Mathematics -III

22/04/2017

1. Complex Variable
2. Laplace

Complex Number System

A Number of the form $a+ib$ is called a complex number when a and b are real numbers and $i=\sqrt{-1}$. Here a is the real part and b is the imaginary part.

$3+5i$, $3i$, 5 etc are complex number.

$3i$ ↑ purely imaginary, 5 ↑ real number

All real numbers are complex number.

$z = a+ib$

conjugate complex, $\bar{z} = a-ib$

$z + \bar{z} =$ real number

$z.\bar{z} =$ real number

* $z_1 = a+ib, \quad z_2 = c+id$

$$z_1 \pm z_2 = (a+ib) \pm (c+id)$$

$$= (a+c) \pm i(b+d)$$

$$z_1 z_2 = (a+ib)(c+id)$$

$$= ac + iad + ibc + i^2 bd$$

$$= ac + i(ad+bc) - bd \quad [i^2 = -1]$$

$$= \underbrace{(ac-bd)}_{\text{real part}} + i \underbrace{(ad+bc)}_{\text{imaginary part}}$$

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)} \cancel{\neq \frac{(ac-bd) + i(ad+bc)}{c^2 - i^2 d^2}}$$

$$= \frac{ac - iad + ibc - i^2bd}{c^2+d^2}$$

$$= \frac{ac+bd + i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\,\frac{bc-ad}{c^2+d^2}$$

$= A + iB$ form

$z_1 = z_2$

$\Rightarrow a+ib = c+id$

$\Rightarrow a = c$ and $b = d$ [requirement]

Ex.1 Seperate real and imaginary form:

$$\frac{(1+i)(2+i)}{(3+i)}$$

$$= \frac{(2+2i+i+i^2)(3-i)}{(3+i)(3-i)}$$

$$= \frac{(2+3i-1)(3-i)}{3^2-i^2}$$

$$= \frac{(1+3i)(3-i)}{9+1}$$

$$= \frac{3-i+9i-3i^2}{10}$$

$$= \frac{3+8i+3}{10}$$

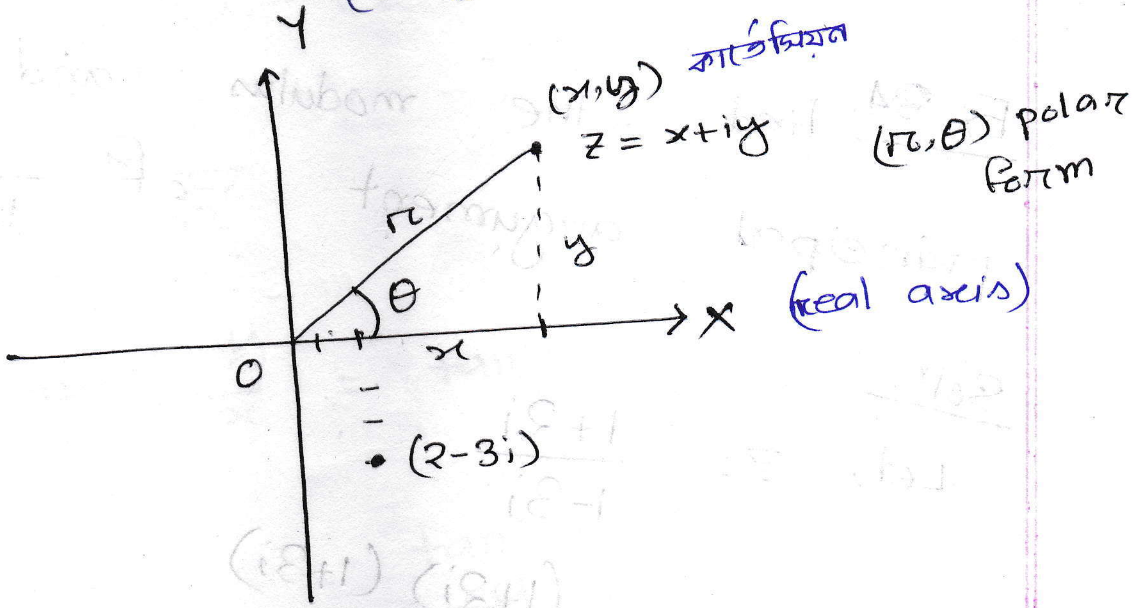
$$= \frac{6+8i}{10}$$

$$= \frac{3+4i}{5} = \frac{3}{5} + i\frac{4}{5}$$

23/04/2017

Geometrical representation of complex number:

$z = x + iy$



$x = r\cos\theta$

$y = r\sin\theta$

$r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\dfrac{y}{x}$

$z = x + iy$

$= r\cos\theta + ir\sin\theta = \underline{r(\cos\theta + i\sin\theta)}$

Another form of complex number

r is the modulus of the complex number and θ is the argument.

Principal argument $= -\pi < \theta \leq \pi$

Ex 04. Find the modulus and principal argument of $\frac{1+2i}{1-3i}$

Soln:

Let, $z = \frac{1+2i}{1-3i}$

$$= \frac{(1+2i)(1+3i)}{(1-3i)(1+3i)}$$

$$= \frac{1+3i+2i+6i^2}{1-9i^2} = \frac{-5+5i}{10}$$

$$= \frac{\cancel{-3}+\cancel{3i}}{\cancel{10}}$$

$$= -\frac{1}{2}+\frac{1}{2}i$$

$$= x+iy$$

$$= r(\cos\theta + i\sin\theta)$$

~~Now, modulus, r =~~

where, $x = -\frac{1}{2}$, $y = \frac{1}{2}$

$$r = \sqrt{x^2+y^2}$$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

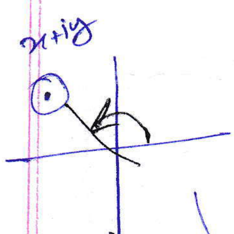
$$= \frac{1}{\sqrt{2}}$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{\frac{1}{2}}{-\frac{1}{2}}$$

$$= \tan^{-1}(-1)$$

$$\cancel{= \tan}$$

$$\cancel{= -\frac{\pi}{4}}$$



$$= \tan^{-1}\left\{\tan\left(\pi - \frac{\pi}{4}\right)\right\}$$

$$= \frac{3\pi}{4}$$

Ex:

$z = 3i$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{3}{0}$$

$$= \tan^{-1}\infty$$

$$= \frac{\pi}{2}$$

Ex: $z = 2$

$$r = \sqrt{2^2+0^2} = 2$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{2}$$

$$= \tan^{-1} 0$$

$$= 0$$

$z = -2$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{-2}$$

$$= \tan^{-1} 0$$

$$= \tan^{-1}\left\{\tan(\pi - 0)\right\}$$

$$= \pi$$

6.2 (topic)

$z = x + iy$

$r = |z| = \sqrt{x^2 + y^2}$

$|z|^2 = z.\bar{z}$

Ex. 10 Prove that,

(i) $|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $|z_1 - z_2| \geq |z_1| - |z_2|$

9.3

Ex 17 What locas is represented by,

$$|z-5-6i| = 4$$

Soln:

Let, $z = x+iy$

$$|z-5-6i| = 4 \rightarrow |z-(5+6i)| = 4$$

$$\Rightarrow |z-z_0| = 4$$

is a circle with radias 4 and center z_0

$$|x+iy-5-6i| = 4$$

or, $|(x-5)+i(y-6)| = 4$

or, $\sqrt{(x-5)^2+(y-6)^2} = 4$

or, $(x-5)^2+(y-6)^2 = 4^2$

which is a circle with center ~~radias~~ at $(5,6)$ and radias of 4.

Ex. 18

what region is represented by

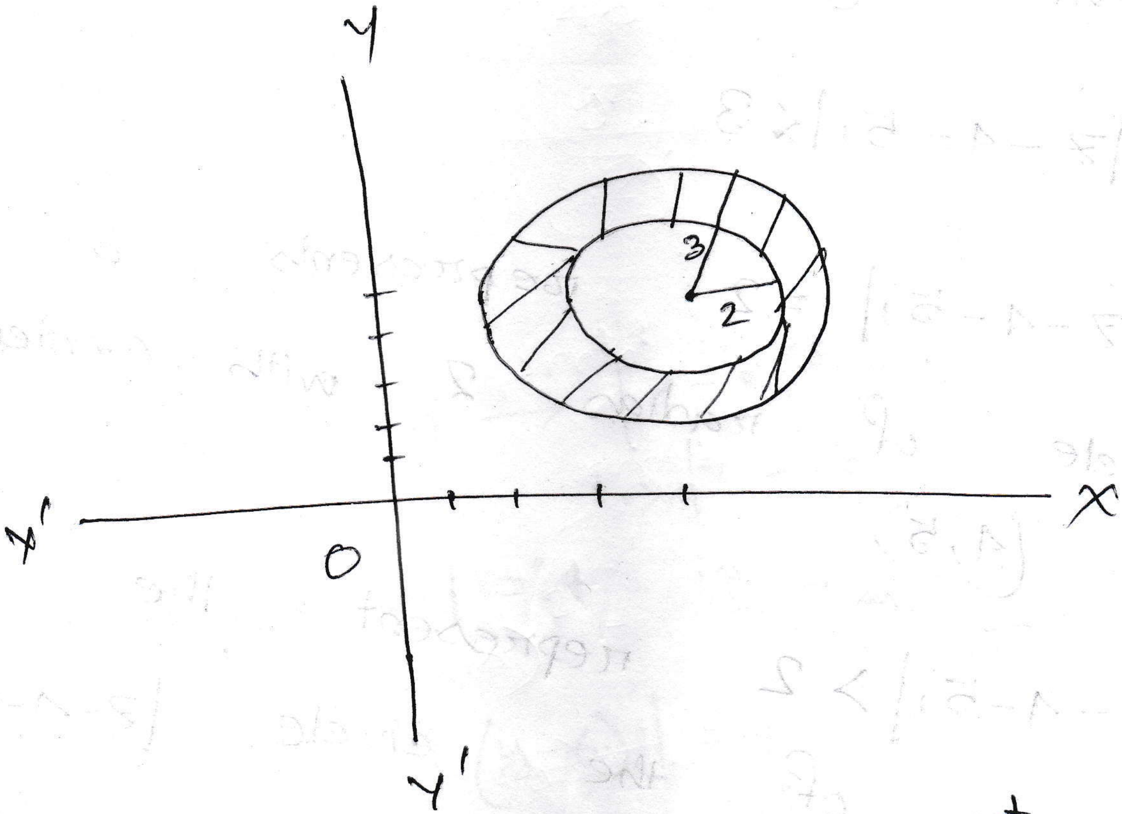
$2 < |z-4-5i| < 3$

$|z-4-5i| = 2$ represents a circle of radias 2 with center at $(4,5)$

$|z-4-5i| > 2$ represent the exterior of the circle $|z-4-5i| = 2$

$|z-4-5i| = 3$ represents a circle of radias 3 with center at $(4,5)$

and $|z-4-5i| < 3$ represents the inner portion of the circle.



$\therefore$ $2 < |z-4-5i| < 3$ represents the region lying between the two circles.

$$|z+2-i| = 3$$
$$\Rightarrow |z-(-2+i)| = 3 \text{ , circle}$$

Exercise 6.3

5 $|z+1| + |z-1| < 3$, what domain it represents?

or, $|x+iy+1| + |x+iy-1| < 3$

or, $\sqrt{(x+1)^2+y^2} + \sqrt{(x-1)^2+y^2} < 3$

or, $\left(\sqrt{(x+1)^2+y^2}\right)^2 < \left(3-\sqrt{(x-1)^2+y^2}\right)^2$

or, $x^2+2x+1+y^2 < 9 - 6\sqrt{(x-1)^2+y^2} + x^2-2x+1+y^2$

or, $\left(6\sqrt{(x-1)^2+y^2}\right)^2 < (9-4x)^2$

or, $36(x^2-2x+1+y^2) < 81-72x+16x^2$

or, $36x^2-72x+36y^2+72x-16x^2 < 81-36$

or, $20x^2+36y^2 < 45$

or, $\frac{20x^2}{45} + \frac{36y^2}{45} < 1$

or $\frac{x^2}{\left(\frac{3}{2}\right)^2} + \frac{y^2}{\left(\frac{\sqrt{5}}{2}\right)^2} < 1$, [~~ell~~ ellipse]

this represents interior of the ellipse. focus, eccentricity, major, minor axis

focus $= \pm(ae, 0)$, eccentricity (e)

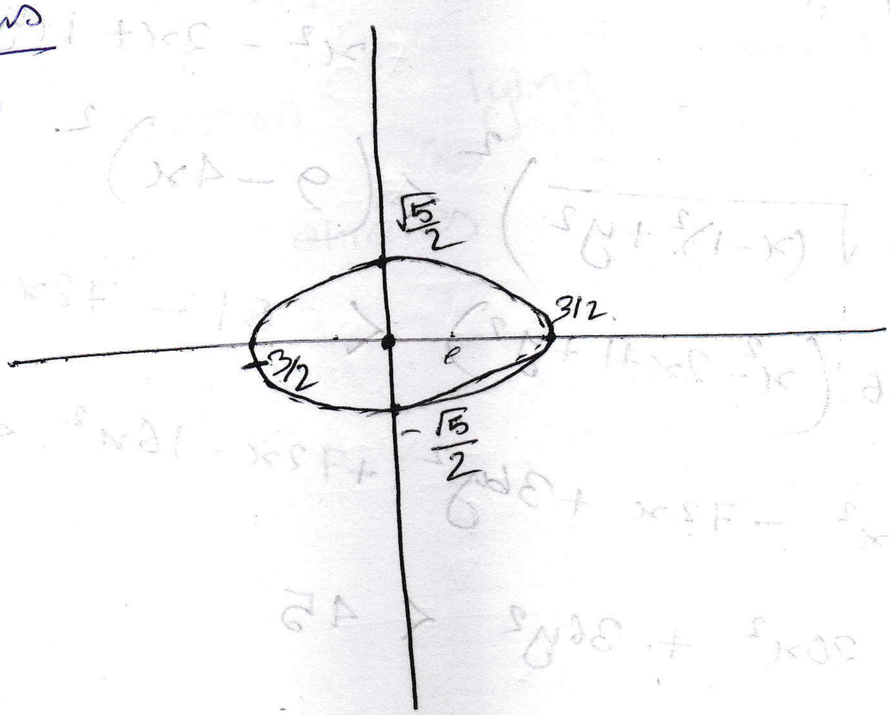
$\cancel{(\text{focus})^2 = a^2 - b^2}$

$(\text{focus})^2 = (\text{major axis})^2 - (\text{minor axis})^2$

[~~try to draw~~]

$\cancel{e = \pm \frac{\text{major axis}}{2}}$

$e = \pm \frac{\text{focus}}{a}$



6)

$$|z+1| = |z-1|$$

$$\Rightarrow |x+iy+1| = |x+iy-1|$$

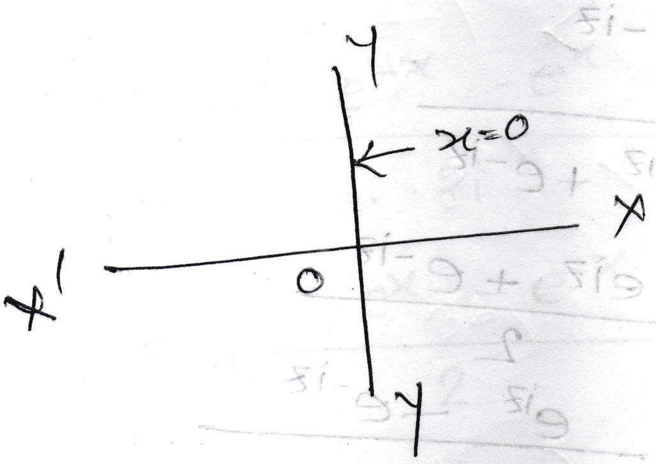
$$\Rightarrow \sqrt{(x+1)^2+y^2} = \sqrt{(x-1)^2+y^2}$$

$$\Rightarrow x^2+2x+1+y^2 = x^2-2x+1+y^2$$

$$\Rightarrow 2x = -2x$$

$$\Rightarrow 4x = 0,$$

$\Rightarrow x=0$, imaginary axis.



[illegible]

$$e^{z} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots$$

$$\cos z + i \sin z = 1 + iz - \frac{z^2}{2!} - i\frac{z^3}{3!} + \frac{z^4}{4!} + i\frac{z^5}{5!} - \cdots$$

$$= 1 + iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \frac{(iz)^4}{4!} + \frac{(iz)^5}{5!} - \cdots$$

$$\therefore \cos z + i \sin z = e^{iz}$$

Similarly

$$\cos z - i \sin z = e^{-iz}$$

$$2 \cos z = e^{iz} + e^{-iz}$$

$$\Rightarrow \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

Similarly, $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$z = x + iy$$

$$= r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta}$$

$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

differentiation + integration

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$

$$\sin ix = \frac{e^{i.ix} - e^{-i.ix}}{2i}$$

$$= \frac{e^{-x} - e^{x}}{2i} \cdot \frac{i}{i}$$

$$= i\,\frac{e^{x} - e^{-x}}{2}$$

$$\therefore \sin ix = i\sinh x$$

Similarly,

$$\cos ix = \cosh x$$

$$\tan ix = i \tanh x$$

$$\cancel{\sinh ix = \frac{e}{\quad}}$$

$$\sinh ix = i \sin x$$

$$\cosh ix = \cos x$$

$$\tanh ix = i \tan x$$

প্রশ্নে exam এ আসবে না।

De Moivre's Theorem:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$

where n is any real number.
(int, fraction etc)

$$(\cos\theta + i\sin\theta)^n = \cancel{e^{in\theta}} \quad e^{in\theta}$$

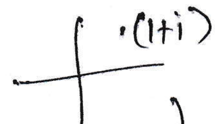
$$\cancel{= \cos nz} = \cos n\theta + i\sin n\theta$$

6.10

Exmpl: 29

$(1+i)^{1/3}$, Find the different values.

$$1+i = r(\cos\theta + i\sin\theta)$$



$$\cancel{r = x^2+y^2}$$

$$r = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\frac{1}{1} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore 1+i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

$\frac{1}{9} \rightarrow$ 9টি value $\qquad \sqrt[3]{}$: 3টি value

or, $(1+i)^{\frac{1}{3}} = \left\{\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right\}^{1/3}$

$$= 2^{1/6}\left\{\cos\left(2n\pi + \frac{\pi}{4}\right) + i\sin\left(2n\pi + \frac{\pi}{4}\right)\right\}^{1/3}$$

where $n = 0, \pm 1, \pm 2, \ldots$ (any integer)

$$= 2^{1/6}\left(\cos\frac{2n\pi + \frac{\pi}{4}}{3} + i\sin\frac{2n\pi + \frac{\pi}{4}}{3}\right)$$

$$= 2^{1/6}\left(\cos\frac{8n\pi + \pi}{12} + i\sin\frac{8n\pi + \pi}{12}\right)$$

Putting $n = 0, 1, 2$ [2 না বসিয়ে -1 বসানো যায়]

$\therefore (1+i)^{1/3} = 2^{1/6}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$,

$2^{1/6}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$, $2^{1/6}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$

$n=3$ বসালে,

$$\cos\frac{25\pi}{12} = \cos 2\pi + \frac{\pi}{12}$$

same
মানই
আসবে

30/04/2017

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Find all the roots of $x^{12}-1=0$.

Soln

$x^{12}-1=0$

or, $x^{12}=1$

or, $x=(1)^{1/12}$

Here,

$r=\sqrt{x^2+y^2} = \sqrt{1+0} = 1$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}0 = 0$

$\therefore (1)^{1/12} = (\cos 0 + i\sin 0)^{1/12}$

$$= \left(\cos 2n\pi + i\sin 2n\pi\right)^{1/12}$$

$$[n = 0, \pm 1, \pm 2, \cdots\cdots]$$

$$= \cos\frac{2n\pi}{12} + i\sin\frac{2n\pi}{12}$$

$$\therefore (1)^{1/2} = \cos\frac{n\pi}{6} + i\sin\frac{n\pi}{6}$$

Putting $n = 0, 1, 2, 3, \cdots\cdots\cdots, 11$

$$\therefore x = (1)^{1/2} = \cos 0 + i\sin 0,\ \cos\frac{\pi}{6} + i\sin\frac{\pi}{6},$$

$$\cos\frac{\pi}{3} + i\sin\frac{\pi}{3},\ \cos\frac{\pi}{2} + i\sin\frac{\pi}{3}$$

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6},$$

$$\cos\pi + i\sin\pi,\ \cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6},$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3},\ \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2},$$

$$\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3},\ \cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}$$

(Ans:)

$x^{12} + 1 = 0$

$\Rightarrow x^{12} = -1$

$\theta = \pi$

$x^{12} + i = 0$

$x^{12} = -i$

$\theta = -\frac{\pi}{2}$

$x^{12} + 1 + i = 0$

$x^{12} = -1 - i$

$\theta =$

$x^{12} + 2 + i = 0$

$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{1}{2}$

$= \cos(2n\pi + \theta) + i\sin(2n\pi + \theta)$

উল্লেখ করতে হবে

[where, $-\pi \le \theta \le -\frac{\pi}{2}$]

[বাকি সব জায়গায় θ লিখে রাখব। শেষে উল্লেখ করব, $\theta = \tan^{-1}\frac{1}{2}$ $\left(-\pi \le \theta \le -\frac{\pi}{2}\right)$]

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$x^4 - x^3 + x^2 - x + 1 = 0$, solve using De Moivre's thm.

or, $(x+1)(x^4 - x^3 + x^2 - x + 1) = 0$

or, $x^5 - x^4 + x^3 - x^2 + x + x^4 - x^3 + x^2 - x + 1 = 0$

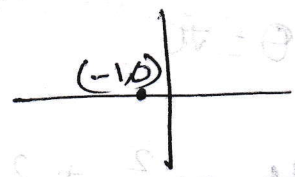
or, $x^5 + 1 = 0$

or, $x^5 = -1$

or, $x = (-1)^{1/5}$

Here,.

$$(-1)^{1/5} = r\,(\cos\theta + i\sin\theta)$$



$$r = \sqrt{(-1)^2 + 0^2} = 1$$

$$\theta = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{0}{-1} = \tan^{-1} 0$$

$$= \pi$$

$$\therefore (-1)^{1/5} = (\cos\pi + i\sin\pi)^{1/5}$$

$$= \left\{\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)\right\}^{1/5}$$

$$[n = 0, \pm1, \pm2, \pm3 \ldots\ldots]$$

$$\therefore x = \cos\frac{2n\pi+\pi}{5} + i\sin\frac{2n\pi+\pi}{5}$$

Putting $n = 0, 1, 2, 3, 4$

$$x = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5},\ \cancel{\cos 35}$$

$$\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5},$$

$\cos \pi + i \sin \pi$, $\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5}$,

$\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}$

Here,
$\cos \pi + i \sin \pi = -1$, which is rejected as it is corresponding to $x+1=0$.

Hence the require roots are

$= \cos \frac{\pi}{5} + i \sin \frac{\pi}{5}$, $\cos \frac{3\pi}{5} + i \sin \frac{3\pi}{5}$,

$\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5}$, $\cos \frac{9\pi}{5} + i \sin \frac{9\pi}{5}$

$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$

$(x+1)$

$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$

$(x-1)$

Exercise 6.5

(1-6) ⇒ easy. s.s.

7

Use De Moivre's th^m to solve,

$$x^9 - x^5 + x^4 - 1 = 0$$

or, $x^5(x^4-1) + 1(x^4-1) = 0$

or, $(x^4-1)(x^5+1) = 0$

$x^4 - 1 = 0$

or, $x^4 = 1$

or, $x = (1)^{1/4}$

or, $x^5 + 1 = 0$

or, $x^5 = -1$

or, $x = (-1)^{1/5}$

Chapter - 7

Function of a Complex Variable

Complex variable, $z = x+iy$

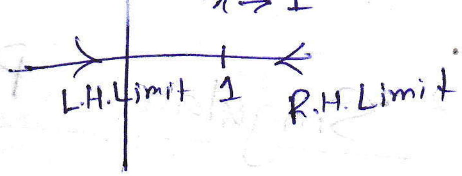
Complex function, $w = f(z)$ | $w = u+iv$

or, $u+iv = f(x+iy)$

or, $u(x,y) + i\,v(x,y)$

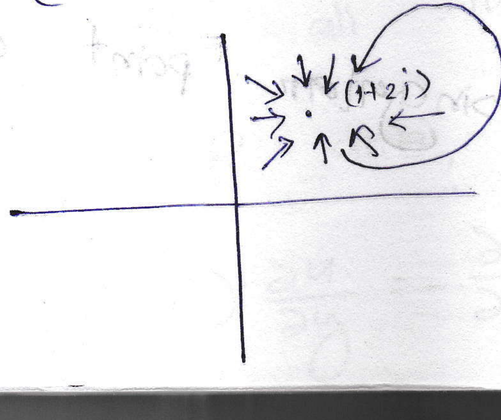
Limit of a function of a complex variable :

$\lim_{x \to 1} f(x) =$



$\lim_{z \to z_0} f(z)$

$z \to 1+2i$



∴ L.H.L এবং R.H.L আসবে না

Analytic

Limit, continuous, ~~Derivativ~~ differential হ্যাঁ → এটা প্রমাণ করা যাবে না, না → এটা প্রমাণ করতে হবে।

Analytic Function:

A function $f(z)$ which is differentiable at $z = z_0$ is said to be analytic at the point $z = z_0$ (or in a region)

Singular Point:

The point at which the function is not differentiable is called a singular point of the function.

06/05/2017

The necessary condition for $f(z) = u + iv$ to be Analytic at all the points in a region R are,

Proof লিখতে হবে না

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ are known as Cauchy-Riemann equation. (C-R eqⁿ)

Theorem 2: The sufficient condition for a function $f(z) = u + iv$ to be Analytic at all the points in a region R are,

(i) $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ (ii) $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

(iii) $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous function of x and y in the region R.

3 Show that,

$f(z) = |z|^2$ is differentiable only in the origin.

Soln: $f(z) = |z|^2$

$\Rightarrow u+iv = |x+iy|^2$

$\Rightarrow u+iv = x^2+y^2$

$\therefore u = x^2+y^2$

and $v = 0$

$$\frac{\partial u}{\partial x} = 2x \quad , \quad \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = 2y \quad , \quad \frac{\partial v}{\partial y} = 0$$

If $f(z)$ is differentiable then,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or, $2x = 0$

$\therefore x = 0$

or, $2y = 0$

$\therefore y = 0$

CR eq^n ~~ares~~ satisfied only when $x = 0$ and $y = 0$

Hence, the given function is ~~on~~ analytic only in the origin.

[x, y এর যে কোন মান বসালে যদি value পাওয়া সম্ভব হয় তবে সেটা continous.]

$\sqrt{x}$ হলে -1 হবে না

$\frac{1}{x}$ হলে 0 " "

3

$$f(z) = e^z$$

$$= e^{x+iy}$$

$$= e^x \cdot e^{iy}$$

$$u+iv = e^x(\cos y + i \sin y)$$

$\therefore u = e^x \cos y$, $v = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$, $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$, $\frac{\partial v}{\partial y} = e^x \cos y$

Here we see that,

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

so, CR eq^n are satisfied

we see tha $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

are continous.

Hence the given function is analytic everywhere.

$$f(z) = u + iv$$

$$f'(z) = \frac{d}{dz}(fz) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\text{or, } f'(z) = \frac{d}{dz}f(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

$$f(z) = u + iv = e^x \cos y + i e^x \sin y$$

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= e^x \cos y + i e^x \sin y$$

$$= e^x (\cos y + i \sin y)$$

$$= e^x e^{iy}$$

$$= e^{x+iy}$$

$$= e^z$$

4. $f(z) = z^3$

$= (x+iy)^3$

$= x^3 + i^3y^3 + 3x^2iy - 3xy^2$

$u+iv = x^3 - 3xy^2 - iy^3 + i3x^2y$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$u = x^3 - 3xy^2$

$v = 3x^2y - y^3$

$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 \qquad \frac{\partial v}{\partial x} = 6xy$

$\frac{\partial u}{\partial y} = -6xy \qquad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$

Here we see that,

$\frac{\partial u}{\partial v} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

so, CR eq^n are satisfied

so, analytic.

continuous $\rightarrow$ so analytic everywher.

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= 3x^2 - 3y^2 + i(6xy)$$

$$= 3(x^2 - y^2 + i2xy)$$

$$= 3(x^2 + 2xiy + i^2y^2)$$

$$= 3(x+iy)^2$$

$$= 3z^2$$

$f(z) = \sin z = \sin(x+iy)$

$= \sin x \cos iy + \sin iy \cos x$

$= \sin x \cosh y + \cancel{\cos}\, i \sinh y \cos x$

$\therefore u = \sin x \cosh y, \quad v = \sinh y \cos x$

$\frac{\partial u}{\partial x} = \cos x \cosh y, \quad \frac{\partial v}{\partial x} = -\sinh y \sin x$

$\frac{\partial u}{\partial y} = \cancel{\cos x \cos}\ \sin x \sinh y, \quad \frac{\partial v}{\partial y} = \cos x \sinh y$

$\therefore$ CR satisfied

continuous

$$f'(z) = \frac{d}{dz} \sin z = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$= \cos x \cosh y - i \sinh y \sin x$$

$$= \cos x \cos iy - \sin iy \sin x$$

$$= \cos (x + iy)$$

$$= \cos z$$

$f(z) = \cos z$

07/05/2017

6. $f(z) = \ln z$

~~u+iv~~

~~x ln~~

$\Rightarrow u+iv = \ln(x+iy)$

$= \ln r(\cos\theta + i\sin\theta)$ [where $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}\frac{y}{x}$]

$= \ln r e^{i\theta}$

$= \ln r + \ln e^{i\theta}$

$= \ln\sqrt{x^2+y^2} + i\theta \ln e$

$= \frac{1}{2}\ln(x^2+y^2) + i\tan^{-1}\frac{y}{x} \cdot 1$

$\therefore u = \frac{1}{2}\ln(x^2+y^2), \quad v = \tan^{-1}\frac{y}{x}$

$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x$, $\quad \frac{\partial v}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{-y}{x^2}$

$= -\frac{y}{x^2+y^2}$

$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y \quad \frac{\partial v}{\partial y} = \frac{1}{1+\frac{y}{x}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2}$

Here,
we see that,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ CR eq^{n}s are satisfied.

Hence, $f(z) = \ln z$ is an analytic function except $x^2+y^2=0$

or, $x = y = 0$ $\boxed{\text{origin}}$

7 → S.S.

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$$W = z|z|$$

$$\Rightarrow u+iv = (x+iy)\,|x+iy|$$

$$= (x+iy)\sqrt{x^2+y^2}$$

$$= x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$$

$\therefore u = x\sqrt{x^2+y^2}$, $v = y\sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{x \cdot 1}{2\sqrt{x^2+y^2}} \cdot 2x + 1 \cdot \sqrt{x^2+y^2}$$

$$= \frac{x^2+y^2+x^2}{\sqrt{x^2+y^2}}$$

$$= \frac{2x^2+y^2}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = x \frac{1}{2\sqrt{x^2+y^2}} \cdot 2y$$

$$= \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial x} = y \frac{2x}{2\sqrt{x^2+y^2}} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial v}{\partial y} = 1 \sqrt{x^2+y^2} + y \frac{1}{2\sqrt{x^2+y^2}} 2y$$

$$= \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

Here,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{when} \quad x = y$$

But $$\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x} \quad \text{when} \quad x = y$$

$\therefore$ CR eqns are not satisfied ~~anywhere~~ at any point.

Hence, the given func is not analytic anywhere.

Q. Show that,

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

Soln:

$$u + iv = \begin{cases} \dfrac{x^3 - y^3}{x^2+y^2} + i\,\dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\therefore u = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$v = \begin{cases} \dfrac{x^3+y^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

মূল নিয়মে অন্তরীকরণ,

$$\frac{\partial u(0,0)}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\overset{\overset{y \text{ fixed}}{\downarrow}}{0}) - u(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{h^3-0}{h^2+0} - 0}{h}$$

$$= \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

x constant

$$\frac{\partial u(0,0)}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k) - u(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{0-k^3}{0+k^2} - 0}{k}$$

$$= \lim_{k \to 0} \frac{-k}{k}$$

$$= \lim_{k \to 0} (-1)$$

$$= -1$$

$$\frac{\partial v(0,0)}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{h^3+0}{h^2+0} - 0}{h}$$

$$= \lim_{h \to 0} \frac{h}{h} = \lim_{h \to 0} 1 = 1$$

$$\frac{\partial v(0,0)}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{\frac{0+k^3}{0+k^2} - 0}{k}$$

$$= \lim_{k \to 0} \frac{k}{k} = \lim_{k \to 0} 1 = 1$$

Here,

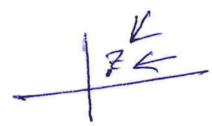
$$\frac{\partial u(0,0)}{\partial x} = \frac{\partial v(0,0)}{\partial y} \text{ and } \frac{\partial u(0,0)}{\partial y} = -\frac{\partial v(0,0)}{\partial x}$$

$\therefore$ CR eq^ns are satisfied at $z = 0$

$$\boxed{f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}}$$

$$f'(0) = \lim_{z\to 0} \frac{f(z) - f(0)}{z}$$

$$= \lim_{z\to 0} \frac{\dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}}{x+iy}$$



Now let,

$z \to 0$ along $x = y$, $x = 2y$ etc....

$$f'(0) = \lim_{x\to 0} \frac{x^3(1+i) - x^3(1-i)}{(x^2+x^2)(x+ix)}$$

$$= \lim_{x\to 0} \frac{x^3(1+i-1+i)}{x^3\, 2(1+i)}$$

$$= \lim_{x\to 0} \frac{i}{1+i}$$

$$= \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{1+i}{2}$$

Again let $z \to 0$ along $y=0$

$$f'(0) = \lim_{x \to 0} \frac{x^3(1+i) - 0}{(x^2+0)(x+0)}$$

$$= \lim_{x \to 0} (1+i)$$

$$= 1+i$$

~~$\therefore f'(0)$~~

We see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z=0$.

Exercise 7.1

13/05/17

Harmonic Function

Def^n: The function that ~~satisf~~ satisfies the Laplace eq^n is called a Harmonic Function.

$f(x_1, x_2, \ldots, x_n)$
Laplace's eq^n for function f,

$$\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2} = 0$$

$f(z) = u + iv$

$u(x,y)$, $v(x,y)$

Harmonic if,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

Theory: If $f(z) = u + iv$ is an analytic function then u and v are both harmonic function.

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \quad \text{and} \Rightarrow \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

again $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$$\Rightarrow \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 v}{\partial y^2} \quad \Rightarrow$$

Such function ~~~~ u and v are called conjugate harmonic function as function.

u+iv is also analytic function.

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$$u = x^2 - y^2$$

$$v = \frac{y}{x^2+y^2}$$

prove that u and v are harmonic function but are not conjugate harmonic.

Soln:

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial u}{\partial y} = -2y$$

$$\frac{\partial^2 u}{\partial x^2} = 2 \qquad \frac{\partial^2 u}{\partial y^2} = -2$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 - 2 = 0$$

Hence u satisfies Laplace's eqn

u is harmonic

$$\frac{\partial v}{\partial x} = \frac{-y}{(x^2+y^2)^2}\,2x = -\frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial x^2} = -\frac{(x^2+y^2)^2\,2y - 2xy\cdot 2(x^2+y^2)\,2x}{(x^2+y^2)^4}$$

$$= -\frac{2y(x^2+y^2)^2(1-4x^2)}{(x^2+y^2)^4}$$

$$= -\frac{2y(x^2+y^2)\,(x^2+y^2-4x^2)}{(x^2+y^2)^4}$$

$$= -\frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{(x^2+y^2)\,1 - y\,2y}{(x^2+y^2)^2}$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2y(x^2+y^2)\cancel{()} - (x^2-y^2)\, 2(x^2+y^2)\, 2y}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)(-x^2-y^2-2x^2-2y^2)}{(x^2+y^2)^4}$$

$$= \frac{2y(y^2-3x^2)}{(x^2+y^2)^3}$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

v satisfies the ~~to~~ Laplace's eqn.

Hence, v is harmonic.

$$\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$$

C-R eqn are not satisfied.

$\therefore$ u+iv is not analytic function.

Hence, u and v are not harmonic conjugate.

Method to find conjugate harmonic function:

$$\frac{dv}{dx} = \frac{\partial v}{\partial x}\frac{dx}{dx} + \frac{\partial v}{\partial y}\frac{dy}{dx}$$

$$= -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy$$

$Mdx + Ndy = 0$

exact হবে যখন

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$-\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial x^2}$

$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

$$\int dv = \underbrace{\int\left(-\frac{\partial u}{\partial y}\right)dx}_{y\ \text{constant}} + \underbrace{\int\left(\frac{\partial u}{\partial x}\right)dy}_{\text{without terms of } x}$$

~~#12~~ 10 $u = x^2 - y^2$

$$\int dv = \underbrace{\int \left(-\frac{\partial u}{\partial y}\right) dx}_{y \text{ constant}} + \underbrace{\int \frac{\partial u}{\partial x}\, dy}_{\text{without terms of } x}$$

$$\Rightarrow v = \int 2y \, dx + \int 2x \, dx$$

$$= 2xy + c$$

∴ The analytic function,

$f(z) = u + iv$

$= x^2 - y^2 + i(2xy + c)$

$= x^2 + i^2 y + 2xiy + ci$

$= (x+iy)^2 + ci$

$= z^2 + ci$

12 if $f(z) = u+iv$ is analytic and

$u = 3x - 2xy$

find v and $f(z)$ in terms of z.

$$\frac{\partial u}{\partial x} = 3-2y$$

$$\frac{\partial u}{\partial y} = -2x$$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

$$= -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

$$= 2x\, dx + (3-2y)\, dy$$

~~this is an exact differen~~

$$\int dv = \int 2x\, dx + \int (3-2y)\, dy$$

$$\Rightarrow v = x^2 + 3y - y^2 + c$$

$$f(z) = u + iv$$

$$= 3x - 2xy + i(x^2 + 3y - y^2 + c)$$

$$= 3x + i^2 2xy + ix^2 + 3yi + i^2 iy^2 + ci$$

$$= 3(x + iy) + i(x^2 + 2xiy + i^2y^2) + ci$$

$$= 3(x+iy) + i(x+iy)^2 + ci$$

$$= 3z + iz^2 + ci$$

$$= iz^2 + 3z + ci$$

13, 14

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Construct an analytic function $f(z)$ of which the real part is $e^x \cos y$

Soln:

$u = e^x \cos y$

$\frac{\partial u}{\partial x} = e^x \cos y$ $\qquad$ $\frac{\partial u}{\partial y} = -e^x \sin y$

$\frac{\partial^2 u}{\partial x^2} = e^x \cos y$ $\qquad$ $\frac{\partial^2 u}{\partial y^2} = -e^x \cos y$

Here,

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

satisfies Laplace's eqn.

So, u is a harmonic function.

$$dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy$$

$$= -\frac{\partial u}{\partial y}\,dx + \frac{\partial u}{\partial x}\,dy \quad [\because \text{C-R eq}^n]$$

$$\frac{\partial u}{x} =$$

$$= e^x \sin y\,dx + e^x \cos y\,dy$$

This is an exact differential eq^n

$$\therefore\ v = \underbrace{\int e^x \sin y\,dx}_{y\text{-constant}} + \underbrace{\int \boxed{e^x \cos y}\,dy}_{\text{ignoring terms containing } x}$$

পুরো আকারে ; পুরোটা একটা term

$$\cancel{= e^x \sin y + \sin y + c}$$

$$= e^x \sin y + \int 0\,dy$$

$$= e^x \sin y + c$$

$\cancel{f(z)} =$

$$f(z) = u+iv$$

$$= e^x \cos y + i(e^x \sin y + c)$$

$$= e^x(\cos y + i\sin y) + ci$$

$$= e^x e^{iy} + ci$$

$$= e^{x+iy} + ci$$

$$= e^z + ci$$

17. $u - v = (x-y)(x^2+4xy+y^2)$

$f(z) = u+iv$ is an analytic function.

Find $f(z)$ in terms of z

Soln:

$$u+iv = f(z) \quad \text{--- (1)}$$

$$i(u+iv) = i f(z)$$

$\Rightarrow iu - v = i f(z)$ ...... (ii)

(i) + (ii)

$$u - v + i(u+v) = (1+i) f(z)$$

or, $U + iV = F(z)$

where, $U = u - v = (x-y)(x^2+4xy+y^2)$

$$= x^3 + 4x^2y + xy^2 - yx^2 - 4xy^2 - y^3$$

$$= x^3 + 3x^2y - 3xy^2 - y^3$$

$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$; $\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$

and $F(z) = (1+i) f(z)$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy$$

$$= -\frac{\partial U}{\partial y} dx + \frac{\partial U}{\partial x} dy$$

$$= (-3x^2 + 6xy + 3y^2) dx + (3x^2 + 6xy - 3y^2) dy$$

This is an exact differential eq^n.

$$\therefore V = \int_{y\text{-constant}} \left(-3x^2 + 6xy + 3y^2\right) dx + \int_{\text{without terms of } x} \left(3x^2 + 6xy - 3y^2\right) dy$$

~~ignoin~~

$$= -x^3 + 3x^2y + 3y^2x - y^3 + C$$

$$F(z) = U + iV$$

$$\Rightarrow (1+i)\, f(z) = x^3 + 3x^2y - 3xy^2 - y^3 + i\left(-x^3 + 3x^2y + 3y^2x - y^3 + c\right)$$

$$= x^3(1-i) + 3x^2y\,(1+i) - 3xy^2(1-i)$$

$$= -y^3(1+i) + c_i$$

$$= (1-i)\left\{x^3 + 3x^2y\frac{(1+i)}{(1-i)} - 3xy^2 - y^3\frac{(1+i)}{(1-i)}\right\} + ci$$

$$= (1-i)\left(x^3 + i3x^2y - 3xy^2 - iy^3\right) + ci$$

$$= (1-i)\left\{x^3 + 3x^2iy + 3x(iy)^2 + (iy)^3\right\} + ci$$

$$= (1-i)(x+iy)^3 + ci$$

$$\therefore f(z) = \frac{1-i}{1+i}(x+iy)^3 + \frac{ci}{1+i}$$

$$= \frac{(1-i)(1+i)}{(1+i)(1+i)}(x+iy)^3 + \frac{ci}{1+i}$$

$$= -iz^3 + \frac{ci}{1+i}$$

18 → S.S.

Milne Thomson Method:

(i) If u is given then

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

where, $\phi_1(x,y) = \frac{\partial u}{\partial x}$

and $\phi_2(x,y) = \frac{\partial u}{\partial y}$

(ii) If v is given then

$$f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz$$

where, $\psi_1(x,y) = \frac{\partial v}{\partial y}$

and $\psi_2(x,y) = \frac{\partial v}{\partial x}$

20/05/2017

Exercise 7·2 (4 - 11)

5.

$$u = \ln\sqrt{x^2+y^2} = \ln(x^2+y^2)^{\frac{1}{2}}$$

$$= \frac{1}{2}\ln(x^2+y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2x = \frac{x}{x^2+y^2} = \phi_1(x,y)$$

$$\frac{\partial u}{\partial y} = \frac{1}{2}\frac{1}{x^2+y^2}\cdot 2y = \frac{y}{x^2+y^2} = \phi_2(x,y)$$

By Milne Thomson method,

$$f(z) = \int \phi_1(z,0)\,dz - i\int \phi_2(z,0)\,dz$$

$$= \cancel{\int \frac{x}{x^2+y}}$$

$$= \int \frac{z}{z^2+0}\,dz - i\int \frac{0}{z^2+0}\,dz$$

$$= \int \frac{1}{z} dz - i \int (0) dz$$

$$= \ln z + C$$

which is the analytic function.

in exam Q. will be to prove Harmonic first

12 - 15

12

$$v = \ln(x^2+y^2) + x - 2y$$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 = \psi_2(x,y)$$

$$\frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} - 2 = \psi_1(x,y)$$

By Milne Thomson method,

$$f(z) = \int \psi_1(z,0)\, dz + i \int \psi_2(z,0)\, dz$$

$$= \int \left(\frac{0}{z^2+0} - 2\right) dz + i \int \left(\frac{2z}{z^2+0} + 1\right) dz$$

$$= \int (\cancel{\theta} 0 - 2)\, dz + i\int\left(\frac{2}{z} + 1\right) dz$$

$$= \cancel{\ln z}\, 0 - 2z + i2\ln z + iz + C$$

$$= -2z + i2\ln z + iz + C$$

$$= 2i\ln z + z(i-2) + C$$

which is the analytic function.

$$f(z) = 2i\ln(re^{i\theta}) + (x+iy)(i-2) + C$$

$$\left[r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\frac{y}{x}\right]$$

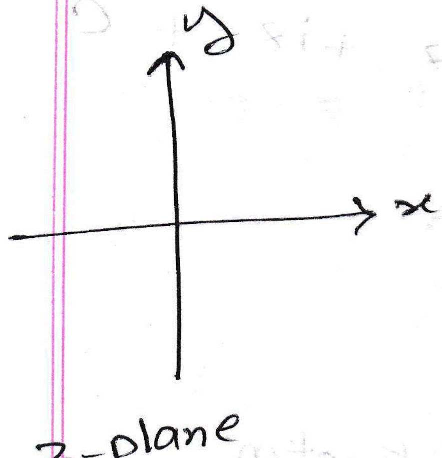
$$= 2i\ln r + 2i\ln e^{i\theta} + xi - 2x - y - 2iy + C$$

$$= 2i\ln\sqrt{x^2+y^2} + 2i.i\theta\ln e + xi - 2x - y - 2iy + C$$

$$u + iv = i\{\ln(x^2+y^2) + x - 2y\} - 2\tan^{-1}\frac{y}{x} - 2x - y + C$$

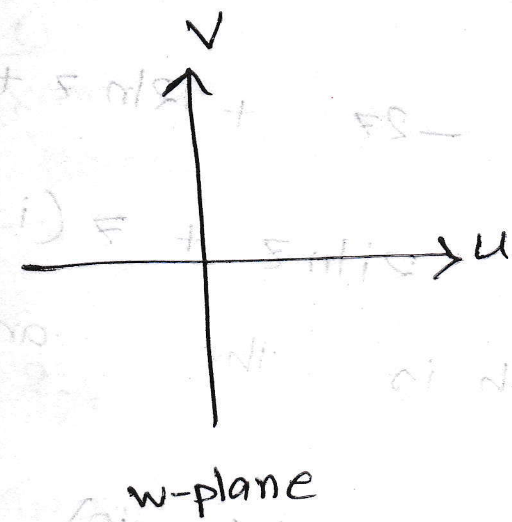
$$\therefore u = -2\tan^{-1}\frac{y}{x} - 2x - y + C$$

Transformation



$w = f(z)$

$\Rightarrow u + iv = f(x + iy)$



25 Transform the curve $x^2 - y^2 = 4$ under the mapping $w = z^2$

hyperbolla

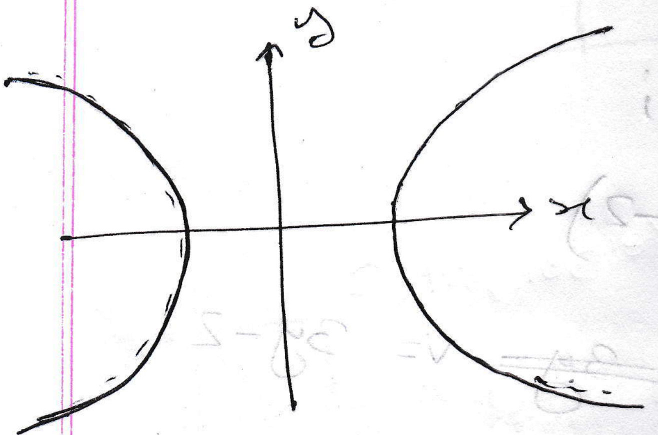
Soln:

$w = z^2$

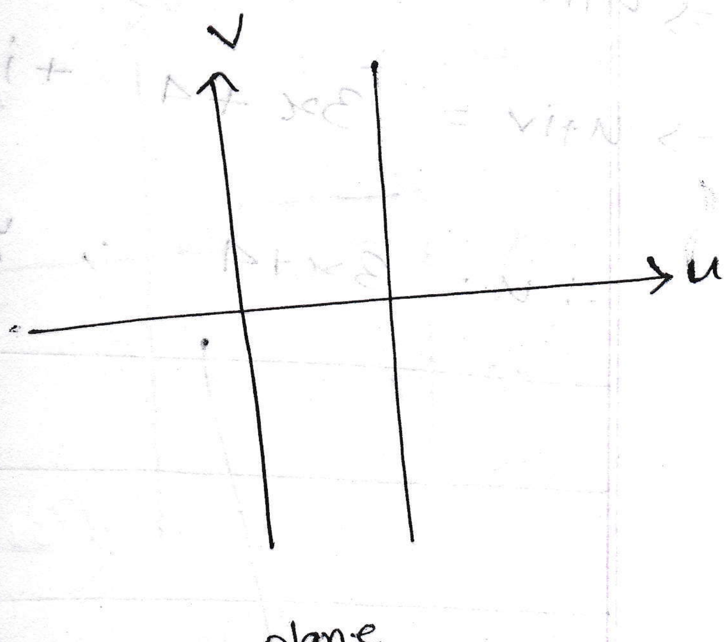
$\Rightarrow u + iv = (x+iy)^2 = x^2 - y^2 + i\,2xy$

$u = x^2 - y^2, \quad v = 2xy$

x	2	2.5	3	3.5	4	
$\pm\sqrt{x^2-4}=y$	0	± 1.5	± 2.2	± 2.9	± 3.5	
$x^2-y^2=u$	4	4	4	4	4	
$2xy=v$	0	± 7.5	± 13.2	± 20.3	± 28	



z-plane



w-plane

26 Find image of the triangle with vertex $i;\ 1+i,\ 1-i$

(i) $w = 3z + 4 - 2i$

(ii) $w = e^{\frac{5\pi i}{3}} z - 2 + 4i$

(i) Sol^n:

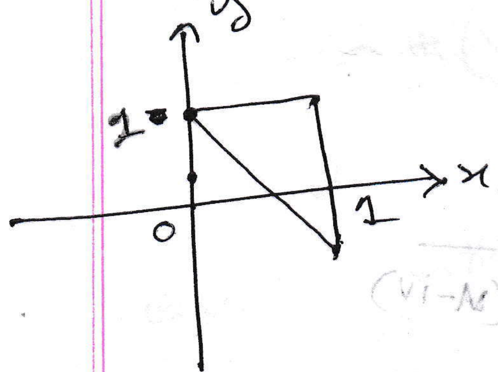
$$w = 3z + 4 - 2i$$

$$\Rightarrow u + iv = 3(x+iy) + 4 - 2i$$

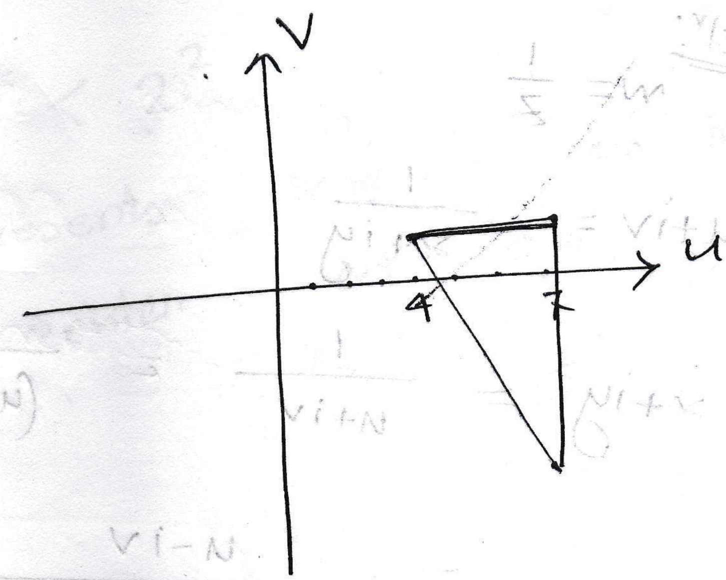
$$\Rightarrow u + iv = 3x + 4 + i(3y - 2)$$

$$\therefore u = 3x + 4, \quad \cancel{y = 3y} \quad v = 3y - 2$$

$z\ (x,y)$	$w\ (u,v)$
(i) $z=i$ $(x=0, y=1)$	$w=4+i$ $(u=4,\ v=1)$
(ii) $z=1+i$ $(x=1, y=1)$	$w=7+i$ $(u=7,\ v=1)$
(iii) $z=1-i$ $(x=1, y=-1)$	$w=\ =7-5i$ $(u=7,\ v=-5)$



z - plane



w-plane

27

Find the image of the infinite st

(i) $\frac{1}{4} < y < \frac{1}{2}$ (ii) $0 < y < \frac{1}{2}$

under the mapping $w = \frac{1}{z}$

Soln:

$$w = \frac{1}{z}$$

$$\Rightarrow u + iv = \frac{1}{x+iy}$$

$$\Rightarrow x + iy = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)(u-iv)}$$

$$= \frac{u-iv}{u^2+v^2}$$

$$= \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$$

$$\therefore x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$$

(i) $\frac{1}{4} < y < \frac{1}{2}$

or, $\frac{1}{4} < -\frac{v}{u^2+v^2} < \frac{1}{2}$

Now,

$$\frac{1}{4} < -\frac{v}{u^2+v^2}$$

or, $u^2+v^2 < -4v$

or, $u^2+v^2+4v < 0$

or, $u^2+(v+2)^2 < 2^2$

which represents interior of the circle with center at $(0,-2)$ and radius 2.

Now,

$$-\frac{v}{u^2+v^2} < \frac{1}{2}$$

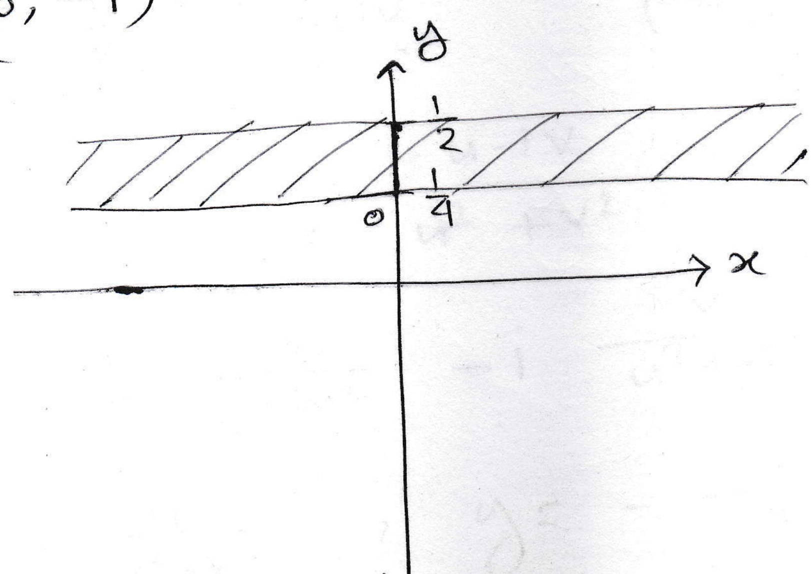
or, $-2v < u^2+v^2$

or, $0 < u^2+v^2+2v$

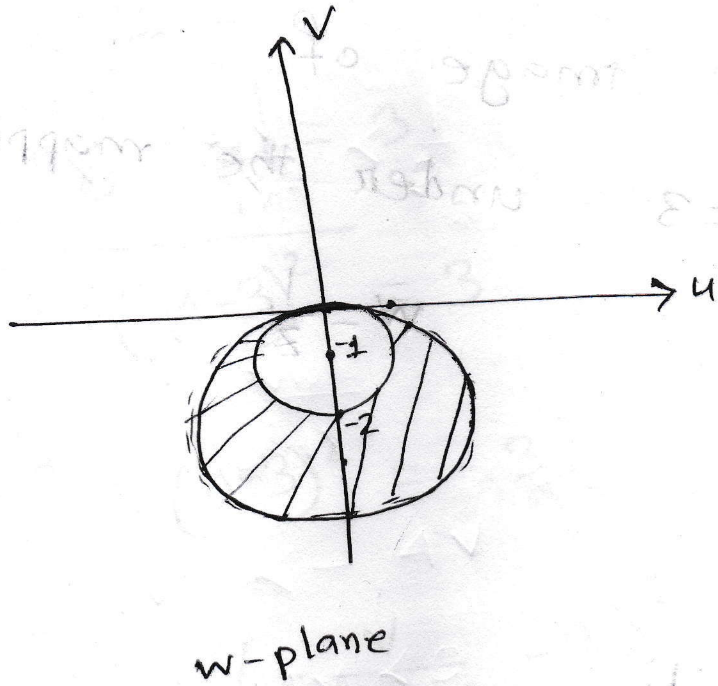
or, $u^2+v^2+2v > 0$

or, $u^2+(v+1)^2 > 1^2$

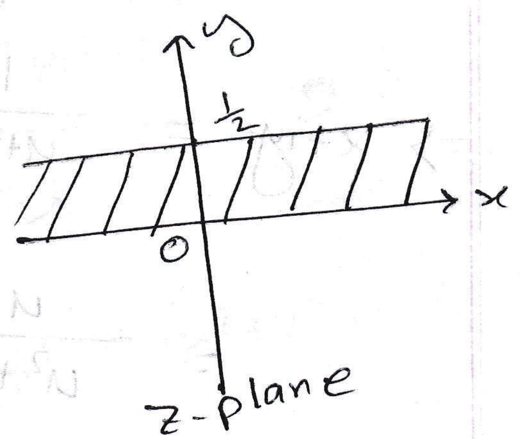
which represents the exterior of the circle with center at $(0,-1)$ and radius 1



z-plane



ⓘⓘ $0 < y < \frac{1}{2}$



or, $0 < -\frac{v}{u^2+v^2} < \frac{1}{2}$

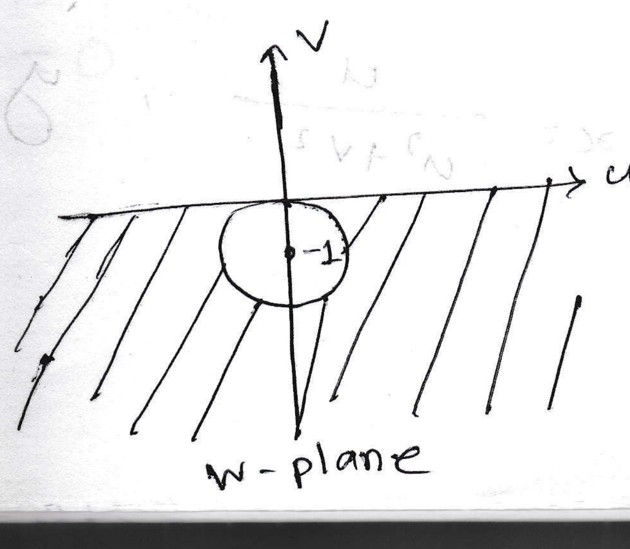
$\therefore \ 0 < -\frac{v}{u^2+v^2}$

or, $0 < -v$

or, $0 > v$

or, $v < 0$

or,



28

Find the image of

$|z-3i|=3$ under the mapping

$w = \frac{1}{z}$

Solⁿ:

Soln:

$w = \frac{1}{z}$

$\Rightarrow u+iv = \frac{1}{x+iy}$

$\Rightarrow x+iy = \frac{1}{u+iv} = \frac{(u-iv)}{(u+iv)(u-iv)}$

$= \frac{u}{u^2+v^2} - i\frac{v}{u^2+v^2}$

$\therefore x = \frac{u}{u^2+v^2}, \quad y = -\frac{v}{u^2+v^2}$

$$|z - 3i| = 3$$

$$\Rightarrow |x + iy - 3i| = 3$$

$$\Rightarrow \sqrt{x^2 + (y-3)^2} = 3$$

$$\Rightarrow x^2 + (y-3)^2 = 3^2$$

$$\Rightarrow \left(\frac{u}{u^2+v^2}\right)^2 + \left(-\frac{v}{u^2+v^2} - 3\right)^2 = 3^2$$

$$\Rightarrow \frac{u^2}{(u^2+v^2)^2} + \frac{\{v + 3(u^2+v^2)\}^2}{(u^2+v^2)^2} = 9$$

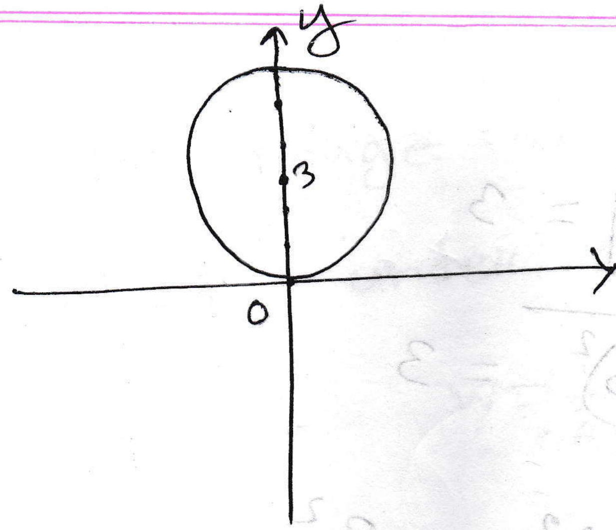
$$\Rightarrow u^2 + v^2 + 6v(u^2+v^2) + 9(u^2+v^2)^2 = 9(u^2+v^2)^2$$

$$\Rightarrow (u^2+v^2)(1+6v) = 0$$

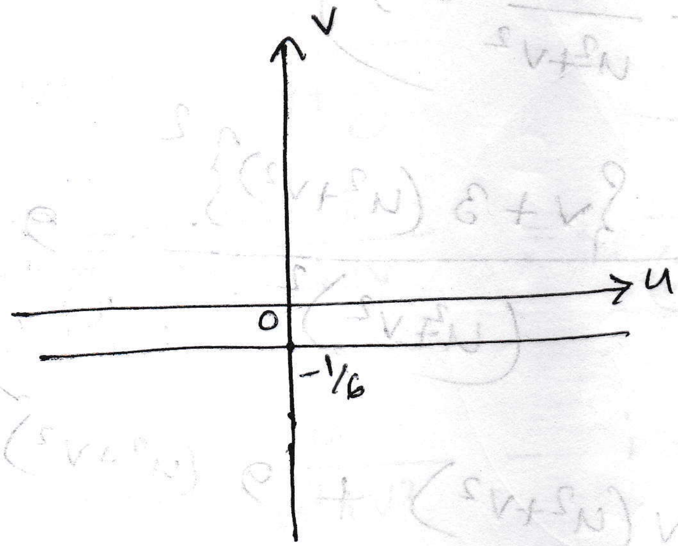
$$\because u^2 + v^2 \neq 0$$

$$\therefore 1 + 6v = 0$$

$$\Rightarrow v = -\frac{1}{6}$$



z-plane



w-plane

$|z-3i|=2$, $w=\frac{1}{z}$

32

Find the image of the mapping of the region $1 \le x \le 2$, $2 \le y \le 3$ under the mapping of $w = e^z$

Solⁿ:

$w = e^z$

$\Rightarrow u + iv = e^{x+iy} = e^x \cdot e^{iy}$

$\Rightarrow R e^{i\varphi} = e^x e^{iy}$

$$\boxed{\begin{aligned} R &= \sqrt{u^2+v^2} \\ \varphi &= \tan^{-1}\frac{v}{u} \end{aligned}}$$

Comparing both sides,

$R = e^x$

and $\varphi = y$

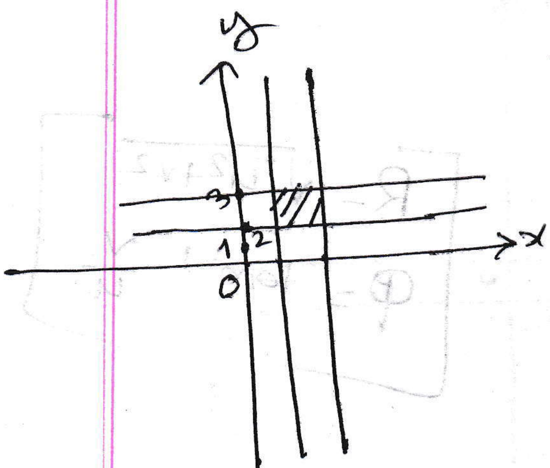
Here, $1 \le x \quad \therefore R \ge e$

and $x \le 2 \quad \therefore R \le e^2$

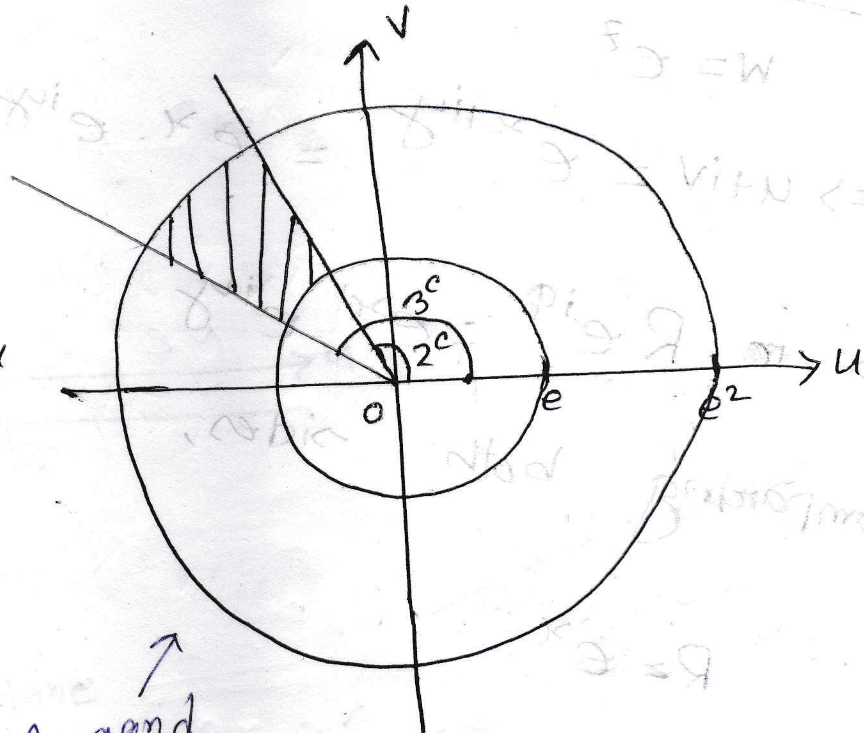
$\therefore \quad e \le R \le e^2$

and $\quad 2 \le \phi \le 3$

degree দেয়া নাই, ∴ Redian



z-plane



Argand

w-plane

Exercise: 7.3

Complex Integration

32

Find the value of,

$$\int_C (x+y)\,dx + x^2 y\,dy$$

ⓐ along $y = x^2$ $(0,0)$ to $(3,9)$

ⓑ along $y = 3x$ $(0,0)$ to $(3,9)$

Solⁿ:

ⓐ along $y = x^2$ $\therefore dy = 2x\,dx$

$$\int_C (x+\cancel{x^2}y)\,dx + x^2 y\,dy$$

$$= \int_0^3 (x+x^2)\,dx + x^2 . x^2 . 2x\,dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{3} + 2\frac{x^6}{6} \right]_0^3$$

$$= \left(\frac{9}{2} + 9 + 3^5 \right)$$

$$= 256 \cdot 5$$

ⓑ along $y = 3x \quad \therefore dy = 3\,dx$

$$\int_C (x+y)\,dx + x^2 y\,dy$$

$$= \int_0^3 (x+3x)\,dx + \frac{x^2 . 3dx}{3x . x^2 3dx}$$

where is $y'' = 3x$?

$$= \int_0^3 4x\,dx + 3x^2\,dx = \left[4\frac{x^2}{2} + 3\frac{x^3}{3} \right]_0^3$$

$$= \left[2x^2 + x^3\right]_0^3$$

$$= 2 \cdot 3^2 + 3^3$$

$$= 18 + 27$$

$$= 45$$

27/05/2017

40 evaluate,

$$\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + 3y^2)\, dx + 2(x^2 + 3xy + 4y^2)\, dy$$

(a) along $y^2 = x$

(b) along $y = x^2$

(c) along $y = x$

Soln:

(a) Along $x = y^2$ $\therefore dx = 2y\,dy$

$$\int_{(0,0)}^{(1,1)} (3x^2 + 4xy + 3y^2)\,dx + 2(x^2 + 3xy + 4y^2)\,dy$$

$$= \int_0^1 (3y^4 + 4y^3 + 3y^2)\,2y\,dy + 2(y^4 + 3y^3 + 4y^2)\,dy$$

$$= \left[6\frac{y^6}{6} + 8\frac{y^5}{5} + 6\frac{y^4}{4} + 2\frac{y^5}{5} + 6\frac{y^4}{4} + 8\frac{y^3}{3} \right]_0^1$$

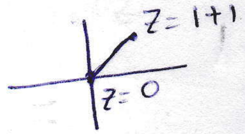
$$= 1 + \frac{8}{5} + \frac{3}{2} + \frac{2}{5} + \frac{3}{2} + \frac{8}{3}$$

$$= \frac{26}{3}$$

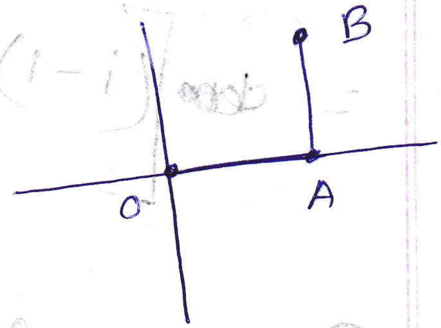
41 Find the value of,

$$\int_0^{1+i} (x - y + ix^2)\, dz$$

(a) along the straight line $z=0$ to $z = 1+i$



(b) along real axis



Soln:

(a) ~~(0,0)~~, (1,1) along the straight line $z=0$ to $z=1+i$ that is, $y = x$ $\therefore dy = dx$

$$\therefore dz = dx + i\,dy$$

$$= dx + i\,dx$$

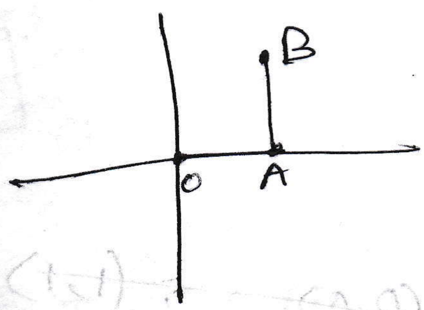
$$= (1+i)\,dx$$

$$\int_0^{1+i} (x - y + ix^2)\, dz$$

$$= \int_0^1 (x - x + ix^2)(1+i)\, dx$$

$$= \left[(i-1)\frac{x^3}{3}\right]_0^1 = \frac{i-1}{3}$$

ⓑ Along OA,



$y = 0 \quad \therefore dy = 0$

$dz = dx + i\,dy = dx$

$$\int_{OA} (x - y + ix^2)\, dz = \int_0^1 (x + ix^2)\, dx$$

$$= \left[\frac{x^2}{2} + i\frac{x^3}{3}\right]_0^1 = \frac{1}{2} + \frac{i}{3}$$

Along AB,

$x = 1 \quad \therefore dx = 0$

$\therefore dz = dx + i\,dy = i\,dy$

$$\int_{AB} (x - y + ix^2)\,dz = \int_0^1 (1 - y + i)\,i\,dy$$

$$= \left[iy - i\frac{y^2}{2} - y \right]_0^1$$

$$= i - \frac{i}{2} - 1$$

$$= -1 + \frac{i}{2}$$

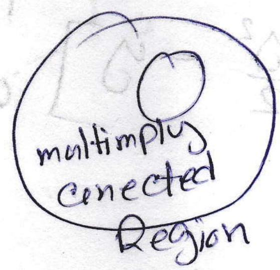
$$\therefore \int_0^{1+i} (x - y + ix^2)\,dz = \int_{OA} (x - y + ix^2)\,dz + \int_{AB} (x - y + ix^2)\,dz$$

$$= \frac{1}{2} + \frac{i}{3} - 1 + \frac{i}{2}$$

$$= -\frac{1}{2} + \frac{5}{6} i$$

Exercise 7.5

Cauchy's Integral Theorem:



If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve ~~simply con~~ C

then,

$$\int_C f(z)\, dz = 0$$

প্রমাণ লাগবে না।

42 Find the value of

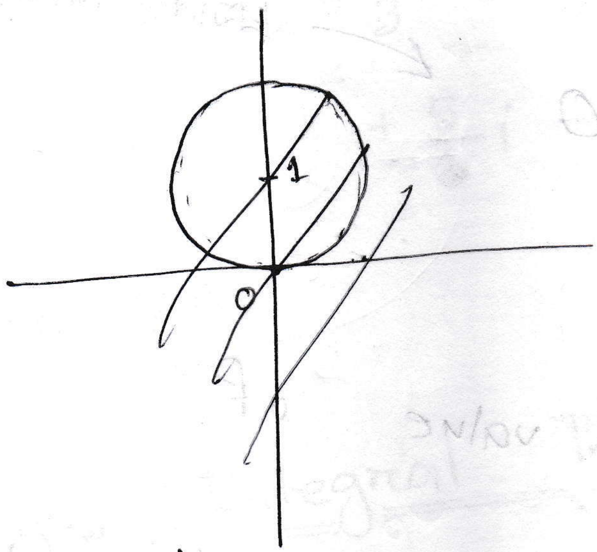
$$\int_C \frac{z+4}{z^2+2z+5}\, dz \quad \text{if } C : |z+1| = 1$$

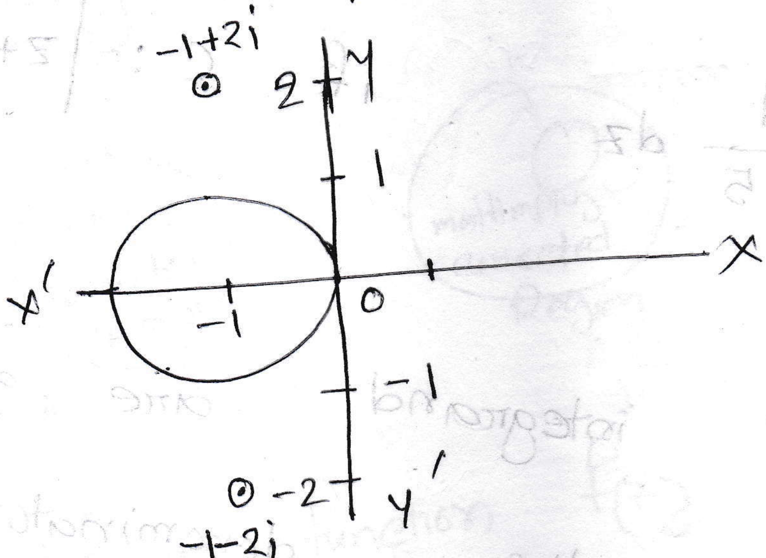
Soln:
Poles of integrand are given by putting the denominator equal to zero.

$$z^2+2z+5=0$$

$$\therefore z = \frac{-2 \pm \sqrt{2^2-4\cdot 5}}{2}$$

$$= -1 \pm 2i$$





The integral has two poles at $z = -1+2i$ and $z = -1-2i$ both poles are outside the given circle with center at $(-1, 0)$ and radius 1.

∴ The given integral is analytic within the given circle.

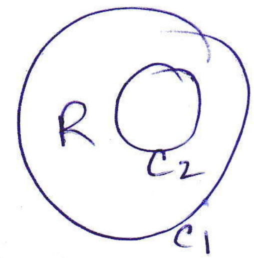
∴ By Cauchy's Integral theorem,

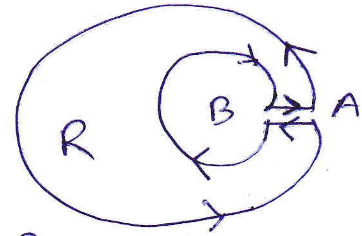
$$\int_C \frac{z+4}{z^2+2z+5}\, dz = 0$$

Extention of Cauchy's Integral theorem:

If a function $f(z)$ is analytic in the region R between two simple closed curve C_1 and C_2 then

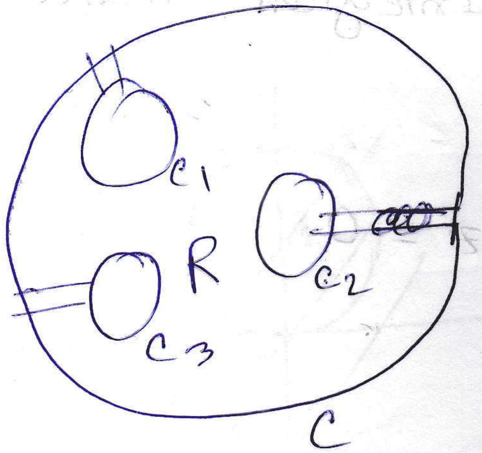
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$$





$$C_1 \quad \int_{C_1} f(z)\,dz + \int_{AB} f(z)\,dz - \int_{C_2} f(z)\,dz - \int_{AB} f(z)\,dz = 0$$

$$\int_{C_1} = \int_{C_2}$$



$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \cdots$$

$$\cdots\cdots + \int_{C_n} f(z)\,dz$$