

PODS Lab 2: Linear Algebra

Hamza Alshamy

Center for Data Science, NYU
ha2486@nyu.edu

01/31/25

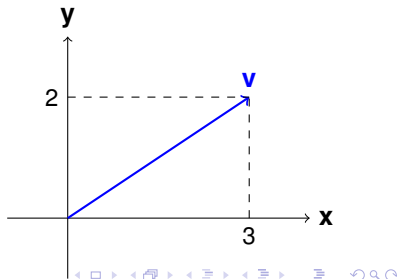
- ➊ **Vectors**
- ➋ **Vector Norms**
- ➌ **Vector Projections**
- ➍ **Quiz + Discussion**
- ➎ **Pain Relief** (Questions?)

Definition of Vectors

- ▶ **Vectors:** Ordered lists of numbers.
 - ▶ **Meaning:** Each entry provides a direction, and the magnitude of its value tells you its effect in that direction.
- ▶ **Dimensionality:** The number of components in a vector.
- ▶ **Orientation:** Column vs. Row vector.
 - ▶ **Transpose:** Converts a row vector to a column vector (and vice versa).

Example: A vector in \mathbb{R}^2 :

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \quad \mathbf{v} \in \mathbb{R}^2$$



Examples of Vectors in Different Spaces

A vector in \mathbb{R}^3 :

$$\mathbf{v} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}, \quad \mathbf{v} \in \mathbb{R}^3$$

A vector in \mathbb{R}^5 :

$$\mathbf{w} = \begin{bmatrix} 1 \\ -3 \\ 0 \\ 5 \\ 2 \end{bmatrix}, \quad \mathbf{w} \in \mathbb{R}^5$$

Dimensions:

- ▶ $\mathbb{R}^3 \rightarrow$ A three-dimensional vector.
- ▶ $\mathbb{R}^5 \rightarrow$ A five-dimensional vector.

Transpose of a Vector

Column Vector:

$$\mathbf{x} = \begin{bmatrix} 2 \\ -5 \\ 8 \end{bmatrix}, \quad \mathbf{x} \in \mathbb{R}^3$$

Row Vector (Transpose):

$$\mathbf{x}^T = [2 \quad -5 \quad 8]$$

Key Idea: Transposing flips the dimensions of a vector.

Vector Operations

1. Vector Addition:

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

2. Scalar Multiplication:

$$3\mathbf{a} = 3 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$$

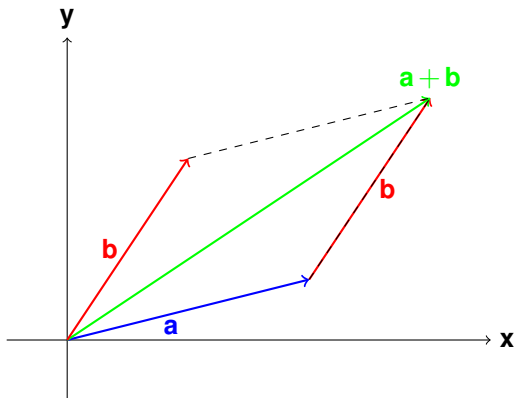
3. Dot Product:

$$\mathbf{a} \cdot \mathbf{b}^T = \sum_{i=1}^n a_i b_i = (3 \times 2) + (1 \times -4) = 6 - 4 = 2$$

Geometry of Vectors: Addition

Assume

$$\mathbf{a} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$



- ▶ The *blue* vector **a** starts at the origin.
- ▶ The *red* vector **b** is placed at the tip of **a**.
- ▶ The *green* vector represents **a + b**.

Vector Norms

- ▶ **Norms:** Functions used to assign a magnitude/length to a given vector.
 - ▶ Different norms measure magnitude differently.
 - ▶ We choose the most appropriate norm depending on the context.
- ▶ **Definition:**

$$\begin{aligned} ||\mathbf{x}||_p &= \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}} \\ &= \sqrt[p]{|x_1|^p + |x_2|^p + \dots + |x_i|^p} \end{aligned}$$

L1 Norm (AKA Manhattan Norm)

- ▶ The L1 norm measures distance as the sum of absolute values of vector components.
- ▶ Formula:

$$||\mathbf{x}||_1 = \sum_{i=1}^n |x_i|$$

- ▶ Example:

$$\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \quad ||\mathbf{x}||_1 = |3| + |-4| + |5| = 3 + 4 + 5 = 12$$

- ▶ **Interpretation:**
 - ▶ Think of it as the distance traveled on a grid (Manhattan distance).

L2 Norm (AKA Euclidean Norm)

- ▶ The L2 norm measures distance as the straight-line (Euclidean) distance from the origin.
- ▶ Formula:

$$\|\mathbf{x}\|_2 = \sqrt{\sum_{i=1}^n x_i^2}$$

- ▶ Example:

$$\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \quad \|\mathbf{x}\|_2 = \sqrt{3^2 + (-4)^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50} \approx 7.1$$

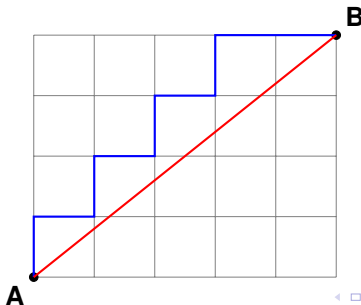
- ▶ **Interpretation:**
 - ▶ Measures the direct shortest distance between two points.

Vector Norms - Geometry

- ▶ **Manhattan Norm** (L_1): Think of an Uber in Manhattan.
 - ▶ Movement is constrained to streets, so no diagonal travel is allowed.
- ▶ **Euclidean Norm** (L_2): Think of a drone taxi.
 - ▶ It can fly directly between two points.

Manhattan (L1) - Grid-Based Movement

Euclidean (L2) - Direct Flight Path



L5 Norm?

- ▶ The L5 norm is a special case of the general Lp norm:

$$\|\mathbf{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{\frac{1}{p}}$$

- ▶ Setting $p = 5$:

$$\|\mathbf{x}\|_5 = \left(\sum_{i=1}^n |x_i|^5 \right)^{\frac{1}{5}}$$

- ▶ Example:

$$\mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}, \quad \|\mathbf{x}\|_5 = (3^5 + (-4)^5 + 5^5)^{\frac{1}{5}} \approx 4.7$$

Norms Comparison

$$\text{Norms of } \mathbf{x} = \begin{bmatrix} 3 \\ -4 \\ 5 \end{bmatrix}$$

Norm	Value
L_1 (Manhattan)	12
L_2 (Euclidean)	7.1
L_5	4.7
L_{30}	5

- **Key point:** As norms increase, the magnitude of a vector converges to the largest value in that vector (e.g. $L_{30} \approx 5$).

Vector Projections: Intuition and Relevance

- ▶ **What is a vector projection?**
 - ▶ The projection of a vector **b** onto another vector **a** is the shadow of **b** when "dropped" onto **a**.
 - ▶ It represents the component of **b** that aligns with **a**.
- ▶ **Why is it relevant in Data Science?**
 - ▶ **Principal Component Analysis (PCA):**
 - ▶ PCA projects data onto principal components to reduce dimensionality while preserving variance.
 - ▶ **Linear Regression:**
 - ▶ Regression finds the best fit line by projecting data points onto a regression line.
 - ▶ **Least Squares Optimization:**
 - ▶ The residuals (errors) in regression are minimized via perpendicular projections.

Derivation of the Vector Projection

Goal: Find the scalar β such that $\beta \mathbf{a}$ is as close as possible to \mathbf{b} .

Since the residual $\mathbf{b} - \beta \mathbf{a}$ is perpendicular to \mathbf{a} , we set:

$$(\mathbf{b} - \beta \mathbf{a})^T \mathbf{a} = 0$$

Expanding the equation:

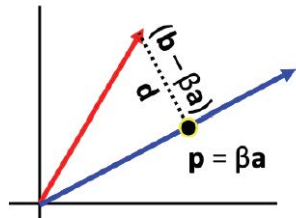
$$\mathbf{a}^T \mathbf{b} - \beta \mathbf{a}^T \mathbf{a} = 0 \implies \mathbf{a}^T \mathbf{b} = \beta \mathbf{a}^T \mathbf{a}$$

Solving for β :

$$\beta = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}}$$

Thus, the projection of \mathbf{b} onto \mathbf{a} is:

$$\text{proj}_{\mathbf{a}}(\mathbf{b}) = \frac{\mathbf{a}^T \mathbf{b}}{\mathbf{a}^T \mathbf{a}} \mathbf{a}$$



Quiz

Quiz Discussion

Any questions?