# PODS Lab 10:
# Non-Parametric Significance Tests

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1. Chi-squared Test ($\chi^2$)
   - Goodness-of-fit
   - Test of independence
2. Mann-Whitney U Test (AKA Wilcoxon Rank-sum Test)
3. Kolmogorov-Smirnov (KS) Test
4. Permutation Test

**Parametric vs. Non-Parametric**

- ▶ **Parametric Tests:** Statistical tests that assume the data follows a specific probability distribution (e.g., Normal distribution).
    - ▶ **Used:** When data can be reduced to sample means.
    - ▶ The underlying assumption is that the **mean** is a sufficient statistic of the data distribution.
    - ▶ **Examples:** Z-tests, t-tests, ANOVA.

**Parametric vs. Non-Parametric**

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    - ▶ The underlying assumption is that the **mean** is a sufficient statistic of the data distribution.
    - ▶ **Examples:** Z-tests, t-tests, ANOVA.

- ▶ **Non-Parametric Tests:** Statistical tests that **do not assume** the data follows a specific distribution or when the distribution is unknown.
    - ▶ **Used:** When the data cannot be reduced to sample means or when the use of the mean is not appropriate or robust.
    - ▶ **Instead,** focus on ranks, medians, and distribution-free metrics.
    - ▶ **Examples:** Chi-squared Test, Mann-Whitney U Test, Kolmogorov-Smirnov (KS) Test, Permutation Test.

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Chi-squared Test ($\chi^2$)
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Mann-Whitney U Test
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Kolmogorov-Smirnov (KS) Test
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Permutation Test
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**Chi-squared Test ($\chi^2$)**

**A significance test for categorical data**

▶ **Purpose:** To assess whether observed frequencies differ from expected frequencies under a null hypothesis.

▶ **Use Cases:**

1. **Goodness-of-fit Test:** Comparing observed categorical data to a theoretical distribution.
   ▶ **E.g.** Zodiac signs and serial killer counts

2. **Test of Independence:** Testing for association between two categorical variables.
   ▶ **E.g.** Preferences for pineapple on pizza across generations

**Chi-squared ($\chi^2$) Steps**
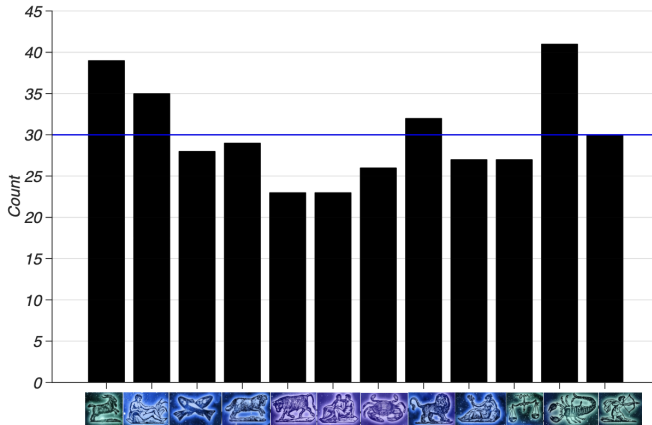
### $\chi^2$ Test Statistic

$$\chi^2 = \sum \frac{(\text{Observed Count} - \text{Expected Count})^2}{\text{Expected Count}}$$

**Steps**

1. Formulate the null hypothesis ($H_0$)

2. Calculate the expected counts for each category based on $H_0$.

3. Compute the test statistic $\chi^2$ using the observed and expected counts.

4. Determine the degrees of freedom and critical value for a chosen significance level $\alpha$.
   - **DF for Goodness-of-fit:** $K - 1$   (K is # of categories)
   - **DF for Test of Independence:** $(r - 1) \cdot (c - 1)$   (rows, columns)

5. If $\chi^2 > \chi^2_{crit}$, reject $H_0$. Otherwise, fail to reject $H_0$.

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Chi-squared Test ($\chi^2$)
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**(1) Goodness-of-fit Test Example: Zodiac Signs and Serial Killers (Full example in lecture!)**

**Question:** Is Zodiac sign predictive of being a serial killer?

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Chi-squared Test ($\chi^2$)
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Permutation Test
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**(1) Goodness-of-fit Test Example: Zodiac Signs and Serial Killers (Full example in lecture!)**

▶ **Idea of goodness-of-fit:** Comparing observed data to a theoretical or expected distribution.

**Question:** Is Zodiac sign predictive of being a serial killer?

▶ **Null Hypothesis:** Serial killer counts are evenly distributed across all 12 Zodiac signs.

▶ **Alternative Hypothesis:** Some signs are disproportionately represented.

▶ **Categories:** 12 Zodiac signs.

▶ **Expected Count:** If $N$ is the total number of serial killers, then each sign is expected to have $\frac{N}{12}$.

▶ **Degrees of Freedom:** $DF = K - 1 = 12 - 1 = \boxed{11}$

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Chi-squared Test ($\chi^2$)
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Permutation Test
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## (1) Goodness-of-fit Test Example: Zodiac Signs and Serial Killers (Full example in lecture!)

| Sign | Observed Count | Expected Count (n/12) | Deviation | Plugged in to equation | Contribution to $\chi^2$ |
|---|---|---|---|---|---|
| Capricorn | 39 | 30 | 9 | $9^2/30$ | 2.7 |
| Aquarius | 35 | 30 | 5 | $5^2/30$ | 0.83 |
| Pisces | 28 | 30 | -2 | $-2^2/30$ | 0.13 |
| Aries | 29 | 30 | -1 | $-1^2/30$ | 0.03 |
| Taurus | 23 | 30 | -7 | $-7^2/30$ | 1.63 |
| Gemini | 23 | 30 | -7 | $-7^2/30$ | 1.63 |
| Cancer | 26 | 30 | -4 | $-4^2/30$ | 0.53 |
| Leo | 32 | 30 | 2 | $2^2/30$ | 0.13 |
| Virgo | 27 | 30 | -3 | $-3^2/30$ | 0.3 |
| Libra | 27 | 30 | -3 | $-3^2/30$ | 0.3 |
| Scorpio | 41 | 30 | 11 | $11^2/30$ | 4.03 |
| Sagittarius | 30 | 30 | 0 | $0^2/30$ | 0 |

$\chi^2$ = 2.7+0.83+0.13+0.03+1.63+1.63+0.53+0.13+0.3+0.3+4.03+0
$\chi^2$ = 12.24

**(1) Goodness-of-fit Test Example: Zodiac Signs and Serial Killers (Full example in lecture!)**

### Gathered information:

- $DF = 11$, $\chi^2 = 12.24$, $\alpha = 0.05$
- **Find** $\chi^2_{crit}$ from table (clickable link!)

| $df$ | $\chi^2_{.995}$ | $\chi^2_{.990}$ | $\chi^2_{.975}$ | $\chi^2_{.950}$ | $\chi^2_{.900}$ | $\chi^2_{.100}$ | $\chi^2_{.050}$ | $\chi^2_{.025}$ | $\chi^2_{.010}$ | $\chi^2_{.005}$ |
| :-- | :-- | :-- | :-- | :-- | :-- | :-- | :-- | :-- | :-- | :-- |
| 1 | 0.000 | 0.000 | 0.001 | 0.004 | 0.016 | 2.706 | 3.841 | 5.024 | 6.635 | 7.879 |
| 2 | 0.010 | 0.020 | 0.051 | 0.103 | 0.211 | 4.605 | 5.991 | 7.378 | 9.210 | 10.597 |
| 3 | 0.072 | 0.115 | 0.216 | 0.352 | 0.584 | 6.251 | 7.815 | 9.348 | 11.345 | 12.838 |
| 4 | 0.207 | 0.297 | 0.484 | 0.711 | 1.064 | 7.779 | 9.488 | 11.143 | 13.277 | 14.860 |
| 5 | 0.412 | 0.554 | 0.831 | 1.145 | 1.610 | 9.236 | 11.070 | 12.833 | 15.086 | 16.750 |
| 6 | 0.676 | 0.872 | 1.237 | 1.635 | 2.204 | 10.645 | 12.592 | 14.449 | 16.812 | 18.548 |
| 7 | 0.989 | 1.239 | 1.690 | 2.167 | 2.833 | 12.017 | 14.067 | 16.013 | 18.475 | 20.278 |
| 8 | 1.344 | 1.646 | 2.180 | 2.733 | 3.490 | 13.362 | 15.507 | 17.535 | 20.090 | 21.955 |
| 9 | 1.735 | 2.088 | 2.700 | 3.325 | 4.168 | 14.684 | 16.919 | 19.023 | 21.666 | 23.589 |
| 10 | 2.156 | 2.558 | 3.247 | 3.940 | 4.865 | 15.987 | 18.307 | 20.483 | 23.209 | 25.188 |
| 11 | 2.603 | 3.053 | 3.816 | 4.575 | 5.578 | 17.275 | 19.675 | 21.920 | 24.725 | 26.757 |
| 12 | 3.074 | 3.571 | 4.404 | 5.226 | 6.304 | 18.549 | 21.026 | 23.337 | 26.217 | 28.300 |

- $\chi^2 \overset{?}{>} \chi^2_{crit} \rightarrow 12.24 < 19.675 \implies$ Fail to reject $H_0$

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Chi-squared Test ($\chi^2$)
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Mann-Whitney U Test
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Kolmogorov-Smirnov (KS) Test
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Permutation Test
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**(2) Test of Independence Example: Pineapple Pizza Preferences**

- ▶ **Idea of test of independence:** Testing whether one categorical variable (X) is associated with another categorical variable (Y).
- ▶ Does knowing the value of X gives you any information about the value of Y?

**(2) Test of Independence Example: Pineapple Pizza Preferences**

- ▶ **Idea of test of independence:** Testing whether one categorical variable (X) is associated with another categorical variable (Y).
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**Question:** Is preference for pineapple pizza independent of generation?

- ▶ **Null Hypothesis:** Preference for pineapple pizza is independent of generation.
- ▶ **Alternative Hypothesis:** Preference varies across generations.
- ▶ **Categories:** Generations (Millennials, Gen Z) vs. Preference (For, Undecided, Against)
- ▶ $DF = (r-1) \cdot (c-1) = (2-1) \cdot (3-1) = 1 \cdot 2 = \boxed{2}$

## (2) Test of Independence Example: Pineapple Pizza Preferences

### Observed Count:

| Observed | For | Undecided | Against | Total |
|----------|-----|-----------|---------|-------|
| Millennials | 92 | 18 | 90 | 200 |
| Gen Z | 68 | 22 | 110 | 200 |
| Total | 160 | 40 | 200 | 400 |

### Expected Count:

| Expectations | For | Undecided | Against |
|--------------|-----|-----------|---------|
| Millennials | 80 | 20 | 100 |
| Gen Z | 80 | 20 | 100 |

$\chi^2 = \frac{(92-80)^2}{80} + \frac{(18-20)^2}{20} + ... + \frac{(110-100)^2}{100} =$
$1.8 + 0.2 + 1 + 1.8 + 0.2 + 1 = \boxed{6}$

**(2) Test of Independence Example: Pineapple Pizza Preferences**

**Gathered Information:**

- $DF = 2$ , $\chi^2 = 6$, $\alpha = 0.05$

- **Find** $\chi^2_{crit}$ from table (clickable link!)

| $df$ | $\chi^2_{.995}$ | $\chi^2_{.990}$ | $\chi^2_{.975}$ | $\chi^2_{.950}$ | $\chi^2_{.900}$ | $\chi^2_{.100}$ | $\chi^2_{.050}$ | $\chi^2_{.025}$ | $\chi^2_{.010}$ | $\chi^2_{.005}$ |
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- $\chi^2 \overset{?}{>} \chi^2_{crit} \rightarrow 6 > 5.991 \implies$ Reject $H_0$

- **Note:** Notice that if $\alpha$ was slightly smaller, the $\chi^2_{crit}$ would've been larger $\implies$ Fail to reject!

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Chi-squared Test ($\chi^2$)
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Mann-Whitney U Test
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Kolmogorov-Smirnov (KS) Test
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Permutation Test
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Scenario and Three Types of Data

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- ▶ **Solution?** An independent t-test might seem reasonable because you are comparing the means of two groups (the two movies).

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- ▶ **Question:** Is it reasonable to reduce movie ratings to means?
- ▶ **Answer:** Probably not. Movie ratings are **ordinal data** where the psychological difference between 3 and 4 may not equal the difference between 4 and 5.

**Scenario and Three Types of Data**

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- ▶ **Answer:** Probably not. Movie ratings are **ordinal data** where the psychological difference between 3 and 4 may not equal the difference between 4 and 5.

**Three Types of Data:**

1. **Nominal:** Unordered categories (e.g., gender, colors).
2. **Ordinal:** Ordered categories **without** consistent differences (e.g., movie ratings, education levels).
   - ▶ Indicates position or rank, but differences between ranks are not necessarily consistent.
3. **Cardinal (Interval/Ratio):** Numerical values where arithmetic operations are meaningful (e.g., temperature, height, age).

**Mann-Whitney U Test (AKA Wilcoxon Rank-sum Test)**

▶ **Mann-Whitney U Test:** A non-parametric test used to see if two samples come from populations with the same **median**.

▶ It is an alternative to the T-test when the assumptions of normality or equal variance are violated.

▶ The t-test compares means, while the Mann-Whitney U Test compares medians (under certain conditions).

**When to Use the Mann-Whitney U Test:**

▶ When it is inappropriate to reduce a dataset to its mean.

▶ For **ordinal data** (e.g. movie ratings) where the difference between values may not be consistent.

▶ When there are outliers, as the median is more robust to extreme values than the mean.

**Mann-Whitney U Test (AKA Wilcoxon Rank-sum Test)**

### Steps

1. Combine all values from both samples and arrange them in rank order (from smallest to largest).
2. Assign ranks to the values, ignoring which sample they belong to. Assign average ranks in case of ties.
3. Calculate the sum of ranks for each sample ($T_A$ and $T_B$).
4. Compute the test statistic $U$:

Test Statistic ($U$)

$$U = n_A n_B + \frac{n_X(n_X + 1)}{2} - T_X$$

### Where:

▶ $n_A, n_B$: Sizes of the two samples.

▶ $n_X$: Size of the sample with the **larger rank sum** (either $n_A$ or $n_B$).

▶ $T_X$: Sum of ranks for the sample with the **larger rank sum**.

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Chi-squared Test ($\chi^2$)
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Permutation Test
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**Kolmogorov-Smirnov (KS) Test**

- ▶ **Kolmogorov-Smirnov (KS) Test:** A non-parametric test used to determine whether two samples come from the same distribution.
- ▶ Unlike tests that compare means or medians, the KS test focuses on the **overall distribution shape**.

**Prupose:**

- ▶ Test whether the underlying distributions of two samples are the same.
- ▶ Serves as a **goodness-of-fit test**, comparing a sample's distribution to a reference.

**KS Test Steps**

**Steps:**

1. Compute the empirical cumulative distribution function (eCDF) for both samples.

2. plot the the eCDF of the two samples.

3. **Test Statistic ($D$):** The value of the largest separation between the two eCDF.



Comparison of Two Empirical CDFs (KS Test)
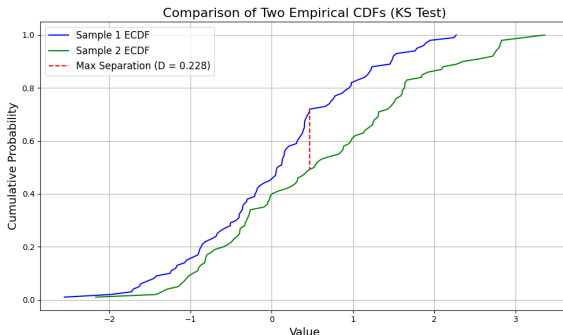
**Permutation Test**

▶ **Permutation Tests:** A non-parametirc tests that works by **shuffling** data and computing the test statistic under many repitions in order to generate a **custom null distribution.**

**Permutation testing is useful under the following conditions:**

1. Data does not meet assumptions for parametric testing.
2. Other non-parametric tests are not applicable.

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**Steps of Permutation Test**

1 **Define the Data Distributions:** Identify the two groups to be compared (e.g., ratings for the first and second Harry Potter movies).

2 **Combine and Shuffle the Data:** Pool the two datasets into one large set and shuffle randomly.

3 **Split the Shuffled Data:** Divide the shuffled data back into two groups of the original sizes.

4 **Compute the Test Statistic:** Calculate a relevant test statistic (e.g., difference in means, medians, etc.) for the new groups.

5 **Generate the Null Distribution:** Repeat Steps 3 and 4 many times to build a distribution of the test statistic under the null hypothesis.

6 **Compute the Test Statistic for Original Data:** Calculate the test statistic for the original (unshuffled) data.

7 **Compare and Compute p-value:** Compare the observed statistic to the null distribution and compute the p-value.

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Chi-squared Test ($\chi^2$)
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Kolmogorov-Smirnov (KS) Test
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Permutation Test
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**Summary of Non-parametric Tests**

| Test | Purpose | Data Type |
|:---:|:---:|:---:|
| **Chi-square** | Test of Independence or Goodness-of-Fit | **Categorical** |
| **Mann-Whitney U** | Comparing Ranks (often interpreted as comparing medians) | **Ordinal** |
| **KS** | Comparing Overall Distribution | **Typically cardinal** |
| **Permutation Test** | General-purpose test depending on resampling | **Any Type\*** |

▶ \*: Depends on test statistic.