 *From the table given below, find out*

(a) the least square regression equation of x_0 and x_1 and

(b) determine x_0 from the given values of x_1 and x_2 and

(c) find the values of x_0 when $x_1 = 54$ and $x_2 = 9$

given

x_0	64	71	53	67	55	58	77	57	56	51	76	68
x_1	57	59	49	62	51	50	55	48	52	42	61	57
x_2	8	10	6	11	8	7	10	9	6	6	12	9

Solution. (*a*) The regression equation of x_0 on x_1 and x_2 is

$$x_0 = a_0 + a_1x_1 + a_2x_2$$

The normal equations to determine a_0, a_1, a_2 are

$$na_0 + a_1\Sigma x_1 + a_2\Sigma x_2 = \Sigma x_0$$

$$a_0\Sigma x_1 + a_1\Sigma x_1^2 + a_2\Sigma x_1x_2 = \Sigma x_0x_1$$

$$a_0\Sigma x_2 + a_1\Sigma x_1x_2 + a_2\Sigma x_2^2 = \Sigma x_0x_2$$

(*b*)

x_0	x_1	x_2	x_1^2	x_2^2	x_0x_1	x_0x_2	x_1x_2
64	57	8	3249	64	3648	512	456
71	59	10	3481	100	4189	710	590
53	49	6	2401	36	2597	318	294

67	62	11	3844	121	4154	737	682
55	51	8	2601	64	2805	440	408
58	50	7	2500	49	2900	406	350
77	55	10	3025	100	4235	770	550
57	48	9	2304	81	2736	513	432
56	52	10	2704	100	2912	560	520
51	42	6	1764	36	2142	306	252
76	61	12	3721	144	4638	912	732
68	57	9	3249	81	3876	612	513
753	643	106	38,843	976	40,830	6,796	5,779

The normal equation are

$$12a_0 + 643a_1 + 106a_2 = 753 \qquad ...(1)$$

$$643a_0 + 34843a_1 + 577a_2 = 40,830 \qquad ...(2)$$

$$106a_0 + 5779a_1 + 976a_2 = 6.796 \qquad ...(3)$$

Solving the equations (1), (2) and (3), we get

$$a_0 = 3.6512, \quad a_1 = 0.8546, \quad a_2 = 1.5063$$

The regression equation is

$$x_0 = 3.6512 + 0.8546x_1 + 1.5063x_2 \qquad ...(4)$$

(*c*) When $x_1 = 54$, $x_2 = 9$, from equation (4),

We get $\quad x_0 = 3.6512 + (0.8546)(54) + (1.5063)(9)$

$\Rightarrow \quad x_0 = 63.356$

The regression equation is $\quad x_0 = 3.6512 + 0.8546x_1 + 1.5063x_2$

and the value of $x_0 = 63.356$ at $x_1 = 54$, $x_2 = 9$

14.14 USES OF REGRESSION ANALYSIS

There are many uses of regression analysis. In many situation, the dependent variable *y* is such that it cannot be measured directly. In such cases, with the help of some auxiliary variables are taken as independent variable in a regression to estimate the value of *y*. Regression equation is often used as a prediction equation. The effect of certain treatments can better be adjudged by estimating the effect of concomitant variables. Regressional analysis is used in predicting yield of a crop, for different doses of a fertilizer, and in predicting future demand of food. Regression analysis is also used to estimate the height of a person at a given age, by finding the regression of height on age.

Exercise 4.1

1. Heights of fathers and sons are given below in inches

Height of father	65	66	67	67	68	69	71	73
Height of son	67	68	64	68	72	70	69	70

form the lines of regression and calculate the expected average height of the son when the height of the father is 67.5 inches.

Hint: let 69 = $\bar{x}$ and 69 = $\bar{y}$ (assumed means), then we have

x	y	u	v	u^2	v^2	uv
65	67	–4	–2	16	4	8
66	68	–3	–1	9	1	3
67	64	–2	–5	4	25	10
67	68	–2	–1	4	1	2
68	72	–1	3	1	9	–3
69	70	0	1	0	1	0
71	69	2	0	4	0	0
73	70	4	1	16	1	4
		–6	–4	54	42	24

$$\bar{x} = \bar{x} + \frac{\Sigma u}{x} = 69 - \frac{6}{8} = 68.25$$

$$\bar{y} = \bar{y} + \frac{\Sigma v}{x} = 69 - \frac{4}{8} = 68.5$$

$$\sigma_x^2 = \frac{54}{8} - \left(\frac{-6}{8}\right)^2 = 6.1875 \Rightarrow \sigma_x = 2.49$$

$$\sigma_y^2 = \frac{42}{8} - \left(\frac{-4}{8}\right)^2 = \frac{42}{8} - \left(\frac{1}{2}\right)^2 = 5 \Rightarrow \sigma_y = 2.23$$

$$r = \frac{24 - 3}{\sqrt{\left(54 - \frac{9}{2}\right)(42 - 2)}} = 0.47$$

The regression equation are $y = 0.421x + 39.77$

$$x = 0.524y + 32.29$$

When fathers height is 67.5. The sons age is 68.19 inches.

2. For the following data, determine the regression lines.

x :	6	2	10	4	8
y :	9	11	5	8	7

3. Find the regression equations for the following data

Age of husband : (x)	36	23	27	28	28	29	30	31	33	35
Age of wife (y) :	29	18	20	22	27	21	29	27	29	28

4. By the method of least squares find the regression of y and x and find the value of y when $x = 4$. Also find the regression equation x on y and find the value of x when $y = 24$: Use the table given below

x :	1	3	5	7	9
y :	15	18	21	23	22

5. Using the method of least squares find the two regression equation for the data given below

x :	5	10	15	20	25
y :	20	40	30	60	50

6. Define regression and find the regression equation of y on x, given

x :	2	6	4	3	2	2	8	4
y :	7	2	1	1	2	3	2	6

7. From the following data, obtain the two regression equations

Sales (x) :	91	97	108	121	67	124	51
Purchase (y) :	71	75	69	97	70	91	39

8. Find the regression equations for the following data

Age of Husband : (x)	36	23	27	28	28	29	30	31	33	35
Age of wife (y) :	29	18	20	22	27	21	29	27	29	28

9. Find the equation of regression lines for the following pairs (x, y) for the variables x and y.

(1, 2), (2, 5), (3, 3), (4, 8), (5, 7)

10. From the following data find the yield of wheat in kg per unit area when the rain fall is 9 inches

	Means	S.D.
Yield of Wheat per unit (in kg)	10	8
Annual rainfall (in inches)	8	2

11. Show that regression coefficients are independent of the change of origin but not the change of scale.

12. Given $\Sigma x_i = 60$, $\Sigma y_i = 40$, $\Sigma x_i y_i = 1{,}150$, $\Sigma x_i^2 = 4{,}160$, $\Sigma y_i^2 = 1,\ 720$, $x = 10$.

Find the regression equation of x or y also find r.

13. Using the data given below find the demand when the price of the quantity is Rs. 12.50

	Price (Rs.)	Demand (000 units)
Means	10	35
Standard deviation	2	5

Coefficient of correlation (r) = 0.8.

14. Find the mean of x_i and y_i ; also find the coefficient of correlation: given

$$2y - x - 50 = 0$$

$$2y - 2x - 10 = 0$$

15. From the following data obtain the two regression equations and calculate the correlation coefficient:

x :	1	2	3	4	5	6	7	8	9
y :	9	8	10	12	11	13	14	16	15

16. From the information given below find

(*a*) the regression equation of y on x

(*b*) the regression equation of x on y

(*c*) the mostly likely value of y when $x = 100$

Answer

2. $y = 11.9 - 0.65x$

$x = 16.4 - 1.3y$

3. $y = -1.739 + 0.8913x$

$x = 11.25 + 0.75y$

4. $y = 15.05 + 0.95x$, the value of y when $x = 4$ is 18.85

$x = -12.58 + 0.888y$ and the value of x when $y = 24$ is 8.73.

5. $y = 16 + 1.6x$ and $x = -1 + 0.4\ y$

6. $y = 4.16 - 0.3x$

7. $y = 15.998 + 0.607x$, $x = 0.081 + 1.286y$

8. $y = -1.739 + 0.8913x$

$x = 11.25 + 0.75y$

9. $y = 1.1 + 1.3x$

$x = 0.5 + 0.5y$

10. 12 kg

12. $x = 3.68 + 0.58y$, $r = 0.37$

13. $y = 15 + 2x$; demand = 40,000 units

14. $\bar{x} = 130$, $\bar{y} = 90$, $r = 0.866$

15. $x = -6.4 + 0.95y$; $y = 7.25 + 0.95\ x$

16. (*a*) $y = -0.532 + 1.422x$ (*b*) $x = 4.4 + 0.2y$ (*c*) 141.668

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