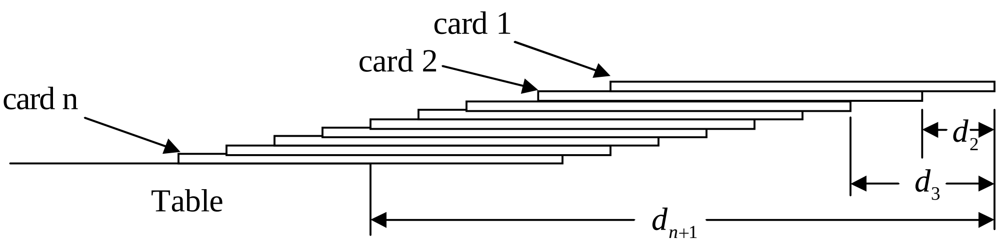
Special Numbers

Harmonic Numbers: $H_n = \frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots\frac{1}{n} = \sum_{k=1}^{n}\frac{1}{k}$, integer $n \ge 0$.

The first few values look like following:

n	0	1	2	3	4	5	6	7	8
H_n	0	1	$\frac{3}{2}$	$\frac{11}{6}$	$\frac{25}{12}$	$\frac{137}{60}$	$\frac{49}{20}$	$\frac{363}{140}$	$\frac{761}{280}$

Cards on the table: Given n cards and a table, we would like to create the largest possible overhang by stacking the cards up over the table's edge, subject to the laws of gravity:



We assume that each card is 2 units long. With one card, we get maximum overhang when its center of gravity is just above the edge of the table. The center of gravity is in the middle of the card, so we can create half a card length or 1 unit of overhang.

With two cards, we get maximum overhang when center of gravity of the top card is just above the edge of the second card and center of gravity of both cards combined is just above the edge of the table. The joint center of gravity of two cards will be in the middle of their common part, so we are able to achieve an additional half unit of overhang.

In general case, the center of gravity of the top *k* cards lies just above the edge of the $(k+1)$ st card. The table plays the role of $(n+1)$ st card. To express this condition algebraically, we can let d_k be the distance from the extreme edge of the top card to the corresponding edge of the *k*-th card. Then $d_1 = 0$ and we want to make d_{k+1} the center of gravity of the first k cards:

$$d_{k+1} = \frac{(d_1+1)+(d_2+1)+\cdots+(d_k+1)}{k} \quad , \qquad \text{for } 1 \le k \le n$$

$$\Rightarrow kd_{k+1} = k + d_1 + d_2 + \cdots + d_{k-1} + d_k \ , \qquad k \ge 0 \quad \cdots \qquad (1)$$

Replacing k by $k-1$ from equation (1), we get

$$(k-1)d_k = k - 1 + d_1 + d_2 + \cdots + d_{k-1} \quad , \qquad k \ge 1 \quad \cdots \qquad (2)$$

Subtracting equation (1) from equation (2) tells us that

$$kd_{k+1} - (k-1)d_k = 1 + d_k$$

$$\Rightarrow kd_{k+1} - kd_k + d_k = 1 + d_k$$
$$\Rightarrow k(d_{k+1} - d_k) = 1$$
$$\Rightarrow d_{k+1} = d_k + \frac{1}{k}$$
$$\Rightarrow d_{k+1} = d_{k-1} + \frac{1}{k-1} + \frac{1}{k}$$
$$\Rightarrow d_{k+1} = d_{k-2} + \frac{1}{k-2} + \frac{1}{k-1} + \frac{1}{k}$$
$$\vdots$$
$$\Rightarrow d_{k+1} = d_1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k} = H_k$$

If we set $k = n$ we get $d_{n+1} = H_n$ as the total overhang when n cards are stacked.

<u>Worm on the rubber band:</u> A slow but persistent worm, *W* starts at one end of a meter-long rubber band and crawls one centimeter per minute toward the other end. At the end of each minute and equally persistent keeper of the band, *K*, whose sole purpose in life is to frustrate *W*, stretches it one meter. Thus after one minute of crawling, *W* is 1 centimeter from the start and 99 from the finish; then *K* stretches it one meter. During the stretching operation *W* maintains his relative position 1% from the start and 99% from the finish; so *W* is now 2cm from starting point and 198cm from the goal. After *W* crawls for another minute the score is 3cm traveled and 197cm to go; but *K* stretches and distance become 4.5cm and 295.5cm and so on. Does the worm ever reach the finish? It keeps moving, but the goal seems to move away even faster. (We are assuming an infinite longevity for *K* and *W*, an infinitely elasticity of the band and an infinitely tiny worm).

When K stretches the band, the fraction of it that W has crawled stays the same. Thus, he crawls 1/100$^{\text{th}}$ of the first minute, 1/200$^{\text{th}}$ the second, 1/300$^{\text{th}}$ the third and so on. After *n* minutes the fraction of the band that he has crawled is

$$\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \cdots + \frac{1}{100n} = \frac{1}{100}\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) = \frac{H_n}{100}\ .$$

So, it reaches the finish if H_n ever surpasses 100.

☺Good Luck☺

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