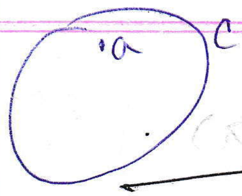


Chauchy Integral Formula

If the function $f(z)$ is analytic within and on a closed curve C and if a is any point withic C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\, dz$$

or, $$\int_C \frac{f(z)}{z-a}\, dz = 2\pi i\, f(a)$$

$$\int_C \frac{f(z)}{(z-a)^2}\, dz = 2\pi i\, f'(a)$$

$$\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{\lfloor 2}\, f''(a)$$

$$\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{\lfloor n-1}\, f^{(n-1)}(a)$$

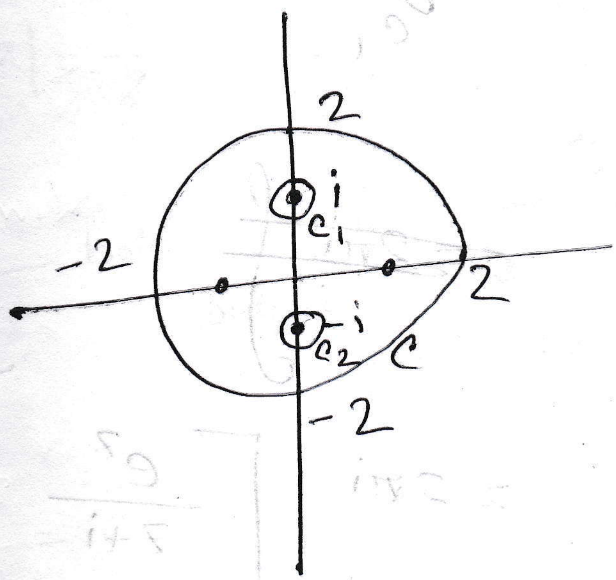
45 $\displaystyle\int \frac{e^z}{z^2+1}\,dz \qquad c: |z| = 2$

$z^2+1=0$

$z^2=-1$

$z=\pm\sqrt{-1}$

$z=\pm i$



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

on, $$\int_C \frac{e^z}{z^2+1}\,dz = \int_{C_1} \frac{e^z}{(z+i)(z-i)}\,dz + \int_{C_2} \frac{e^z}{(z+i)(z-i)}\,dz$$

C_1 এর ভিতর analytic

$$= \int_{C_1} \frac{\frac{e^z}{z+i}}{z-i}\,dz + \int_{C_2} \frac{\frac{e^z}{z-i}}{z+i}\,dz$$

$a=i$

$$\cancel{= 2\pi i \int}$$

$$= 2\pi i \left[\frac{e^z}{z+i}\right]_{z=i} +$$

$$2\pi i \left[\frac{e^z}{z-i}\right]_{z=-i}$$

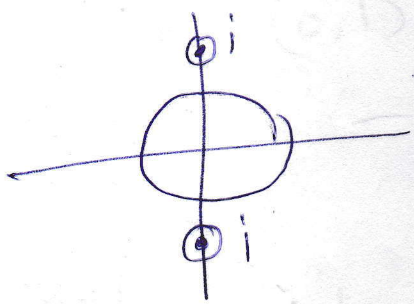
$$= 2\pi i \frac{e^i}{i+i} + 2\pi i \frac{e^{-i}}{-i-i}$$

$$= \pi e^i - \pi e^{-i}$$

$$= 2\pi i \, \frac{e^{i}-e^{-i}}{2i}$$

$$= 2\pi i \sin 1$$

$C : |z| = \frac{1}{2}$



both poles are outside circle

$$\therefore \oint_C f(z)\, dz = 0$$

46

47

$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz \qquad c: |z-i| = 2$$

Soln:

$\therefore$ Poles are,

$(z+1)^2 (z-2) = 0$

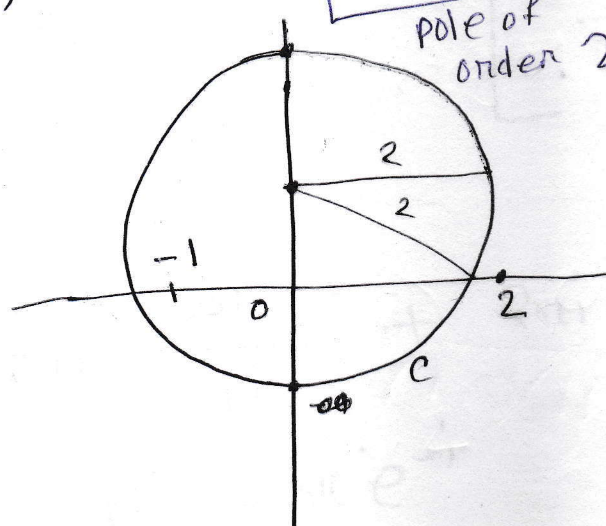
or, $z = \underbrace{-1, -1}_{\text{pole of order 2}}, 2$

ভিতরে না বাইরে বুঝতে না পারলে circle এ z এর মান বসাব

< radius ভিতরে

= " উপরে

> " বাইরে



The ~~integral has~~ integrand has two poles $z=-1$ (2nd order) and $z=2$ of which $z=-1$ is inside the given circle with centre at $(0,1)$, radias 2

$$\int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = \int_C \frac{\frac{z-1}{z-2}}{(z+1)^2}\,dz$$

$$= 2\pi i \left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]_{z=-1}$$

$$= 2\pi i \left[\frac{(z-2).1-(z-1).1}{(z-2)^2}\right]_{z=-1}$$

$$= 2\pi i \frac{-1-2-(-1-1)}{(-1-2)^2}$$

$$= 2\pi i \frac{-1}{9}$$

$$= -\frac{2\pi i}{9}$$

$\boxed{04/06/2017}$

49 $\int_c \frac{4-3z}{z(z-1)^2(z-2)} dz$

$$\int_c \frac{4-3z}{z(z-1)(z-2)} dz \qquad c: |z| = \frac{3}{2}$$

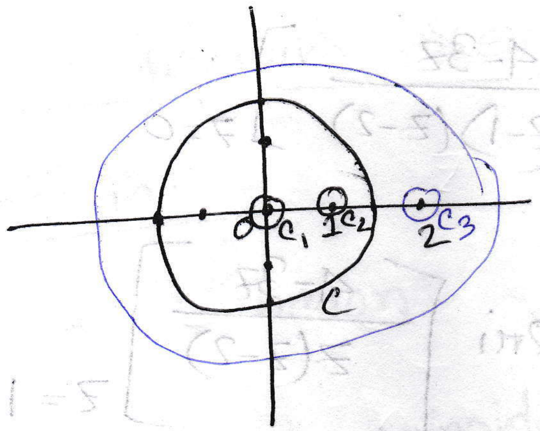
$c: |z| = 3$

Soln:

Poles of integral are given by putting,

$$z(z-1)(z-2) = 0$$

$$\therefore z = 0, 1, 2$$



The integrand has three poles at $z=0$, $z=1$ and $z=2$ of which $z=0$ and $z=1$ are inside the given circle with center at $(0,0)$, radias $\frac{3}{2}$

$$\int_{c} \frac{4-3z}{z(z-1)(z-2)}\,dz = \int_{c_1} \frac{4-3z}{z(z-1)(z-2)}\,dz + \int_{c_2} \frac{4-3z}{z(z-1)(z-2)}\,dz + \int_{c_3}$$

$$= \int_{c_1} \frac{\frac{4-3z}{(z-1)(z-2)}}{z}\,dz + \int_{c_2} \frac{\frac{4-3z}{z(z-2)}}{z-1}\,dz$$

$$= 2\pi i\left[\frac{4-3z}{(z-1)(z-2)}\right]_{z=0}$$

$$+ 2\pi i\left[\frac{4-3z}{z(z-2)}\right]_{z=1}$$

$$= 2\pi i \frac{4}{(-1)(-2)} + 2\pi i \frac{4-3}{1(1-2)}$$

$$= 4\pi i - 2\pi i$$

$$= 2\pi i$$

$z(2z-1)(z-2) = 0$

$z = \frac{1}{2}$

$$\oint \frac{\frac{1}{2}}{z-\frac{1}{2}}$$

Exercise: 7.6

Taylor's Series

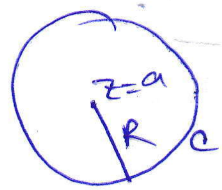
Taylor's Thm:

If a function $f(z)$ is analytic at all points inside a circle c with its center at the point a and radias R then each point z inside

$$f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a)$$

$$+ \frac{(z-a)^3}{\lfloor 2} f'''(a) + \cdots$$

$$\cdots + \frac{(z-a)^n}{\lfloor n} f^n(a) + \cdots$$



radias চাইতে পারে — convergance

52

Find the first four terms,

$$f(z)=\frac{z+1}{(z-3)(z-4)}$$ about $z=2$

find the region of convergance.

not analytic in $z=3,4$ so
circle এর ভিতরে 3, 4 আসা যাবে না।

Soln: If centre of a circle is at $z=2$ then the distance of the singularities $z=3$ and

$z = 4$ from the centre are 1 and 2.

Hence, if a circle is drawn with centre at $z = 2$ and radias 1 then within the circle $|z-2| = 1$ the given function $f(z)$ is analytic.

Hence the region of convergance is $|z-2| = 1$

$f(z)$ কে আংশিক করে differentiate করা যায়।

$$f(z) = \frac{z+1}{(z-3)(z-4)}$$

$$= \frac{3+1}{(z-3)(3-4)} + \frac{4+1}{(4-3)(z-4)}$$

$$= -\frac{4}{z-3} + \frac{5}{z-4}$$

$(1-x)^{-1} = 1+x+x^2+x^3+ \cdots\cdots$

$(1+x)^{-1} = 1-x+x^2-x^3+ \cdots\cdots$

x has a limitation $\Rightarrow |x|<1$

(ii)

$$f(z) = -\frac{4}{(z-2)-1} + \frac{5}{(z-2)-2}$$

$$= \frac{4}{\{1-(z-2)\}} - \frac{5}{2\left(1-\frac{z-2}{2}\right)}$$

$$= 4\{1-(z-2)\}^{-1} - \frac{5}{2}\left(1-\frac{z-2}{2}\right)^{-1}$$

$$= 4\{1+(z-2)+(z-2)^2+(z-2)^3+(z-2)^4+\cdots\cdots\}$$

$$-\frac{5}{2}\left\{1+\frac{z-2}{2}+\left(\frac{z-2}{2}\right)^2+\left(\frac{z-2}{2}\right)^3+\left(\frac{z-2}{2}\right)^4+\cdots\cdots\right\}$$

$$= 4-\frac{5}{2} + (z-2)\left(4-\frac{5}{4}\right) + (z-2)^2\left(4-\frac{5}{8}\right)$$

$$+ (z-2)^3\left(4-\frac{5}{16}\right) + (z-2)^4\left(4-\frac{5}{32}\right)$$

$$+ \cdots$$

$$= \frac{3}{2} + \frac{11}{4}(z-2) + \frac{27}{8}(z-2)^2 + \frac{59}{16}(z-2)^3 + \cdots$$

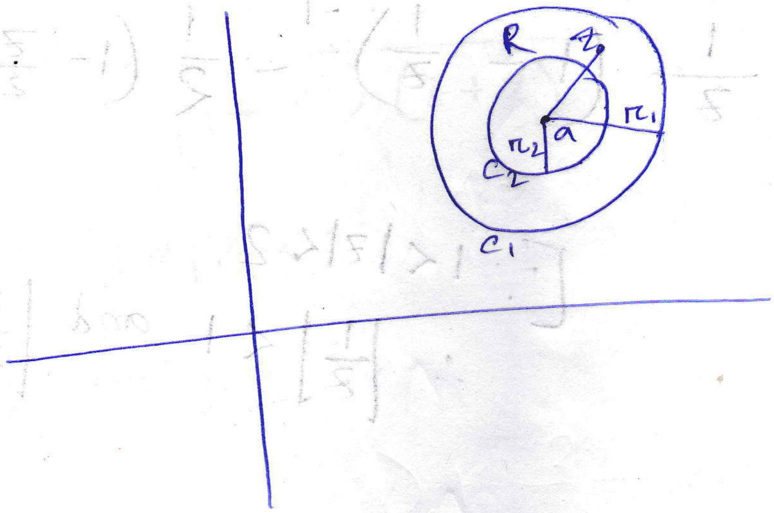
Laurant's Theorem

If $f(z)$ is analytic in the region R bounded by two concentric circle C_1 and C_2 and radii r_1 and r_2 $(r_2 < r_1)$ with center at a then for all z in R,

$$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + a_3(z-a)^3 + \cdots + \frac{b_1}{z-a} + \frac{b_2}{(z-a)^2} + \frac{b_3}{(z-a)^3} + \cdots$$

where, $a_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \frac{f(w)}{(w-a)^{n+1}} \, dw$

and $b_n = \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \frac{f(w)}{(w-a)^{-n+1}} \, dw$.



56 Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ for

$1 < |z| < 2$

Sol^n:

$$f(z) = \frac{1}{(z-1)(z-2)}$$

$$= \frac{1}{(z-1)(1-2)} + \frac{1}{(2-1)(z-2)}$$

$$= -\frac{1}{z-1} + \frac{1}{z-2}$$

$$= -\frac{1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$\left[\begin{array}{l} \because 1<|z|<2 \\ \therefore \left|\frac{1}{z}\right|<1 \text{ and } \left|\frac{z}{2}\right|<1 \end{array}\right]$$

$$= -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right)$$

$$-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)$$

$$= -\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \cdots$$

$$\cdots - \frac{1}{z} - \frac{1}{z^2} - \frac{1}{z^3} - \frac{1}{z^4}$$

57

$$f(z) = \frac{1}{(1+z^2)(z+2)}$$

(i) $1 < |z| < 2$

(ii) $|z| > 2$

Soln:

$$f(z) = \frac{1}{(1+z^2)(z+2)}$$

Let,

$$\frac{1}{(1+z^2)(z+2)} = \frac{Az+B}{1+z^2} + \frac{C}{z+2}$$

or, $1 = (Az+B)(z+2) + C(1+z^2)$ --- (i)

Putting $z = -2$

$\therefore 1 = C(1+4)$

$\Rightarrow C = \frac{1}{5}$

Eq

Equating the coefficient of z^2 and z, in eq^n (i)

$\therefore z^2 \rightarrow 0 = A+C$

$z \longrightarrow 0 = 2A+B$

$\therefore A = -C = -\frac{1}{5}$

$\therefore B = -2A = \frac{2}{5}$

$$f(z) = \frac{-\frac{1}{5}z + \frac{2}{5}}{1+z^2} + \frac{\frac{1}{5}}{z+2}$$

(i) $f(z) = \frac{-\frac{z}{5} + \frac{2}{5}}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{1}{5}\,\frac{1}{2\left(1+\frac{z}{2}\right)}$

$$\therefore = \left(-\frac{1}{5z} + \frac{2}{5z^2}\right)\left(1 + \frac{1}{z^2}\right)^{-1} + \frac{1}{10}\left(1 + \frac{z}{2}\right)^{-1}$$

$$\left[\because 1 < |z| < 2 \quad \therefore \left|\frac{1}{z^2}\right| < 1, \left|\frac{z}{2}\right| < 1\right]$$

$$= \left(-\frac{1}{5z} + \frac{2}{5z^2}\right)\left(1 - \frac{1}{z^2} + \frac{1}{z^4} - \frac{1}{z^6} + \frac{1}{z^8} - \cdots\right) + \frac{1}{10}\left(1 - \frac{z}{2} + \frac{z^2}{4} - \frac{z^3}{8} + \frac{z^4}{16} - \cdots\right)$$

$$= -\frac{1}{5z} + \frac{2}{5z^2} + \frac{1}{5z^3} - \frac{2}{5z^4} - \frac{1}{5z^5} + \frac{2}{5z^6} + \cdots + \frac{1}{10} - \frac{z}{20} + \frac{z^2}{40} - \frac{z^3}{80} + \cdots$$

$=$

(ii)

$$f(z) = \frac{-\frac{1}{5}z + \frac{2}{5}}{z^2\left(1+\frac{1}{z^2}\right)} + \frac{1}{5}\,\frac{1}{z\left(1+\frac{2}{z}\right)}$$

$$\cancel{= -\frac{z}{5} + \frac{2}{5})}$$

$$= \left(-\frac{1}{5z} + \frac{2}{5z^2}\right)\left(1+\frac{1}{z^2}\right)^{-1}$$

$$+ \frac{1}{5z}\left(1+\frac{2}{z}\right)^{-1}$$

$=$

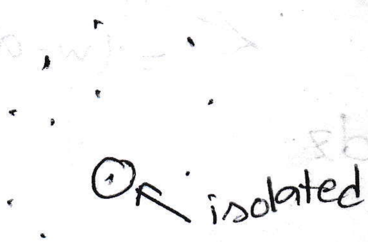
$=$

$$= \frac{1}{z^3} - \frac{2}{z^4} + \frac{3}{z^5} - \frac{6}{z^6} + \frac{13}{z^7} - \cdots\cdots$$

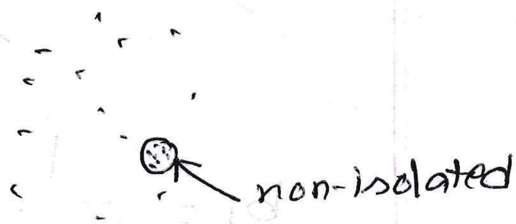
11/06/2017

Singular point: 2 types.

Isolated and Non-Isolated singular point.



$$f(z) = \frac{1}{(z-1)(z-\frac{1}{2})}$$



$$f(z) = \frac{1}{\sin\frac{\pi}{z}}$$

$$z = \underbrace{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots}_{\text{Isolated}} \;\; \dots \;\; 0$$

Isolated

non isolated

Residue:

The co-efficient of $\frac{1}{z-a}$ ~~in~~ that is b_1 in ~~Laurant's expansion~~ Laurant's expansion about an point $z=a$ is isolated singular called the residue of $f(z)$

at $z = a$.

$$\cancel{b_1 = \frac{1}{2\pi i} \int_C \frac{p}{\quad}}$$

$(w-a)^{-1+1}$
$= (w-a)^0 = 1$

$$b_1 = \frac{1}{2\pi i} \int_{C_1} f(z)\, dz$$

$\text{Res} f(a) =$ Residue of $f(z)$ at $(z = a)$

$$= \frac{1}{2\pi i} \int_{C_1} f(z)\, dz$$

Method of finding residue:

① If $f(z)$ has a simple pole at $z=a$ then,

$$\text{Res } f(a) = \lim_{z \to a} (z-a) f(z) \qquad \frac{\cdots}{(z-a)\cdots}$$

② If $f(z)$ ~~k~~ is of the form ~~f(z)~~

$$f(z) = \frac{\phi(z)}{\psi(z)}$$ where $\psi(a) = 0$ but $\phi(a) \neq 0$ then Res of $f(z)$

at $(z=a)$ $= \dfrac{\phi(a)}{\psi'(a)}$

③ If $f(z)$ has a pole of order n at $z=a$ then

$$\cancel{\text{Res}(\text{at } z=a) = \frac{1}{\underline{|n-1}} \frac{d^{n-1}}{dz^{n-1}} f_z}$$

$$\text{Res}(\text{at } z=a) = \frac{1}{\underline{|n-1}} \left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \right]_{z=a}$$

④ Laurant's expansion করে সেখান থেকে b_1 বের করব। $\frac{1}{z-a}$ এর সহগ।

59

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$ determine poles and res at each pole.

Soln:

putting, denominator = 0

$$(z-1)^2(z+2) = 0$$

$$\therefore z = 1, 1, -2$$

The function ~~of~~ has two poles at $z = 1$ of 2nd order and $z = -2$

Res (of fz at $z=-2$)

$$= \lim_{z \to -2} (z+2) \frac{z^2}{(z-1)^2 (z+2)}$$

$$= \lim_{z \to -2} \frac{z^2}{(z-1)^2}$$

$$= \frac{(-2)^2}{(-2-1)^2}$$

$$= \frac{4}{9}$$

Res ~~of~~ (at $z=1$) $= \frac{1}{\lfloor 2-1} \left[\frac{d}{dz} (z-1)^2 \frac{z^2}{(z-1)^2 (z+2)} \right]_{z=1}$

$$= \left[\frac{d}{dz} \frac{z^2}{(z+2)} \right]_{z=1}$$

$$= \left[\frac{(z+2)\,2z - z^2 \cdot 1}{(z+2)^2} \right]_{z=1}$$

$$= \frac{(1+2)2-1}{(1+2)^2} = \frac{5}{9}$$

60

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

Putting denominator $= 0$

$$\sin z = 0$$

$$\therefore z = n\pi, \quad n = 0, \pm 1, \pm 2, \ldots$$

Residue (of $f(z)$ at $z = n\pi$)

$$= \left[\frac{\cos z}{\frac{d}{dz}\sin z}\right]_{z=n\pi}$$

$$= \left[\frac{\cos z}{\cos z}\right]_{z=n\pi} = \left[1\right]_{z=n\pi}$$

$$= 1$$

Exercise 7.8

Residue Theorem:

If $f(z)$ is analytic in a closed curve c except at a finite number of poles within c then

$$\int_c f(z)\,dz = 2\pi i\left(\text{sum of residue at the poles within } c\right)$$

66

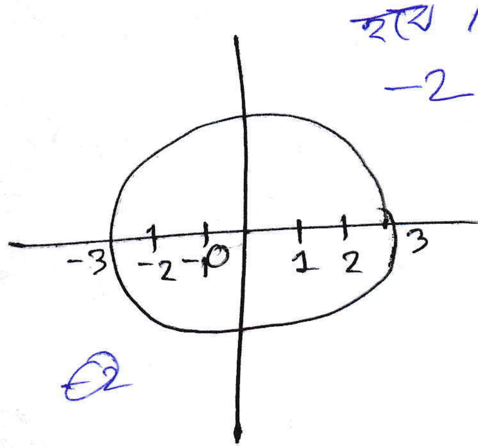
$$\int_c \frac{z^2}{(z-1)^2(z+2)}\,dz \qquad |z| = 3$$

$\frac{3}{2}$

-2 বাইরে হবে [illegible]

-2 আসবে না

poles $z = 1, 1, -2$



[illegible]

Residue [illegible]

By Residue thm,

$$\int_C f(z)\,dz = 2\pi i\,(\text{Sum of residue of poles within } C)$$

$$= 2\pi i\,\left(\underbrace{\text{Res}\,(\text{of } z=1)}_{3/2} + \text{Res}\,(\text{at } z=-2)\right)$$

$$= 2\pi i\left(\frac{4}{9} + \frac{5}{9}\right)$$

$$= 2\pi i$$

08-07-17

65

$$\int_C \frac{\coth z}{z-i}\,dz \qquad c: |z| = 2$$

Soln:

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\therefore \coth z = \frac{\cosh z}{\sinh z}$$

$$= \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$\therefore \int_C \frac{\coth z}{z-i}\,dz = \int_C \frac{e^z + e^{-z}}{(e^z - e^{-z})(z-i)}\,dz$$

$\therefore$ poles are, $(e^z - e^z)(z-i) = 0$

So, $e^{z} - e^{-z} = 0$ or $z - i = 0$

$\Rightarrow z = i$

$\Rightarrow e^{z} = e^{-z}$

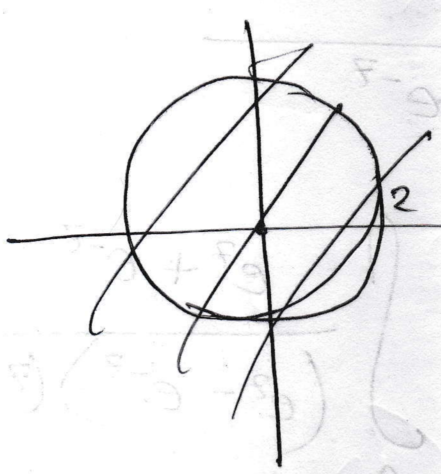
$\Rightarrow e^{z} = \frac{1}{e^{z}}$

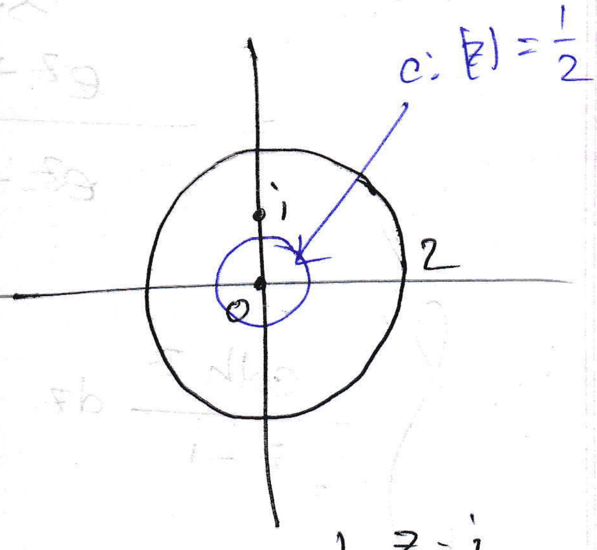
$\Rightarrow e^{z} \cdot e^{z} = 1$

$\Rightarrow e^{2z} = e^{0}$

$\Rightarrow 2z = 0$

$\Rightarrow z = 0$





$\therefore$ ~~both~~ the poles $z = 0$ and $z = i$

are both inside the given

circle, with centre at origin and radius 2.

Residue (at $z=0$) $= \left[\dfrac{(e^z + e^{-z})/(z-i)}{\frac{d}{dz}(e^z - e^{-z})} \right]_{z=0}$

$= \left[\dfrac{\dfrac{e^z + e^{-z}}{z-i}}{e^z + e^{-z}} \right]_{z=0}$

$= \dfrac{\dfrac{1+1}{0-i}}{1+1}$

$= \dfrac{2}{-i} \cdot \dfrac{1}{2} = \dfrac{1 \times i}{-i \times i} = \dfrac{i}{-i^2} = \dfrac{i}{-(-1)}$

$= i$

~~Residue (at $z=i$) $= \lim_{z \to i}$~~

$C: |z| = \frac{1}{2}$ হলে $\downarrow \times$

$$\text{Res}(\text{at } z=i) = \lim_{z \to i} (z-i) \frac{e^z + e^{-z}}{(e^z - e^{-z})(z-i)}$$

$$= \lim_{z \to i} \frac{e^z + e^{-z}}{e^z - e^{-z}}$$

$$= \frac{e^i + e^{-i}}{e^i - e^{-i}}$$

$$= \coth i$$

By Residue th^m,

$$\int_C f(z)\,dz = 2\pi i \left(\begin{array}{c}\text{Sum of residue at} \\ \text{the poles within } C\end{array}\right)$$

$$= 2\pi i \left\{ \text{Res}(\text{at } z=0) + \text{Res}(\text{at } z=i) \right\}$$

$$= 2\pi i \,(i + \coth i)$$

Contour Integration

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

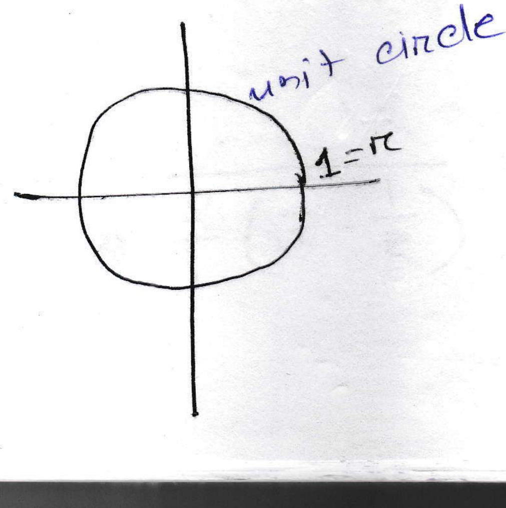
$z = x + iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$

if $r = 1$, $z = e^{i\theta}$

$e^{-i\theta} = \frac{1}{z}$

$$= \frac{z - \frac{1}{z}}{2i} \quad \text{[when } r = 1)$$



real integration

এর complex এ
নিয়ে কাজ করব।

$$\therefore \int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta$$

$$= \int_C f(z)\, dz$$

unit ~~circl~~ circle

$C: |z| = 1$

real integration

69

$$\int_0^{2\pi} \frac{1}{5-4\sin\theta}\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{5-4\,\frac{e^{i\theta}-e^{-i\theta}}{2i}}\, d\theta$$

putting, $z = re^{i\theta}$

$\Rightarrow z = e^{i\theta}$ $[r=1]$

and $e^{-i\theta} = \frac{1}{z}$

Again,

$e^{i\theta} = z$

$\therefore e^{i\theta}\, i\, d\theta = dz$

$\Rightarrow d\theta = \frac{dz}{ie^{i\theta}}$

$= \frac{dz}{iz}$

$$\therefore \int_C \frac{1}{5 - \frac{2}{i}\left(z - \frac{1}{z}\right)} \frac{dz}{iz}$$

[where C is the circle $|z| = 1$]

$$= \int_C \frac{1}{5iz - 2z^2 + 2} \, dz$$

$$= \int_C \frac{1}{-2z^2 + 5iz + 2} \, dz$$

the poles are, $-2z^2 + 5iz + 2 = 0$

$\Rightarrow -2z^2 + 4iz + iz + 2 = 0$

$\cancel{= 2z(2i-1)}$

$\Rightarrow 2z^2 i^2 + 4iz + iz + 2 = 0$

$\Rightarrow 2iz(iz+2) + 1\,(iz+2) = 0$

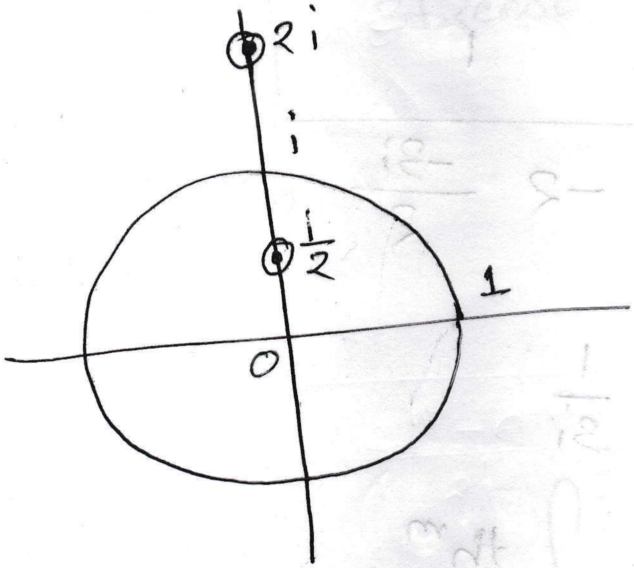
$\Rightarrow (2iz + 1)(iz+2) = 0$

$\therefore 2iz + 1 = 0$ or $iz + 2 = 0$

$\Rightarrow z = -\frac{1}{2i} = \frac{i}{2}$ $\quad \Rightarrow z = -\frac{2}{i} = 2i$

Sir direct use করতে বলেছে

$\therefore\ z = 2i, \quad z = \frac{i}{2}$



the integrand has two poles at $z = \frac{i}{2}$ and $z = 2i$ of ~~between~~ which $z = \frac{i}{2}$ is inside the unit circle.

$$\text{Res}\left(\text{of } f(z) \text{ at } z = \frac{i}{2}\right) = \lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) \frac{1}{-2z^2 + 5iz + 2}$$

$$= \lim_{z \to \frac{i}{2}} \left(z - \frac{i}{2}\right) \frac{1}{-2\left(z - \frac{i}{2}\right)(z - 2i)}$$

$$= \lim_{z \to \frac{i}{2}} \frac{1}{-2(z - 2i)}$$

$$= \frac{1}{-2\left(\frac{i}{2} - 2i\right)}$$

$$= \frac{1}{-2 \, \frac{-3i}{2}}$$

$$= \frac{1}{3i}$$

By Residue th^m,

$$\int_c f(z)\, dz = 2\pi i \left(\text{sum of residue at the poles within } C\right)$$

$$= 2\pi i \ \mathrm{Res}\left(\text{at } z = \frac{i}{2}\right)$$

$$= 2\pi i \, \frac{1}{3i}$$

$$= \frac{2\pi}{3}$$

09/07/2017

70

Evaluate $\int_0^{\pi} \frac{d\theta}{3+2\cos\theta}$ by using conture integration.

Soln:

$$\int_0^{\pi} \frac{d\theta}{3+2\cos\theta} = \int_0^{\frac{1}{2}\cdot 2\pi} \frac{d\theta}{3+2\cos\theta}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\cos\theta}$$

$\int_0^{an} f(x)dx = n\int_0^{a} f(x)dx$

if $f(x) = f(a+x)$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{3+2\,\frac{e^{i\theta}+e^{-i\theta}}{2}}$$

Here, $re^{i\theta} = z$

$\Rightarrow e^{i\theta} = z$ when $r = 1$

$\therefore e^{i\theta}\, i\, d\theta = dz$

$$d\theta = \frac{dz}{iz}$$

$$\therefore \frac{1}{2} \int_{c} \frac{1}{3 + \left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

[where $c : |z| = 1$]

$$= \frac{1}{2i} \int_{c} \frac{1}{z^2 + 3z + 1} dz$$

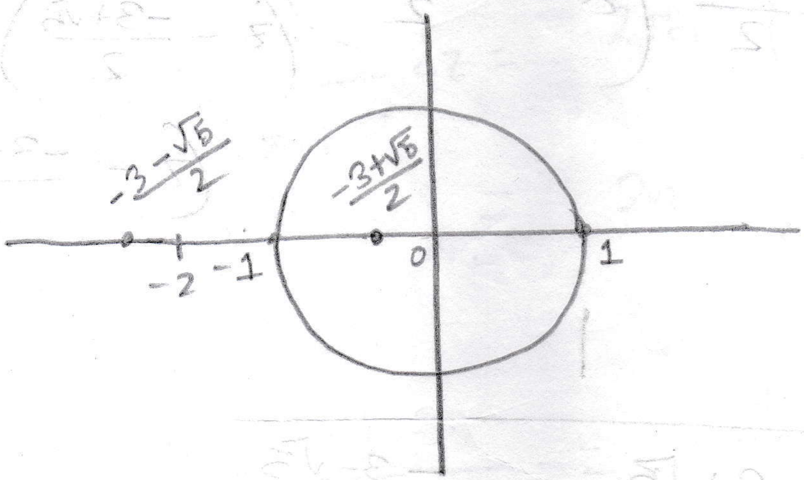
$\therefore$ poles, $z^2 + 3z + 1 = 0$

$$\therefore z = \frac{-3 \pm \sqrt{9-4}}{2}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

$$= \frac{-3+\sqrt{5}}{2}, \quad \frac{-3-\sqrt{5}}{2}$$

$$= \cancel{-0}3.8 \quad , \quad -2.6$$



the integrand has two poles ~~a~~ of which one is inside the circle at

$$z = \cancel{-3} \quad \frac{-3+\sqrt{5}}{2}$$

$$\text{Res}\left(\text{at } z = \frac{-3+\sqrt{5}}{2}\right)$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3+\sqrt{5}}{2}\right) \frac{1}{z^2+3z+1}$$

$$= \lim_{z \to \frac{-3+\sqrt{5}}{2}} \left(z - \frac{-3+\sqrt{5}}{2}\right) \frac{1}{\left(z - \frac{-3+\sqrt{5}}{2}\right)\left(z - \frac{-3-\sqrt{5}}{2}\right)}$$

$$= \frac{1}{\frac{-3+\sqrt{5}}{2} - \frac{-3-\sqrt{5}}{2}}$$

$$= \frac{1}{\sqrt{5}}$$

By Residue th^m,

$$\int_C f(z)\, dz = 2\pi i \text{ (sum of residue of the poles within } C)$$

$$\int_C \frac{1}{z^2+3z+1}\, dz = 2\pi i \left(\text{Res at } z = \frac{-3+\sqrt{5}}{2}\right)$$

$$= 2\pi i \frac{1}{\sqrt{5}}$$

$$\therefore \frac{1}{2i} \int_C \frac{1}{z^2+3z+1} = \frac{2\pi i}{2i} \frac{1}{\sqrt{5}}$$

$$= \frac{\pi}{\sqrt{5}}$$

73

$$\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\, d\theta$$

$e^{i\theta} = \cos\theta + i\sin\theta$

Real part

$$= \text{Real part of} \int_0^{2\pi} \frac{e^{i\theta}}{3+\sin\theta}\, d\theta$$

$$= \text{" " "} \int_0^{2\pi} \frac{e^{i\theta}}{3+\frac{e^{i\theta}-e^{i\theta}}{2i}}\, d\theta$$

$$= \text{" " "} \int_C \frac{z}{3+\frac{1}{2i}\left(z-\frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \text{" " "} \int_C \frac{z}{3iz+\frac{z^2}{2}-\frac{1}{2}}\, dz$$

$$= \text{"} \quad \text{"} \quad \text{"} \int_C \frac{2z}{z^2 + 6iz - 1} \, dz$$

$\therefore$ poles are, $z^2 + 6iz - 1 = 0$

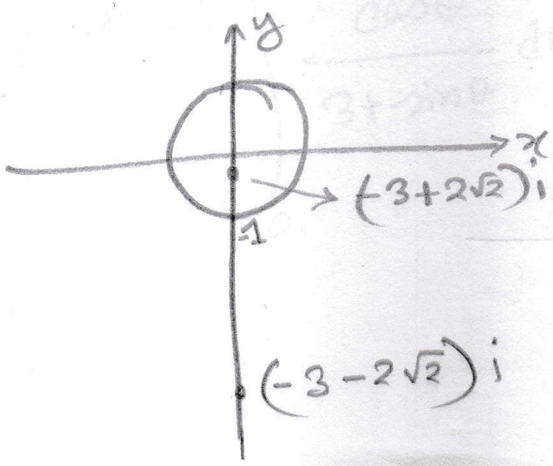
$$\Rightarrow \quad z = \frac{-6i \pm \sqrt{-36+4}}{2}$$

$$= \frac{-6i \pm 4\sqrt{2}i}{2}$$

$$= -3i \pm 2\sqrt{2}i$$

$$\Rightarrow z = (-3+2\sqrt{2})i, \ (-3-2\sqrt{2})i$$

-0.17 $\qquad\qquad$ -5.8



Res $\left(\text{at } z = \left(-3+2\sqrt{2}\right)i\right)$

$$= \lim_{z \to \left(-3+2\sqrt{2}\right)i} \frac{\left\{z - \left(-3+2\sqrt{2}\right)i\right\} 2z}{\left\{z - \left(-3+2\sqrt{2}\right)i\right\}\left\{z - \left(-3-2\sqrt{2}\right)i\right\}}$$

$$= \cancel{\lim_{z \to \left(-3+2\sqrt{2}\right)i}}$$

$$= \frac{2\left(-3+2\sqrt{2}\right)i}{\left(-3+2\sqrt{2}\right)i - \left(-3-2\sqrt{2}\right)i}$$

$$=$$

$$= \frac{-3+2\sqrt{2}}{2\sqrt{2}}$$

By Residue thm,

$$\int_C f(z)\,dz = 2\pi i \left(\text{sum of residue of the poles within } C \right)$$

$$\Rightarrow \int_C \frac{2z}{z^2+6iz-1}\,dz = 2\pi i \left(\text{Res at } z = (-3+2\sqrt{2})i \right)$$

$$= 2\pi i \, \frac{-3+2\sqrt{2}}{2\sqrt{2}}$$

$$= \frac{(-3+2\sqrt{2})\pi i}{\sqrt{2}}$$

$\therefore$ Real part of $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{3+\sin\theta}\,d\theta =$ Real part of $\displaystyle\int_C \frac{2z}{z^2+6iz-1}\,dz$

$= \text{Real part of } \dfrac{(-3+2\sqrt{2})\pi i}{\sqrt{2}}$

$= 0$

Laplace Transformation

Defn: Let $f(t)$ be function defined for all positive values of t then

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

provided that integral exists is called the Laplace transformation of $f(t)$. It is denoted by $L\{f(t)\}$.

$$L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

t = time, তাই value সব positive, কারণ time negative হয় না।

(i) $L\{1\} = \int_0^\infty e^{-st} \, 1 \, dt = \left[\frac{e^{-st}}{-s} \right]_0^\infty$

$= 0 - \frac{1}{-s}$

$= \frac{1}{s}$

(ii) $L\{t^n\} = \int_0^\infty e^{-st} \, t^n \, dt$

γ function use করব।

Let $st = x$

$\Rightarrow t = \frac{x}{s}$

$\therefore dt = \frac{dx}{s}$

$\therefore L\{t^n\} = \int_0^\infty e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$

$$= \frac{1}{s^{n+1}} \int_0^\infty (x)^{(n+1)-1} e^{-x} \, dx$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1) = \frac{\Gamma(n+1)}{s^{n+1}}$$

$$= \frac{\lfloor n}{s^{n+1}}$$ when n is integar.

(iii) $$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt$$

– e common রাখতে হবে তাহলে ∞ বসালে 0 হবে না.

$$= \int_0^\infty e^{-(s-a)t} \, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^\infty$$

$$= 0 - \frac{1}{-(s-a)}$$

$$= \frac{1}{s-a}$$

(iv) $L\{\cosh at\} = \frac{s}{s^2-a^2}$

(v) $L\{\sinh at\} = \frac{a}{s^2-a^2}$

(vi) $L\{\cos at\} = \frac{s}{s^2+a^2}$

(vii) $L\{\sin at\} = \frac{a}{s^2+a^2}$

~~(viii)~~

Properties of Laplace transformation:

1. $L\{a f_1(t) + b f_2(t)\} = a L\{f_1(t)\} + b L\{f_2(t)\}$

2. **First Shifting Theorem:**

If $L\{f(t)\} = F(s)$ then $L\{e^{at} f(t)\} = F(s-a)$

Γ গামা

(i) $L\{e^{at} t^n\}$; $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{n+1}}$

$$= \frac{\Gamma(n+1)}{(s-a)^{n+1}}$$

(ii) $L\{e^{at} \sin bt\}$

$L\{\sin at\} = \dfrac{a}{s^2+a^2}$

$$= \frac{b}{(s-a)^2 + b^2}$$

$L\{\cos at\} = \dfrac{s}{s^2+a^2}$

(iii) $L\{e^{at} \cos bt\}$

$$= \frac{s-a}{(s-a)^2 + b^2}$$

Ex:

$$L\left\{t^{-\frac{1}{2}}\right\} = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{s^{-\frac{1}{2}+1}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{s^{\frac{1}{2}}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} \qquad \boxed{\Gamma\tfrac{1}{2} = \sqrt{\pi}}$$

$\cos^2\theta$, $\cos^3\theta$; $\sin^2\theta$, $\sin^3\theta$] etc ক্ষেত্রে

$$\cos^2 t = \frac{1+\cos 2t}{\cancel{2}}$$

Ex:

$$L\left\{t^{\frac{3}{2}} + e^{-2t}\sin t + \cosh 2t\right\}$$

$$= L\left\{t^{\frac{3}{2}}\right\} + L\left\{e^{-2t}\sin t\right\} + L\left\{\cosh 2t\right\}$$

$$= \frac{\Gamma\left(\frac{3}{2}+1\right)}{s^{\frac{3}{2}+1}} + \frac{1}{(s+2)^2+1} + \frac{s}{s^2-2^2}$$

$$= \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{s^{\frac{5}{2}}} + \text{——} + \text{——}$$

Laplace transform of the derivative of $f(t)$:

$$L\{f'(t)\} = s\,L\{f(t)\} - f(0)$$

$$= sF(s) - f(0)$$

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s)e^{-st} f(t)\,dt$$

$$= 0 - 1f(0) + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= s\,L\{f(t)\} - f(0)$$

$$L\{f''(t)\} = s^2 L\{f(t)\} - s f(0) - f'(0)$$

$$L\{f'''(t)\} = s^3 L\{f(t)\} - s^2 f(0) - s f'(0) - f''(0)$$

$$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0)$$

~~Lap~~

$$L\left\{\int_0^t f(t)\,dt\right\} = \frac{1}{s} L\{f(t)\}$$

$$= \frac{F(s)}{s}$$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

where $F(s) = L\{f(t)\}$

[n = positive integer]

$$L\{t f(t)\} = -\frac{d}{ds} F(s)$$

$\underline{9}$ $L\left\{t^2 e^t \sin 4t\right\}$

$\underline{\text{Soln:}}$

We know,

$$L\{\sin 4t\} = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$$

$$\therefore L\{e^t \sin 4t\} = \frac{4}{(s-1)^2+16}$$

$$= \frac{4}{s^2-2s+17}$$

$$\therefore L\{t e^t \sin 4t\} = (-1)^2 \frac{d^2}{ds^2} L\{e^t \sin 4t\}$$

$$= \frac{d}{ds}\left[\frac{d}{ds} \frac{4}{s^2-2s+17}\right]$$

$$= \frac{d}{ds}\left[\frac{-4(2s-2)}{(s^2-2s+17)^2}\right]$$

$$= -8 \frac{d}{ds} \frac{s-1}{(s^2-2s+17)^2}$$

$$= -8 \frac{(s^2-2s+17)^2 \cdot 1 - (s-1)\,2(s^2-2s+17)(2s-2)}{(s^2-2s+17)^4}$$

$$= -8 \frac{(s^2-2s+17)(s^2-2s+17-4s^2+8s-4)}{(s^2-2s+17)^4}$$

$$= 8 \frac{3s^2-6s-13}{(s^2-2s+17)^3}$$

Exercise – 13.2

9

$$\mathcal{L}\left\{\int_0^t e^{-2t}\, t \sin^3 t \, dt\right\}$$

(i) (ii) (iii)

$$= \frac{1}{s}\, \mathcal{L}\left\{ e^{-2t}\, t \sin^3 t \right\}$$

$L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(s)\,ds$

where $L\{f(t)\} = F(s)$

<u>11</u>

$$\int_0^t \frac{\sin t}{t}\,dt$$

<u>Soln:</u>

We know,

$$L\{\sin t\} = \frac{1}{s^2+1}$$

$$\therefore L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}\,ds$$

$$= \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

$$\therefore L\left\{\int_0^t \frac{\sin t}{t}\,dt\right\}$$

$$= \frac{1}{s} L\left\{\frac{\sin t}{t}\right\}$$

$$= \frac{1}{s}\left[\frac{\pi}{2} - \tan^{-1} s\right]$$

12

$$L\left\{\frac{1-\cos t}{t^2}\right\}$$

Soln:

$$L\{1-\cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{1-\cos t}{t}\right\} = \int_s^\infty L\{(1-\cos t)\}\,ds$$

$$= \int_{s}^{\infty} \left(\frac{1}{s} - \frac{s}{s^2+1} \right) ds$$

$$= \left[\ln s - \frac{1}{2} \ln (s^2+1) \right]_{s}^{\infty}$$

$$= \frac{1}{2} \left[\ln s^2 - \ln (s^2+1) \right]_{s}^{\infty}$$

$$= \frac{1}{2} \left[\ln \frac{s^2}{s^2+1} \right]_{s}^{\infty}$$

$$= \frac{1}{2} \left[\ln \frac{1}{1+\frac{1}{s^2}} \right]_{s}^{\infty}$$

$$= \frac{1}{2} \left[\ln \left(\frac{1}{1+0} \right) - \ln \left(\frac{1}{1+\frac{1}{s^2}} \right) \right]$$

$$= -\frac{1}{2} \ln \frac{s^2}{s^2+1}$$

$$\therefore L\left\{\frac{1-\cos t}{t^2}\right\} = \int_s^\infty L\left\{\frac{1-\cos t}{t}\right\} ds$$

$$= \int_s^\infty \left(-\frac{1}{2}\ln\frac{s^2}{s^2+1}\right) ds$$

Exercise: 13·3

22/07/17

Unit Step Function:

$$u(t-a) = \begin{cases} 0 & \text{when } t < a \\ 1 & \text{when } t \geq a \end{cases}$$

$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

Second Shifting Thm:

If $L\{f(t)\} = F(s)$ then $L\{f(t-a)\,u(t-a)\}$

$$= e^{-as} F(s)$$

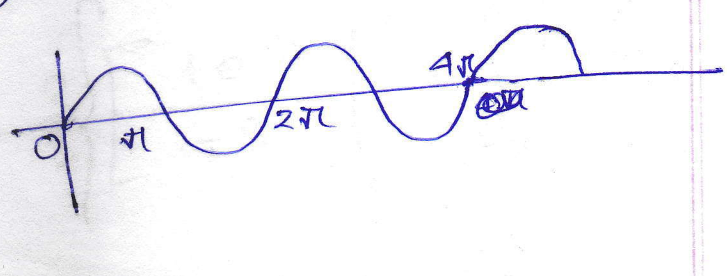
$$L\{f(t)\,u(t-a)\} = e^{-as} L\{f(t+a)\}$$

Periodic Function:

$f(x) = f(x+T)$ T's lowest value is period

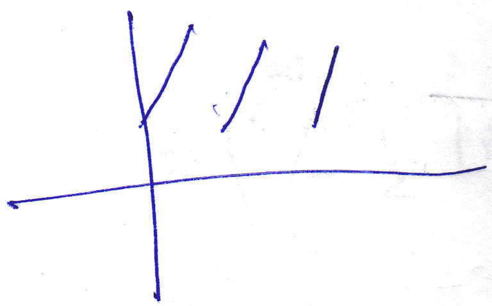
$\sin x = \sin(2\pi + x)$ ~~$t\,2\pi$~~ 2π period

$\sin x = \sin(4\pi + x)$ 2π period



$f(x) = x+1$ and $f(x) = f(x+2)$

$$f(2) = f(0+2)$$
$$= f(0)$$
$$= 1$$



Let, $f(t)$ is a periodic function with period T, then,

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

23, 24

23
$f(t) = \frac{2t}{3}$ $0 \le t \le 3$, Find the Laplace transform of the waveform.

Soln:
We know, the Laplace transform of periodic function,

$$L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}$$

$$= \frac{1}{1-e^{-sT}} \int_0^{3} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-3s}} \int_0^T e^{-st} \frac{2t}{3}\,dt$$

$$= \frac{1}{1-e^{-3s}} \frac{2}{3} \left\{ \left[\frac{te^{-st}}{-s} \right]_0^3 - \int_0^3 1 \cdot \frac{e^{-st}}{-s}\,dt \right\}$$

$$= \frac{2}{3(1-e^{-3s})}\left\{\frac{3e^{-3s}}{-s} - 0 + \left[\frac{e^{-st}}{-s}\right]_0^3\right\}$$

$$= \frac{2}{3(1-e^{-3s})}\left\{-\frac{3}{5}e^{-3s} - \frac{1}{s^2}\left(e^{-3s}-1\right)\right\}$$

$$= \frac{2e^{-3s}}{-s(1-e^{-3s})} + \frac{2}{3s^2}$$

25, 26

Convolution Theorem:

If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$

then $L\left\{\int_0^t f_1(x) f_2(t-x)\, dx\right\} = F_1(s)\, F_2(s)$

Evalution of Integral:

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$

28

$$\int_0^\infty t e^{-3t} \sin t \, dt$$

Soln:

$$\int_0^\infty e^{-st} (t \sin t)\, dt = L\{t \sin t\}$$

Now, $L\{\sin t\} = \frac{1}{s^2+1}$

$$\therefore L\{t \sin t\} = -\frac{d}{ds} L\{\sin t\}$$

$$= -\frac{d}{ds} \frac{1}{s^2+1}$$

$$= -\frac{-1}{(s^2+1^2)^2} 2s$$

$$= \frac{2s}{(s^2+1)^2}$$

$$\therefore \int_0^\infty e^{-st} (t \sin t)\, dt = \frac{2s}{(s^2+1)^2}$$

putting $s = 3$ we get,

$$\int_0^\infty e^{-3t}\, t \sin t\, dt = \frac{2(3)}{(3^2+1)^2}$$

$$= \frac{6}{100}$$

$$= \frac{3}{50}$$

$$\# \int_0^\infty \frac{e^{-3t} \sin t}{dt}\, dt$$

$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty L\{\sin t\}\, ds$$

$$= \int_s^\infty \frac{1}{s^2+1}\, ds = \left[\tan^{-1} s\right]_s^\infty$$

$$= \tan^{-1}\infty - \tan^{-1} s$$

$$= \frac{\pi}{2} - \tan^{-1} s$$

28 , 29

Exercise 13.7

13.8

Inverse Laplace Transform:

If $L\{f(t)\} = F(S)$, the inverse Laplace Transform of $F(S)$ is $f(t)$.

$$L^{-1}\{F(S)\} = f(t)$$

$$L^{-1}\left(\frac{1}{s}\right) = 1$$

$$L^{-1}\left\{\frac{1}{s^n}\right\} = \frac{t^{n-1}}{\Gamma n}$$

$$L^{-1}\left\{\frac{1}{(S-a)}\right\} = e^{at}$$

$$L^{-1}\left\{\frac{a}{s^2+a^2}\right\} = \sin at$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cancel{a}\ \cos at$$

$$L\{e^{at} f(t)\} = F(s-a)$$

$$L\{e^{at} \cos bt\} = \frac{s-a}{(s-a)^2+b^2}$$

$\boxed{23/07/2017}$

Ex: $L^{-1}\left\{\frac{4s+2}{s^2+2s+3}\right\}$

$$= L^{-1}\left\{\frac{4s+2}{(s+1)^2+2}\right\}$$

$$= L^{-1}\left\{\frac{4(s+1)-2}{(s+1)^2+(\sqrt{2})^2}\right\}$$

$$= L^{-1}\left\{\frac{4(s+1)}{(s+1)^2+(\sqrt{2})^2} - \frac{2}{(s+1)^2+(\sqrt{2})^2}\right\}$$

$$= 4L^{-1}\left\{\frac{s+1}{(s+1)^2+(\sqrt{2})^2}\right\} - \sqrt{2}\,L^{-1}\left\{\frac{\sqrt{2}}{(s+1)^2+(\sqrt{2})^2}\right\}$$

$= 4e^{-t}\cos t - \sqrt{2}\,e^{-t}\sin t$ $\left[\because L\{e^{at}f(t)\} = F(s-a)\right]$

$L^{-1}\{sF(s)\} = \frac{d}{dt}f(t) + f(0)\,\delta(t)$

<u>Ex: 32</u>

① $L^{-1}\left\{\frac{s}{s^2+1}\right\} = L^{-1}\left\{s \cdot \frac{1}{s^2+1}\right\}$

$= \frac{d}{dt}\sin t + \sin 0\,\delta(t)$

$= \cos t + 0$

$= \cos t$

$L^{-1}\left\{\frac{F(s)}{(s)}\right\} = \int_0^t L^{-1}\{F(s)\}\,dt$

$= \int_0^t f(t)\,dt$

33

(ii) $L^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

$$= L^{-1}\left\{\frac{\frac{1}{s^2+1}}{s}\right\}$$

$$= \int_0^t L^{-1}\left\{\frac{1}{s^2+1}\right\} dt$$

$$= \int_0^t \sin t \, dt$$

$$= \left[-\cos t\right]_0^t$$

$$= -(\cos t - \cos 0)$$

$$= 1 - \cos t$$

partial fraction করছি

$$\# \quad \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

$$\Rightarrow \quad 1 = A(s^2+1) + (Bs+C)s$$

putting $s=0$ $\quad \therefore A = 1$

equating the co-effient of s^2 and s,

we get,

$$0 = A + B \quad \therefore B = -1$$

$$0 = C$$

$$\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\therefore L^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s}\right\} - L^{-1}\left\{\frac{s}{s^2+1}\right\}$$

$$= 1 - \cos t$$

if $L^{-1}\{F(s)\} = f(t)$ then $L^{-1}\{F(s+a)\} = e^{-at} f(t)$

<u>34</u>

(ii) $L^{-1}\left\{ \dfrac{s}{s^2+4s+13} \right\}$

$$= L^{-1}\left\{ \frac{(s+2)-2}{(s+2)^2+9} \right\}$$

$$= L^{-1}\left\{ \frac{s+2}{(s+2)^2+3^2} \right\} - 2L^{-1}\left\{ \frac{1}{(s+2)^2+3^2} \right\}$$

$$= L^{-1}\left\{ \frac{s+2}{(s+2)^2+3^2} \right\} - \frac{2}{3} L^{-1}\left\{ \frac{3}{(s+2)^2+3^2} \right\}$$

$$= e^{-2t} \cos 3t + \frac{2}{3} e^{-2t} \sin 3t$$

Ex:

$$L^{-1}\left\{\frac{s}{s^2-4s+1}\right\}$$

$$= L^{-1}\left\{\frac{s}{(s-2)^2 -4+1}\right\}$$

$$= L^{-1}\left\{\frac{s}{(s-2)^2-3}\right\}$$

$$= L^{-1}\left\{\frac{(s-2)+2}{(s-2)^2-(\sqrt{3})^2}\right\}$$

$$= L^{-1}\left\{\frac{s-2}{(s-2)^2-(\sqrt{3})^2}\right\} + \frac{2}{\sqrt{3}} L^{-1}\left\{\frac{\sqrt{3}}{(s-2)^2-(\sqrt{3})^2}\right\}$$

$$= e^{2t}\cosh\sqrt{3}t + \frac{2}{\sqrt{3}} e^{2t} \sinh\sqrt{3}t$$

$\tan^{-1}(\,), \ln(\,), \cos^{-1}(\,)$

$$\# \quad L^{-1}\left\{\frac{d}{ds} F(s)\right\} = -t \; L^{-1}\left\{F(s)\right\}$$

$$= -t f(t)$$

or, $$L^{-1}\left\{F(s)\right\} = -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} F(s)\right\}$$

<u>37</u> $$L^{-1}\left\{\tan^{-1}\frac{1}{s}\right\}$$

$$= -\frac{1}{t} \; L^{-1}\left\{\frac{d}{ds} \tan^{-1}\frac{1}{s}\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{1+\left(\frac{1}{s}\right)^2}\left(\frac{-1}{s^2}\right)\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{-1}{s^2+1}\right\}$$

$$= \frac{1}{t} \; L^{-1}\left\{\frac{1}{s^2+1}\right\} = \frac{1}{t} \sin t = \frac{\sin t}{t}$$

38

$$L^{-1}\left\{\ln \frac{s^2-1}{s^2}\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds} \ln \frac{s^2-1}{s^2}\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{d}{ds}\left[\ln (s^2-1) - \ln s^2\right]\right\}$$

$$= -\frac{1}{t} L^{-1}\left\{\frac{1}{s^2-1} \cdot 2s - \frac{1}{s^2} \cdot 2s\right\}$$

$$= -\frac{2}{t} L^{-1}\left\{\frac{s}{s^2-1} - \frac{1}{s}\right\}$$

$$= -\frac{2}{t}(\cosh t - 1)$$

$$= \frac{2(1-\cosh t)}{t}$$

Exercise : 13.3

$L^{-1}\left\{\int_s^{\infty} F(s)\,ds\right\} = \frac{1}{t} L^{-1}\left\{F(s)\right\} = \frac{f(t)}{t}$

or, $L^{-1}\left\{F(s)\right\} = t\,L^{-1}\left\{\int_s^{\infty} F(s)\,ds\right\}$

40

$$L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = t\,L^{-1}\left\{\int_s^{\infty} F(s)\,ds\right\} \ldots\ldots (1)$$

$$\int_s^{\infty} F(s)\,ds = \int_s^{\infty} \frac{2s}{(s^2+1)^2}\,ds$$

let, $s^2+1 = x$

$2s\,dx = dx$

$$\therefore \int \frac{2s}{(s^2+1)^2}\,ds = \int \frac{dx}{x^2} = -\frac{1}{x} = \frac{-1}{s^2+1}$$

$$\therefore \int_s^{\infty} \frac{2s}{(s^2+1)^2}\,ds = \left[-\frac{1}{s^2+1}\right]_s^{\infty}$$

$$= \left[-0 - \frac{1}{s^2+1}\right] = \frac{1}{s^2+1}$$

(11) $\rightarrow$

From eqn (i),

$$L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} = t\, L^{-1}\left\{\frac{1}{s^2+1}\right\}$$

$$= t\ \sin t$$

Inverse Laplace transform by Convolution th^m:

$L\left\{\int_0^t f_1(x)\, f_2(t-x)\, dx\right\} = F_1(s).F_2(s)$

$$\therefore L^{-1}\left\{F_1(s).F_2(s)\right\} = \int_0^t f_1(x)\, f_2(t-x)\, dx$$

where,

$L^{-1}\left\{F_1(s)\right\} = f_1(t)$

$L^{-1}\left\{F_2(s)\right\} = f_2(t)$

45

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} \quad ; a \neq b$$

Sol^n:

$$L^{-1}\left\{\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+b^2}\right\}$$

Now,

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at = f_1(t)$$

$$L^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt = f_2(t)$$

By convolution thm

$$L^{-1}\{F_1(s).\,F_2(s)\} = \int_0^t f_1(x)\, f_2(t-x)\, dx$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\cdot\frac{s}{s^2+b^2}\right\} = \int_0^t \cos ax \cos b(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t 2\cos ax \cos b(t-x)\, dx$$

$$= \frac{1}{2}\int_0^t \{\cos(ax+bt-bx) + \cos(ax-bt+bx)\}\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(ax+bt-bx)}{a-b} + \frac{\sin(ax-bt+bx)}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(at+bt-bt)}{a-b} + \frac{\sin(at-bt+bt)}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin(-bt)}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{\sin at - \sin bt}{a-b} + \frac{\sin at + \sin bt}{a+b}\right]$$

$$=$$

$$= \frac{a \sin at - b \sin bt}{a^2-b^2}$$

45

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} \qquad a \neq b$$

আংশিক ভগ্নাংশ

$$\text{Let} \quad \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{As+B}{s^2+a^2} + \frac{Cs+D}{s^2+b^2}$$

সহগ সমীকৃতকরণ

46

$$L^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = L^{-1}\left\{\frac{1}{s}\cdot\frac{1}{s^2+a^2}\right\}$$

Now,

$$L^{-1}\left\{\frac{1}{s}\right\} = 1 = f_1(t)$$

$$L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a} = f_2(t)$$

By convolution thm,

$$L^{-1}\{F_1(s)\cdot F_2(s)\} = \int_0^t f_1(x)\, f_2(t-x)\,dx$$

$$= \int_0^t 1\,\frac{\sin a(t-x)}{a}\,dx$$

$$= -\frac{1}{a}\left[\frac{\cos a(t-x)}{-a}\right]_0^t$$

$$= \frac{1}{a^2}\left[\cos a(t-t) - \cos a(t-0)\right]$$

$$= \frac{1}{a^2}(1-\cos at)$$

30/07/17

Solution of Differential equation using Laplace transform:

47

initial value problem.

$y'' - 4y' + 4y = 64\sin 2t$ ---- (1) , $y(0) = 0$, $y'(0) = 1$

Taking Laplace transform of both sides of eq^n (1) we get,

$$L\{y'' - 4y' + 4y\} = 64\, L\{\sin 2t\}$$

or, $s^2 L\{y\} - s\,y(0) - y'(0) - 4\{s\, L\{y\} - y(0)\} + 4\, L\{y\} = 64 \dfrac{2}{s^2+4}$

or, $s^2 L\{y\} - s.0 - 1 - 4\{s\, L\{y\} - 0\} + 4\, L\{y\} = \dfrac{128}{s^2+4}$

or, $L\{y\}\left(s^2-4s+4\right) = \frac{128}{s^2+4}+1$

or, $L\{y\}\left(s-2\right)^2 = \frac{128+s^2+4}{s^2+4}$

or, $L\{y\} = \frac{s^2+132}{\left(s^2+4\right)\left(s-2\right)^2}$

Taking inverse Laplace transform of both side we get,

$$y = L^{-1}\left\{\frac{s^2+132}{\left(s^2+4\right)\left(s-2\right)^2}\right\}$$

আংশিক ভগ্নাংশ করব

Let,

$$\frac{s^2+132}{\left(s^2+4\right)\left(s-2\right)^2} = \frac{As+B}{s^2+4}+\frac{C}{(s-2)}+\frac{D}{(s-2)^2} \quad \text{---} (11)$$

or, $s^2 + 132 = (As+B)(s-2)^2 + C(s^2+4)(s-2) + D(s^2+4)$

putting $s = 2$ we get,

$4 + 132 = D(4+4)$ $\therefore D = \frac{136}{8} = 17$

Equating the co-efficient of both sides of s^3, s^2 and s of ~~eqn~~ (11),

eqn এর দুই

$0 = A + C$

$1 = -4A + B - 2C + D$

$0 = 4A - 4B + 4C$ $\therefore B = 0$ $[\because A + C = 0]$

or, $0 = A - B + C$

$1 = -4A - 2C + 17$

or, $4A + 2C = 17 - 1$ $\left[\because A + C = 0, \therefore C = -A\right]$

or, $4A + 2(-A) = 16$

or, $A = 8$ $\therefore C = -8$

$$y = L^{-1}\left\{\frac{8s}{s^2+4} - \frac{8}{s-2} + \frac{17}{(s-2)^2}\right\}$$

$y(t)$

$$= 8\cos 2t - 8e^{2t} + 17\, t\, e^{2t}$$

$\boxed{L\{y\} = Y(s)}$

48

$y'' + 25y = 10\cos 5t \quad , \quad y(0)=2,\ y'(0)=0$

Taking Laplace transform of both side,

$$L\{y''\} + 25L\{y\} = 10L\{\cos 5t\}$$

or, $s^2 L\{y\} - s\,y(0) - y'(0) + 25L\{y\} = 10L\{\cos 5t\}$

or, $s^2 L\{y\} - 2s - 0 + 25L\{y\} = 10\,\dfrac{s}{s^2+25}$