LUCKY BOOKS

KNOWLEDGE IS POWER

Name: Fahim Ahsan Khan

School/College: AUST

Class: Section:

Subject: MATH Roll No: 122

Mathematics-II

Math : 1219

1) Integral calculus.
2) Differential ~~calculus~~ Equations

Class Attendane - 10
Quiz - 20 → 3 quizs best 2
Final - 70

Total : 100

22 Nov
রবিবার

Integration:-

The process of finding an integer of a function of x is called integration and the operation is indicated by writing the integeral sign '$\int$' before the given function, the differen--tial dx indicating that x is the variable of integration and $f(x)$ is called the integrand.
That is.

$$\int f(x)\,dx$$

differential
variable
integrand
integral sign

$$\int (x^2+x)\,dx = \frac{x^3}{3} + \frac{x^2}{2} + c$$

$$\int (\cos x + \sin x + 1)\,dx = \sin x - \cos x + x + c$$

Method of substitution:-

Form 1:

$$\int \frac{dx}{ax^2+bx+c} \;;\; a \neq 0$$

$$= \int \frac{dx}{a(x^2+\frac{b}{a}x+\frac{c}{a})}$$

$$= \frac{1}{a}\int \frac{dx}{x^2+2.x.\frac{b}{2a}+\frac{b^2}{4a^2}+\frac{c}{a}-\frac{b^2}{4a^2}}$$

$$= \frac{1}{a}\int \frac{dx}{(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a^2}}$$

Ex:- Evaluate $\int \frac{dx}{4x^2+4x+5}$

$$= \int \frac{dx}{4(x^2+x+5/4)}$$

$$= \frac{1}{4}\int \frac{dx}{x^2+2\cdot x\cdot \frac{1}{2}+\frac{1}{4}+\frac{5}{4}-\frac{1}{4}}$$

$$= \frac{1}{4}\int \frac{dx}{(x+\frac{1}{2})^2+(1)^2} \quad \text{—— (i)}$$

Let $x+\frac{1}{2} = z$

$\Rightarrow \frac{d}{dx}(x+\frac{1}{2}) = \frac{dz}{dx}$

$\Rightarrow 1 = \frac{dz}{dx}$

$\Rightarrow dx = dz$

Now from (1) we get

$$\frac{1}{4}\int \frac{dz}{z^2+1^2}$$

$$= \frac{1}{4}\cdot\frac{1}{1}\tan^{-1}\frac{z}{1}+c$$

$$= \frac{1}{4}\tan^{-1}(x+\tfrac{1}{2})+c \quad \text{(Ans)}$$

Exercise:-

(i) $\int \frac{dx}{6x^2+7x+2}$

(ii) $\int \frac{dx}{x^2+x+1}$

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}+c$$

Form 2:-

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}\ ; \ a\neq 0$$

$$= \int \frac{dx}{\sqrt{a(x^2+\frac{b}{a}\cdot x+\frac{c}{a})}}$$

$$= \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{(x+\frac{b}{2a})^2+\frac{4ac-b^2}{4a^2}}}$$

Ex 1:-
Evaluete: $\int \frac{dx}{\sqrt{x^2+x-2}}$

$$= \int \frac{dx}{\sqrt{x^2+2.x.\frac{1}{2}+(\frac{1}{2})^2-2-\frac{1}{4}}}$$

$$= \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2-\frac{9}{4}}}$$

$$= \int \frac{dx}{\sqrt{(x+\frac{1}{2})^2-(\frac{3}{2})^2}} \quad \text{——— (1)}$$

Let $x+\frac{1}{2} = z$

$\Rightarrow 1 = \frac{dz}{dx}$

$\therefore dx = dz$

Now from (1), we get,

$$\int \frac{dx}{\sqrt{z^2-(\frac{3}{2})^2}}$$

$$* \int \frac{dx}{\sqrt{x^2-a^2}} = \ln|x+\sqrt{x^2-a^2}|$$

$$= \log|z+\sqrt{z^2-\frac{9}{4}}|+c$$

$$= \log|x+\frac{1}{2}+\sqrt{x^2+x+\frac{1}{4}-\frac{9}{4}}|+c$$

$$= \log|x+\frac{1}{2}+\sqrt{x^2+x+-2}|+c$$

(Ans)

Exercise:-

Evaluate:- (1) $\int \frac{dx}{\sqrt{x^2+3x+3}}$;

(2) $\int \frac{dx}{\sqrt{x^2-7x+12}}$

Form 3:-

$$\int \sqrt{ax^2+bx+c}\, dx$$

Ex 1:- Evaluate $\int \sqrt{4+3x-2x^2}\, dx$

Solution:-

Here, $4+3x-2x^2$

$= 2(-x^2+\frac{3}{2}x+2)$

$= -2\left[x^2-2.x.\frac{3}{4}+\frac{9}{16}-2-\frac{9}{16}\right]$

$= -2\left[(x-\frac{3}{4})^2-\frac{41}{16}\right]$

$= 2\left[\left(\frac{\sqrt{41}}{4}\right)^2-(x-\frac{3}{4})^2\right]$

Now,

$$\int \sqrt{2\left\{\left(\frac{\sqrt{41}}{4}\right)^2-(x-\frac{3}{4})^2\right\}}\, dx$$

$= \sqrt{2}\int \sqrt{\left\{\left(\frac{\sqrt{41}}{4}\right)^2-(x-\frac{3}{4})^2\right\}}\, dx$ — (1)

Let,

$x-\frac{3}{4} = z$

$\Rightarrow \frac{d}{dx}(x-\frac{3}{4}) = \frac{dz}{dx}$

$\Rightarrow 1 = \frac{dz}{dx}$

$\Rightarrow dx = dz$

Then the integral (1) reduces to,

$$\sqrt{2}\int \sqrt{\left\{\left(\frac{\sqrt{41}}{4}\right)^2-z^2\right\}}\, dz$$

$$= \sqrt{2}\left[\frac{(x-3/4)\sqrt{(\sqrt{41}/4)^2-(x-3/4)^2}}{2} + \frac{\left(\frac{\sqrt{41}}{4}\right)^2}{2}\sin^{-1}\frac{x-3/4}{\sqrt{41}/4}\right]+c$$

Ex 2: Evaluate $\int\sqrt{x^2+x+1}\,dx$

$\int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}x/a$

$\int\sqrt{x^2+a^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}x/a$

Here $\int\sqrt{x^2+x+1}\,dx$

$$= \int\sqrt{x^2+2.x.\tfrac{1}{2}+\tfrac{1}{4}+1-\tfrac{1}{4}}\,dx$$

$$= \int\sqrt{\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}}\,dx$$

$$= \int\sqrt{\left(x+\tfrac{1}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}\right)^2}\,dx$$

hyperbole

$$\cosh x = \tfrac{1}{2}(e^x+e^{-x})$$

$$\sinh x = \tfrac{1}{2}(e^x-e^{-x})$$

Form 4:-

$$\int\frac{px+q}{ax^2+bx+c}\,dx$$

numerator

denominator

$$= \int\frac{\frac{p}{2a}(2ax+b)+q-\frac{pb}{2a}}{ax^2+bx+c}\,dx$$

Ex.1 Evaluate $\int\frac{7x-9}{x^2-2x+35}\,dx$

$$= \int\frac{\frac{7}{2}(2x-2)-2}{x^2-2x+35}\,dx$$

$\frac{7}{2.1}(2.1x-2)-9-\frac{7(-2)}{2.1}$

$$= \frac{7}{2}\int\frac{2x-2}{x^2-2x+35}\,dx-2\int\frac{dx}{x^2-2x+35}$$

$$= \frac{7}{2}\ln\left|(x^2-2x+35)\right|-2\int\frac{dx}{x^2-2x+35}$$

$$\int\frac{f'(x)}{f(x)}\,dx$$

$$= \ln|f(x)|+c$$

$$= \frac{7}{2}\ln\left[(x^2-2x+35)\right| - 2\int \frac{dx}{x^2-2.x.1+1^2+35-1}$$

$$= \frac{7}{2}\ln\left[(x^2-2x+35)\right| - 2\int \frac{dx}{(x-1)^2+(\sqrt{34})^2}$$

$$= \frac{7}{2}\ln\left[(x^2-2x+35)\right] - 2.\frac{1}{\sqrt{34}}\tan^{-1}\frac{x-1}{\sqrt{34}} + c$$

Ans

Exercise:-

Evaluate

(i) $\int \frac{4x+3}{3x^2+3x+1}\,dx$

(ii) $\int \frac{x+1}{x^2+4x+5}\,dx$

Form 5:-

$$\int \frac{dx}{(ax+b)\sqrt{cx+d}} \; ; \; a \neq 0, \; c \neq 0.$$

Let,

$cx+d = z^2 \longrightarrow \therefore x = \frac{z^2-d}{c}$

$\Rightarrow \frac{d}{dx}(cx+d) = \frac{d}{dx}(z^2)$

$\Rightarrow c = 2z\frac{dz}{dx}$

$\Rightarrow c\,dx = 2z\,dz$

$\therefore dx = \frac{2}{c}z\,dz$

Ex:- Evaluate $\int \frac{dx}{(1-x)\sqrt{x}}$

Let $x = z^2$ $\quad\therefore z = \sqrt{x}$

$\Rightarrow \frac{d}{dx}(x) = \frac{d}{dx}(z^2)$

$\Rightarrow 1 = 2z.\frac{dz}{dx}$

$\therefore dx = 2z\,dz$

Then the given integral becomes

$$\int \frac{2z\,dz}{(1-z^2)\,z} = 2\int \frac{dz}{1-z^2}$$

$$= 2 \cdot \frac{1}{2.1} \cdot \log\left|\frac{1+z}{1-z}\right| + c$$

$$= \log\left|\frac{1+\sqrt{x}}{1-\sqrt{x}}\right| + c.$$

Exercise:

(i) $\displaystyle\int \frac{dx}{(2+x)\sqrt{1+x}}$ (ii) $\displaystyle\int \frac{da}{(2a+1)\sqrt{4a+3}}$

* $\displaystyle\int \frac{dx}{a^2-x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right|$.

24-11-15

মঙ্গলবার -

Form 6:- $\displaystyle\int \frac{dx}{(px+q)\sqrt{ax^2+bx+c}}$; Let $px+q = z$ or $1/z$

Ex:- Evaluate $\displaystyle\int \frac{dx}{(2x+3)\sqrt{x^2+3x+2}}$

Sol:- Let $2x+3 = z$ $\quad \therefore x = \frac{z-3}{2}$

$$\Rightarrow \frac{d}{dx}(2x+3) = dz/dx$$

$$\Rightarrow 2 = \frac{dz}{dx}$$

$$\Rightarrow dx = \frac{dz}{2}$$

Then from the given integral, $\displaystyle\int \frac{dz/2}{z\sqrt{\left(\frac{z-3}{2}\right)^2 + 3\left(\frac{z-3}{2}\right) + 2}}$

$$= \frac{1}{2}\int \frac{dz}{z\sqrt{\frac{z^2-6z+9}{4} + \frac{3z-9}{2} + 2}}$$

$$= \frac{1}{2}\int \frac{dz}{z\sqrt{\frac{z^2-\cancel{6z}+9+\cancel{6z}-18+8}{4}}}$$

$$= \frac{2}{2}\int \frac{dz}{z\sqrt{z^2-1}}$$

$= \frac{1}{1}\sec^{-1}\left(\frac{z}{1}\right)+c$

$= \sec^{-1}(2x+3)+c$

(Ans)

$* \int \frac{dx}{x\sqrt{x^2-a^2}}$

$= \frac{1}{a}\sec^{-1}\frac{x}{a}$

Exercise:-

Evaluate (i) $\int \frac{dx}{(1+x)\sqrt{1+2x-x^2}}$

(ii) $\int \frac{dx}{(x-3)\sqrt{x^2-6x+8}}$

Form 7:- $\int \frac{a\cos x+b\sin x}{p\cos x+q\sin x}\,dx$

Ex:- Evaluate $\int \frac{2\sin x+3\cos x}{3\sin x+4\cos x}\,dx$ —— (1)

Sol:- Let $2\sin x+3\cos x = \lambda$ (denominator) + m(differential co-efficient of denominator)

$\Rightarrow 2\sin x+3\cos x = \lambda(3\sin x+4\cos x)+m(3\cos x-4\sin x)$

$= (3\lambda-4m)\sin x+(4\lambda+3m)\cos x$

Now equating the co-efficient of sinx and cosx from both sides we get,

$3\lambda-4m=2$ —— (2) $\times 4$

$4\lambda+3m=3$ —— (m) $\times 3$ (−)

Solving (2) and (3) we get,

$\lambda = \frac{18}{25}$ and $m = \frac{1}{25}$

Hence, $2\sin x+3\cos x = \frac{18}{25}(3\sin x+4\cos x)+\frac{1}{25}(3\cos x-4\sin x)$

Now from (1), the given integral becomes,

$\frac{18}{25}\int \frac{(3\sin x+4\cos x)}{(3\sin x+4\cos x)}\,dx+\frac{1}{25}\int \frac{(3\cos x-4\sin x)}{(3\sin x+4\cos x)}\,dx$

$= \frac{18}{25}x+\frac{1}{25}\ln|(3\sin x+4\cos x)|+c.$

(Ans)

$* \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c$

Exercise:-

Evaluate:- $\int \frac{11\cos x-16\sin x}{2\cos x+5\sin x}\,dx.$

Integration by partial fraction:-

Ex:- Evaluate $\int \frac{dx}{(x-3)(x+3)}$

Sol:- Given, $\int \frac{dx}{(x-3)(x+3)}$ ——— (1)

Let,

$$\frac{1}{(x-3)(x+3)} = \frac{A}{(x-3)} + \frac{B}{(x+3)} \quad \text{——— (2)}$$

Multiplying both sides by $(x-3)(x+3)$ we get,

$$1 = A(x+3) + B(x-3) \quad \text{——— (3)}$$

Put $x = -3$ in (3) we get

$$1 = B(-6) \quad \therefore B = -\frac{1}{6}$$

Again put $x = 3$ in (3) we get,

$$1 = A(6) \quad \therefore A = \frac{1}{6}$$

Then from (2) we get,

$$\frac{1}{(x-3)(x+3)} = \frac{1}{6(x-3)} - \frac{1}{6(x+3)}$$

Now from (1),

$$\int \frac{dx}{(x-3)(x+3)} = \int \frac{dx}{6(x-3)} - \int \frac{dx}{6(x+3)}$$

$$= \frac{1}{6}\left(\ln(x-3) - \ln(x+3)\right) + c$$

$$= \frac{1}{6} \ln \frac{(x-3)}{(x+3)} + c.$$

29 Nov (রবিবার)

Ex:- Evaluate $\int \frac{(x-1)(x-2)(x-3)}{(x-5)(x-6)(x-7)} dx$

Here the numerator is of the same digree of the denominator.

power (sum) $\frac{6}{6}$

if same add

Sol

Let, $\frac{(x-1)(x-2)(x-3)}{(x-5)(x-6)(x-7)} = 1 + \frac{A}{(x-5)} + \frac{B}{(x-6)} + \frac{C}{(x-7)}$ —— (1)

multiplying both sides by $(x-5)(x-6)(x-7)$ we get

$(x-1)(x-2)(x-3) = (x-5)(x-6)(x-7) + A(x-6)(x-7)$

$+ B(x-5)(x-7) + C(x-5)(x-6)$ —— (2)

put $x=5$ in (2) we get

$(\underset{4}{5-1})(\underset{3}{5-2})(\underset{2}{5-3}) = 0 + A(-1)(-2) + 0$

$\Rightarrow 2A = 24$

$\Rightarrow A = 12$

put $x=7$;

$(7-1)(7-2)(7-3) = 0 + 0 + 0 + C(7-5)(7-6)$

$\Rightarrow 6 \times 5 \times \cancel{4}^{2} = \cancel{2}C$

$\Rightarrow C = 60$

put $x=6$

$(5)(4)(3) = 0 + 0 + B(1)(-1) + 0$

$\Rightarrow 60 = B(-1)$

$\Rightarrow B = -60$

$\therefore$ from (1) $\frac{(x-1)(x-2)(x-3)}{(x-5)(x-6)(x-7)} = 1 + \frac{12}{(x-5)} - \frac{60}{(x-6)} + \frac{60}{(x-7)}$

$\Rightarrow \int (\quad) dx = \int dx + 12\int \frac{dx}{(x-5)} - 60\int \frac{dx}{(x-6)} + 60\int \frac{dx}{x-7}$

$+ 2 \ln(x-5)$

Definite integrals:-

$$\int_0^2 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_0^2 = \left[\left(\frac{8}{3}+2\right)-(0)\right]$$

$$= \frac{14}{3} \quad \text{Ans}$$

Ex:- Evaluate: $\int_0^a \sqrt{a^2-x^2}\,dx$

Let $x = a\sin\theta$

$\therefore \frac{dx}{d\theta} = \frac{d}{d\theta}(a\sin\theta)$

$= a\cos\theta$

$\therefore dx = a\cos\theta\, d\theta$

$$= \int_0^{\pi/2} \sqrt{a^2-a^2\sin^2\theta}\; a\cos\theta\, d\theta$$

x	0	a
θ	0	$\pi/2$

$$= a^2\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 2\cos^2\theta\, dx \quad \checkmark$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 2\cos^2\theta\, d\theta \quad \alpha$$

$$= \frac{a^2}{2}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[(\pi/2+0)-(0+0)\right]$$

$$= a^2\,\pi/4 \quad \text{Ans.}$$

Exercise:-

Evaluate $\int_0^{2a} \sqrt{2ax-x^2}\, dx$

$\cos 2\theta = 2\cos^2\theta - 1$

Ex: Evaluate $\int_0^1 \frac{dx}{(x^2+1)^2}$

Let $x = \tan\theta$

$\therefore dx = \sec^2\theta\, d\theta$

x	0	1
θ	0	$\pi/4$

$$= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{(\tan^2\theta+1)^2}$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^4\theta}$$

$$= \int_0^{\pi/4} \cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} 2\cos^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{\pi}{4} + 0 + 0 + \frac{1}{2}\right)$$

$$= \frac{\pi}{8} + \frac{1}{4}$$

$= \cancel{\frac{\pi+2}{8}}$

Ex: Evaluate $\int_0^{\pi/2} (\sec\theta - \tan\theta)\, d\theta$

$= \int_0^{\pi/2} \sec\theta\, d\theta - \int_0^{\pi/2} \tan\theta\, d\theta$

$= \left[\ln(\sec\theta + \tan\theta) - \ln(\sec\theta)\right]_0^{\pi/2}$

$= \left[\ln \frac{(\sec\theta + \tan\theta)}{\sec\theta}\right]_0^{\pi/2}$

$= \left[\ln(1+\sin\theta)\right]_0^{\pi/2}$

$= \left[\ln(2) - \ln(1)\right]$

$= \ln 2$ A

Ex: Evaluate $\int_0^{\pi/2} \frac{dx}{a+b\cos x}$

Let $\tan\frac{x}{2} = z$

$\Rightarrow \sec^2\frac{x}{2} \cdot \frac{1}{2} \frac{dx}{dz} = 1$

$\Rightarrow \sec^2\frac{x}{2}\, dx = 2\,dz$

$= \int_0^{\pi/2} \frac{dx}{a(\underbrace{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}}_{1}) + b(\cos^2\frac{x}{2} - \sin^2\frac{x}{2})}$

$= \int_0^{\pi/2} \frac{dx}{\cos^2\frac{x}{2}(a+b) + (a-b)\sin^2\frac{x}{2}}$

x	0	$\frac{\pi}{2}$
z	0	1

divided by $\cos^2\frac{x}{2}$

$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}\, dx}{(a+b) + (a-b)\tan^2\frac{x}{2}}$

divided by $a-b$

$= \frac{1}{a-b} \int_0^1 \frac{\sec^2\frac{x}{2}\, dx}{\frac{a+b}{a-b} + \tan^2\frac{x}{2}}$

Again let $A^2 = \frac{a+b}{a-b}$

$= \frac{1}{a-b} \int_0^1 \frac{2\,dz}{A^2 + z^2}$

$$= \frac{2}{a-b}\left[\frac{1}{A}\tan^{-1}\frac{z}{A}\right]_0^1$$

$$= \frac{2}{a-b}\left[\sqrt{\frac{a-b}{a+b}}\tan^{-1}\frac{z}{A}\right]_0^1$$

$$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\frac{1}{A}-\tan^{-1}0\right]$$

$$= \frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\sqrt{\frac{a-b}{a+b}} \quad (A$$

<u>Exercise</u> Evaluate —

$$\int_0^{\pi/2}\frac{dx}{1+\sin x}$$

<u>Improper integrals:-</u>

<u>Ex:</u> Evaluate $\int_{-\alpha}^{3}\frac{dx}{\sqrt{7-x}}$

<u>Sol:</u> Given $\int_{-\alpha}^{3}\frac{dx}{\sqrt{7-x}}$

$$= \lim_{p\to 0}\int_{-1/p}^{3}\frac{dx}{\sqrt{7-x}} \text{ ——— (1)}$$

$\frac{1}{0} = \alpha$

Now, $\int\frac{dx}{\sqrt{7-x}}$ Let $7-x = z$

$\Rightarrow -dx = dz$

$$= \int\frac{dz}{\sqrt{\cancel{7}-z-\cancel{7}}} \quad ?$$

$$= \int\frac{dz}{-\sqrt{z}}$$

$$= -2\sqrt{z} = -2\sqrt{7-x}$$

Now from (1) $\lim_{p\to 0}\left[-2\sqrt{7-x}\right]_{-1/p}^{3}$

$\Rightarrow \lim_{p\to 0} -2\left[\sqrt{7-x}\right]_{-1/p}^{3}$

$\Rightarrow \lim_{p\to 0} -2\left[\sqrt{4}-\sqrt{7+1/p}\right]$

$\Rightarrow -2(2-\alpha)$

Hence $= \alpha$ the value the given integral does not exist.

__Ex-2:__ Evaluate $\int_0^{\alpha} \frac{x}{x^4+1}dx$

$\Rightarrow \lim_{p\to 0}\int_0^{1/p}\frac{x}{x^4+1}dx$

$= \lim_{p\to 0}\int_0^{1/p}\frac{1}{2}\frac{2x}{x^4+1}dx$ —— (1)

Now, $\int \frac{2x}{(x^2)^2+1}dx$

Let $x^2 = z$

$\therefore 2x\,dx = dz$

$\int \frac{dz}{z^2+1}$

$= \tan^{-1}z$

$= \tan^{-1}x^2$

$\therefore \int_0^{1/p}\frac{2x}{x^4+1}dx$

$= \left[\tan^{-1}x^2\right]_0^{1/p}$

$= \tan^{-1}\frac{1}{p^2}$

Now (1) becomes

$$\int_0^{\alpha} \frac{x\,dx}{x^4+1}$$

$$= \lim_{p\to 0} \int_0^{1/p} \frac{1}{2} \cdot \frac{2x\,dx}{(x^2)^2+1}$$

$$= \lim_{p\to 0} \frac{1}{2} \tan^{-1} \frac{1}{p^2}$$

$$= \frac{1}{2} \tan^{-1} \alpha$$

$$= \frac{1}{2} \tan^{-1} \tan \frac{\pi}{2}$$

$$= \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{4}$$

Ex:- Evaluate $\int_{-\alpha}^{\alpha} \frac{dx}{x^2+4}$

$$= \lim_{p\to 0} \int_{-1/p}^{1/p} \frac{dx}{x^2+4}$$

Let $x^2 = z$

$\Rightarrow 2x\,dx = dz$

Now, $\int \frac{dx}{x^2+4}$

$$= \int \frac{dz}{z^2+4}$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2}$$

$$= \frac{1}{2} \tan^{-1} \frac{x}{2}$$

$\frac{1}{2}\int_{-1/p}^{1/p} \tan^{-1}\frac{x}{2}$

$= \frac{1}{2}\left[\tan^{-1}\frac{x}{2}\right]_{-1/p}^{1/p}$

$= \frac{1}{2}\left[\tan^{-1}\frac{1}{2p} - \tan^{-1}\frac{1}{-2p}\right]$

$= \frac{1}{2}\left(\tan^{-1}\tan\frac{\pi}{2} + \tan^{-1}\tan\frac{\pi}{2}\right)$

$= \frac{1}{2}\left(\cancel{\tan^{-1}}\frac{\pi}{2} + \frac{\pi}{2}\right)$

$= \frac{\pi}{2}$

Exercise: Evaluate

① $\int_0^\infty x e^{-x^2}dx$ → $\frac{1}{2}$ ✓

② $\int_0^\infty x e^{-x}dx$

Integration by parts.

$= \int uv\,dx = u\int v\,dx - \int\left\{\frac{du}{dx}\int v\,dx\right\}dx$

1 Dec → মঙ্গলবার

Reduction formula:-

Reduction formula for $\int x^n e^{ak}dx$.

Ex:- Establish a reduction formula

Sol:- Let, $I_n = \int x^n . e^{ax}dx$

$= x^n\int e^{ax}.dx - \int\left\{\frac{d}{dx}(x^n)\int e^{ax}dx\right\}dx$

$= x^n\frac{e^{ax}}{a} - \int n x^{n-1}\frac{e^{ax}}{a}dx$

$= \frac{1}{a}x^n . e^{ax} - \frac{n}{a}\int x^{n-1}.e^{ax}.dx$

$= \frac{1}{a}x^n e^{ax} - \frac{n}{a}I_{n-1}$, which is our required reduction formula.

(if Hence ↓ Previous solution use find value)

Ex 2:- Obtain a reduction formula for $\int \sin^n x\,dx$.

Hence find $\int \sin^7 x\,dx$.

Sol:- Let $I_n = \int \sin^n x\,dx$

$= \int \sin^{n-1} x . \sin x\,dx$

$= \sin^{n-1} x \int \sin x\,dx - \int \left\{ \frac{d}{dx}(\sin^{n-1} x) \int \sin x\,dx \right\} dx$

$= \sin^{n-1} x(-\cos x) - \int \{ (n-1) \sin^{n-2} x . \cos x . (-\cos x) \} dx$

$= -\sin^{n-1} x . \cos x + (n-1) \int \sin^{n-2} x \cos^2 x\,dx$

$= -\cos x . \sin^{n-1} x + (n-1) \int \sin^{n-2} x (1-\sin^2 x) dx$

$I_n = -\cos x . \sin^{n-1} x + (n-1) \int \sin^{n-2} x\,dx - (n-1) \int \sin^n x\,dx$

$I_n = -\cos x . \sin^{n-1} x + (n-1) I_{n-2} - (n-1) I_n$

$\Rightarrow (1+n-1) I_n = -\cos x . \sin^{n-1} x + (n-1) I_{n-2}$

$\Rightarrow n I_n = -\cos x . \sin^{n-1} x + (n-1) I_{n-2}$

$\therefore I_n = -\frac{\cos x . \sin^{n-1} x}{n} + \frac{(n-1)}{n} I_{n-2}$ —— (1) Ans

$I_7 = \int \sin^7 x\,dx$

$= -\frac{\cos x . \sin^6 x}{7} + \frac{6}{7} I_5$ —— (2)

$\therefore I_5 = -\frac{\cos x . \sin^4 x}{5} + \frac{4}{5} I_3$ —— (3)

$\therefore I_3 = -\frac{\cos x . \sin^2 x}{3} + \frac{2}{3} I_1$ —— (4)

$\therefore I_1 = \int \sin x\,dx$

$= -\cos x.$

Then (2) becomes-

$$I_7 = -\frac{\cos x \cdot \sin^6 x}{7} + \frac{6}{7}\left[-\frac{\sin^4 x \cdot \cos x}{5} + \frac{4}{5}\left\{-\frac{\sin^2 x \cdot \cos x}{3} + \frac{2}{3}(-\cos x)\right\}\right]$$

$$= -\frac{\sin^6 x \cdot \cos x}{7} - \frac{6}{35}\sin^4 x \cdot \cos x - \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{1}{3}\sin^2 x \cdot \cos x - \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}\cos x$$

(Ans)

Ex-2: Obtain a reduction formula for $\int \cos^n x\,dx$

Ex: Obtain a reduction formula for $\int \tan^n x\,dx$.

$I_n = \int \tan^n x\,dx$

Hence- $\int \tan^5 x\,dx$.

Let, $I_n = \int \tan^{n-2} x \cdot \tan^2 x\,dx$

$$= \int \tan^{n-2} x \cdot (\sec^2 x - 1)\,dx$$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \int \tan^{n-2} x \cdot \sec^2 x\,dx - I_{n-2} \quad \text{— (1)}$$

Now, $\int \tan^{n-2} x \cdot \sec^2 x\,dx$

$$= \tan^{n-2} x \int \sec^2 x\,dx - \int \left\{\frac{d}{dx}(\tan^{n-2} x)\int \sec^2 x\,dx\right\}dx$$

$$= \tan^{n-2} x \cdot \tan x - \int \left\{(n-2)\tan^{n-3} x \cdot \sec^2 x \cdot \tan x\right\}dx$$

$$= \tan^{n-1} x - (n-2)\int \cancel{(n-2)}\tan^{n-2} x \cdot \sec^2 x\,dx.$$

$$\Rightarrow \int \tan^{n-2} x \cdot \sec^2 x\,dx = \tan^{n-1} x - (n-2)\int \tan^{n-2} x \sec^2 x\,dx$$

$$\Rightarrow (1+n-2)\int \tan^{n-2} x \cdot \sec^2 x\,dx = \tan^{n-1} x$$

$$\therefore \int \tan^{n-2} x \cdot \sec^2 x\,dx = \frac{\tan^{n-1} x}{(n-1)}$$

Now, from (1) $I_n = \frac{\tan^{n-1} x}{(n-1)} - I_{n-2}$

$I_5 = \int \tan^5 x\,dx$

$$= \int \tan^3 x \cdot \sec^2 x\,dx - I_3 \quad \text{— (2)}$$

$$I_3 = \int \tan x \cdot \sec^2 x\,dx - I_1 \quad \text{— (3)}$$

$$I_1 = \int \tan x\,dx$$

$$= \sec^2 x\,dx$$

$$I_5 = \frac{\tan^4 x}{4} - I_3 \quad \text{—(2)}$$

$$I_3 = \frac{\tan^2 x}{2} - I_1 \quad \text{—(3)}$$

$$I_1 = \int \tan x\, dx$$

$$= \sec^2 x\, \cancel{dx}$$

Sat - 11:20 - 1:00
sun - (,)
wed - (.)
Thu - 10:30 - 11:20

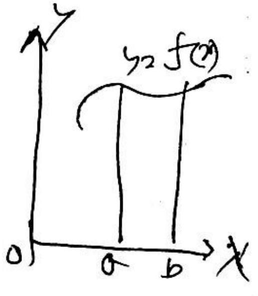
curve part

6 Dec → রবিবার

Arc Length:

If $f(x)$ be a smoth curve on $[a,b]$ then the the arc lengths of the curve $y = f(x)$ from a to b is given by,

$$S = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx.$$



Ex: Find the arc length of the curve $y = x^{3/2}$ from $x=1$ to $x=2$.

Solution:-

Given, $y = x^{3/2}$

$\therefore \frac{dy}{dx} = \frac{3}{2}x^{1/2}$

$\therefore$ Arc length, $l = \int_1^2 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\, dx$

$= \int_1^2 \sqrt{1+\frac{9}{4}x}\, dx$

$= \int_1^2 \sqrt{\frac{4+9x}{4}}\, dx$

$= \frac{1}{2}\int_1^2 \sqrt{4+9x}\, dx$

$= \frac{1}{18}\int_{13}^{22} \sqrt{z}\, dz$

Let, $4+9x = z$
$\therefore 9dx = dz$
$\Rightarrow dx = \frac{dz}{9}$

$= \frac{1}{18}\left[\frac{z^{3/2}}{3/2}\right]_{13}^{22}$

$= \frac{1}{18}\left[(22)^{3/2} - (13)^{3/2}\right]$

x	1	2
z	13	22

$= 22.08$ Ans.

If both x and y are expressed in terms of a common variable parameter t and s is also a function of t then the length of the curve between two points on it for which $t = t_1$ and $t = t_2$ respecting are given by.

$$s = \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Ex 1: Determine the length of an arc of the curve $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ measured from $\theta = 0$ to $\theta = \pi$.

Solution:

The required arc length of the given ~~cour~~ curve from $\theta=0$ to $\theta=\pi$ is

$$S = \int_0^{\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta \quad \text{--- (1)}$$

Given, $x = a(\theta + \sin\theta)$ and $y = a(1-\cos\theta)$

$\therefore \frac{dx}{d\theta} = a + a\cos\theta$ | $\frac{dy}{d\theta} = a\sin\theta$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (a + a\cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2 + 2a^2\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta$$

$$= a^2(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta)$$

$$= a^2(2 + 2\cos\theta)$$

$$= 2a^2 \cdot 2\cos^2\theta/2$$

$$= 4a^2\cos^2\theta/2$$

$$\therefore \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = 2a\cos\theta/2$$

Hence from (1), $s = \int_0^{\pi} 2a\cos\theta/2\, d\theta$

$$= 2a\left[\frac{\sin\theta/2}{1/2}\right]_0^{\pi} = 4a\left[\sin\pi/2 - \sin 0\right]$$

$$= 4a$$

Exercise:

Find the length of the arc of the curve $x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ from $\theta = 0$ to $\theta = \pi$

Intrinsci equation to a curve:

If s denotes the length of an arc of a plane curve measured from some fixed point A on it to an arbitrary point P, and if ψ be the inclination of the ~~length~~ tangent to the curve at P to any fixed line on the plane (for example, the x-axis), the relation between s and ψ is called the intrinsic equation of the curve.

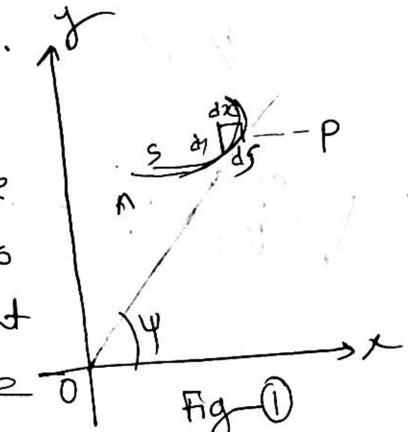


Fig (1)

Here in Fig (1) $\frac{dx}{ds} = \cos\psi$ and $\frac{dy}{ds} = \sin\psi$,

* Intrinsic equation derived from Cartesian equation:

Let the cartesian equation to the curve be $y = f(x)$. Then from fig (1), ψ denoting the angle between the tangent at any point p and the x-axis.

$$\therefore \tan\psi = \frac{dy}{dx} = f'(x) \quad \text{—— (1)}$$

Also, $s =$ arc $AP = \int_a^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_a^x \sqrt{1+\{f'(x)\}^2}\, dx$$

$$= F(x), \text{ say} \quad \text{—— (2)}$$

'a' denoting the abscissa of A, and (x) that of P.

Now, the x-element between ① and ② (which will be a relation between s and ψ), will be the required intrinsic equation of the curve.

\# If the equation to the curve be given in the parametric form $x = f(t)$, $y = \phi(t)$, we can write

$$\tan\psi = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)} \quad \text{—} (3)$$

Also $s = \int_{t_1}^{t} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

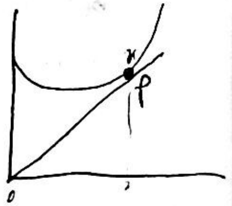
$$= \int_{t_1}^{t} \sqrt{\{f'(t)\}^2 + \{\phi'(t)\}^2}\, dt$$

$= F(t)$, say —(4)

where t_1 is the value of the parameter t at A. The t-eliminant between (3) and (4) will be the required intrinsic equation to the curve.

Ex: 8 Dec ⇒ মঙ্গলবার

Ex: Obtain the ~~in~~ intrinsic equation of the catenary $y = c\cosh\frac{x}{c}$. in the form $s = c\tan\psi$.



Sol:- Here, $\tan\psi = \frac{dy}{dx}$

$$= \frac{d}{dx}(c\cosh x/c)$$

$$= c.\sinh x/c . 1/c$$

$$= \sinh x/c \quad \text{--- (1)}$$

Also, measuring s from the vertex, where $x=0$

$$s = \int_0^x \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= \int_0^x \sqrt{1+\sinh^2 x/c}\,dx$$

$$= \int_0^x \sqrt{\cosh^2 x/c}\,dx$$

$$= \int_0^x \cosh x/c\,dx$$

$$= \left[\frac{\sinh x/c}{1/c}\right]_0^x$$

$$= c\sinh\frac{x}{c}$$

Hence using (1) we get $s = c\tan\psi$. (Proved)

hyperbolic

* $\cosh x = \frac{1}{2}(e^x + e^{-x})$
* $\sinh x = \frac{1}{2}(e^x - e^{-x})$
* $\cosh^2 x - \sinh^2 x = 1$

Ex:-2:- Obtain the intrinsic equation of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1-\cos\theta)$ taking the vertex as the fixed point and the tangent at that point. at the fixed line.

$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta)$

Sol:- The length of the arc of the above cycloid measured from the vertex is

$$s=\int_0^{\theta}\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta \quad \rightarrow \text{parametric}$$

$$=\int_0^{\theta} 2a\cos\theta/2\,d\theta \quad \left[\sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}=2a\cos\theta/2\right]$$

previous $\rightarrow$

$$=4a\sin\theta/2 \quad \text{—} (1)$$

Also, $\tan\psi=\frac{dy}{dx}$

$$=\frac{dy/d\theta}{dx/d\theta}=\frac{a(\sin\theta)}{a(1+\cos\theta)}$$

$$=\frac{2\sin\theta/2\cdot\cos\theta/2}{2\cos^2\theta/2}$$

$$=\tan\theta/2$$

$$\therefore \psi=\frac{\theta}{2}$$

why?

Hence from (1), $s=4a\sin\psi$, which is the required intrinsic equation. (Ans)

Ex:- Find the cartesian equation of the curve for which the intrinsic equation is $s=a\psi$.

Sol:- Here $\frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}$

$$=\cos\psi\cdot a$$

$$\therefore dx=a\cos\psi\,dx$$

Integrating both sides we get,

$$x=a\sin\psi+c$$

$$\Rightarrow (x-c)=a\sin\psi \quad \text{—} (1)$$

Again, $\frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\sin\psi\cdot a$

$$\therefore dy=a\sin\varphi\,d\psi$$

Integrating both sides we get.

$$y=-a\cos\psi+d.$$

$$\Rightarrow (y-d)=-a\cos\psi \quad \text{—} (2)$$

From ① and ②, eliminating ψ, we get,

$(x-c)^2+(y-d)^2=a^2(\sin^2\psi+\cos^2\psi)=a^2$, which is the required cartesian equation.

(Ans)

Quiz-1

Supto Intrinsic equation.

22.12.15.

Gamma and Beta Functions:-

Gamma Function: $\int_0^\alpha e^{-x}x^{n-1}dx$ is called Gama function of n. It is also written as $\Gamma(n)$.

$=\int_0^\alpha e^{-x}x^{n-1}dx, [n>0]$

Beta Function:- $\int_0^1 x^{l-1}(1-x)^{m-1}dx$ is called the Beta function of l, m. It is also written as $B(l,m)=\int_0^1 x^{l-1}(1-x)^{m-1}dx, [l>0, m>0]$

Relation between Gamma and Beta functions:

~~Let~~

$1-x \Rightarrow x = \sin^2\theta$
$1+x \Rightarrow x =$

$$B(l,m) = \frac{\Gamma(l).\Gamma(m)}{\Gamma(l+m)}$$

Ex. show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$

Proof: We know that, $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx$ — (1)

Let $x = \sin^2\theta$

$\Rightarrow dx = 2\sin\theta.\cos\theta\, d\theta$

and $1-x = 1-\sin^2\theta = \cos^2\theta$

x	0	1
θ	0	$\pi/2$

Then (1) becomes, $\beta(m,n) = \int_0^{\pi/2} \sin^{2m-2}\theta.(1-\sin^2\theta)^{n-1}.2\sin\theta.\cos\theta\, d\theta$

$$\Rightarrow \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta.\cos^{2n-1}\theta\, d\theta. \quad \text{— (2)}$$

Let $2m-1 = p$ and $2n-1 = q$

$\therefore m = \frac{p+1}{2}$ | $n = \frac{q+1}{2}$

Formula
(i) $\Gamma(n+1) = n\Gamma(n)$
(ii) $\Gamma(n+1) = n!$
(iii) $\Gamma(1) = 1$.

Then from (2),
$$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = 2\int_0^{\pi/2} \sin^p\theta.\cos^q\theta\, d\theta$$

$$\therefore \int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \text{ (Proved)}$$

$\Gamma(4) = \Gamma(3+1)$

$= 3\Gamma 3$

$= 3\Gamma(2+1)$

$= 3.2\Gamma 2$

$= 3.2\Gamma(1+1)$

$= 3.2.1.1.\Gamma 1 = 3.2.1.! = 6$

$\Gamma(3+1)$

$= 3!$

$= 3.2$

$= 6.$

Ex 2:- Show that $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{1}{2}\Gamma(\frac{1}{4})\Gamma(\frac{3}{4})$

Proof:- We know that, $\int_0^{\pi/2} \sin^p\theta . \cos^q\theta \, d\theta$.

$$= \frac{\Gamma(\frac{p+1}{2})\Gamma(\frac{q+1}{2})}{2\Gamma(\frac{p+q+2}{2})} \quad \text{--- (1)}$$

Now

$$\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \int_0^{\pi/2} \frac{\cos^{1/2}\theta}{\sin^{1/2}\theta} . d\theta$$

$$= \int_0^{\pi/2} \cos^{1/2}\theta . \sin^{-1/2}\theta . d\theta$$

Now applying formula (1) we get,

$$\int_0^{\pi/2} \sin^{-1/2}\theta . \cos^{1/2}\theta . d\theta = \frac{\Gamma(\frac{-1/2+1}{2})\Gamma(\frac{1/2+1}{2})}{2\Gamma(\frac{-1/2+1/2+2}{2})}$$

$$= \frac{\Gamma(1/4)\Gamma(3/4)}{2\Gamma(1)} = \frac{1}{2}\Gamma(1/4)\,\Gamma(3/4)$$

(Proved)

Formula:- (i) $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

(ii) $\beta(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}} dx = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

(iii) $\int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$

~~Evaporator~~

Ex 3:- Prove that $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

Proof:- We know that, $\beta(m,n) = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$

$$\Rightarrow \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

put $m+n=1 \Rightarrow m = 1-n$, then from above we get,

$$\frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)} = \int_0^{\infty} \frac{x^{n-1}}{(1+x)^1} dx$$

$$\Rightarrow \Gamma(1-n)\,\Gamma(n) = \int_0^{\infty} \frac{x^{n-1}}{1+x} dx$$

$$\therefore \Gamma(1-n)\,\Gamma(n) = \frac{\pi}{\sin n\pi} \quad \left[\because \int_0^{\infty} \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}\right]$$

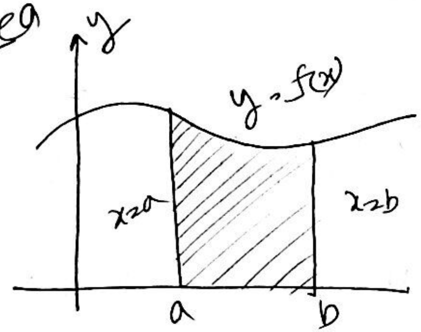
(Proved)

Exercise: Prove that $\int_0^{\pi/2} \tan^p\theta \, d\theta = \pi/2$

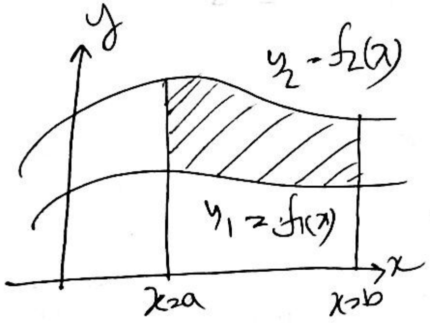
$\sec \frac{p\pi}{2}$

New chapter

Area

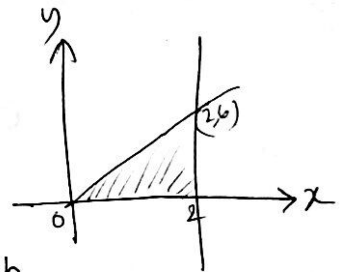


$$\text{Area}, A = \int_a^b y\,dx$$



$$\text{Area}; A = \int_a^b (y_2 - y_1)\,dx$$

Ex:- Find the Area enclosed by $y = 3x$, x-axis and $x = 2$.



$$\text{Area} = \int_a^b (y_2 - y_1)\,dx$$

$$= \int_0^2 3x\,dx$$

$$= 3\left[\frac{x^2}{2}\right]_0^2$$

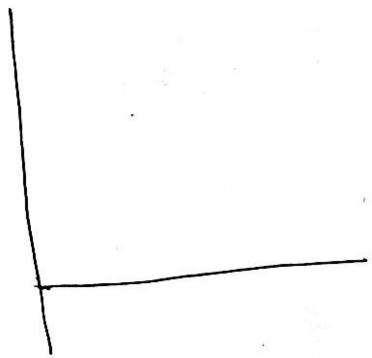
$$= 6.$$

$$\int_0^2 y\,dx$$

$$= \int_0^2 3x\,dx$$

$$= 3\left[\frac{x^2}{2}\right]_0^2$$

$$= 6$$



20/20

Ex2: Find the Area between the curves $y^2=4x$, $y=x$ from $x=0$ to $x=4$

$$\text{Area} = \cancel{\int_0^4 (4x-x)\,dx}$$

$$= 2\int_0^4 \sqrt{x}\,dx - \int_0^4 x\,dx$$

$$= 2\cdot\tfrac{2}{3}\,x^{3/2} - \left[\frac{x^2}{2}\right]_0^4$$

$$= \tfrac{4}{3}\cdot 8 - 8$$

$$= 8\left(\tfrac{4}{3}-1\right)$$

$$= \frac{8}{3}$$

$x^2=4x$

$\Rightarrow x^2-4x=0$

$\Rightarrow x(x-4)=0$

$\therefore x=0,4$

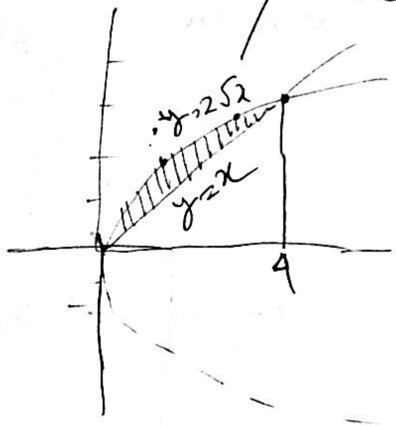
figure must be

or

$y^2=4x$

$\therefore y=2\sqrt{x}$

x	0	1	4
y	0	2	4



$$\text{Area} = \int_0^4 (y_2-y_1)\,dx$$

$$= \left[2\cdot\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^4$$

$$= \left[\tfrac{4}{3}(4)^{3/2} - 8\right] - 0$$

$$= \tfrac{4}{3}\cdot 8 - 8$$

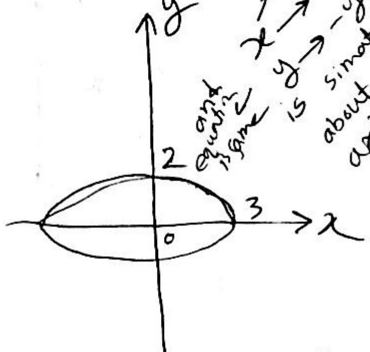
$$= \frac{32-24}{3} \text{ sq. unit}$$

$$= 8/3$$

15-12-2015 // → মঙ্গলবার

Ex: Find the area of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ by using integration

Solution:



$x \to -x$ } is replaced by $y \to -y$ and equation is same / is simatric about the y axis / x-axis

Given,

$$\frac{x^2}{9}+\frac{y^2}{4}=1$$

$$\Rightarrow \frac{y^2}{4}=1-\frac{x^2}{9}$$

$$=\frac{9-x^2}{9}$$

$$y^2=\frac{4}{9}(9-x^2)$$

$$\therefore y=\frac{2}{3}\sqrt{9-x^2}$$

Let $x=3\sin\theta$

$\therefore dx=3\cos\theta\, d\theta$

x	0	3
θ	0	$\pi/2$

∴ The required Area is, $A=4\int_0^3 y\,dx$ → each portion same

$$=4\int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$$

$$=\frac{8}{3}\int_0^{\pi/2}\sqrt{9-9\sin\theta}\;3\cos\theta\,d\theta$$

$$=\frac{8}{3}\times 3\times 3\int_0^{\pi/2}\sqrt{1-\sin\theta}\;\cos\theta\,d\theta$$

$$=24\int_0^{\pi/2}\cos^2\theta\,d\theta$$

$$=12\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta$$

$$=12\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$=12\left[\left(\frac{\pi}{2}+\frac{1}{2}\sin\pi\right)-\left(0+\frac{1}{2}\cdot 0\right)\right]$$

$$=12\times\frac{\pi}{2}=6\pi \text{ square unit.}$$

Exercise: Find the area of the circle $x^2+y^2=a^2$ by using integration.

Ex: Find the area of the curve $ay^2 = x^3(2a-x)$

Sol: Given $ay^2 = x^3(2a-x)$ —— (1)

If $y=0$, then $x^3(2a-x)=0$

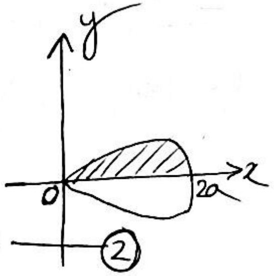
$\therefore x = 0, 2a$

Again replace y by $(-y)$ in (1), it follows that,

$ay^2 = x^3(2a-x)$

So (1) is the symmentric about x-axis

Hence, area $A = 2\int_0^{2a} y\,dx$



$= 2\int_0^{2a} \frac{1}{a} x^{3/2}\sqrt{2a-x}\,dx$ —— (2)

Let $x = 2a\sin^2\theta$

$\therefore dx = 4a\sin\theta\cdot\cos\theta\,d\theta$

x	0	$2a$
θ	0	$\pi/2$

Then (2) becomes

$A = \frac{2}{a}\int_0^{\pi/2} (2a\sin^2\theta)^{3/2}\sqrt{2a-2a\sin^2\theta}\cdot 4a\sin\theta\cdot\cos\theta\,d\theta$

$= \frac{2}{a}\cdot\sqrt{2a}\,(2a)^{3/2}\cdot 4a\int_0^{\pi/2}\sin^3\theta\cdot\sin\theta\cdot\cos^2\theta\cdot d\theta$

$= 32a^2\int_0^{\pi/2}\sin^4\theta\cdot\cos^2\theta\,d\theta$

$\beta\Gamma$ rules $\Rightarrow$

$= 32a^2\,\frac{\Gamma(5/2)\Gamma(3/2)}{2\Gamma(4)}$

$\left[\because \int_0^{\pi/2}\sin^m\theta\cdot\cos^n\theta\cdot d\theta = \frac{\Gamma(\frac{m+1}{2})\Gamma(\frac{n+1}{2})}{2\Gamma(\frac{m+n+2}{2})}\right]$

$= 32a^2\,\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2\cdot3\cdot2\cdot1}$

$= \pi a^2$ (Ans)

$\Gamma(5/2)$
$= \Gamma(3/2+1)$
$= \frac{3}{2}\Gamma(3/2)$
$= \frac{3}{2}\Gamma(\frac{1}{2}+1)$
$= \frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$
$= \frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}$

$n! = \Gamma(n+1) = n\Gamma n$

$\Gamma(1/2) = \sqrt{\pi}$

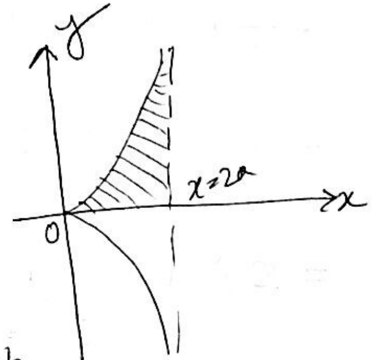
20 Dec রবিবার

Ex:- Find the area between the curve $y^2(2a-x)=x^3$ and its asymptote.

Sol:-

Given, $y^2(2a-x)=x^3$

$$\Rightarrow y^2 = \frac{x^3}{2a-x} \quad \text{--- (1)}$$



If we replace y by $(-y)$ then (1) is ~~asymptote~~ symmetric about x-axis.

$$\therefore A = 2\int_0^{2a} y\,dx$$

$$= 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a-x}}\,dx \quad \text{--- (ii)}$$

$$\therefore A = 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{2a}\cos\theta} 4a\sin\theta.\cos\theta\,d\theta$$

Let $x=2a\sin^2\theta$

$\therefore dx = 2a.2\sin\theta.\cos\theta\,d\theta$

$= 4a\sin\theta.\cos\theta\,d\theta$

x	0	$2a$
θ	0	$\pi/2$

$$= 2\int_0^{\pi/2} \frac{(2a)\sqrt{2a}\sin^4\theta}{\sqrt{2a}} 4a\,d\theta$$

$$= 16a^2\int_0^{\pi/2} \sin^4\theta.\cos^0\theta.d\theta$$

$$= 16a^2 \frac{\Gamma(5/2)\,\Gamma(1/2)}{2\Gamma(3)} \quad \left[\because \int_0^{\pi/2} \sin^m\theta.\cos^n\theta\,d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}\right]$$

$$= 16a^2 \frac{3/2.1/2\sqrt{\pi}.\sqrt{\pi}}{2.2.1}$$

$$= 3\pi a^2 \text{ Ans}$$

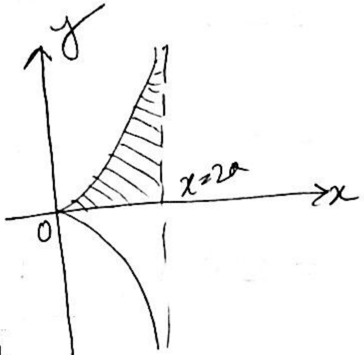
20 Dec → রবিবার

Ex: Find the area between the curve $y^2(2a-x)=x^3$ and its asymptote.

Sol:

Given, $y^2(2a-x)=x^3$

$$\Rightarrow y^2 = \frac{x^3}{2a-x} \quad \text{---(1)}$$



If we replace y by $(-y)$ then (1) is ~~asymptote~~ symmetric about x-axis.

$$\therefore A = 2\int_0^{2a} y\,dx$$

$$= 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{2a-x}}\,dx \quad \text{---(ii)}$$

Let $x = 2a\sin^2\theta$

$\therefore dx = 2a\cdot 2\sin\theta\cdot\cos\theta\, d\theta$

$= 4a\sin\theta\cdot\cos\theta\, d\theta$

x	0	$2a$
θ	0	$\pi/2$

$$\therefore A = 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{2a}\cos\theta}\, 4a\sin\theta\cdot\cos\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \frac{(2a)\sqrt{2a}\,\sin^4\theta}{\sqrt{2a}}\, 4a\, d\theta$$

$$= 16a^2\int_0^{\pi/2} \sin^4\theta\cdot\cos^0\theta\, d\theta$$

$$= 16a^2\, \frac{\Gamma(5/2)\,\Gamma(1/2)}{2\Gamma(3)} \qquad \left[\because \int_0^{\pi/2}\sin^m\theta\cdot\cos^n\theta\, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}\right.$$

$$= 16a^2\, \frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}}{2.2.1}$$

$$= 3\pi a^2 \quad \text{Ans}$$

* Find the area bounded by the curve.

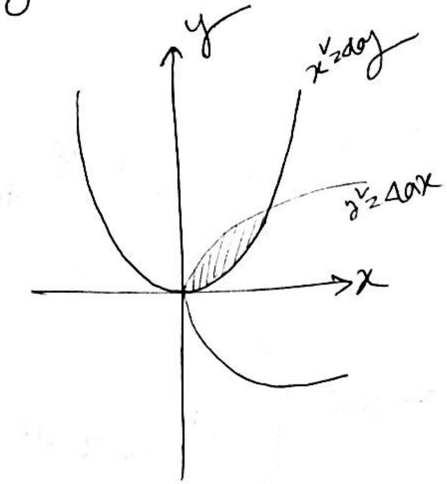
$y^2=4ax, \; x^2=4ay$

Sol:-

Given,

$y^2=4ax$ —— (1)

$x^2=4ay$ —— (2)



From (2)

$y=\frac{x^2}{4a}$

From (1)

$\frac{x^4}{16a^2}=4ax$

$\Rightarrow x^4=64a^3x$

$\Rightarrow x(x^3-64a^3)=0$

$\therefore x=0, 4a$

Hence, ~~area~~ Area, $A=\int_0^{4a}(y_1-y_2)\,dx$

$=\int_0^{4a}\left(2\sqrt{a}\sqrt{x}-\frac{x^2}{4a}\right)dx$

$=2\sqrt{a}\int_0^{4a}\sqrt{x}\,dx-\frac{1}{4a}\int_0^{4a}x^2\,dx$

$=2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a}-\frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$

$=\frac{4}{3}\sqrt{a}(4a)^{3/2}-\frac{1}{12a}(4a)^3$

$=\frac{32}{3}a^2-\frac{16}{3}a^2=\frac{16}{3}a^2$

Rules // Volume:

* if $y=f(x)$ then

volume, $V=\int_a^b \pi y^2\,dx$

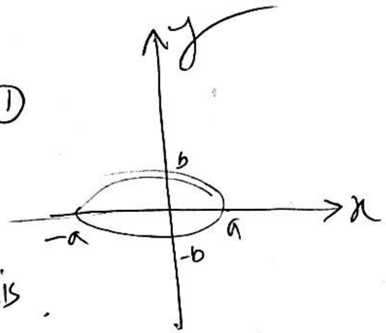
Ex: Find the volume of the eclipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ that is revolved ~~that~~ about x-axis.

ঘুরানোর

Solution

Given

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{--- (1)}$$



The equation is symmentric about both axis $-x$ and y axis.

$\therefore$ volume, $V=\int_{-a}^{a}\pi y^2\,dx$

$$=\int_{-a}^{a}\pi\cdot\frac{b^2}{a^2}(a^2-x^2)\,dx$$

$$=\frac{\pi b^2}{a^2}\int_{-a}^{a}(a^2-x^2)\,dx$$

$$=2\pi\cdot\frac{b^2}{a^2}\int_{0}^{a}(a^2-x^2)\,dx$$

$$=2\pi\,\frac{b^2}{a^2}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=2\pi\cdot\frac{b^2}{a^2}\cdot\left[a^3-\frac{a^3}{3}\right]$$

$$=\frac{4}{3}\pi ab^2\ \text{(Ans)}$$

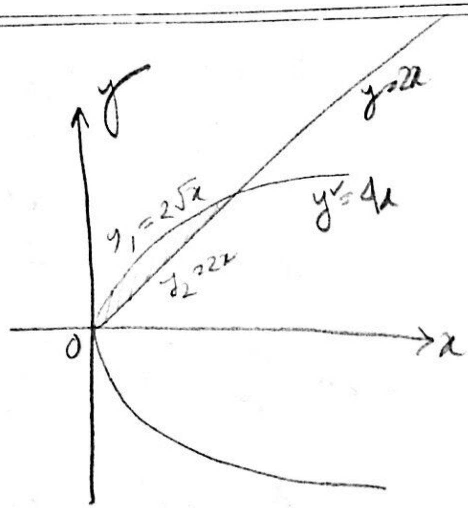
Ex2: Find the volume of the solid generated when the regiona between the curve $y^2=4x$ and $y=2x$ which is revolved about x-axis.

Sol:

Sol

$y^2 = 4x$ — (1)

$y = 2x$ — (ii)



(1) $\Rightarrow 4x^2 = 4x$

$\Rightarrow 4x(x-1) = 0$

$\Rightarrow x = 0, 1$

Hence vol, $V = \int_0^1 \pi (y_1^2 - y_2^2)\, dx$

$= \pi \int_0^1 (4x - 4x^2)\, dx$

$= \pi \left[4 \cdot \frac{x^2}{2} - 4 \cdot \frac{x^3}{3} \right]_0^1$

$= \frac{2\pi}{3}$ (Ans)

Ex: A loop of a curve $(x-4a)y^2 = ax(x-3a)$ is revolved about the x-axis, then find its volume.

Sol:

Given, $(x-4a)y^2 = ax(x-3a)$

where x-cuts $\Rightarrow y^2 = \frac{ax(x-3a)}{(x-4a)}$ — (1)

If $y = 0$, it follows that $ax(x-3a) = 0$

$\therefore x = 0, 3a$.

Hence volume, $V = \int_0^{3a} \pi y^2\, dx$

$= \int_0^{3a} \pi \frac{ax(x-3a)}{(x-4a)}\, dx$

$= \int_0^{3a} \pi a \frac{x(x-4a+a)}{(x-4a)}\, dx$

$= \pi a \left[\int_0^{3a} x\, dx + a \int_0^{3a} \frac{x\, dx}{x-4a} \right]$

$$= \pi a\left[\int_0^{3a} x\,dx + a\int_0^{3a} \frac{x-4a+4a}{x-4a}\,dx\right]$$

$$= \pi a\left[\left[\frac{x^2}{2}\right]_0^{3a} + a\int_0^{3a}\left(1+\frac{4a}{x-4a}\right)dx\right]$$

$$= \pi a\left[\frac{9a^2}{2} + a\left[x+4a\ln(x-4a)\right]_0^{3a}\right]$$

$$= \frac{9}{2}\pi a^3 + \pi a^2(3a) + 4\pi a^3\ln\left(\frac{-a}{-4a}\right) \quad \frac{1}{4} \quad 2^{-2}$$

$$= \frac{9}{2}\pi a^3 + 3\pi a^3 + 4\pi a^3\left[-2\ln 2\right]$$

$$= \pi a^3\left[\frac{9}{2}+3-8\ln 2\right]$$

$$= \pi a^3\left(\frac{15}{2}-8\ln 2\right)$$

Exercise:-

(1) Find the volume of the solid that is obtained when the region under the curve $y=\sqrt{x}$ over the interval $[1,4]$ is revolved about the x-axis.

(2) Find the volume of the solid generated when the regions between the curves $y=\frac{1}{2}+x^2$ and $y=x$ over $[0,2]$ is revolved about x-axis

27 Dec:

Differential Equation:

An equation involving derivation of one or more dependent variable with respect to one or more independant variable is called a differential equation. As for example

$$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^2 = 0 \text{ --- } (1)$$

$$\frac{d^4x}{dx^4} + 5\frac{d^2x}{dt^2} + 5x = \sin t \text{ --- } (2)$$

$$\frac{\partial v}{\partial s} + \frac{\partial v}{\partial t} = v \text{ --- } (3)$$

Equation (1) and (2) are ordinary differential equation. and equation (3) is a partial differential equation.

Ordinary differential equation (ODE):

A differential equation involving ordinary derivatives of one or more dependent variable with respect to a single independent variable is called an ordinary differential equation.

Partial differential equation (PDE):

A differential equation involving partial derivatives of one or more dependent variables with respect to more than one independent variable is called a partial differential equation.

dependent

$$\frac{d^2y}{dx^2} + xy\frac{dy}{dx} + 3y = 0$$

independent

$y = x^2$

$\frac{dy}{dx} = 2x$

Order of the differential equation:

The order of the highest ordered derivative involved in a differential equation is called the order of the differential equation.

As for example,

1st degree

$\frac{d^2y}{dx^2} + xy\left(\frac{dy}{dx}\right)^6 = 0$ is of $\boxed{\text{2nd order}}$ differential equation.

1st degree

$\Rightarrow \frac{d^4x}{dt^4} + 5\frac{d^2x}{dt^2} + 3x = \sin t$ is of $\boxed{\text{4th order}}$ differential equation.

Degree of a differential equation:

The degree of the highest ordered derivative involved in a differential equation is called the degree of the differential equation.

As for example,

highest derivative

$$\left(\frac{d^2y}{dx^2}\right)^3 + 5x\left(\frac{dy}{dx}\right)^5 - 16\frac{dy}{dx} + xy = 0$$

$\boxed{\text{2nd order}}$

is of $\boxed{\text{3rd degree}}$ diff. equation.

Solution of differential equation:-

Ex: Solve: $\frac{dy}{dx} = 2e^{2x} + e^x - \cos x + 3x^2$; $y(0) = 1$

Sol: Given $\frac{dy}{dx} = 2e^{2x} + e^x - \cos x + 3x^2$

$\Rightarrow dy = 2e^{2x}\,dx + e^x\,dx - \cos x\,dx + 3x^2\,dx$

Integrating both sides, we get,

$$\int dy = 2\int e^{2x}dx + \int e^x dx - \int \cos x\,dx + 3\int x^2 dx$$

$$\Rightarrow y = 2\cdot\frac{e^{2x}}{2} + e^x - \sin x + 3\cdot\frac{x^3}{3} + c$$

$$= e^{2x} + e^x - \sin x + x^3 + c, \quad \text{—(1)}$$ ~~which~~

where c is the arbitrary constant.

না থাকলে this is the general — — — — equat

Using the initial condition $y(0)=1$ in ① we get,

$1 = e^0 + e^0 - \cancel{0} + 0 + c$

$\Rightarrow 1 = 1 + 1 + c$

$\Rightarrow c = -1$

Hence the required general solution.

$y = e^{2x} + e^x - \sin x + x^3 - 1$. (Ans)

Ex: $y'' = 2$, $y'(0) = 1$, $y(0) = 1$

Sol: Given, $y'' = 2$ → order 2

$\Rightarrow \frac{d^2y}{dx^2} = 2$

$\Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right) = 2$

$\Rightarrow d\left(\frac{dy}{dx}\right) = 2dx$

$\Rightarrow \int d\left(\frac{dy}{dx}\right) = 2\int dx$

$\Rightarrow \frac{dy}{dx} = 2x + c_1$ — ①

$\Rightarrow dy = 2x\,dx + c_1\,dx$

$\Rightarrow \int dy = 2\int x\,dx + c_1 \int dx$

$\Rightarrow y = 2 \cdot \frac{x^2}{2} + c_1 x + c_2$

$\Rightarrow y = x^2 + c_1 x + c_2$ — ②, where c_1 and c_2 are arbitrary constants.

Using the initial condition $y'(0) = 1$ in ① we get,

$1 = 0 + c_1 \quad \therefore \boxed{c_1 = 1}$

using the initial condition $y(0) = 1$ in ② we get

$1 = 0 + 1 \cdot 0 + c_2 \quad \boxed{\therefore c_2 = 1}$

Hence the required general solution is,

$y = x^2 + x + 1$.

Ex:- $x\cancel{y'}+y=e^x+\cos x-2x$.

Sol:-

Given:-

$$x\frac{dy}{dx}+y=e^x+\cos x-2x$$

$$\Rightarrow \frac{d}{dx}(xy)=e^x+\cos x-2x$$

$$\Rightarrow d(xy)=e^x dx+\cos x\,dx-2x\,dx$$

$$\Rightarrow \int d(xy)=\int e^x dx+\int \cos x\,dx-2\int x\,dx$$

$\Rightarrow xy=e^x+\sin x-x^2+c$, which is the required general solution.

Ex:-2

solution $x^2y'+2xy=3e^{3x}+4x^3-2\sin 2x$

Given,

$$x^2\frac{dy}{dx}+2xy=3e^{3x}+4x^3-2\sin 2x$$

$\Rightarrow$ ~~$x^2\frac{dy}{dx}+2xy$~~

$$\Rightarrow \frac{d}{dx}(x^2y)=3e^{3x}+4x^3-2\sin 2x.$$

$$\Rightarrow \int d(x^2y)=3\int e^{3x}dx+4\int x^3dx-2\int \sin 2x\,dx$$

Ans: $x^2y=e^{3x}+x^4+\cos 2x+c$.

Variable separable:-

Some standard integrals:-

① $\int \tan x\,dx = \ln|\sec x|$

② $\int \cot x\,dx = \ln|\sin x|$

(iii) $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$.

The solution of the differential equation of the form

$M(x)\,dx + N(y)\,dy = 0$ is $\int M(x)\,dx + \int N(y)\,dy = c$

Ex:- Solve:- $\tan x\,dx = \cot y\,dy$

$\Rightarrow \int \tan x\,dx = \int \cot y\,dy$

$\Rightarrow \ln|\sec x| = \ln|\sin y| + \ln c$

স্থিতর

$\Rightarrow \ln(\sec x) = \ln(c.\sin y)$

$\Rightarrow \sec x = c\sin y$

$\Rightarrow \sec x = c.\frac{1}{\operatorname{cosec} y}$

$\Rightarrow \sec x.\operatorname{cosec} y = c$ which is the required solution.

Ex2:- Solve: $\ln\left(\frac{dy}{dx}\right) = ax+by$

Sol:- Given, $\ln\left(\frac{dy}{dx}\right) = ax+by$

$\Rightarrow e^{\ln(dy/dx)} = e^{ax+by}$

$\Rightarrow \frac{dy}{dx} = e^{ax}.e^{by}$

$\Rightarrow \frac{dy}{dx} = \frac{e^{ax}}{e^{-by}}$

$\Rightarrow e^{ax}\,dx = e^{-by}.dy$

$\Rightarrow e^{ax}\,dx - e^{-by}\,dy = 0$

$\Rightarrow \int e^{ax}\,dx - \int e^{-by}\,dy = 0$ [integrating both sides]

$$\Rightarrow \frac{e^{ax}}{a} - \frac{e^{-by}}{(-b)} + c = 0$$

$\Rightarrow \frac{1}{a} e^{ax} + \frac{1}{b} e^{-by} + c = 0$, which is the required general solution.

Exercise solve: $\frac{dy}{dx} + 1 = y^2$

$$\Rightarrow \ln\left(\frac{dy}{dx} + 1\right) = \ln(y^2)$$

$$\Rightarrow \ln \frac{dy}{dx} =$$

$$\int dx = \int \frac{dy}{y^2 - 1}$$

$$\Rightarrow x = \frac{1}{2 \cdot 1} \ln\left|\frac{y-1}{y+1}\right| + c$$

$$\Rightarrow 2x = \ln\left(\frac{y-1}{y+1}\right) + c$$

$$\ln \frac{y-1}{y+1} = 2x + c$$

29 Dec ⇒ মঙ্গলবার

* The $\boxed{\text{solution}}$ of the differential equation of the form $M(y)\,dx + N(x)\,dy = 0$ is

$$\int \frac{dx}{N(x)} + \int \frac{dy}{M(y)} = c$$

Ex: solve the differential equation $y\,dx = x\,dy$

Sol: Given, $y\,dx = x\,dy$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\Rightarrow \ln x = \ln y + \boxed{\ln c}$$ → for calculation advantage

$$\Rightarrow \ln x = \ln(c \cdot y)$$

$$\Rightarrow e^{\ln x} = e^{\ln(c \cdot y)}$$

$\Rightarrow x = cy$. which is the required solution.

(Ans)

Ex: 2:

Solve: $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

Given,

$$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

$$\Rightarrow \frac{dy}{dx} + \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow dy = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} . dx$$

$$\Rightarrow \frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \int \frac{dy}{\sqrt{1-y^2}} + \int \frac{dx}{\sqrt{1-x^2}} = 0$$

$\Rightarrow \sin^{-1} y + \sin^{-1} x = c$, which is the required general solution.

Exercise:

Solve:

(i) $\tan x \, dy = \cot y \, dx$

(ii) $\frac{dy}{dx} = e^{3x-2y} + x^2 e^{-2y}$

Homogeneous differential equations:-

An equation of the form $\frac{dy}{dx} = \frac{f_1(x,y)}{f_2(x,y)}$ in which $f_1(x,y)$ and $f_2(x,y)$ are homogeneous functions of x and y of the same degree can be reduced to an equation in which variables are separable by putting $\boxed{y = v.x}$

$$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$$

all degree 2

Ex: Solve: $(x^2+y^2)\,dx + 2xy\,dy = 0$.

$f(x,y) = x^2+y^2+z^2+xy$

$f(x,y) = x^3+y^3+2xy^2$

Sol: Given,

$(x^2+y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow (x^2+y^2)\,dx = -2xy\,dy$

$\Rightarrow \frac{dy}{dx} = -\frac{(x^2+y^2)}{2xy}$ (homogeneous) — (1)

$\frac{2\ \ 2}{(1+1)(2)}$

So let, $y = v.x$

(uv) Rules → $\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$, then, equation (1) becomes.

$v + x\frac{dv}{dx} = -\frac{x^2+v^2x^2}{2x.vx} = -\frac{1+v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v$

$\Rightarrow x\frac{dv}{dx} = -\frac{1+v^2+2v^2}{2v}$

$\Rightarrow x\frac{dv}{dx} = -\frac{1+3v^2}{2v}$

(x) (v)

$\Rightarrow \frac{dx}{x} = -\left(\frac{2v}{1+3v^2}\right)dv$

$\Rightarrow \frac{dx}{x} = -\frac{1}{3}\left(\frac{6v}{1+3v^2}\right)dv$

$\Rightarrow \int\frac{dx}{x} = -\frac{1}{3}\int\frac{6v}{1+3v^2}\cdot dv$ ←

$$\boxed{\int\frac{f'(x)}{f(x)}dx = \ln|f(x)|}$$

$\Rightarrow \ln x = -\frac{1}{3}\ln(1+3v^2) + \ln c$

$\Rightarrow \ln x + \ln(1+3v^2)^{1/3} = \ln c$

$\Rightarrow \ln\{x(1+3v^2)^{1/3}\} = \ln c$

$\Rightarrow x\left(1+3\frac{y^2}{x^2}\right)^{1/3} = c$ Ans

Ex2: Solve: $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

Sol: Given, $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$

$\Rightarrow xy\frac{dy}{dx} - x^2\frac{dy}{dx} = y^2$ — (i)

(x) get this $\frac{dy}{dx}$ in left side

$\Rightarrow \frac{dy}{dx}(xy - x^2) = y^2$

$\Rightarrow \frac{dy}{dx} = \frac{y^2}{xy - x^2}$ (Homogeneous) — (1)

Let, $y = v.x$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$

Then from ① we get,

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2} = \frac{v^2}{v-1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{v-1} - v$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v-1}{v}\cdot dv$$ [variable separable]

$$\Rightarrow \frac{dx}{x} = \left(1 - \frac{1}{v}\right) dv$$

$$\Rightarrow \int \frac{dx}{x} = \int \left(1 - \frac{1}{v}\right) dv$$

$$\Rightarrow \ln x = v - \ln v + \ln c$$

$$\Rightarrow \ln x + \ln v = v + \ln c$$

$$\Rightarrow \ln(v.x) = v + \ln c$$

$$\Rightarrow e^{\ln(v.x)} = e^{(v + \ln c)}$$

$$\Rightarrow v.x = \cancel{v + \ln c}\ e^{v}\cdot c$$

$$\Rightarrow \cancel{\quad} y/x \cdot x = e^{y/x}\cdot c \quad [\because y = vx]$$

$\Rightarrow y = ce^{y/x}$, which is our required general solution.

Exercise 1 Ex. 2 (P.10) $\Rightarrow$ Part ①

$\rightarrow$ step by step

Page — differential equation Sharma Book

12 Jan 2016

Ex 3: Solve: $\underline{(x-2e^y)}_{N}\,dy + \underline{(y+x\sin x)}_{M}\,dx = 0$

Sol:- Here, $M = y + x\sin x$ and $N = x - 2e^y$

Now, $\dfrac{\partial M}{\partial y} = 1$ and $\dfrac{\partial N}{\partial x} = 1$

same result

since, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ ⇒ so the given equation is $\boxed{\text{exact.}}$

1st step

Now, $\int M\,dx = \int (y + x\sin x)\,dx = \int y\,dx + \int x\sin x\,dx$

(y as constant; function, operation)

$= yx + \left[x\int \sin x\,dx - \int\left\{\frac{d}{dx}(x)\int \sin x\,dx\right\}dx\right]$

$= xy + \left[x(-\cos x) - \int 1.(-\cos x)\,dx\right]$

$= xy + [-x\cos x + \sin x]$

$= xy - x\cos x + \sin x$

2nd step

In N, terms free from x is $-2e^y$.

So, $\int -2e^y\,dy = -2e^y$

$M\,dx + N\,dy = 0$

$\Rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

3rd step

Hence, the general solution is

$xy - x\cos x + \sin x - 2e^y = c$

Ex 4: Solve $\underline{(1+e^{x/y})}_{M}\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$

Sol: Here $M = 1 + e^{x/y}$ and $N = e^{x/y}\left(1 - \frac{x}{y}\right)$

Now $\dfrac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right)$

and $\dfrac{\partial N}{\partial y} = e^{x/y}\left(-\frac{1}{y}\right) + \left(1-\frac{x}{y}\right)\cdot e^{x/y}\left(\frac{1}{y}\right)$

$= -\frac{x}{y^2}\cdot e^{x/y}$.

Since, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, so the given equation is exact.

Now,

$$\int M dx = \int (1+e^{x/y})\, dx$$

$$= x + \frac{e^{x/y}}{(1/y)} = x + y\, e^{x/y}$$

In N, there is no x free term.

Hence the general solution is,

$$x + y\, e^{x/y} = c \quad \text{(Ans)}$$

Exercise:- Ex 8. (P. 40)

Linear differential equations:-

A differential equation of the form $\frac{dy}{dx} + Py = Q$ — (1)
where P, Q are functions of x or constants, is called the linear.
differential equation of first order

To solve this equation, multiply both sides by $e^{\int P dx}$

Then it becomes,

$$e^{\int P dx} \frac{dy}{dx} + P y e^{\int P dx} = Q . e^{\int P dx}$$

$$\Rightarrow \frac{d}{dx}\left[y e^{\int P dx}\right] = Q . e^{\int P dx}$$

Integrating both the side, with respect to x, we get—

$$y e^{\int P dx} = \int \left[Q\, e^{\int P dx}\right] dx + c,$$ which

is the required solution.

Integrating Factor (I.F): It will be noticed that for solving (9), we multiplied it by a factor $e^{\int P dx}$ and the equation become read (directly) integrable. Such a factor is called the integrating factor.

Note:- If the linear differential equation is of the form $\frac{dx}{dy} + Px = Q$, where, ~~tand~~ Q are functions of y or constants rely the integrating factor is $e^{\int P dy}$ and the solution is.

$$x e^{\int P dy} = \int [Q e^{\int P dy}] dy + c,$$

Find out ————

solution ————

12- JAN - রবি

Ex:-

Solve: $x\frac{dy}{dx} + 2y = x^2 \ln x$

Sol Given,

$$x\frac{dy}{dx} + 2y = x^2 \ln x$$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x} y = x \ln x$$

$\frac{dy}{dx} + Py = Q$

I.F $= e^{\int P dx}$

$y e^{\int P dx} = \int Q e^{\int P dx} dx + c$

Now, $I.F = e^{\int (2/x) dx} = e^{2\ln x} = e^{\ln x^2} = x^2$

Hence the solution, $y \cdot x^2 = \int x^2 \cdot x \ln x \, dx + c$

$$= \int x^3 \ln x \, dx + c$$

$$= \left[\ln x \int x^3 dx - \int \left\{ \frac{d}{dx}(\ln x) \int x^3 dx \right\} dx + c\right.$$

$$\Rightarrow y x^2 = \left[\ln x \cdot \frac{x^4}{4} - \int \frac{1}{x} \cdot \frac{x^3}{4} dx\right] + c$$

$\Rightarrow yx^2 = \frac{1}{4}x^4 \ln x - \frac{1}{4}\cdot\frac{x^4}{4} + c$

$= \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4 + c$

$\Rightarrow y = \frac{1}{4}x^2 \ln x - \frac{1}{16}x^2 + cx^{-2}$

$= \frac{1}{4}x^2(\ln x - \frac{1}{4}) + cx^{-2}$

Ex-2:- If $\frac{dy}{dx} + 2y\tan x = \sin x$ and if $y = 0$ when $x = \pi/3$, express y in terms of x.

Sol:- The given differential equation is linear.

So, I.F $= e^{\int 2\tan x\, dx}$

$= e^{2\ln(\sec x)}$

$= e^{\ln(\sec x)^2}$

$= \sec^2 x$

$\rightarrow \frac{dy}{dx} + Py = Q$

I.F $= e^{\int P dx}$

So the general solution is,

$y.\sec^2 x = \int \sin x . \sec^2 x . dx + c$

$= \int \sec x . \tan x . dx + c$

$\Rightarrow y.\sec^2 x = \sec x + c$ —— (1)

where, $x = \pi/3$, then $y = 0$, then from (1) we get

$0 = \sec \pi/3 + c$.

$\Rightarrow c + 2 = 0 \therefore \boxed{c = -2}$

Hence from (1)

$y\sec^2 x = \sec x - 2$.

$\Rightarrow y = \cos x - 2\cos^2 x$, which is the required solution

Exercise solution

① $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

② $(1+x)\cdot\frac{dy}{dx} + 3y = \frac{1+x+x^2}{(1+x)^3}$

Ex: Solve: $dx + x\,dy = e^{-y}\ln y\ dy$

Sol:

Given, $dx + x\,dy = e^{-y}\ln y\ dy$

$\Rightarrow \frac{dx}{dy} + x = e^{-y}\ln y$

Now, IF. $= \cancel{e^{\int x dx}} \cdot e^{\int P dy}$ [dependent = $\frac{dy}{dx} + Py = 0$, I.F $= e^{\int P dx}$]

$= e^{\int 1 dy} = e^{y}$

Hence the general solution is

$x \cdot e^{y} = \int e^{y} \ln y\, e^{-y} \cdot dy + c$

$= \int \ln y\, dy + c$

$= \ln y \int dy - \int \left\{ \frac{d}{dx}(\ln y) \int dy \right\} dy + c$

$= y\ln y - \int \frac{1}{y} \cdot y\ dy + c$

$\Rightarrow x \cdot e^{y} = y\ln y - y + c$

Exercise

Solve: $(1+y^2)\,dx + (x - \tan^{-1}y)\,dy = 0$

Exercise solution

① $x(x-1)\frac{dy}{dx} - y = x^2(x-1)^2$

② $(1+x)\cdot\frac{dy}{dx} + 3y = \frac{1+x+x^2}{(1+x)^3}$

Ex:- Solve: $dx + x\,dy = e^{-y} \ln y \, dy$

Sol:

Given, $dx + x\,dy = e^{-y} \ln y \, dy$

$\Rightarrow \frac{dx}{dy} + x = e^{-y} \ln y$

Now, IF. ~~$e^{\int x dx}$~~ $e^{\int P dy}$ [dependent = dy]

$\frac{dy}{dx} + Py = 0$

$= e^{\int 1 dy} = e^{y}$

$I.F = e^{\int P dx}$

Hence the general solution is

$x.e^{y} = \int e^{-y} \ln y \, e^{y} \cdot dy + c$

$= \int \ln y \, dy + c$

$= \ln y \int dy - \int \left\{ \frac{d}{dx}(\ln y) \int dy \right\} dy + c$

$= y \ln y - \int \frac{1}{y} \cdot y \, dy + c$

$\Rightarrow x.e^{y} = y \ln y - y + c$

Exercise

Solve: $(1+y^2)\,dx + (x - \tan^{-1} y)\,dy = 0$

Equations reducible to linear form:-

Bernoulil Equation:-

$$\frac{dy}{dx}+Py=Qy^n \quad \text{---} (1)$$ where P and Q are functions of x or constants.

Dividing both ~~series~~ sides of (1) by y^n we get

$$y^{-n}\frac{dy}{dx}+Py^{1-n}=Q \quad \text{---} (ii)$$

Now put $y^{1-n}=v$

$$\therefore (1-n)y^{-n}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow y^{-n}\frac{dy}{dx}=\frac{1}{1-n}\frac{dv}{dx}$$

Then equ^n (ii) becomes—

$$\frac{1}{1-n}\frac{dv}{dx}+Pv=Q.$$

$$\Rightarrow \frac{dv}{dx}+P(1-n)v=(1-n)Q.$$ which is a linear equation in v and x

Ex1: Solve: $\frac{dy}{dx}=x^3y^3-xy$

Given,

$$\frac{dy}{dx}=x^3y^3-xy$$

$$\Rightarrow \frac{dy}{dx}+xy=x^3y^3$$

Dividing by y^3 we get,

$$\frac{1}{y^3}\frac{dy}{dx}+x\frac{1}{y^2}=x^3 \quad \text{---} (i)$$

put $1/y^2=v$

$$\Rightarrow (-2)y^{-3}\frac{dy}{dx}=\frac{dv}{dx} \quad [\text{Diff both side}]$$

$$\Rightarrow \frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\cdot\frac{dv}{dx}$$

Then equ^n. (ii) becomes.

$$-\frac{1}{2}\cdot\frac{dv}{dx}+vx=x^3$$

$$\Rightarrow \frac{dv}{dx}-2vx=-2x^3,$$ which is a linear in v and x

Now,

$I.F = e^{\int -2x\,dx} = e^{-x^2}$ $\qquad e^{\int P dx}$ v

Hence the general solution is. $\quad \to x e^{\int P dy} = \int [Q \cdot e^{\int P dy}]\,dy + c$

$v e^{-x^2} = \int -2x^3 e^{-x^2} dx + c$

$= \int x^2 (-2x) e^{-x^2} dx + c$

$= \int -t e^{t} dt + c$ [where, $t = -x^2$
$\therefore dt = -2x\,dx$]

$= -t \int e^t dt - \int \left\{ \frac{d}{dt}(-t) \int e^t dt \right\} dt + c$

$= -t e^t - \int (-1) e^t dt + c$

$\therefore v e^{-x^2} = -t e^t + e^t + c$

$\Rightarrow v e^{-x^2} = x^2 e^{-x^2} + e^{-x^2} + c$

$\Rightarrow v = x^2 + 1 + c e^{x^2}$

$1/y^2 = 1 + x^2 + c e^{x^2}$ Ans

Ex: Ex.3 (P.24)
E.4 (P.24).

Quiz ⇒ From Gamma-Beta function
- Linear differential Equation.
on. 26.01.16 → মঙ্গলবার

18 Jan → মঙ্গলবার

(4) Equation $f'(y)\frac{dy}{dx} + Pf(y) = Q$ —— (1) where

P and Q are functions of x or constants.

Then put $f(y) = v$.

$\therefore f'(y)\frac{dy}{dx} = \frac{dv}{dx}$ [diff both side].

$\therefore$ Equation (1) becomes, $\frac{dv}{dx} + Pv = Q$, which is a linear equation in v and x.

Ex: Solve: $(x-y^2)\,dx + 2xy\,dy = 0$

Sol: Given, $(x-y^2)\,dx + 2xy\,dy = 0$

$\Rightarrow 2xy\frac{dy}{dx} + x - y^2 = 0$

$\Rightarrow \underbrace{2y}_{f'(y)}\frac{dy}{dx} - \frac{1}{x}\underbrace{y^2}_{f(y)} = -1$ —— (1)

Now, put $y^2 = v$.

$\therefore 2y\frac{dy}{dx} = \frac{dv}{dx}$ [Diff. Both sides]

Then the eq^n (1) becomes, $\frac{dv}{dx} - \frac{1}{x}v = -1$ (linear)

So, the I.F $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = x^{-1} = \frac{1}{x}$

Hence, the general solution is,

$v \cdot \frac{1}{x} = \int (-1)\frac{1}{x}\,dx + c$

$\Rightarrow \frac{v}{x} = -\ln x + c$

$\Rightarrow \frac{y^2}{x} = c - \ln x$. Ans

Exercise:

Ex. 12 (P. 28)

Ex. 15 (P. 29)

$e^{mx} \rightarrow$ trial solution

Linear differential equations with constant co-efficient

first order linear $\frac{dy}{dx}+Py=Q$

A differential equation of the form.

$$\frac{d^n y}{dx^n}+P_1\frac{d^{n-1}y}{dx^{n-1}}+P_2\frac{d^{n-2}y}{dx^{n-2}}+\dots+P_n y=X,$$

where $P_1, P_2 \dots P_n$. and X are functions of x or constants, is called a linear differential equation of nth order.

If $P_1, P_2, \dots, P_n$ are all constants (not function of x) and X is some function of x, then the equation is a linear differential equation with constant co-efficients.

As for example,

$$\frac{d^3y}{dx^3}\underbrace{-13}_{P_1}\frac{dy}{dx}\underbrace{-12}_{P_2}y=0$$

Roots of Auxillary Equation	complete Solution
Case I: All roots $m_1, m_2, \dots m_n$ are real and different	$y=c_1 e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}+\dots+c_ne^{m_nx}$
Case II: $m_1=m_2$, but other roots are real and different. $\left[1, \overbrace{2, 2}^{same}, 3\right] \rightarrow$	$y=(c_1+c_2x)e^{m_1x}+c_3e^{m_3x}+\dots+c_ne^{m_nx}$ $\boxed{y=(c_1+c_2x)e^{2x}+c_3e^{1.x}+c_4e^{3x}}$
Case III: (Imaginary Roots)	
1. $\alpha \pm i\beta$, a pair of imaginary roots.	corresponding part of the general solution is, $e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ or, $c_1e^{\alpha x}\cos(\beta x+c_2)$ or, $c_1e^{\alpha x}\sin(\beta x+c_2)$
2 $(\alpha \pm i\beta), (\alpha \pm i\beta)$ related twice. $[3 \pm i2] \Rightarrow$	corresponding part of the general solution is $y=\boxed{e^{\alpha x}(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x}$

24 JAN ⇒ রবি ⇒

$D = \frac{d}{dx}$

Ex:- $D^2y - 3Dy + 2y = 0$

$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$

Sol:- Given, $D^2y - 3Dy + 2y = 0$ —(1)

Let $y = e^{mx}$ be the trial solution of (1).

Then from (1) we get,

$D^2 e^{mx} - 3D e^{mx} + 2e^{mx} = 0$

$\Rightarrow m^2 e^{mx} - 3me^{mx} + 2e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 - 3m + 2) = 0$

trial solution ~~হবে না~~

So the auxiliary equation (A.E) is

$m^2 - 3m + 2 = 0$

$\Rightarrow m^2 - 2m - m + 2 = 0$

$\Rightarrow m(m-2) - 1(m-2) = 0$

$\Rightarrow (m-2)(m-1) = 0$

$\therefore m = 1, 2$

So the general solution (G.S) is

$y = c_1 e^{1 \cdot x} + c_2 e^{2x}$ ⇏ [previous case]

$= c_1 e^{x} + c_2 e^{2x}$ (Ans)

Ex2: Solution:

$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$ —(1)

Let $y = e^{mx}$ be the trial ~~equation~~ solution of (1)

Then from (1) we get

$\frac{d^2}{dx^2}(e^{mx}) - 6\frac{d}{dx}(e^{mx}) + 9e^{mx} = 0$

$\Rightarrow m^2 e^{mx} - 6me^{mx} + 9e^{mx} = 0$

$\Rightarrow e^{mx}(m^2 - 6m + 9) = 0$

So the auxiliary equation (A.E) is

$m^2 - 6m + 9 = 0$

$\Rightarrow m^2 - 3m - 3m + 9 = 0$

$\Rightarrow m(m-3) - 3(m-3) = 0$

$\Rightarrow m = 3, 3.$

G.S is $y = (c_1 + c_2x)e^{3x}$ Ans.

if $m = 3, 3, 3$

$y = (c_1 + c_2x + c_3x^2)e^{3x}$

$m = 2, 2, 3, 3$

$y = (c_1 + c_2x)e^{2x} + (c_3 + c_4x)e^{3x}$

$\left(D^2+6D+4\right)y=0$

$\frac{d^2}{dx^2}(e^{mx}$

So the auxiliary equation (A.E) is

$$m^2+6m+4=0$$

$$=\frac{-6\pm\sqrt{36-4.1.4}}{2.1}$$

$$=\frac{-6\pm\sqrt{20}}{2}$$

$$=-3\pm\sqrt{5}$$

So, the general solution (G.S) is

$$y=c_1e^{(-3+\sqrt{5})x}+c_2e^{(-3-\sqrt{5})x}$$

$$=c_1e^{-3x}\cdot e^{\sqrt{5}x}+c_2\cdot e^{-3x}\cdot e^{-\sqrt{5}x}$$

$$=e^{-3x}\left(c_1e^{\sqrt{5}x}+c_2e^{-\sqrt{5}x}\right)\text{ A}$$

$m=2\pm3i$ $\quad \alpha\pm i\beta$

$y=e$

So the general solution (G.S) is

$$y=e^{2x}(c_1\cos3x+c_2\sin3x)\text{ A.}$$

$(D^2+2)^2y=0$

$y=e^{mx}$

So the a. eq (A.E) is $(m^2+2)^2=0$

$\Rightarrow m^2+2=0$ | $m^2+2=0$

$\Rightarrow m=-\sqrt{2}$ | $m=-\sqrt{2}$

| $m=0\pm\sqrt{2}i$

$\therefore m=\pm\sqrt{-2}$

$=\pm\sqrt{2}.i$

$=\pm\sqrt{2}.i$

So the general solution (G.S) is

$y=e^{x}\left[(c_1+c_2x)\cos\sqrt{2}x+(c_3+c_4x)\sin\sqrt{2}x\right]$

$=(c_1+c_2x)\cos\sqrt{2}x+(c_3+c_4x)\sin\sqrt{2}x=0$ Ans

Exercise:

Solve $y'''-4y''+y'+6y=0$

Rules:-

① $(1-D)^{-1}=1+D+D^2+\cdots$

② $(1+D)^{-1}=1-D+D^2-\cdots$

③ $(1-D)^{-2}=1+2D+3D^2+\cdots$

④ $(1+D)^{-2}=1-2D+3D^2-\cdots$

Ex: Solve: $(D^2-4D)y=5$ → constant

$\Rightarrow$ Given, $(D^2-4D)y=5$ — ①

Let, $y=e^{mx}$ be the trial solution of

$(D^2-4D)y=0$ — ② → consider as 0

$(D^2+2)^2 y = 0$

$y = e^{mx}$

So the a. eq (A.E) is $(m^2+2)^2 = 0$

$\Rightarrow m^2 + 2 = 0$ | $m^2+2=0$

$\Rightarrow m^2 = -2$ | $m^2 = -2$

$m = \pm\sqrt{-2}$ | $m = 0 \pm \sqrt{2} i$

$= \pm\sqrt{2}.i$

$= \pm\sqrt{2}.i$

So the general solution (G.S) is

$y = e^{0x}\left[(c_1+c_2x)\cos\sqrt{2}x + (c_3+c_4x)\sin\sqrt{2}x\right]$

$= (c_1+c_2x)\cos\sqrt{2}x + (c_3+c_4x)\sin\sqrt{2}x$ Ans

Exercise:

Solve $y''' - 4y'' + y' + 6y = 0$

Rules:-

① $(1-D)^{-1} = 1 + D + D^2 + \dots$

② $(1+D)^{-1} = 1 - D + D^2 - \dots$

③ $(1-D)^{-2} = 1 + 2D + 3D^2 + \dots$

④ $(1+D)^{-2} = 1 - 2D + 3D^2 - \dots$

Ex: Solve: $(D^2-4D)y = 5$ → constant

$\Rightarrow$ Given, $(D^2-4D)y = 5$ —①

Let, $y = e^{mx}$ be the trial solution of

$(D^2-4D)y = 0$ —② consider as 0

From (2) $\Rightarrow (D^2-4D)e^{mx}=0$

$\Rightarrow (m^2-4m)e^{mx}=0$

So. the AE is $m^2-4m=0$

$\Rightarrow m(m-4)=0$

$\therefore m=0,4.$

not 0

So, the complimentary function (C.F) is

$y_c = c_1 e^{0.x} + c_2 e^{4x}$

$= c_1 + c_2 e^{4x}$

Now, the particular integral (P.I) is,

$y_p = \frac{1}{(D^2-4D)} \cdot 5$

$= \frac{-1}{4D-D^2} \cdot 5$

$= \frac{-1}{4D\left(1-\frac{D^2}{4D}\right)} \cdot 5$

$= -\frac{1}{4D}\left(1-\frac{D}{4}\right)^{-1} 5$

to get rules

(A)

$= -\frac{1}{4D}\left(1+\frac{D}{4}+\frac{D^2}{16}+\ldots\right)5$

$= -\frac{1}{4}\left(\frac{1}{D}+\frac{1}{4}+\frac{D}{16}+\ldots\right)5$

$= -\frac{1}{4}\left(5x+\frac{5}{4}\right)$

before getting zero

$= -\frac{5}{4}x - \frac{5}{16}$

$$\boxed{\begin{array}{l} D = \frac{d}{dx} \\ \frac{1}{D}(5) \\ \rightarrow \int 5\,dx = 5x \end{array}}$$

So the general solution (G.S) is

$y = y_c + y_p$

$\Rightarrow y = c_1 + c_2 e^{4x} - \frac{5}{4}x - \frac{5}{16}$ Ans

$\frac{D}{16}5 = \frac{1}{16}\frac{d}{dx}(5)$

$= 0$

Ex: Solve $(D^2+4)y = x^2$

Let $y = e^{mx}$ be the trial solution of $(D^2+4)y=0$ ——(ii)

From (ii) $\Rightarrow (D^2+4)e^{mx} = 0$

$\Rightarrow (m^2+4)e^{mx} = 0$

So, the A.E is $m^2+4=0$

$\Rightarrow m^2 = -4$

$\Rightarrow m = \sqrt{-4} = \sqrt{4}\, i = \pm 2i$

So, the complimentary function (C.F) is

$y_c =$

Now the particular integral (P.I) is $y_p = \frac{1}{D^2+4}x^2$

$= \frac{1}{4(1+\frac{D^2}{4})}x^2$

$= \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2$

$= \frac{1}{4}\left(1-\frac{D^2}{4}+\frac{D^4}{16}-\cdots\right)x^2$

$= \frac{1}{4}\cdot\left(x^2-\frac{2}{4}+0\right)$

$= \frac{1}{4}\left(x^2-\frac{1}{2}\right)$

$= \frac{1}{8}(2x^2-1)$

So, the general solution (G.S) is

$y = y_c + y_p$

$\Rightarrow y = c_1\cos 2x + c_2\sin 2x + \frac{1}{8}(2x^2-1)$ Ans

Exercise

Solve $(D^2+a^2)y = x^2$

Ex:
$$\frac{d^2y}{dx^2}+y=x^2,\quad y(0)=0,\ y'(1)=1$$

Sol: $(D^2+1)y=x^2$

Let, $y=e^{mx}$

$$(m^2+1)e^{mx}=0$$

$\Rightarrow m^2+1=0$

$\Rightarrow m=\pm 1$

$$y_c=c_1\cos x+c_2\sin x.$$

Now, the particular integral (P.I) is $y_p=\frac{1}{(D^2+1)}x^2$

$=$

G.S is $y=y_c+y_p$

$=c_1\cos x+c_2\sin x+x^2-2 \quad \text{—(A)}$

$y'=-c_1\sin x+c_2\cos x+2x \quad \text{—(B)}$

$y(0)=c_1-2$

$\Rightarrow 0=c_1-2$

$\Rightarrow c_1=2.$

$y'(1)=1$, in (B),

$1=-2\sin 1+c_2\cos 1+2$

$$c_2=\frac{2\sin 1-1}{\cos 1}$$

Rules:

① $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$, where $f(a) \neq 0$

② $\frac{1}{f(D)} \cdot e^{ax} \cdot v(x) = e^{ax} \frac{1}{f(D+a)} v(x)$, where

$v(x)$ is a function of x.

③ $\frac{1}{f_1(D^2)} \cdot \sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}$, when $f_1(-a^2) \neq 0$

④ $\frac{1}{f_1(D^2)} \cos(ax+b) = \frac{\cos(ax+b)}{f_1(-a^2)}$, when, $f_1(\cancel{a^2}) \neq 0$

Ex: Solve, $(D+4)^2 y = e^{4x}$

$(m+4)^2 e^{mx} = 0$

$m^2 + 2.4m + 16 = 0$

$\Rightarrow m^2 + 8m + 16 = 0$

$\Rightarrow m^2 + 4m + 4m + 16 = 0$

$\Rightarrow (m+4)(m+4) = 0$

$\Rightarrow$ $\cancel{(m+4)^2 = 0}$

$\Rightarrow m = -4, -4$ //

?

$y_c = (c_1 + c_2 x) e^{-4x}$

$y_p = \frac{1}{(D+4)^2} e^{4x}$

$= \frac{e^{4x}}{(4+4)^2} = \frac{e^{4x}}{64}$.

So, the G.S is $y = y_c + y_p$

$= (c_1 + c_2 x) e^{-4x} + \frac{1}{64} e^{4x}$

Ex: Solve: $(D^2 - 36) y = \cos 6x$

$(m^2 - 36) e^{mx} = 0$

$m^2 - 36 = 0$

$\Rightarrow m = \pm 6$

$y_c = c_1 e^{6x} + c_2 e^{-6x}$

Rules:

① $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$, where $f(a) \neq 0$

② $\frac{1}{f(D)} \cdot e^{ax} \cdot v(x) = e^{ax} \frac{1}{f(D+a)} v(x)$, where

$v(x)$ is a function of x.

③ $\frac{1}{f_1(D^2)} \cdot \sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}$, when $f_1(-a^2) \neq 0$

④ $\frac{1}{f_1(D^2)} \cos(ax+b) = \frac{\cos(ax+b)}{f_1(-a^2)}$, when $f_1(-a^2) \neq 0$

Ex: Solve $(D+4)^2 y = e^{4x}$

$(m+4)^2 e^{mx} = 0$

$m^2 + 2.4m + 16 = 0$

$\Rightarrow m^2 + 8m + 16 = 0$

$\Rightarrow m^2 + 4m + 4m + 16 = 0$

$\Rightarrow (m+4)(m+4) = 0$

~~$(m+4)^2 = 0$~~

$\Rightarrow m = -4, -4$

?

$y_c = (c_1 + c_2 x) e^{-4x}$

$y_p = \frac{1}{(D+4)^2} e^{4x}$

$= \frac{e^{4x}}{(4+4)^2} = \frac{e^{4x}}{64}$.

So, the GS is $y = y_c + y_p$

$= (c_1 + c_2 x) e^{-4x} + \frac{1}{64} e^{4x}$

Ex:

Solve: $(D^2 - 36) y = \cos 6x$

$(m^2 - 36) e^{mx} = 0$

$m^2 - 36 = 0$

$\Rightarrow m = \pm 6$

$y_c = c_1 e^{6x} + c_2 e^{-6x}$

Rules:

① $\frac{1}{f(D)} e^{ax} = \frac{e^{ax}}{f(a)}$, where $f(a) \neq 0$

② $\frac{1}{f(D)} \cdot e^{ax} \cdot v(x) = e^{ax} \frac{1}{f(D+a)} v(x)$, where $v(x)$ is a function of x.

③ $\frac{1}{f_1(D^2)} \cdot \sin(ax+b) = \frac{\sin(ax+b)}{f(-a^2)}$, when $f_1(-a^2) \neq 0$

④ $\frac{1}{f_1(D^2)} \cos(ax+b) = \frac{\cos(ax+b)}{f_1(-a^2)}$, when $f_1(-a^2) \neq 0$

Ex: Solve, $(D+4)^2 y = e^{4x}$

$(m+4)^2 e^{mx} = 0$

$m^2 + 2.4m + 16 = 0$

$\Rightarrow m^2 + 8m + 16 = 0$

$\Rightarrow m^2 + 4m + 4m + 16 = 0$

$\Rightarrow (m+4)(m+4) = 0$

$\cancel{(m+4)^2 = 0}$

$\Rightarrow m = -4, -4$ //

? =

$y_c = (c_1 + c_2 x) e^{-4x}$

$y_p = \frac{1}{(D+4)^2} e^{4x}$

$= \frac{e^{4x}}{(4+4)^2} = \frac{e^{4x}}{64}$.

So, the G.S is $y = y_c + y_p$

$= (c_1 + c_2 x) e^{-4x} + \frac{1}{64} e^{4x}$ Ans.

Ex:-

Solve: $(D^2 - 36) y = \cos 6x$

$(m^2 - 36) e^{mx} = 0$

$m^2 - 36 = 0$

$\Rightarrow m = \pm 6$

$y_c = c_1 e^{6x} + c_2 e^{-6x}$

31-JAN

$$D^2y - Dy - 6y$$

Ex: Solve: $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 5\sin 2x$

Sol: Given,

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 5\sin 2x \quad \text{---} (1)$$

Let, $y = e^{mx}$ be the trial solution of $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6y = 0$

from (2) $\Rightarrow \dfrac{d^2}{dx^2}(e^{mx}) - \dfrac{d}{dx}(e^{mx}) - 6e^{mx} = 0$

$$\Rightarrow (m^2 - m - 6)e^{mx} = 0$$

So the A.E is $m^2 - m - 6 = 0$

$$\Rightarrow m^2 - 3m + 2m - 6 = 0$$

$$\Rightarrow m(m-3) + 2(m-3) = 0$$

$$\Rightarrow (m+2)(m-3) = 0$$

$$\therefore m = -2, 3.$$

so, $y_c = c_1 e^{-2x} + c_2 e^{3x}$

Now, the P.I. is $y_p = \frac{1}{D^2 - D - 6} 5\sin 2x$

$= 5 \frac{1}{-2^2 - D - 6} \sin 2x$

$= -5 \frac{1}{D+10} \sin 2x$

need D^2

$= -5 \frac{(D-10)}{(D+10)(D-10)} \sin 2x$

$= -5 \frac{(D-10)}{D^2 - 100} \sin 2x$

$= -5 \frac{(D-10)}{-2^2 - 100} \sin 2x$

differentiation

$= \frac{5}{104} (D-10) \sin 2x$

$= \frac{5}{104} (2\cos 2x - 10\sin 2x)$

$= \frac{5}{52} (\cos 2x - 5\sin 2x)$

so, the G.S is $y = y_c + y_p$

$\Rightarrow y = c_1 e^{-2x} + c_2 e^{3x} + \frac{5}{52} (\cos 2x - 5\sin 2x)$

Ans

* $\frac{1}{f_1(D^2)} \sin(ax+b) = \frac{\sin(ax+b)}{f_1(-a^2)}$; $f_1(-a^2) \neq 0$.

$(D^2 - 4D + 4)$

Ex: Sol: $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x$ — (1)

Let $y = e^{mx}$ be the trial solution of ——

from (2) $\frac{d^2}{dx^2}(e^{mx}) - 4\frac{d}{dx}(e^{mx}) + 4e^{mx} = 0$

$\Rightarrow m^2 e^{mx} - 4m e^{mx} + 4e^{mx} = 0$

$\Rightarrow (m^2 - 4m + 4) e^{mx} = 0$

$\Rightarrow (m^2 - 2m - 2m + 4) e^{mx} = 0$

$\Rightarrow m(m-2) - 2(m-2) = 0$

$\Rightarrow (m-2)(m-2) e^{mx} = 0$

$\Rightarrow m = 2, 2$

$m = 2, 2$

$\therefore y_c = (c_1 + c_2 x) e^{2x}$

Now, $y_p = \frac{1}{D^2 - 4D + 4} 3x^2 e^{2x} \sin 2x$

$= \frac{1}{(D-2)^2} \cdot 3x^2 \cdot e^{2x} \sin 2x$

$= 3e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x$

$= 3e^{2x} \frac{1}{D^2} \cdot x^2 \sin 2x$

is $x^2 \cos 2x$

R.H.S

$= \text{I.P of } 3e^{2x} \frac{1}{D^2} x^2 e^{2ix} \; [\because e^{2ix} = \cos 2x + i \sin 2x]$

$= \text{I.P of } 3e^{2x} \cdot e^{2ix} \frac{1}{(D+2i)^2} x^2$

$= \text{I.P of } 3e^{2x} \cdot e^{2ix} \cdot \frac{1}{D^2 + 4iD - 4} x^2$

$= \text{I.P of } -3e^{2x} \cdot e^{2ix} \frac{1}{4 - 4iD - D^2} x^2.$

$= \text{I.P of } -3e^{2x} \cdot e^{2ix} \frac{1}{4} \left(1 - iD - \frac{1}{4}D^2\right)^{-1} x^2$

$= \text{I.P of } -3e^{2x} \cdot e^{2ix} \frac{1}{4} \left\{1 - \left(iD + \frac{1}{4}D^2\right)\right\}^{-1} x^2$

$= \text{I.P of } -3e^{2x} \cdot e^{2ix} \frac{1}{4} \left\{1 + \left(iD + \frac{1}{4}D^2\right) + \left(iD + \frac{1}{4}D^2\right)^2 + \cdots\right\} x^2$

$= \text{I.P of } -3e^{2x} \cdot e^{2ix} \cdot \frac{1}{4} \left\{1 + iD + \frac{1}{4}D^2 + i^2 D^2 + \cdots\right\} x^2$

$= \text{I.P of } -3e^{2x} \cdot e^{2ix} \frac{1}{4} \left(x^2 + 2ix + \frac{1}{2} - 2\right)$

$= \text{I.P of } -3e^{2x} (\cos 2x + i \sin 2x) \cdot \frac{1}{4} \left(x^2 + 2ix - \frac{3}{2}\right)$

$\Rightarrow$ I.P of $-\frac{3}{4} \cdot e^{2x}(\cos 2x + i\sin 2x)\left(\frac{2x^2+4ix-3}{2}\right)$

$= $ I.P of $-\frac{3}{8}\, e^{2x}(\cos 2x + i\sin 2x)\{(2x^2-3)+4ix\}$

$= -\frac{3}{8}\, e^{2x}[(2x^2-3)\sin 2x + 4x\cos 2x]$.

Hence the G.S is $y = y_c + y_p$

$= (c_1 + c_2 x)e^{2x} - \frac{3}{8} e^{2x}[(2x^2-3)\sin 2x + 4x\cos 2x]$

Exercise

Ex. 1 (P. 80)

Ex: 4 (P. 82)

Ex. 7 (P. 83)

$e^{ix} = \cos x + i\sin x$

$\frac{1}{f(D)} e^{ax} V(x) = e^{ax} \frac{1}{f(D+a)} V(x)$.

Rule: Exceptional case $\frac{1}{f(D)} e^{ax}$ when $f(a) = 0$

$$\frac{1}{f(D)} e^{ax} = x \cdot \frac{1}{f'(D)} e^{ax} = \frac{x e^{ax}}{f'(a)}$$

Again if $f'(a) = 0$ and $f''(a) \neq 0$ then $(D-a)$ is a fuction repeated twice, and applying the above resuet once again, we get.

$$\frac{1}{f(D)} e^{ax} = x^2 \frac{1}{f''(D)} e^{ax} \text{ and so on.}$$

Ex: Solve: $(D^2+4D+3)y = e^{-3x}$

$\frac{1}{f(D)}e^{ax} = \frac{e^{ax}}{f(a)};\ f(a) \neq 0$

$(-3)^2 + 4(-3) + 3$
$= 12 - 12 = 0$

$m = -1, -3$

$y_c = c_1 e^{-x} + c_2 e^{-3x}$

Now $y_p = \frac{1}{D^2+4D+3} e^{-3x}$ (case of failure)

$$= x \frac{1}{2D+4} e^{-3x}$$

$$= x \frac{e^{-3x}}{2(-3)+4}$$

$$= -\frac{1}{2} x e^{-3x}$$

So, the G.S. is $y = y_c + y_p$

$$= c_1 e^{-x} + c_2 e^{-3x} - \frac{1}{2} x e^{-3x}$$ Ans

Ex2: $\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = e^{-x}$

$m^3 + 3m^2 + 3m + 1 = 0$

$\Rightarrow m = -1, -1, -1$

$y_c = (c_1 + c_2 x + c_3 x^2) e^{-x}$

Now, $y_p = \frac{1}{D^3+3D^2+3D+1} \cdot e^{-x}$

$$= \frac{1}{(D+1)^3} e^{-x} \text{ (case of failure)}$$

$$= x \frac{1}{3(D+1)^2} e^{-x} \quad (\text{''})$$

$$= x^2 \frac{1}{6(D+1)} e^{-x} \quad (\text{''})$$

$$= x^3 \frac{1}{6} e^{-x}$$

$$= \frac{1}{6} x^3 e^{-x}$$

so, the G.S is $y = y_c + y_p$

$$\Rightarrow y = (c_1 + c_2 x + c_3 x^2) e^{x} + \frac{1}{48} x^3 e^{-x} \quad A$$

Exercise: Ex. 4 (P. 72)

Ex. 5 (P. 73)

Rule

Exceptional case of (i) $\frac{1}{f(D^2)} \sin ax$, when $f(-a^2) = 0$

(ii) $\frac{1}{f(D^2)} \cos ax$, when $f(-a^2) = 0$

Then,

(I) $\frac{1}{f(D^2)} \sin ax = x \frac{1}{f'(D^2)} \sin ax$

(II) $\frac{1}{f(D^2)} \cos ax = x \frac{1}{f'(D^2)} \cos ax$

In case $f'(-a^2) = 0$ and $f''(-a^2) \neq 0$, then the above result once again we get,

$$\frac{1}{f(D^2)} \cdot \sin ax = x^2 \frac{1}{f''(D^2)} \sin ax$$

$$\frac{1}{f(D^2)} \cos ax = x^2 \frac{1}{f''(D^2)} \cdot \cos ax$$

Ex:

Solve $\frac{d^6 y}{dx^6} + y = \sin \frac{3}{2} x \cdot \sin \frac{1}{2} x$

Sol:

A.E is $m^6 + 1 = 0$

$\Rightarrow (m^2)^3 + (1)^3 = 0$

$\Rightarrow (m^2 + 1)(m^4 - m^2 + 1) = 0$

$\Rightarrow (m^2 + 1)\{(m^2 + 1)^2 - 3m^2\} = 0$

$\Rightarrow (m^2+1)\{(m^2+1)^2-(\sqrt{3}m)^2\}=0$

$\Rightarrow (m^2+1)(m^2+1+\sqrt{3}m)(m^2+1-\sqrt{3}m)=0$

$\Rightarrow (m^2+1)(m^2+\sqrt{3}m+1)(m^2-\sqrt{3}m+1)=0$

$\therefore m^2+1=0$

$\therefore m=\pm\sqrt{-1}$

$=0\pm i$

$m^2+\sqrt{3}m+1=0$

$\therefore m=\frac{-\sqrt{3}\pm\sqrt{3-4.1.1}}{2.1}$

$=\frac{-\sqrt{3}\pm\sqrt{-1}}{2}$

$=\frac{-\sqrt{3}}{2}\pm\frac{i}{2}$

$m^2-\sqrt{3}m+1=0$

$\therefore m=\frac{\sqrt{3}}{2}\pm\frac{i}{2}$

Hence, $y_c=c_1\cos x+c_2\sin x+e^{-\frac{\sqrt{3}}{2}x}\left(c_3\cos\frac{x}{2}+c_4\sin\frac{x}{2}\right)+$

$e^{\frac{\sqrt{3}}{2}x}\left(c_5\cos\frac{x}{2}+c_6\sin\frac{x}{2}\right)$.

Now $\sin\frac{3}{2}x\cdot\sin\frac{x}{2}=\frac{1}{2}\cdot 2\sin\frac{3}{2}x\cdot\sin\frac{1}{2}x$

$=\frac{1}{2}(\cos x-\cos 2x)$

Now

$y_p=\frac{1}{D^6+1}\cdot\left\{\frac{1}{2}(\cos x-\cos 2x)\right\}$

$=\frac{1}{2}\frac{1}{D^6+1}\cos x-\frac{1}{2}\frac{1}{D^6+1}\cdot\cos 2x$ [first theorem case of failure]

$(D^2)^3+1$

$=(-1)^3+1=0$.

$=\frac{1}{2}x\frac{1}{6D^5}\cos x-\frac{1}{2}\frac{1}{(D^2)^3+1}\cdot\cos 2x$

$=\frac{1}{2}x\frac{1}{6(D^2)^2 D}\cos x-\frac{1}{2}\frac{\cos 2x}{(-2^2)^3+1}$

$=\frac{x}{2}\frac{1}{6(-1)^2 D}\cos x+\frac{1}{126}\cos 2x$

$\frac{1}{D}$ integrate

$=\frac{x}{12}\sin x+\frac{1}{126}\cos 2x$

Eq. G.S. $y = y_c + y_p$

Quiz-3

up to today

from Bernoulli's eqn - today

on 09.02.16

2 Feb ⇒ মঙ্গলবার-

Homogeneous linear equation:

An equation of the form $x^n \frac{d^n y}{dx^n} + P_1 x^{n-1} \frac{d^{n-1}}{dx^{n-1}} + \cdots + P_n y = x$ —(1)

order

where $P_1, P_2, \ldots P_n$ are constants and x is a function of x, is called the homogeneous linear equation.

Important substitution:

If we put $x = e^t$ or $z = \ln x$, the equation (1) is is transformed into an equation with constant co-efficient changing the dependent variable x to z.

Thus, if $x = e^z$, or $z = \ln x$ $\therefore \frac{dz}{dx} = \frac{1}{x}$ —(2)

$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz}$, or, $x \frac{dy}{dx} = \frac{dy}{dz}$ —(3)

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$= \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right)$

$= \frac{d}{dx}\left(\frac{\frac{dy}{dz}}{x}\right)$

$$= \frac{x \cdot \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{dy}{dz}}{x^2} = \frac{x \cdot \frac{d^2y}{dz^2} \cdot \frac{1}{x} \cdot \frac{dy}{dz}}{x^2}$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} \quad \text{———} (4)$$

Similarly,

$$x^3 \frac{d^3y}{dx^3} = \frac{d^3y}{dz^3} - 3\frac{d^2y}{dz^2} + 2\frac{dy}{dx} \quad \text{———} (5)$$

Thus if we put.

$x \frac{d}{dx} = \frac{d}{dz} = D$, (3), (4), (5), etc. can be

put as $x \frac{dy}{dx} = Dy$,

$$x^2 \frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

and $x^n \frac{d^ny}{dx^n} = D(D-1)(D-2) \text{———} (D-n+1)y$

Ex :- 1 Solve $x^2 \frac{d^2y}{dx^2} + y = 3x^2$

Sol: Let $x = e^z$ and $D = \frac{d}{dz}$, the equation becomes,

$$D(D-1)y + y = 3e^{2z}$$

$$\Rightarrow (D^2 - D + 1)y = 3e^{2z}$$

The. A.E is

$$m^2 - m + 1 = 0$$

$$\Rightarrow m = \frac{-(-1) \pm \sqrt{(-1)^2 - 4.1.1}}{2.1}$$

$$= \frac{1 \pm \sqrt{3}\,i}{2}$$

$$y_c = e^{z/2}\left(c_1 \cos\frac{\sqrt{3}}{2}z + c_2 \sin\frac{\sqrt{3}}{2}z\right)$$

Now, the P.I is $y_p = \frac{1}{D^2 - D + 1} 3e^{2z}$

$$= 3\frac{1}{D^2 - D + 1} e^{2z}$$

$$= 3\frac{e^{2z}}{4 - 2 + 1} = e^{2z}$$

So, the G.S is $y = y_c + y_p$.

16-12-9
1-16

$y = e^{z/2}\left(c_1 \cos \frac{\sqrt{3}}{2} z + c_2 \sin \frac{\sqrt{3}}{2} z\right) + e^{2z}$

$\Rightarrow y = e^{\frac{1}{2}\ln x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right]$

$+ e^{2\ln x}$

$= x^{\frac{1}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\ln x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\ln x\right)\right] + x^2$

Ex.2: $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$ $\Rightarrow$ find is homogeneous? no co-efficient constant.

Let,

$x = e^z$, and $D = \frac{d}{dz}$ the equation

becomes,

$\cancel{D(D-1)y \quad -2D = e^{4z}}$

$e^{2z} D^2 y - 2e^z Dy - 4y = e^{4z}$

$\Rightarrow De^{2z}(D-1)y - 4y = e^{4z}$

$\Rightarrow D^2 - 3D - 4$

$\therefore m = -4, -1.$

$y_c = c_1 e^{4z} + c_2 e^{-z}$

$y_p = \frac{1}{D^2 - 3D - 4} \cdot e^{4z}$ [case of failure]

$= z \frac{1}{2D-3} e^{4z}$

$= z \frac{1}{8-3} e^{4z}$

$= \frac{ze^{4z}}{5}$

$y = y_c + y_p$

$= c_1 e^{4z} + c_2 e^{-z} + \frac{1}{5} z e^{4z}$

$= c_1 x^4 + c_2 \cdot \frac{1}{x} + \frac{1}{5}(\ln x) \cdot x^4$ as $x = e^z$ A

7 Feb ⇒ রবিবার

Ex 3: Solve: $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10(x + 1/x)$

Sol Given

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10(x + 1/x) \quad \text{--- (1)}$$

put $x = e^z$ and $D = \frac{d}{dz}$, then the above equation becomes

$$D(D-1)(D-2)y + 2D(D-1)y + 2y = 10(e^z + e^{-z})$$

$$\Rightarrow (D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

So, the A.E is, $m^3 - m^2 + 2 = 0$

$D \to m$ $\Rightarrow m^2(m+1) - 2m(m+1) + 2(m+1) = 0$

$$\Rightarrow (m+1)(m^2 - 2m + 2) = 0$$

$\therefore m+1 = 0$ | $m^2 - 2m + 2 = 0$

$\Rightarrow m = -1$ | $\Rightarrow m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2.1} = \frac{2 \pm 2i}{2}$

$$= (1 \pm i)$$

So, the C.F is $y_c = c_1 e^{-z} + e^z(c_2 \cos z + c_3 \sin z)$

$$\therefore y_c = c_1 x^{-1} + x\{c_2 \cos(\ln x) + c_3 \sin(\ln x)\}$$

Now, the P.I. is $y_p = \frac{1}{D^3 - D^2 + 2} 10(e^z + e^{-z})$

$$\Rightarrow y_p = 10 \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{1}{D^3 - D^2 + 2} e^{-z} \quad [\text{2nd term case of failure}]$$

$$= 10 \frac{e^z}{1 - 1 + 2} + 10 z \frac{1}{3D^2 - 2D} e^{-z} \quad [2$$

$$= 5e^z + 10z \frac{e^{-z}}{3+2} \quad [\text{হলে 0 হয় না}]$$

$$= 5e^z + 2ze^{-z}$$

$$= 5x + \frac{2}{x}\ln x \text{ as } x = e^z.$$

So the G.S is $y = y_c + y_p$

$$\Rightarrow y = c_1 x^{-1} + x\{c_1 \cos(\ln x) + c_2 \sin(\ln x)\} + 5x + \frac{2}{x}\ln x \quad \text{Ans.}$$

Exercise:

Ex 2 (P. 96)

Ex 6(a) (P. 98)

Ex. 10 (P. 100).

* Clairaut's Equation $y = px + f(p)$; where $p = \frac{dy}{dx}$

To solve $y = px + f(p)$ —— (1), differentiate it w.r.t respect to x, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\Rightarrow p = p + x\frac{dp}{dx} + f'(p).\frac{dp}{dx}$$

$$\Rightarrow [x + f'(p)]\frac{dp}{dx} = 0$$

$$\int dp = \int 0 \cdot dx$$
$$p = 0 + c$$

Neglecting $x + f'(p) = 0$, we get $\frac{dp}{dx} = 0$

$\Rightarrow p = c$ [integrating both sides]

put $p = c$ in (1), we get the required solution;

$y = cx + f(c)$.

Solve: $(y - px)(p - 1) = p$

Given: $(y - px)(p - 1) = p$

$$\Rightarrow y - px = \frac{p}{p-1}$$

$$\Rightarrow y = px + \frac{p}{p-1}$$, which is in clairauts form.

Here, put c for p. Hence the required general solution is

$$y = cx + \frac{c}{c-1}$$ Ans.

Ex2: Solve $\left(\frac{dy}{dx}\right)^2(x^2-a^2) - 2\left(\frac{dy}{dx}\right)xy + y^2 - b^2 = 0$

Sol: The given equation can be written as

$p^2(x^2-a^2) - 2pxy + y^2 - b^2 = 0$

$\Rightarrow p^2x^2 - 2pxy + y^2 = a^2p^2 + b^2$

$\Rightarrow y^2 - 2ypx + p^2x^2 = a^2p^2 + b^2$

$\Rightarrow (y-px)^2 = a^2p^2 + b^2$

$\therefore y - px = \pm\sqrt{a^2p^2+b^2}$

$\therefore y = px \pm \sqrt{a^2p^2+b^2}$

Both of these are in clairaut's ~~equation~~ form. Hence the solution is

$y = cx \pm \sqrt{a^2c^2+b^2}$

Exercise: Ex 7 (P.132)

Examples reducible to clairaut's form.

Ex: Solve: $x^2(y-px) = yp^2$

Sol $x^2(y-px) = yp^2$ ——— (1)

put $x^2 = u \rightarrow 2x\,dx = du$ ——— (2)

and $y^2 = v, \rightarrow 2y\,dy = dv$ ——— (3)

from (3) $\div$ (2), we get,

$\frac{y}{x}\frac{dy}{dx} = \frac{dv}{du}$

$\Rightarrow \frac{y}{x}p = \frac{dv}{du}$

$\Rightarrow p = \frac{x}{y}\frac{dv}{du}$.

Then the equation (1) becomes.

$x^2\left(y - \frac{x}{y}\cdot\frac{dv}{du}x\right) = y\frac{x^2}{y^2}\left(\frac{dv}{du}\right)^2$

$\Rightarrow \left(\frac{y^2 - x^2\frac{dv}{du}}{y}\right) = \frac{1}{y}\left(\frac{dv}{du}\right)^2$

independent variable.

$y = pu + f(p)$

$\Rightarrow \left(y^2 - x^2 \frac{dv}{du}\right) = \left(\frac{dv}{du}\right)^2$

$\Rightarrow v = u\frac{dv}{du} + \left(\frac{dv}{du}\right)^2$, which is in clairauts form.

Hence, the solution is,

$v = cu + c^2$

$\Rightarrow y^2 = cx^2 + c^2$.

Ex 2: Solve: $e^{3x}(p-1) + p^3 e^{2y} = 0$

Given, $e^{3x}(p-1) + p^3 e^{2y} = 0$ —— (1)

Let, $e^x = u$ and $e^y = v$, so that

$\therefore e^x dx = du \mid \therefore e^y dy = dv$

$\therefore \frac{dv}{du} = \frac{e^y}{e^x} \frac{dy}{dx} = \frac{v}{u} \cdot \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{u}{v} \cdot \frac{dv}{du}$

$\Rightarrow p = \frac{u}{v} P$ [where $p = \frac{dy}{dx}$ and $P = \frac{dv}{du}$]

Then the equation (1) becomes,

$u^3\left(\frac{u}{v}P - 1\right) + \frac{u^3}{v^3} P^3 v^2 = 0$

$\Rightarrow \left(\frac{u}{v}P - 1\right) + \frac{P^3}{v} = 0$

$\Rightarrow uP - v + P^3 = 0$

$\Rightarrow v = uP + P^3$, which is in clairauts form.

Hence, the solution is, $y = px + f(p)$

$v = uc + c^3$

$\Rightarrow e^y = ce^x + c^3$

Ex 3: Reduce the differential equation $(px-y)(x-yp)=2p$ to ~~calcut~~ clairaut's form by the substitution $x^2=u,\ y^2=v$ and find its complete primitive

Sol: given $(px-y)(x-yp)=2p$ —— ①

$x^2=u,\quad y^2=v$

put $x^2=u$ $\quad 2x\,dx=du$ —— ②

$y^2=v$ $\quad 2y\,dy=dv$ —— ③

from ② ÷ ③

$\frac{y}{x}\frac{dy}{dx}=\frac{dv}{du}$

$\Rightarrow \frac{y}{x}p=\frac{dv}{du}$

$\Rightarrow p=\frac{x}{y}\frac{dv}{du} \Rightarrow p=\frac{x}{y}P$ (say)

Then ① becomes

$\cancel{\left(\frac{x}{y}\frac{dv}{du}\cdot x-y\right)\left(x-y\cdot\frac{x}{y}\frac{dv}{du}\right)=2p}$

$\cancel{\Rightarrow \left(\frac{x\frac{dv}{du}x-y^2}{y}\right)\left(\frac{xy-}{y}\right)=2p}$

$\left(\frac{x}{y}P\cdot x-y\right)\left(x-y\cdot\frac{x}{y}P\right)=2\frac{x}{y}P$

$\Rightarrow \left(\frac{x^2P-y^2}{y}\right)(x-xP)=\frac{2x}{y}P$

$\Rightarrow \cancel{(x^2P-y^2)(x-xP)=2xP}$

$\Rightarrow \left(\frac{Px^2-y^2}{y}\right)x(1-P)=2\frac{x}{y}P$

$\Rightarrow (Px^2-y^2)(1-P)=2P$

$\Rightarrow Px^2-y^2=\frac{2P}{1-P}$

$\Rightarrow y^2=Px^2-\frac{2P}{1-P} \Rightarrow v=Pu-\frac{2P}{1-P}$

$v = pu \cdot \frac{2p}{1-p}$ (Clairaut's form)

Hence, the solution is, $v = cu - \frac{2c}{1-c}$

$\Rightarrow y^2 = cx^2 - \frac{2c}{1-c}$

$\Rightarrow (1-c)\, y^2 = (1-c)\, cx^2 - 2c$

$\Rightarrow y^2 - y^2 c = cx^2 - c^2x^2 - 2c$

$\Rightarrow c^2x^2 - c(x^2 + y^2 - 2) + y^2 = 0$

Exercise

Ex. 7 (P. 132)
13 (P. 134)
16 (P. 134)
18 (P. 135)

14 Feb ⇒ রবিবার

Linear partial differential equations of order one:

Equation $Pp + Qq = R$ is the standard form of the linear partial differential equation of order one, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$ and P, Q, R are functions of x, y and z.

Lagrange's Method:- The general solution of the linear partial differential equation $Pp + Qq = R$ is $\phi(u, v) = 0$ where ϕ is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are solution of equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where P, Q, R are function of x, y and z.

Lagrange's auxiliary equation

Ex:- solve:- $\frac{y^2 z}{x} p + xzq = y^2$

Sol:- Given, $\frac{y^2 z}{x} p + xzq = y^2$

$\Rightarrow y^2 z p + x^2 z q = xy^2$

The auxiliary equation are,

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z} = \frac{dz}{xy^2}$$

From first two,

$$\frac{dx}{y^2 z} = \frac{dy}{x^2 z}$$

$\Rightarrow \frac{dx}{y^2} = \frac{dy}{x^2}$

$\Rightarrow x^2 dx = y^2 dy$

$\Rightarrow \frac{x^3}{3} = \frac{y^3}{3} + c$ [Integrating both sides]

$\Rightarrow x^3 - y^3 = 3c = c_1$,

Now, from first and third, $\frac{dx}{y^2 z} = \frac{dz}{xy^2}$

$\Rightarrow \frac{dx}{z} = \frac{dz}{x}$

$\Rightarrow x\,dx = z\,dz$

$\Rightarrow \frac{x^2}{2} = \frac{z^2}{2} + c$ [I. Both sides]

$\Rightarrow x^2 - z^2 = 2c = c_2$

Hence $\phi(x^3 - y^3, x^2 - z^2) = 0$ is the complete solution.

Ex2:- solve:- $(y+z)p + (z+x)q = x+y$

Sol:- Lagrange's auxiliary equations are

$$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} = \frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)} \quad (1)$$

Now, from, $\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y}$

$\Rightarrow \frac{dy-dx}{y-x} = \frac{dz-dy}{z-y} \Rightarrow \frac{d(y-x)}{(y-x)} = \frac{d(z-y)}{(z-y)}$

$\Rightarrow \ln(y-x) = \ln(z-y) + \ln c_1$ (Integrating)

$\Rightarrow \ln\left(\frac{y-x}{z-y}\right) = \ln c_1$

$\Rightarrow \frac{y-x}{z-y} = c_1$

Again from,

$$\frac{dy-dz}{z-y} = \frac{dx+dy+dz}{2(x+y+z)}$$

$$\Rightarrow 2\,\frac{dy-dz}{y-z} + \frac{dx+dy+dz}{x+y+z} = 0 \qquad \frac{d(x+y+z)}{(x+y+z)}$$

$\Rightarrow 2\ln(y-z) + \ln(x+y+z) = \ln c_2$

$\Rightarrow \ln\{(y+z)^2 (x+y+z)\} = \ln c_2$

$\Rightarrow (y-z)^2 (x+y+z) = c_2$

Therefore solution of the given equation

$$\phi\left[\frac{y-x}{z-y}, (y-z)^2 (x+y+z)\right] = 0$$

Ex 3: Solve: $(y^2+z^2-x^2)p - 2xyq + 2xz = 0$.

Sol: Given $(y^2+z^2-x^2)p - 2xyq = -2xz$

The auxiliary equations are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}$$

From the last two,

$$\frac{dy}{-2xy} = \frac{dz}{-2xz}$$

$\Rightarrow \frac{dy}{y} = \frac{dz}{z}$

$\Rightarrow \ln y = \ln z + \ln c_1$

$\Rightarrow \ln(y/z) = \ln c_1$

$\therefore c_1 = \frac{y}{z}$

Next using x, y, z as multipliers, we get,

$$\frac{dx}{x^2-y^2-z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} = \frac{x\,dx+y\,dy+z\,dz}{x^3-xy^2-xz^2+2xy^2+2xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^3 + xy^2 + xz^2}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

From the last two of these

$$\frac{dz}{2xz} = \frac{x\,dx + y\,dy + z\,dz}{x(x^2 + y^2 + z^2)}$$

$$\Rightarrow \frac{dz}{z} = \frac{2(x\,dx + y\,dy + z\,dz)}{x^2 + y^2 + z^2}$$

$$\Rightarrow \int \frac{d(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)} = \int \frac{dz}{z}$$

$$\Rightarrow \ln(x^2 + y^2 + z^2) = \ln z + \ln c_2 = \ln(z \cdot c_2)$$

$$\Rightarrow x^2 + y^2 + z^2 = z\,c_2$$

$$\therefore c_2 = \frac{x^2 + y^2 + z^2}{z}$$

Hence $\varphi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0$ is the required solution.

Exercise:

Ex 7(a). (P 9)

Ex 11 (P 11)

Ex 12 (P 11)

What do you