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Calculus With Analytic Geometry

Our Effort To Surve You Better

Calculus
With
Analytic Geometry

By
S.M Yusuf

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Chapter 4 TECHNIQUES OF INTEGRATION

Exercise Set 4.1 (Page 132)

Write down the indefinite integral of each of the following:

1. 0

Sol. A constant.

2. $\sqrt{x}$

Sol. $\int \sqrt{x}\, dx = \frac{x^{3/2}}{3/2} = \frac{2}{3} x^{3/2}$

3. $\frac{1+x}{x}$

Sol. $\int \frac{1+x}{x}\, dx = \int \left(\frac{1}{x} + 1\right) dx = \ln|x| + x$

4. $\frac{x^2-1}{x^2+1}$

Sol. $\int \frac{x^2-1}{x^2+1}\, dx = \int \left\{1 - \frac{2}{x^2+1}\right\} dx$

$$= \int 1 \, . \, dx - 2\int \frac{dx}{x^2+1} = x - 2 \arctan x$$

5. $\tan^2 x$

Sol. $\int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x$

6. $\cot^2 x$

Sol. $\int \cot^2 x\, dx = \int (\operatorname{cosec}^2 x - 1)\, dx = -\cot x - x$

7. $\cos^2 x$

Sol. $\int \cos^2 x\, dx = \frac{1}{2}\int 2\cos^2 x\, dx = \frac{1}{2}\int (1 + \cos 2x)\, dx$

$$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]$$

8. $\sin^2 x$

Sol. $\int \sin^2 x\, dx = \frac{1}{2}\int 2\sin^2 x\, dx = \frac{1}{2}\int (1 - \cos 2x)\, dx$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]$$

9. $\sqrt{1 - \cos x}$

Sol. $\int \sqrt{1 - \cos x}\, dx = \int \sqrt{2\sin^2 \frac{x}{2}}\, dx = \sqrt{2}\int \sin\frac{x}{2}\, dx = \sqrt{2}\left(\frac{-\cos\frac{x}{2}}{\frac{1}{2}}\right)$

$$= \sqrt{2}\left(-2\cos\frac{x}{2}\right) = -2\sqrt{2}\cos\frac{x}{2}$$

10. $\sqrt{4 - x^2}$

Sol. $\int \sqrt{4 - x^2}\, dx = \frac{x}{2}\sqrt{4 - x^2} + \frac{(2)^2}{2}\arcsin\frac{x}{2}$

$$= \frac{x\sqrt{4 - x^2}}{2} + 2\arcsin\frac{x}{2}$$

11. $\sqrt{4 + x^2}$

Sol. $\int \sqrt{4 + x^2}\, dx = \frac{x\sqrt{4 + x^2}}{2} + \frac{(2)^2}{2}\ln\left|\frac{x + \sqrt{4 + x^2}}{2}\right|$

$$= \frac{x\sqrt{4 + x^2}}{2} + 2\ln\left|\frac{x + \sqrt{4 + x^2}}{2}\right|$$

12. $\sqrt{x^2 - 4}$

Sol. $\int \sqrt{x^2 - 4}\, dx = \frac{x\sqrt{x^2 - 4}}{2} - \frac{(2)^2}{2}\ln\left|\frac{x + \sqrt{x^2 - 4}}{2}\right|$

$$= \frac{x}{2}\sqrt{x^2 - 4} - 2\ln\left|\frac{x + \sqrt{x^2 - 4}}{2}\right|$$

Exercise Set 4.2 (Page 135)

Evaluate (Problems 1 – 22):

1. $\int \frac{dx}{\sqrt{a^2 + x^2}}$

Sol. Put $x = a\sinh\theta$ or $dx = a\cosh\theta\, dx$

Now $\int \frac{dx}{\sqrt{a^2 + x^2}} = \int \frac{a\cosh\theta\, d\theta}{\sqrt{a^2 + a^2\sinh^2\theta}} = \int \frac{a\cosh\theta}{a\cosh\theta} d\theta$

$$= \int 1\,.\, d\theta = \theta = \sinh^{-1}\frac{x}{a}$$

2. $\int \frac{dx}{\sqrt{x^2 - a^2}}$

Sol. Put $x = a\cosh\theta$, $dx = a\sinh\theta\, d\theta$. Then

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \int \frac{a\sinh\theta\, d\theta}{\sqrt{a^2\cosh^2\theta - a^2}} = \int \frac{a\sinh\theta}{a\sinh\theta} d\theta$$

$$= \int d\theta = \theta = \cosh^{-1}\frac{x}{a}$$

3. $\int \tan x\, dx$

Sol. $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx$

$$= -\ln|\cos x| = \ln|\cos x|^{-1} = \ln|\sec x|$$

4. $\int \cot x\, dx$

Sol. $\int \cot x\, dx = \int \frac{\cos x}{\sin x} dx = \ln|\sin x|$

5. $\int \sec x\, dx$

Sol. $\int \sec x\, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx = \int \frac{\sec^2 x + \sec x\tan x}{(\sec x + \tan x)} dx$

$$= \ln|\sec x + \tan x| = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right|$$

$$= \ln \left|\frac{1+\sin x}{\cos x}\right| = \ln \left|\frac{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}\right|$$

$$= \ln \left|\frac{\cos\frac{x}{2}+\sin\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right| = \ln \left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$$

6. $\int \csc x\, dx$

Sol. $\int \csc x\, dx = \int \frac{\operatorname{cosec} x\,(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x}\, dx$

$$= \ln |\operatorname{cosec} x - \cot x| = \ln \left|\frac{1}{\sin x} - \frac{\cos x}{\sin x}\right| = \ln \left|\frac{1-\cos x}{\sin x}\right|$$

$$= \ln \left|\frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right| = \ln \left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right| = \ln \left|\tan\frac{x}{2}\right|$$

7. $\int (ax^2 + 2bx + c)^n (ax + b)dx$

Sol. Let $t = ax^2 + 2bx + c$ so that

$$dt = 2(ax+b)\,dx \quad \text{or} \quad (ax+b)\,dx = \frac{1}{2}dt$$

Now $\int (ax^2 + 2bx + c)^n (ax+b)dx = \frac{1}{2}\int t^n \,.\, dt$

$$= \frac{1}{2}\frac{t^{n+1}}{n+1} = \frac{1}{2}\cdot\frac{1}{n+1}\cdot(ax^2+2bx+c)^{n+1}$$

8. $\int \sqrt{\frac{1+x}{1-x}}\,dx$

Sol. $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}\sqrt{1+x}}{\sqrt{1-x^2}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$

$$= \int \frac{dx}{\sqrt{1-x^2}}\,dx + \int \frac{x\,dx}{\sqrt{1-x^2}}\,dx$$

$$= \arcsin x - \frac{1}{2}\int (1-x^2)^{-1/2}(-2x)\,dx$$

$$= \arcsin x - \frac{1}{2}\frac{(1-x)^{1/2}}{\frac{1}{2}} = \arcsin x - \sqrt{1-x^2}$$

9. $\int \frac{dx}{a+\sqrt{bx+c}}$ $\quad (a > 0)$

Sol. Let $bx + c = z^2$ so that $b\,dx = 2z\,dz$ or $dx = \frac{2}{b}z\,dz$

Now, $\int \frac{dx}{a+\sqrt{bx+c}} = \frac{2}{b}\int\frac{z\,dz}{a+z} = \frac{2}{b}\int\left(1-\frac{a}{a+z}\right)dz$

$$= \frac{2}{b}[z - a\ln(a+z)] = \frac{2}{b}\left[\sqrt{bx+c} - a\ln(a+\sqrt{bx+c})\right]$$

10. $\int \frac{dx}{(1+x^2)\arctan x}$

Sol. $I = \int \frac{dx}{(1+x^2)\arctan x} = \int \frac{1}{\arctan x}\cdot\frac{1}{1+x^2}\,dx$

Put $\arctan x = t$ so that $\frac{1}{1+x^2}dx = dt$. Then

$$I = \int\frac{dt}{t} = \ln|t| = \ln|\arctan x|$$

11. $\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx$

Sol. Put $\sin x - \cos x = t$ so that $(\cos x + \sin x)\,dx = dt$. Then

$$\int \frac{\sin x + \cos x}{\sin x - \cos x}\,dx = \int\frac{dt}{t} = \ln|t| = \ln|\sin x - \cos x|$$

12. $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$

Sol. $I = \int \sin\sqrt{x}\,.\,\frac{1}{\sqrt{x}}\,dx$

Put $\sqrt{x} = t$ so that $\frac{1}{2}x^{1/2}\,dx = dt$ *i.e.*, $\frac{dx}{\sqrt{x}} = 2\,dt$. Then

$$I = 2\int \sin t\,dt = -2\cos t = -2\cos\sqrt{x}$$

13. $\int \sqrt{e^{2x}+e^{3x}}\,dx$

Sol. $\int \sqrt{e^{2x}+e^{3x}}\,dx = \int (\sqrt{1+e^x})\cdot e^x\,dx$

Let $1 + e^x = z$ so that $e^x\,dx = dz$

and $\int(\sqrt{1+e^x})\,e^x\,dx = \int\sqrt{z}\,dz = \frac{2}{3}z^{3/2} = \frac{2}{3}(1+e^x)^{3/2}$

14. $\displaystyle\int \frac{dx}{e^x + e^{-x}}$

Sol. Put $e^x = t$ so that $e^x\,dx = dt$. Then

$$I = \int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1} = \int \frac{dt}{t^2+1} = \arctan t = \arctan e^x.$$

15. $\displaystyle\int \frac{e^{2x}\,dx}{\sqrt{e^x - 1}}$

Sol. Let $e^x - 1 = z^2$ so that $e^x\,dx = 2z\,dz$ and

$$\int \frac{e^{2x}\,dx}{\sqrt{e^x-1}} = \int \frac{e^x}{\sqrt{e^x-1}}.e^x dx = \int \frac{(1+z^2)\,2z\,dz}{z} = 2\left(z + \frac{z^3}{3}\right)$$

$$= 2z\left(1 + \frac{z^2}{3}\right) = 2\sqrt{e^x-1}\left(1 + \frac{e^x-1}{3}\right) = \frac{2}{3}\sqrt{e^x-1}\,(2+e^x)$$

16. $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

Sol. Let $\ln x = t$ so that $\frac{1}{x}dx = dt$. Then

$$\int \frac{\cos(\ln x)}{x}\,dx = \int \cos t\,dt = \sin t = \sin(\ln x)$$

17. $\displaystyle\int \frac{2x+5}{\sqrt{x^2+5x+7}}\,dx$

Sol. Let $x^2 + 5x + 7 = t$ so that $(2x+5)\,dx = dt$ and

$$\int \frac{2x+5}{\sqrt{x^2+5x+7}}\,dx = \int (x^2+5x+7)^{-1/2}(2x+5)\,dx$$

$$= \int t^{-1/2}\,dt = 2\,t^{1/2} = 2\sqrt{x^2+5x+7}$$

18. $\displaystyle\int \frac{(x+2)dx}{\sqrt{2x^2+8x+5}}$

Sol. Let $2x^2 + 8x + 5 = t$ so that $(4x+8)\,dx = dt$

or $(x+2)\,dx = \frac{1}{4}dt$. Then

$$\int \frac{(x+2)dx}{\sqrt{2x^2+8x+5}} = \int (2x^2+8x+5)^{-1/2}(x+2)\,dx = \int t^{-1/2}.\frac{1}{4}dt$$

$$= \frac{1}{4}.\frac{t^{1/2}}{\frac{1}{2}} = \frac{1}{2}t^{1/2} = \frac{1}{2}\sqrt{2x^2+8x+5} = \frac{\sqrt{2x^2+8x+5}}{2}$$

19. $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx$

Sol. Put $x = a\sec\theta$, so that $dx = a\sec\theta\tan\theta\,d\theta$. Then

$$\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx = \int \frac{\sqrt{a^2\sec^2\theta - a^2}}{a^4\sec^4\theta}.a\sec\theta\tan\theta.d\theta$$

$$= \int \frac{a\tan\theta.\tan\theta}{a^3\sec^3\theta}\,d\theta = \frac{1}{a^2}\int \frac{\sin^2\theta}{\cos^2\theta}\times\cos^3\theta\,d\theta$$

$$= \frac{1}{a^2}\int \sin^2\theta\times\cos\theta\,d\theta$$

$$= \frac{1}{a^2}.\frac{\sin^3\theta}{3} \qquad \left(\text{As } \cos\theta = \frac{a}{x}, \text{ so } \sin^2\theta = 1 - \cos^2\theta\right.$$

$$= \frac{1}{3a^2}.\frac{(x-a^2)^{3/2}}{x^3} \qquad = 1 - \frac{a^2}{x^2}$$

$$= \frac{(x^2-a^2)^{3/2}}{3a^2x^3} \qquad \left.= \frac{x^2-a^2}{x^2}\right)$$

20. $\displaystyle\int \cos^6 x\sin^3 x\,dx$

Sol. $I = \int -\cos^6 x\,.\sin^2 x\,.(-\sin x\,dx) = -\int \cos^6 x\,(1-\cos^2 x)\,(-\sin x\,dx)$

Putting $\cos x = t$ and $-\sin x\,dx = dt$ in I, we have

$$I = -\int t^6(1-t^2)\,dt = \int t^8\,dt - \int t^6\,dt$$

$$= \frac{t^9}{9} - \frac{t^7}{7} = \frac{1}{9}\cos^9 x - \frac{1}{7}\cos^7 x$$

21. $\displaystyle\int \tan^3\theta\sec^3\theta\,d\theta$

Sol. We have

$$I = \int \tan^3\theta\sec^3\theta\,d\theta = \int \tan^2\theta\sec^2\theta\,(\tan\theta\sec\theta)\,d\theta$$

$$= \int (\sec^2\theta - 1)\sec^2\theta\tan\theta\sec\theta\,d\theta$$

Now let $\sec\theta = z$ so that $\sec\theta\tan\theta\,d\theta = dz$

and $I = \int (z^2-1)z^2\,dz = \int (z^4 - z^2)\,dz = \frac{z^5}{5} - \frac{z^3}{3} = \frac{\sec^5\theta}{5} - \frac{\sec^3\theta}{3}$

22. $\int \cot^3 x \csc^4 x\, dx$

Sol. Let $\cot x = z$ so that $-\csc^2 x\, dx = dz$ and

$$\int \cot^3 x \csc^4 x\, dx = -\text{v}\int \cot^3 x, \csc^2 x\,(-\csc^2 x)\, dx$$

$$-\int z^3(1+z^2)\, dz = -\int (z^3+z^5)\, dz = -\left(\frac{z^4}{4}+\frac{z^6}{6}\right) = -\left(\frac{\cot^4 x}{4}+\frac{\cot^6 x}{6}\right)$$

Find an antiderivative of each of the following (Problems 23 – 40):

23. $\dfrac{1}{\sqrt{2x^2+3x+4}}$

Sol. $$\int \frac{dx}{\sqrt{2}\sqrt{x^2+\frac{3}{2}x+2}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2+2-\left(\frac{3}{4}\right)^2}} \quad \text{adding and subtracting } \left(\frac{3}{4}\right)^2$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\left(x+\frac{3}{4}\right)^2+\left(\frac{\sqrt{23}}{4}\right)^2}} = \frac{1}{\sqrt{2}}\sinh^{-1}\frac{x+\frac{3}{4}}{\frac{\sqrt{23}}{4}}$$

$$= \frac{1}{\sqrt{2}}\sinh^{-1}\frac{4x+3}{\sqrt{23}}$$

24. $\sqrt{a^2-x^2}$

Sol. Put $x = a\sin\theta$ so that $dx = a\cos\theta\, d\theta$ and

$$\int\sqrt{a^2-x^2}\, dx = \int a\cos\theta\, a\cos\theta\, d\theta = \frac{a^2}{2}\int(1+\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\frac{\sin 2\theta}{2} = \frac{a^2}{2}\theta + \frac{a^2}{2}\cdot\frac{2\sin\theta\cos\theta}{2}$$

$$= \frac{a^2}{2}\theta + \frac{a^2}{2}\sin\theta\sqrt{1-\sin^2\theta}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2}$$

25. $(2x+3)\sqrt{2x+1}$

Sol. Let $2x+1 = z^2$. Then $2\, dx = 2z\, dz$

or $dx = z\, dz$

Substituting, we get

$$\int(2x+3)\sqrt{2x+1}\, dx = \int z\,(z^2+2)\, z\, dz = \int(z^4+2z^2)\, dz$$

$$= \frac{z^5}{5}+\frac{2z^3}{3} = \frac{(2x+1)^{5/2}}{5}+\frac{2\,(2x+1)^{3/2}}{3}$$

26. $(1+x^2)^{-3/2}$

Sol. Let $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$

Substituting, we have

$$\int\frac{dx}{(1+x^2)^{3/2}} = \int\frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int\cos\theta\, d\theta = \sin\theta$$

Since $\tan\theta = x$, $\sin\theta = \dfrac{x}{\sqrt{1+x^2}}$

Therefore, $$\int\frac{dx}{(1+x^2)^{3/2}} = \frac{x}{\sqrt{1+x^2}}$$

27. $\dfrac{x^2}{\sqrt{x^2+1}}$

Sol. $$\int\frac{x^2}{\sqrt{x^2+1}}\, dx = \int\frac{(x^2+1)-1}{\sqrt{x^2+1}}\, dx = \int\sqrt{x^2+1}\, dx - \int\frac{dx}{\sqrt{x^2+1}}$$

But $$\int\sqrt{x^2+1}\, dx = \frac{x\sqrt{x^2+1}}{2}+\frac{1}{2}\sinh^{-1}x$$

Hence $$\int\frac{x^2}{\sqrt{x^2+1}}\, dx = \frac{x\sqrt{x^2+1}}{2}+\frac{1}{2}\sinh^{-1}x-\sinh^{-1}x$$

$$= \frac{x\sqrt{x^2+1}}{2}-\frac{1}{2}\sinh^{-1}x.$$

28. $(2x+4)\sqrt{2x^2+3x+1}$

Sol. $\int(2x+4)(2x^2+3x+1)^{1/2}\, dx$

$$= \frac{1}{2}\int(4x+3+5)(2x^2+3x+1)^{1/2}\, dx$$

$$= \frac{1}{2}\int(4x+3)(2x^2+3x+1)^{1/2}\, dx + \frac{5}{2}\int(2x^2+3x+1)^{1/2}\, dx$$

$$= \frac{(2x^2+3x+1)^{3/2}}{3}+\frac{5}{\sqrt{2}}\int\left\{x^2+\frac{3}{2}x+\left(\frac{3}{4}\right)^2+\frac{1}{2}-\left(\frac{3}{4}\right)^2\right\}^{1/2}dx$$

$$= \frac{(2x^2+3x+1)^{3/2}}{3}+\frac{5}{\sqrt{2}}\int\left\{\left(x+\frac{3}{4}\right)^2+\frac{1}{2}-\frac{9}{16}\right\}^{1/2}dx$$

$$= \frac{(2x^2 + 3x + 1)^{3/2}}{3} + \frac{5}{\sqrt{2}}\int\left\{\left(x + \frac{3}{4}\right)^2 - \left(\frac{1}{4}\right)^2\right\}^{1/2} dx \qquad (1)$$

By the formula

$\int\sqrt{x^2 - a^2} = \frac{x\sqrt{x^2 - a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a}$, we have

$$\int\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}\,dx = \frac{\left(x+\frac{3}{4}\right)\sqrt{\left(x+\frac{3}{4}\right)^2-\left(\frac{1}{4}\right)^2}}{2} - \frac{\left(\frac{1}{4}\right)^2}{2}\cosh^{-1}\frac{x+\frac{3}{4}}{\frac{1}{4}}$$

$$= \frac{\left(\frac{4x+3}{4}\right)\sqrt{x^2 + \frac{3}{2}x + \frac{9}{16} - \frac{1}{16}}}{2} - \frac{1}{32}\cosh^{-1}\left(\frac{4x+3}{1}\right)$$

$$= \frac{4x+3}{8}\sqrt{x^2 + \frac{3}{2}x + \frac{1}{2}} - \frac{1}{32}\cosh^{-1}(4x+3)$$

$$= \frac{4x+3}{8\sqrt{2}}\sqrt{2x^2 + 3x + 1} - \frac{1}{32}\cosh^{-1}(4x+3) \qquad (2)$$

Putting from (2) into (1), we get

$$\int(2x+4)\sqrt{2x^2+3x+1}\,dx = \frac{(2x^2+3x+1)}{3}$$

$$+ \frac{5}{\sqrt{2}}\left[\frac{4x+3}{8\sqrt{2}}\sqrt{2x^2+3x+1} - \frac{1}{32}\cosh^{-1}(4x+4)\right]$$

$$= \frac{(2x^2+3x+1)^{3/2}}{3} + \frac{5(4x+3)}{16}\sqrt{2x^2+3x+1} - \frac{5}{32\sqrt{2}}\cosh^{-1}(4x+3)$$

29. $\dfrac{1}{3\sin x + 4\cos x}$

Sol. $\displaystyle\int\frac{dx}{3\sin x + 4\cos x}$

Put $3 = r\cos\theta, 4 = r\sin\theta$, we get $r = 5$ and $\tan\theta = \frac{4}{3}$

Now $\displaystyle\int\frac{dx}{3\sin x + 4\cos x} = \int\frac{dx}{5(\sin x\cos\theta + \cos x\sin\theta)}$

$$= \int\frac{dx}{5\sin(x+\theta)} = \frac{1}{5}\int\operatorname{cosec}(x+\theta)\,dx$$

$$= \frac{1}{5}\ln\left|\tan\left(\frac{x}{2}+\frac{\theta}{2}\right)\right| = \frac{1}{5}\ln\left|\tan\left(\frac{x}{2}+\frac{1}{2}\arctan\frac{4}{3}\right)\right|$$

30. $\dfrac{\tan x}{\cos x + \sec x}$

Sol. $\displaystyle I = \int\frac{\tan x}{\cos x + \sec x}dx = \int\frac{\frac{\sin x}{\cos x}}{\cos x + \frac{1}{\cos x}}dx = \int\frac{\sin x\,dx}{\cos^2 x + 1}$

Put $\cos x = \theta$. Then $-\sin x\,dx = d\theta$ or $\sin x\,dx = -d\theta$ and

$$I = -\int\frac{d\theta}{\theta^2+1} = -\arctan\theta = -\arctan(\cos x).$$

31. $\dfrac{1}{\sin(x-a)\sin(x-b)}$

Sol. $\displaystyle\int\frac{dx}{\sin(x-a)\sin(x-b)}$

$a - b = (x-b) - (x-a)$

or $\sin(a-b) = \sin\{(x-b)-(x-a)\}$ (1)

Thus $\displaystyle\int\frac{dx}{\sin(x-a)\sin(x-b)} = \frac{1}{\sin(a-b)}\int\frac{\sin(a-b)}{(\sin(x-a)\sin(x-b))}dx$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin[(x-b)-(x-a)]}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\frac{\sin(x-b)\cos(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$- \frac{1}{\sin(a-b)}\int\frac{\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\int\frac{\cos(x-a)}{\sin(x-a)}dx - \frac{1}{\sin(a-b)}\int\frac{\cos(x-b)}{\sin(x-b)}dx$$

$$= \frac{1}{\sin(a-b)}\ln|\sin(x-a)| - \frac{1}{\sin(a-b)}\ln|\sin(x-b)|$$

$$= \frac{1}{\sin(a-b)}\ln\left|\frac{\sin(x-a)}{\sin(x-b)}\right|$$

32. $\tan x\ln(\sec x)$

Sol. Put $\ln\sec x = t$. Then $\dfrac{(\sec x\tan x)}{\sec x}dx = dt$ or $\tan x\,dx = dt$

$$\int(\ln(\sec x))\tan x\,dx = \int t\,dt = \frac{t^2}{2} = \frac{1}{2}(\ln\sec x)^2$$

33. $\dfrac{1}{(3\tan x+1)\cos^2 x}$

Sol. Put $3\tan x+1=t$ Then $3\sec^2 x\,dx=dt$ or $\sec^2 x\,dx=\frac{1}{3}dt$

$$\int\frac{\sec^2 x\,dx}{3\tan x+1}=\int\frac{\frac{1}{3}dt}{t}=\frac{1}{3}\ln|t|=\frac{1}{3}\ln|3\tan x+1|$$

34. $e^{\sin x}\cos x$

Sol. Put $\sin x=t$. Then $\cos x\,dx=dt$ and

$$\int e^{\sin x}\cos x\,dx=\int e^t\,dt=e^t=e^{\sin x}.$$

35. $\sqrt{1+3\cos^2 x}\,\sin 2x$

Sol. Put $1+3\cos^2 x=t$. Then $-6\cos x\sin x\,dx=dt$

or $-3\sin 2x\,dx=dt$ or $\sin 2x\,dx=-\frac{1}{3}dt$

$$\int\sqrt{1+3\cos^2 x}\,.\sin 2x\,dx=-\frac{1}{3}\int t^{1/2}dt=-\frac{2}{9}t^{3/2}$$
$$=-\frac{2}{9}(1+3\cos^2 x)^{3/2}$$

36. $\dfrac{\sin 2x}{\sqrt{1+\cos^2 x}}$

Sol. Put $1+\cos^2 x=t$ so that $-2\cos x\sin x\,dx=dt$

or $-\sin 2x\,dx=dt$ or $\sin 2x\,dx=-dt$

$$\int\frac{\sin 2x\,dx}{\sqrt{1+\cos^2 x}}=-\int\frac{dt}{\sqrt{t}}=-\int t^{-1/2}dt$$
$$=-2t^{1/2}=-2\sqrt{1+\cos^2 x}$$

37. $\dfrac{1}{2\sin^2 x+3\cos^2 x}$

Sol. $I=\displaystyle\int\frac{dx}{2\sin^2 x+3\cos^2 x}$

Dividing Num. and Denom. by $\cos^2 x$, we have

$I=\displaystyle\int\frac{\sec^2 x\,dx}{2\tan^2 x+3}$ Put $\tan x=t$ so that $\sec^2 x\,dx=dt$

and $I=\displaystyle\int\frac{dt}{2t^2+3}=\frac{1}{2}\int\frac{dt}{t^2+\frac{3}{2}}=\frac{1}{2}\cdot\frac{\sqrt{2}}{\sqrt{3}}\arctan\frac{t}{\sqrt{\frac{3}{2}}}$

$$=\frac{1}{\sqrt{6}}\arctan\sqrt{\frac{2}{3}}\,t=\frac{1}{\sqrt{6}}\arctan\left(\sqrt{\frac{2}{3}}\tan x\right)$$

38. $\dfrac{1}{\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}$

Sol. $I=\displaystyle\int\frac{1}{\sqrt{x}}\sec\sqrt{x}\tan\sqrt{x}\,dx=\int(\sec\sqrt{x}\tan\sqrt{x})\frac{1}{\sqrt{x}}dx$

Put $\sqrt{x}=u$ so that $\dfrac{1}{2\sqrt{x}}dx=du$ or $\dfrac{1}{\sqrt{x}}dx=2du$

Then $I=2\int\sec u\tan u\,du=2\sec u=2\sec\sqrt{x}$

39. $[\pi^{\sin x}+(\sin x)^{\pi}]\cos x$

Sol. $I=\int[\pi^{\sin x}+(\sin x)^{\pi}]\cos x\,dx$

Put $\sin x=u$ so that $\cos x\,dx=du$ and

$$I=\int(\pi^u+u^{\pi})\,du=\frac{\pi^u}{\ln\pi}+\frac{u^{\pi+1}}{\pi+1}$$
$$=\frac{\pi^{\sin x}}{\ln\pi}+\frac{(\sin x)^{\pi+1}}{\pi+1}$$

40. $\dfrac{\cos x}{3\sin x+4\sqrt{\sin x}}$

Sol. $I=\displaystyle\int\frac{\cos x}{3\sin x+4\sqrt{\sin x}}dx$

Put $\sqrt{\sin x}=z$ or $\sin x=z^2$. Therefore, $\cos x\,dx=2z\,dz$

and $I=\displaystyle\int\frac{2z\,dz}{3z^2+4z}=\int\frac{2\,dz}{4+3z}=\frac{2}{3}\int\frac{1}{4+3z}\cdot 3dz$

$$=\frac{2}{3}\ln|4+3z|=\frac{2}{3}\ln(4+3\sqrt{\sin x})$$

Exercise Set 4.3 (Page 142)

Evaluate (Problems 1 – 20):

1. $\int x\sec^2 x\,dx$

Sol. $I=\int x\sec^2 x\,dx$

Integrating by parts, regarding x as first function, we have

$$I=x\tan x-\int\tan x\,dx=x\tan x+\int\frac{1}{\cos x}\cdot(-\sin x)\,dx$$

$$= x\tan x + \ln|\cos x|$$

2. $\int x \csc^2 x\, dx$

Sol. $\int x\csc^2 x\, dx = -\int x(-\csc^2 x)\, dx = -[x\cot x - \int(\cot x) - 1\, dx]$

$$= -x\cot x + \int \frac{\cos x}{\sin x} dx = -x\cot x + \ln|\sin x|$$

3. $\int x^n \ln x\, dx$

Sol. $\int x^n \ln x\, dx = \int (\ln x)\, x^n\, dx$

$$= (\ln x)\cdot\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}dx, \text{ (Integrating by parts)}$$

$$= \ln x\cdot\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\int x^n\, dx = \ln x\cdot\frac{x^{n+1}}{n+1} - \frac{1}{n+1}\cdot\frac{x^{n+1}}{n+1}$$

$$= \frac{x^{n+1}}{n+1}\cdot\ln x - \frac{x^{n+1}}{(n+1)^2}$$

4. $\int x^2 \arctan x\, dx$

Sol. $\int x^2\arctan x\, dx = \int (\arctan x)\, x^2\, dx = (\arctan x)\frac{x^3}{3} - \int \frac{x^3}{3}\cdot\frac{1}{1+x^2}dx$

$$= \frac{x^3}{3}\arctan x - \frac{1}{3}\int\frac{x^3}{1+x^2}dx \qquad (1)$$

Now $\int \frac{x^3}{1+x^2}dx = \int\left(x - \frac{x}{x^2+1}\right)dx = \frac{x^2}{2} - \frac{1}{2}\int\frac{2x}{x^2+1}dx$

$$= \frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)$$

Putting this value into (1), we have

$$\int x^2\arctan x\, dx = \frac{x^3}{3}\arctan x - \frac{1}{3}\left[\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1)\right]$$

$$= \frac{x^3}{3}\arctan x - \frac{1}{6}x^2 + \frac{1}{6}\ln(x^2+1)$$

5. $\int \sec^3 x\, dx$

Sol. $\int \sec^3 x\, dx = \int \sec x \sec^2 x\, dx = \sec x\tan x - \int \tan x\sec x\tan x\, dx$

$$= \sec x\tan x - \int \sec x(\sec^2 x - 1)\, dx$$

$$= \sec x\tan x - \int\sec^3 x\, dx + \int \sec x\, dx$$

or $\quad 2\int\sec^3 x\, dx = \sec x\tan x + \int\sec x\, dx$

$$= \sec x\tan x + \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$$

or $\quad \int\sec^3 x\, dx = \frac{\sec x\tan x}{2} + \frac{1}{2}\ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right|$

6. $\int \csc^3 x\, dx$

Sol. $\int\csc^3 x\, dx = \int \csc x\,.\csc^2 x\, dx$

$$= \csc x(-\cot x) - \int(-\cot x)(-\csc x\cot x)\, dx$$

$$= -\csc x\cot x - \int(\csc^2 x - 1)\csc x\, dx$$

$$= -\csc x\cot x - \int\csc^3 x\, dx + \int\csc x\, dx$$

or $\quad 2\int\csc^3 x\, dx = -\csc x\cot x + \int\csc x\, dx$

$$= -\csc x\cot x + \ln\left|\tan\frac{x}{2}\right|$$

$$\int\csc^3 x\, dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|$$

7. $\int\frac{x-\sin x}{1-\cos x}dx$

Sol. $\int\frac{x-\sin x}{1-\cos x}dx = \int\frac{x - 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}dx$

$$= \int x\left(\csc^2\frac{x}{2}\right)\frac{1}{2}dx - \int\cot\frac{x}{2}dx$$

$$= x\left(-\cot\frac{x}{2}\right) - \int\left(-\cot\frac{x}{2}\right).1\, dx - \int\cot\frac{x}{2}dx$$

$$= -x\cot\frac{x}{2} + \int\cot\frac{x}{2}dx - \int\cot\frac{x}{2}dx = -x\cot\frac{x}{2}$$

8. $\int x\arcsin x\, dx$

Sol. $I = \int x\arcsin x\, dx = \int(\arcsin x)\, x\, dx$

Integrating by parts taking arcsin x as first function, we have

$$I = (\arcsin x)\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx \qquad (1)$$

Now $\displaystyle\int \frac{-x^2}{\sqrt{1-x^2}}\,dx = \int \frac{1-x^2-1}{\sqrt{1-x^2}}\,dx = \int\sqrt{1-x^2}\,dx - \int\frac{1}{\sqrt{1-x^2}}\,dx$

$$= x\cdot\frac{\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x - \arcsin x$$

$$= x\cdot\frac{\sqrt{1-x^2}}{2} - \frac{1}{2}\arcsin x.$$

Putting this value into (1), we have

$$I = \frac{x^2}{2}\arcsin x + \frac{1}{4}x\cdot\sqrt{1-x^2} - \frac{1}{4}\arcsin x$$

$$= \frac{2x^2-1}{4}\arcsin x + \frac{1}{4}x\cdot\sqrt{1-x^2}$$

9. $\int x^3\sqrt{x^2+1}\,dx$

Sol. Let $u = x^2$ and $v = \sqrt{x^2+1}\,(2x)$.

Integrating by parts, we get

$$\int x^3\sqrt{x^2+1}\,dx = \frac{1}{2}\int x^2\sqrt{x^2+1}\,(2x)\,dx$$

$$= \frac{1}{2}\left[x^2\,\frac{2}{3}(x^2+1)^{3/2} - \int\frac{2}{3}(x^2+1)^{3/2}\,2x\,dx\right]$$

$$= \frac{1}{3}x^2(x^2+1)^{3/2} - \frac{1}{3}\cdot\frac{2}{5}(x^2+1)^{5/2}$$

$$= \frac{1}{3}x^2(x^2+)^{3/2} - \frac{2}{15}(x^2+1)^{5/2}$$

10. $\displaystyle\int e^x\frac{1+x\ln x}{x}\,dx$

Sol. $\displaystyle\int e^x\left(\frac{1+x\ln x}{x}\right)dx = \int e^x\cdot\frac{1}{x}\,dx + \int e^x\ln x\,dx$

$$= e^x\ln x - \int \ln x\,.\,e^x\,dx + \int e^x\ln x\,dx = e^x\ln x$$

11. $\displaystyle\int e^x\cdot\frac{1-\sin x}{1-\cos x}\,dx$

Sol. $\displaystyle\int e^x\cdot\frac{1-\sin x}{1-\cos x}\,dx = \int e^x\cdot\frac{1-2\sin\frac{x}{2}\cos\frac{x}{2}}{2\sin^2\frac{x}{2}}$

$$= \int e^x\left(\frac{1}{2}\csc^2\frac{x}{2}\right)dx - \int e^x\cot\frac{x}{2}\,dx$$

$$= e^x\left(-\cot\frac{x}{2}\right) + \int\left(\cot\frac{x}{2}\right)e^x\,dx - \int e^x\cot\frac{x}{2}\,dx$$

$$= -e^x\cot\frac{x}{2}$$

12. $\displaystyle\int \arctan\left(\sqrt{\frac{1-x}{1+x}}\right)dx$

Sol. Put $x = \cos\theta$ or $dx = -\sin\theta\,d\theta$

$$I = \int\arctan\left(\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)(-\sin\theta)\,d\theta$$

$$= \int\arctan\left(\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)(-\sin\theta)\,d\theta$$

$$= \int\arctan\left(\tan\frac{\theta}{2}\right)(-\sin\theta)\,d\theta$$

$$= \int\frac{\theta}{2}(-\sin\theta)\,d\theta = \frac{1}{2}\left[\theta\cos\theta - \int\cos\theta\,d\theta\right]$$

$$= \frac{1}{2}[\theta\cos\theta - \sin\theta] = \frac{1}{2}\left[x\arccos x - \sqrt{1-x^2}\right]$$

13. $\displaystyle\int\arcsin\left(\sqrt{\frac{x}{x+a}}\right)dx$

Sol. Put $x = a\tan^2\theta$ or $dx = 2a\tan\theta\sec^2\theta\,d\theta$

$$I = \int\arcsin\sqrt{\frac{a\tan^2\theta}{a\sec^2\theta}}\,.\,2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int\arcsin\left(\frac{\tan\theta}{\sec\theta}\right).\,2a\tan\theta\sec^2\theta\,d\theta$$

$$= \int\arcsin(\sin\theta)\,.\,2a\tan\theta\,.\sec^2\theta\,d\theta$$

$$= 2a\int\theta\,.(\tan\theta\sec^2\theta)\,d\theta$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\,d\theta\right]$$

$$= a\,\theta\tan^2\theta - a\int\tan^2\theta\,d\theta = a\theta\tan^2\theta - a\int(\sec^2\theta - 1)\,d\theta$$

$$= a\,\theta\tan^2\theta - a\,(\tan\theta - \theta)$$

$$= (a\tan^2\theta)\,\theta - a\tan\theta + a\theta$$

$$= x\arctan\sqrt{\frac{x}{a}} - a\sqrt{\frac{x}{a}} + a\arctan\sqrt{\frac{x}{a}}$$

$$= (x + a)\arctan\sqrt{\frac{x}{a}} - \sqrt{ax}$$

14. $\int e^{ax}\sin(bx + c)\,dx$

Sol. Taking e^{ax} as first function and $\sin(bx + c)$ as the second function and then integrating by parts, we have

$$\int e^{ax}\sin(bx + c)\,dx = e^{ax}\frac{-\cos(bx + c)}{b} - \int -\frac{\cos(bx + c)}{b}e^{ax}.\,a\,dx$$

$$= -\frac{e^{ax}}{b}\cos(bx + c) + \frac{a}{b}\int e^{ax}\cos(bx + c)\,dx$$

$$= -\frac{1}{b}e^{ax}\cos(bx + c) + \frac{a}{b}\left[e^{ax}\frac{\sin(bx + c)}{b} - \int\frac{\sin(bx + c)}{b}\,a\,.\,e^{ax}\,dx\right]$$

$$= -\frac{1}{b}e^{ax}\cos(bx + c) + \frac{a}{b^2}e^{ax}\sin(bx + c) - \frac{a^2}{b^2}\int e^{ax}\sin(bx + c)\,dx$$

or $\left(1 + \frac{a^2}{b^2}\right)\int e^{ax}\sin(bx + c)\,dx = e^{ax}\left[-\frac{\cos(bx + c)}{b} + \frac{a\sin(bx + c)}{b^2}\right]$

or $\left(\frac{a^2 + b^2}{b^2}\right)\int e^{ax}\sin(bx + c)\,dx = \frac{e^{ax}}{b^2}[a\sin(bx + c) - b\cos(bx+c)]$

i.e., $\int e^{ax}\sin(bx+c)\,dx = \frac{e^{ax}}{a^2 + b^2}[a\sin(bx+c) - b\cos(bx+c)$ (1)

Put $a = r\cos\theta$ and $b = r\sin\theta$

then $r = \sqrt{a^2 + b^2}$ and $\tan\theta = \frac{b}{a}$

Making these substitutions in the R.H.S. of (1), we get

$$\int e^{ax}\sin(bx + x) = \frac{e^{ax}}{\sqrt{a^2 + b^2}}[\sin(bx + c)\cos\theta - \cos(bx + c)\sin\theta]$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin(bx + c - \theta)$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}}\sin\left(bx + c - \arctan\frac{b}{a}\right)$$

15. $\int\ln(x + \sqrt{1 + x^2})\,dx$

Sol. $\int\ln(x + \sqrt{1 + x^2})\,dx = \int(\ln(x + \sqrt{1 + x^2}))\,.\,1\,dx$

$$= \ln(x + \sqrt{1 + x^2})\,.\,x - \int x\cdot\frac{1 + \frac{x}{\sqrt{1 + x^2}}}{x + \sqrt{1 + x^2}}\,dx$$

$$= x\ln(x + \sqrt{1 + x^2}) - \int\frac{x}{\sqrt{1 + x^2}}\,dx$$

$$= x\ln(x + \sqrt{1 + x^2}) - \int(1 + x^2)^{-1/2}\,.\,x\,dx$$

$$= x\ln(x + \sqrt{1 + x^2}) - \frac{1}{2}\frac{(1 + x^2)^{1/2}}{\frac{1}{2}}$$

$$= x\ln(x + \sqrt{1 + x^2}) - \sqrt{1 + x^2}$$

16. $\int\frac{x^2 + 1}{(x + 1)^2}e^x\,dx$

Sol. $\int\frac{x^2 + 1}{(x + 1)^2}e^x\,dx = \int\frac{(x + 1)^2 - 2x}{(x + 1)^2}e^x\,dx = \int\left[1 - \frac{2x}{(x + 1)^2}\right]e^x\,dx$

$$= \int\left[1 - 2\frac{(x + 1 - 1)}{(x + 1)^2}\right]e^x\,dx$$

$$= \int\left[1 - 2\left\{\frac{1}{x + 1} - \frac{1}{(x + 1)^2}\right\}\right]e^x\,dx$$

$$= \int e^x\,dx - 2\int\frac{e^x\,dx}{x + 1}\,dx + 2\int\frac{e^x\,dx}{(x + 1)^2}$$

$$= e^x - 2\left[\frac{1}{x + 1}e^x - \int e^x\left(\frac{-1}{(x + 1)^2}\right)dx\right] + 2\int\frac{e^x\,dx}{(x + 1)^2}$$

(Integrating the first and second terms and leaving the third term as it is)

$$= e^x - \frac{2}{x + 1}e^x - 2\int\frac{e^x\,dx}{(x + 1)^2} + 2\int\frac{e^x}{(x + 1)^2}\,dx$$

$$= e^x - \frac{2}{x + 1}e^x = \left(\frac{x - 1}{x + 1}\right)e^x$$

17. $\int\cos(\ln x)\,dx$

Sol. Take $\cos(\ln x)$ as first function, 1 as second function and integrate by parts to get

$$I = x\,(\cos(\ln x)) - \int x\left(-\frac{\sin(\ln x)}{x}\right)dx$$
$$= x\cos(\ln x) + \int \sin(\ln x)\,dx$$
$$= x\cos(\ln x) + x\sin(\ln x) - \int \frac{\cos(\ln x)}{x}\,x\,dx$$
$$= x\cos(\ln x) + x\sin(\ln x) - I$$
or $\quad 2I = x\cos(\ln x) + x\sin(\ln x)$

or $\quad I = \frac{1}{2}[x\cos(\ln x) + x\sin(\ln x)]$

18. $\int \sqrt{x}\, e^{-\sqrt{x}}\,dx$

Sol. $I = \int \sqrt{x}\, e^{-\sqrt{x}}\,dx$

Put $\sqrt{x} = z$ or $\frac{1}{2\sqrt{x}}dx = dz$ or $dx = 2z\,dz$

$$I = \int z\,.\,e^{-z}\,.\,2z\,dz = 2\int z^2 e^{-z}\,dz = 2\left[z^2\,.\,\frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1}\,.\,2z\,dz\right]$$
$$= 2\left[z^2\frac{e^{-z}}{-1} + 2\int z e^{-z}\,dz\right] = 2\left[-z^2 e^{-z} + 2\left\{z\frac{e^{-z}}{-1} - \int \frac{e^{-z}}{-1}dz\right\}\right]$$
$$= 2\left[-z^2 e^{-z} - 2ze^{-z} + 2\int e^{-z}\,dz\right] = 2\,[-z^2 e^{-z} - 2\,ze^{-z} - 2e^{-z}]$$
$$= -2\,(x\,e^{-\sqrt{x}} + 2\sqrt{x}\,e^{-\sqrt{x}} + 2e^{-\sqrt{x}})$$

19. $\int x^3 e^{2x}\,dx$

Sol. $\int x^3 e^{2x}\,dx = x^3\,.\,\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,.\,3x^2\,dx = \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\int x^2 e^{2x}\,dx$

$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{2}\left[x^2\,.\,\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\,.\,2x\,dx\right]$$
$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\int x\,e^{2x}\,dx$$
$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{2}\left[x\cdot\frac{e^{2x}}{2} - \int \frac{e^{2x}}{2}\cdot 1\,dx\right]$$
$$= \frac{1}{2}x^3 e^{2x} - \frac{3}{4}x^2 e^{2x} + \frac{3}{4}x\,e^{2x} - \frac{3}{4}\cdot\frac{e^{2x}}{2}$$
$$= \frac{e^{2x}}{8}(4x^3 - 6x^2 + 6x - 3)$$

20. $\int x^5 e^{x^3}\,dx$

Sol. We take the first function as $u = x^3$ and the second function as $v = x^2 e^{x^3}$. Then $\int v\,dx = \int x^2 e^{x^3}\,dx = \frac{1}{3}e^{x^3}$

Hence using the by-parts formula, we have

$$\int x^5 e^{x^3}\,dx = \int x^3\,.\,e^{x^3}\,.\,x^2\,dx = x^3\,.\,\frac{e^{x^3}}{3} - \int 3x^2\,.\,\frac{e^{x^3}}{3}\,dx$$
$$= \frac{1}{3}x^3 e^{x^3} - \int e^{x^3}\,.\,x^2\,dx = \frac{1}{3}x^3 e^{x^3} - \frac{1}{3}e^{x^3}$$

21. Show that

$$\int x^n \arctan x\,dx = \frac{x^{n+1}}{n+1}\arctan x - \frac{1}{n+1}\int \frac{x^{n+1}}{1+x^2}\,dx$$

Hence evaluate $\int x^3 \arctan x\,dx$.

Sol. Integrate by parts with

$u = \arctan x$ and $v = x^n$ so that

$$\int (\arctan x)\,x^n\,dx = (\arctan x)\frac{x^{n+1}}{n+1} - \int \frac{x^{n+1}}{n+1}\,.\,\frac{dx}{1+x^2}$$
$$= \frac{x^{n+1}}{n+1}\arctan x - \frac{1}{x+1}\int \frac{x^{n+1}}{1+x^2}dx \quad \text{as required.}$$

Setting $n = 3$ in the above equation, we get

$$\int x^3 \arctan x\,dx = \frac{x^4}{4}\arctan x - \frac{1}{4}\int \frac{x^4}{1+x^2}\,dx$$

Now $\int \frac{x^4}{1+x^2}\,dx = \int\left(x^2 - 1 + \frac{1}{1+x^2}\right) = \frac{x^3}{3} - x + \arctan x$

Therefore,

$$\int x^3 \arctan x\,dx = \frac{x^4}{4}\arctan x - \frac{x^3}{12} + \frac{x}{4} - \frac{\arctan x}{4}$$
$$= \frac{1}{4}(x^4 - 1)\arctan x - \frac{x^3}{12} + \frac{x}{4}$$

22. Find a reduction formula for $\int x^n e^{ax}\,dx$ and apply it to evaluate $\int x^3 e^{ax}\,dx$.

Sol. $\int x^n e^{ax}\,dx = x^n\,.\,\frac{e^{ax}}{a} - \int \frac{e^{ax}}{a}\,.\,nx^{n-1}\,dx$

$$= x^n \cdot \frac{e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx \qquad (1)$$

is the required reduction formula.
Taking $n = 3$

$$\int x^3 e^{ax}\,dx = \frac{e^3 e^{ax}}{a} - \frac{3}{a}\int x^2 e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a}\left[x^2 \cdot \frac{e^{ax}}{a} - \frac{2}{a}\int xe^{ax}\,dx\right] \quad \left(\begin{array}{l}\text{Putting } n = 2\\ \text{in the formula (1)}\end{array}\right)$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^2}\int xe^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^2}\left[x \cdot \frac{e^{ax}}{a} - \frac{1}{a}\int e^{ax}\,dx\right]$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3}{a^2}x^2 e^{ax} + \frac{6}{a^3}x\,e^{ax} - \frac{6}{a^3}\int e^{ax}\,dx$$

$$= \frac{x^3 e^{ax}}{a} - \frac{3x^2 e^{ax}}{a^2} + \frac{6}{a^3}x\,e^{ax} - \frac{6}{a^4}e^{ax}$$

$$= e^{ax}\left[\frac{x^3}{a} - \frac{3x^2}{a^2} + \frac{6x}{a^3} - \frac{6}{a^4}\right] = \frac{e^{ax}}{a^4}\,[a^3x^3 - 3a^2x^2 + 6ax - 6]$$

23. Find reduction formulas for $\int \sin^n x\,dx$ and $\int \cos^n x\,dx$, where n is a positive integer.

Sol. $I_n = \int \sin^n x\,dx$

We write $I_n = \int \sin^{n-1}x \sin x\,dx$ and integrate by parts taking $\sin^{n-1}x$ as the first function.

$$I_n = \sin^{n-1}x\,(-\cos x) - \int (n-1)\sin^{n-2}x \cos x\,(-\cos x)\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \cos^2 x \sin^{n-2} x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int (1 - \sin^2 x)\sin^{n-2}x\,dx$$

$$= -\cos x \sin^{n-1}x + (n-1)\int \sin^{n-2}x\,dx - (n-1)\,I_n$$

Hence $\quad I_n\,(n - 1 + 1) = -\cos x \sin^{n-1}x + (n-1)\,I_{n-2}$

or $\quad I_n = -\dfrac{\cos x \sin^{n-1}x}{n} + \dfrac{n-1}{n}I_{n-2}$

which is the required reduction formula.
Next, let us write

$$I_n = \int \cos^n x\,dx = \int \cos^{n-1} x \cos x\,dx$$

Integrating by parts, we have

$$I_n = \cos^{n-1}x \sin x - \int (n-1)\cos^{n-2}x\,(-\sin x)\sin x\,dx$$

$$= \cos^{n-1}x \sin x + (n-1)\int \cos^{n-2}x\,(1 - \cos^2 x)\,dx$$

$$= \sin x \cos^{n-1}x - (n-1)\,I_n + (n-1)\,I_{n-2}$$

$$nI_n = \sin x \cos^{n-1}x + (n-1)\,I_{n-2}$$

or $\quad I_n = \dfrac{\sin x \cos^{n-1}x}{n} + \dfrac{n-1}{n}I_{n-2}$

24. Find a reduction formula for $\int x^n \sin ax\,dx$, where $n > 1$ is an integer. Hence evaluate $\int x^4 \sin 4x\,dx$.

Sol. Let $u = x^n$ and $v = \sin ax$ so that the by-parts formula gives

$$\int x^n \sin ax\,dx = x^n\left(-\frac{\cos ax}{a}\right) - \int nx^{n-1}\left(-\frac{\cos ax}{a}\right)dx$$

$$= -\frac{1}{a}x^n \cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx \qquad (1)$$

Again, $\int x^{n-1}\cos ax\,dx = x^{n-1}\left(\dfrac{\sin ax}{a}\right) - \int (n-1)\,x^{n-2}\left(\dfrac{\sin ax}{a}\right)dx$

$$= \frac{1}{a}x^{n-1}\sin ax - \frac{n-1}{a}\int x^{n-2}\sin ax\,dx \qquad (2)$$

From (1) and (2), we obtain

$$\int x^n \sin ax\,dx = -\frac{1}{a}x^n \cos ax + \frac{n}{a^2}x^{n-1}\sin ax - \frac{n(n-1)}{a^2}\int x^{n-2}\sin ax\,dx$$

which is the required reduction formula.

Now $\int x^4 \sin 4x = -\dfrac{1}{4}x^4 \cos 4x + \dfrac{4}{16}x^3 \sin 4x - \dfrac{4 \times 3}{16}\int x^2 \sin 4x\,dx \qquad (3)$

Again,

$$\int x^2 \sin 4x\,dx = -\frac{x^2}{4}\cos 4x + \frac{2}{16}x \sin 4x - \frac{2}{16}\int \sin 4x\,dx \qquad (4)$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{1}{32}x^4.$$

From (3) and (4), we have

$$\int x^4 \sin 4x\,dx = -\frac{1}{4}x^4 \cos 4x + \frac{1}{4}x^3 \sin 4x - \frac{3}{4}\left[-\frac{1}{4}x^2 \cos 4x + \frac{1}{8}x \sin 4x + \frac{1}{32}\cos 4x\right]$$

$$= -\frac{1}{4}x^4 \cos 4x + \frac{1}{4}x^3 \sin 4x + \frac{3}{16}x^2 \cos 4x - \frac{3}{32}x \sin 4x - \frac{3}{128} \times \cos 4x$$

25. Find a reduction formula for $\int x^m (\ln x)^n\,dx$, $m \neq -1$ and n is an integer greater than 1. Hence evaluate $\int x^3 (\ln x)^2\,dx$.

Sol. Let $u = (\ln x)^n$ and $v = x^m$. Then the by-part rule gives

$$\int (\ln x)^n \cdot x^m\,dx = (\ln x)^n \cdot \frac{x^{m+1}}{m+1} - \int \frac{x^{m+1}}{m+1} \cdot \frac{n(\ln x)^{n-1}}{x}\,dx$$

$$= \frac{x^{m+1}}{m+1}(\ln x)^n - \frac{n}{m+1}\int x^m (\ln x)^{n-1}\,dx$$

as the required reduction formula.

In the above reduction formula putting $m = 3$ and $n = 2$, we have

$$\int x^3 (\ln x)^2\,dx = \frac{x^4}{4}(\ln x)^2 - \frac{2}{4}\int x^3 \ln x\,dx$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{1}{2}\left[(\ln x)\frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x}\,dx\right]$$

$$= \frac{x^4}{4}(\ln x)^2 - \frac{x^4}{8}\ln x + \frac{1}{32}x^4$$

Exercise Set 4.4 (Page 149)

Integrate each of the following with respect to x:

1. $\dfrac{x}{(x-1)(x-2)}$

Sol. Here $\dfrac{x}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{2}{x-2}$

$$\int \frac{x\,dx}{(x-1)(x-2)} = -\int \frac{dx}{x-1} + \int \frac{dx}{x-2} = -\ln|x-1| + 2\ln|x-2|$$

$$= \ln\frac{(x-2)^2}{|x-1|}$$

2. $\dfrac{2x-3}{(x^2-1)(2x+3)}$

Sol. Here $\dfrac{2x-3}{(x^2-1)(2x+3)} = \dfrac{2x-3}{(x-1)(x+1)(2x+3)}$

$$= \frac{-1}{10(x-1)} + \frac{-5}{-2(x+1)} + \frac{-6}{\left(\frac{-5}{2}\right)\left(\frac{-1}{2}\right)(2x+3)}$$

Integrating, $\displaystyle\int \frac{2x-3}{(x2-1)(2x+3)}\,dx$

$$= -\frac{1}{10}\int \frac{dx}{x-1} + \frac{5}{2}\int \frac{dx}{x+1} - \frac{24}{5}\int \frac{dx}{2x+3}$$

$$= -\frac{1}{10}\ln|x-1| + \frac{5}{2}\ln|x+1| - \frac{24}{5} \cdot \frac{1}{2}\ln|2x+3|$$

$$= \frac{5}{2}\ln|x+1| - \frac{1}{10}\ln|x-1| - \frac{12}{5}\ln|2x+3|$$

3. $\dfrac{x+1}{x^2+4x+5}$

Sol. $\displaystyle\int \frac{x+1}{x^2+4x+5}\,dx = \frac{1}{2}\int \frac{2x+2}{x^2+4x+5}\,dx = \frac{1}{2}\int \frac{(2x+4)-2}{x^2+4x+5}\,dx$

$$= \frac{1}{2}\int \frac{2x+4}{x^2+4x+5}\,dx - \frac{1}{2}\int \frac{2}{(x^2+4x+5)}\,dx$$

$$= \frac{1}{2}\ln(x^2+4x+5) - \int \frac{dx}{x^2+4x+5}$$

$$= \frac{1}{2}\ln(x^2+4x+5) - \int \frac{dx}{(x+2)^2+1}$$

$$= \frac{1}{2}\ln(x^2+4x+5) - \arctan(x+2)$$

4. $\dfrac{2x^2+3x+1}{x^2+2x+2}$

Sol. $\dfrac{2x^2+3x+1}{x^2+2x+2} = 2 - \dfrac{x+3}{x^2+2x+2}$

Therefore,

$$\int \frac{2x^2+3x+1}{x^2+2x+2}\,dx = 2\int 1\,.\,dx - \int \frac{x+3}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\int \frac{(2x+2)+4}{x^2+2x+2} = 2x - \frac{1}{2}\int \frac{(2x+2)+4}{x^2+2x+2}\,dx$$

$$= 2x - \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}\,dx - 2\int \frac{dx}{x^2+2x+2}$$

$$= 2x - \frac{1}{2}\ln(x^2 + 2x + 2) - 2\int \frac{dx}{(x+1)^2 + 1}$$

$$= 2x - \frac{1}{2}\ln(x^2 + 2x + 2) - 2\arctan(x+1).$$

5. $\dfrac{x^2}{(x-1)^3(x+1)}$

Sol. $\dfrac{x^2}{(x-1)^3(x+1)} = \dfrac{1}{2(x-1)^3} + \dfrac{3}{4(x-1)^2} + \dfrac{1}{8(x+1)} - \dfrac{1}{8(x+1)}$

Now, integrating

$$\int \frac{x^2\,dx}{(x-1)^3(x+1)} = \frac{1}{2}\int \frac{dx}{(x-1)^3} + \frac{3}{4}\int \frac{dx}{(x-1)^2} + \frac{1}{8}\int \frac{dx}{x-1} - \frac{1}{8}\int \frac{dx}{x+1}$$

$$= \frac{1}{2}\cdot\frac{(x-1)^{-2}}{-2} + \frac{3}{4}\frac{(x-1)^{-1}}{-1} + \frac{1}{8}\ln|x-1| - \frac{1}{8}|x+1|$$

$$= -\frac{1}{4}\cdot\frac{1}{(x-1)^2} - \frac{3}{4}\frac{1}{x-1} + \frac{1}{8}\ln|x-1| - \frac{1}{8}\ln|x+1|$$

$$= -\frac{1}{4}\cdot\frac{1}{(x-1)^2} - \frac{3}{4}\frac{1}{x-1} + \frac{1}{8}\ln\left|\frac{x-1}{x+1}\right|$$

6. $\dfrac{1}{x(x+1)^3}$

Sol. $\dfrac{1}{x(x+1)^3} = -\dfrac{1}{(x+1)^3} - \dfrac{1}{(x+1)^2} - \dfrac{1}{x+1} + \dfrac{1}{x}$

Integrating, we have

$$\int \frac{dx}{x(x+1)^3} = -\int \frac{dx}{(x+1)^3} - \int \frac{dx}{(x+1)^2} - \int \frac{dx}{x+1} + \int \frac{dx}{x}$$

$$= -\frac{(x+1)^{-2}}{-2} - \frac{(x+1)^{-1}}{-1} - \ln|x+1| + \ln x$$

$$= \frac{1}{2(x+1)^2} + \frac{1}{x+1} - \ln|x+1| + \ln|x|$$

$$= \ln|x| - \ln|x+1| + \frac{1}{x+1} + \frac{1}{2(x+1)^2}$$

7. $\dfrac{x+1}{(x-1)^2(x+2)^2}$

Sol. Resolving $\dfrac{x+1}{(x-1)^2(x+2)^2}$ into partial fractions, we have

$$\frac{x+1}{(x-1)^2(x+2)^2} = \frac{-\frac{1}{27}}{x-1} + \frac{\frac{2}{9}}{(x-1)^2} + \frac{\frac{1}{27}}{x+2} - \frac{\frac{1}{9}}{(x+2)^2}\ \text{Therefore,}$$

$$\int \frac{x+1}{(x-1)^2(x+2)^2}dx = -\frac{1}{27}\int\frac{dx}{x-1} + \frac{2}{9}\int\frac{dx}{(x-1)^2} + \frac{1}{27}\int\frac{dx}{x+2} - \frac{1}{9}\int\frac{dx}{(x+2)^2}$$

$$= -\frac{1}{27}\ln|x-1| - \frac{2}{9}\cdot\frac{1}{x-1} + \frac{1}{27}\ln|x+2| + \frac{1}{9}\cdot\frac{1}{(x+2)}$$

$$= \frac{1}{27}\ln\left|\frac{x+2}{x-1}\right| - \frac{2}{9}\cdot\frac{1}{x-1} + \frac{1}{9}\cdot\frac{1}{x+2}$$

8. $\dfrac{1}{1-x^3}$

Sol. Resolving $\dfrac{1}{1-x^3}$ into partial fractions, we have

$$\frac{1}{1-x^3} = \frac{1}{3}\cdot\frac{1}{1-x} + \frac{1}{3}\cdot\frac{x+2}{x^2+x+1}.\ \text{Hence}$$

$$\int \frac{1}{1-x^3}dx = \frac{1}{3}\int\frac{dx}{1-x} + \frac{1}{3}\int\frac{(x+2)\,dx}{x^2+x+1}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{3}\cdot\frac{1}{2}\int\frac{2x+4}{x^2+x+1}dx$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\int\frac{(2x+1)+3}{x^2+x+1}dx$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\int\frac{2x+1}{x^2+x+1}dx + \frac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\int\frac{dx}{x^2+x+1}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}$$

$$= -\frac{1}{3}\ln|1-x| + \frac{1}{6}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)$$

9. $\dfrac{x^2+1}{x^3+1}$

Sol. Resolving $\dfrac{x^2+1}{x^3+1}$ into partial fractions, we find that

$$\frac{x^2+1}{x^3+1} = \frac{\frac{2}{3}}{x+1} + \frac{\frac{1}{3}x + \frac{1}{3}}{x^2-x+1}.\ \text{Therefore,}$$

$$\int\frac{x^2+1}{x^3+1}dx=\frac{2}{3}\int\frac{dx}{x+1}+\frac{1}{3}\int\frac{(x+1)\,dx}{x^2-x+1}$$
$$=\frac{2}{3}\ln|x+1|+\frac{1}{6}\int\frac{2x+2}{x^2-x+1}dx$$
$$=\frac{2}{3}\ln|x+1|+\frac{1}{6}\int\frac{(2x-1)+3}{x^2-x+1}dx$$
$$=\frac{2}{3}\ln|x+1|+\frac{1}{6}\int\frac{(2x-1)\,dx}{x^2-x+1}+\frac{1}{2}\int\frac{dx}{x^2-x+1}$$
$$=\frac{2}{3}\ln|x+1|+\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$
$$=\frac{2}{3}\ln|x+1|+\frac{1}{6}\ln(x^2-x+1)+\frac{1}{2}\cdot\frac{2}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$
$$=\frac{2}{3}\ln|x+1|+\frac{1}{6}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\arctan\frac{2x-1}{\sqrt{3}}$$

10. $\dfrac{1}{(x-1)(x^2+4)}$

Sol. Resolving $\dfrac{1}{(x-1)(x^2+4)}$ into partial fractions, we have

$$\frac{1}{(x-1)(x^2+4)}=\frac{\frac{1}{5}}{x-1}-\frac{\frac{1}{5}x+\frac{1}{5}}{x^2+4}.\text{ Thus,}$$
$$\int\frac{dx}{(x-1)(x^2+4)}=\frac{1}{5}\int\frac{dx}{x-1}-\frac{1}{5}\int\frac{x+1}{x^2+4}dx$$
$$=\frac{1}{5}\ln|x-1|-\frac{1}{10}\int\frac{2x}{x^2+4}dx-\frac{1}{5}\int\frac{dx}{x^2+4}$$
$$=\frac{1}{5}\ln|x-1|-\frac{1}{10}\ln(x^2+4)-\frac{1}{10}\left(\arctan\frac{x}{2}\right)$$

11. $\dfrac{2x^2-3x-3}{(x-1)(x^2-2x+5)}$

Sol. Let $\dfrac{2x^2-3x-3}{(x-1)(x^2-2x+5)}=\dfrac{A}{x-1}+\dfrac{Bx+C}{x^2-2x+5}=-\dfrac{1}{x-1}$

$+\dfrac{3x-2}{x^2-2x+5}$, after finding the values of A, B, C. Then

$$\int\frac{2x^2-3x-3}{(x-1)(x^2-2x+5)}dx=-\int\frac{dx}{x-1}+\int\frac{3x-2}{x^2-2x+5}dx$$

$$=-\ln|x-1|+3\int\frac{x-\frac{2}{3}}{x^2-2x+5}dx$$
$$=-\ln|x-1|+\frac{3}{2}\int\frac{2x-\frac{4}{3}}{x^2-2x+5}dx$$
$$=-\ln|x-1|+\frac{3}{2}\int\frac{(2x-2)+\left(2-\frac{4}{3}\right)}{x^2-2x+5}dx$$
$$=-\ln|x-1|+\frac{3}{2}\int\frac{2x-2}{x^2-2x+5}dx+\frac{3}{2}\cdot\frac{2}{3}\int\frac{dx}{x^2-2x+5}$$
$$=-\ln|x-1|+\frac{3}{2}\ln(x^2-2x+5)+\int\frac{dx}{(x-1)^2+4}$$
$$=-\ln|x-1|+\frac{3}{2}\ln(x^2-2x+5)+\frac{1}{2}\arctan\left(\frac{x-1}{2}\right)$$

12. $\dfrac{1}{x^4+1}$

Sol. $\displaystyle\int\frac{dx}{x^4+1}=\int\frac{1}{x^4+1}dx=\frac{1}{2}\int\frac{2}{x^4+1}dx$

$$=\frac{1}{2}\int\frac{(x^2+1)-(x^2-1)}{x^4+1}dx=\frac{1}{2}\int\frac{x^2+1}{x^4+1}dx-\frac{1}{2}\int\frac{x^2-1}{x^4+1}dx$$
$$=\frac{1}{2}(I_1-I_2), \tag{1}$$

where, $I_1=\displaystyle\int\frac{x^2+1}{x^4+1}dx$

$$=\int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx.$$ Put $x-\frac{1}{x}=t$, or $\left(1+\frac{1}{x^2}\right)dx=dt$

and $x^2+\frac{1}{x^2}-2=t^2$, *i.e.*, $x^2+\frac{1}{x^2}=t^2+2$

$$I_1=\int\frac{dt}{t^2+2}=\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}=\frac{1}{\sqrt{2}}\arctan\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}$$
$$=\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}\,x} \tag{2}$$

and $I_2 = \int \frac{x^2-1}{x^4+1}\,dx = \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}}\,dx$

Put $x+\frac{1}{x}=t$, or $\left(1-\frac{1}{x^2}\right)dx = dt$

and $x^2+\frac{1}{x^2}+2=t^2$, *i.e.*, $x^2+\frac{1}{x^2}=t^2-2$

$$I_2 = \int\frac{dt}{t^2-2} = \frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| = \frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right|$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| \qquad (3)$$

Hence from (1), (2) and (3), we have

$$\int\frac{dx}{x^4+1} = \frac{1}{2}\left[\frac{1}{\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right]$$

$$= \frac{1}{2\sqrt{2}}\arctan\frac{x^2-1}{\sqrt{2}x} - \frac{1}{4\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|$$

13. $\frac{x^4}{x^4+2x^2+1}$

Sol. $\frac{x^4}{x^4+2x^2+1} = 1+\frac{-2x^2-1}{(x^2+1)^2} = 1-\frac{2}{1+x^2}+\frac{1}{(1+x^2)^2}$

Therefore,

$$\int\frac{x^4}{x^4+2x^2+1}\,dx = \int\left(1-\frac{2}{1+x^2}+\frac{1}{(1+x^2)^2}\right)dx$$

$$= x-2\arctan x+\int\frac{dx}{(x^2+1)^2} \qquad (1)$$

In $\int\frac{dx}{(x^2+1)^2}$, put $x=\tan\theta$ so that $dx=\sec^2\theta\,d\theta$

Then $\int\frac{dx}{(x^2+1)^2} = \int\frac{\sec^2\theta\,d\theta}{(\sec\theta)^2} = \int\cos^2\theta\,d\theta = \frac{1}{2}\int(1+\cos 2\theta)\,d\theta$

$$= \frac{1}{2}\theta+\frac{1}{4}\sin 2\theta = \frac{1}{2}\theta+\frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\arctan x+\frac{1}{2}\frac{x}{\sqrt{1+x^2}}\cdot\frac{1}{\sqrt{1+x^2}}$$

Putting this value in (1), we get

$$\int\frac{x^4}{x^4+2x^2+1}\,dx = x-\frac{3}{2}\arctan x+\frac{1}{2}\cdot\frac{x}{1+x^2}$$

14. $\frac{x^2+1}{x^4-x^2+1}$

Sol. $\int\frac{x^2+1}{x^4-x^2+1}\,dx = \int\frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx$

$= \int\frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-1}\,dx$ Put $x-\frac{1}{x}=t$ so that $\left(1+\frac{1}{x^2}\right)dx=dt$

and $x^2+\frac{1}{x^2}-2=t^2$ or $x^2+\frac{1}{x^2}=t^2+2$

$$= \int\frac{dt}{t^2+2-1}$$

$$= \int\frac{dt}{t^2+1} = \arctan t = \arctan\left(x-\frac{1}{x}\right) = \arctan\left(\frac{x^2-1}{x}\right)$$

15. $\frac{1}{(e^x-1)^2}$

Sol. Put $e^x=z$ so that $e^x\,dx=dz$ or $dx=\frac{dz}{z}$. Then

$$\int\frac{dx}{(e^x-1)^2} = \int\frac{dz}{z(z-1)^2}$$

Now $\frac{1}{z(z-1)^2} = \frac{1}{(z-1)^2}-\frac{1}{z-1}+\frac{1}{z}$

Therefore, $\int\frac{dz}{z(z-1)^2} = \int\frac{dz}{(z-1)^2}-\int\frac{dz}{z-1}+\int\frac{dz}{z}$

$$= -\frac{1}{z-1}-\ln(z-1)+\ln z$$

i.e., $\int\frac{dx}{(e^x-1)^2} = -\frac{1}{e^x-1}-\ln(e^x-1)+\ln e^x$

$$= \frac{-1}{e^x-1}-\ln(e^x-1)+x = x-\frac{1}{e^x-1}-\ln(e^x-1)$$

16. $\frac{1}{(1+e^x)(1+e^{-x})}$

Sol. $\displaystyle\int\frac{dx}{(1+e^x)(1+e^{-x})}=\int\frac{dx}{(1+e^x)\left(1+\frac{1}{e^x}\right)}=\int\frac{e^x\,dx}{(1+e^x)(e^x+1)}$

$\displaystyle=\int\frac{e^x}{(e^x+1)^2}dx$. Put $e^x+1=t$ or $e^x\,dx=dt=\displaystyle\frac{dt}{t^2}=-\frac{1}{t}=-\frac{1}{e^x+1}$

17. $\displaystyle\frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}$

Sol. Put $\sin x=t$ or $\cos x\,dx=dt$. Then

$$\int\frac{\cos x}{(1+\sin x)(2+\sin x)(3+\sin x)}dx \qquad (\sin x\neq -1)$$

$$=\int\frac{1}{(1+t)(2+t)(3+t)}dt=\int\left[\frac{\frac{1}{2}}{1+t}+\frac{-1}{t+2}+\frac{\frac{1}{2}}{t+3}\right]dt$$

$$=\frac{1}{2}\int\frac{dt}{t+1}-\int\frac{dt}{t+2}+\frac{1}{2}\int\frac{dt}{t+3}$$

$$=\frac{1}{2}\ln(t+1)-\ln(t+2)+\frac{1}{2}\ln(t+3)$$

$$=\frac{1}{2}\ln(1+\sin x)-\ln(2+\sin x)+\frac{1}{2}\ln(3+\sin x)$$

18. $\displaystyle\frac{\sec x}{1+\csc x}$

Sol. $\displaystyle\int\frac{\sec x\,dx}{1+\csc x}=\int\frac{\frac{1}{\cos x}}{1+\frac{1}{\sin x}}dx=\int\frac{\sin x\,dx}{\cos x\,(1+\sin x)}=\int\frac{\cos x\sin x\,dx}{\cos^2 x\,(1+\sin x)}$

$$=\int\frac{\cos x\sin x\,dx}{(1-\sin^2 x)(1+\sin x)} \qquad (-1<\sin x<1)$$

$$=\int\frac{\cos x\sin x\,dx}{(1-\sin x)(1+\sin x)^2}, \qquad \text{Put } \sin x=t \text{ or } \cos x\,dx=dt$$

$$=\int\frac{t\,dt}{(1-t)(1+t)^2}$$

Now, $\displaystyle\frac{t}{(1-t)(1+t)^2}=\frac{1}{4(1-t)}+\frac{1}{4(1+t)}-\frac{1}{2(1+t)^2}$

Therefore,

$$\int\frac{t}{(1-t)(1+t)^2}dt=\frac{1}{4}\int\frac{dt}{1-t}+\frac{1}{4}\int\frac{dt}{1+t}-\frac{1}{2}\int\frac{dt}{(1+t)^2}$$

$$=\frac{1}{4}\ln(1-t)+\frac{1}{4}\ln(1+t)+\frac{1}{2}\cdot\frac{1}{1+t}$$

Hence $\displaystyle\int\frac{\sec x\,dx}{1+\csc x}=-\frac{1}{4}\ln(1-\sin x)+\frac{1}{4}\ln(1+\sin x)\frac{1}{2}\cdot\frac{1}{1+\sin x}$

$$=\frac{1}{4}\ln\left(\frac{1+\sin x}{1-\sin x}\right)+\frac{1}{2}\cdot\frac{1}{1+\sin x}$$

19. $\displaystyle\frac{\sin x}{\sin 3x}$

Sol. $\displaystyle I=\int\frac{\sin x\,dx}{\sin 3x}=\int\frac{\sin x\,dx}{3\sin x-4\sin^3 x}=\int\frac{dx}{3-4\sin^2 x}$

$$=\int\frac{dx}{3(\sin^2 x+\cos^2 x)-4\sin^2 x}$$

$$=\frac{dx}{3\cos^2 x-\sin^2 x}=\int\frac{\sec^2 x\,dx}{3-\tan^2 x}$$

Put $\tan x=t$ or $\sec^2 x\,dx=dt$. Then

$$I=\int\frac{dt}{3-t^2}=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|=\frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$$

20. $\displaystyle\frac{\cot x-3\cot 3x}{3\tan 3x-\tan x}$

Sol. We know that

$$\tan 3x=\frac{3\tan x-\tan^3 x}{1-3\tan^2 x} \text{ and } \cot 3x=\frac{1-3\tan^2 x}{3\tan x-\tan^3 x}$$

Therefore, $\displaystyle\frac{\cot x-3\cot 3x}{3\tan 3x-\tan x}=\frac{\frac{1}{\tan x}-3\cdot\frac{1-3\tan^2 x}{3\tan x-\tan^3 x}}{3\frac{3\tan x-\tan^3 x}{1-3\tan^2 x}-\tan x}$

$$=\frac{\frac{3-\tan^2 x-3+9\tan^2 x}{3\tan x-\tan^3 x}}{\frac{9\tan x-3\tan^3 x-\tan x+3\tan^3 x}{1-3\tan^2 x}}$$

$$=\frac{8\tan^2 x}{3\tan x-\tan^3 x}\cdot\frac{1-3\tan^2 x}{8\tan x}$$

$$=\frac{1-3\tan^2 x}{3-\tan^2 x}=3-\frac{8}{3-\tan^2 x}$$

Therefore, $\displaystyle\int\frac{\cot x-3\cot x}{3\tan 3x-\tan x}dx=3x-8\int\frac{dx}{3-\tan^2 x} \qquad (1)$

Put $\tan x = t$, so that $\sec^2 x\,dx = dt \Rightarrow dx = \dfrac{dt}{\sec^2 x} = \dfrac{dt}{1+t^2}$

Now $\displaystyle\int \frac{dx}{3-\tan^2 x} = \int \frac{dt}{(3-t^2)(1+t^2)}$

Let $\dfrac{1}{(3-t^2)(1+t^2)} = \dfrac{A}{\sqrt{3}+t} + \dfrac{B}{\sqrt{3}-t} + \dfrac{Ct+D}{t^2+1}$

$$= \frac{1}{8\sqrt{3}(\sqrt{3}+t)} + \frac{1}{8\sqrt{3}(\sqrt{3}-t)} + \frac{1}{4(t^2+1)},$$

after finding the values of A, B, C and D. Therefore,

$$\int \frac{dt}{(3-t^2)(1+t^2)} = \frac{1}{8\sqrt{3}}\int \frac{dt}{\sqrt{3}+t} + \frac{1}{8\sqrt{3}}\int \frac{dt}{\sqrt{3}-t} + \frac{1}{4}\int \frac{dt}{t^2+1}$$

$$= \frac{1}{8\sqrt{3}}\ln|\sqrt{3}+t| - \frac{1}{8\sqrt{3}}\ln|\sqrt{3}-t| + \frac{1}{4}\arctan t$$

$$= \frac{1}{8\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + \frac{1}{4}\arctan(\tan x)$$

$$= \frac{1}{8\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + \frac{1}{4}x$$

Putting it in (1), we get

$$\int \frac{\cot x - 3\cot 3x}{3\tan 3x - \tan x}dx = 3x - 8\left[\frac{1}{8\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| + \frac{1}{4}x\right]$$

$$= 3x - \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right| - 2x = x - \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}+\tan x}{\sqrt{3}-\tan x}\right|$$

21. $\dfrac{\cos x}{\sin^2 x + 4\sin x - 5}$

Sol. We make the substitution $\sin x = u$ so that $\cos x\,dx = du$ and

$$\int \frac{\cos x\,dx}{\sin^2 x + 4\sin x - 5} = \int \frac{du}{u^2+4u-5} = \int \frac{du}{(u+5)(u-1)}$$

$$= -\frac{1}{6}\int \frac{du}{u+5} + \frac{1}{6}\int \frac{du}{u-1}$$

$$= -\frac{1}{6}\ln|u+5| + \frac{1}{6}\ln|u-1|$$

$$= \frac{1}{6}\ln\left|\frac{u-1}{u+5}\right| = \frac{1}{6}\ln\left|\frac{\sin x - 1}{\sin x + 5}\right|$$

22. $\dfrac{\sec^2 x}{\tan^3 x - \tan^2 x}$

Sol. Here we put $\tan x = u$. Therefore, $\sec^2 x\,dx = du$ and

$$\int \frac{\sec^2 x}{\tan^3 x - \tan^2 x} = \int \frac{du}{u^2(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u} - \int \frac{du}{u^2}$$

$$= \ln|u-1| - \ln|u| + \frac{1}{u} = \ln\left|\frac{4-1}{u}\right| + \frac{1}{u} = \ln\left|\frac{\tan x - 1}{\tan x}\right| + \cot x.$$

23. $\dfrac{x^2+1}{(x^2+2x+3)^2}$

Sol. It is easy to see that

$$\frac{x^2+1}{(x^2+2x+3)^2} = \frac{1}{x^2+2x+3} - \frac{2x+2}{(x^2+2x+3)^2}$$

Hence $\displaystyle\int \frac{x^2+1}{(x^2+2x+3)^2}dx = \int \frac{dx}{x^2+2x+3} - \int \frac{2x+2}{(x^2+2x+3)^2}dx$

$$= \int \frac{dx}{(x+1)^2+(\sqrt{2})^2} + \frac{1}{x^2+2x+3}$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x+1}{\sqrt{2}}\right) + \frac{1}{x^2+2x+3}$$

24. $\dfrac{x^3+2x^2-3}{(x^2+9)^2}$

Sol. We have $\dfrac{x^3+2x^2-3}{(x^2+9)^2} = \dfrac{Ax+B}{x^2+9} + \dfrac{Cx+D}{(x^2+9)^2}$

Therefore, $x^3 + 2x^2 - 3 = (Ax+B)(x^2+9) + Cx + D$ (1)

is an identity.

Equating coefficients of like terms in (1), we get

Coefficient of x^3: $1 = A$

Coefficient of x^2: $2 = B$

Coefficient of x: $0 = 9A + C = 9 + C \Rightarrow C = -9$

Constant terms: $-3 = 9B + D = 18 + D \Rightarrow D = -21$

Now, $\displaystyle\int \frac{x^3+2x^2-3}{(x^2+9)^2}dx = \int \frac{(x+2)\,dx}{x^2+9} + \int \frac{-9x-21}{(x^2+9)^2}dx$

$$= \frac{1}{2}\int \frac{2x}{x^2+9}dx + \int \frac{2\,dx}{x^2+9} - \frac{9}{2}\int \frac{2x\,dx}{(x^2+9)^2} - 21\int \frac{dx}{(x^2+9)^2}$$

$$= \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{1}{x^2+9} - 21\int \frac{dx}{(x^2+9)^2} \quad (2)$$

In $\displaystyle\int \frac{dx}{(x^2+9)^2}$, put $x = 3\tan\theta$ so that $dx = 3\sec^2\theta\,d\theta$ and

$$\int \frac{dx}{(x^2+9)^2} = \int \frac{3\sec^2\theta\, d\theta}{9^2 \sec^4\theta} = \frac{1}{27}\int \cos^2\theta\, d\theta = \frac{1}{27}\int \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{27}\left(\frac{1}{2}\theta + \frac{\sin\theta\cos\theta}{2}\right) = \frac{1}{54}\left[\arctan\left(\frac{x}{3}\right) + \frac{3x}{x^2+9}\right] \qquad (3)$$

From (2) and (3), we get $\int \frac{x^3+2x^2-3}{(x^2+9)}\,dx$

$$= \ln\sqrt{x^2+9} + \frac{2}{3}\arctan\left(\frac{x}{3}\right) + \frac{9}{2}\cdot\frac{1}{x^2+9} - \frac{21}{54}\left[\arctan\frac{x}{3} + \frac{3x}{x^2+9}\right]$$

$$= \ln\sqrt{x^2+9} + \frac{5}{18}\arctan\left(\frac{x}{3}\right) + \frac{27-7x}{6(x^2+9)}$$

25. $\dfrac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}$

Sol. $$\frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2} = \frac{A}{x+2} + \frac{Bx+C}{x^2+3} + \frac{Dx+E}{(x^2+3)^2}$$

$$= \frac{1}{x+2} + \frac{2x}{x^2+3} + \frac{4x}{(x^2+3)^2}$$

after finding the values of the constant A, B, C, D and E.

Therefore, $\int \frac{3x^4+4x^3+16x^2+20x+9}{(x+2)(x^2+3)^2}\,dx$

$$= \int\frac{dx}{x+2} + \int\frac{2x\,dx}{x^2+3} + \frac{4x\,dx}{(x^2+3)^2} = \ln|x+2| + \ln(x^2+3) - \frac{2}{x^2+3}$$

Exercise Set 4.5 (Page 156)

Integrate each of the following with respect to x:

1. $x^2\sqrt{25-x^2}$

Sol. We make the substitution

$x = 5\sin\theta$ or $dx = 5\cos\theta\, d\theta$. Then

$$\int x^2\sqrt{(25-x^2)}\,dx = \int 25\sin^2\theta\sqrt{25-25\sin^2\theta}\,.\,5\cos\theta\, d\theta$$

$$= 625\int \sin^2\theta\cos^2\theta\, d\theta = 625\int\frac{\sin^2 2\theta}{4}\, d\theta$$

$$= \frac{625}{4}\int\frac{1-\cos 4\theta}{2}\, d\theta = \frac{625}{8}\left[\theta - \frac{\sin 4\theta}{4}\right] d\theta$$

$$= \frac{625}{8}\left[\theta - \frac{2\sin 2\theta\cos 2\theta}{4}\right]$$

$$= \frac{625}{8}[\theta - \sin\theta\cos\theta(\cos^2 - \sin^2\theta]$$

$$= \frac{625}{8}[\theta - \sin\theta\cos^3\theta + \sin^3\theta\cos\theta]$$

$$= \frac{625}{8}\left[\arcsin\left(\frac{x}{5}\right) - \frac{x}{5}\left(1-\frac{x^2}{25}\right)^{3/2} + \frac{x^3}{125}\sqrt{1-\frac{x^2}{25}}\right]$$

$$= \frac{625}{8}\arcsin\left(\frac{x}{5}\right) - \frac{x}{8}(25-x^2)^{3/2} + \frac{x^3}{8}\sqrt{25-x^2}$$

2. $x(x+4)^{1/3}$

Sol. Put $(x+4)^{1/3} = z$ or $x+4 = z^3$ or $dx = 3z^2\,dz$

$$\int x(x+4)^{1/3}\,dx = \int (z^3-4)\,z\,.\,3z^2\,dz = 3\int (z^6 - 4z^3)\,dz$$

$$= \frac{3}{7}z^7 - 3z^4 = \frac{3}{7}(x+4)^{7/3} - 3(x+4)^{4/3}$$

3. $e^x\sqrt{1-e^{2x}}$

Sol. Put $e^x = \sin\theta$, so that $e^x\,dx = \cos\theta\, d\theta$

$$\int e^x\sqrt{(1-e^{2x})}\,dx = \int\cos^2\theta\, d\theta = \int\frac{1+\cos 2\theta}{2}\, d\theta$$

$$= \frac{1}{2}\theta + \frac{\sin 2\theta}{4} = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\arcsin e^x + \frac{1}{2}e^x\,.\sqrt{1-e^{2x}}$$

4. $\dfrac{x}{(1-x^2)^{3/2}}$

Sol. $$\int\frac{x}{(1-x^2)^{3/2}}\,dx = -\frac{1}{2}\int (1-x^2)^{-3/2}\,.\,(-2x)\,dx$$

$$= -\frac{1}{2}\cdot\frac{(1-x^2)^{-3/2+1}}{-\frac{3}{2}+1} = (1-x^2)^{-1/2} = \frac{1}{\sqrt{1-x^2}}$$

5. $\dfrac{x^2-3}{x\sqrt{x^2+4}}$

Sol. $$I = \int\frac{x^2-3}{x\sqrt{x^2+4}}\,dx = \frac{1}{2}\int\frac{2x}{\sqrt{x^2+4}}\,dx - \int\frac{3}{x\sqrt{x^2+4}}\,dx$$

$$= \sqrt{x^2+4} - \int\frac{3}{x\sqrt{x^2+4}}\,dx$$

Put $x = 2\tan\theta$ or $dx = 2\sec^2\theta\, d\theta$

Then $\displaystyle\int\frac{3}{x\sqrt{x^2+4}}dx = \int\frac{6\sec^2\theta\,d\theta}{2\tan\theta.2\sec\theta}$

$$= \frac{3}{2}\int\frac{\sec\theta}{\tan\theta}d\theta = \frac{3}{2}\int\csc\theta\,d\theta$$

$$= \frac{3}{2}\ln|\csc\theta - \cot\theta|$$

$$= \frac{3}{2}\ln\left|\frac{\sqrt{4+x^2}}{x} - \frac{2}{x}\right|$$

Hence $I = \sqrt{x^2+4} - \frac{3}{2}\ln\left|\frac{\sqrt{4+x^2}-2}{x}\right|$

6. $\sqrt{3x^2-4x+1}$

Sol. $\displaystyle\int\sqrt{3x^2-4x+1}\,dx = \sqrt{3}\int\sqrt{x^2-\frac{4}{3}x+\frac{1}{3}}\,dx$

$$= \sqrt{3}\sqrt{\left(x-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}\,dx$$

$$= \sqrt{3}\left[\frac{\left(x-\frac{2}{3}\right)\sqrt{\left(x-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}}{2} - \frac{\left(\frac{1}{3}\right)^2}{2}\cosh^{-1}\left(\frac{x-\frac{2}{3}}{\frac{1}{3}}\right)\right]$$

$$= \sqrt{3}\left[\frac{(3x-2)\sqrt{x^2-\frac{4}{3}x+\frac{1}{3}}}{6} - \frac{1}{18}\cosh^{-1}\left(\frac{3x-2}{1}\right)\right]$$

$$= \frac{3x-2}{6}.\sqrt{3x^2-4x+1} - \frac{\sqrt{3}}{18}\cosh^{-1}(3x-2)$$

7. $\sqrt{x^2+2x+3}$

Sol. $\displaystyle\int\sqrt{x^2+2x+3}\,dx = \int\sqrt{(x+1)^2+(\sqrt{2})^2}\,dx$

$$= \frac{(x+1)\sqrt{(x+1)^2+(\sqrt{2})^2}}{2} + \frac{(\sqrt{2})^2}{2}\sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

$$= \frac{(x+1)\sqrt{x^2+2x+3}}{2} + \sinh^{-1}\left(\frac{x+1}{\sqrt{2}}\right)$$

8. $\dfrac{x}{\sqrt{4+3x-2x^2}}$

Sol. $\displaystyle\int\frac{x\,dx}{\sqrt{4+3x-2x^2}} = -\frac{1}{4}\int\frac{-4x\,dx}{\sqrt{4+3x-2x^2}} = -\frac{1}{4}\int\frac{\{(3-4x)-3\}\,dx}{\sqrt{4+3x-2x^2}}$

$$= -\frac{1}{4}\int(3-4x)(4+3x-2x^2)^{-1/2}\,dx + \frac{3}{4}\int\frac{dx}{\sqrt{4+3x-x^2}}$$

$$= -\frac{1}{4}\frac{(4+3x-2x^2)^{1/2}}{\frac{1}{2}} + \frac{3}{4\times\sqrt{2}}\int\frac{dx}{\sqrt{2+\frac{3}{2}x-x^2}}$$

$$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\int\frac{dx}{\sqrt{\frac{9}{16}+2-\left(x-\frac{3}{4}\right)^2}}$$

$$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\int\frac{dx}{\sqrt{\left(\frac{41}{4}\right)^2-\left(x-\frac{3}{4}\right)^2}}$$

$$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\arcsin\left(\frac{x-\frac{3}{4}}{\frac{\sqrt{41}}{4}}\right)$$

$$= -\frac{1}{2}\sqrt{4+3x-2x^2} + \frac{3}{4\sqrt{2}}\arcsin\left(\frac{4x-3}{\sqrt{41}}\right)$$

9. $\dfrac{1}{\sqrt{3x^2-4x+1}}$

Sol. $\displaystyle\frac{dx}{\sqrt{3x^2-4x+1}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{x^2-\frac{4}{3}x+\frac{1}{3}}} = \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x-\frac{2}{3}\right)^2+\frac{1}{3}-\frac{4}{9}}}$

$$= \frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{\left(x-\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2}} = \frac{1}{\sqrt{3}}\cosh^{-1}\frac{x-\frac{2}{3}}{\frac{1}{3}} = \frac{1}{\sqrt{3}}\cosh^{-1}(3x-2)$$

10. $\dfrac{x+1}{\sqrt{x^2+2x+4}}$

Sol. $\displaystyle\int\frac{x+1}{\sqrt{x^2+2x+4}}dx = \frac{1}{2}\int\frac{2x+2}{\sqrt{x^2+2x+3}}dx$

$$= \frac{1}{2}\int(x^2+2x+3)^{-1/2}(2x+2)\,dx$$

$$= \frac{1}{2}\frac{(x^2+2x+3)^{-1/2}}{\frac{1}{2}} = \sqrt{x^2+2x+3}$$

11. $\dfrac{x^2+2x+3}{\sqrt{x^2+x+1}}$

Sol. $I = \displaystyle\int \frac{x^2+2x+3}{\sqrt{x^2+x+1}}\,dx = \int \frac{(x^2+x+1)+(x+2)}{\sqrt{x^2+x+1}}\,dx$

$$= \int \sqrt{x^2+x+1}\,dx + \int \frac{x+2}{\sqrt{x^2+x+1}}\,dx$$

$$= \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} + \frac{1}{2}\int \frac{2x+4}{\sqrt{x^2+x+1}}\,dx = I_1 + I_2$$

$$I_1 = \int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\left(x+\frac{1}{2}\right)\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}{2} + \frac{\left(\frac{\sqrt{3}}{2}\right)^2}{2}\sinh^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{(2x+1)\sqrt{x^2+x+1}}{4} + \frac{3}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \qquad (1)$$

and $I_2 = \dfrac{1}{2}\displaystyle\int \frac{2x+4}{\sqrt{x^2+x+1}}\,dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}} + \frac{3}{2}\int \frac{dx}{\sqrt{x^2+x+1}}\,dx$

$$= \sqrt{x^2+x+1} + \frac{3}{2}\int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= \sqrt{x^2+x+1} + \frac{3}{2}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) \qquad (2)$$

Adding (1) and (2), we get

$$I = \left(\frac{(2x+1)}{4}+1\right)\sqrt{x^2+x+1} + \left(\frac{3}{8}+\frac{3}{2}\right)\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

$$= \left(\frac{2x+5}{4}\right)\sqrt{x^2+x+1} + \frac{15}{8}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

12. $\dfrac{1}{(2x+3)\sqrt{x+5}}$

Sol. Put $\sqrt{x+5} = t$ or $x = t^2-5$ or $dx = 2t\,dt$

$$\int \frac{dx}{(2x+3)\sqrt{x+5}} = \int \frac{2t\,dt}{(2t^2-7)\,t} \qquad (2x+3 = 2(t^2-5)+3))$$

$$= 2\int \frac{dt}{2t^2-7} = \int \frac{dt}{t^2-\frac{7}{2}}$$

$$= \frac{1}{2}\sqrt{\frac{2}{7}}\int\left(\frac{1}{t-\sqrt{\frac{7}{2}}} - \frac{1}{t+\sqrt{\frac{7}{2}}}\right)dt = \frac{1}{2}\cdot\sqrt{\frac{2}{7}}\ln\left|\frac{t-\sqrt{\frac{7}{2}}}{t+\sqrt{\frac{7}{2}}}\right|$$

$$= \frac{1}{\sqrt{14}}\ln\left|\frac{\sqrt{x+5}-\sqrt{\frac{7}{2}}}{\sqrt{x+5}+\sqrt{\frac{7}{2}}}\right|$$

13. $\dfrac{1}{(1-2x)\sqrt{1+4x}}$

Sol. Put $1+4x = t^2$ or $x = \dfrac{t^2-1}{4} \Rightarrow dx = \dfrac{t\,dt}{2}$

$$\int \frac{dx}{(1-2x)\sqrt{1+4x}} = \frac{1}{2}\int \frac{t\,dt}{\left(1-\frac{t^2-1}{2}\right).t} = \frac{1}{2}\int \frac{2\,dt}{(2-t^2+1)} = \int \frac{dt}{3-t^2}$$

$$= \frac{1}{2\sqrt{3}}\int\left(\frac{1}{\sqrt{3}+t} + \frac{1}{\sqrt{3}-t}\right)dt = \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right|$$

$$= \frac{1}{2\sqrt{3}}\ln\left|\frac{\sqrt{3}+\sqrt{1+4x}}{\sqrt{3}-\sqrt{1+4x}}\right|$$

14. $\dfrac{x\sqrt{1+x}}{\sqrt{1-x}}$

Sol. $\displaystyle\int \frac{x\sqrt{1+x}}{\sqrt{1-x}}\,dx = \int \frac{x\,(1+x)}{\sqrt{1-x}\sqrt{1+x}}\,dx = \int \frac{x}{\sqrt{1-x^2}}\,dx + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$$= -\frac{1}{2}\int (-2x)\,(1-x^2)^{-1/2}\,dx - \int \frac{(1-x^2)-1}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{1}{2}\frac{(1-x^2)^{1/2}}{\frac{1}{2}} - \int\sqrt{1-x^2}\,dx + \int \frac{dx}{\sqrt{1-x^2}}$$

$$= -\sqrt{1-x^2} - \left[\frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x\right] + \arcsin x$$

$$= -\sqrt{1-x^2} - \frac{x\sqrt{1-x^2}}{2} + \frac{1}{2}\arcsin x$$

15. $\dfrac{x^4}{(x-1)\sqrt{x+2}}$

Sol. Put $x+2=t^2$ or $x=t^2-2$ or $dx=2t\,dt$

$$\int\frac{x^4}{(x-1)\sqrt{x+2}}dx = \int\frac{(t^2-2)^4 \,.\, 2t\,dt}{(t^2-3)\,t} = 2\int\frac{(t^2-2)^4}{t^2-3}dt$$

$$= 2\int\frac{t^8-8t^6+24t^4-32t^2+16}{t^2-3}dx$$

$$= 2\int\left(t^6-5t^4+9t^2-5+\frac{1}{t^2-3}\right)dx$$

$$= 2\left[\frac{t^7}{7}-t^5+3t^3-5t+\frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|\right]$$

$$= \frac{2}{7}t^7-2t^5+6t^3-10t+\frac{1}{\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right|$$

$$= \frac{2}{7}(x+2)^{7/2}-2(x+2)^{5/2}+6(x+2)^{3/2}-10\sqrt{x+2}$$

$$+\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x+2}-\sqrt{3}}{\sqrt{x+2}+\sqrt{3}}\right|$$

16. $\dfrac{1}{(x^2-2x+2)\sqrt{x-1}}$

Sol. Put $x-1=t^2$ so that $dx=2t\,dt$. Then

$$\int\frac{dx}{(x^2-2x+2)\sqrt{x-1}} = \int\frac{2t\,dt}{[(t^2+1)^2-2(t^2+1)+2]\,.\,t}$$

$$= 2\int\frac{dt}{t^4+2t^2+1-2t^2-2+2} = 2\int\frac{dt}{t^4+1}$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{t^2-1}{\sqrt{2}t}\right)-\frac{1}{2\sqrt{2}}\ln\left|\frac{t^2-\sqrt{2}\,t+1}{t^2+\sqrt{2}\,t+1}\right| \quad \text{(By Q.12 Ex. 4.4)}$$

$$= \frac{1}{\sqrt{2}}\arctan\left(\frac{x-2}{\sqrt{2(x-1)}}\right)-\frac{1}{2\sqrt{2}}\ln\left|\frac{x-\sqrt{2(x-1)}}{x+\sqrt{2(x-1)}}\right|$$

17. $\dfrac{1}{(x^2+4x+5)\sqrt{x+2}}$

Sol. Put $x+2=t^2$ so that $x=t^2-2 \Rightarrow dx=2t\,dt$. Then

$$\int\frac{dx}{(x^2+4x+5)\sqrt{x+2}} = \int\frac{2t\,dt}{[(t^2-2)^2+4(t^2-2)+5]\,t}$$

$$= 2\int\frac{dt}{t^4-4t^2+4+4t^2-8+5} = 2\int\frac{dt}{t^4+1}$$

$$= \frac{1}{\sqrt{2}}\arctan\frac{t^2-1}{\sqrt{2}\,t}-\frac{1}{2\sqrt{2}}\ln\left|\frac{t^2-\sqrt{2}\,t+1}{t^2+\sqrt{2}\,t+1}\right| \quad \text{(By Q. 12 Ex. 4.4)}$$

$$= \frac{1}{2}\arctan\left(\frac{x+1}{\sqrt{2(x+2)}}\right)-\frac{1}{2\sqrt{2}}\ln\left|\frac{x+3-\sqrt{2(x+2)}}{x+3+\sqrt{2(x+2)}}\right|$$

18. $\dfrac{1}{(x-1)\sqrt{x^2+1}}$

Sol. Put $x-1=\frac{1}{t}$ i.e., $dx=-\frac{1}{t^2}$ and $x^2=\left(\frac{1}{t}+1\right)^2=\frac{1}{t^2}+\frac{2}{t}+1$

$$\text{Now} \int\frac{dx}{(x-1)\sqrt{x^2+1}} = \int\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}+\frac{2}{t}+2}} = -\int\frac{dt}{\sqrt{2t^2+2t+1}}$$

$$= -\frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{t^2+t+\frac{1}{2}}} = -\frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\frac{t+\frac{1}{2}}{\frac{1}{2}} = -\frac{1}{\sqrt{2}}\sinh^{-1}(2t+1)$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{2}{x-1}+1\right) = -\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{x+1}{x-1}\right)$$

19. $\dfrac{1}{(x+1)\sqrt{x^2-1}}$

Sol. Put $x+1=\frac{1}{t}$ i.e., $x=\frac{1}{t}-1$, $dx=-\frac{1}{t^2}dt$ and $x^2=\frac{1}{t^2}-\frac{2}{t}t+1$

$$\frac{dx}{(x+1)\sqrt{x^2-1}} = \int\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{2}{t}}} = -\int\frac{dt}{\sqrt{1-2t}}$$

$$= -\int(1-2t)^{-1/2}\,dt = -\frac{(1-2t)^{1/2}}{-2\left(\frac{1}{2}\right)} = \sqrt{1-2t}$$

$$= \sqrt{1-\frac{2}{x+1}} = \sqrt{\frac{x+1-2}{x+1}} = \sqrt{\frac{x-1}{x+1}}$$

20. $\dfrac{1}{ax^n + bx}$

Sol. $I = \displaystyle\int \frac{dx}{ax^n + bx} = \int \frac{dx}{x^n(a + bx^{-n+1})} = \int \frac{1}{a + bx^{n-1}} \cdot x^{-n}\, dx$

Put $a + bx^{-n+1} = z$. Then $b(-n+1)x^{-n}\, dx = dz$

or $x^{-n} \cdot dx = \dfrac{dz}{b(1-n)}$

$$I = \int \frac{1}{z} \cdot \frac{dz}{b(1-n)} = \frac{1}{b(1-n)} \int \frac{dz}{z} \cdot \frac{1}{(x_1 - n)} \cdot \ln |z|$$

$$= \frac{1}{b(1-n)} \ln |a + bx^{-n+1}|$$

21. $\dfrac{x^2 + 2x + 3}{(x+2)\sqrt{x^2+1}}$

Sol. We have $\dfrac{x^2 + 2x + 3}{x + 2} = x + \dfrac{3}{x+2}$

and $\dfrac{x^2+2x+3}{(x+2)\sqrt{x^2+1}} = \dfrac{x}{\sqrt{x^2+1}} + \dfrac{3}{(x+2)\sqrt{x^2+1}}$. Therefore,

$$\int \frac{x^2+2x+3}{(x+2)\sqrt{x^2+1}}\, dx = \int \frac{x}{\sqrt{x^2+2}}\, dx + 3 \int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \frac{1}{2} \int (2x)(x^2+1)^{-1/2}\, dx + 3 \int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \frac{1}{2} \frac{(x^2+1)^{1/2}}{\frac{1}{2}} + 3 \int \frac{dx}{(x+2)\sqrt{x^2+1}}$$

$$= \sqrt{x^2+1} + 3 \int \frac{dx}{(x+2)\sqrt{x^2+1}} \qquad (1)$$

To evaluate, $\displaystyle\int \frac{dx}{(x+2)\sqrt{x^2+1}}$, put $x + 2 = \dfrac{1}{t}$

or $x = \dfrac{1}{t} - 2$ i.e., $dx = -\dfrac{1}{t^2} dx$ and $x^2 = \dfrac{1}{t^2} - \dfrac{4}{t} + 4$

or $x^2 + 1 = \dfrac{1}{t^2} - \dfrac{4}{t} + 5$ or $\sqrt{x^2+1} = \dfrac{\sqrt{1 - 4t + 5t^2}}{t}$. Thus

$$\int \frac{dx}{(x+2)\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t} \cdot \frac{\sqrt{1-4t+5t^2}}{t}} = -\int \frac{dt}{\sqrt{1-4t+5t^2}}$$

$$= -\frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{t^2 - \frac{4}{5}t + \frac{1}{5}}} = \frac{1}{-\sqrt{5}} \int \frac{dt}{\sqrt{\left(t - \frac{2}{5}\right)^2 + \frac{1}{5} - \frac{4}{25}}}$$

$$= -\frac{1}{\sqrt{5}} \int \frac{dt}{\sqrt{\left(t - \frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2}} = -\frac{1}{\sqrt{5}} \sinh^{-1} \frac{t - \frac{2}{5}}{\frac{1}{5}}$$

$$= -\frac{1}{\sqrt{5}} \sinh^{-1}(5t - 2) = -\frac{1}{\sqrt{5}} \sinh^{-1}\left\{5 \cdot \frac{1}{x+2} - 2\right\}$$

$$= -\frac{1}{\sqrt{5}} \sinh^{-1}\left(\frac{5 - 2x - 4}{x+2}\right) = -\frac{1}{\sqrt{5}} \sinh^{-1}\left(\frac{1-2x}{x+2}\right)$$

Putting in (1), we get

$$\int \frac{(x^2+2x+3)\, dx}{(x+2)\sqrt{x^2+1}} = \sqrt{x^2+1} - \frac{3}{\sqrt{5}} \sinh^{-1} \frac{1-2x}{x+2}$$

22. $\dfrac{1}{x^2\sqrt{x^2+1}}$

Sol. Putting $x = \dfrac{1}{t}$, we get $dx = -\dfrac{1}{t^2} dt$

Therefore, $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{-\frac{1}{t^2}\, dt}{\frac{1}{t^2}\sqrt{\frac{1}{t^2} + 1}} = -\int \frac{t}{\sqrt{1+t^2}}\, dx$

$$= -\frac{1}{2} \int 2t(1+t^2)^{-1/2}\, dt = -\frac{1}{2} \frac{(1+t^2)^{1/2}}{\frac{1}{2}}$$

$$= -\sqrt{1+t^2} = -\sqrt{1 + \frac{1}{x^2}} = -\frac{\sqrt{1+x^2}}{x}$$

23. $\dfrac{1}{(1+x^2)\sqrt{1-x^2}}$

Sol. Putting $x = \dfrac{1}{t}$, we get $dx = -\dfrac{1}{t^2} dt$. Therefore,

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{-\frac{1}{t^2}\,dt}{\left(1+\frac{1}{t^2}\right)\sqrt{1-\frac{1}{t^2}}} = -\int \frac{t\,dt}{(t^2+1)\sqrt{t^2-1}}$$

Again set $t^2-1=u^2$ *i.e.*, $2t\,dt = 2u\,du$ or $t\,dt = u\,du$

Also $t^2 = u^2+1$ or $t^2+1 = u^2+2$

Hence $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = -\int \frac{u\,du}{(u^2+2)u} = -\int \frac{du}{u^2+2}$

$$= -\frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}} = -\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{t^2-1}}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{\frac{1}{x^2}-1}}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\cdot\sqrt{\frac{1-x^2}{x^2}}\right)$$

$$= -\frac{1}{\sqrt{2}}\arctan\left(\frac{\sqrt{1-x^2}}{\sqrt{2}\,x}\right)$$

24. $\dfrac{1}{(1-2x^2)\sqrt{1-x^2}}$

Sol. Let $I = \displaystyle\int \frac{dx}{(1-2x^2)\sqrt{1-x^2}}$

Put $x = \frac{1}{t}$ or $dx = -\frac{1}{t^2}dt$ and $1-x^2 = 1-\frac{1}{t^2} = \frac{t^2-1}{t^2}$

Therefore, $I = \displaystyle\int \frac{-\frac{1}{t^2}\,dt}{\left(1-\frac{2}{t^2}\right)\frac{\sqrt{t^2-1}}{t}} = -\int \frac{t\,dt}{(t^2-2)\sqrt{t^2-1}}$

New set $t^2-1=u^2$ or $t\,dt = u\,du$ and $t^2 = u^2+1$

$$I = \int \frac{u\,du}{(u^2-1)u} = -\int \frac{du}{u^2-1} = \int \frac{du}{1-u^2} = \frac{1}{2}\cdot\int\left(\frac{1}{1+u}+\frac{1}{1-u}\right)dx$$

$= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = \frac{1}{2}\ln\left|\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right|$ Put $x = \cos\theta$. Then

Alternative Answer:

$$\frac{1}{2}\ln\left|\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right| = \frac{1}{2}\ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| = \frac{1}{2}\ln\left|\frac{1+\tan\theta}{1-\tan\theta}\right|$$

$$= \frac{1}{2}\ln\left|\tan\left(\frac{\pi}{4}+\arccos x\right)\right|$$

25. $\dfrac{1}{(2x^2-3x+1)\sqrt{3x^2-2x+1}}$

Sol. We have $\dfrac{1}{2x^2-3x+1} = \dfrac{1}{(x-1)(2x-1)} = \dfrac{-2}{2x-1}+\dfrac{1}{x-1}$

So, $\displaystyle\int \frac{dx}{(2x^2-3x+1)\sqrt{3x^2-2x+1}}$

$$= -\int \frac{2\,dx}{(2x-1)\sqrt{3x^2-2x+1}} + \int \frac{dx}{(x-1)\sqrt{3x^2-2x+1}}$$

$= I_1 + I_2$ (say)

Now $I_1 = \displaystyle\int \frac{-2\,dx}{(2x-1)\sqrt{3x^2-2x+1}}$

Put $2x-1 = \frac{1}{t}$ or $2x = 1+\frac{1}{t}$

or $2dx = -\frac{1}{t^2}dt \Rightarrow -2dx = \frac{1}{t^2}dt$

and $2x = 1+\frac{1}{t} = \frac{t+1}{t}$ or $x = \frac{t+1}{2t}$

$$I_1 = \int \frac{\frac{1}{t^2}\,dx}{\frac{1}{t}\sqrt{3\left(\frac{t+1}{2t}\right)^2 - 2\left(\frac{t+1}{2t}\right)+1}}$$

$$= \int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{3}{4t^2}(t^2+2t+1)-\frac{t+1}{t}+1}}$$

$$= \int \frac{2\,dt}{\sqrt{3(t^2+2t+1)-4t(t+1)+4t^2}}$$

$$= 2\int \frac{dt}{\sqrt{3t^2+6t+3-4t}} = 2\int \frac{dt}{\sqrt{3t^2+2t+3}}$$

$$= \frac{2}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+\frac{2}{3}t+1}} = \frac{2}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{3}\right)^2+\left(1-\frac{1}{9}\right)}}$$

$$= \frac{2}{\sqrt{3}} \int \frac{dt}{\sqrt{\left(t + \frac{1}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2}} = \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{t + \frac{1}{3}}{\frac{2\sqrt{2}}{3}}\right)$$

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{3t+1}{2\sqrt{2}}\right) = \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{3\left(\frac{1}{2x-1}\right)+1}{2\sqrt{2}}\right)$$

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{3+2x-1}{2\sqrt{2}\,(2x-1)}\right) = \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{2x+2}{2\sqrt{2}\,(2x-1)}\right)$$

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{x+1}{\sqrt{2}\,(2x-1)}\right)$$

Again $I_2 = \int \frac{dx}{(x-1)\sqrt{3x^2 - 2x + 1}}$

Putting $x - 1 = \frac{1}{t}$ or $x = \frac{1}{t} + 1$ or $dx = -\frac{1}{t^2}\,dt$, we have

$$I_2 = \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{3\left(\frac{1}{t}+1\right)^2 - 2\left(\frac{1}{t}+1\right) + 1}}$$

$$= \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{3\left(\frac{1}{t^2}+\frac{2}{t}+1\right) - \frac{2}{t} - 2 + 1}}$$

$$= \int \frac{-dx}{\sqrt{3(1+2t+t^2) - 2t - t^2}} = -\int \frac{dt}{\sqrt{2t^2+4t+3}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{t^2+2t+\frac{3}{2}}} = -\frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{(t+1)^2 + \left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= -\frac{1}{\sqrt{2}} \sinh^{-1} \frac{t+1}{\frac{1}{\sqrt{2}}} = -\frac{1}{\sqrt{2}} \sinh^{-1} \sqrt{2}\,(t+1)$$

$$= -\frac{1}{\sqrt{2}} \sinh^{-1} \sqrt{2}\left[\frac{1}{x-1} + 1\right] = -\frac{1}{\sqrt{2}} \sinh^{-1} \frac{\sqrt{2}\,x}{x-1}$$

Hence the required integral

$$= \frac{2}{\sqrt{3}} \sinh^{-1}\left(\frac{x+1}{\sqrt{2}\,(2x-1)}\right) - \frac{1}{\sqrt{2}} \sinh^{-1}\left(\frac{\sqrt{2}\,x}{x-1}\right)$$

26. $\frac{\sqrt{x}}{1 + \sqrt[3]{x}}$

Sol. $I = \int \frac{\sqrt{x}}{1 + x^{1/3}}\,dx$

Here $x^{1/2}$ and $x^{1/3}$ are involved

We put $x^{1/6} = z$ *i.e.*, $x = z^6$ or $dx = 6z^5\,dz$

$$I = \int \frac{z^3}{1+z^2} \cdot 6z^5\,dz = 6\int \frac{z^8\,dz}{1+z^2}$$

$$= 6\int \left(z^6 - z^4 + z^2 - 1 + \frac{1}{1+z^2}\right) dz$$

$$= 6\left[\frac{z^7}{7} - \frac{z^5}{5} + \frac{z^3}{3} - z + \arctan z\right]$$

$$= 6\left(\frac{x^{7/6}}{7} - \frac{x^{5/6}}{5} + \frac{x^{1/2}}{3} - x^{1/6} + \arctan x^{1/6}\right)$$

27. $\frac{x^3}{\sqrt{1+x^2}}$

Sol. Put $x = \tan\theta$ or $dx = \sec^2\theta\,d\theta$

$$\int \frac{x^3}{\sqrt{1+x^2}}\,dx = \int \frac{\tan^3\theta}{\sec\theta}\sec^2 d\theta = \int \tan^2\theta(\sec\theta\tan\theta)\,d\theta$$

$= \tan^2\theta\sec\theta \int (2\tan\theta\sec^2\theta) \,.\, \sec\theta\,d\theta$ (integrating by parts).

$$= \tan^2\theta\sec\theta - 2\int (\tan\theta\sec\theta)\sec^2\theta\,d\theta$$

$$= \tan^2\theta\sec\theta - 2\frac{\sec^3\theta}{3} = x^2\sqrt{1+x^2} - \frac{2}{3}(1+x^2)^{3/2}$$

28. $\frac{1}{\sqrt{x} + 2x^{1/3}}$

Sol. Put $x = z^6$ or $dx = 6z^5\,dz$

$$\int \frac{dx}{\sqrt{x} + 2x^{1/3}} = \int \frac{6z^5\,dz}{z^3 + 2z^2} = 6\int \frac{z^3}{z+2}\,dz$$

$$= 6\int \left(z^2 - 2z + 4 - \frac{8}{z+2}\right) dz = 6\left(\frac{z^3}{3} - z^2 + 4z - 8\ln|z+2|\right)$$

$$= 2\sqrt{x} - 6x^{1/3} + 24x^{1/6} - 48\ln|x^{1/6} + 2|.$$

Exercise Set 4.6 (Page 163)

Integrate with respect to x (Problems 1 – 4):

1. $\sin^5 x$

Sol. $\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx$, Put $\cos x = t$ or $-\sin x\,dx = dt$

$$= \int (1-\cos^2 x)^2 \sin x\,dx$$

$$= -\int (1-t^2)^2\,dt = -\int (1-2t^2+t^4)\,dt$$

$$= -\left[t - 2\frac{t^3}{3} + \frac{t^5}{5}\right] = -t + \frac{2}{3}t^3 - \frac{t^5}{5}$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x$$

2. $\cos^7 x$

Sol. Putting $\sin x = t$ or $\cos x\,dx = dt$, we have

$$\int \cos^7 x\,dx = \int (1-\sin^2 x)^3 \cos x\,dx = \int (1-t^2)^3\,dt$$

$$= \int (1-3t^2+3t^4-t^6)\,dt = t - t^3 + 3\cdot\frac{t^5}{5} - \frac{t^7}{7}$$

$$= \sin x - \sin^3 x + \frac{3}{5}\sin^5 x - \frac{1}{7}\sin^7 x$$

3. $\sin^8 x$

Sol. By the reduction formula

$$\int \sin^8 x\,dx = -\frac{\cos x \sin^7 x}{8} + \frac{7}{8}\int \sin^6 x\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} + \frac{7}{8}\left[-\frac{\cos x \sin^5 x}{6} + \frac{5}{6}\int \sin^4 x\,dx\right]$$

(Repeating the formula)

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x + \frac{35}{48}\int \sin^4 x\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x + \frac{35}{48}\left[-\frac{\cos x \sin^3 x}{4} + \frac{3}{4}\int \sin^2 x\,dx\right]$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^5 x + \frac{35}{64}\int \sin^2 x\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^3 x + \frac{35}{128}\int (1-\cos 2x)\,dx$$

$$= -\frac{\cos x \sin^7 x}{8} - \frac{7}{48}\cos x \sin^5 x - \frac{35}{192}\cos x \sin^3 x + \frac{35}{128}x - \frac{35}{128}\sin x \cos x$$

4. $\cos^6 x$

Sol. By the reduction formula

$$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} dx, \text{ we get}$$

$$\int \cos^6 x\,dx = \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\int \cos^4 x\,dx$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{6}\left[\frac{\sin x \cos^3 x}{4} + \frac{3}{4}\int \cos^2 x\,dx\right]$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{8\times 2}\int (1+\cos 2x)\,dx$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}\left[x + \frac{\sin 2x}{2}\right]$$

$$= \frac{\sin x \cos^5 x}{6} + \frac{5}{24}\sin x \cos^3 x + \frac{5}{16}x + \frac{5}{16}\sin x \cos x.$$

Find a reduction formula for each of the following (Problems 5 – 8): (n > 1 is an integer).

5. $\int \tan^n x\,dx$

Sol. $\int \tan^n x\,dx = \int \tan^{n-2} x\,.\,(\sec^2 x - 1)\,dx$

$$= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx$$

$$= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$$

which is the required reduction formula.

6. $\int \sec^n x\,dx$

Sol. $\int \sec^n x\,dx = \int \sec^{n-2} x\,.\,\sec^2 x\,dx$

$$= \sec^{n-2} x.\tan x - \int \tan x\,.\,(n-2)\sec^{n-3} x\,.\sec x \tan x\,dx$$

(Integrating by parts).

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x \tan^2 x \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \sec^{n-2} x \tan x - (n-2) \int \sec^n x \, dx + (n-2) \int \sec^{n-2} x \, dx$$

or $\quad (1 + n - 2) \int \sec^n x \, dx = \sec^{n-2} x \tan x + (n-2) \int \sec^{n-2} x \, dx$

or $\quad \int \sec^n x \, dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1} \int \sec^{n-2} x \, dx$

7. $\int \cot^n x \, dx$

Sol. $\int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx$

$$= \int \cot^{n-2} x . (\csc^2 x - 1) \, dx = \int \cot^{n-2} \csc^2 x \, dx - \int \cot^{n-2} x \, dx$$

$$= -\frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx$$

which is the required reduction formula.

8. $\int \csc^n x \, dx$

Sol. $\int \csc^n x \, dx = \int \csc^{n-2} x \csc^2 x$

$$= -\csc^{n-2} x \cot x - \int (-\cot x)(n-2) \csc^{n-3} x (-\csc x \cot x) \, dx$$

(Integrating by parts)

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^{n-2} x \cot^2 x \, dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^{n-2} x (\csc^2 x - 1) \, dx$$

$$= -\csc^{n-2} x \cot x - (n-2) \int \csc^n x \, dx + (n-2) \int \csc^{n-2} x \, dx$$

or $(1 + n - 2) \int \csc^n x \, dx = -\csc^{n-2} x \cot x + (n-2) \int \csc^{n-2} x \, dx$

or $\quad \int \csc^n x \, dx = -\dfrac{\csc^{n-2} x \cot x}{n-1} + \dfrac{n-2}{n-1} \int \csc^{n-2} x \, dx$

which is the required reduction formula.

Evaluate (Problems 9 – 12):

9. $\int \tan^6 x \, dx$

Sol. By the reduction formula, we have

$$\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx \qquad (1)$$

Putting $n = 6$ in (1), we get

$$\int \tan^6 x \, dx = \frac{\tan^5 x}{5} - \int \tan^4 x \, dx \qquad (2)$$

Again putting $n = 4$ in (1), we get

$$\int \tan^4 x \, dx = \frac{\tan^3 x}{3} - \int \tan^2 x \, dx = \frac{\tan^3 x}{3} - \int (\sec^2 x - 1) \, dx$$

$$= \frac{\tan^3 x}{3} - |\tan x - x| = \frac{\tan^3 x}{3} - \tan x + x \qquad (3)$$

Putting the value of $\int \tan^4 x \, dx$ from (3) into (2), we have

$$\int \tan^6 x \, dx = \frac{\tan^5 x}{5} - \left[\frac{\tan^3 x}{3} - \tan x + x\right]$$

$$= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x$$

10. $\int \cot^5 x \, dx$

Sol. By the reduction formula, we have

$$\int \cot^n x \, dx = -\int \frac{\cot^{n-1} x}{n-1} - \int \cot^{n-2} x \, dx \qquad (1)$$

Putting $n = 5$ in (1), we get

$$\int \cot^5 x \, dx = -\frac{\cot^4 x}{4} - \int \cot^3 x \, dx \qquad (2)$$

Again putting $n = 3$, in (1), we get

$$\int \cot^3 x \, dx = -\frac{\cot^2 x}{2} - \int \cot x \, dx$$

$$= -\frac{\cot^2 x}{2} - \ln |\sin x| \qquad (3)$$

Now putting the value of $\int \cot^3 x \, dx$ from (3) into (2), we get

$$\int \cot^5 x \, dx = -\frac{\cot^4 x}{4} - \left[-\frac{\cot^2 x}{2} - \ln |\sin x|\right]$$

$$= -\frac{\cot^4 x}{4} + \frac{\cot^2 x}{2} + \ln |\sin x|$$

11. $\int \sec^5 x \, dx$

Sol. By the reduction formula, we have

$$\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx \qquad (1)$$

Putting $n = 6$, in (1), we get

$$\int \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\int \sec^4 x\, dx \qquad (2)$$

Again putting $n = 4$, in (1), we have

$$\int \sec^4 x\, dx = \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\int \sec^2 x\, dx$$

$$= \frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x$$

Putting the value of $\int \sec^4 x\, dx$ in (2), we have

$$\int \sec^6 x\, dx = \frac{\sec^4 x \tan x}{5} + \frac{4}{5}\left[\frac{\sec^2 x \tan x}{3} + \frac{2}{3}\tan x\right]$$

$$= \frac{\sec^4 x \tan x}{5} + \frac{4}{15}\sec^2 x \tan x + \frac{8}{15}\tan x$$

12. $\int \csc^5 x\, dx$

Sol. By the reduction formula, we have

$$\int \csc^n x\, dx = -\frac{\csc^{n-2} x \cot x}{n-1} + \frac{n-2}{n-1}\int \csc^{n-2} x\, dx \qquad (1)$$

Putting $n = 5$, in (1), we get

$$\int \csc^5 x\, dx = -\frac{\cot x \csc^3 x}{4} + \frac{3}{4}\int \csc^3 x\, dx \qquad (2)$$

Again putting $n = 3$ in (1), we get

$$\int \csc^3 x\, dx = -\frac{\cot x \csc x}{2} + \frac{1}{2}\int \csc x\, dx$$

$$= -\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|$$

Now putting the value of $\int \csc^3 x\, dx$ in (2), we get

$$\int \csc^5 x\, dx = -\frac{1}{4}\cot x \csc^3 x + \frac{3}{4}\left[-\frac{1}{2}\cot x \csc x + \frac{1}{2}\ln\left|\tan\frac{x}{2}\right|\right]$$

$$= -\frac{1}{4}\cot x \csc^3 x - \frac{3}{8}\cot x \csc x + \frac{3}{8}\ln\left|\tan\frac{x}{2}\right|$$

Integrate with respect to x (Problems 13 – 24):

13. $\dfrac{1}{a + b\sin x}$

Sol. $I = \displaystyle\int \frac{1}{a + b\sin x}\, dx$

$$= \int \frac{1}{a\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 2b\sin\frac{x}{2}\cos\frac{x}{2}}\, dx$$

$$= \int \frac{\sec^2\frac{x}{2}}{a + a\tan^2\frac{x}{2} + 2b\tan\frac{x}{2}}\, dx, \quad \text{dividing numerater and denominator by } \cos^2 x/2$$

Now put $\tan\frac{x}{2} = z$, so that (1)

$$\left(\sec^2\frac{x}{2}\right) \times \frac{1}{2}dx = dz \qquad \text{or} \qquad \sec^2\frac{x}{2}dx = 2\,dz$$

$$I = \int \frac{2\,dz}{a + az^2 + 2bz} = 2\int \frac{1}{a\left[\left(z^2 + 2\frac{b}{a}z\right) + 1\right]}dx$$

$$= \frac{2}{a}\int \frac{1}{\left(z^2 + 2\frac{b}{a}z + \frac{b^2}{a^2}\right) - \frac{b^2}{a^2} + 1}dz$$

$$= \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 + \frac{a^2 - b^2}{a^2}}dz \qquad (2)$$

Case I: If $a^2 > b^2$, then from (2), we get

$$I = \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 + \left(\frac{\sqrt{a^2 - b^2}}{a}\right)^2}dz = \frac{2}{a}\frac{1}{\frac{\sqrt{a^2 - b^2}}{a}}\arctan\frac{z + \frac{b}{a}}{\frac{\sqrt{a^2 - b^2}}{a}}$$

$$= \frac{2}{\sqrt{a^2 - b^2}}\arctan\frac{a\tan\frac{x}{2} + b}{\sqrt{a^2 - b^2}}, \text{ [from (1)].}$$

Case II: If $a^2 < b^2$, then from (2), we have

$$I = \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 - \frac{b^2 - a^2}{a^2}}dz = \frac{2}{a}\int \frac{1}{\left(z + \frac{b}{a}\right)^2 - \left(\frac{\sqrt{b^2 - a^2}}{a}\right)^2}dz$$

$$= \frac{2}{a}\frac{1}{2\frac{\sqrt{b^2 - a^2}}{a}}\ln\frac{z + \frac{b}{a} - \frac{\sqrt{b^2 - a^2}}{a}}{z + \frac{b}{a} + \frac{\sqrt{b^2 - a^2}}{a}}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \ln \frac{a \tan\frac{x}{2} + b - \sqrt{b^2-a^2}}{a \tan\frac{x}{2} + b + \sqrt{b^2-a^2}}$$

Alternative Method:

$$I = \int \frac{1}{a + b \sin x}$$

Put $x = \frac{\pi}{2} + z$ then $dx = dz$ and $z = x - \frac{\pi}{2} = -\left(\frac{\pi}{2} - x\right)$

$$I = \int \frac{1}{a + b \sin\left(\frac{\pi}{2} + z\right)} dz = \int \frac{1}{a + b \cos z} dz$$

Case I: If $a^2 > b^2$, then from Example 40, we have

$$I = \frac{1}{\sqrt{a^2-b^2}} \arccos \frac{b + a\cos z}{a + b \cos z}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \arccos \frac{b + a\cos\left[-\left(\frac{\pi}{2}-x\right)\right]}{a + b\cos\left[-\left(\frac{\pi}{2}-x\right)\right]} \qquad \left[\begin{aligned} \text{since } \cos\left[-\left(\tfrac{\pi}{2}-x\right)\right] &= \cos\left(\tfrac{\pi}{2}-x\right) \\ &= \sin x \end{aligned}\right]$$

$$= \frac{1}{\sqrt{a^2-b^2}} \arccos \frac{b + a \sin x}{a + b \sin x}$$

Case II: If $a^2 < b^2$, then from Example 40, we have

$$I = \frac{1}{\sqrt{b^2-a^2}} \cosh^{-1} \frac{b + a\cos z}{a + b\cos z}$$

$$= \frac{1}{\sqrt{b^2-a^2}} \cosh^{-1} \frac{b + a\cos\left[-\left(\frac{\pi}{2}-x\right)\right]}{a + b\cos\left[-\left(\frac{\pi}{2}-x\right)\right]} \qquad \left[\begin{aligned} \text{since } \cos\left[-\left(\tfrac{\pi}{2}-x\right)\right] &= \cos\left(\tfrac{\pi}{2}-x\right) \\ &= \sin x \end{aligned}\right]$$

$$= \frac{1}{\sqrt{b^2-a^2}} \cosh^{-1} \frac{b + a \sin x}{a + b \sin x}$$

14. $\dfrac{1}{a + b \cosh x}$

Sol. $I = \displaystyle\int \frac{1}{a + b\cosh x} dx$

$$= \int \frac{1}{a\left(\cosh^2\frac{x}{2} - \sinh^2\frac{x}{2}\right) + b\left(\cosh^2\frac{x}{2} + \sinh^2\frac{x}{2}\right)} dx$$

$$= \int \frac{1}{(a+b)\cosh^2\frac{x}{2} + (b-a)\sinh^2\frac{x}{2}} dx$$

Divide the numerator and denominator by $\cosh^2\frac{x}{2}$, then

$$I = \int \frac{\operatorname{sech}^2\frac{x}{2}}{(a+b) + (b-a)\tanh^2\frac{x}{2}} dx$$

Put $\tanh\frac{x}{2} = z$, so that (1)

$$\left(\operatorname{sech}^2\frac{x}{2}\right) \times \frac{1}{2} dx = dz \quad \text{or} \quad \operatorname{sech}^2\frac{x}{2} dx = 2\,dz$$

$$I = \int \frac{2\,dz}{(a+b) + (b-a)z^2} \qquad (2)$$

(Assume that $a + b$ is positive)

Case I: If $a^2 > b^2$ i.e., $a^2 - b^2 > 0$ or $(a+b)(a-b) > 0$ as $a + b > 0$ so $a - b > 0$, then from (2), we have

$$I = 2\int \frac{1}{(a-b)\left(\frac{a+b}{a-b} - z^2\right)} dz = \frac{2}{a-b} \int \frac{1}{\left(\sqrt{\frac{a+b}{a-b}}\right)^2 - z^2} dz$$

$$= \frac{2}{a-b} \frac{1}{2\sqrt{\frac{a+b}{a-b}}} \ln \frac{\sqrt{\frac{a+b}{a-b}} + z}{\sqrt{\frac{a+b}{a-b}} - z}$$

$$= \frac{1}{\sqrt{a^2-b^2}} \ln \frac{\sqrt{a+b} + \sqrt{a-b}\tanh\frac{x}{2}}{\sqrt{a+b} - \sqrt{a-b}\tanh\frac{x}{2}}$$

Case II: If $a^2 < b^2$

$a^2 - b^2$ is –ve and $a + b$ is +ve, i.e.,
$a - b$ is –ve or $b - a$ is +ve

then from (2), we get

$$I = 2\int \frac{1}{(b-a)\left[\frac{b+a}{b-a} + z^2\right]} dz = \frac{2}{(b-a)} \int \frac{1}{\left(\sqrt{\frac{b+a}{b-a}}\right)^2 + z^2} dz$$

$$= \frac{2}{b-a}\,\frac{1}{\sqrt{\dfrac{b+a}{b-a}}}\arctan\frac{z}{\sqrt{\dfrac{b+a}{b-a}}}$$

$$= \frac{2}{\sqrt{b^2-a^2}}\arctan\left[\sqrt{\frac{b-a}{b+a}}\tanh\frac{x}{2}\right], \qquad \text{from (1)}$$

15. $\dfrac{\cot x}{1+\sin x}$

Sol. $I = \displaystyle\int\frac{\cot x\,dx}{1+\sin x} = \int\frac{\cos x\,dx}{\sin x\,(1+\sin x)}$

Put $\sin x = z$ so that $\cos x\,dx = dz$ and on substitution,

$$I = \int\frac{dz}{z\,(1+z)} = \int\left(\frac{1}{z}-\frac{1}{1+z}\right)dz = \ln[z] - \ln[1+z]$$

$$= \ln[\sin x] - \ln[1+\sin x]$$

$$= \ln\left[\frac{\sin x}{1+\sin x}\right] = \ln\left[\frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)^2}\right] = \ln\left[\frac{2\tan\frac{x}{2}}{\left(1+\tan\frac{x}{2}\right)^2}\right]$$

$$= \ln\left[2\tan\frac{x}{2}\right] - 2\ln\left[1+\tan\frac{x}{2}\right]$$

16. $\dfrac{2-\cos x}{2+\cos x}$

Sol. $I = \displaystyle\int\frac{2-\cos x}{2+\cos x}\,dx$

Let $\tan\frac{x}{2} = z$ so that $\left(\sec^2\frac{x}{2}\right)\frac{1}{2}\,dx = dz$ or $dx = \dfrac{z}{1+z^3}\,dz$

$$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-z^2}{1+z^2}$$

$$I = \int\frac{2-\dfrac{1-z^2}{1+z^2}}{2+\dfrac{1-z^2}{1+z^2}}\cdot\frac{2\,dz}{1+z^2} = z\int\frac{1+3z^2}{(3+z^2)\,(1+z^2)}\,dz$$

$$= \int\left(\frac{8}{3+z^2}-\frac{2}{1+z^2}\right)dz = \frac{8}{\sqrt{3}}\arctan\frac{2}{\sqrt{3}} - 2\arctan z$$

$$= \frac{8}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - 2\arctan\left(\tan\frac{x}{2}\right)$$

$$= \frac{8}{\sqrt{3}}\arctan\left(\frac{1}{\sqrt{3}}\tan\frac{x}{2}\right) - x$$

17. $\dfrac{1}{1+\sin x+\cos x}$

Sol. $I = \displaystyle\int\frac{dx}{1+\sin x+\cos x} = \int\frac{dx}{2\cos^2\left(\frac{x}{2}\right)+2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}$

$$= \int\frac{\frac{1}{2}\sec^2\left(\frac{x}{2}\right)}{1+\tan\left(\frac{x}{2}\right)}\,dx$$

Now put $\tan\left(\frac{x}{2}\right) = z$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$. Then

$$I = \int\frac{dz}{1+z} = \ln|1+z| = \ln\left|1+\tan\left(\frac{x}{2}\right)\right|$$

18. $\dfrac{\cos x}{2-\cos x}$

Sol. $I = \displaystyle\int\frac{\cos x}{2-\cos x}\,dx$

Put $\tan\left(\frac{x}{2}\right) = z$ so that $\frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx = dz$ or $dx = \dfrac{2\,dz}{1+z^2}$

and $\cos x = \dfrac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)} = \dfrac{1-z^2}{1+z^2}$

Then $I = \displaystyle\int\frac{\dfrac{1-z^2}{1+z^2}}{2-\dfrac{1-z^2}{1+z^2}}\cdot\frac{2}{1+z^2}\,dz = \int\frac{1-z^2}{1+3z^2}\cdot\frac{2}{1+z^2}\,dz$

$$= 2\int\left(\frac{2}{1+3z^2}-\frac{1}{1+z^2}\right)dz$$

$$= \frac{4}{3}\int\frac{dz}{\left(\frac{1}{\sqrt{3}}\right)^2+z^2} - \int\frac{2}{1+z^2}\,dz$$

$$= \frac{4}{3}\cdot\frac{1}{\frac{1}{\sqrt{3}}}\arctan\left(\frac{z}{\frac{1}{\sqrt{3}}}\right) - 2\arctan z$$

$$= \frac{4}{\sqrt{3}} \arctan\left(\sqrt{3}\tan\frac{x}{2}\right) - 2\arctan\left(\tan\frac{x}{2}\right)$$

$$= \frac{4}{\sqrt{3}} \arctan\left[\sqrt{3}\tan\left(\frac{x}{2}\right)\right] - x$$

19. $\dfrac{1}{4\sin x - 3\cos x}$

Sol. $I = \displaystyle\int \frac{dx}{4\sin x - 3\cos x}$

Put $\tan\left(\frac{x}{2}\right) = z$ so that $dx = \dfrac{2}{1+z^2}dz$

$\sin x = \dfrac{2z}{1+z^2}$, $\cos x = \dfrac{1-z^2}{1+z^2}$

$$I = \int \frac{1}{\frac{8z}{1+z^2} - \frac{3-3z^2}{1+z^2}} \cdot \frac{2}{1+z^2}dz = \int \frac{2}{3z^2+8z-3}dz$$

$$= \frac{2}{3}\int \frac{dz}{z^2 + \frac{8}{3}z - 1} = \frac{2}{3}\int \frac{dz}{\left(z+\frac{4}{3}\right)^2 - \left(\frac{5}{3}\right)^2} = \frac{2}{3}\cdot\frac{3}{10}\ln\left|\frac{z+\frac{4}{3}-\frac{5}{3}}{z+\frac{4}{3}+\frac{5}{3}}\right|$$

$$= \frac{1}{5}\ln\left|\frac{3z-1}{3z+9}\right| = \frac{1}{5}\ln\left|\frac{3\tan\left(\frac{x}{2}\right)-1}{3\tan\left(\frac{x}{2}\right)+9}\right|$$

Alternative Method:

$$\int \frac{2}{3z^2+8z-3}dz = \int\left(\frac{\frac{3}{5}}{3z-1} + \frac{-\frac{1}{5}}{z+3}\right)dz$$

$$= \frac{1}{5}\left[\int\frac{1}{3z-1}3dz - \int\frac{1}{z+3}dz\right] = \frac{1}{5}\left[\ln|3z-1| - \ln|z+3|\right]$$

$$= \frac{1}{5}\ln\left|\frac{3z-1}{z+3}\right| = \frac{1}{5}\ln\left|\frac{3\tan\frac{x}{2}-1}{\tan\frac{x}{2}+3}\right|$$

20. $\dfrac{1}{\tan x - \sin x}$

Sol. $I = \displaystyle\int \frac{dx}{\tan x - \sin x} = \int \frac{\frac{2}{1+z^2}dz}{\frac{2z}{1-z^2} - \frac{2z}{1+z^2}}$, where $z = \tan\left(\frac{x}{2}\right)$ and $dx = \dfrac{2}{1+z^2}dz$

$$= \int \frac{1-z^2}{2z^3}dz = \int\left(\frac{1}{2}\cdot z^{-3} - \frac{1}{2}\cdot\frac{1}{z}\right)dz = \frac{1}{2}\cdot\frac{z^{-2}}{-2} - \frac{1}{2}\ln|z|$$

$$= -\frac{1}{4}\left[\tan\left(\frac{x}{2}\right)\right]^{-2} - \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right|$$

21. $\dfrac{1}{2\cosh x + \sinh x}$

Sol. $I = \displaystyle\int \frac{dx}{2\cosh x + \sinh x}$

Here we make the substitution $\tanh\left(\frac{x}{2}\right) = z$

Therefore, $\frac{1}{2}\operatorname{sech}^2\left(\frac{x}{2}\right)dx = dz$

and $dx = \dfrac{2\,dz}{\operatorname{sech}^2\left(\frac{x}{2}\right)} = \dfrac{2\,dz}{1-\tanh^2\left(\frac{x}{2}\right)} = \dfrac{2\,dz}{1-z^2}$

$$\sinh x = \frac{2\tanh\left(\frac{x}{2}\right)}{1-\tanh^2\left(\frac{x}{2}\right)} = \frac{2z}{1-z^2}$$

$$\cosh x = \frac{\cosh^2\left(\frac{x}{2}\right)+\sinh^2\left(\frac{x}{2}\right)}{\cosh^2\left(\frac{x}{2}\right)-\sinh^2\left(\frac{x}{2}\right)} = \frac{1+z^2}{1-z^2}$$

$$I = \int \frac{2\,dz}{(1-z^2)\left[2\frac{1+z^2}{1-z^2} + \frac{2z}{1-z^2}\right]} = \int\frac{dz}{z^2+z+1}$$

$$= \int \frac{dz}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(z+\frac{1}{2}\right)^2} = \frac{\sqrt{2}}{3}\arctan\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)$$

$$= \frac{\sqrt{2}}{3}\arctan\left(\frac{2z+1}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\arctan\left[\frac{2\tanh\left(\frac{x}{2}\right)+1}{\sqrt{3}}\right]$$

22. $\dfrac{\sin x + \cos x}{\tan x}$

Sol. $\displaystyle\int \frac{\sin x + \cos x}{\tan x}\,dx = \left(\cos x + \frac{\cos^2 x}{\sin x}\right)dx$

In $\displaystyle\int \frac{\cos^2 x}{\sin x}\,dx$, put $\cos x = z$ so that $-\sin x\,dx = dz$

$$\int \frac{\cos^2 x}{\sin x}\,dx = \int \frac{\cos^2 x}{\sin^2 x}\cdot \sin x\,dx = \int \frac{-z^2}{-z^2+1}\,dz$$

$$= \int\left(1 - \frac{1}{-z^2+1}\right)dz = z - \frac{1}{2}\ln\left|\frac{z+1}{-z+1}\right|$$

$$= z + \frac{1}{2}\ln\left|\frac{-z+1}{z+1}\right|$$

$$I = \sin x + \cos x + \frac{1}{2}\ln\left|\frac{-\cos x + 1}{\cos x + 1}\right| \qquad (\cos x \neq 1)$$

$$= \sin x + \cos x + \ln\left|\sqrt{\frac{1-\cos x}{1+\cos x}}\right|$$

$$= \sin x + \cos x + \ln\left|\tan\frac{x}{2}\right|$$

23. $\cos x\sqrt{1-\cos x}$

Sol. Put $\cos x = z$ so that $-\sin x\,dx = dz$

$$dx = \frac{-dz}{\sqrt{1-z^2}}$$

$$I = -\int \frac{z\sqrt{1-z}}{\sqrt{1-z^2}}\,dz = -\int \frac{z\,dz}{\sqrt{1+z}}$$

Now put $1 + z = t^2$ so that $dz = 2t\,dt$

$$I = -2\int \frac{t^2-1}{t}\,t\,dt = -2\int (t^2-1)\,dt = -2\frac{t^3}{3} + 2t$$

$$= -\frac{2}{3}(1+z)^{3/2} + 2\sqrt{1+z} = 2\sqrt{1+z}\left[1 - \frac{1+z}{3}\right]$$

$$= 2\sqrt{1+z}\left(\frac{2-z}{3}\right) = \frac{2}{3}(2-\cos x)\sqrt{1+\cos x}$$

24. $\sqrt{a + \sec^2 x}$

Sol. Put $z = \sec x$ so that $dz = \sec x\tan x\,dx$ or $dx = \dfrac{dz}{z\sqrt{z^2-1}}$

$$I = \int\sqrt{a+z^2}\cdot\frac{dz}{z\sqrt{z^2-1}} = \int\sqrt{a+z^2}\cdot\frac{1}{\sqrt{z^2-1}}\cdot\frac{1}{z}\,dz$$

Now put $z^2 = t$, so that $2z\,dz = dt \Rightarrow \dfrac{1}{z}\,dz = \dfrac{1}{2t}\,dt$

$$I = \int\sqrt{a+t}\,\frac{1}{\sqrt{t-1}}\cdot\frac{1}{2t}\,dt = \frac{1}{2}\int\sqrt{\frac{a+t}{t-1}}\cdot\frac{1}{t}\,dt$$

In this integral, set

$$\frac{a+t}{t-1} = \omega^2,\ a + t = \omega^2 t - w^2, \quad \text{or} \quad t = \frac{a+\omega^2}{\omega^2-1}$$

$$\frac{t-1-(a+t)}{(t-1)^2}\,dt = 2\omega\,d\omega \quad \text{or} \quad \frac{-1-a}{(t-1)^2}\,dt = 2\omega\,d\omega$$

$$dt = \frac{-2\omega}{1+a}\left[\frac{a+\omega^2}{\omega^2-a} - 1\right]^2 d\omega = \frac{-2\omega}{1+a}\left(\frac{a+1}{\omega^2-1}\right)^2 d\omega$$

$$= \frac{-2w(a+1)}{(w^2-1)^2}\,dw$$

$$I = \frac{1}{2}\int \omega\,\frac{\omega^2-1}{\omega^2+a}\cdot\frac{-2w(a+1)}{(w^2-1)^2}\,dw$$

$$= -\int\frac{(a+1)\,\omega^2}{(\omega^2+a)(\omega^2-1)}\,d\omega = -\int\left(\frac{a}{\omega^2+a} + \frac{1}{\omega^2-1}\right)d\omega$$

$$= -a\,.\,\frac{1}{\sqrt{a}}\arctan\left(\frac{\omega}{\sqrt{a}}\right) - \frac{1}{2}\ln\left|\frac{\omega-1}{\omega+1}\right|$$

$$= -\sqrt{a}\arctan\sqrt{\frac{a+t}{a(t-1)}} - \frac{1}{2}\ln\left|\frac{\sqrt{\frac{a+t}{t-1}} - 1}{\sqrt{\frac{a+t}{t-1}} + 1}\right|$$

$$= -\sqrt{a}\arctan\sqrt{\frac{a+\sec^2 x}{a\tan^2 x}} - \frac{1}{2}\ln\left|\frac{\sqrt{\frac{a+\sec^2 x}{\tan^2 x}} - 1}{\sqrt{\frac{a+\sec^2 x}{\tan^2 x}} + 1}\right|$$

$$= -\sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right) - \frac{1}{2}\ln\left|\frac{\sqrt{a+\sec^2 x} - \tan x}{\sqrt{a+\sec^2 x} + \tan x}\right|$$

$$= \ln\left|\frac{\sqrt{a+\sec^2 x}+\tan x}{\sqrt{a+\sec^2 x}-\tan x}\right|^{1/2} - \sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right)$$

$$= \ln|\sqrt{a\sec^2 x}+\tan x| - \frac{1}{2}\ln|a+1|$$

$$-\sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right)$$

$$= \ln|\sqrt{a+\sec^2 x}+\tan x| - \sqrt{a}\arctan\left(\frac{\sqrt{a+\sec^2 x}}{\sqrt{a}\tan x}\right)$$

25. Evaluate $I = \int\sqrt{\frac{1-\cos\theta}{\cos\alpha-\cos\theta}}\,d\theta$, α constant and $0<\alpha<\theta<\pi$.

Sol. $I = \int\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\theta}{2}}}\,d\theta$, $\left(\begin{array}{l}\text{since } (1+\cos\alpha)-(1+\cos\theta)\\ = 2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\theta}{2}\end{array}\right)$

$$= \int\frac{\tan\frac{\theta}{2}}{\sqrt{\cos^2\frac{\alpha}{2}\sec^2\frac{\theta}{2}-1}}\,d\theta = \frac{1}{\cos\frac{\alpha}{2}}\int\frac{\tan\frac{\theta}{2}}{\sqrt{\sec^2\frac{\theta}{2}-\sec^2\frac{\alpha}{2}}}\,d\theta$$

Now put $\sec^2\frac{\theta}{2}-\sec^2\frac{\alpha}{2}=u^2$

or $\sec^2\frac{\theta}{2}\tan\frac{\theta}{2}\,d\theta = 2u\,du$

or $d\theta = \dfrac{2u\,du}{\left(u^2+\sec^2\frac{\alpha}{2}\right)\tan\frac{\theta}{2}}$

$$I = \sec\frac{\alpha}{2}\int\frac{\tan\frac{\theta}{2}\cdot 2u\,du}{u\left(u^2+\sec^2\frac{\alpha}{2}\right)\tan\frac{\theta}{2}} = 2\sec\frac{\alpha}{2}\int\frac{du}{u^2+\sec^2\frac{\alpha}{2}}$$

$$= 2\sec\frac{\alpha}{2}\cdot\frac{1}{\sec\frac{\alpha}{2}}\arctan\left(\frac{u}{\sec\frac{\alpha}{2}}\right)$$

$$= 2\arctan\left[\frac{\sqrt{\sec^2\frac{\theta}{2}-\sec^2\frac{\alpha}{2}}}{\sec\frac{\alpha}{2}}\right]$$

$$= 2\arctan\left[\frac{\sqrt{\cos^2\frac{\alpha}{2}-\cos^2\frac{\theta}{2}}}{\cos\frac{\alpha}{2}\cos\frac{\theta}{2}\cdot\sec\frac{\alpha}{2}}\right] = 2\arctan\left[\frac{\sqrt{\cos\alpha-\cos\theta}}{\sqrt{2}\cos\frac{\theta}{2}}\right]$$

26. By using the substitution $z=\tan\left(\frac{x}{2}\right)$, show that

(i) $\int\sec x\,dx = \ln\left|\frac{1+\tan\left(\frac{x}{2}\right)}{1-\tan\left(\frac{x}{2}\right)}\right| = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$

(ii) $\int\csc dx = \frac{1}{2}\left|\frac{1-\cos x}{1+\cos x}\right|$

Sol.

(i) Let $I = \int\sec x\,dx = \int\frac{dx}{\cos x} = \int\frac{dx}{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}} = \int\frac{\sec^2\frac{x}{2}\,dx}{1-\tan^2\frac{x}{2}}$

Put $\tan\frac{x}{2}=z$ so that $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz \Rightarrow \sec^2\frac{x}{2}\,dx = 2dz$

$$I = \int\frac{2\,dz}{1-z^2} = \int\left(\frac{1}{1+z}+\frac{1}{1-z}\right)dz = \ln\left|\frac{1+z}{1-z}\right|$$

$$= \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|$$

$$\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|^2 = \left[\frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\cos\frac{x}{2}-\sin\frac{x}{2}}\right]^2 = \frac{1+\sin x}{1-\sin x}$$

$$\Rightarrow \left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| = \left|\frac{1+\sin x}{1-\sin x}\right|^{1/2}$$

Thus $I = \ln\left|\left(\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right)\right| = \ln\left|\frac{1+\sin x}{1-\sin x}\right|^{1/2} = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|$.

(ii) $\int\csc x\,dx = \int\frac{dx}{\sin x}$