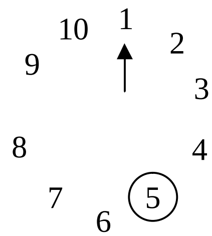
Recurrent Problems

Josephus Problem: We start with n people numbered 1 to n around a circle and we eliminate every second remaining person until only one survives. For example, here is the starting configuration for n = 10.



The elimination order is $2 \to 4 \to 6 \to 8 \to 10 \to 3 \to 7 \to 1 \to 9$, so 5^{th} person survives. The problem is to determine the survivor's number, $J(n)$. In this example $J(10) = 5$.

Consider, we have $2n$ people, then after 1^{st} round of elimination, we are left with n people and 3^{rd} person will be the next person waiting for elimination. This is just like starting with n people, except that every person's number has been doubled and decreased by 1. Similarly, if we start with $2n+1$ people, then after 1^{st} round of elimination number 1 will be eliminated. Then, we almost have the same situation with n people like previous one, but this time their numbers are doubled and increased by 1.

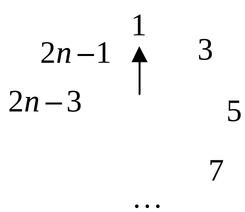


Fig: Start with $2n$ people

$J(2n) = 2J(n) - 1$, for $n \ge 1$

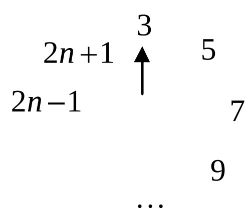


Fig: Start with $2n+1$ people

$J(2n+1) = 2J(n) + 1$, for $n \ge 1$

Write the Recurrence for Josephus Problem

Establish the logic for these Recurrences

Combining these equations with $J(1) = 1$ gives us a recurrence that defines J in all cases:

$$
\begin{aligned}
&J(1) = 1 \\
&J(2n) = 2J(n) - 1 , \qquad \text{for } n \ge 1 \qquad \cdots(1) \\
&J(2n+1) = 2J(n) + 1 , \qquad \text{for } n \ge 1
\end{aligned}
$$

Find J(100) using thess recurrences:

J(100) = 2 J(50) -1
= 4 J(25) -2 -1 = 8J(12) + 4 - 2 -1
=16 J(6) -8 +4 -2 -1
= 32 J(3) - 16 -8 + 4 -2 -1
= 64 J(1) + 32 -16 -8 +4 -2 -1 = 64 + 32 -16-8+4-2-1 = 73 (ANS)

Our recurrence makes it possible to build a table of small values which can helps us to guess the *closed form* of Josephus problem.

N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

$J(n)$ is always 1 at the beginning of a group and it increases by 2 within a group. So, if we write n in the form $n = 2^m + l$, where 2^m is the largest power of 2 not exceeding n where $l = n - 2^m$ satisfies $0 \le l < 2^{m+1} - 2^m = 2^m$, the solution to our recurrence would be

J(**n**) = $J(2^m + l) = 2l + 1$, for $m \ge 0$ and $0 \le l < 2^m$

To illustrate the solution let's compute $J(100)$.

$J(100) = J(2^6 + 36) = 2 \times 36 + 1 = 73$

Try to prove, $J(2^m) = 1$

Binary Property of the Josephus Problem:

Every solution to a problem can be generalized so that it applies to a wider class of problems. Power of 2 played an important role in our recurrence solution. Thus it's natural to look at the radix 2 representation of n and $J(n)$.

$n = (b_m b_{m-1} b_{m-2} \cdots b_2 b_1 b_0)_2$ i.e. $n = b_m 2^m + b_{m-1} 2^{m-1} + b_{m-2} 2^{m-2} + \cdots + b_2 2^2 + b_1 2 + b_0$

where, each bit b_i is either 0 or 1 and the leading bit b_m is 1. Recalling that $n = 2^m + l$, we have, successively,

$$n = (1b_{m-1}b_{m-2}\cdots b_2 b_1 b_0)_2$$
$$l = (0b_{m-1}b_{m-2}\cdots b_2 b_1 b_0)_2$$
$$2l = (b_{m-1}b_{m-2}\cdots b_2 b_1 b_0 0)_2$$
$$2l+1 = (b_{m-1}b_{m-2}\cdots b_2 b_1 b_0 1)_2$$
$$J(n) = (b_{m-1}b_{m-2}\cdots b_2 b_1 b_0 b_m)_2$$

Q: Derive the Binary Property of the Josephus problem

We have proved that, $J(n) = J((b_m b_{m-1} b_{m-2} \cdots b_2 b_1 b_0)_2) = (b_{m-1} b_{m-2} \cdots b_2 b_1 b_0 b_m)_2$

For example, $J(100) = J((1100100)_2) = (1001001)_2 = 64 + 8 + 1 = 73$

Question: Find J(100)by using the Binary Property of the Josephus Problem

We can find out special cases solution of Josephus problem if required. For example, we going to find when $J(n) = \frac{n}{2}$ is true.

Q: Find the minimum three values of n for which J(n) = n/3
Q: Find threee values of n for which J(n) = n
see the file : "Josephus Add On.docx" for such problems
l = 1/5 (2^m-3)

$$J(n) = \frac{n}{2}$$
$$2l + 1 = \frac{2^m + l}{2}$$
$$l = \frac{1}{3}(2^m - 2)$$ l must be integer

For all m, if $l = \frac{1}{3}(2^m - 2)$ is an integer, then solution exists. We can verify this equation

m	*l=1/3*(2^m-2)*	$n = 2^m + l$	$J(n) = 2l + 1 = n/2$
1	0	2	1
3	2	10	5
5	10	42	21
7	42	170	85

Now, we are going to find out the closed form of more general recurrence of Josephus problem introducing constants α, β and γ.

This is called General Recurrence of the Josephus Problem

$$f(1) = \alpha$$
$$f(2n) = 2f(n) + \beta, \quad \text{for } n \ge 1 \qquad \cdots(2)$$
$$f(2n+1) = 2f(n) + \gamma, \quad \text{for } n \ge 1$$

We can construct the following general table for small values of *n*.

n	$f(n)$
1	α
2	$2\alpha + \beta$
3	$2\alpha + \gamma$
4	$4\alpha + 3\beta$
5	$4\alpha + 2\beta + \gamma$
6	$4\alpha + \beta + 2\gamma$
7	$4\alpha + 3\gamma$
8	$8\alpha + 7\beta$
9	$8\alpha + 6\beta + \gamma$

Q: Write the General Recurrence of Josephus Problem. Solve the recurrence (Or, find the closed form expression of the recurrence) by Using Repertoire Method.

Thus, we can express $f(n)$ as following form

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \qquad \cdots(3)$$

where $A(n)$, $B(n)$ and $C(n)$ are coefficients of α, β and γ respectively.

Considering the special case $\alpha = 1, \beta = \gamma = 0$, we get $f(n) = A(n)$ and recurrence becomes

$A(1) = 1$

$A(2n) = 2A(n)$, for $n \geq 1$

$A(2n+1) = 2A(n)$, for $n \geq 1$

Solving the above recurrence we get $A(n) = A(2^m + l) = 2^m$

Plugging the constant function $f(n) = 1$ into equation (2), we get

$1 = \alpha$

$1 = 2 \times 1 + \beta \Rightarrow \beta = -1$

$1 = 2 \times 1 + \gamma \Rightarrow \gamma = -1$

Putting the values of α, β and γ in equation (3), we get

$$1 = A(n) - B(n) - C(n) \qquad \cdots(4)$$

Similarly, we can plug in $f(n) = n$ into equation (2).

$1 = \alpha$

$2n = 2n + \beta \Rightarrow \beta = 0$

$2n + 1 = 2n + \gamma \Rightarrow \gamma = 1$

Putting the values of α, β and γ in equation (3), we get

$$n = A(n) + C(n)$$
$$\Rightarrow C(n) = n - A(n)$$
$$= 2^m + l - 2^m$$
$$= l$$

Putting the values of $A(n)$ and $C(n)$ into equation (4), we can find $B(n)$.

$$B(n) = A(n) - C(n) - 1$$
$$= 2^m - l - 1$$

$\therefore f(n) = 2^m\alpha + (2^m - l - 1)\beta + l\gamma$

The above method of solving recurrence problems is called *repertoire method*. In this method first we find settings of general parameters for which we know the solution, this gives us a repertoire of special cases that we can solve. Then we obtain the general case by combining the special cases.

We can write the generalized recurrence of equation (2) as

$f(1) = \alpha$

$f(2n + j) = 2f(n) + \beta_j$, for $j = 0$, 1 and $n \geq 1$

where $\beta_0 = \beta$ and $\beta_1 = \gamma$, and this recurrence unfolds, binary-wise

$$\begin{aligned}
f((b_m b_{m-1} b_{m-2} \cdots b_2 b_1 b_0)_2) &= 2f((b_m b_{m-1} b_{m-2} \cdots b_2 b_1)_2) + \beta_{b_0} \\
&= 2(2f((b_m b_{m-1} b_{m-2} \cdots b_3 b_2)_2) + \beta_{b_1}) + \beta_{b_0} \\
&= 2^2 f((b_m b_{m-1} b_{m-2} \cdots b_3 b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\
&= 2^2 (2f((b_m b_{m-1} b_{m-2} \cdots b_3)_2) + \beta_{b_2}) + 2\beta_{b_1} + \beta_{b_0} \\
&= 2^3 f((b_m b_{m-1} b_{m-2} \cdots b_3)_2) + 2^2\beta_{b_2} + 2\beta_{b_1} + \beta_{b_0} \\
&\vdots \\
&= 2^m f((b_m)_2) + 2^{m-1}\beta_{b_{m-1}} + 2^{m-2}\beta_{b_{m-2}} + \cdots + 2^2\beta_{b_2} + 2\beta_{b_1} + \beta_{b_0} \\
&= 2^m\alpha + 2^{m-1}\beta_{b_{m-1}} + 2^{m-2}\beta_{b_{m-2}} + \cdots + 2^2\beta_{b_2} + 2\beta_{b_1} + \beta_{b_0} \\
&= (\alpha \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_2} \beta_{b_1} \beta_{b_0})_2
\end{aligned}$$

$f(1) = \alpha$
$f(2n+0) = 2f(n) + \beta_0$
$f(2n+1) = 2f(n) + \beta_1$

in **Josephus Problem** =>
$\alpha = 1$, $\beta_0 = -1$, $\beta_0 = +1$

To verify our new general formula, we are going to check the solution with previously known value of Josephus problem. For example, when $n = 100 = (1100100)_2$, our original Josephus value $\alpha = 1, \beta_0 = \beta = -1$ and $\beta_1 = \gamma = 1$ yield **(must write this: alpha, beta0, beta1)**

Question: Find J(100) by using the Radix Based Property of the ****Generalized**** *Josephus Problem*

$n =$	(1	1	0	0	1	0	$0)_2$	=	100
$n =$	((1	1	0	0	1	0	$0)_2$)		
$f(n) =$	(α	β_1	β_0	β_0	β_1	β_0	$\beta_0)_2$		
=	(1	1	−1	−1	1	−1	$-1)_2$		
=	+64	+32	−16	−8	+4	−2	−1	=	73

We can generalize even more. The recurrence

$f(j) = \alpha_j$, for $1 \leq j < d$

$f(dn + j) = cf(n) + \beta_j$, for $0 \leq j < d$ and $n \geq 1$

***Q: Write The Radix Based Property Of the Generalized JosephusRecurrence. Use This Property to Find J(100)**

is the same as the previous one except that we start with numbers in radix d and produce values in radix c i.e. it has the radix-changing solution.

$$f((b_m b_{m-1} b_{m-2} \cdots b_2 b_1 b_0)_d) = (\alpha_{b_m} \beta_{b_{m-1}} \beta_{b_{m-2}} \cdots \beta_{b_2} \beta_{b_1} \beta_{b_0})_c$$

*Question: Solve following recurrence using *Radix Based Properties* of the Generalized Josephus Problem*

For example, suppose we have the given recurrence

$f(1) = 34$

$f(2) = 5$

$f(3n) = 10f(n) + 76$, for $n \geq 2$

$f(3n+1) = 10f(n) - 2$, for $n \geq 2$

$f(3n+2) = 10f(n) + 8$, for $n \geq 2$

and suppose we want to compute $f(19)$.

Here, we have $d = 3$ and $c = 10$.

$$f(19) = f((201)_3)$$
$$= (\alpha_2\beta_0\beta_1)_{10}$$
$$= (5\ 76\ -2)_{10} = 5\times10^2 + 76\times10^1 - 2\times10^0 = 500 + 760 - 2 = 1258$$

which is our answer.

Read this Section from the Scanned Class Lecture Pages 25 - 29 (more explanation and examples)

Though Easy, dont do like below (but you can always verify your answer using this):

*f(19) = f(3*6+1) =10f(6) - 2 = 10f(3*2) - 2*

=10 [10f(2) + 76] - 2*

=10 [10*5 + 76] - 2*

*= 100*5 +760 -2 ==> 1258*

21 Suppose there are $2n$ people in a circle; the first n are "good guys" and the last n are "bad guys." Show that there is always an integer m (depending on n) such that, if we go around the circle executing every mth person, all the bad guys are first to go. (For example, when $n = 3$ we can take $m = 5$; when $n = 4$ we can take $m = 30$.)

1.21 We can let m be the least (or any) common multiple of $2n$, $2n-1$, $\ldots$, $n+1$. ~~[A non-rigorous argument suggests that a "random" value of m will succeed with probability~~

for n=3 (i.e., 6 persons, first 3 Good, next 3 Bad) => m= LCM(4,5,6) = 60

for n=4 (i.e., 8 persons, first 4 Good, next 4 Bad) => m= LCM(5,6,7,8) = 840

c) Suppose there are $3n$ people in a circle; the first n are "good guys", the middle n are "mixed of good and bad guys" and the last n are "bad guys". Find an integer m (in terms of n) such that if we go around the circle executing every m-th person, then all bad guys are to go first, all the middle n guys are next to go, and finally all the good guys are to go. Justify your answer. 2

m = LCM of (3n, 3n-1, ... , n+1)