# PODS Lab 12: Confidence Intervals, Bootstrapping, and Bayes' Theorem

Hamza Alshamy

Center for Data Science, NYU
ha2486@nyu.edu

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1. Confidence Intervals (CI)
2. Bootstrapping
3. Bayes' Theorem
4. **Quiz + Discussion**

## Moving From a Point Estimate to an Interval Estimate

- **So far,** we have focused on **point estimates:**
  - **Examples:** Effect size, test statistic, p-value, power, sample mean $\bar{X}$
- **Now,** we shift to **interval estimates.**

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- **Option 1:** Take a single sample and compute the sample mean $\bar{X}$.
- **Option 2:** Construct a **range of plausible values** – an **interval**.

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### Definition: Confidence Interval (CI)

An interval estimate that captures plausible values for the true population parameter.

## Idea of Confidence Intervals

- **Sampling Error:** Every time we sample from a population, our sample statistic (e.g., $\bar{X}$) can differ due to random variability.
- **Consequence:** Different samples would give different estimates.
- **Big Idea:**
  - If we repeatedly sample from the same population and construct a confidence interval for each sample,
  - Then approximately 95% of those intervals would contain the true population parameter (e.g., the population mean), **assuming no sampling bias**.

## Clarifying Points

- **Question:** How can we build a confidence interval for something we don't know (population parameter)?
- **Answer:** The **Central Limit Theorem (CLT)** gives us the structure to do it.
  - **CLT:** If we repeatedly sample from a population and compute the sample mean $\bar{X}$ each time, then the distribution of the sample means will tend toward a **normal distribution**, regardless of the shape of the original population, provided the sample size is **large enough**.

**In short:** The sampling distribution of a sample statistic (e.g. $\bar{X}$) becomes approximately normal even if the population itself is not.

## How to Calculate Confidence Intervals

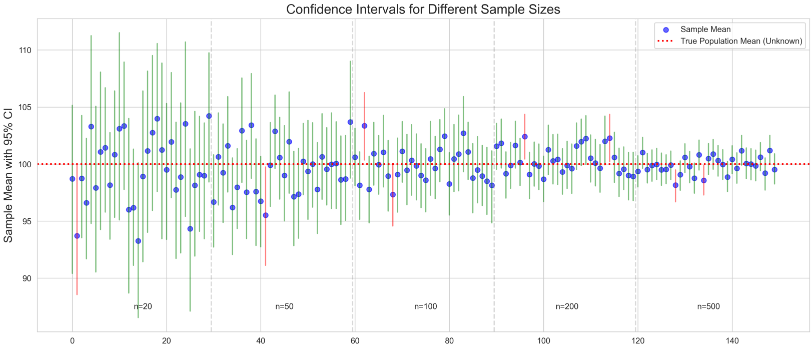
1. Declare a **confidence level:** 99.9%, 99%, 95%, 50%
   - Find the **critical value** $z$ that corresponds to your confidence level
2. Pick a **sample size** $n$
3. **Measure** the sample statistic (e.g., sample mean $\bar{X}$)
4. Plug into the confidence interval formula

### Confidence Interval Formula

$$\text{CI} = \bar{X} \pm (z_{\alpha/2} \times \text{SE}) \qquad \text{,where}$$

- **SE** $= \frac{\sigma}{\sqrt{n}}$ or $\frac{s}{\sqrt{n}}$ if $\sigma$ is unknown.
- $z$ **(Confidence level):** 99.9% $\rightarrow$ 3.29; 99% $\rightarrow$ 2.58; 95% $\rightarrow$ 1.96; 90% $\rightarrow$ 1.645; 80% $\rightarrow$ 1.28; 50% $\rightarrow$ 0.674 (From z-table)

# Confidence Intervals Visualization and interpretation



Each point represents a sample mean with its 95% confidence interval.
Green intervals contain the true population mean, red intervals do not.
Note how larger sample sizes produce narrower confidence intervals.

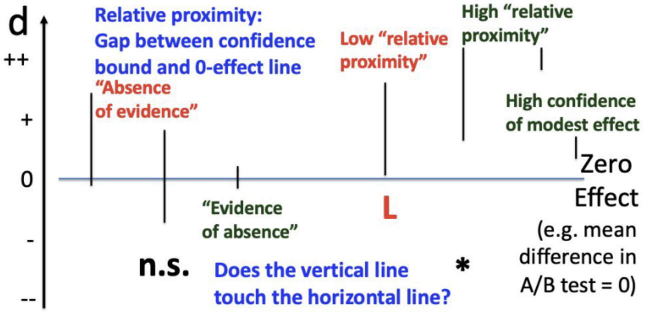
**Observe:**

1. At the same confidence level (95%) and sample size, **confidence intervals may vary** due to sampling variability.
2. **Larger samples yield narrower intervals** because standard error decreases as $n$ increases.

## How to Interpret Confidence Intervals

- **A 95% confidence interval means:**
  - If we repeated sampling many times and constructed a confidence interval each time,
  - About 95% of those intervals would contain the true population parameter.
- **Important:**
  - It does **not** mean there is a 95% probability that the true parameter is inside **this particular** interval.
  - The true parameter is fixed; the interval is random.
- **Tradeoff Between Confidence and Precision:**
  - Higher confidence levels (e.g., 99%) result in wider intervals, as they must include more possible values.
  - Narrower intervals offer greater precision but are less likely to include the true parameter.

## How to Interpret Confidence Intervals




| Interval | Interpretation |
|---|---|
| Contains 0 | No significant difference between groups |
| Entirely positive (+, +) | Group 1 $>$ Group 2 (Significant) |
| Entirely negative $(-, -)$ | Group 1 $<$ Group 2 (Significant) |

## Assumptions of Confidence Intervals

1. Random Sampling
2. Independence of Observations
3. Sufficiently Large Sample Size
4. No Sampling Bias
5. **Normality of the Population Distribution**:
   - For **small sample sizes**, many confidence interval methods assume that the population from which the sample is drawn is normally distributed.
   - For **larger sample sizes**, the **Central Limit Theorem (CLT)** ensures that the sampling distribution of the statistic becomes approximately normal, even if the population itself is skewed.

- **Problem:** Not all statistics naturally distribute normally – especially for small samples, ratios, or rare events.
  - **Solution:** Bootstrap!

## Bootstrapping

- **Bootstrapping:** A resampling technique to approximate the sampling distribution of a statistic.

1. Obtain one sufficiently large and representative sample (the **original sample**).
2. Sample **with replacement** from the original sample to create many **bootstrap samples**.
   - We treat the original sample as if it were the entire population.
3. Calculate the sample statistic (e.g., mean, median, effect size) for each bootstrap sample.
4. Build a sampling distribution from these bootstrap statistics.

## Two Options for Building Confidence Intervals (Bootstrapping)

- After generating bootstrap samples, you have two common ways to construct a confidence interval:

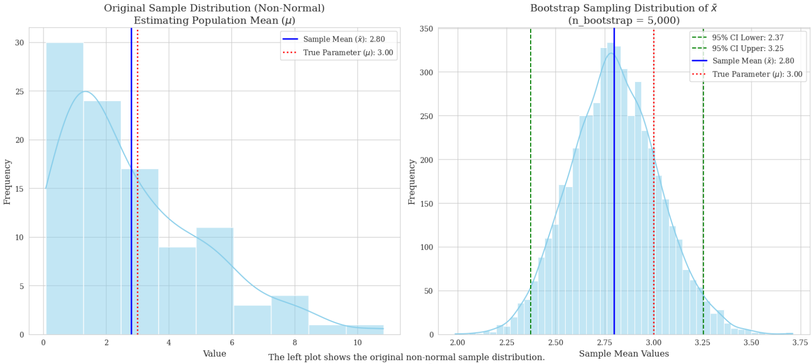
1. **Percentile Method:** (Our Method)

$$\text{CI}_{95\%} = [\text{2.5th percentile},\ \text{97.5th percentile}]$$

2. **Standard Error Method** (similar to before):

$$\text{CI} = \bar{x}_{\text{original}} \pm (z \times \text{SE}_{\text{bootstrapped}})$$

# Bootstrapping and Confidence Interval Visualization




The left plot shows the original non-normal sample distribution.
The right plot shows the bootstrap sampling distribution of $\bar{x}$ with 95% confidence interval for $\mu$ (green lines).
Note how the bootstrap distribution approximates normality even though the original data is skewed.

- From the original sample, we drew 5,000 bootstrap samples (sampling with replacement).
- We calculated the sample mean for each bootstrap sample to build the sampling distribution.
- The 95% confidence interval is obtained by slicing the bootstrap distribution at the 2.5th and 97.5th percentiles.

## Reminder! Conditional Probability: Updating Beliefs

**Key Idea:** Conditional probability helps us revise our beliefs when new information is available.

### Conditional Probability Formula

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

**In words:** What is the probability $B$ Given that $A$ has happened? how does this affect our belief in $B$?

## From Conditional Probability to Bayes' Rule

- **Bayes' Rule:** Allows us to "reverse" conditional probabilities.

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)}$$

**Rewriting:**

$$P(A \cap B) = P(B \mid A) \times P(A)$$

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### Deriving Bayes' Rule

$$P(B \mid A) \times P(A) = P(A \mid B) \times P(B)$$

$$\implies \quad P(A \mid B) = \frac{P(B \mid A) \times P(A)}{P(B)}$$

## Some Terminology

### Bayes' Rule with Interpretation

$$\underbrace{P(A \mid B)}_{\text{Posterior}} = \frac{\overbrace{P(B \mid A)}^{\text{Likelihood}} \times \overbrace{P(A)}^{\text{Prior (of } A)}}{\underbrace{P(B)}_{\text{Prior (of } B)}}$$

- **Prior:** Our belief about event $A$ before seeing the data (e.g., how likely we thought $A$ was initially).
- **Likelihood:** The probability of observing the data $B$ assuming $A$ is true.
- **Posterior:** Our updated belief about $A$ after seeing the data $B$.

## Example: Updating Beliefs About a Coin

**Scenario:**

- **Prior Belief:** The coin is fair.
  - $P(\text{Heads}) = 0.5$
  - $P(\text{Tails}) = 0.5$
- **New Evidence:** We flip the coin 60 times and observe heads 70% of the time (42 heads).
- **Updated Belief (Posterior):**
  - Based on the data, we update our belief: The coin may be biased toward heads.

**Key Idea:** Prior $\rightarrow$ New Data $\rightarrow$ Posterior

# Quiz + Discussion