Random Variables

Continuous Random Variable: We say that X is a continuous random variable if there exists a nonnegative function $f(x)$, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers

***** Must See the Document: "PDF and CDF.docx" for More Insights! see Book (Ross) also**

$$P\{X \in B) = \int_B f(x)dx$$

The function f(x) is called the *probability density function* of the random variable X. The relationship between the cumulative distribution $F(.)$ and the probability density function $f(x)$ is expressed by

$$F(a) = P\{X \in (-\infty, a)\} = \int_{-\infty}^{a} f(x)dx$$

P{ a <= X <= b} = Integrate f(x) dx from a to b

Uniform Random Variable: A random variable is said to be uniformly distributed over the interval (0,1) if its probability density function is given by

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that the preceding is a density function since $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)dx = \int_0^1 dx = 1$$

we say that X is a uniform random variable on the interval (α, β) if its probability density function is given by

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, & \text{if } \alpha < x < \beta \\ 0, & \text{otherwise} \end{cases}$$

Example 8: If X is uniformly distributed over (0,10), calculate the probability that (a) $X < 3$, (b) $X > 7$, (c) $1 < X < 6$.

Solution:

The PDF f(x) for the Random Variable that is Uniformly Distributed over (0, 10) is: $\frac{1}{10 - 0} = \frac{1}{10}$ if X is the range (0,10);

Otherwise, 0 (that is, x is outside the range of (0,10)

$$P\{X < 3\} = \frac{\int_0^3 dx}{10} = \frac{3}{10}$$

$$P\{X > 7\} = \frac{\int_7^{10} dx}{10} = \frac{10-7}{10} = \frac{3}{10}$$

$$P\{1 < X < 6\} = \frac{\int_1^6 dx}{10} = \frac{6-1}{10} = \frac{1}{2}$$

Exponential Random Variables: A continuous random variable whose probability density function is given for some $\lambda > 0$, by

$$\boxed{f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{if } x < 0 \end{cases}}$$

is said to be an exponential random variable with parameter λ.

We can check the validity of exponential distribution by following way.

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = \lambda \frac{1}{\lambda}\left[e^{-\lambda x}\right]_0^{\infty} = -1\cdot(0-1) = 1$$

Cumulative distribution function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} \lambda e^{-\lambda y}dy = \int_{0}^{x} \lambda e^{-\lambda y}dy = -\lambda\frac{1}{\lambda}\left[e^{-\lambda y}\right]_0^{x} = -(e^{-\lambda x}-1) = 1-e^{-\lambda x}$$

On the other hand, $P(X > x) = 1-(1-e^{-\lambda x}) = e^{-\lambda x}$

Properties of Exponential Distribution: A random variable X is said to be without memory; or *memoryless*, if $P\{X > s+t \mid X > t\} = P\{X > s\}$ for all $s, t \ge 0$.

If we think of X as being the lifetime of some instrument, then the probability that the instrument lives for at least *s* + *t* hours given that it has survived *t* hours is the same as the initial probability that it lives for at least *s* hours. That is, the instrument does not remember that it has already been in use for a time *t*.

From Bayes equation, $P(A/B) = \dfrac{P(AB)}{P(B)}$

$$\Rightarrow P\{X > s\} = \frac{P\{x > s+t, x > t\}}{P\{x > t\}} = \frac{P\{x > s+t\}}{P\{x > t\}}$$

$$\therefore P\{x > s+t\} = P\{X > s\}P\{x > t\}$$

$$\Rightarrow e^{-\lambda(s+t)} = e^{-\lambda s}e^{-\lambda t}$$

Thus, exponential distribution is memoryless.

because exponentiar distribution's Mean is (1/Lambda) => See this in next Lecture (i.e., Lec-15)
{But, for Poisson distribution, Mean = Lambda}

Example 9: Suppose that the amount of time one spends in a bank is exponentially distributed with mean ten minutes, that is $\lambda = 1/10$. What is the probability that a customer will spend more than fifteen minutes in the bank? What is the probability that a customer will spend more than fifteen minutes in the bank given that she is still in the bank after ten minutes?

******* Important ***** (5 STARS)**

***** Answer below is Incomplete! See this example from Book (Ross) ex. 5.2 (page 295 or 312)**

Solution: If X represents the amount of time that the customer spends in the bank, then the first probability is just $P\{X > 15\} = e^{-15\lambda} = e^{-3/2} \approx 0.223$ (Ans.)

Example 10: Suppose that the amount of time that a light bulb works before burning itself out is exponentially distributed with mean ten hours. Suppose that a person enters a room in which a light bulb is burning. If this person desire to work for five hours, then what is the probability that she will be able to complete her work without the bulb burning out?

lambda = 1/10 P{ X>x } = exp (-x*lamdda)

Solution: Since the bulb is burning when the person enters the room it follows bye memoryless property of the exponential, that its remaining lifetime is exponential with mean ten. Hence the desired probability is

$P\{\text{remaining lifetime} > 5\} = 1 - F(5) = e^{-5\lambda} = e^{-1/2} \approx 0.607$ (Ans.)

☺Good Luck☺