'0' তম fibonacci এর last digit 0,

∴ 1500 " " " " 0 হবে

Example:

f(1510) এর last digit = 1510 % 60

= 10th fibo এর last digit

= 5

Last digit of f(1401) = 1401 % 60

= Last digit of 21th fibonacci

= ~~1~~ 6

4. Fibonacci Numbers are factors of fibonacci numbers:

২ দ্বারা যেগুলো ভাগ যাচ্ছে ✓

3rd fibonacci num

fib(i)	0	1	1	2	3	5	8	13	21	34	55	89	144
fib(3) = 2				✓			✓			✓			✓
fib(4) = 3					✓				✓				✓
fib(5) = 5						✓					✓		
fib(6) = 8							✓						✓

Every 3rd Fibonacci number is a multiple of F(3) [counting start from 0,1,2,3 → এভাবে]
?

Every 4th Fibonacci number is a multiple of F(4)

⋮

Every 6th Fibonacci number is a multiple of F(6)

∴ Every kth Fibonacci number is a multiple of F(k)

Every positive integer can be written uniquely as a product over many primes. [This is also called the Fundamental Theorm of Arithmetic
↳ See the proof from Lec-7 (Number Theory)
↳ [V. Imp]]

$12 = 2 \times 2 \times 3 = 2^2 \times 3^1$

$18 = 2 \times 3 \times 3 = 2^1 \times 3^2$

Read this part from pdf lecture: "Lec7-Number Theory.pdf" * Besides, this Lec-7 has some additional topics, like euclid number, mersenne number - etc.**

$n = \prod p^{np}$ here,

sign of multiplication

p is a prime

np = (power of p)

↳ np = Prime exponent

$12 = 2^2 \times 3^1 \times 5^0 \times 7^0 \times \ldots\ldots$

Suppose

$2^2 \times 3^1$

$p = 2$ $\quad p = 3$

$np = 2$ $\quad np = 1$

np (may or may not prime) but p is must prime.

In Prime exponent form,

1st prime no (2) এর frequency

2nd " " (3) "

$\overset{m}{12} = \{2, 1, 0, 0 \ldots\ldots\}$

m_2 m_3 m_5 ← frequency

[mp]

$\overset{n}{18} = \{1, 2, 0, 0 \ldots\}$

n_2 n_3 n_3 → np

$12 \times 18 = \{\underbrace{3, 3, 0, 0 \ldots\ldots}\}$

$2^3 \times 3^3 = 8 \times 27 = 216 = 12 \times 18$

[12 & 18 এর prime exponent form এর যোগফল

$(2+1) = 3, (1+2) = 3$]

Write and Prove the Fundamental Theorem of Arithmetic: Any Integer n>1 Can Be Represented as a Unique Product of Prime Numbers Only. This unique product is called the Unique Prime Facorization of the integer n.

*** FIVE Questions in Lec7-Number Theory (involving Prime Exponent Representation, Euclid Prime, Mersenne Prime, Prove or Disprove that ... ==> They are MUST SEE short Questions (last page of "Lec7-Number Theory.pdf")

$$m_{12} = \{2, 1, 0, 0 \ldots \}$$

$$n_{18} = \{1, 2, 0, 0 \ldots \ldots \}$$

$$\gcd(m,n) = \prod_p \min(m_p, n_p)$$

$$\gcd(12, 18) = 2^{\min(2,1)} \times 3^{\min(1,2)}$$

$$= 2^1 \times 3^1$$

$$= 6$$

$$LCM(m,n) = \prod_p \max(m_p, n_p)$$

$$LCM(12, 18) = 2^{\max(2,1)} \times 3^{\max(1,2)}$$

$$= 2^2 \times 3^2$$

$$= 36$$

Example: $\gcd(12, 31)$ & $LCM(12, 31) = ?$

$$12 = 2 \times 2 \times 3 = 2^2 \times 3^1 \times 5^0 \times \ldots \ldots$$

$31 = 2^0 \times 3^0 \times 5^0 \times 7^0 \times 11^0 \times 13^0 \times 17^0 \times 19^0 \times 23^0$
$\times 29^0 \times 31^1 \times 37^0 \times \ldots$

$12 = \{2, 1, 0, 0 \ldots \ldots 0\}$

$31 = \{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0 \ldots\}$

$gcd(12, 31) = 2^{min(2,0)} \times 3^{min(1,0)} \times 31^{min(0,1)}$

$= 2^0 \times 3^0 \times 31^0$

$= 1$

$LCM(12, 31) = 2^{max(2,0)} \times 3^{max(1,0)} \times 31^{max(0,1)}$

$= 2^2 \times 3^1 \times 31^1$

$= 12 \times 31$

$= 372.$

(*) Also Study Euclid Numbers, Mersenne Primes from Lec-7 (Number Theory) $\longrightarrow$ Important as ShortQues

Write and Prove the Fundamental Theorem of Arithmetic: Any Integer n>1 Can Be Represented as a Unique Product of Prime Numbers Only. This unique product is called the Unique Prime Facorization of the integer n.
***** FIVE Questions in Lec7-Number Theory (involving Prime Exponent Representation, Euclid Prime, Mersenne Prime, Prove or Disprove that ... ==> They are MUST SEE short Questions (last page of "Lec7-Number Theory.pdf")**

[More detailed → here
but u Organized : in Lec-8 — Number Theory]

Factorial Factors:

$$n! = 1.2.3 \dots \dots n = \prod_{K=1}^{n} K$$

n	0	1	2	3	4	5	10
n!	1	1	2	6	24		3628800

big value

certain range এর পর বের করা যাবে না তাই string হিসাবে নিতে হবে।

What is the highest power of Prime P that divides n!?

the largest power of p that divides n!

$\mathcal{E}_p(n!)$

prime

$\mathcal{E}_2?(10!)$

$10! = 3628800$

2 এর কোন highest power টা 10! কে ভাগ করতে পারে

$5! = 120$

$\mathcal{E}_2?(5!)$ ← 2^3 Ans : 3

$\mathcal{E}_2(100!) \Rightarrow$ 100! টা বের করাই ঝামেলা

$\boxed{}$ $\varepsilon_2(10!)$ → n!

1 ton

power of 2	1	2	3	4	5	6	7	8	9	10	
2		X		X		X		X		X	$5 = \lfloor \frac{10}{2} \rfloor$
4				X				X			$2 = \lfloor \frac{10}{4} \rfloor$
8								X			$1 = \lfloor \frac{10}{8} \rfloor$
	0	1	0	2	0	1	0	3	0	1	8

→ 10

n এর চেয়ে নিচে or সমান এমন power p এর

X এর সংখ্যা

$8 = (0+1+0+2+0+1+0+3+0+1)$

maximum 2^8 দিয়ে 10! divide করা যাবে।

⊛ $\varepsilon_2(17!)$ হলে উপরে (1 - 17)

বামে (2 - 16)

Technical proof:

$\varepsilon_2(100!)$ বললে 100 পর্যন্ত লিখতে হলে এই prob avoid করতে

$$\varepsilon_2(10!) = \left\lfloor \frac{10}{2} \right\rfloor + \left\lfloor \frac{10}{4} \right\rfloor + \left\lfloor \frac{10}{8} \right\rfloor \; ; \; 2^k \leq n$$

$\mathcal{E}_2(100!) = ?$

$$\left\lfloor \frac{100}{2} \right\rfloor + \left\lfloor \frac{100}{4} \right\rfloor + \left\lfloor \frac{100}{8} \right\rfloor + \left\lfloor \frac{100}{16} \right\rfloor + \left\lfloor \frac{100}{32} \right\rfloor + \left\lfloor \frac{100}{64} \right\rfloor$$

$= 50 + 25 + 12 + 6 + 3 + 1$

$= 97$

[$\frac{100}{128}$ লেখা যাবে না

$\because$ $128 > 100$

$2^7 > 100$]

So, general Formula is

$$\mathcal{E}_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

$$= \sum_{k \geqslant 1} \frac{n}{p^k} \; ; \; p^k \geqslant n$$

// just Formula বসিয়ে math আসতে পারে exam এ আর যদি formula derive করতে বলা হয় তাহলে যেকোন n ধরে (যদি n বলা না থাকে) table form, then eq^n then value.....

⊞ $\mathcal{E}_p{}^{n!} \longrightarrow$ how large?

$$\mathcal{E}_p(n!) \leqslant \frac{n}{p} + \frac{n}{p^2} + \frac{n}{p^3} + \cdots\cdots$$

$$\mathcal{E}_p(n!) \leqslant \frac{n}{p}\left(1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\cdots\right)$$

↑ infinite series.

$$\mathcal{E}_p(n!) \leqslant \frac{n}{p}\left(\frac{p}{p-1}\right)$$

solution of this type infinite series.

$$\mathcal{E}_p(n!) \leqslant \frac{n}{p-1}$$

Verification:

$\mathcal{E}_2(100!) \longrightarrow$ how large? (upper bound)

$$\mathcal{E}_2(100!) \leqslant \frac{100}{2-1}$$

$$\mathcal{E}_2(100!) \leqslant 100$$

→ আসলে Ans এসেছিল 97 which is less than 100

$$\mathcal{E}_3(1000!) = ?$$

$$\frac{1000}{3-1} = 500 \quad \therefore \mathcal{E}_3(1000!) \leqslant 500$$

[This section much better in Lec-9]

So, you should read it (*selectively*)from "Lec9-Number Theory.pdf" ... especially the 3 Proofs

Relatively Prime:

m, n are Relatively prime if $\gcd(m,n)=1$

$\underset{m}{8}, \underset{n}{31}$

(8, 31) — 8 not prime, 31 prime — but they are relatively prime as $\gcd(8,31)=1$

$(1,3), (3,22) \longleftrightarrow$ relatively prime

Notation: $m \perp n$, means m, n relatively prime

$m \perp n \iff m, n$ are integers and $\gcd(m,n)=1$

$m \perp n \iff \min(m_p, n_p) = 0$ for all p.

$12 = \{2, 1, 0, 0, \ldots\ldots\}$ — m_p

$31 = \{0, 0, 0 \ldots\ldots 1\}$ — n_p

$\underset{n}{12}, \underset{n}{31}$ relatively prime

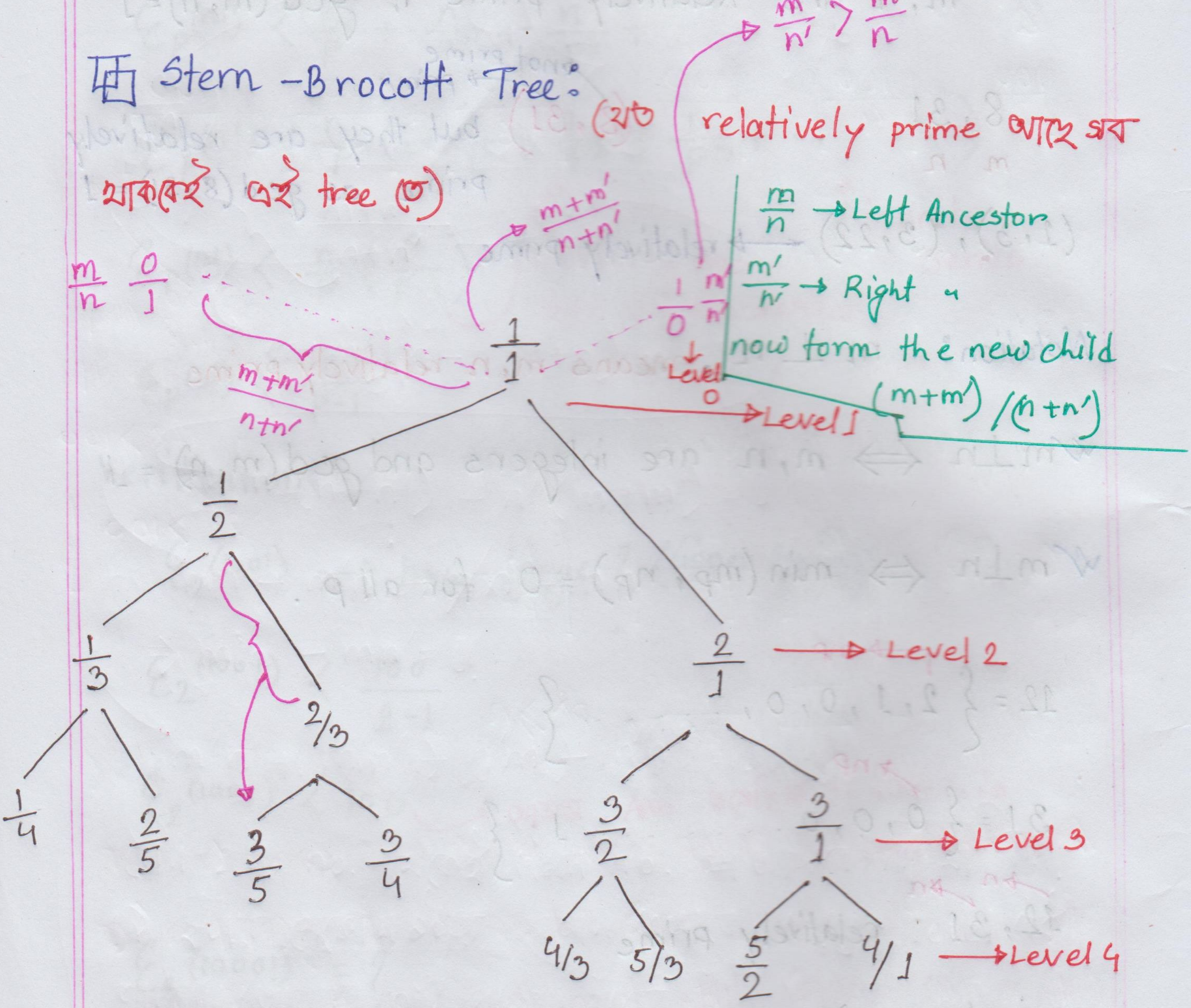
$\min(m_p, n_p) = 0$

$18 = \{1, 2, 0 \ldots\ldots 0 \ldots\ldots 0 \ldots\ldots\}$

$\min(m_p, n_p) \neq 0$, $\therefore$ 18, 12 not relatively prime.

$\frac{2}{9}, \frac{9}{2}$ (2,9) fraction relatively prime.

Stern-Brocott Tree: (যত relatively prime আছে সব থাকবেই এই tree তে)



//exam এ যেই level বলা থাকবে সেই level পর্যন্ত আঁকতে হবে

* এই tree তে কোন fraction কি ২ বার আসতে পারে?

↳ No.

verify:

$\frac{m}{n}$ $\qquad$ $\frac{m'}{n}$

Pick 2 consecutive no.

$\frac{m'}{n'}$ $\frac{1}{0}$ $\qquad$ $\frac{m}{n}$ $\frac{1}{1}$

$m'n - n'm = 1.1 - 1.0 = 1$ → always.

যেকোন ২টা consecutive no. always হবে এভাবে
right one is always $\frac{m'}{n'}$
left one ,, ,, $\frac{m}{n}$

$\frac{3}{2}\left(\frac{m}{n}\right)$ $\qquad$ $\frac{2}{1}\left(\frac{m'}{n'}\right)$ $\qquad$ $m'n - n'm = 4 - 3 = 1$

tree এর যেকোন ২ জায়গায় ১ই no. আসলে এই property violate হয়ে যাচ্ছে

Violate হয় নাকি check:

$\frac{m'}{n} / \frac{m}{n} = \frac{m+m'}{n+n'}$ কিনা

দেখালেই হবে

P.T.O.

$$\left(\frac{m'}{n'}\right)$$

$$\left(\frac{m}{n}\right)$$

$$\frac{m+m'}{n+n'}$$

always

$$(m+m')n - (n+n')m = 1$$

$$m'(n+n') - n'(m+m') = 1$$

$$\cancel{m}n + m'n - \cancel{m}n - mn'$$

$$\Rightarrow m'n - mn' = 1$$

$$m'n + \cancel{m'n'} - n'm - \cancel{m'n'} = 1$$

কোন 2টা no. থেকেই এমন no. আসবে না যা কিনা ঐ ২টার যেকোন ১টার সমান এভাবে প্রত্যেক level এই আলাদা আলাদা no. পাওয়া যাবে

(verified)

verification এর জন্য property.

Chapter - 5

Binomial co-efficient

$\binom{n}{k} \rightarrow {}^nC_k$ means — n choose k

— number of ways to choose k element subset from n-element set.

Ex:

$\overset{n}{4}$ choose $\overset{k}{2}$ {1, 2, 3, 4}

$\binom{4}{2} = ?$

{1, 2} {1, 3}, {1, 4}
{2, 1} {2, 3} {2, 4}
{3, 1} {3, 2} {3, 4}
{4, 1} {4, 2} {4, 3}

permutation নিলে

$${}^nP_r = n! / (n-r)!$$

permutation → 1, 2, 3 → n!
1, 3, 2
2, 1, 3
2, 3, 1 . . . .

distinct item হবে

{1, 2} {1, 3} {1, 4} {2, 3} {2, 4} {3, 4}

Ans : 6 দিতে হবে।

$${}^nC_r = n! / \{ r! * (n-r)! \}$$

total no of way

$$\frac{4\times 3}{2!} = \frac{12}{2} = 6$$

distinct no. of ways

* $\binom{5}{3} = ?$ $\xrightarrow{\text{Ans}}$ $\frac{5\times 4\times 3}{3!}$

* $\binom{4}{3} = \frac{4\times 3\times 2}{3!}$

* $\binom{4}{2} = \frac{4\times 3}{2!}$

* $\binom{4}{4} = \frac{4\times 3\times 2\times 1}{4!} = 1$

$$\boxed{\binom{n}{k} = \frac{n.(n-1)\cdot(n-2)\cdots(n-k+1)}{k!}} \text{ —— (I)}$$

* $\binom{5}{3} = \frac{5\times 4\times 3}{3!}$ → $3 = 5-3+1$ $(n-k+1)$

$$\boxed{\binom{n}{k} = \frac{n!}{k!\,(n-k)!}} \text{ —— (II)}$$

to show (i) = (ii) :

$$\underline{L.H.S} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)(n-k)(n-k-1)\cdots 1}{k!\,(n-k)!}$$

$$= \frac{n!}{k!\,(n-k)!} = \underline{R.H.S.}$$

* $\binom{n}{0} = 1$

// $\binom{n}{k}$ না হলে ~~n~~ $\binom{r}{k}$ বলা হবে generally in this chapter

r can be '+' / '−'

$$\binom{r}{k} = \begin{cases} \dfrac{r(r-1)(r-2)\cdots(r-k+1)}{k!} & ; \text{if } (k \geq 0) \\ 0 & ; \text{if } (k < 0) \end{cases}$$

upper index

lower index (which can't be less than 0) হলে value = 0 হবে।

[when k is '−ve']

$\binom{-1}{3} = ?$

$$\frac{(-1)\cdot(-2)\ \cdots\cdots\ (-1-3+1)}{3!}$$

$$= \frac{(-1)^3 \cdot 1 \cdot 2 \cdot 3}{1 \cdot 2 \cdot 3}$$

$$= (-1)^3$$

$$= -1$$

$\binom{-7}{3} = ?$

$(-7-3+1)$ i.e. -9 পর্যন্ত যাবে

$$\frac{(-7)(-8)\ (-9)}{3\times 2\times 1} = \frac{(-1)^3 \cdot 7 \cdot \cancel{8}^{4} \cdot \cancel{9}^{3}}{1 \cdot \cancel{2} \cdot \cancel{3}}$$

$$= -84$$

$\binom{r}{0} = 1$

$\binom{r}{2} = \frac{r(r-1)}{2\times 1} = \frac{r(r-1)}{2!}$

$\binom{r}{1} = r$

If lower inden ('+'ve) > upper inden

* Remember the coefficient in the Binomial Expansion formula??

value will be zero.

Ex:

$$\binom{1}{2} = 0 \qquad // \quad \frac{1 \cdot 0 \cdot \ldots}{2!} = 0$$

$\binom{r}{r} = 1$

Order of rules:

$\binom{r}{k}$ $= 0$ when $k<0$
$= 1$ when $k=0$
$= 0$ when $k>r$
$=$ apply formula (page 96) for other values of k

n	$\binom{n}{0}$	$\binom{n}{1}$	$\binom{n}{2}$	$\binom{n}{3}$	$\binom{n}{4}$	$\binom{n}{5}$	$\binom{n}{6}$	$\binom{n}{7}$
0	1							
1	1	1						
2	1	2	1					
3	1	3	3	1				
4	1	4	6	4	1			
5	1	5	10	10	5	1		
6	1	6	15	20	15	6	1	
7	1	7	21	35	35	21	7	1

ঘরে সবই 0 coz lower index > upper index

Symmetry identity:

$$\binom{n}{k} = \binom{n}{r-k}$$

$$\binom{4}{3} = \binom{4}{4-3} = \binom{4}{1}$$

$$\binom{7}{3} = \binom{7}{7-3} = \binom{7}{4}$$

can be verified from chart.

Mathematical proof:

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\text{L.H.S} = \binom{n}{k} = \frac{n}{k!\,(n-k)!}$$

$$= \frac{n!}{(n-(n-k))!\,(n-k)!}$$

consider as k then

$$= \binom{n}{n-k}$$

(If $r<0$ then) or

For negetive r, whether this symmetry identity holds?

⇒ Let $r = -1$

whether $\binom{-1}{k} = \binom{-1}{-1-k}$ holds?

if $k = 0$

L.H.S $= \binom{-1}{0} = 1$ R.H.S $= \binom{-1}{-1-0} = \binom{-1}{-1} = 0$

($\because$ lower index is negetive)

$\therefore$ L.H.S $\neq$ R.H.S.

if $k > 0$

$$\text{L.H.S} = \binom{-1}{k} = \frac{-1 \cdot -2 \cdot -3 \cdot \ldots \cdot (\cancel{-1} - k + \cancel{1})}{k!}$$

$$= \frac{-1 \cdot -2 \cdot -3 \cdot \ldots \cdot -k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}$$

$$= \frac{(-1)^k \cdot 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot k}$$

$$= (-1)^k = \pm 1$$

$$R.H.S = \binom{-1}{-1-K}$$

lower index is ~~more~~ negetive

$$\therefore \binom{-1}{-1-K} = 0 \quad \therefore L.H.S \neq R.H.S.$$

if $K<0$

$$L.H.S = \binom{-1}{K} = 0$$ // $\because K$ is '–'ve

[From here, we exploit the notation by letting $k = |k|$ (magnitude of k)

As k is actually negetive, $k = -k$]

$$R.H.S = \binom{-1}{-1-(-K)}$$

$$= \binom{-1}{-1+K}$$

$$= \frac{-1.-2.-3. \ldots \ldots (-1+1-k+1)}{(K-1)!}$$

$$= \frac{-1.-2.-3. \ldots \ldots -(K-1)}{1.2.3 \ldots \ldots (K-1)}$$

$$= \frac{(-1)^{K-1}.1.2.3 \ldots \ldots (K-1)}{1.2.3 \ldots \ldots (K-1)}$$

$= (-1)^{k-1}$

$= \pm 1$

L.H.S $\neq$ R.H.S

$\therefore$ Symmetry identity doesn't hold if n is negetive.

Absorption Identity:

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}$$

$$\binom{n}{k} = \binom{n}{n-k}$$

symmetric identity

$$\binom{5}{3} = \frac{5}{3} \times \binom{4}{2}$$

$= \frac{5}{3} \times 6$ from chart $\left[\binom{4}{2} = 6\right]$

$= 10$

☐ Prove or Show that:

$$(r-k)\binom{r}{k} = r\binom{r-1}{k}$$

$$L.H.S = (r-k)\binom{r}{k}$$

$$= (r-k)\binom{r}{r-k} \quad \text{[using symmetry identity]}$$

$$= \cancel{(r-k)}\,\frac{r}{\cancel{(r-k)}}\binom{r-1}{r-k-1} \quad \text{[using Absorption identity]}$$

$$= r\binom{r-1}{r-k-1}$$

$$= r\binom{r-1}{\cancel{r}-\cancel{1}-\cancel{r}+k+\cancel{1}} \quad \text{[using symmetry identity]}$$

$$= r\binom{r-1}{k}$$

$$= R.H.S$$

[Showed]

Addition Formula:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}$$

Ex.

$$\binom{7}{3} = \binom{6}{3} + \binom{6}{2}$$

$= 20 + 15$

$= 35$

$(7 \times 6 \times 5)/(1 \times 2 \times 3) = 35$

$6 \times 5 \times 4/(1 \times 2 \times 3) = 20$

$6 \times 5 /(1 \times 2) = 15$

$35 == 20 + 15$

Show that, the addition formula exists:

/* $\left[\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}\right.$ (absorption identity]

$\Rightarrow k\binom{r}{k} = r\binom{r-1}{k-1}\left.\right]$ */

from Absorption identity, we get following (i)

$$k\binom{r}{k} = r\binom{r-1}{k-1} \quad \text{———} \quad (i)$$

$$(r-k)\binom{r}{k} = r\binom{r-1}{k} \quad \text{———} \quad (ii)$$

[From previous proof]

① + ⑪ ⇒

$$k\binom{r}{k} + (r-k)\binom{r}{k} = r\binom{r-1}{k-1} + r\binom{r-1}{k}$$

$$\Rightarrow \cancel{k\binom{r}{k}} + r\binom{r}{k} - \cancel{k\binom{r}{k}} = r\binom{r-1}{k-1} + r\binom{r-1}{k}$$

$$\Rightarrow r\binom{r}{k} = r\binom{r-1}{k} + r\binom{r-1}{k-1}$$

$$\therefore \binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1},$$

So, the addition formula exists.

⊞ Binomial no. এর summation এর closed form: —

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$ // addition formula

$$= \binom{4}{3} + \binom{3}{2} + \binom{3}{1}$$

[বড়টা fixed রেখে ছোটটা expand (−) আমলে আমলাম]

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{2}{0}$$

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0} + \binom{1}{-1}^{*}$$

↳ this term = 0

$$= \binom{4}{3} + \binom{3}{2} + \binom{2}{1} + \binom{1}{0}$$

$$\sum_{k=0}^{n} \binom{r+k}{k}$$ ***where n=3***

Here, $r = 1$

$K = 0$

$$\binom{1+0}{0} + \binom{2}{1} + \binom{3}{2} + \binom{4}{3} = \binom{r+n+1}{n} = \binom{1+3+1}{3}$$

n=3

$$= \binom{5}{3}$$

* এই summation এর result $\binom{5}{3}$

$$\boxed{\therefore \sum_{k \leq n} \binom{r+k}{k} = \binom{r+n+1}{n}}$$

Ex:

$$\sum_{k \leq 6} \binom{3+k}{k} = \binom{r+n+1}{n}$$

$$= \binom{3+6+1}{6} = \binom{10}{6}$$

এবার ছোটটা fixed রেখে বড়টা ভাঙবো:

$$\binom{5}{3} = \binom{4}{3} + \binom{4}{2}$$

$$= \binom{3}{3} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{2}{3} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

[actually এখানেই 0 হয় তবু 3 expand করছি]

$$= \underbrace{\binom{1}{3}}_{0} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

$$= \binom{0}{3} + \binom{0}{2} + \binom{1}{2} + \binom{2}{2} + \binom{3}{2} + \binom{4}{2}$$

m=2
n=4

$$= \sum_{0 \leq k \leq n} \binom{k}{m}$$

//m = constant
here m = 2

এই series এর result $\binom{5}{3}$

So closed form is

$$\boxed{\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}} *$$

m=2
n=4
So: n + 1 = 5
m + 1 = 3

Thus: $\binom{5}{3}$

***** Must Solve Maths using these formula from file: "Practise Question 03.docx" ******

Proof that,

instead of r

here $m+k \geqslant 0$

$$\sum_{k \leqslant n} \binom{m+k}{k} = \binom{m+n+1}{n}$$

for m+k >= 0

Here, $m+k \geqslant 0$

$k \geqslant -m$ //k এর lower limit derive করে নিলাম

L.H.S =

$$\therefore \sum_{-m \leqslant k \leqslant n} \binom{m+k}{k}$$

$$\Rightarrow \sum_{-m \leqslant k \leqslant n} \binom{m+k}{m+k-k}$$ [using symmetry identity]

$$\Rightarrow \sum_{-m \leqslant k \leqslant n} \binom{m+k}{m}$$

Replace k by k−m

$$\Rightarrow \sum_{-m \leqslant k-m \leqslant n} \binom{m+k-m}{m}$$

$-m \leqslant k - m$

$0 \leqslant k$

$k-m \leqslant n$

$k \leqslant m+n$

$$= \sum_{0 \leqslant k \leqslant (m+n)} \binom{k}{m}$$

$$* = \binom{m+n+1}{m+1}$$ [using * from previous page]

** write here the formula from previous page*

$$= \binom{m+n+1}{m+n+1-m-1} \quad \text{[using symmetry identity]}$$

$$= \binom{m+n+1}{n}$$

$$= R.H.S.$$

[verified]

Show that,

$$\binom{r}{m}\binom{m}{k} = \binom{r}{k}\binom{r-k}{m-k}$$

L.H.S :

$$\binom{r}{m}\binom{m}{k} = \frac{r!}{\cancel{m!}\,(r-m)!} \times \frac{\cancel{m!}}{k!\,(m-k)!}$$

$$= \frac{r!}{(r-m)!\,(m-k)!\,k!}$$

$$= \frac{r!}{k!\,(r-k)!} \times \frac{(r-k)!}{(m-k)!\,(r-m)!}$$

$$= \binom{r}{k}\left(\frac{(r-k)!}{(m-k)!\,(r-k-m+k)}\right)$$

$$= \binom{r}{k}\binom{r-k}{m-k}$$

$= R.H.S.$

Find a closed form for:

$$\sum_{k=0}^{m} \binom{m}{k} \Big/ \binom{n}{k} \text{ ——— (I)}$$

Always, n >= m

This Derivation: Excluded from Syllabus

*** But Application (Math): Included in Syllabus ***

We know,

$$\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$$

$$\Rightarrow \binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k} \text{ [replace n by n]}$$

divide both side by $\binom{n}{k}\binom{n}{m}$,

$$\Rightarrow \frac{\cancel{\binom{n}{m}}\binom{m}{k}}{\binom{n}{k}\cancel{\binom{n}{m}}} = \frac{\cancel{\binom{n}{k}}\binom{n-k}{m-k}}{\cancel{\binom{n}{k}}\binom{n}{m}}$$

$$\Rightarrow \binom{m}{k} \Big/ \binom{n}{k} = \binom{n-k}{m-k} \Big/ \binom{n}{m}$$

∵ k নাই তাই const ধরা যাবে

Now eqn (I) becomes,

$$\sum_{k=0}^{m} \binom{n-k}{m-k} \Big/ \binom{n}{m} \text{ ——— (II)}$$

Now,

$$\sum_{k\geq 0}\binom{n-k}{m-k}$$

[upper limit (m) না লিখলে prob নাই। ∵ m এর উপরে k এর value এমনিতেই যাবে না (−) হয়ে 0 হয়ে যাবে বলে]

replace k by $(m-k)$

We get,

$$\sum_{m-k\geq 0}\binom{n-m+k}{m-m+k}$$

$m-k \geq 0$

$\Rightarrow m \geq k$

$$=\sum_{k\leq m \to n}\binom{\overbrace{n-m}^{r}+k}{k}$$

$$=\binom{n-m+m+1}{m}$$

$$\left[\sum_{k\leq n}\binom{r+k}{k} = \binom{r+n+1}{n}\right]$$

$$=\binom{n+1}{m}$$

Now eq^n (ii) becomes,

$$\binom{n+1}{m}\Big/\binom{n}{m}$$

see after absorption identity

$$(r-k)\binom{r}{k}=r\binom{r-1}{k}$$

$$\Rightarrow \binom{r-1}{k}\Big/\binom{r}{k}=\frac{r-k}{r}$$

$$\frac{(n+1)!}{m! * (n+1-m)!} \Big/ \frac{n!}{m! * (n-m)!} = \frac{n+1}{n+1-m}$$

$$= \frac{n+1}{n+1-m}$$

$$\binom{r}{k} \Big/ \binom{r-1}{k} = \frac{r}{r-k}$$

$$\therefore \boxed{\sum_{k=0}^{m} \binom{m}{k} \Big/ \binom{n}{k} = \frac{n+1}{n+1-m}}$$

(closed form proof করে then এটা value বের করতে হতে পারে) এখন

Ex:

$$\sum_{k=0}^{10} \binom{3}{k} \Big/ \binom{4}{k} = ?$$

Think: **How to solve if *k* starts from 1 or 2 or 3, instead of 0**
(may see the practise question doc file)

$$= \frac{4+1}{4+1-3}$$

$$= \frac{5}{2}$$

[Ans]

\# $\binom{0}{1} + \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots + \binom{n}{1}$

$$0 + 1 + 2 + 3 + 4 + \cdots + n = \frac{n(n+1)}{2}$$

resembles with,

$$\sum_{0 \leq k \leq n} \binom{k}{m} = \binom{n+1}{m+1}$$

So,

$$\binom{0}{1} + \binom{1}{1} + \binom{2}{1} + \binom{3}{1} + \cdots\cdots \binom{n}{1} = \binom{n+1}{1+1}$$

$$= \binom{n+1}{2}$$

$$= \frac{(n+1)n}{2!}$$

$$\left[\because \binom{r}{k} = \frac{r.(r-1).(r-2)\ldots(r-k+1)}{k!} \right]$$

Probability Theory

/* ১টা coin throw করলে $S = \{H, T\}$ → outcome

Set যাকে sample space বলে

dice throw করলে $S = \{\underbrace{1, 2, 3, 4, 5, 6}_{\text{outcome}}\}$ ← sample space

probability (head),

$P(H) = \frac{1}{2}$

Sample Space S = Set of All Possible Outcomes
Event E = Set of Some Outcomes = Subset of S
We have to find Probabilit of Event E
= P(E) = size of Set E divided by size of Set S

dice এ জোড় সংখ্যা পড়ার probability $= \frac{3}{6} = \frac{1}{2}$

Event of Getting Even number
Even of Fetting Greater than 2

2 > পড়ার " $= 4/6$

2টা coin toss করলে ~~no. of events~~ *Sample space (=all possible outcomes):*

$\{(H,H), (H,T), (T,H), (T,T)\}$

$2 \times 2 = 4$ টা ~~event~~ *Outcome in Sample Space*

In 2 coins throw, 1st time H পড়ার Probability $2/4$

S = {(H,H), (H,T), (T,H), (T,T)}
E = {(H,H), (H,T)}
P(E) = |E| / |S| = 2 / 4

Fair coin:

H/T পড়বে কিনা তার সম্ভাবনা even (unbiased) হলে।

Unfair coin:

সম্ভাবনা biased হলে

Example:

Biased Coin => 2টা H পড়লে 1টা tale এমন হলে then sample space {H, [redacted] T}

$P(H) = \frac{2}{3}$

$P(T) = 1/3$

2টি dice throw করলে:

$$
S = \left\{ \begin{array}{llll}
(1,1), & (1,2) & \ldots\ldots & (1,6) \\
(2,1), & \ldots & \ldots & \ldots \\
(3,1) & \ldots & \ldots & \ldots \\
(4,1) & \ldots & \ldots & \ldots \\
(5,1) & \ldots & \ldots & \ldots \\
(6,1) & \ldots & \ldots & (6,6)
\end{array} \right\}
$$

*

Event A = The first Die is TWO
Event B = Both the Dice are EVEN
(check grammar: singular = Die, plural = Dice)

T: 36 events

শুধু ১মটা 2 পড়ার সম্ভাবনা $\frac{6}{36} = \frac{1}{6}$

P(A) =
P{ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6) }
= 6/36 = 1/6

২টা even " "

22 42
24 44
26 62
46 64
66

$\frac{9}{36}$

P (B)= P{both even}
= {(2,2), (2,4), (2,6), (4,2), (4,4), (4,6), (6,2), (6,4), (6,6) }
= 9/36
= 1/4

১ম dice > 3 & 2nd dice < 2

$p = \frac{3}{36}$ *

*

Event E = 1st die > 3, 2nd die <2
= {(4,1), (5,1), (6,1)}

Sample Space S
= {...write 36 outcomes...}

Set operation: ***E*** and ***F***: events

$E = \{H\}$ $F = \{T\}$

E OR F = $E \cup F = \{H, T\}$

E AND F = $E \cap F = \{\emptyset\}$
or EF

$E = \{1, 3, 5\}$

$F = \{1, 3, 6\}$

E OR F = $E \cup F = \{1, 3, 5, 6\}$

E AND F = * $EF / E \cap F = \{1, 3\}$

means intersection

***** Event = EF = E AND F**
==> AND / intersection hobe

//dice এর ক্ষেত্রে :

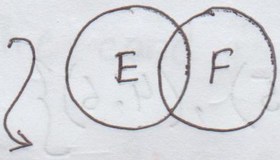
$$S = \{1, 2, 3, 4, 5, 6\}$$

$$P(2) = \frac{1}{6} \; \ldots$$

$$P(1) = \frac{1}{6}$$

$$P(S) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = 1$$

$\boxed{\therefore P(S) = 1}$ ← always



E, F sample space

Both E and F are Sets

Any Set or Operations on Set can be viewed as *Venn Diagram*

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= P(E) + P(F) - P(EF)$$

$P(E \cup F) = P(E) + P(F)$ if E and F are disjoint Sets (Mutually Exclusive events)

☐ Conditional Probability:

\# 2 dice thrown. What is the probability that the sum of two dice will be six, given that first dice is a 4.

Solⁿ:

E = the sum of two dice is 6 = ?

F = the first one is 4. (given)

$$P(\underset{\text{what?}}{E} \mid \underset{\text{given}}{F}) = \frac{P(EF)}{P(F)}$$

P(E|F): Probability of E Given F (i.e., it is given that F has already happened!)
P(EF)=P(E AND F)=> intersection use korba

E ={ (1,5), (2,4), (3,3), (4,2), (5,1) }

F = { (4,1) (4,2) (4,3), (4,4), (4,5), (4,6) } → ঘটে গেছে

F

So, EF = E AND F = E intersect F = {(4,2)}

{ (4,2) } ⟶ desired o/p

EF

// ২টা ঘটনা dependent হলে (F ঘটে গেছে) then E এর ঘটার probability,

$$P(E \mid F) = \frac{P(EF)}{P(F)}$$

P(EF) ← E, F একসাথে ঘটার সম্ভাবনা

P(F) → শুধু F ঘটার সম্ভাবনা

$P(EF) = \frac{1}{36}$ P(EF) = P{ (4,2) } = 1/36

$P(F) = \frac{6}{36}$ P(F) = P { (4,1), (4,2), ..., (4,6) } = 6/36 = 1/6

$$\therefore P(E|F) = \frac{P(EF)}{P(F)} = \frac{1/36}{6/36} = \frac{1}{6}$$

Book example 1.4 >>

\# 1, 2, ... 10 (1-10টা card দেয়া আছে)

What is the probability that a card drawn is a 10, given that it is atleast five.

Solnº

F = the card is atleast five (given)

E = The card is 10 (what?)

$F = \{5, 6, 7, 8, 9, 10\}$ **P(F) = 6/10**

$E = \{10\}$ **P(E)= 1/10** EF = E AND F = E (intersect) F = {10} **so, P(EF)=1/10**

$$P(E|F) = \frac{P(EF)}{P(F)} = \frac{1/10}{6/10}$$

$$= \frac{1}{6}$$

Book example 1.5 >>

Given that

১টি family তে ২টি child. At least one of them is a boy. (~~given~~) → Both of them boys

Find probability that

E

Soln:

F = atleast 1 of them is a boy

$= \{(b,g), (g,b), (b,b)\}$

$E = \{(b,b)\}$

EF = E (intersect) F
= {(b,b)}

$S = \{bb, bg, gb, gg\}$

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$= \frac{1/4}{3/4} = \frac{1}{3} \text{ [Ans]}$$

Book example 1.6 >>

Bev can take computers or chemistry.

computer ⇒ probability of getting A $\frac{1}{2}$

chemistry ⇒ " " " A $\frac{1}{3}$

Bev takes her desicion based on the flip of a Fair coin.

What is the probability that Bev will get

A in chemistry ? // কোথাও বলা নাই chem নিয়ে নিয়েছে

Actually, You have to find P {Bev will get A in Chemistry}
= P {Bev takes Chemistry AND gets A in it}
= P (FE) = P (EF)

Sol^n:

F = Bev takes chemistry

E = " gets A in chemistry

$$P(E|F) = \frac{P(EF)}{P(F)}$$

what?

EF = Bev gets A after taking chemistry

$$\Rightarrow P(EF) = P(E|F) * P(F)$$

$$= \frac{1}{3} * \frac{1}{2}$$

(∵ coin এর উপর decision নিচ্ছে so probability $\frac{1}{2}$)

$$= \frac{1}{6}$$

$\boxed{1 \cdot 7}$

5W
7B

□ Draw 2 ball without replacement. Both balls drawn are black ?

What is the probabilty that Both ...

E: First Ball drawn is Black
F: Second ball drawn is Black

// $P(EF) = P(E) * P(F)$ when E,F independent

$P(EF) = P(E) * P(F|E)$ - OR - $P(F)*P(E|F)$ when E,F are NOT independent / dependant

// $P(E|F) = \frac{P(EF)}{P(F)}$ " " " dep ?

F = First one is black // and বললে '$\cap$' এবং

E = 2nd one is black OR বললে '$\cup$' চিন্তা করতে হবে

$$P(E|F) = \frac{P(EF)}{P(F)}$$

$$\therefore P(EF) = P(E|F) * P(E)$$

$$= \frac{6}{11} * \frac{7}{12}$$

$$= \frac{42}{132}$$

Another way : (using more text/explanation)

P (Both Balls Black)

= P (1st Ball Black AND 2nd Ball Black)

= P (1st Ball Black) * P(2nd Ball Black | 1st Ball already Black)

= 7/12 * 6/11

= 42 / 132

What is the probability that no one ...

Three man throw their hat. No one of them gets his own hat.

Soln:

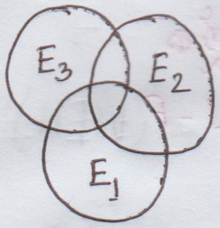
E_1 = First man gets his own hat

E_2 = 2nd " " " " "

E_3 = 3rd " " " " "

$P(E_1 \cup E_2 \cup E_3)$ = কমপক্ষে ১জন hat খুঁজে পেয়েছে

$\therefore \underline{1 - P(E_1 \cup E_2 \cup E_3)}$ → ১জনও খুঁজে পায়নি এমন সম্ভাবনা



1. What is the probability All THREE get their correct hats? => $P(E_1E_2E_3)$

2. Solve both problems for FOUR persons also ...

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1E_2) - P(E_2E_3) - P(E_1E_3) + P(E_1E_2E_3)$$

$$= \frac{1}{3} + \frac{1}{3} + \frac{1}{3} - \frac{1}{6} - \frac{1}{6} - \frac{1}{6} + \frac{1}{6}$$

$$= \frac{2}{3} \quad \cancel{\boxed{\text{Ans}}}$$

Answer: = 1 - 2/3 = 1/3

/* P(E1) = 1/3 P(E2) = 1/3 P(E3) = 1/3

$$P(E_1E_2) = P(E_1) * P(E_2 | E_1)$$

$$= \frac{1}{3} * \frac{1}{2} = \frac{1}{6}$$

To understand,
VISUALIZE this scenario!!!

P(E2|E1)
= P(2nd person gets his hat | 1st person already got his hat)

$$P(E_1E_2E_3) = P(E_1E_2) * P(E_3 | (E_1E_2))$$

$$= \frac{1}{6} * 1 = \frac{1}{6}$$

∵ E_1, E_2 নিজের নিজের hat ঠিক ঠিক মত নিয়ে গেছে।
∴ E_3 এর hat পাওয়ার probability 100%. (only his hat is remained!)

* Also, Solve the previous Hat problem for FOUR persons.
=> see class lecture for four persons solution

* For both Three hat/ Four Hat problem, also find =>
Probability that at least 1 person finds his hat and P(at least 2 persons find hats)
Probability (EXACTLY 2 persons ...) Probability that All of them find their hats

Baye's Formula:

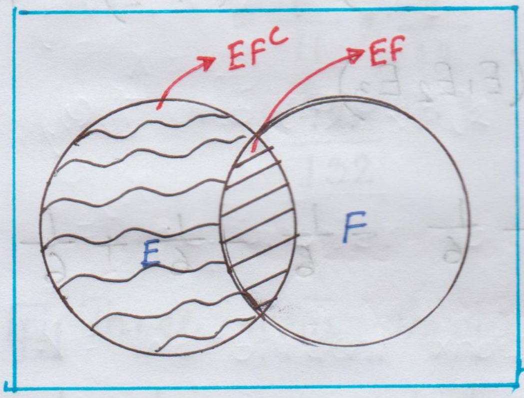
$$E = EF \cup EF^c$$

$F^c \rightarrow F$ complement আসে- F বাদে সবটুকু

$F^c = 1 - F$

$$\Rightarrow P(E) = P(EF) + P(EF^c)$$

$$\boxed{P(E) = P(E|F)\,P(F) + P(E|F^c)\,P(F^c)}$$



// $P(E|F) = \frac{P(EF)}{P(F)}$

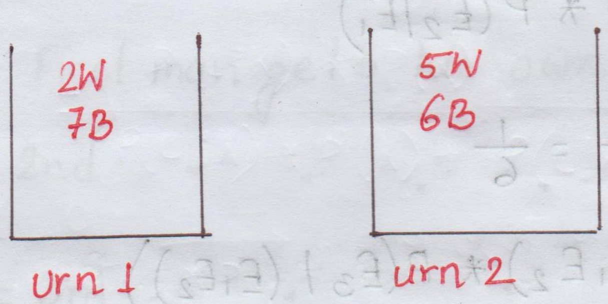
// $P(F|E) = \frac{P(FE)}{P(E)}$

// $P(EF) = P(E/F) * P(F)$

$= P(F/E) * P(E)$

Toss Coin ... Choose One of the Urns Based on Head or Tail

#



head $\rightarrow$ 1st urn
tail $\rightarrow$ 2nd "

What is the probability that Outcome of the toss is head. given a white ball is selected

P(White | Head) = EASY!
P(Head | White) = Tough!

So Convert the Tough One into Easy one by Bayes formula !

W = White ball is selected

H = outcome of coin toss is head = ?

$$P(H|W) = \frac{P(HW)}{P(W)}$$

[P(H|W) * P(W) লিখলে হবেনা ∵ P(H|W) = ?]

karon, er Ulta chinta korata difficult, P(H|W) chinta korao jai na, but opposite ta P(W|H) super easy to think

Converted to Easy One

$$= \frac{P(W|H) * P(H)}{P(W|H)\,P(H) + P(W|H^c)\,P(H^c)}$$

[according to baye's formula]

$$= \frac{\frac{2}{9} * \frac{1}{2}}{\frac{2}{9} * \frac{1}{2} + \frac{5}{11} * \frac{1}{2}}$$

[∵ H^c means tail occured]

$$= \frac{22}{67}$$ [Ans]

P.T.O.

$\boxed{1.13}$

\# P → knows answer

1-P → gueses

m = number of multiple choice question.

What is the Probability that the student knows the answer given that the answer is correct,

(জেনে ans দেয়া হয়েছে এর P = ?)

C = the answer is correct

K = the student knowns the answer

$$P(K|C) = \frac{P(KC)}{P(C)}$$

P(K|C) = Tough !
P(C|K) = EASY = 1.0
convert the tough one into easy by Bayes formula !!!

$$= \frac{P(C|K) * P(K)}{P(C|K) * P(K) + P(C|K^c) * P(K^c)}$$

∵ ans জেনে correct ans. দেয়ার probability **= 1.0**

$$= \frac{1 * P}{1 * P + \frac{1}{m} * (1-P)}$$

ans Na jeneo correct korar probability = 1/m (jehetu there are m options in the MCQ)

$$= \frac{P}{P + \frac{1}{m}(1-P)} = \frac{mP}{1+(m-1)P}$$ *

Exercise: 18, 19, 21, 31, ~~38~~, 36, 44

*[এখানে m & P এর Value বলা থাকলে এখান thk $\underset{4}{m}$, $\underset{0.4}{P}$
P(K|C) বের করা যাবে)

[বাকী 4% হয়তো detect ই করতে পারছি না]

Laboratory blood test is 95% effective in detecting a disease, 1% false positive result, 0.5% of population has the disease.

What is the Probability that A person has disease, given the result is positive.

$P(E \mid D) = P(\text{positive} \mid \text{disease})$
$= 0.95$

$P(E \mid D^c) = P(\text{positive} \mid \text{No Disease})$
$= P(\text{false positive}) = 0.01$

$P(D) = P(\text{disease}) = 0.005$

$P(D^c) = 1 - P(D) = 0.995$

E = the result is positive

D = A person has disease

$$P(D|E) = \frac{P(DE)}{P(E)}$$

// $\frac{95}{100} = 0.95$

// $\frac{0.5}{100} = 0.005$

$$= \frac{P(E|D) * P(D)}{P(E|D) * P(D) + P(E|D^c) * P(D^c)}$$

// $D^c = 99.5\%$
$= 0.995$

using baye's formula

$$= \frac{0.95 * 0.005}{0.95 * 0.005 + \cancel{0.001}\,0.01 * 0.995}$$

// $E|D^c = \frac{1}{100}$
$= \cancel{0.001}$
$= 0.01$

$= 0.323$ [Ans]

$\boxplus$ Chapter → 2

This chapter best Explained in the TextBook (my Soft Copy Book) (Ross - Chapter 2) see my pdf version (I added many comments + text in Ross ch 2 to guide you. Also: the example problems are better explained in the Book soft copy (I added some comments) + some example r also extended by me *** Must see the textbook (with my added comments in the Dropbox folder soft copy book)

$\boxplus$ Random variables:

Example 2.1:

X is a random variable **which is**

Sum of two dice

$$
\begin{cases}
P(X=2) = \frac{1}{36} \\
P(X=4) = \frac{3}{36} \\
\vdots \\
P(X=12) = \frac{1}{36}
\end{cases}
$$

Ex 2.2:

random variable Y = Number of head **when Tossing TWO coins**

$$P\{Y=0\} = \frac{1}{4}$$

$$P\{Y=1\} = \frac{2}{4}$$

$$P\{Y=2\} = \frac{1}{4}$$

Ex 2·3:

heap = P, tail = 1-P

কয়বার toss করলে head পেতে পারি

1st head পাওয়ার জন্য কয়বার toss = ধরি সেই random variable : N.

$P\{N=1\} = P\{H\} = P$

$P\{N=2\} = P\{T, H\} = (1-P) \cdot P$

N : Geometric Random Variable

$N=4 = (1-P)(1-P)(1-P) \cdot P$

$= (1-P)^3 P$

[Example 2·4 এর আগের proof]

See: 2.2.2

Binomial Random Variable:

$n \rightarrow$ number of trial

$P \rightarrow$ success

$1-P \rightarrow$ failure.

X is Binomial RV (Random Variables) that stores the number of success in n.

no. of trials

trials with parameter (n, p)

success হওয়ার probability.

$n = 3$, success $= 2$

হলে $p.p.(1-p) = p^2(1-p)$

trial $= n$, number of success i হলে,

probability,

যতভাবে achieve করা যায়

$$\binom{n}{i} p^i . (1-p)^{n-i}$$

$(n-i)$ বার failure.

i বার success

$\therefore$ এই probability $\binom{n}{i}$ way তে হতে পারে।

n সংখ্যক trial থেকে i সংখ্যক success

$$\therefore \boxed{P(i) = \binom{n}{i} p^i . (1-p)^{n-i}} *$$

Geometric random variable এ ১০ বার throw করে first time /1 time only H পাওয়ার probability বের করা হচ্ছে। ১ বার H পেলেই শেষ but binomial rv total no. of success বের করি।

Ex: 2.6

[১টা coin 4 বার throw করা যা mean করে
4 " " ≠1 " " " " " সেই " " 1]

2h 2T — ?

H − $\frac{1}{2}$ ⟶ P(success)

T − $\frac{1}{2}$ ⟶ (1−P) (failure)

$$P(2) = \binom{4}{2} \cdot \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2$$

[১টা coin 4 বার throw করলে 2টা H পাওয়ার P]

Ex: 2.7

defective item produce করে এই $P = 0.1$

∴ ঠিক " " " " $P = 0.9$

3 item produced, atmost ১টা defective

at most

$P\{X=0\} + \{X=1\}$

(∵ দুটোই non defective হতে পারে)

$P(0) = \binom{3}{0} \cdot (0.1)^0 (0.9)^3 + \binom{3}{1} (0.1)^1 (0.9)^2$
+P(1)

(i defective হবে এমন = P(i))

যদি 6টা item থাকে ~~atleast~~ at most 3 defective = P(0) + P(1) + P(2) + P(3)

= 1 - P(4) + P(5) + P(6) . . . . . [6টা random]

Ex: 2·8

engine fail : 1−P

fail করে না : P

atleast 50% engine ঠিক থাকতে হবে। P এর কোন value – এর জন্য ১টা প্লেনের ২ এর বদলে 4টা engine নিয়ে উড়বে? For which value of p, a 4-engine plane will be more relaibe than a 2-engine plane ?

যদি 4টা engine থাকে then success =

$P\{X=2\} + P\{X=3\} + P\{X=4\}$

$\binom{4}{2}(P)^2(1-P)^2 + \binom{4}{3}(P)^3(1-P)^1 + \binom{4}{4}(P)^4(1-P)^0$

যদি 2টা engine থাকে then successful flight,

$$P\{Y=1\}+P\{Y=2\}$$

$$\binom{2}{1}p^1(1-p)^1+\binom{2}{2}p^2(1-p)^0$$

→ 4টা engine এর জন্য ≥ 2টা engine এর জন্য হলে 4টা engine নিয়ে fly করা safe.

$p=\frac{2}{3}$ [from book]

∴ fail করার probability $=\left(1-\frac{2}{3}\right)=\frac{1}{3}$

Section 2.4

⊞ Expectation of a R.V.:

Expectation

on avg dice এ কি পাওয়া যেতে পারে / on avg দোকানে কতজন লোক আসছে এটাই expectation.

equation:

$$E[X]=\sum_{p(x)>0} x\,p(x)$$

outcome

Dice $\rightarrow X \rightarrow \{1, 2, 3, 4, 5, 6\}$

$$\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$$

$$E[X] = 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6} = \frac{7}{2}$$

এটাই হল expected or avg value যা কিনা হতে পারে unlike probability.

$H \leftarrow 1$, $T \leftarrow 2$ (coin throw করলে)

$\{1, 2\}$

$$E(X) = 1 \times \frac{1}{2} + 2 \times \frac{1}{2} = 1.5$$

only জানতে হবে expectation, not math.

Chapter – 8

Queuing Theory

L → Average number of customer in a system

LQ → " " " " " " waiting queue.

W → " amount of time a customer spends in System

WQ → " " " " " " " Queue.

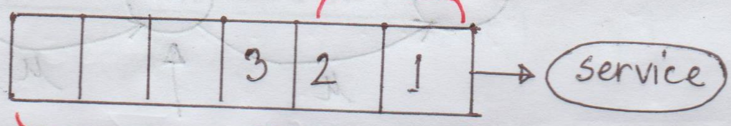
λ_a = avg arrival rate in time t

$$\lambda_a = \frac{N(t)}{t} = \frac{\text{number of customer entered}}{\text{in certain ammount of time}}$$

$$\boxed{L = \lambda_a W}$$

arrival rate.

customer



Queue (infinite size)

//service time + waiting time = system spend time

1 customer / 1 sec =

P.T.O

1 customer / 1 sec =

sec → 1 2 3 4 5 6 7

per → ~~1~~ ~~2~~ 3 4 5 6 7

5

$\lambda_a = 1$, $W = 5$,

$L = 1 \times 5$

State Diagram:

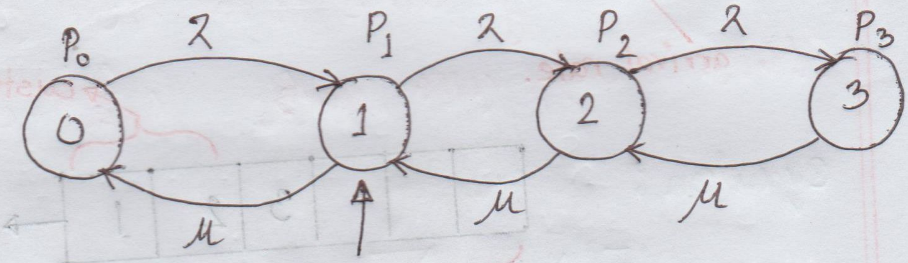
P_0 → The proportion af time the system has 0 customer

:

P_n → " " " " n "

λ = arrival rate

μ = service "



Comparing with $L = \lambda_a W$

$P_1 = \lambda P_0$ ← যেই rate ঢুকলাম

μP_1 ← যেই rate এ বের হলাম

arrival rate & leaving rate সমান হলে,

$$\lambda P_0 = \mu P_1 \quad \text{——— (i)}$$

$$P_1 = \frac{\lambda}{\mu} P_0 \quad \text{——— (ii)}$$

$\lambda P_0 + \mu P_2 \leftarrow$ state 1 এ আসার way

$\mu P_1 + \lambda P_1 \leftarrow$ " " এ বের হওয়ার rate

$$\lambda P_0 + \mu P_2 = \mu P_1 + \lambda P_1$$

$$\Rightarrow (\lambda + \mu) P_1 = \lambda P_0 + \mu P_2$$

$\vdots$

$$(\lambda + \mu) P_2 = \lambda P_1 + \mu P_3$$

$\vdots$

$$(\lambda + \mu) P_n = \lambda P_{n-1} + \mu P_{n+1}$$

$$\Rightarrow P_{n+1} = \frac{(\lambda + \mu) P_n - \lambda P_{n-1}}{\mu}$$

$$= \frac{\lambda P_n + \mu P_n - \lambda P_{n-1}}{\mu}$$

$$= \frac{\lambda}{\mu} P_n + P_n - \frac{\lambda}{\mu} P_{n-1}$$

$$\boxed{P_{n+1} = \frac{\lambda}{\mu} P_n + \left(P_n - \frac{\lambda}{\mu} P_{n-1}\right)}$$

$$P_2 = \frac{\lambda}{\mu} P_1 + \left(P_1 - \frac{\lambda}{\mu} P_0\right)$$

$$= \frac{\lambda}{\mu} \cdot \frac{\lambda}{\mu} P_0 + (P_1 - P_1) \quad [\text{from (ii)}]$$

$$= \left(\frac{\lambda}{\mu}\right)^2 P_0$$

$$P_3 = \frac{\lambda}{\mu} P_2 + \left(P_2 - \frac{\lambda}{\mu} P_1\right)$$

$$P_3 = \left(\frac{\lambda}{\mu}\right)^3 P_0$$

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n P_0}$$

$P_0 \quad P_1 \quad P_2 \ldots \ldots \ldots \ldots \ldots P_n$

$$\sum_{n=0}^{\infty} P_n = 1$$ (যদি proportion of time কে probability consider করি)

$$\Rightarrow \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_0 = P_0 \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n$$

Now, $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

So, $P_0 \left(\frac{1}{1-\frac{\lambda}{\mu}} \right) = 1$; here, $x = \frac{\lambda}{\mu}$

$\Rightarrow P_0 = 1 - \frac{\lambda}{\mu}$

$\therefore P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

$$\boxed{P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)}$$

Ex:

$P_5 = ?$ λ, μ given

// $E[x] = \sum x P_x$

$L = \sum_{n=0}^{\infty} n P_n$

$= \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

$= \left(1 - \frac{\lambda}{\mu}\right) \sum_{n=0}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n$

$$\boxed{\sum_{n=0}^{\infty} n x^n = \frac{x}{(1-x)^2}}$$

$$= \left(1-\frac{\lambda}{\mu}\right)\left[\frac{\lambda/\mu}{\left(1-\frac{\lambda}{\mu}\right)^2}\right]$$

$$= \frac{\frac{\lambda}{\mu}}{\left(1-\frac{\lambda}{\mu}\right)} = \frac{\lambda}{\mu}\left[\frac{1}{\frac{\mu-\lambda}{\mu}}\right]$$

$$= \frac{\lambda}{\mu}\left[\frac{\mu}{\mu-\lambda}\right]$$

$$\boxed{L = \frac{\lambda}{\mu-\lambda}}$$

Again, $L = \lambda W$

$$\therefore W = \frac{L}{\lambda}$$

$$= \frac{\lambda}{\mu-\lambda} \times \frac{1}{\lambda}$$

$$\boxed{W = \frac{1}{\mu-\lambda}}$$

$W_Q = W -$ Average service time.

$$= \frac{1}{\mu-\lambda} - \frac{1}{\mu}$$

[$\because$ μ total service rate

$\mu = 8/\text{sec}$

$\frac{1}{\mu} = \frac{1}{8}$]

[Chapter-2 theke binomial theory type math থাকবে]

$$\boxed{W_Q = \frac{\lambda}{\mu(\mu-\lambda)}}$$

$L = \lambda_a W$

$L_Q = \lambda_a W_Q$

$= \lambda . W_Q$

$= \lambda . \frac{\lambda}{\mu(\mu-\lambda)}$

$$\boxed{L_Q = \frac{\lambda^2}{\mu(\mu-\lambda)}}$$

Ex:

System এ 12/sec rate এ ঢুকে, 5/sec এ বের হয়।

$P_7 = ?$ $L, L_Q, W, W_Q = ?$

Application:

Network এর মধ্যে Queuing theory imp. কত data আসবে, যাবে, wait করবে, device এ কি পরিমাণ load handle করতে পারবে etc. . . . এভাবে device design.

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Ex 8.2:

$$\lambda = \frac{1}{12}, \quad \mu = \frac{1}{8}$$

08/01/2016