Markov Chains

The Gambler's Ruin Problem: Consider a gambler who at each play of the game has probability p of winning one unit and probability $q = 1 - p$ of losing one unit. Assuming that successive plays of the game are independent, what is the probability that starting with i units, the gambler's fortune will reach N before reaching 0?

If we let X_n denote the player's fortune at time n, then the process $\{X_n, n = 0, 1, 2, \cdots\}$ is a Markov chain with transition probabilities

$$P_{00} = P_{NN} = 1$$
$$P_{i,i+1} = p = 1 - P_{i,i-1}, \qquad i = 1, 2, \cdots, N-1$$

This Markov chain has three classes namely $\{0\}, \{1, 2, \cdots, N-1\}$ and $\{N\}$; the first and third class being recurrent and the second transient. Since each transient state is visited only finitely often, it follows that, after some finite amount of time, the gambler will either attain his goal of N or go broke.

Let $P_i, i = 0, 1, \cdots, N$, denote the probability that, starting with i, the gambler's fortune will eventually reach N. By conditioning on the outcome of the initial play of the game we obtain

N = Total no. of Coins in the System
= Total no. of Coins of Player1 + Player 2

$$P_i = pP_{i+1} + qP_{i-1}, \qquad i = 1, 2, \cdots, N-1$$
$$\Rightarrow (p+q)P_i = pP_{i+1} + qP_{i-1} \quad \textbf{\{as p+q=1\}}$$
$$\Rightarrow pP_i + qP_i = pP_{i+1} + qP_{i-1}$$
$$\Rightarrow p(P_{i+1} - P_i) = q(P_i - P_{i-1})$$
$$\Rightarrow \boxed{P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}),} \qquad i = 1, 2, \cdots, N-1 \qquad \cdots \qquad (1)$$

Hence, since $P_0 = 0$, we obtain from preceding line that

Putting i = 1 in equation (1), we get, $P_2 - P_1 = \frac{q}{p}(P_1 - P_0) = \frac{q}{p} P_1$

Putting i = 2 in equation (1), we get, $P_3 - P_2 = \frac{q}{p}(P_2 - P_1) = \left(\frac{q}{p}\right)^2 P_1$

P4 - P3 = ...

$\vdots$

Putting i = i in equation (1), we get, $P_i - P_{i-1} = \frac{q}{p}(P_{i-1} - P_{i-2}) = \left(\frac{q}{p}\right)^{i-1} P_1$

$\vdots$

Putting i = N in equation (1), we get, $P_N - P_{N-1} = \frac{q}{p}(P_{N-1} - P_{N-2}) = \left(\frac{q}{p}\right)^{N-1} P_1$

Adding the first $i-1$ of these equations yields, $P_i - P_1 = P_1\left[\left(\frac{q}{p}\right)+\left(\frac{q}{p}\right)^2+\cdots+\left(\frac{q}{p}\right)^{i-1}\right]$

$$\Rightarrow P_i = P_1\left[1+\left(\frac{q}{p}\right)+\left(\frac{q}{p}\right)^2+\cdots+\left(\frac{q}{p}\right)^{i-1}\right]$$

If $\frac{q}{p}\neq 1$, then $P_i = P_1\left[\frac{1-\left(q/p\right)^i}{1-\left(q/p\right)}\right]$, $\left[\because 1+x+x^2+\cdots+x^{n-1}=\frac{1-x^n}{1-x}\right]$

If $\frac{q}{p}=1$, then $P_i = P_1[\underbrace{1+1+\cdots+1}_{i}] = iP_1$

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These formulas are OK
even when p < q OR p> q OR p = q
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$$\text{Thus, } P_i = \begin{cases}\left[\frac{1-\left(q/p\right)^i}{1-\left(q/p\right)}\right]P_1, & \text{if } \frac{q}{p}\neq 1 \\ iP_1, & \text{if } \frac{q}{p}=1\end{cases} \qquad \cdots \quad (2)$$

Partial proof may appear in exam !!!

Or, p = q = 1/2 Or, p == q (i.e., when winning chance = loosing chance. example: fair coin toss => head=win, tail = loss or vice versa)

$$\text{Putting } i = N \text{ in equation (2), we get, } P_N = \begin{cases}\left[\frac{1-\left(q/p\right)^N}{1-\left(q/p\right)}\right]P_1, & \text{if } \frac{q}{p}\neq 1 \\ NP_1, & \text{if } \frac{q}{p}=1\end{cases} \qquad \cdots \quad (3)$$

Now, using the fact that $P_N = 1$ in equation (3), we obtain that

If $\frac{q}{p}\neq 1$, then $P_N = \left[\frac{1-\left(q/p\right)^N}{1-\left(q/p\right)}\right]P_1$

$$\Rightarrow P_1 = \left[\frac{1-\left(q/p\right)}{1-\left(q/p\right)^N}\right]$$

If $\frac{q}{p}=1$, then $1 = NP_1$

$$\Rightarrow P_1 = \frac{1}{N}$$

$$\therefore P_1 = \begin{cases} \left[\dfrac{1-\left(q/p\right)}{1-\left(q/p\right)^N} \right] & \text{if } p \neq \dfrac{1}{2} \\ \dfrac{1}{N}, & \text{if } p = \dfrac{1}{2} \end{cases}$$

Putting the value of P_1 in equation (2), we get

$$\text{If } p \neq \frac{1}{2}, \text{ then } P_i = \left[\frac{1-\left(q/p\right)^i}{1-\left(q/p\right)} \right] \cdot \left[\frac{1-\left(q/p\right)}{1-\left(q/p\right)^N} \right] = \frac{1-\left(q/p\right)^i}{1-\left(q/p\right)^N}$$

$$\text{If } p = \frac{1}{2}, \text{ then } P_i = i\frac{1}{N} = \frac{i}{N}$$

p = winning probability (of the person)
q = 1- p = losing probability
i = starting number of coins (of the person)
N = total coins in the system

This is full proof
Full proof may appear in exam, Too!

$$\boxed{\text{Finally, we get } P_i = \begin{cases} \dfrac{1-\left(q/p\right)^i}{1-\left(q/p\right)^N}, & \text{if } p \neq \dfrac{1}{2} \\ \dfrac{i}{N}, & \text{if } p = \dfrac{1}{2}, \ q = 1/2 \end{cases} \quad \cdots} \quad (4)$$

(or, p = q)

Example: Suppose Max and Patty decide to flip pennies; the one coming closest to the wall wins. Patty, being the better player, has a probability 0.6 of winning on each flip.
(a) If Patty starts with five pennies and Max with ten, then what is the probability that patty will wipe Max out? ***This is why p = 0.6, q =0.4, i=5 (If we ask "Max will wipe Patty out", then p=0.4, q=0.6, i=10)***
(b) What if Patty starts with ten and Max with twenty?

Solution: (a) Let, $i = 5$, $N = 5 + 10 = 15$, $p = 0.6$, $q = 1 - p = 1 - 0.6 = 0.4$

$$\text{Hence the desired probability is } P_5 = \frac{1-\left(0.4/0.6\right)^5}{1-\left(0.4/0.6\right)^{15}} \approx 0.87 \quad \text{(Ans.)}$$

(b) Let, $i = 10$, $N = 10 + 20 = 30$, $p = 0.6$, $q = 1 - p = 1 - 0.6 = 0.4$

$$\text{Hence the desired probability is } P_{10} = \frac{1-\left(0.4/0.6\right)^{10}}{1-\left(0.4/0.6\right)^{30}} \approx 0.87 \quad \text{(Ans.)}$$

Think: How to solve when p = q = 0.5

**** Also: see similar questions in the Practise Questions 03.doc ****

☺Good Luck☺