Queuing Theory

A single-Server Exponential Queuing System Having Finite Capacity: In the previous model, we assumed that there was no limit on the number of customers that could be in the system at the same time. However, in reality there is always a finite system capacity *N*, in the sense that there can be no more than *N* customers in the system at any time. By this, we mean that if an arriving customer finds that there are already *N* customers present, then he does not enter the system.

We let $P_n, 0 \le n \le N$, denote the limiting probability that there are *n* customers in the system. The rate-equality principle yields the following set of balance equations:

State	*Rate at which the process leaves = rate at which it enters*
0	$\lambda P_0 = \mu P_1$
$1 \le n \le N-1$	$(\lambda+\mu)P_n = \lambda P_{n-1} + \mu P_{n+1}$
N	$\mu P_N = \lambda P_{N-1}$

The equation for state 0 to *N*-1 is similar to single-server exponential queuing system with infinite capacity. But, we have new equation for state *N* for finite capacity *N*. State *N* can only be left via a departure since an arriving customer will not enter the system when it is in state *N*; also state *N* can now only be entered from state $N-1$ via an arrival.

To solve, we again rewrite the preceding system of equations:

$$P_1 = \frac{\lambda}{\mu} P_0$$

$$P_{n+1} = \frac{\lambda}{\mu} P_n + (P_n - \frac{\lambda}{\mu} P_{n-1}), \qquad 1 \le n \le N-1$$

$$P_N = \frac{\lambda}{\mu} P_{N-1}$$

which, solving in terms of P_0, yields

Putting *n* = 0, we get $P_1 = \frac{\lambda}{\mu} P_0$

Putting *n* = 1, we get $P_2 = \frac{\lambda}{\mu} P_1 + \left(P_1 - \frac{\lambda}{\mu} P_0 \right) = \frac{\lambda}{\mu} P_1 = \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu} P_0 = \left(\frac{\lambda}{\mu} \right)^2 P_0$

Putting *n* = 2, we get $P_3 = \frac{\lambda}{\mu} P_2 + \left(P_2 - \frac{\lambda}{\mu} P_1 \right) = \frac{\lambda}{\mu} P_2 = \frac{\lambda}{\mu} \cdot \left(\frac{\lambda}{\mu} \right)^2 P_0 = \left(\frac{\lambda}{\mu} \right)^3 P_0$

$$\vdots$$

Putting *n* = *N*-1, we get $P_{N-1} = \frac{\lambda}{\mu} P_{N-2} + \left(P_{N-2} - \frac{\lambda}{\mu} P_{N-3} \right) = \frac{\lambda}{\mu} P_{N-1} = \frac{\lambda}{\mu} \cdot \left(\frac{\lambda}{\mu} \right)^{N-2} P_0 = \left(\frac{\lambda}{\mu} \right)^{N-1} P_0$

Putting *n* = *N*-1, we get $P_N = \frac{\lambda}{\mu} P_{N-1} = \frac{\lambda}{\mu} \cdot \left(\frac{\lambda}{\mu} \right)^{N-1} P_0 = \left(\frac{\lambda}{\mu} \right)^{N} P_0$

By using the fact $\sum_{n=0}^{N} P_n = 1$, we obtain

$$1 = P_0 \sum_{n=0}^{N} \left(\frac{\lambda}{\mu}\right)^n = P_0 \left[\frac{1-\left(\lambda/\mu\right)^{N+1}}{1-\lambda/\mu}\right], \qquad \left[\because \sum_{i=0}^{n} x^i = 1 + x + x^2 + \cdots + x^n = \frac{1-x^{n+1}}{1-x}\right]$$

$$\Rightarrow P_0 = \frac{\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}}$$

Hence, $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n \cdot \frac{\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}} = \frac{\left(\lambda/\mu\right)^n \left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}}, \quad n = 0,1,\cdots,N$

Now we can find out L putting the value of P_n.

$$L = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} n \frac{\left(\lambda/\mu\right)^n \left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}}$$

$$= \frac{\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}} \sum_{n=0}^{N} n \left(\lambda/\mu\right)^n$$

We can solve $\sum_{n=0}^{N} n \left(\lambda/\mu\right)^n$ using perturbation technique.

Though $\sum_{k=0}^{n} kx^k = \frac{x-(n+1)x^{n+1}+nx^{n+2}}{(1-x)^2}$ (See Lec-5)

Similarly, $\sum_{n=0}^{N} n \left(\lambda/\mu\right)^n = \frac{\lambda/\mu - (N+1)\left(\lambda/\mu\right)^{N+1} + N\left(\lambda/\mu\right)^{N+2}}{\left(1-\lambda/\mu\right)^2}$

$$L = \frac{\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}} \cdot \frac{\lambda/\mu - (N+1)\left(\lambda/\mu\right)^{N+1} + N\left(\lambda/\mu\right)^{N+2}}{\left(1-\lambda/\mu\right)^2}$$

$$= \frac{\lambda/\mu - (N+1)\left(\lambda/\mu\right)^{N+1} + N\left(\lambda/\mu\right)^{N+2}}{\left[1-\left(\lambda/\mu\right)^{N+1}\right]\left[(\mu-\lambda)/\mu\right]} = \frac{\lambda\left[1 + N\left(\lambda/\mu\right)^{N+1} - (N+1)\left(\lambda/\mu\right)^{N}\right]}{(\mu-\lambda)\left[1-\left(\lambda/\mu\right)^{N+1}\right]}$$

In deriving *W*, the expected amount of time a customer spends in the system, we must be little careful. If we have full capacity *N* customer in the system, then extra customers cannot enter the system for service and they will not spend their time and money in the system. Thus we should only consider those customers who get the chance to get service. Since the fraction of arrivals that actually enter the system is $1-P_N$, it follows that $\lambda_a = \lambda\left(1-P_N\right)$. Now, *W* can be obtained from the following equation

$$W = \frac{L}{\lambda_a} = \frac{\lambda\left[1+N\left(\lambda/\mu\right)^{N+1}-\left(N+1\right)\left(\lambda/\mu\right)^{N}\right]}{\left(\mu-\lambda\right)\left[1-\left(\lambda/\mu\right)^{N+1}\right]\lambda\left[1-\dfrac{\left(\lambda/\mu\right)^{N}\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}}\right]}$$

$$= \frac{\left[1+N\left(\lambda/\mu\right)^{N+1}-\left(N+1\right)\left(\lambda/\mu\right)^{N}\right]}{\left(\mu-\lambda\right)\left[1-\left(\lambda/\mu\right)^{N+1}\right]\left[\dfrac{1-\left(\lambda/\mu\right)^{N+1}-\left(\lambda/\mu\right)^{N}\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}}\right]}$$

$$= \frac{\left[1+N\left(\lambda/\mu\right)^{N+1}-\left(N+1\right)\left(\lambda/\mu\right)^{N}\right]}{\left(\mu-\lambda\right)\left[1-\left(\lambda/\mu\right)^{N+1}-\left(\lambda/\mu\right)^{N}\left(1-\lambda/\mu\right)\right]}$$

We can also find out L_Q and W_Q similarly like single-server exponential model with infinite system.

<u>Example:</u> Suppose that it costs $c\mu$ dollars per hour to provide service at a rate μ. Suppose also that we incur a gross profit of *A* dollar for each customer served. If the system has a capacity *N*, what service rate μ maximizes our total profit?

<u>Solution:</u> Let, potential customers arrive at λ rate. However, a certain proportion of them do not join the system; namely, those who arrive when there are *N* customers already in the system. Hence, since P_N is the proportion of time that the system is full, it follows that entering customer arrive at a rate of $\lambda(1-P_N)$. Since each customer pays \$A, it follows that money come in at an hourly rate of $\lambda(1-P_N)A$ and since it goes out at an hourly rate of $c\mu$, it follows that our hourly profit per hour is given by

profit per hour = $\lambda(1-P_N)A - c\mu$

$$= \lambda A\left[1-\frac{\left(\lambda/\mu\right)^N\left(1-\lambda/\mu\right)}{1-\left(\lambda/\mu\right)^{N+1}}\right]-c\mu$$

$$= \lambda A\left[\frac{1-\left(\lambda/\mu\right)^{N+1}-\left(\lambda/\mu\right)^N+\left(\lambda/\mu\right)^{N+1}}{1-\left(\lambda/\mu\right)^{N+1}}\right]-c\mu = \frac{\lambda A\left[1-\left(\lambda/\mu\right)^N\right]}{1-\left(\lambda/\mu\right)^{N+1}}-c\mu$$

For instance if $N=2, \lambda=1, A=10, c=1$, then

$$\text{profit per hour } = \frac{10\left[1-\left(1/\mu\right)^2\right]}{1-\left(1/\mu\right)^3}-\mu = \frac{10\left(\mu^2-1\right)}{\mu^2}\times\frac{\mu^3}{\mu^3-1}-\mu = \frac{10\left(\mu^3-\mu\right)}{\mu^3-1}-\mu$$

in order to maximize profit we differentiate to obtain

$$\frac{d}{d\mu}\left[\text{profit per hour}\right] = \frac{10\left[\left(\mu^3-1\right)\left(3\mu^2-1\right)-\left(\mu^3-\mu\right)\left(3\mu^2\right)\right]}{(\mu^3-1)^2}-1, \quad \left[\because \frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{d}{dx}(u)-u\frac{d}{dx}(v)}{v^2}\right]$$

$$= \frac{10\left(3\mu^5-\mu^3-3\mu^2+1-3\mu^5+3\mu^3\right)}{(\mu^3-1)^2}-1 = 10\frac{\left(2\mu^3-3\mu^2+1\right)}{(\mu^3-1)^2}-1$$

The value of μ that maximizes our profit now can be obtained by equating to zero and solving numerically.

☺Good Luck☺