*For each Topic in this PDF, see both the PDF book (added comments) and this PDF too*

# Random Variables

Expectation of a Random Variable: If $X$ is a discrete random variable having a probability function $p(x)$, then the expectation value of $X$ is defined by

$$\boxed{E[X] = \sum_{x,p(x)>0} x \cdot p(x)}$$

For example, if the probability mass function of $X$ is given by $p(1) = 1/3$ , $p(2) = 2/3$ ,

then $E[X] = 1 \cdot \left(\frac{1}{3}\right) + 2 \cdot \left(\frac{2}{3}\right) = \frac{5}{3}$ = 1.667

If p(1)=0.5 and p(2)=0.5, then E[X] = 1.5, which is intuitive!!!

Example 11: Find $E[X]$ where X is the outcome when we roll a fair die.

** (Two STAR) → Find E[X] when X is sum of two fair dice; Find E[Y], Here Y is a R.V. indicating the no. of Heads from two Fair Coins. (do: three fair coins!)

Solution: Since $p(1) = p(2) = p(3) = p(4) = p(5) = p(6) = \frac{1}{6}$ , we obtain

$$E[X] = 1 \cdot \left(\frac{1}{6}\right) + 2 \cdot \left(\frac{1}{6}\right) + 3 \cdot \left(\frac{1}{6}\right) + 4 \cdot \left(\frac{1}{6}\right) + 5 \cdot \left(\frac{1}{6}\right) + 6 \cdot \left(\frac{1}{6}\right) = \frac{21}{6} = \frac{7}{2} \text{ (Ans.)}$$

Example 12: Calculate $E[X]$ when $X$ is a bernoulli random variable with parameter $p$.

Solution: Since $p(0) = 1 - p,\ p(1) = p$ , we have $E[X] = 0 \cdot (1-p) + 1 \cdot (p) = p$ (Ans.)

Example 13: Calculate $E[X]$ when $X$ is a binomial distribution with parameter $n$ and $p$.

**Q. Find E[X], where X denotes the no. of heads obtained from flipping a fair coin n times**
**Ans: Same as Below!**

Solution:

$$E[X] = \sum_{i=0}^{n} i \cdot p(i) = \sum_{i=0}^{n} i \cdot \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=1}^{n} \frac{i \cdot n!}{(n-i)!\,i!} p^i (1-p)^{n-i}$$

**Important**
**** (two stars)**

$$= \sum_{i=1}^{n} \frac{n!}{(n-i)!(i-1)!} p^i (1-p)^{n-i}$$

$$= np \sum_{i=1}^{n} \frac{(n-1)!}{(n-i)!(i-1)!} p^{i-1} (1-p)^{n-i}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{(n-1-k)!\,k!} p^k (1-p)^{n-1-k}, \quad [\text{Let, } k = i-1]$$

$$= np \sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}$$

$$= np[p + (1-p)]^{n-1} = np$$

Example 14: Calculate $E[X]$ when $X$ is a geometric random variable with parameter $p$.

**Q. Find E[X], where X denotes the no. of coin Flips required to get the firsrt Head.**

Solution: **Ans: Same as Below!**

**Important**
**** (two stars)**

$$E[X]=\sum_{n=1}^{\infty} np(1-p)^{n-1}=p\sum_{n=1}^{\infty} nq^{n-1}, \quad [\text{where } q=1-p]$$
$$=p\sum_{n=1}^{\infty}\frac{d}{dq}\left(q^n\right)$$
$$=p\frac{d}{dq}\left(\sum_{n=1}^{\infty} q^n\right)$$
$$=p\frac{d}{dq}\left(\frac{q}{1-q}\right)=\frac{p}{(1-q)^2}=\frac{p}{[1-(1-p)]^2}=\frac{1}{p}$$

E[X] always means the Mean value of the random variable X.V For the following reason (examples 15, 17), The mean value of a Poisson R.V. is just LAMBDA, but for an exponential R.V., the mean is 1/LAMBDA. In other words, if the MEAN is given, you can find the value of LAMBDA for your math.

Example 15: Calculate $E[X]$ when $X$ is a poison random variable with parameter $\lambda$ .

Solution:

**Important**
**** (two stars)**

$$E[X]=\sum_{i=0}^{\infty}\frac{ie^{-\lambda}\lambda^i}{i!}=\sum_{i=1}^{\infty}\frac{e^{-\lambda}\lambda^i}{(i-1)!}=\lambda e^{-\lambda}\sum_{i=1}^{\infty}\frac{\lambda^{i-1}}{(i-1)!}=\lambda e^{-\lambda}\sum_{k=0}^{\infty}\frac{\lambda^k}{k!}, \quad [\because \sum_{i=0}^{\infty}\frac{x^i}{i!}=e^x]$$
$$=\lambda e^{-\lambda}e^{\lambda}=\lambda$$

ClassTest #2 Upto This Point (because all these are Discrete R.V.)

Example 16: Calculate the expectation of a random variable uniformly distributed over $(\alpha,\beta)$ .

Solution: $$E[X]=\int_{\alpha}^{\beta}\frac{x}{\beta-\alpha}dx=\frac{\beta^2-\alpha^2}{2(\beta-\alpha)}=\frac{\beta+\alpha}{2}$$

These Two (uniform, exponential) are Continuous R.V.

*****
**important**
*****
**(5 STARS)**

Example 17: Let $X$ be exponentially distributed with parameter $\lambda$ . Calculate $E[X]$.

Solution:

$$E[X]=\int_0^{\infty} x\lambda e^{-\lambda x}dx=\lambda\int_0^{\infty} xe^{-\lambda x}dx, \qquad \left[\because \int uvdx=u\int vdx-\int\left[\frac{du}{dx}\int vdx\right]dx\right]$$
$$=\left[-xe^{-\lambda x}\right]_0^{\infty}+\int_0^{\infty} e^{-\lambda x}dx \;=0-\left[\frac{e^{-\lambda x}}{\lambda}\right]_0^{\infty}=\frac{1}{\lambda}$$

<u>Stochastic Process:</u> A *stochastic process* $\{X(t), t \in T\}$ is a collection of random variables. That is, for each $t \in T$, $X(t)$ is a random variable. The index $t$ is often interpreted as time and as a result, we refer to $X(t)$ as the state of the process at time $t$. for example, $X(t)$ might equal the total number of customers that have entered a supermarket by time $t$; or the number of customers in the supermarket at time $t$; or the total amount of sales that have been recorded in the market by time $t$; etc.
The set $T$ is called the *index set* of the process. When $T$ is a countable set the stochastic process is said to be *discrete-time* process. If $T$ is an interval of the real line, the stochastic process is said to be a *continuous-time* process.

☺Good Luck☺

$$\int_0^\infty x\lambda e^{-\lambda x}\,dx \longrightarrow$$

$u = x$

$v = \lambda e^{-\lambda x}$

সূত্র:

$$\cancel{\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\left(\int v\,dx\right)\right]dx}$$

সূত্র:

$$\boxed{\int uv\,dx = u\int v\,dx - \int\left[\frac{du}{dx}\int v\,dx\right]dx}$$

$$x\int_0^\infty \lambda e^{-\lambda x}\,dx - \int_0^\infty\left[\frac{d}{dx}(x)\cdot\int_0^\infty \lambda e^{-\lambda x}\,dx\right]dx$$

$$= \left[x\left[-e^{-\lambda x}\right]\right]_0^\infty - \int_0^\infty -e^{-\lambda x}\,dx$$

$$= 0 + \int_0^\infty e^{-\lambda x}\,dx$$

$$= \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^\infty = -\frac{1}{\lambda}\left[\frac{1}{e^{\lambda x}}\right]_0^\infty = -\frac{1}{\lambda}\left[0 - \frac{1}{e^0}\right]$$

$$= -\frac{1}{\lambda}(-1) = \boxed{\frac{1}{\lambda}}$$