Sums

Manipulation of Sums: Let P be any finite set of integers. Sums over the elements of P can be transformed by using three simple rules:

Distributive Law: $\sum_{k\in P} ca_k = c\sum_{k\in P} a_k \qquad \cdots \qquad (1)$

Associative Law: $\sum_{k\in P}(a_k + b_k) = \sum_{k\in P} a_k + \sum_{k\in P} b_k \qquad \cdots \qquad (2)$

Commutative Law: $\sum_{k\in P} a_k = \sum_{m\in P} a_m \qquad \cdots \qquad (3)$

For example, if $K = \{-1, 0, 1\}$ and $m = -k$ these three laws tell us respectively that

$ca_{-1} + ca_0 + ca_{-1} = c(a_{-1} + a_0 + a_1)$ [Distributive Law]

$(a_{-1} + b_{-1}) + (a_0 + b_0) + (a_1 + b_1) = (a_{-1} + a_0 + a_1) + (b_{-1} + b_0 + b_1)$ [Associative Law]

$a_{-1} + a_0 + a_1 = a_1 + a_0 + a_{-1}$ [Commutative Law]

Suppose, we want to compute the general sum, $S = \sum_{0\le k\le n}(a + bk)$

By using commutative law, we can replace k by $n-k$, obtaining

$$S = \sum_{0\le (n-k)\le n}(a + b(n-k)) = \sum_{0\le k\le n}(a + bn - bk)$$

These two equations can be added by using associative law:

$$2S = \sum_{0\le k\le n}(a + bk) + \sum_{0\le k\le n}(a + bn - bk) = \sum_{0\le k\le n}(a + bk + a + bn - bk) = \sum_{0\le k\le n}(2a + bn)$$

Now, we can apply distributive law to evaluate the sum.

$$2S = \sum_{0\le k\le n}(2a + bn) = (2a + bn)\sum_{0\le k\le n} 1 = (2a + bn)(n+1)$$

$$\Rightarrow S = \frac{(2a + bn)(n+1)}{2} = (a + \frac{bn}{2})(n+1)$$

Perturbation Method: This method is used to evaluate a sum in closed form. The operation of splitting off a term is the basis of this method. The idea is to start with an unknown sum and call it S_n .

$$S_n = \sum_{0\le k\le n} a_k$$

Then, we rewrite S_{n+1} in two ways, by splitting off both its last term and its first term:

$$S_n + a_{n+1} = \sum_{0\le k\le n+1} a_k = a_0 + \sum_{1\le k\le n+1} a_k = a_0 + \sum_{1\le k+1\le n+1} a_{k+1} = a_0 + \sum_{0\le k\le n} a_{k+1}$$

Now, we try to represent the last in terms of S_n and if we succeed then we obtain an equation whose solution is the sum we seek.

For example, let's use this technique to find out the sum, $S_n = \sum_{0\le k\le n} ax^k$.

$$S_n + ax^{n+1} = ax^0 + \sum_{1\le k\le n+1} ax^k = a + \sum_{1\le k+1\le n+1} ax^{k+1} = a + x\sum_{0\le k\le n} ax^k = a + xS_n$$

$\Rightarrow S_n - xS_n = a - ax^{n+1}$

$\Rightarrow S_n = \dfrac{a(1-x^{n+1})}{1-x}$, for $x \neq 1$

Let's try perturbation technique for another example. Evaluate $\sum_{0\le k\le n} k \cdot 2^k$.

Let, $S_n = \sum_{0\le k\le n} k \cdot 2^k$

$S_n + (n+1)\cdot 2^{n+1} = 0\cdot 2^0 + \sum_{0\le k\le n} (k+1)\cdot 2^{k+1} = \sum_{0\le k\le n} k\cdot 2^{k+1} + \sum_{0\le k\le n} 2^{k+1} = 2\sum_{0\le k\le n} k\cdot 2^k + 2\sum_{0\le k\le n} 2^k$

Let, $\sum_{0\le k\le n} 2^k = R_n$. Now, first we find out the sum of R_n , using the result we get S_n finally.

$R_n + 2^{n+1} = 2^0 + \sum_{0\le k\le n} 2^{k+1} = 1 + 2\sum_{0\le k\le n} 2^k = 1 + 2R_n$

$\Rightarrow R_n + 2^{n+1} = 1 + 2R_n$

$\Rightarrow R_n = 2^{n+1} - 1$

Now, $S_n + (n+1)\cdot 2^{n+1} = 2\sum_{0\le k\le n} k\cdot 2^k + 2\sum_{0\le k\le n} 2^k = 2S_n + 2(2^{n+1}-1) = 2S_n + 2^{n+2} - 2$

$\Rightarrow S_n = n2^{n+1} + 2^{n+1} - 2^{n+2} + 2 = n2^{n+1} + 2^{n+1}(1-2) + 2 = n2^{n+1} - 2^{n+1} + 2 = 2 + (n-1)2^{n+1}$

Try to prove that $\sum_{k=0}^{n} kx^k = \dfrac{x-(n+1)x^{n+1} + nx^{n+2}}{(1-x)^2}$ using perturbation technique.

☺Good Luck☺