Queuing Theory

A Shoeshine Shop: Consider a shoeshine shop consisting of two chairs. Suppose that an entering customer first will go to chair 1. When his work is completed in chair 1, he will go either to chair 2 if that chair is empty or else wait in chair 1 until chair 2 becomes empty. Suppose that a potential customer will enter this shop as long as chair 1 is empty. (Thus, for instance, a potential customer might enter even if there is a customer in chair 2.)
If we suppose that potential customers arrive in accordance with a Poisson process at rate λ and that the serving time for two chairs independent and have respective exponential rates μ_1 and μ_2, then

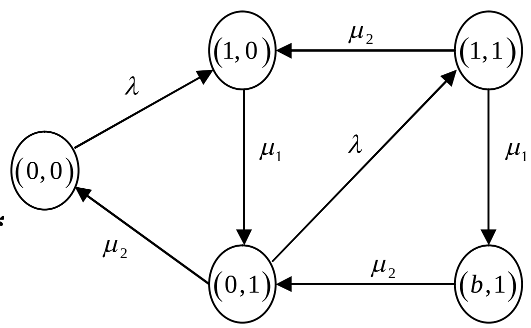
(a) What proportion of potential customers enters the system? **What proportion does not Enter?**
(b) What is the mean number of customers in the system?
(c) What is the average amount of time that an entering customer spends in the system?

What proportion of customers has to wait on chair 1 after their service at chair 1 is over?

State	*Interpretation*
(0,0)	There are no customers in the system.
(1,0)	There is one customer in the system and he is in chair 1.
(0,1)	There is one customer in the system and he is in chair 2.
(1,1)	There are two customers in the system and both are presently being served.
(b,1)	There are two customers in the system but the customer in the first chair has completed his work in that chair and is waiting for the second chair to become free.

It should be noted that when the system is in state (b,1), the person in chair 1, though not being served is nevertheless "blocking" potential arrivals from entering the system. The transition diagram given below shows all transition between above mentioned 5 states.

****** This is called the State Transition Diagram for the Shoeshine Shop Model ******



To write the balance equations we equate the sum of the arrows (multiplied by the probability of the states where they originate) coming into a state with the sum of arrows (multiplied by the probability of the state) going out of that state. This gives

State	*Rate that the process leaves = rate that it enters*
$(0,0)$	$\lambda P_{00} = \mu_2 P_{01}$
$(1,0)$	$\mu_1 P_{10} = \lambda P_{00} + \mu_2 P_{11}$
$(0,1)$	$(\lambda + \mu_2) P_{01} = \mu_1 P_{10} + \mu_2 P_{b1}$
$(1,1)$	$(\mu_1 + \mu_2) P_{11} = \lambda P_{01}$
$(b,1)$	$\mu_2 P_{b1} = \mu_1 P_{11}$

These along with the equation,

$$P_{00}+P_{10}+P_{01}+P_{11}+P_{b1}=1$$

may be solved to determine the limiting probabilities.

(a) Since a potential customer will enter the system when the state is either (0,0) or (0,1), it follows that the proportion of customers entering the system is $P_{00}+P_{01}$.

(b) Since there is one customer in the system whenever the state is (0,1) or (1,0) and two customer in the system whenever the state is (1,1) or (b,1), it follows that L, the average number of customer in the system is given by

$$L=P_{01}+P_{10}+2\left(P_{11}+P_{b1}\right)$$

Average (x) = E[x]
$=\sum$x.Px
= 0*P00 + 1*P01 + 1*P10 + 2*P11 + 2*Pb1

(c) To derive the average amount of time that an entering customer spends in the system, we use the relationship $W=L/\lambda_a$. Since a potential customer will enter the system when in state (0,0) or (0,1), it follows that $\lambda_a=\lambda\left(P_{00}+P_{01}\right)$ and hence

$$W=\frac{P_{01}+P_{10}+2\left(P_{11}+P_{b1}\right)}{\lambda\left(P_{00}+P_{01}\right)}$$

W = L * lambda won't work directly because all of lambda (i.e., arrivals) not effective (i.e., some ppl not entering)

<u>Example:</u> (a) If $\lambda=1, \mu_1=1, \mu_2=2$, then calculate the preceding quantities of interest.
(b) If $\lambda=1, \mu_1=2, \mu_2=1$, then calculate the preceding.

<u>Solution:</u> (a) Putting the values of $\lambda=1, \mu_1=1, \mu_2=2$ in probability equations, we get

$$P_{00}=2P_{01} \quad \cdots \quad (1)$$
$$P_{10}=P_{00}+2P_{11} \quad \cdots \quad (2)$$
$$3P_{01}=P_{10}+2P_{b1} \quad \cdots \quad (3)$$
$$3P_{11}=P_{01} \quad \cdots \quad (4)$$
$$2P_{b1}=P_{11} \quad \cdots \quad (5)$$
$$P_{00}+P_{10}+P_{01}+P_{11}+P_{b1}=1 \quad \cdots \quad (6)$$

From equation (6), we get

$$P_{00}+\left(P_{00}+2P_{11}\right)+\frac{1}{2}P_{00}+P_{11}+\frac{1}{2}P_{11}=1, \quad \text{[using equation (1), (2) and (5)]}$$

$$\Rightarrow \frac{5}{2}P_{00}+\frac{7}{2}\times\frac{1}{3}P_{01}=1, \quad \text{[using equation (4)]}$$

$$\Rightarrow \frac{5}{2}P_{00}+\frac{7}{6}\times\frac{1}{2}P_{00}=1, \quad \text{[using equation (1)]}$$

$$\Rightarrow \frac{30+7}{12}P_{00}=1$$

$$\therefore P_{00}=\frac{12}{37}$$

Putting the value of P_{00} in equation (1), we get

$$\frac{12}{37} = 2P_{01} \Rightarrow P_{01} = \frac{6}{37}$$

Putting the value of P_{01} in equation (4), we get

$$3P_{11} = \frac{6}{37} \Rightarrow P_{11} = \frac{2}{37}$$

Putting the value of P_{00} and P_{11} in equation (2), we get

$$P_{10} = \frac{12}{37} + 2\times\frac{2}{37} = \frac{16}{37}$$

Putting the value of P_{11} in equation (5), we get

$$2P_{b1} = \frac{2}{37} \Rightarrow P_{b1} = \frac{1}{37}$$

Hence, $P_{00} + P_{01} = \frac{12}{37} + \frac{6}{37} = \frac{18}{37}$ (Ans.)

$$L = P_{01} + P_{10} + 2\left(P_{11} + P_{b1}\right) = \frac{6}{37} + \frac{16}{37} + 2\left(\frac{2}{37} + \frac{1}{37}\right) = \frac{28}{37} \quad \text{(Ans.)}$$

$$W = \frac{P_{01} + P_{10} + 2\left(P_{11} + P_{b1}\right)}{\lambda\left(P_{00} + P_{01}\right)} = \frac{\frac{6}{37} + \frac{16}{37} + 2\left(\frac{2}{37} + \frac{1}{37}\right)}{\left(\frac{12}{37} + \frac{6}{37}\right)} = \frac{28}{37}\times\frac{37}{18} = \frac{14}{9} \quad \text{(Ans.)}$$

Solution (b) is similar to (a). Try it yourself and check the answers are given below

$$P_{00} = \frac{3}{11}, \quad P_{01} = \frac{3}{11}, \quad P_{11} = \frac{1}{11}, \quad P_{10} = \frac{2}{11}, \quad P_{b1} = \frac{2}{11}$$

* Hence, $P_{00} + P_{01} = \frac{6}{11}$, $L = 1$, $W = \frac{11}{6}$

** There are many possible Questions for the exam, because Many Possible Quantities You may have to find ! Following are some examples ...

The Probability (or, Fraction of Time) Both the Chairs are Empty (ans: find P
00
)

The Probabilty that Chair 1 is empty (find P00+P01)

The Fraction of Time Chair 2 is empty (ans: find P00 + P10)

The Probability Chair 2 is Filled (ans: find P01 + P11 + Pb1)

The fraction of time Either or Both Chairs Filled = (1 - P00) OR P01+P10+P11+Pb1

The Probabilty that Both the chairs are filled = P11 + Pb1