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Course No : Math 2101

Course Title : Mathematics – III

Residue at $z = z_0$

$$= \lim_{z \to z_0} \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}} f(z)\right]$$

* Advanced Engineering Mathematics
by Sherin Mariam Alex

$$\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$\overline{(x_1 + iy_1) + (x_2 + iy_2)} = \overline{(x_1 + x_2) + i(y_1 + y_2)}$$

$$= x_1 + x_2 - i(y_1 + y_2)$$

$$= (x_1 - iy_1) + (x_2 - iy_2)$$

$$= \bar{z}_1 + \bar{z}_2$$

30.04.16

Complex Number :—

Definition :

An ordered pair of real number such as (x,y) is termed as a complex number if we write $z=x+iy$ or (x,y) where $i=\sqrt{-1}$, then x is called the real part of and imaginary part of the complex number z and denoted by $x = Re(z)$ or $R(z)$ or R_z, $y = I(z)$ or $I_m(z)$ or I_z.

Historical Note : (1777 – 1855)

~~Carl Fredrichrich Gauss~~

Carl Fredrich Gauss 1832

William Rowan Hamilton. 1835 (1805 – 65)

$$\text{principal argument } \theta = \tan^{-1}\frac{y}{x}$$

* General Argument := $2n\pi$ $(0, \pm 1, \pm 2)$

* ~~Sat~~ State De–Moires Theorem — দি-মোয়ার্স

For all values of n

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

1. Find the modulas and first five General argument of $\frac{-2}{1+i\sqrt{3}}$

Soln:

Here, $\frac{-2}{1+i\sqrt{3}} = \frac{-2(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-2+2i\sqrt{3}}{1+3}$

$= -\frac{2}{4} + \frac{2i\sqrt{3}}{4} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}$

$\therefore$ Modulas of $\frac{-2}{1+i\sqrt{3}} = \left|\frac{-2}{1+i\sqrt{3}}\right| = \sqrt{(-1/2)^2 + (\sqrt{3}/2)^2}$

$= \sqrt{1/4 + \frac{3}{4}} = 1$

So, the principal argument is, $\theta = \tan^{-1}\frac{\sqrt{3}/2}{-1/2}$

$= \tan^{-1}(-\sqrt{3})$

$= \tan^{-1}\tan\frac{2\pi}{3} = \frac{2\pi}{3}$

So, the General argument is, $2n\pi + P.A(\theta)$ $n = 0, \pm 1, \pm 2$

So, the five G.A $= \frac{2\pi}{3}, \frac{8\pi}{3}, -\frac{4\pi}{3}, \frac{14\pi}{3}, \frac{-10\pi}{3}$

2. Find all the values of $(1+i)^{1/4}$

Soln : $1 = r\cos\theta \quad , \quad 1 = r\sin\theta$

$\therefore \; r = \sqrt{2} \quad , \quad \theta = \frac{\pi}{4}$

$\therefore \; (1+i)^{1/4} = (r\cos\theta + ir\sin\theta)^{1/4}$

$= (r)^{1/4}(\cos\pi/4 + i\sin\pi/4)^{1/4}$

$= (\sqrt{2})^{1/4}\left\{\cos\left(2n\pi+\frac{\pi}{4}\right) + i\sin\left(2n\pi+\frac{\pi}{4}\right)\right\}^{1/4},$

$n = 0, 1, 2, 3, \ldots$

$= 2^{1/8}\left\{\cos\frac{1}{4}\left(2n\pi+\frac{\pi}{4}\right) + i\sin\frac{1}{4}\left(2n\pi+\frac{\pi}{4}\right)\right\}$ "

$= 2^{1/8}\left\{\cos\frac{8n+1}{16}\pi + i\sin\frac{8n+1}{16}\pi\right\}$ "

3. Show that, $|z|\sqrt{2} \geq |Re(z)| + |Im(z)|$, where z is any complex number.

Soln : Let. $z = x+iy$ then $Re(z) = x$, $Im(z) = y$

For any two positive real number we know that.

$\frac{a^m+b^m}{2} \geq \left(\frac{a+b}{2}\right)^m$, where ~~miss~~ m is any real number except $0 < m < 1$

Taking $a = |x|$, $b = |y|$ and $m = 2$ we have,

$$\frac{|x|^2 + |y|^2}{2} \geqslant \left(\frac{|x| + |y|}{2}\right)^2$$

$$\Rightarrow \quad x^2 + y^2 \geqslant \frac{(|x| + |y|)^2}{2}$$

$$\Rightarrow \quad \left(\sqrt{x^2+y^2}\right)^2 \geqslant \left(\frac{|x| + |y|}{\sqrt{2}}\right)^2$$

$$\Rightarrow \quad \sqrt{x^2+y^2} \geqslant \frac{|x| + |y|}{\sqrt{2}}$$

$$\Rightarrow \quad |z|\sqrt{2} \geqslant |R_e(z)| + |I_m(z)|$$

[showed]

11.05.16

(2) Prove that if sum and product of two complex numbers are both real then. the two numbers must either be real or conjugate.

$\Rightarrow$ Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers.

Then, their sum, $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

and product, $z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2)$

$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

sum will be real if $y_1 + y_2 = 0 \Rightarrow y_2 = -y_1$ —(1)

and $y_1 = y_2 = 0$ —(2)

Product will be real if $x_1 y_2 + x_2 y_1 = 0$

$$\Rightarrow x_1 y_2 = -x_2 y_1$$

$$\Rightarrow \frac{x_1}{x_2} = -\frac{y_1}{y_2} \quad \text{—(3)}$$

(1) & (3) gives

$$\frac{x_1}{x_2} = 1$$

$$\Rightarrow x_1 = x_2$$

Now, when $y_1 = y_2 = 0$, then $z_1 = x_1$, $z_2 = x_2$

then, in this case z_1 & z_2 both are real.

when, $x_1 = x_2$ and $y_2 = -y_1$

$$z_1 = x_1 + iy_1 = x_2 - iy_2 = \overline{x_2 + iy_2} = \bar{z}_2$$

$$z_2 = x_2 + iy_2 = x_1 - iy_1 = \overline{x_1 + iy_1} = \bar{z}_1$$

Thus if sum and product of two complex number are both real, then the two numbers must either be real or conjugate.

* Solve the equ^n, $|z| - z = 2 + i$

Sol^n: Let, $z = x + iy$, then $|z| = \sqrt{x^2 + y^2}$

Given that, $|z| - z = 2 + i$

$$\Rightarrow \sqrt{x^2+y^2} - (x + iy) = 2 + i$$

Equating real and imaginary parts,

$$\sqrt{x^2+y^2} - x = 2 \quad \text{——} \; (1)$$

$$-y = 1 \quad \text{——} \; (2)$$

$$\Rightarrow y = -1$$

using (II) in (1) we get.

$$\sqrt{x^2+1} - x = 2$$

$$\Rightarrow \sqrt{x^2+1} = \cancel{2x}\; x + 2$$

$$\Rightarrow x^2 + 1 = x^2 + 4x + 4$$

$$\Rightarrow 4x = 1 - 4 = -3$$

$$x = -\frac{3}{4}$$

$$\therefore z = -3/4 - i$$

* complex num graph এ representation কে Argand diagram বলে।

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* Describe geometrically the region of $|2z+3| > 4$

Solⁿ: Let $z = x+iy$,

then, $|2z+3| > 4 = |2(x+iy)+3| > 4$

$$\Rightarrow |(2x+3)+2iy| > 4$$

$$\Rightarrow \sqrt{(2x+3)^2+(2y)^2} > 4$$

$$\Rightarrow 4x^2+12x+9+4y^2 > 16$$

$$\Rightarrow 4x^2+4y^2+12x > 7$$

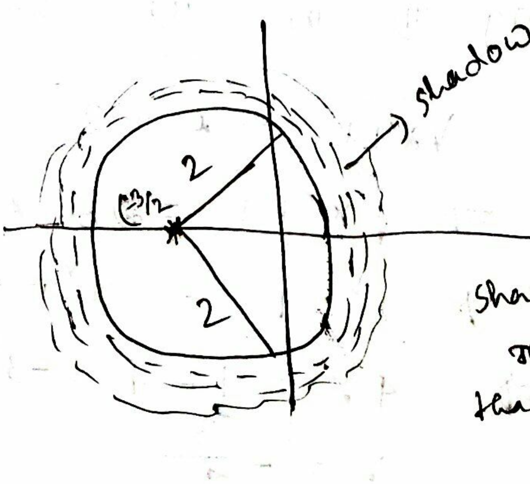
$$\Rightarrow x^2+y^2+3x > \frac{7}{4}$$

$$\Rightarrow x^2+2\cdot x\cdot\frac{3}{2}+\frac{9}{4}+y^2 > \frac{7}{4}+\frac{9}{4}$$

$$\Rightarrow \cancel{(x+3/2)}\ (x+\tfrac{3}{2})^2+(y-0)^2 > 2^2$$

$$\Rightarrow \{x-(-3/2)\}^2+(y-0)^2 > 2^2$$

The region is the set of all external points of the circle, ~~whose~~ whose centre is $(-3/2, 0)$ and radius is 2.



circle touch করবে না।

shadow touch করবে যদি greater than equal

14.05.16

Ex-1, 2, 3(ii), 4, 12, 18; 28, 33, 34, 35

Limit of a function :—

$\lim_{x \to a} f(x) = L$ → x এর মান a এর খুব কাছাকাছি হলে f(x) এর মান defined হয় / finite value আছে।

differentiation → একটির সাপেক্ষে অন্যটির পরিবর্তনের/ পরিবর্তনের হার

function → x এর একটি মানের জন্য y এর একটি মান পাওয়া

* define limit of function.

Limit of a function :—

Let $f(z)$ be a single valued function in a region R and let z_0 be a limit point of R. Then L is said to be the limit of $f(z)$ at z_0 if for any $\epsilon > 0$ there exist a positive δ such that

$$|f(z) - L| < \epsilon \quad \text{whenever} \quad |z - z_0| < \delta$$

We denote this limit as $\lim_{z \to z_0} f(z) = L$

Continuity of a function :-

Let $f(z)$ be a single valued function in a region R and let z_0 be a point in the region if

i) $\lim_{z \to z_0} f(z)$ exist

ii) $\lim_{z \to z_0} f(z) = f(z_0)$

Then $f(z)$ said to be continuous at $z = z_0$

* Differentiability of a given point :-

Let $f(z)$ be a single valued function in a region R and let z_0 be a point in the region. Then the function $f(z)$ is said to be differentiable at z_0 if the function is continuous at z_0 and $\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$ exist and is denoted by $f'(z)$

Analytic $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

18.06.16

Neighbourhood :— The neighbourhood of a point z_0 is the set of all points z such that $|z - z_0| < \delta$ where δ is any positive. A deleted neighbourhood of the point z_0 is the set of $0 < |z - z_0| < \delta$.

* function , limit, continuity , differentiable → Analytic function

* **Analytic (or regular or holomorphic) functions :—**

A single valued function $f(z)$ is said to be analytic at the point z_0 , if its possesses derivatives not only at z_0 but also in the neighbourhood of z_0 -

A single valued function $f(z)$ is said to be analytic in a region R if its analytic at every point of the region R .

* যে Point এর জন্য function analytic না — singular point

* **singular point / singularity :** If a function $f(z)$ fails to be analytic at a point z_0 but in every neighbourhood of z_0 . there exist at least one point where the function is analytic , then z_0 said to be a singular point or singularity of $f(z)$.

* $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

* Show that $\frac{d}{dz}(\bar{z})$ does not exist any where -

Soln: By defn where,

$$\frac{d}{dz}(\bar{z}) = \lim_{\Delta z \to 0} \frac{\overline{z+\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\overline{\Delta x + i\Delta y}}{\Delta x + i\Delta y}$$

$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

Along real axis, $\Delta x \to 0$, $\Delta y = 0$

$$\therefore \frac{d}{dz}(\bar{z}) = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} = 1$$

Along imaginary axis, $\Delta x = 0$, $\Delta y \to 0$

$$\therefore \frac{d}{dz}(\bar{z}) = \lim_{\Delta y \to 0} \frac{-i\Delta y}{i\Delta y} = -1$$

This shows that limit depends on manner in which $\Delta z \to 0$, Here $\frac{d}{dz}(\bar{z})$ does not exist anywhere, thus $f(z) = (\bar{z})$ is not analytic anywhere,

H.W

* State$^{(E)}$ and prove necessary and sufficent condition of Cauchy - riemann (C-R) equations.

$$\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y} \quad , \quad \frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$$

$$f(z) = u(x,y) + iv(x,y)$$

* ~~f(z~~ if ~~f~~ $f(z) = u+iv$ is a analytic function of $z = x+iy$ and ϕ is any function of x and y with differential co-efficient of first order, then show that, $\left(\frac{\delta\phi}{\delta x}\right)^2 + \left(\frac{\delta\phi}{\delta y}\right)^2 = \left\{\left(\frac{\delta\phi}{\delta u}\right)^2 + \left(\frac{\delta\phi}{\delta v}\right)^2\right\} |f'(z)|^2$

Soln: we have, $\phi = \phi(x,y)$

$$\therefore \frac{\delta\phi}{\delta x} = \frac{\delta\phi}{\delta u}\frac{\delta u}{\delta x} + \frac{\delta\phi}{\delta v}\frac{\delta v}{dx} \quad \text{—— (I)}$$

$$\frac{\delta\phi}{\delta y} = \frac{\delta\phi}{\delta u}\frac{\delta u}{\delta y} + \frac{\delta\phi}{\delta v}\frac{\delta v}{\delta y} \quad \text{—— (II)}$$

*

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From C-R equation we have, $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$, $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$ —— (3)

By (3), (2) becomes, $\frac{\delta \phi}{\delta y} = \frac{\delta \phi}{\delta u}\left(-\frac{\delta v}{\delta x}\right) + \frac{\delta \phi}{\delta v}\left(\frac{\delta u}{\delta x}\right)$ —— (4)

$(1)^2 + (4)^2$, we get.

$$\left(\frac{\delta \phi}{\delta x}\right)^2 + \left(\frac{\delta \phi}{\delta y}\right)^2$$

$$= \left(\frac{\delta \phi}{\delta u}\right)^2\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta \phi}{\delta v}\right)^2\left(\frac{\delta v}{\delta x}\right)^2 + 2\frac{\delta \phi}{\delta u}\frac{\delta u}{\delta x}\cdot\frac{\delta \phi}{\delta v}\cdot\frac{\delta v}{\delta x}$$

$$+ \left(\frac{\delta \phi}{\delta u}\right)^2 + \left(\frac{\delta v}{\delta x}\right)^2 + \left(\frac{\delta \phi}{\delta v}\right)^2\left(\frac{\delta u}{\delta x}\right) - 2\cdot\frac{\delta \phi}{\delta u}\cdot\frac{\delta v}{\delta x}\cdot\frac{\delta \phi}{\delta v}\cdot\frac{\delta u}{\delta x}$$

$$= \left(\frac{\delta \phi}{\delta u}\right)^2\left\{\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x}\right)^2\right\} + \left(\frac{\delta \phi}{\delta u}\right)^2\left\{\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x}\right)^2\right\}$$

Given $f(z) = u + iv$

$$f'(z) = \frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}$$

$$\Rightarrow |f'(z)| = \sqrt{\left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x}\right)^2}$$

$$\Rightarrow |f'(z)|^2 = \left(\frac{\delta u}{\delta x}\right)^2 + \left(\frac{\delta v}{\delta x}\right)^2$$ —— (5)

From ⑤ and ⑥ we get,

$$\left(\frac{\delta\phi}{\delta x}\right)^2 + \left(\frac{\delta\phi}{\delta y}\right)^2 = \left\{\left(\frac{\delta\phi}{\delta u}\right)^2 + \left(\frac{\delta\phi}{\delta u}\right)^2\right\} |f'(z)|^2$$

* Show that the function $f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0 & z = 0 \end{cases}$ is not analytic through C-R equations are satisfied at origin.

Sol^n: Let $f(z) = u + iv = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$

$$= \frac{x^3 - y^3}{x^2+y^2} + i\,\frac{(x^3+y^3)}{x^2+y^2}$$

$\therefore\ u(x,y) = \frac{x^3-y^3}{x^2+y^2}$, $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$

At the origion $(0,0)$

$$\frac{\delta u}{\delta x} = \lim_{h\to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{h^3/h^2 - 0}{h}$$

$$= 1$$

$$\frac{\delta u}{\delta y} = \lim_{k\to 0} \frac{u(0,0+k) - u(0,0)}{k} = \lim_{k\to 0} \frac{-\frac{k^3}{k^2} - 0}{k} = -1$$

$$\frac{\delta v}{\delta x} = \lim_{h\to 0} \frac{v(0+h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1$$

$$\frac{\delta v}{\delta y} = \lim_{k\to 0} \frac{v(0,0+k) - v(0,0)}{k} = \lim_{k\to 0} \frac{k^3/k^2 - 0}{k} = 1$$

So, we get, $\frac{\delta u}{\delta x} = \frac{\delta v}{\delta y}$, $\frac{\delta u}{\delta y} = -\frac{\delta v}{\delta x}$

Hence C-R equations satisfied at origin.

Again, $f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}$

$$= \lim_{z \to 0} \frac{\frac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2} - 0}{x + iy}$$

$$= \lim_{z \to 0} \frac{(x^3 - y^3) + i(x^3 + y^3)}{(x + iy)(x^2 + y^2)}$$

Along $y = 0$, $f'(0) = \lim_{x \to 0} \frac{x^3(1+i)}{x^3} = 1 + i$

Along $y = x$, $f'(0) = \lim_{x \to 0} \frac{2ix^3}{2x^3(1+i)} = \frac{i}{1+i}$

Therefore $f'(0)$ has different values along different curves which implies that the limit does not exist/unique.

Hence, $f(z)$ is not analytic through C-R equations are satisfied at the origin.

Harmonic Function :—

A real valued function of two variables x and y that possesses a continuous second order partial derivative in x and y are satisfy the Laplace equation is called Harmonic function.

* Laplace equation : $\Delta^2 \phi = 0$

$$= \frac{\delta^2 \phi}{\delta x^2} + \frac{\delta^2 \phi}{\delta y^2} = 0$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 \longrightarrow \text{satisfied করলে Harmonic হবে ।}$$

* Show that $u(x,y) = \frac{1}{2} \log(x^2+y^2)$ is harmonic and find its conjugatte.

Sol^n : Given, $u = \frac{1}{2}\log(x^2+y^2)$

$$\frac{\delta u}{\delta x} = \frac{1}{2} \cdot \frac{2x}{x^2+y^2}$$

$$\frac{\delta^2 u}{\delta x^2} = \frac{(x^2+y^2)\cdot 1 - x \cdot 2x}{(x^2+y^2)^2}$$

$$= \frac{y^2 - x^2}{(x^2+y^2)^2} \quad \text{—— (1)}$$

$$\frac{\delta w}{\delta y} = \frac{1}{2} \frac{2y}{(x^2+y^2)}$$

$$\frac{\delta^2 u}{\delta y^2} = \frac{(x^2+y^2)\cdot 1 - y \cdot 2y}{(x^2+y^2)^2}$$

* কোনোর function ~~যদি~~ harmonic হলে , ঐ function → analytic .

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$$= \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$= \frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}$$

$$= 0$$

$\therefore$ $u(x,y)$ satisfies Laplace equation.

$\therefore$ u is harmonic.

$$\frac{\delta u}{\delta x} = \frac{x}{x^2+y^2} = \phi_1(x,y) \quad \text{---} \ (A)$$

$$\frac{\delta u}{\delta y} = \frac{y}{x^2+y^2} = \phi_2(x,y) \quad \text{---} \ (B)$$

By miline method, taking $x = z$ and $y = 0$ in $\phi_1(x,y)$ & $\phi_2(x,y)$.

$$\therefore \phi_1(z,0) = \frac{z}{z^2} = \cancel{\frac{1}{z}} \ \frac{1}{z}$$

$\frac{\delta u}{\delta y} = \frac{d}{dy}(x \sin y)$
$= -e^x$

$\frac{d}{dx}(x\cos y - y\sin y) =$

$\phi_2(z,0) = 0$

$\therefore$ By Miline method,

$f'(z) = \phi_1^*(z,0) - i\,\phi_2(z,0)$

$= 1/z - 0$

Integrating in both side,

$f(z) = \int 1/z \, dz = \ln z + c$

$\Rightarrow u + iv = \ln z + c$

* Show that the function $u = e^x(x\cos y - y\sin y)$ is a harmonic function and find the corresponding analytic function $f(z) = u + iv$, from it find v.

Sol^n: We have, $u = e^x(x\cos y - y\sin y)$

$$\frac{\delta u}{\delta x} = e^x(x\cos y - y\sin y) + e^x\cos y = \phi_1(x,y) \quad \text{--- (1)}$$

$$\frac{\delta^2 u}{\delta x^2} = e^x(x\cos y - y\sin y) + e^x.\cos y + e^x\cos y \quad \text{--- (2)}$$

$$\frac{\delta u}{\delta y} = -e^x\, x.\sin y - e^x(\sin y + y\cos y) = \phi_2(x,y) \quad \text{--- (3)}$$

$$\frac{\delta^2 u}{\delta y^2} = -e^x.x\cos y - e^x\cos y - e^x(\cos y - y\sin y) \quad \text{--- (4)}$$

$\int u,v\,dx$ {LIATE D→Derivaties

② + ④,, we get,

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0$$

$\therefore$ u satisfies Laplace equation.

$\therefore$ u is harmonic.

putting $x = z$, $y = 0$ in ① & ③,

$$\phi_1(z,0) = e^z(z+1) \quad , \quad \phi_2(z,0) = 0$$

By Miline method,

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$
$$= e^z \cdot z + e^z - 0$$

Integrating on both sides,

$$f(z) = \int e^z z\,dz + \int e^z\,dz + C$$

$$\Rightarrow u + iv = z e^z - e^z + e^z + c$$
$$= (x+iy)e^{x+iy} + c_1 + ic_2$$
$$= e^x(x+iy)\cdot(\cos y + i\sin y) + c_1 + ic_2$$
$$= e^x(x\cos y - y\sin y) + ie^x(y\cos y + x\sin y) + (c_1 + ic_2)$$
$$= \{e^x(x\cos y - y\sin y) + c_1\} + i\{e^x(x\sin y + y\cos y) + c_2\}$$

$\therefore V = e^x(x\sin y + y\cos y) + c_2$

* If $w = \phi + i\psi$ represent the complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$. Determine the function ϕ.

Sol^n: Given, $w = \phi + i\psi$ and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$

$$\therefore \frac{\partial \psi}{\partial x} = 2x + \frac{y^2 - x^2}{(x^2+y^2)^2}$$

$$\frac{\partial \psi}{\partial y} = -2y - \frac{2xy}{(x^2+y^2)^2}$$

$$\therefore d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$= \frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy \qquad \left[\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \right]$$

$$= \left\{ -2y - \frac{2xy}{(x^2+y^2)^2} \right\} dx - \left\{ 2x + \frac{y^2 - x^2}{(x^2+y^2)^2} \right\} dy$$

This is an exact D.E.

Integrating M w.r.to x keeping y as const + Int. N w.r.t y (free x term)

$$\therefore \phi = -2xy + \frac{y}{x^2+y^2} + C$$

01·06·16

* If $f(z)$ is an analytic function in any domain, prove that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p^2 |f'(z)|^2 |f(z)|^{p-2}$$

Solⁿ: let $f(z) = u+iv$ and $f'(z) = u_x+iv_x$

$\Rightarrow |f(z)| = \sqrt{u^2+v^2}$
$= (u^2+v^2)^{1/2}$

$\Rightarrow f(z)^p = (u^2+v^2)^{p/2}$

$\Rightarrow f(z)^{p-2} = (u^2+v^2)^{\frac{p-2}{2}}$

$\Rightarrow |f'(z)| = \sqrt{u_x^2+v_x^2}$

$\Rightarrow |f'(z)|^2 = u_x^2+v_x^2$

~~$u_x =$~~ $u_x = v_y$, $u_y = -v_x$

$u_{xx}+u_{yy} = 0$, $v_{xx}+v_{yy}=0$

Now, $\frac{\partial}{\partial x}|f(z)|^p = \frac{\partial}{\partial x}(u^2+v^2)^{p/2} = \frac{p}{2}(u^2+v^2)^{\frac{p}{2}-1}(2uu_x+2vv_x)$

$$\frac{\partial^2}{\partial x^2}|f(z)|^p = \frac{p}{2}\left(\frac{p}{2}-1\right)(u^2+v^2)^{\frac{p}{2}-2}(2uu_x+2vv_x)^2+\frac{p}{2}(u^2+v^2)^{p/2-1}\cdot 2(uu_{xx}+v.v_{xx}+v_x^2+u_x^2).$$

$$= \frac{p(p-2)}{4}(u^2+v^2)^{p/2-2}.\,4(u^2u_x^2+v^2v_x^2+2uvu_xv_x)+p(u^2+v^2)^{p/2-1}.(uu_{xx}+v.v_{xx}+u_x^2+v_x^2) \quad \text{— (1)}$$

Similarly,

$$\frac{\partial^2}{\partial y^2}|f(z)|^p = p(p-2)(u^2+v^2)^{\frac{p}{2}-2}.(u^2u_y^2+v^2v_y^2+2uvu_yv_y)+p(u^2+v^2)^{\frac{p}{2}-1}(u.u_{yy}+v.v_{yy}+u_y^2+v_y^2) \quad \text{— (2)}$$

(1) + (2) $\Rightarrow$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^p = p(p-2)(u^2+v^2)^{\frac{p}{2}-2}(u^2u_x^2+u^2u_y^2+v^2v_x^2+v^2v_y^2+2uvu_xv_x+2uvu_yv_y)+p(u^2+v^2)^{p/2-1}\{u(u_{xx}+v_{yy})+v(v_{xx}+v_{yy})+u_x^2+u_y^2+v_x^2+v_y^2\}$$

* Alex – P- 412 – (7.2) → theorem

* 426 → 1, 2, 3, 4, 5, 6, 8, 9, 13, 16

$$= p(p-2)(u^2+v^2)^{p/2-2}(u^2u_x^2+u^2v_x^2+v^2v_x^2+v^2u_x^2+2uvu_xv_x - 2uvu_xv_x) + p(u^2+v^2)^{p/2-1}(u_x^2+v_x^2+v_x^2+u_x^2)$$

$$= p(p-2)(u^2+v^2)^{\frac{p}{2}-2}(u^2+v^2)(u_x^2+v_x^2)+p(u^2+v^2)^{\frac{p}{2}-1}\cdot 2(u_x^2+v_x^2)$$

$$= \left[p(p-2)(u^2+v^2)^{p/2-1}+2p(u^2+v^2)^{p/2-1}\right](u_x^2+v_x^2)$$

$$= (u^2+v^2)^{p/2-1}\left[p^2-2p+2p\right]|f'(z)|^2$$

$$= p^2|f(z)|^{p-2}\cdot|f'(z)|^2$$

def^n

Conformal mapping

↑

Isogonal mapping

↑

mapping

08.06.16

$x = r\cos\theta$
$y = r\sin\theta$

Def^n

Conformal mapping → magnitude + (sense of orientation) change → angle

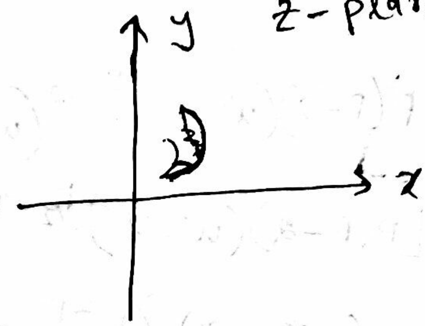
↑

Isogonal mapping → মানের ওপর change

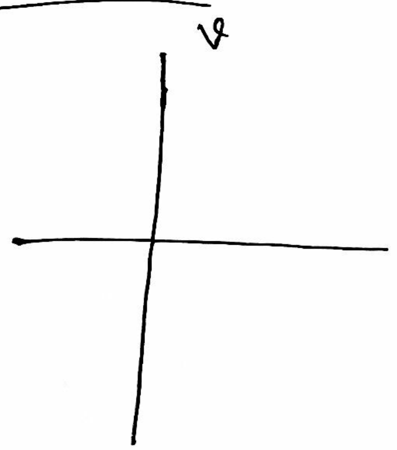
↑

Mapping → 2D কে 4D তে transfer করার Process

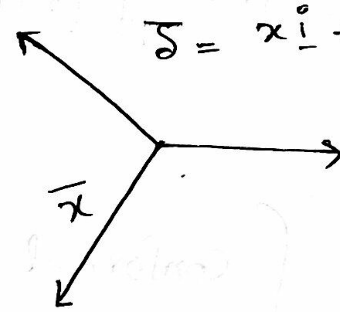
z-plane



W-plane



$\bar{s} = x\underline{i} + y\underline{j} + z\underline{k}$



$w = u(x, y) + iv(x, y)$

Page ~~381~~ 446
6, 7, 9, 34

$z \to 2D$
$w \to 4D$

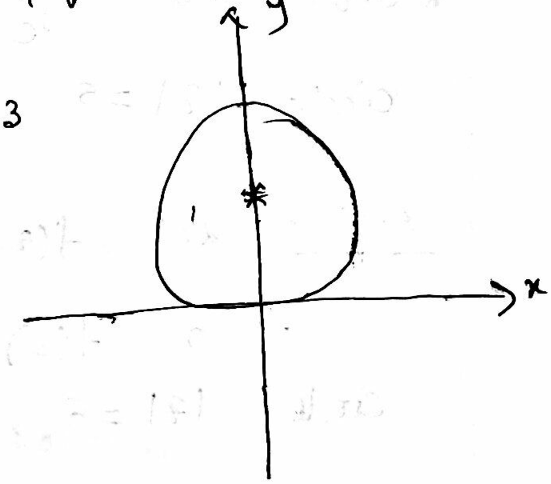
Quiz → 18.06.18
1st to today

* Find the image of the circle $|z-3i| = 3$ under the transformation $w = \frac{1}{z}$. (fig must)

Sol^n: The given transformation is, $w = \frac{1}{z}$

$$\Rightarrow z = \frac{1}{w} \Rightarrow x+iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$$



The given curve is, $|z-3i| = 3$

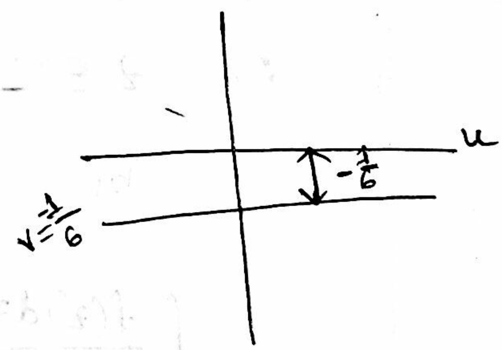
$$\Rightarrow |x+iy-3i| = 3$$

$$\Rightarrow |x+i(y-3)| = 3$$

$$\Rightarrow \sqrt{x^2+(y-3)^2} = 3$$

$$\Rightarrow x^2+(y-3)^2 = 3^2$$

$$\Rightarrow x^2+y^2-6y+9 = 9$$



$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} + \frac{6v}{u^2+v^2} = 0$$

$$\Rightarrow \frac{u^2+v^2}{(u^2+v^2)^2} + \frac{6v}{u^2+v^2} = 0$$

$$\Rightarrow \frac{1}{u^2+v^2} + \frac{6v}{u^2+v^2} = 0$$

$$\Rightarrow 1+6v = 0$$

$\Rightarrow v = -\frac{1}{6}$ which is a straight line.

U-1 3

11.06.16

statement

* Cauchy's integral theorem : $\int f(z)\,dz = 0$ (outside region)

* " " formula : $f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-a}$

* " " " for derivatives : $f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}}$

* Show that $\oint_C \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i$; where C is the circle $|z|=5$.

Sol^n : Let $f(z) = \sin 3z$

then $f(z) = \sin 3z$ is analytic inside and on the circle $|z|=5$, here $z = -\frac{\pi}{2}$

$\Rightarrow |z| = |-\frac{\pi}{2}| = \frac{\pi}{2} = \frac{3.14}{2} = 1.57 < 5$

$\therefore z = -\frac{\pi}{2}$ lies in the circle $|z|=5$

Hence by cauchy's integral formula,

$$\int_C \frac{f(z)\,dz}{(z-a)} = 2\pi i f(a)$$

$$\int_C \frac{\sin 3z}{z+\pi/2}\,dz = 2\pi i\, f(-\pi/2)$$

$= 2\pi i \sin(-3\pi/2)$

$= -2\pi i \sin\frac{3\pi}{2}$

$= -2\pi i \sin(\pi+\frac{\pi}{2})$

$= -2\pi i(-\sin\frac{\pi}{2})$

$= -2\pi i(-1)$

$= 2\pi i$

$\left.\begin{array}{l} * \sin n\pi = 0 \\ * \cos n\pi = (-1)^n \end{array}\right\}$ পূর্ণসংখ্যার

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* Show that $\displaystyle\int_c \frac{e^{3z}}{z-\pi i}\,dz = \begin{cases} -2\pi i & \text{if } c \text{ is the circle } |z-1| = 4 \\ 0 & \text{if } c \text{ is the ellipse } |z-2|+|z+2| = 6 \end{cases}$

Sol^n: $f(z) = e^{3z}$ is analytic in side and on the circle $|z-1| = 4$

$\Rightarrow |x+iy-1| = 4 \quad \Rightarrow |(x-1)+iy| = 4$

$\Rightarrow \sqrt{(x-1)^2+y^2} = 4$

$\Rightarrow (x-1)^2+y^2 = 4^2$

$\therefore z = \pi i \Rightarrow |z| = |\pi i| = \pi = 3.14 < 4$

$\therefore z = \pi i$ lies inside the circle $|z-1| = 4$

Hence by Cauchy's integral formula,

$$\int \frac{e^{3z}}{z-\pi i}\,dz = 2\pi i f(\pi i) = 2\pi i e^{3i\pi}$$

$$= 2\pi i(\cos 3\pi + i\sin 3\pi)$$

$$= 2\pi i(-1+0)$$

$$= -2\pi i$$

2nd part

$|z-2| + |z+2| = 6$

$\Rightarrow |(x-2)+iy| + |(x+2)+iy| = 6$

$\Rightarrow \sqrt{(x-2)^2+y^2} + \sqrt{(x+2)^2+y^2} = 6$

$\Rightarrow (x-2)^2+y^2 = 36 - 12\sqrt{(x+2)^2+y^2} + (x+2)^2+y^2$

$\Rightarrow x^2 - 4x + 4 + y^2 = 36 - 12\sqrt{(x+2)^2 + y^2} + x^2 + 4x + 4 + y^2$

$$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$$

$$\Rightarrow \frac{x^2}{3^2} + \frac{y^2}{(\sqrt{5})^2} = 1$$

$\therefore z = \pi i$ lies outside the ellipse. ~~so~~

So by,

Cauchy's integral theorm,

$$\int_C \frac{e^{3z}}{z - \pi i} dz = 0$$

[showed]

*

14.06.16

* show that. $\frac{1}{2\pi i}\int_C \frac{ze^{tz}}{(z+1)^3}\,dz = \left(t-\frac{1}{2}t^2\right)e^{-t}$, C is any simple closed curve $|z|=2$, $t>0$

Soln: Let $f(z) = ze^{tz}$

?????? $\therefore$ $z=-1$ lies in $|z|=2$

we know, the cauchy's integral formula,

$$f^n(a) = \frac{n!}{2\pi i}\int \frac{f(z)\,dz}{(z-a)^{n+1}}$$

$$\Rightarrow \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-a)^{n+1}} = \frac{f^n(a)}{n!}$$

Here, $n=2$

$$\therefore \frac{1}{2\pi i}\int_C \frac{ze^{tz}}{(z+1)^{2+1}} = \frac{f''(-1)}{2!}$$

Here,

$$f(z) = ze^{tz}$$

$$f'(z) = e^{tz} + tze^{tz}$$

$$f''(z) = te^{tz} + t^2ze^{tz} + te^{tz}$$

$$= 2te^{tz} + t^2.ze^{tz}$$

$$\therefore \frac{f''(-1)}{2!} = \frac{2te^{-t} - t^2e^{-t}}{2} = \frac{2}{2}\left(t-\frac{1}{2}t^2\right)e^{-t}$$

$$= \left(t-\tfrac{1}{2}t^2\right)e^{-t}$$

[showed

statement

· Cauchy's ~~residdce~~ residue Theorem

$\uparrow$

def^n $\Bigg\{$ ~~Residence~~ residue

$\uparrow$

Poles

* z এর প্রত্যেকটির মানকে pole বলা হয়,

$$f(z) = \frac{e^z}{(z-1)(z-2)^2(z-3)^2} = 0$$

$z = 1, \longrightarrow$ simple pole

$z = 2, 2 \rightarrow$ double pole

$z = 3, 3, 3 \rightarrow$ triple pole

* Residue of $f(z)$ at $z = z_0$

$$= \lim_{z \to z_0} \frac{1}{(n-1)!}\left[\frac{d^{n-1}}{dz^{n-1}}\left[(z-z_0)^n f(z)\right]\right]$$

* Residue $= 2\pi i *$ sum of the residue

* $\displaystyle\int_c \frac{e^{tz}}{z^2+1}\,dz = 2\pi i \sin t \qquad |z| = 3,\ t > 0$

$\hookrightarrow (z+i)(z-i) \quad \Rightarrow \displaystyle\frac{1}{2i}\int\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\}e^{tz}\,dz$

* $\frac{e^{it}-e^{-it}}{2i} = \sin t$ $\frac{1}{(n-1)!}\cdot\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right] dz$

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* Show that $\int_C \frac{e^{tz}}{(z^2+1)^2} dz = \pi i(\sin t - t\cos t)$ where c is the circle $|z|=3$, $t>0$

* $\cancel{\int_C \frac{e^{tz}}{z^2+1} dz = 2\pi i \sum i}$

<u>Sol^n:</u> Here the circle $|z|=3$. The poles of $\frac{e^{tz}}{(z^2+1)^2}$ are obtained by solving the equ^n $(z^2+1)^2=0$

$$\Rightarrow (z+i)^2(z-i)^2=0$$

$\therefore z=-i,-i$ & $z=i,i$

$\therefore z=-i$ & i both are double poles., & lies $|z|=3$

Residue of $z=i$ is.

$$R_1 = \lim_{z\to i} \frac{1}{(2-1)!}\left\{\frac{d}{dz}\left\{\cancel{(z-i)^2}\frac{e^{tz}}{(z+i)^2\cancel{(z-i)^2}}\right\}\right\}$$

$$= \lim_{z\to i} \frac{(z+i)^2 t\cdot e^{tz} - e^{tz}\cdot 2(z+i)}{(z+i)^4}$$

$$= \lim_{z\to i} \frac{z\cdot t\cdot e^{tz} + ite^{tz} - 2e^{tz}}{(z+i)^3}$$

$$= \frac{ite^{it} + it\,e^{it} - 2e^{it}}{(2i)^3} = \frac{2ite^{it} - 2e^{it}}{-8i} = \frac{ite^{it} - e^{it}}{-4i}$$

$R_2 =$ Residue of z at $z = -i$

$$= \frac{-ite^{-it} - e^{-it}}{4i}$$

So by cauchy's residue theorem,

$$\oint \frac{e^{tz}}{(z^2+1)^2} dz = 2\pi i \times \text{sum of the residue}.$$

$$= 2\pi i \left(\frac{ite^{it} - e^{it}}{-4i} + \frac{-ite^{-it} - e^{-it}}{4i} \right)$$

$$= \frac{2\pi i}{-4i} \left(ite^{it} - e^{it} + ite^{-it} + e^{-it} \right)$$

$$= \frac{2\pi i}{-4i} \left\{ \left(it\left(e^{it} + e^{-it}\right) - \left(e^{it} - e^{-it}\right) \right) \right\}$$

$$= \pi i \left[-it \frac{\left(e^{it} + e^{-it}\right)}{2i} + \frac{e^{it} - e^{-it}}{2i} \right]$$

???????$= \pi i \left[-t \cos t + \sin t \right]$

$$= \pi i \left[\sin t - t\cos t \right]$$

$(z^2-\pi^2 i^2)$
$= (z-\pi i)^2(z+\pi i)^2$

15.06.16

~~Taylor's Theorem~~

~~If $f(z)$ be a analytic inside and on a simple~~

Show that $\displaystyle\int \frac{e^z}{(z^2+\pi^2)^2}\,dz = \frac{i}{\pi}$, $|z|=4$

?????????$\Rightarrow$ $z=\pi i$, $-\pi i$ both are double pole.

Residue at $z=\pi i$

??????? $$R_1 = \lim_{z\to\pi i} \frac{d}{dz}\left\{ (z-\pi i)^2 \frac{e^z}{(z+\pi i)^2(z-\pi i)^2} \right\}$$

?????????? $$= \lim_{z\to\pi i} \frac{(z+\pi i)^2\cdot e^z - e^z\, 2(z+\pi i)}{(z+\pi i)^4}$$

$$= \lim_{z\to\pi i} \frac{z\cdot e^z + \pi i e^z - \cancel{2e^z}\ 2e^z}{(z+\pi i)^3}$$

$$= \frac{\pi i e^{\pi i} + \pi i e^{\pi i} - 2e^{\pi i}}{(\pi i+\pi i)^3}$$

??????? $$= \frac{2\pi i e^{\pi i} - 2e^{\pi i}}{-8\pi^3 i}$$

$$= \frac{(\pi i - 1)e^{\pi i}}{-4\pi^3 i}$$

$$R_2 = \frac{-(\pi i - 1)e^{\pi i}}{4\pi^3 i}$$

So, by cauchy's residue theorem,

$$\oint \frac{e^z}{(z^2+\pi^2)^2}\,dz = 2\pi i \times \text{sum of the residue}$$

$$= 2\pi i\left(\frac{(\pi i-1)e^{\pi i}}{-4\pi^3 i} + \frac{-(\pi i-1)e^{\pi i}}{4\pi^3 i}\right)$$

$$= \frac{2\pi i}{4\pi^3 i}\left(-(\pi i-1)e^{\pi i} - (\pi i-1)e^{\pi i}\right)$$

$$= \frac{i}{2\pi^2 i} \times \left(-2(\pi i-1)e^{\pi i}\right)$$

$$= -\pi^2(\pi i-1)e^{\pi i}$$

Taylors Theorem:

if $f(z)$ be a analytic inside and on a simple closed curved c let a and $a+h$ be two points inside c.

then,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots \quad (1)$$

or, writing, $z = a+h$, $\Rightarrow h = z-a$

$$f(z) = f(a) + (z-a)f'(a) + \cancel{\frac{z}{1}}\frac{(z-a)^2}{2}f''(a) + \cdots$$
$$+ \frac{(z-a)^n}{n!}f^n(a) + \cdots \quad (2)$$

* State Taylor Theorem for analytic function.

* Find the first four terms of the Taylor series expansion of the complex variable function

$$f(z) = \frac{z+1}{(z-3)(z-4)} \text{ about } z = 2$$

Sol^n : Given function, $f(z) = \dfrac{z+1}{(z-3)(z-4)}$

Let us convert the given fraction into partial fractions,

$$\therefore f(z) = \frac{z+1}{(z-3)(z-4)} = \frac{A}{z-3} + \frac{B}{z-4}$$

$$\Rightarrow (z+1) = A(z-4) + B(z-3)$$

if $z = 3$, $4 = -A \Rightarrow A = -4$

if $z = 4$, $5 = B \Rightarrow B = 5$

$$\therefore f(z) = \frac{5}{z-4} - \frac{4}{z-3}$$

$$\therefore f(2) = -\frac{5}{2} + 4 = \frac{3}{2}$$

$$\therefore f'(z) = -\frac{5}{(z-4)^2} + \frac{4}{(z-3)^2} \qquad \therefore f'(2) = \frac{11}{4}$$

$$\therefore f''(z) = \frac{10}{(z-4)^3} - \frac{8}{(z-3)^3} \qquad \therefore f''(2) = \frac{27}{4}$$

$$\therefore f'''(z) = \frac{-30}{(z-4)^4} + \frac{24}{(z-3)^4} \qquad \therefore f''(2) = \frac{177}{8}$$

$\therefore$ The Taylor's series is,

$$f(z) = f(a) + (z-a)f'(a) + \frac{(z-a)^2}{2!} f''(a) + \frac{(z-a)^3}{3!} f'''(a) + \cdots$$

about $z = 2$

$$\frac{z+1}{(z-3)(z-4)} = \frac{3}{2} + (z-2)\frac{11}{4} + \frac{(z-2)^2}{2!} \cdot \frac{27}{4} + \frac{(z-2)^3}{3!} \frac{177}{8} + \cdots$$

22.06.16

* Laurent's Series :

* Expand $f(z) = \frac{z^2}{(z-1)(z-2)}$ in a Laurent's series for the region (i) $1 < |z| < 2$
(ii) $0 < |z| < 1$

Solution :

Here, $f(z) = \frac{z^2}{(z-1)(z-2)} = 1 + \frac{A}{z-1} + \frac{B}{z-2}$ (say)

$$\Rightarrow z^2 = (z-1)(z-2) + A(z-2) + B(z-1)$$

if, $z = 1$, $A = -1$
if, $z = 2$, $B = 4$

$$\therefore f(z) = 1 - \frac{1}{z-1} + \frac{4}{z-2}$$

1) $1 < |z| < 2$

$|z| > 1$ $\Rightarrow \frac{1}{|z|} < 1$ | $|z| < 2$ $\Rightarrow \frac{|z|}{2} < 1$

$$\therefore f(z) = 1 - \frac{1}{z-1} + \frac{4}{z-2}$$

$$= 1 - \frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{4}{-2\left(1-\frac{z}{2}\right)}$$

$$= 1 - \frac{1}{z}\left(1 - 1/z\right)^{-1} - 2\left(1-\frac{z}{2}\right)^{-1}$$

$$= 1 - \frac{1}{z}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right) - 2\left(1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right)$$

(ii) $0 < |z| < 1$

$\Rightarrow |z| > 0 \quad \Big| \quad |z| < 1$

$$\therefore f(z) = 1 - \frac{1}{z-1} + \frac{4}{z-2}$$

$$= 1 - \frac{1}{-(1-z)} + \frac{4}{-2\left(1-\frac{z}{2}\right)}$$

$$= 1 + (1-z)^{-1} - 2\left(1-\frac{z}{2}\right)^{-1}$$

$$= 1 + (1 + z + z^2 + z^3 + \cdots) - 2\left(1 + z/2 + \frac{z^2}{4} + \cdots\right)$$

* Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent's series.

(i) $1 < |z| < 3$ (ii) $|z| > 3$ (iii) $0 < |z+1| < 2$

(iv) $|z| < 1$

Here, $f(z) = \dfrac{1}{(z+1)(z+3)}$

$$\therefore f(z) = \frac{1}{2}\frac{1}{(z+1)} - \frac{1}{2}\frac{1}{(z+3)}$$

i) $1 < |z| < 3$

$\Rightarrow |z| > 1 \quad \Big| \quad |z| < 3$

$\Rightarrow \dfrac{1}{|z|} < 1 \quad \Big| \quad \dfrac{|z|}{3} < 1$

$$\therefore f(z) = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{3\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \cdots\right) - \frac{1}{6}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \cdots\right)$$

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(ii) $|z| > 3$

$$\Rightarrow \frac{3}{|z|} < 1$$

$$\therefore f(z) = \frac{1}{2} \cdot \frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2} \cdot \frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right) - \frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\cdots\right)$$

(iii) $0 < |z+1| < 2$

Let $u = z+1 \Rightarrow z = u-1$

$$\therefore \quad 0 < |u| < 2$$

???????????$\Rightarrow \quad u > 0 \quad \Big| \quad u < 2 \Rightarrow \frac{|u|}{2} < 1$

???????$\therefore f(u-1) = \frac{1}{2u} - \frac{1}{2(u+2)}$

$$= \frac{1}{2u} - \frac{1}{4}\left(1+\frac{u}{2}\right)^{-1}$$

$$= \frac{1}{2u} - \frac{1}{4}\left(1-\frac{u}{2}+\frac{u^2}{4}-\frac{u^3}{8}+\cdots\right)$$

$$\therefore f(z) = \frac{1}{2(z+1)} - \frac{1}{4}\left[1-\frac{(z+1)}{2}+\frac{(z+1)^2}{4}-\frac{(z+1)^3}{8}+\cdots\right]$$

$\frac{|z|}{1} < 1$

$\frac{|z|}{3} < 1$

(iv) $|z| < 1$

????? $f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$

$= \frac{1}{2}(1+z)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$

$= \frac{1}{2}\left(1-z+z^2-z^3+\cdots\right) - \frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$

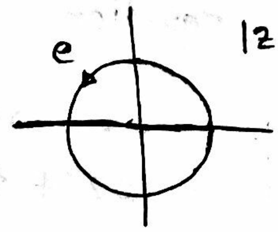
* যেকোনো ত্রিকোণমিতিক curve এ Max $= +1$ ও min $= -1$

<u>Next Quiz</u>

<u>23.07.16</u>

<u>16.07.16</u>

* Evaluate $\displaystyle\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$ by using contour integration :-



Sol^n: Let us, Consider the unit circle $|z| = 1$ as the Contour c.

Then,

$$z = e^{i\theta} \Rightarrow dz = ie^{i\theta}\,d\theta \Rightarrow dz = iz\,d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

$[0 \le \theta \le 2\pi]$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+\frac{1}{z}}{2} = \frac{z^2+1}{2z}$$

$$\therefore \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \text{Real part of} \int_0^{2\pi} \frac{\cos 2\theta + i\sin 2\theta}{5+4\cos\theta}\,d\theta$$

$$= \text{R.P of} \int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta$$

$$= \text{R.P of} \int_c \frac{z^2}{5+4\,\frac{z^2+1}{2z}}\,\frac{dz}{iz}$$

$$= \text{R.P of } \frac{1}{i}\int_c \frac{z}{\frac{10z+4z^2+4}{2z}}\,dz$$

$$= \text{R.P of } \frac{1}{i}\int_c \frac{z^2}{2z^2+5z+2}\,dz$$

$$= \text{R.P of } \frac{1}{i}\int_c f(z)\,dz, \quad \text{where } f(z) = \frac{z^2}{2z^2+5z+2} \qquad \text{—— (1)}$$

1 time হলে diff লাগবে না।
2 times হলে diff লাগবে।

Page - 472 - Complex Integration 1-9, 11 | Page - 505 1-7 (contour)
Page - 481 - (1-15) (taylor, laurant)
498 - (4-11) - (residue)

The poles of $f(z)$ are obtained by ~~solu~~ solving the equation,

$$2z^2+5z+2=0$$

$$\Rightarrow 2z^2+4z+z+1=0$$

$$\Rightarrow 2z(z+2)+1(z+2)=0$$

$$\Rightarrow (z+2)(2z+1)=0$$

$$\therefore z=-2,\ -\frac{1}{2}$$

$$\therefore |-2| = 2 > 1 \quad \text{and} \quad |-1/2| = \frac{1}{2} < 1$$

The only pole $z=-\frac{1}{2}$ lies inside the contour which is simple pole.

$\therefore$ Residue at $z=-\frac{1}{2}$ is, $\lim_{z\to -\frac{1}{2}} (z+\frac{1}{2}) f(z)$

$$= \lim_{z\to -\frac{1}{2}} \left(z+\frac{1}{2}\right) \frac{z^2}{2(z+1)(z+\frac{1}{2})} = \frac{(-1/2)^2}{2(-1/2+2)}$$

$$= \frac{1/4}{2\cdot\frac{3}{\cancel{4}\,2}} = \frac{1}{12}$$

$\therefore$ by Cauchy's residue Theorem,

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = R.P \text{ of } \frac{1}{i}\left[2\pi i \times \text{Residue at } z=\frac{-1}{2}\right]$$

$$= \frac{1}{i}\cdot 2\pi i\cdot \frac{1}{12} = \frac{\pi}{6}$$

$L\{1\} = \frac{1}{s}$, $L^{-1}\{\frac{1}{s}\} = 1$

19.07.16

Laplace Transformation

Ex Def^n :- The definite integral $\int_0^\infty e^{-st} F(t)\,dt$ is called the laplace transform of $F(t)$. It is denoted by $L\{F(t)\}$ or $\mathcal{L}\{F(t)\}$ and define $L\{F(t)\} = \int_0^\infty e^{-st} F(t)dt = f$

* Find i) $L\{1\}$ ii) $L\{t\}$ iii) $L\{e^{at}\}$ iv) $L\{\sin at\}$ or L
v) $L\{\sinh at\}$ or $L\{\cosh at\}$

Sol^n:

By the def^n of Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} F(t)\,dt$$

i) $\therefore$ $L\{1\} = \int_0^\infty e^{-st} . 1\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = -\frac{1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{s}[0-1] = \frac{1}{s}$$

ii) $L\{t\} = \int_0^\infty e^{-st} . t\,dt$

$$= \left[t \int e^{-st} dt - \int\left(\frac{d}{dt}(t) \times \int e^{-st} dt\right) dt\right]$$

e^{-st+at}

$$= \left[-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}\right]_0^{\infty}$$

$$= 0 - 0 + 0 + \frac{1}{s^2}$$

$$\therefore L\{t\} = \frac{1}{s^2}$$

$$L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

iii) $$L\{e^{at}\} = \int_0^{\infty} e^{-st} \cdot e^{at}\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\, dt$$

$$= \left[-\frac{e^{-(s-a)t}}{(s-a)}\right]_0^{\infty}$$

$$= -\frac{1}{(s-a)}\left[0-1\right]$$

$$L\{e^{at}\} = \frac{1}{s-a}$$

$$\therefore L^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$$

* $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$
* $\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

* $\sinh x = \frac{e^x - e^{-x}}{2}$
* $\cosh x = \frac{e^x + e^{-x}}{2}$

iv) $L\{\sin at\} = \int_0^\infty e^{-st} \sin at\, dt$

$$= \left[\frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\right]_0^\infty$$

$$= 0 + \frac{a}{s^2+a^2}$$

$\therefore$ $L\{\sin at\} = \frac{a}{s^2+a^2}$

$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$

~~v) L{sin h at}~~

v) $L\{\cosh at\} = \int_0^\infty e^{-st} \cdot \cosh at\, dt$

$$= \int_0^\infty e^{-st}\left(\frac{e^{at}+e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\, dt + \frac{1}{2}\int_0^\infty e^{-(s+a)t}\, dt$$

$$= \frac{1}{2}\left[-\frac{e^{-(s-a)t}}{(s-a)}\right]_0^\infty + \frac{1}{2}\left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= 0 + \frac{1}{2}\frac{1}{(s-a)} + 0 + \frac{1}{2(s+a)}$$

$$= \frac{s+a+s-a}{2(s^2-a^2)} = \frac{s}{s^2-a^2}$$

20.07.16

vi) $L\{t^n\} = \int_0^\infty e^{-st} t^n \, dt$

$= \int_0^\infty e^{-y} \frac{y^n}{s^n} \cdot \frac{1}{s} \, dy$

$= \frac{1}{s^{n+1}} \int_0^\infty e^{-y} y^n \, dy$

Let,

$st = y$

$\Rightarrow s\,dt = dy$

$\Rightarrow dt = \frac{1}{s} dy$

t	0	∞
y	0	∞

???????? $L\{t^n\} = \frac{\sqrt{n+1}}{s^{n+1}} = \frac{n!}{s^{n+1}}$

$\Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\underline{|n}}$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx$$

??????????? $\sqrt{n} = \int_0^\infty e^{-x} \cdot x^{n-1} \, dx$

$\Rightarrow \sqrt{n+1} = \int_0^\infty e^{-x} \cdot x^n \, dx$

$= \int_0^\infty e^{-y} y^n \, dy$

* $L\{t\cos at\}$

* $L\{t\sin at\} = \int_0^\infty e^{-st}\, t \cdot \sin at \, dt$

$$= \int_0^\infty t\,(e^{-st}\sin at)\,dt$$

$$= \left[t \cdot \frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2} \right]_0^\infty - \int_0^\infty \frac{e^{-st}(-s\sin at - a\cos at)}{s^2+a^2}\,dt$$

$$= 0 - 0 + \frac{s}{s^2+a^2}\int_0^\infty e^{-st} \cdot \sin at + \frac{a}{s^2+a^2}\int_0^\infty e^{-st}\cos at\,dt$$

$$= \frac{s}{s^2+a^2} \cdot \frac{a}{s^2-a^2} + \frac{a}{s^2+a^2} \cdot \frac{s}{s^2+a^2}$$

* $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

* Find the Laplace transform of the function $F(t) = \begin{cases} t, & 0<t<2 \\ 3, & t>2 \end{cases}$

Sol^n: By the def^n of Laplace transform,

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\, dt = \int_0^2 e^{-st} F(t)\, dt + \int_2^\infty e^{-st} F(t)\, dt$$

$$= \int_0^2 e^{-st} \cdot t\, dt + \int_2^\infty e^{-st} \cdot 3\, dt$$

$$= \left[t \cdot \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_0^2 + 3 \left[\frac{e^{-st}}{-s} \right]_2^\infty$$

$$= -\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + 0 + \frac{1}{s^2} - 0 + \frac{3e^{-2s}}{s}$$

$$= \frac{1}{s^2} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s}$$

Law:

* If $L\{F(t)\} = f(s)$ then $L\{e^{at} F(t)\} = f(s-a)$ — 1st translation or shifting property

* If $L\{F(t)\} = f(s)$ and $G(t) = \begin{cases} F(t-a), & t>a \\ 0, & t<a \end{cases}$ then

$$L\{G(t)\} = e^{-as} f(s)$$

2nd shifting property

Soln :

By the defn of Laplace transform,

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt = f(s)$$

$$\therefore L\{G(t)\} = \int_0^\infty e^{-st} G(t)\,dt = \int_0^a e^{-st} G(t)\,dt + \int_a^\infty e^{-st} G(t)\,dt$$

let, $t - a = u$

$dt = du$

t	a	∞
u	0	∞

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} \cdot F(t-a)\,dt$$

$$= 0 + \int_0^\infty e^{-s(u+a)} F(u)\,du = e^{-as} \int_0^\infty e^{-su} F(u)\,du$$

$$= e^{-as} \int_0^\infty e^{-st} F(t)\,dt$$

$$= e^{-as} L\{F(t)\}$$

$$= e^{-as} f(s)$$

Theorem: If $L\{F(t)\} = f(s)$, Then $L\{F(at)\} = \frac{1}{a} f\left(\frac{s}{a}\right)$

Prove

H.W - $L\{F'''(t)\}$

If $L\{F(t)\} = f(s)$, then $L\{F''(t)\} = s^2 f(s) - sF(0) - F'(0)$

Solⁿ: ~~Derivation~~ Derivative of Laplace transform,

By the defⁿ, $L\{F(t)\} = \int_0^\infty e^{-st} F(t)\,dt$

$$\therefore L\{F''(t)\} = \int_0^\infty e^{-st} F''(t)\,dt$$

$$= \left[e^{-st} \int F''(t)\,dt - \int \left\{ \frac{d}{dt}(e^{-st}) \int F''(t)\,dt \right\} dt \right]_0^\infty$$

$$= \left[e^{-st} \cdot F'(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} F'(t)\,dt$$

$$= 0 - F'(0) + s\left[\left(e^{-st} \cdot F(t)\right)_0^\infty + s\int_0^\infty e^{-st} F(t)\,dt\right]$$

$$= -F'(0) - sF(0) + s^2 f(s)$$

$$= s^2 f(s) - sF(0) - F'(0)$$

23.07.16

Law: If $L\{F(t)\} = f(s)$, then $L\{t^n F(t)\} = (-1)^n \frac{d^n}{ds^n} f(s)$

State:

Periodic Function :- If $F(t)$ has period $T>0$ then

$$L\{F(t)\} = \frac{\int_0^T e^{-st} F(t)}{1-e^{-sT}} dt$$

Proof: If $F(t)$ is a periodic function with period T.

?????? Then ~~$L\{F(t)\}$~~ $F(t+T) = F(t)$.

Now, by defn of laplace transform, $L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt$

$$= \int_0^T e^{-st} F(t) dt + \int_T^{2T} e^{-st} F(t) dt + \int_{2T}^{3T} e^{-st} F(t) dt + \cdots$$

In the 1st integral, $t = u \Rightarrow dt = du$

t	0	T
u	0	T

$t = u+T$, $dt = du$

t	T	$2T$
u	0	T

$t = u+2T$, $dt = du$

t	$2T$	$3T$
u	0	T

$$\therefore L\{F(t)\} = \int_0^T e^{-su} F(u) du + \int_0^T e^{-s(u+T)} F(u+T) du + \int_0^T e^{-s(u+2T)} F(u+2T) du + \cdots$$

$$= \int_0^T e^{-su} F(u) du + e^{-sT} \int_0^T e^{-su} F(u) du + e^{-2sT} \int_0^T e^{-su} F(u) du + \cdots$$

$$= \left(1 + e^{-sT} + e^{-2sT} + \cdots\right) \int_0^T e^{-su} F(u)\, du$$

$$= \left(1 - e^{-sT}\right)^{-1} \int_0^T e^{-st} F(t)\, dt$$

$$= \frac{\int_0^T e^{-st} F(t)\, dt}{1 - e^{-sT}}$$

[Proved]

* Find $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

$$= 3L\{t^4\} - 2L\{t^3\} + 4L\{e^{-3t}\} - 2L\{\sin 5t\} + 3L\{\cos 2t\}$$

$$= 3 \cdot \frac{4!}{s^5} - 2 \cdot \frac{3!}{s^4} + 4 \cdot \frac{1}{s+3} - 2 \cdot \frac{5}{s^2+25} + 3 \cdot \frac{s}{s^2+4}$$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}$$

* Find the laplace transform of $F(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & , \pi < t < 2\pi \end{cases}$

Soln: If $F(t)$ has period $T > 0$ then

$$L\{F(t)\} = \frac{\int_0^T e^{-st} F(t)\,dt}{1 - e^{-sT}} \quad , \text{ here } T = 2\pi$$

$$= \frac{1}{1 - e^{-2\pi s}} \int_0^{2\pi} e^{-st} F(t)\,dt$$

$$= \frac{1}{1 - e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\,dt + \int_{\pi}^{2\pi} e^{-st} \cdot 0\,dt \right]$$

$$= \frac{1}{1 - (e^{-\pi s})^2} \left[\frac{e^{-st}(-s \cdot \sin t - \cos t)}{s^2 + 1} \right]_0^{\pi} + 0$$

$$= \frac{1}{(1 - e^{-\pi s})(1 + e^{-\pi s})} \left[\frac{e^{-\pi s}}{s^2 + 1} + \frac{1}{s^2 + 1} \right]$$

$$= \frac{(1 + e^{-\pi s})}{(1 - e^{-\pi s})(1 + e^{-\pi s})(1 + s^2)} = \frac{1}{(1 - e^{-\pi s})(1 + s^2)}$$

[Ans]

* $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \frac{n(n-1)(n-2)}{3!} x^3 + \cdots$

Laplace transform of Bessel's Function :

* Prove that $L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$

Sol^n: By the def^n of Bessel's function of order n, we have,

$$J_n(t) = \frac{t^n}{2^n \cdot \Gamma(n+1)}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right]$$

putting, $n=0$,

$$J_0(t) = \frac{1}{1}\left[1 - \frac{t^2}{4} + \frac{t^4}{2\cdot 4\cdot 2\cdot 4} - \cdots\right]$$

Taking Laplace on both side,

$$\therefore L\{J_0(t)\} = L\{1\} - \frac{1}{4} L\{t^2\} + \frac{1}{64} L\{t^4\} - \cdots$$

$$= \frac{1}{s} - \frac{1}{4}\cdot\frac{2}{s^3} + \frac{1}{64}\cdot\frac{24}{s^5} - \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1\cdot 3}{2\cdot 4}\left(\frac{1}{s^4}\right) - \cdots\right]$$

$$= \frac{1}{s}\cdot\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{s}\cdot\left(\frac{1+s^2}{s^2}\right)^{-1/2}$$

$$= \frac{1}{s}\cdot\frac{(s^2)^{1/2}}{(1+s^2)^{1/2}} = \frac{1}{\sqrt{1+s^2}}$$ [proved]

27.07.16

* Find $L\{F(t)\}$ if $F(t) = \begin{cases} \sin t & , 0 < t < \pi \\ 0 & , \pi < t < 2\pi \end{cases}$

Soln: If $f(t)$ is a period $T > 0$ then,

$$L\{F(t)\} = \int_0^T \frac{e^{-st} F(t)\, dt}{1 - e^{-st}}$$

Here, $T = 2\pi$

$$\therefore \quad L\{F(t)\} = \frac{\int_0^{2\pi} e^{-st}\, F(t)\, dt}{1 - e^{-2\pi s}}$$

$$= \frac{1}{1-e^{-2\pi s}} \left[\int_0^{\pi} e^{-st} \sin t\, dt + \int_0^{2\pi} e^{-st} \cdot 0\, dt \right]$$

$$= \frac{1}{1-(e^{-\pi s})^2} \left[\frac{e^{-st}(-s \sin t - \cos t)}{s^2 + 1^2} \right]_0^{\pi}$$

$$= \frac{1}{(1-e^{-\pi s})(1+e^{-\pi s})} \left(\frac{e^{-\pi s} + 1 + 0}{1^2 + s^2} \right)$$

$$= \frac{1}{(1-e^{-\pi s})(1+s^2)}$$

* For laplace transform of Bessel's function prove that

$$- L\{J_0(t)\} = \frac{1}{\sqrt{s^2+1}}$$

Soln: By the defn of Bessel's function.

$$J_n(t) = \frac{t^n}{2^n \Gamma n+1}\left[1 - \frac{t^2}{2(2n+2)} + \frac{t^4}{2\cdot 4(2n+2)(2n+4)} - \cdots\right] \quad (1)$$

putting $n=0$ and taking laplace on both sides of (1).

$$L\{J_0(t)\} = L\{1\} - L\left\{\frac{t^2}{4}\right\} + L\left\{\frac{t^4}{64}\right\} - \cdots$$

$$= \frac{1}{s} - \frac{1}{4}\cdot\frac{2}{s^3} + \frac{1}{64}\,\frac{24}{s^5} - \cdots$$

$$= \frac{1}{s}\left[1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1.3}{2\cdot 4}\left(\frac{1}{s^4}\right) - \cdots\right]$$

$$= \frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2} = \frac{1}{s}\left(\frac{s^2+1}{s^2}\right)^{1/2} = \frac{1}{s}\,\frac{s}{\sqrt{1+s^2}}$$

$$= \frac{1}{\sqrt{1+s^2}}$$

* $\int_0^\infty t e^{-st} \sin at \, dt = \frac{2as}{(s^2+a^2)^2}$ — H.W

* Prove that $\int_0^\infty t e^{-st} \cos at \, dt = \frac{s^2 - a^2}{(s^2+a^2)^2}$

Solⁿ: By the defⁿ of laplace transform, we have

$$L\{F(t)\} = \int_0^\infty e^{-st} F(t) dt = f(s)$$

$$L.H.S = \int_0^\infty t e^{-st} \cos at \, dt = \int_0^\infty e^{-st} (t \cos at) \, dt$$

$$= L\{t \cos at\}$$

$$= (-1)^1 \frac{d}{ds} \left(\frac{s}{s^2+a^2} \right)$$

$$= -\frac{(s^2+a^2) \cdot 1 - s \cdot 2s}{(s^2+a^2)^2}$$

$$= \frac{-s^2 - a^2 + 2s^2}{(s^2+a^2)^2}$$

$$= \frac{s^2 - a^2}{(s^2+a^2)^2}$$

$$\Gamma\tfrac{7}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\tfrac{1}{2} = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}$$

$$e^x = 1 + x + x^2/2 + \cdots$$

* Prove that $L\{\sin\sqrt{t}\} = \dfrac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-1/4s}$

Proof: From trigonometric series,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (1)$$

Putting, $x = \sqrt{t}$, $\sin\sqrt{t} = \sqrt{t} - \dfrac{(\sqrt{t})^3}{3!} + \dfrac{(\sqrt{t})^5}{5!} - \dfrac{(\sqrt{t})^7}{7!} + \cdots$

taking laplace on both side,

$$L\{\sin\sqrt{t}\} = L\{t^{1/2}\} - \frac{1}{3!}L\{t^{3/2}\} + \frac{1}{5!}L\{t^{5/2}\} - \frac{1}{7!}L\{t^{7/2}\} + \cdots$$

$$= \frac{\Gamma 3/2}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma 5/2}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma 7/2}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma 9/2}{s^{9/2}} + \cdots$$

$$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{1}{3!}\frac{3/2\cdot\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{5!}\frac{5/2\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{1}{7!}\frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{9/2}}$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{2^2 s} + \frac{1}{2!}\left(\frac{1}{2^2 s}\right)^2 - \frac{1}{3!}\left(\frac{1}{2^2 s}\right)^3 + \cdots\right]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\cdot e^{-\frac{1}{2^2 s}} = \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}$$

Also prove that, $L\left\{\frac{\cos\sqrt{t}}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

Solnº: Let, $F(t) = \sin\sqrt{t}$

$\therefore F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$ and $F(0) = 0$

From the laplace transform of derivatives;

We know, $L\{F'(t)\} = sL\{F(t)\} - F(0)$

$\therefore L\left\{\frac{\cos\sqrt{t}}{2\sqrt{t}}\right\} = s\,L\{\sin\sqrt{t}\} - 0$

$= s \cdot \frac{\sqrt{\pi}}{2 \cdot s^{3/2}}\, e^{-1/4s}$

?????????????????????????? $= \frac{2\sqrt{\pi}}{2\, s^{1/2}}\, e^{-1/4s} = \cancel{\sqrt{\pi}}\sqrt{\frac{\pi}{s}}\, e^{-1/4s}$

* $L\{\sin at\} = \frac{a}{s^2+a^2}$
$\Rightarrow L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

30.07.16

The Inverse Laplace Transform

Prove that: $L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} = 2e^{2t}(3\cos 4t + \sin 4t)$

$$L.H.S = L^{-1}\left\{\frac{6(s-2)+8}{(s-2)^2+4^2}\right\}$$

$$= 6L^{-1}\left\{\frac{(s-2)}{\underbrace{(s-2)}_{s}{}^2+\underbrace{4}_{a}{}^2}\right\} + 2L^{-1}\left\{\frac{4}{(s-2)^2+4^2}\right\}$$

????????????$= 6e^{2t}\cos 4t + 2\cdot e^{2t}\cdot \sin 4t$

$$= 2e^{2t}(3\cos 4t + \sin 4t)$$

Theorem: $L^{-1}\left\{\frac{f(s)}{s}\right\} = \int_0^t F(u)\,du$

* Evaluate $L^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\}$

Soln:

Now,

$$L^{-1}\left\{\frac{1}{s^2+4}\right\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{\sin 2t}{2} = F(t)$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = L^{-1}\left\{\frac{\frac{1}{(s^2+4)}}{s}\right\} = \int_0^t \frac{\sin 2u}{2}\,du$$

$$= \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t = -\frac{1}{4}\cos 2t + \frac{1}{4} = 1/4($$

$$L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = L^{-1}\left\{\frac{\frac{1}{s(s^2+4)}}{s}\right\} = \int_0^t \frac{1}{4}(1-\cos 2u)\,du$$

$$= \frac{1}{4}\left[u - \frac{\sin 2u}{2}\right]_0^t = \frac{1}{4}\left(t - \frac{\sin 2t}{2} - 0 + 0\right]$$

$$= \frac{1}{4}t - \frac{\sin 2t}{8}$$

$$L^{-1}\left\{\frac{1}{s^3(s^2+4)}\right\} = L^{-1}\left\{\frac{\frac{1}{s^2(s^2+4)}}{s}\right\} = \int_0^t \left(\frac{1}{4}u - \frac{\sin 2u}{8}\right)du$$

$$= \left(\frac{1}{4}\cdot\frac{u^2}{2} + \frac{\cos 2u}{16}\right)_0^t = \frac{t^2}{8} + \frac{\cos 2t}{16} - 0 - \frac{1}{16}$$

$$= t^2/8 + \frac{\cos 2t}{16} - \frac{1}{16}$$

* Prove that, i) $L^{-1}\left\{\frac{1}{s^{3/2}}\right\} = 2\sqrt{\frac{t}{\pi}}$

ii) $L^{-1}\left\{\frac{1}{s^{7/2}}\right\} = \frac{8}{15\sqrt{\pi}}\,t^{5/2}$

Solⁿ: ii) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{n+1}}$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^{n+1}}\right\} = \frac{t^n}{\Gamma(n+1)}$$

$$\Rightarrow L^{-1}\left\{\frac{1}{s^{7/2}}\right\} = L^{-1}\left\{\frac{1}{s^{5/2+1}}\right\} = \frac{t^{5/2}}{\Gamma(7/2)} = \frac{t^{5/2}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{8}{15\sqrt{\pi}} t^{5/2}$$

* Prove $L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = \frac{2}{t}(1-\cos t)$

$\Rightarrow$ We know, $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\therefore \log\left(1+\frac{1}{s^2}\right) = \frac{1}{s^2} - \frac{1}{2s^4} + \frac{1}{3.s^6} - \frac{1}{4s^8} + \cdots$$

$$\therefore L^{-1}\left\{\log\left(1+\frac{1}{s^2}\right)\right\} = L^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{2}L^{-1}\left\{\frac{1}{s^4}\right\} + \frac{1}{3}L^{-1}\left\{\frac{1}{s^6}\right\} - \frac{1}{4}L^{-1}\left\{\frac{1}{s^8}\right\} + \cdots$$

$$= t - \frac{1}{2}\cdot\frac{t^3}{3!} + \frac{1}{3}\cdot\frac{t^5}{5!} - \frac{1}{4}\cdot\frac{t^7}{7!} + \cdots$$

$$= \frac{2t^2}{2t} - \frac{2}{t}\cdot\frac{t^4}{4.3!} + \frac{2}{t}\cdot\frac{t^6}{6.5!} - \frac{2}{t}\cdot\frac{t^8}{8.7!} + \cdots$$

$$= \frac{2}{t}\left[\frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + \cdots\right]$$

$$= \frac{2}{t}\left[1 - 1 + \frac{t^2}{2!} - \frac{t^4}{4!} + \frac{t^6}{6!} - \frac{t^8}{8!} + \cdots\right]$$

$$= \frac{2}{t}\left[1 - \cos t\right]$$

[Proved]

* Evaluate : $L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\}$

Soln : Let, $\dfrac{5s+3}{(s-1)(s^2+2s+5)} = \dfrac{A}{s-1} + \dfrac{Bs+C}{s^2+2s+5}$ - -

$\Rightarrow 5s+3 = A(s^2+2s+5) + (Bs+C)(s-1)$

$\Rightarrow 5s+3 = As^2 + 2As + 5A + Bs^2 - Bs + Cs - C$

Putting. $s=1$, $8A = 8 \Rightarrow A = 1$.

Equating the ~~cof~~ co-efficient of s^2 and constant term,

$A+B=0 \Rightarrow B = -A = -1$, $5A - C = 3 \Rightarrow 5.1 - 3 = C$
$\Rightarrow C = 2$

$\therefore L^{-1}\left\{\dfrac{5s+3}{(s-1)(s^2+2s+5)}\right\} = L^{-1}\left\{\dfrac{1}{s-1}\right\} + L^{-1}\left\{\dfrac{-1.s+2}{s^2+2s+5}\right\}$

$= e^t + L^{-1}\left\{\dfrac{-(s+1)}{(s+1)^2+2^2}\right\} + \dfrac{3}{2} L^{-1}\left\{\dfrac{2}{(s+1)^2+2^2}\right\}$

$= e^t - e^{-t}\cos 2t + \dfrac{3}{2} e^{-t}.\sin 2t$

03.08.16

Def^n of Convolution :– Let $F(t)$ and $G(t)$ be two functions of a class A. Than the convolution of the two functions $F(t)$ and $G(t)$ denoted by $F * G$ and ~~denoted~~ defined by the relation $F * G = \int_0^t F(u)\, G(t-u)\, du$

Statement :–

If $L^{-1}\{f(s)\} = F(t)$ and $L^{-1}\{g(s)\} = G(t)$ then

$$L^{-1}\{f(s), g(s)\} = \int_0^t F(u)\, G(t-u)\, du.$$

* Evaluate : $L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\}$ by use of the convolution theorem.

Sol^n : We can write $\frac{1}{s^2(s^2+4)} = \frac{1}{s^2+4} \cdot \frac{1}{s^2}$

Let $f(s) = \frac{1}{s^2+4}$ $\therefore F(t) = L^{-1}\{f(s)\} = L^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \frac{1}{2}\sin 2t$

$\Rightarrow F(u) = \frac{1}{2}\sin 2u$

?????? Let $g(s) = \frac{1}{s^2}$ $\therefore G(t) = L^{-1}\left\{\frac{1}{s^2}\right\} = t$ $\therefore G(t-u) = t-u$

By convolution theorem,

$$L^{-1}\left\{\frac{1}{s^2(s^2+4)}\right\} = \int_0^t \frac{1}{2}\sin 2u \cdot (t-u)\, du$$

$$= \frac{1}{2}t\int_0^t \sin 2u\, du - \frac{1}{2}\int_0^t u\sin 2u\, du$$

$$= \frac{1}{2}t\left[-\frac{\cos 2u}{2}\right]_0^t - \frac{1}{2}\left[u \cdot \frac{\cos 2u}{-2} + \frac{\sin 2u}{4}\right]_0^t$$

$$= \frac{1}{4}t\left[-\cos 2t + 1\right] - \left[-\tfrac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t\right]$$

$$= \frac{1}{4}t - \tfrac{1}{4}\cos 2t + \tfrac{1}{4}t\cos 2t - \frac{1}{8}\sin 2t = \frac{1}{4}t - \frac{1}{8}\sin 2t$$

* Prove that – i) $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$

ii) $\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$

iii) $\int_0^\infty \sin x^2\, dx = \frac{1}{2}\sqrt{\pi/2}$

using laplace transform,

i) proof

Let $G(t) = \int_0^\infty e^{-tx^2}\, dx$

By def^n of laplace transform,

$$L\{G(t)\} = \int_0^\infty e^{-st}\, G(t)\, dt$$

$$= \int_0^\infty e^{-st} \int_0^\infty e^{-tx^2}\, dx\, dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cdot e^{-tx^2}\, dt \right] dx$$

$$= \int_0^\infty L\{e^{-tx^2}\}\, dx$$

$$= \int_0^\infty \frac{1}{s+x^2}\, dx$$

Quiz - 3
17.08.16

$$= \int_0^{\infty} \frac{1}{x^2+(\sqrt{s})^2}\, dx$$

$$= \left[\frac{1}{\sqrt{s}} \tan^{-1} \frac{x}{\sqrt{s}}\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{s}} \tan^{-1}\infty - \frac{1}{\sqrt{s}} \tan^{-1} 0$$

$$= \frac{1}{\sqrt{s}} \tan^{-1} \tan \frac{\pi}{2} - 0$$

$$= \frac{\pi}{2\sqrt{s}}$$

$$\therefore L\{G(t)\} = \frac{\pi}{2\sqrt{s}}$$

$$\Rightarrow G(t) = \frac{\pi}{2} L^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \frac{\pi}{2} L^{-1}\left\{\frac{1}{s^{-1/2+1}}\right\}$$

$$= \frac{\pi}{2} \cdot \frac{t^{-1/2}}{\Gamma(-1/2+1)}$$

$$= \frac{\pi}{2} \cdot \frac{1}{\sqrt{\pi} \cdot t^{1/2}}$$

$$\Rightarrow G(t) = \frac{1}{2}\sqrt{\pi/t}$$

$$\therefore \int_0^{\infty} e^{-tx^2}\, dx = \frac{1}{2}\sqrt{\pi/t}$$

putting $t=1$, $\int_0^{\infty} e^{-x^2}\, dx = \frac{1}{2}\sqrt{\pi} = \frac{\sqrt{\pi}}{2}$ (P)

06.08.16

* Prove that $\int_0^\infty \cos x^2\, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$ by use of laplace transform.

Sol^n: Let $G(t) = \int_0^\infty \cos t x^2\, dx$

By the defn of laplace transform,

$$L\{G(t)\} = \int_0^\infty e^{-st} \cancel{\cos} G(t)\, dt$$

$$= \int_0^\infty e^{-st} \int_0^\infty \cos t x^2\, dx\, dt$$

?????$$= \int_0^\infty \left[\int_0^\infty e^{-st} \cos x^2\, dt\right] dx$$

$$= \int_0^\infty L\{\cos x^2\}\, dx$$

$$= \int_0^\infty \frac{s}{s^2 + x^4}\, dx$$

Let $x^2 = s\tan\theta$

$\Rightarrow x = \sqrt{s}\cdot\sqrt{\tan\theta}$

$dx = \sqrt{s}\cdot\frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\, d\theta$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{s\cdot\sqrt{s}\cdot\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}\,(s^2 + s^2\tan^2\theta)}$$

$$= \int_0^{\pi/2} \frac{s^{3/2}\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}\; s^2\sec^2\theta}$$

* $\frac{1}{2}\beta(m,n) = \int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma m \Gamma n}{2\Gamma m+n}$
* $\Gamma p \, \Gamma 1-p = \frac{\pi}{\sin p\pi}$

$$= \int_0^{\pi/2} \frac{1}{2\sqrt{s}} (\tan\theta)^{-1/2} \, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta \, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{1/2-1}\theta \cos^{3/2-1}\theta \, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{2\cdot\frac{1}{4}-1}\theta \cos^{2\cdot\frac{3}{4}-1}\theta \, d\theta$$

$$= \frac{1}{2\sqrt{s}} \cdot \frac{1}{2}\beta\left(\frac{1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\Gamma 1/4 \, \Gamma 3/4}{\Gamma 1/4 + 3/4}$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\Gamma 1/4 \cdot \Gamma 1-1/4}{\Gamma 1}$$

??????? $\Rightarrow L\{G(t)\} = \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{\sin \pi/4}$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{1/\sqrt{2}} = \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

$$\therefore G(t) = L^{-1}\left\{\frac{\pi\sqrt{2}}{4\sqrt{s}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} L^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} L^{-1}\left\{\frac{1}{s^{-1/2+1}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} \cdot \frac{t^{-1/2}}{\Gamma(-1/2+1)}$$

$$= \frac{\pi\sqrt{2}}{4} \cdot \frac{1}{\sqrt{\pi} \cdot \sqrt{t}}$$

$$= \int_0^\infty \cos tx^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2} \cdot \sqrt{t}}$$

putting $t = 1$,

$$\int_0^\infty \cos tx^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2} \cdot 1}$$

$$= \frac{1}{2}\sqrt{\pi/2}$$

(proved

* prove that $\int_0^\infty \sin x^2 \, dx = \frac{1}{2}\sqrt{\pi/2}$ by use of laplace transform.

Sol^n: Let $G(t) = \int_0^\infty \sin t x^2 \, dx$

By the def^n of laplace transform.

$$L\{G(t)\} = \int_0^\infty e^{-st} \, G(t) \, dt$$

$$= \int_0^\infty e^{-st} \cdot \int_0^\infty \sin tx^2 \, dx \, dt$$

$$= \int_0^\infty \left[\int_0^\infty e^{-st} \sin tx^2 \, dt \right] dx$$

$$= \int_0^\infty L\{\sin t\, x^2\}\, dx$$

$$= \int_0^\infty \frac{x^2}{s^2 + x^4}\, dx$$

Let . $x^2 = s\tan\theta$

$\Rightarrow x = \sqrt{s}\sqrt{\tan\theta}$

$dx = \sqrt{s} \cdot \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta\, d\theta$

x	0	∞
θ	0	$\pi/2$

$$= \int_0^{\pi/2} \frac{s\cdot\tan\theta\cdot\sqrt{s}\cdot\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}\,(s^2 + s^2\tan^2\theta)}$$

$$= \int_0^{\pi/2} \frac{s\cdot\tan\theta\cdot\sqrt{s}\cdot\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}\cdot s^2\cdot\sec^2\theta}$$

$$= \int_0^{\pi/2} \frac{1}{2\sqrt{s}} (\tan\theta)^{1/2}\, \cancel{d\theta}\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2\sqrt{s}} \sin^{1/2}\theta \cos^{-1/2}\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2\sqrt{s}} \sin^{3/2-1}\theta\ \cancel{\cos^{1/2}}\ \cos^{1/2-1}\theta\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \int_0^{\pi/2} \sin^{2\cdot\frac{3}{4}-1}\theta \cdot \cos^{2\cdot\frac{1}{4}-1}\theta\, d\theta$$

$$= \frac{1}{2\sqrt{s}} \frac{1}{2}\beta\left(3/4, 1/4\right)$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\Gamma 3/4 \cdot \Gamma 1/4}{\Gamma 3/4 + 1/4}$$

$$= \frac{1}{4\sqrt{3}} \frac{\Gamma 3/4 \cdot \Gamma\left(1-\frac{3}{4}\right)}{\Gamma 1}$$

$$\Rightarrow L\{G(t)\} = \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{\sin\frac{3\pi}{4}}$$

$$= \frac{1}{4\sqrt{s}} \cdot \frac{\pi}{1/\sqrt{2}}$$

$$= \frac{\pi\sqrt{2}}{4\sqrt{s}}$$

$$\therefore G(t) = L^{-1}\left\{\frac{\pi\sqrt{2}}{4\sqrt{s}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} L^{-1}\left\{\frac{1}{s^{1/2}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} L^{-1}\left\{\frac{1}{s^{-1/2+1}}\right\}$$

$$= \frac{\pi\sqrt{2}}{4} \cdot \frac{\cancel{t^{-1/2}}\ t^{-1/2}}{\Gamma\left(-1/2+1\right)}$$

$$= \frac{\pi\sqrt{2}}{4} \cdot \frac{1}{\sqrt{\pi} \cdot \sqrt{t}}$$

$$\Rightarrow \int_0^\infty \sin t x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2} \cdot \sqrt{t}}$$

putting $t = 1$

$$\int_0^\infty \sin t x^2 \, dx = \frac{\sqrt{\pi}}{2\sqrt{2} \cdot 1}$$

$$= \frac{1}{2}\sqrt{\frac{\pi}{2}} \quad \text{(Proved)}$$

13.08.16

* Solve the following simultaneous differential equation by the ~~do~~ laplace transform.

$$\frac{dx}{dt} + y = \sin t \quad \text{--- (1)}, \qquad \frac{dy}{dt} + x = \cos t, \quad \text{--- (2)} \qquad x(0) = 2,\ y(0) = 0$$

Soln: Let $L\{x\} = X$ and $L\{y\} = Y$

Taking the laplace transform on both side of (1) & (2)

$$L\left\{\frac{dx}{dt}\right\} + L\{y\} = L\{\sin t\}$$

$$\Rightarrow L\{x'(t)\} + L\{y\} = \frac{1}{1+s^2}$$

$$\Rightarrow sX - x(0) + Y = \frac{1}{1+s^2}$$

$$\Rightarrow sX + Y = 2 + \frac{1}{1+s^2} = \frac{3+2s^2}{1+s^2}$$

$$\Rightarrow sX + Y - \frac{2s^2+3}{1+s^2} = 0 \quad \text{--- (3)}$$

$$L\left\{\frac{dy}{dt}\right\} + L\{x\} = L\{\cos(t)\}$$

$$\Rightarrow L\{y'(t)\} + X = \frac{s}{1+s^2}$$

$$\Rightarrow sY - y(0) + X = \frac{s}{1+s^2}$$

$$\Rightarrow X + sY - \frac{s}{1+s^2} = 0 \quad \text{--- (4)}$$

Solving (3) and (4) for X and Y

$$sX + Y - \frac{2s^2+3}{1+s^2} = 0$$

$$X + sY - \frac{s}{1+s^2} = 0$$

$$\frac{X}{-\frac{s}{1+s^2} + \frac{2s^3+3s}{1+s^2}} = \frac{Y}{-\frac{2s^2+3}{1+s^2} + \frac{s^2}{1+s^2}} = \frac{1}{s^2-1}$$

$$\Rightarrow \quad \frac{X}{\frac{2s^3+2s}{1+s^2}} = \frac{Y}{\frac{-s^2-3}{1+s^2}} = \frac{1}{s^2-1}$$

Now,

$$\frac{X}{\frac{2s(1+s^2)}{(1+s^2)}} = \frac{1}{s^2-1}$$

$$\Rightarrow \quad X = \frac{2s}{s^2-1} = \frac{1}{s+1} + \frac{1}{s-1} \quad \text{——} (5)$$

$$\frac{Y}{\frac{-(s^2+3)}{1+s^2}} = \frac{1}{s^2-1}$$

$$\Rightarrow \quad Y = \frac{-(s^2+3)}{(s^2-1)(1+s^2)} = \frac{-(s^2+1)-2}{(1+s^2)(s^2-1)}$$

$$= -\left[\frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\cdot\frac{1}{s+1}\right] + \frac{1}{s^2+1} - \frac{1}{s^2-1}$$

$$= -\frac{1}{2}\cdot\frac{1}{s-1} + \frac{1}{2}\frac{1}{s+1} + \frac{1}{s^2+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\cdot\frac{1}{s+1}$$

$$= \frac{1}{s+1} - \frac{1}{s-1} + \frac{1}{s^2+1} \quad \text{——— (6)}$$

Taking inverse laplace on both sides of (5) & (6)

$$L^{-1}\{x\} = L^{-1}\left\{\frac{1}{s+1}\right\} + L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$\Rightarrow x = e^{-t} + e^{t} \quad \text{——— (7)}$$

$$L^{-1}\{Y\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\} + L^{-1}\left\{\frac{1}{1+s^2}\right\}$$

$$\Rightarrow y = e^{-t} - e^{t} + \sin t \quad \text{——— (8)}$$

* laplace + inverse → 1 set
* laplace inverse + application — 1 set

* 1st – cauchy এর আগে পর্যন্ত — 1/1.5 set
* ~~contour~~ mapping — .5 set
* cauchy → 1/.5 set
* contour + series → .5/1 set

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579 (1) — (3)

588 — (1) — (5)

606 — (1) — (14)

615 — (2) — (14)