Sums

Notation: We will be working with sums of the general form

$a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$

This form of representing sums is called *three-dot notation*. Other alternative notations are also available to represent sums.

$\sum_{1 \le k \le n} a_k$ is called *Sigma-notation* because it uses the Greek letter $\sum$ (sigma).

$\sum_{k=1}^{n} a_k$ is called *delimited* form to represent sums.

For example, we can express the sum of the squares of all odd positive integers below 100 as follows:

$\sum_{\substack{1 \le k < 100 \\ k \text{ odd}}} k^2$. The delimited equivalent of this sum is $\sum_{k=0}^{49} (2k+1)^2$.

The biggest advantage of Sigma-notation is that we can manipulate it more easily than the delimited form. For example, suppose we want to change the index k to k+1.

$\sum_{1 \le k \le n} a_k = \sum_{1 \le k+1 \le n} a_k$ it is easy to see what is going one. But for delimited form, we have

$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1}$ it is harder to see what is happed and more likely to make mistake. But delimited form is nice and tidy, we can write it quickly.

People are often tempted to write $\sum_{k=2}^{n-1} k(k-1)(n-k)$ instead of $\sum_{k=0}^{n} k(k-1)(n-k)$ because the terms for k = 0, 1 and n in this sum are zero. But it is more helpful to keep upper and lower bounds on an index of summation as simple as possible, because sum can be manipulated much more easily when the bounds are simple. Indeed the form $\sum_{k=2}^{n-1}$ can even be ambiguous, because it's meaning is not clear when n = 0 or n = 1.

Sums and Recurrences: There is an intimate relation between sums and recurrences. The sum $S_n = \sum_{k=0}^{n} a_k$ is equivalent to the recurrence

$S_0 = a_0$

$S_n = S_{n-1} + a_n$, for $n > 0$ $\cdots$ (1)

Therefore, we can evaluate sums by using methods we learned earlier to solve recurrence in closed form. For example, if a_n is equal to a constant plus a multiple of n, the recurrence takes the following form:

$R_0 = \alpha$

$R_n = R_{n-1} + \beta + \gamma n$, for $n > 0$ $\cdots$ (2)

If we look after small cases to solve the above recurrence then we get

$$R_1 = R_0 + \beta + \gamma = \alpha + \beta + \gamma$$
$$R_2 = R_1 + \beta + 2\gamma = \alpha + \beta + \gamma + \beta + 2\gamma = \alpha + 2\beta + 3\gamma \quad ,$$
$$R_3 = R_2 + \beta + 3\gamma = \alpha + 2\beta + 3\gamma + \beta + 3\gamma = \alpha + 3\beta + 6\gamma$$

and so on, in general the solution can be written in the form $R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$ where A(n), B(n) and C(n) are coefficients of dependence on the general parameters α, β and γ. Setting $R_n = 1$ in equation (2), we get

$1 = \alpha$

$1 = 1 + \beta + \gamma n \Rightarrow \beta + \gamma n = 0 \Rightarrow \beta = 0$ and $\gamma = 0$

hence $A(n) = 1$.

Setting $R_n = n$ in equation (2), we get

$0 = \alpha$

$n = n - 1 + \beta + \gamma n \Rightarrow \beta + \gamma n = 1 \Rightarrow \beta = 1$ and $\gamma = 0$

hence $B(n) = n$.

Again, setting $R_n = n^2$ in equation (2), we get

$0 = \alpha$

$n^2 = (n-1)^2 + \beta + \gamma n \Rightarrow n^2 = n^2 - 2n + 1 + \beta + \gamma n \Rightarrow \beta + \gamma n = 2n - 1 \Rightarrow \beta = -1$ and $\gamma = 2$

hence $2C(n) - B(n) = n^2 \Rightarrow 2C(n) = n^2 + n \Rightarrow C(n) = \frac{n}{2}(n+1)$

Putting the values of $A(n)$, $B(n)$ and $C(n)$ we get the solution of equation (2)

$$R_n = \alpha + n\beta + \frac{n}{2}(n+1)\gamma$$

Therefore if we want to evaluate $\sum_{k=0}^{n}(a + bk)$, then we can write the sum as follows

$S_0 = a$

$S_n = S_{n-1} + a + bn \quad , \quad$ for $n > 0$

Comparing with general equation (2) with the above sum, we get $\alpha = a, \beta = a$ and $\gamma = b$.

Thus $\sum_{k=0}^{n}(a + bk) = a + na + \frac{n}{2}(n+1)b = a(n+1) + \frac{n}{2}(n+1)b = (n+1)(a + \frac{nb}{2})$

Conversely, many recurrences can be reduced to sums. For example, we are going to transfer the Tower of Hanoi recurrence solution into sums.

$T_0 = 0$

$T_n = 2T_{n-1} + 1 \quad , \quad$ for $n > 0$

Dividing both side by 2^n , we get

$T_0 / 2^0 = 0 / 2^0$

$T_n / 2^n = 2T_{n-1} / 2^n + 1 / 2^n = T_{n-1} / 2^{n-1} + 2^{-n} \quad , \quad$ for $n > 0$

Setting $S_n = T_n / 2^n$, we get

$$S_0 = 0$$

$$\begin{aligned}
S_n &= S_{n-1} + 2^{-n} , \qquad \text{for } n > 0 \\
&= S_{n-2} + 2^{-(n-1)} + 2^{-n} \\
&= S_{n-3} + 2^{-(n-2)} + 2^{-(n-1)} + 2^{-n} \\
&\quad \vdots \\
&= S_{n-n} + 2^{-(n-n+1)} + 2^{-(n-n+2)} + 2^{-(n-n+3)} + \cdots + 2^{-(n-2)} + 2^{-(n-1)} + 2^{-n} \\
&= S_0 + 2^{-1} + 2^{-2} + 2^{-3} + \cdots + 2^{-(n-2)} + 2^{-(n-1)} + 2^{-n} \\
&= \sum_{k=1}^{n} 2^{-k}
\end{aligned}$$

We have derived sum from the Tower of Hanoi recurrence solution. Adding 2^0 with both side of the sum, we get

$$S_n + 2^0 = \sum_{k=1}^{n} 2^{-k} + 2^0$$

$$\Rightarrow S_n + 1 = 1/2^0 + 1/2^1 + 1/2^2 + \cdots + 1/2^{n-2} + 1/2^{n-1} + 1/2^n$$

$$\Rightarrow S_n = \frac{1 - 1/2^{n+1}}{1 - 1/2} - 1$$

$$\Rightarrow S_n = 2(1 - 1/2^{n+1}) - 1$$

$$\Rightarrow T_n / 2^n = 1 - 1/2^n$$

$$\Rightarrow T_n = 2^n - 1$$

The trick of dividing both-side by 2^n in Tower of Hanoi recurrence to get S_n from T_n is a special case of a general technique that can reduce virtually any recurrence of the form

$$a_n T_n = b_n T_{n-1} + c_n \qquad \cdots \qquad (3)$$

to a sum. The idea is to multiple both sides by a *summation faction*, s_n :

$$s_n a_n T_n = s_n b_n T_{n-1} + s_n c_n$$

This factor s_n is clearly chosen to make $s_n b_n = s_{n-1} a_{n-1}$

Then if we write $S_n = s_n a_n T_n$, we have a sum-recurrence

$$S_n = s_n b_n T_{n-1} + s_n c_n = s_{n-1} a_{n-1} T_{n-1} + s_n c_n = S_{n-1} + s_n c_n$$

Hence,

$$\begin{aligned}
S_n &= S_{n-1} + s_n c_n = S_{n-2} + s_{n-1} c_{n-1} + s_n c_n \\
&\quad \vdots \\
&= S_0 + s_1 c_1 + s_2 c_2 + \cdots + s_{n-1} c_{n-1} + s_n c_n \\
&= S_0 + \sum_{k=1}^{n} s_k c_k = s_0 a_0 T_0 + \sum_{k=1}^{n} s_k c_k = s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k
\end{aligned}$$

The solution of the general recurrence is $T_n = \dfrac{S_n}{s_n a_n} = \dfrac{1}{s_n a_n}(s_1 b_1 T_0 + \sum_{k=1}^{n} s_k c_k) \qquad \cdots \qquad (4)$

For example, when $n = 1$, then $T_1 = {(s_1 b_1 T_0 + s_1 c_1)}\big/{s_1 a_1} = {(b_1 T_0 + c_1)}\big/{a_1}$. The value of s_1 cancels out, so it can be anything but zero. To find out the right summation factor, s_n we can unfold the relation $s_n = \dfrac{s_{n-1} a_{n-1}}{b_n} = \dfrac{s_{n-2} a_{n-2} a_{n-1}}{b_{n-1} b_n} \cdots = \dfrac{s_1 a_1 \cdots a_{n-2} a_{n-1}}{b_2 \cdots b_{n-1} b_n}$

Thus, $\boxed{s_n = \dfrac{a_1 \cdots a_{n-2} a_{n-1}}{b_2 \cdots b_{n-1} b_n}}$ or any convenient constant multiple of this value, will be a suitable summation factor.

To understand the whole process, we are going to apply these ideas to solve the average number of comparison steps required by quick-sort when it is applied to n items

$$C_0 = 0$$

$$C_n = (n+1) + \frac{2}{n} \sum_{k=0}^{n-1} C_k \ , \qquad \text{for } n > 0 \qquad \cdots \qquad (5)$$

We can find out some small cases from the above sum to guess the solution.

$$C_1 = (1+1) + 2C_0 = 2$$

$$C_2 = (2+1) + \frac{2}{2}(C_0 + C_1) = 3 + 2 = 5$$

With the intent to simplify the equation (5), we can multiply both sides by n

$$nC_n = n(n+1) + 2\sum_{k=0}^{n-1} C_k = n^2 + n + 2\sum_{k=0}^{n-1} C_k \ , \quad \text{for } n > 0 \qquad \cdots \qquad (6)$$

hence, if we replace n by (n-1), we get

$$(n-1)C_{n-1} = (n-1)^2 + (n-1) + 2\sum_{k=0}^{n-2} C_k \ , \qquad \text{for } n-1 > 0 \qquad \cdots \qquad (7)$$

Now, we can subtract equation (7) from equation (6) to get rid of $\sum$ sign

$$\begin{aligned} nC_n - (n-1)C_{n-1} &= n^2 + n + 2\sum_{k=0}^{n-1} C_k - (n^2 - 2n + 1) - (n-1) - 2\sum_{k=0}^{n-2} C_k \ , && \text{for } n > 1 \\ &= n^2 + n + 2\sum_{k=0}^{n-2} C_k + 2C_{n-1} - n^2 + 2n - 1 - n + 1 - 2\sum_{k=0}^{n-2} C_k \ , && \text{for } n > 1 \\ &= 2n + 2C_{n-1} \ , \quad \mathbf{1} && \text{for } n > 1 \end{aligned}$$

Now, we are going to find out a suitable summation faction for the above recurrence. Here, $a_n = n$, $b_n = n+1$ and $c_n = 2n$.

Thus the summation faction, $s_n = \dfrac{a_{n-1} a_{n-2} \cdots a_2 a_1}{b_n b_{n-1} \cdots b_3 b_2} = \dfrac{(n-1)\cdot(n-2)\cdots 2\cdot 1}{(n+1)\cdot n \cdots 4\cdot 3} = \dfrac{2}{(n+1)n}$.

The solution to our recurrence according to equation (4), is therefore

$$C_n = \frac{n(n+1)}{2n}\left[\frac{2}{(1+1)\times 1}(1+1)C_0 + \sum_{k=1}^{n} \frac{2}{k(k+1)} 2k\right] = 2(n+1)\sum_{k=1}^{n} \frac{1}{k+1} \qquad \cdots \qquad (8)$$

We can represent C_n using *Harmonic number*, $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$

To find out the closed form of quick-sort recurrence, we have to find out the summation portion of the equation.

$$\sum_{k=1}^{n} \frac{1}{k+1} = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} = (1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}) - 1 + \frac{1}{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \frac{-n-1+1}{n+1} = H_n - \frac{n}{n+1}$$

Putting the value of summation in equation (8), we get

$$C_n = 2(n+1)[H_n - \frac{n}{n+1}] = 2(n+1)H_n - 2n \ .$$

We can check small cases to verify our solution

$$C_1 = 2(1+1)H_1 - 2\times 1 = 4(1) - 2 = 4 - 2 = 2$$

$$C_2 = 2(2+1)H_2 - 2\times 2 = 6(1 + \frac{1}{2}) - 4 = 6\times\frac{3}{2} - 4 = 5$$

$$C_3 = 2(3+1)H_3 - 2\times 3 = 8(1 + \frac{1}{2} + \frac{1}{3}) - 6 = 8\times\frac{11}{6} - 6 = \frac{44}{3} - 6 = \frac{26}{3}$$

☺Good Luck☺