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Calculus With Analytic Geometry

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Analytic Geometry

By
S.M Yusuf

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Chapter 2

THE DERIVATIVE

Exercise Set 2.1 (Page 51)

1. Show that the function $f(x) = |x| + |x-1|$ is continuous for every value of x but is not differentiable at $x = 0$ and $x = 1$.

Sol. Let a be an arbitrary real number. We check the continuity of $f(x)$ at $x = a$.

$$\lim_{x \to a} f(x) = \lim_{x \to a} (|x| + |x-1|)$$
$$= \lim_{x \to a} |x| + \lim_{x \to a} |x-1|$$
$$= |a| + |a-1|.$$
$$f(a) = |a| + |a-1|.$$

Thus $f(a) = \lim_{x \to a} f(x)$ and so $f(x)$ is continuous. But a is any real number. Therefore, the given function is continuous for every real value of x.

Now $Lf'(0) = \lim_{x \to 0^-} \dfrac{f(x)-f(0)}{x-0} = \lim_{x \to 0^-} \dfrac{|x| + |x-1| - |-1|}{x}$

$$= \lim_{x \to 0^-} \left(\frac{-x}{x} + \frac{-x+1}{x} - \frac{1}{x}\right) = \lim_{x \to 0^-} \left(-1 - 1 + \frac{1}{x} - \frac{1}{x}\right)$$
$$= -2$$

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0^+} \frac{|x| + |x-1| - |-1|}{x}$$
$$= \lim_{x \to 0^+} \left(\frac{x}{x} + \frac{1-x}{x} - \frac{1}{x}\right) = 0$$

Thus $Lf'(0) \neq Rf'(0)$ and so the function is not differentiable at $x = 0$

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{|x| + |x-1| - 1}{x-1}$$
$$= \lim_{x \to 1^-} \frac{x+1-x-1}{x-1} = 0$$
$$Rf'(1) = \lim_{x \to 1^+} \frac{|x| + |x-1| - 1}{x-1} = \lim_{x \to 1^+} \frac{x+x-1-1}{x-1}$$
$$= \lim_{x \to 1^+} \frac{2(x-1)}{x-1} = 2$$

Therefore, $Lf'(1) \neq Rf'(1)$ and so the function is not differentiable at $x = 1$.

2. Let $f(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } 1 < x \le 2. \end{cases}$

Discuss the continuity and differentiability of f at $x = 1$.

Sol. $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (2x-1) = 1$

Also $f(1) = 1$

Thus Left hand limit = Right hand limit
= Value of $f(x)$ at $x = 1$

Hence $f(x)$ is continuous at $x = 1$

For its differentiability at $x = 1$, we have

$$Lf'(1) = \lim_{x \to 1^-} \frac{f(x)-f(1)}{x-1} = \lim_{x \to 1^-} \frac{x-1}{x-1} = 1$$

and $Rf'(1) = \lim_{x \to 1^+} \dfrac{f(x)-f(1)}{x-1} = \lim_{x \to 1^+} \dfrac{2x-1-1}{x-1} = 2$

Thus $Lf'(1) \neq Rf'(1)$

Hence f is not differentiable at $x = 1$.

3. Let $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$

Show that f is continuous and differentiable at $x = 0$.

Sol. $|f(x) - f(0)| = \left|x^2 \sin\frac{1}{x} - 0\right| = \left|x^2 \sin\frac{1}{x}\right|$

$$\le x^2 < \delta^2 = \varepsilon, \left(\text{since } \left|\sin\frac{1}{x}\right| \le 1\right)$$

whenever $|x| < \delta$

Thus $f(x)$ is continuous at $x = 0$

Now, $Rf'(0) = \lim_{h \to 0} \dfrac{(0+h)^2 \sin\left|\frac{1}{0+h}\right| - 0}{h}$

$$= \lim_{h \to 0} h \sin\frac{1}{h} = 0$$

$$Lf'(0) = \lim_{h \to 0} \frac{(0-h)^2 \sin\frac{1}{0-h} - 0}{-h} = \lim_{h \to 0} \frac{h^2 \sin\left(-\frac{1}{h}\right)}{-h}$$

$$= \lim_{h \to 0} h \sin\left(\frac{1}{h}\right) = 0$$

Thus $Rf'(0) = Lf'(0) = 0$

Hence f is differentiable at $x = 0$

4. Is the function f defined by $f(x) = \begin{cases} (x-a)\sin\left(\dfrac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = a. \end{cases}$

continuous and differentiable at $x = a$?

Sol. $|f(x) - f(a)| = \left|(x-a)\sin\dfrac{1}{x-a} - 0\right|$

$$= \left|(x-a)\sin\frac{1}{x-a}\right| = |x-a|\left|\sin\frac{1}{x-a}\right|$$

$$\leq |x-a|, \left\{\text{since } \left|\sin\frac{1}{x-a}\right| \leq 1\right\}$$

$$< \delta = \varepsilon, \text{ whenever } |x-a| < \delta$$

Hence the function is continuous at $x = a$.

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{h\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \sin\frac{1}{h}, \text{ which does not exist.}$$

Hence the function is not differentiable at $x = a$.

5. Let $f(x) = \begin{cases} x \arctan\left(\dfrac{1}{x}\right) & \text{if } x \neq a \\ 0 & \text{if } x = 0. \end{cases}$

Discuss the continuity and differentiability of f at $x = 0$.

Sol. $|f(x) - f(0)| = \left|x\arctan\dfrac{1}{x} - 0\right| = \left|x\arctan\dfrac{1}{x}\right|$

$$= |x|\left|\arctan\frac{1}{x}\right| < \varepsilon\frac{\pi}{2}, \text{ wherever } |x| < \delta$$

Hence $f(x)$ is continuous at $x = 0$,

$$f'(0) = \lim_{h \to 0} \frac{(h)\arctan\left(\frac{1}{h}\right) - 0}{h}$$

$$= \lim_{h \to 0} \arctan\left(\frac{1}{h}\right), \text{ which does not exist.}$$

Hence $f(x)$ is not differentiable at $x = 0$.

6. Find $Lf'(2)$ and $Rf'(2)$ for the function $f(x) = |x^2 - 4|$.

Sol. $Lf'(2) = \lim\limits_{x \to 2^-} \dfrac{f(x) - f(2)}{x-2} = \lim\limits_{x \to 2^-} \dfrac{|x^2 - 4|}{x-2}$

$$= \lim_{x \to 2^-} \frac{|x-2|(x+2)}{x-2} = \lim_{x \to 2^-} \frac{(2-x)(x+2)}{x-2} = -4$$

Similarly,

$$Rf'(2) = \lim_{x \to 2^+} \frac{f(x) - f(2)}{x-2} = \lim_{x \to 2^+} \frac{|x^2-4|}{x-2}$$

$$= \lim_{x \to 2^+} \frac{(x-2)(x+2)}{x-2} = 4$$

Thus $Lf'(2) \neq Rf'(2)$ and so f is not differentiate at $x = 2$

7. Find the values of a and b so that the function f is continuous and differentiable at $x = 1$, where

$$f(x) = \begin{cases} x^3 & \text{if } x < 1 \\ ax + b & \text{if } x \geq 1. \end{cases}$$

Sol. $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x^3 = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (ax + b) = a + b$

For continuity at $x = 1$, we must have

$f(1) = a + b = 1$ (1)

Now, $Lf'(1) = \lim\limits_{x \to 1^-} \dfrac{f(x) - f(1)}{x-1} = \lim\limits_{x \to 1^-} \dfrac{x^3 - 1}{x-1}$

$$= \lim_{x \to 1^-} \frac{(x-1)(x^2+x+1)}{x-1} = 3$$

$$Rf'(1) = \lim_{x \to 1^+} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1^+} \frac{ax + b - (a+b)}{x-1} = a$$

For differentiability at $x = 1$, we must have

$Lf'(1) = Rf'(1)$, i.e., $a = 3$

Now we have

$a + b = 1$ and $a = 3$

Therefore, $b = 1 - 3 = -2$

8. Let $f(x) = \begin{cases} \sin 2x & \text{if } 0 < x \leq \dfrac{\pi}{6} \\ ax + b & \text{if } \dfrac{\pi}{6} < x \leq 1. \end{cases}$

Derive the values of a and b so that f is continuous and differentiable at $x = \dfrac{\pi}{6}$

Sol. $\lim\limits_{x \to \frac{\pi}{6}^-} f(x) = \lim\limits_{x \to \frac{\pi}{6}^-} \sin 2x = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$

$$\lim_{x \to \frac{\pi}{6}^+} f(x) = \lim_{x \to \frac{\pi}{6}^+} ax + b = \frac{\pi}{6}a + b$$

For continuity at $x = \frac{\pi}{6}$, we must have

$$\frac{\pi}{6}a + b = \frac{\sqrt{3}}{2} = f\left(\frac{\pi}{6}\right)$$

For differentiability at $x = \frac{\pi}{6}$, we have

$$Lf'\left(\frac{\pi}{6}\right) = \lim_{x \to \frac{\pi}{6}^-} \frac{f(x) - f\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = \lim_{x \to \frac{\pi}{6}^-} \frac{\sin 2x - \sin\left(2 \cdot \frac{\pi}{6}\right)}{x - \frac{\pi}{6}}$$

$$= \lim_{x \to \frac{\pi}{6}^-} \frac{2\cos\left(x + \frac{\pi}{6}\right)\sin\left(x - \frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = 2\cos\frac{\pi}{3} = 1$$

$$Rf'\left(\frac{\pi}{6}\right) = \lim_{x \to \frac{\pi}{6}^+} \frac{f(x) - f\left(\frac{\pi}{6}\right)}{x - \frac{\pi}{6}} = \lim_{x \to \frac{\pi}{6}^+} \frac{ax + b - a\frac{\pi}{6} - b}{x - \frac{\pi}{6}} = a$$

Thus $f(a)$ is differentiable at $x = \frac{\pi}{6}$ if $a = 1$

Now we have, $\frac{\pi}{6}a + b = \frac{\sqrt{3}}{2}$ and $a = 1$

Therefore, $a = 1$ and $b = \frac{\sqrt{3}}{2} - \frac{\pi}{6} = \frac{3\sqrt{3} - \pi}{6}$

9. Let $f(x) = x \tanh\left(\frac{1}{x}\right)$ if $x \neq 0$ and $f(0) = 0$. Discuss the continuity and differentiability of f at $x = 0$.

Sol. Here $f(x) = x \tanh\frac{1}{x} = x\,\frac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}} = x\,\frac{e^{2/x} - 1}{e^{2/x} + 1}$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-}\left(x \cdot \frac{e^{2/x} - 1}{e^{2/x} + 1}\right) = 0 \cdot (-1) = 0$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x\,\frac{e^{2/x} - 1}{e^{2/x} + 1} = \lim_{x \to 0^+} x\,\frac{1 - e^{-2/x}}{1 + e^{-2/x}} = 0.1 = 0$$

Hence f is continuous at $x = 0$

For differentiability, we have

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{e^{2/x} - 1}{e^{2/x} + 1} = -1$$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{1 - e^{-2/x}}{1 + e^{-2/x}} = 1$$

$f'(0)$ does not exist, so f is not differentiable at $x = 0$.

Find the slope of the tangent line to the given curve at the indicated point (Problems 10 – 12):

10. $y = x^2$ at $(2, 4)$

Sol. $y = x^2$

$$\frac{dy}{dx} = 2x$$

$$\left[\frac{dy}{dx}\right]_{(2,4)} = 4, \text{ which is the required slope.}$$

11. $y = \frac{1}{x}$ at $(1, 1)$

Sol. $\frac{dy}{dx} = -\frac{1}{x^2}$

$$\left[\frac{dy}{dx}\right]_{(1,1)} = -\frac{1}{(1)^2} = -1, \text{ which is the required slope.}$$

12. $y = x^2 - 7x + 3$ at $(7, 3)$

Sol. $\frac{dy}{dx} = 2x - 7$

$$\left[\frac{dy}{dx}\right]_{(7,3)} = 14 - 7 = 7, \text{ which is the required slope.}$$

13. Let v be the velocity of a particle at any given time t. Deduce that the acceleration of the particle at this instant is $\frac{dv}{dt}$.

Sol. Let v be the velocity at P after time t and $v + \delta v$ be the velocity at Q after a time $t + \delta t$.

The change in velocity $= \delta v$; Change in time $= \delta t$

$$\text{Acceleration} = \lim_{\delta t \to 0} \frac{\delta v}{\delta t} = \frac{dv}{dt}$$

Find the velocity and acceleration at $t = 0, 1, 2$ in each of the following (Problems 14 – 16):

14. $s = \dfrac{1}{t+1}$

Sol. $\dfrac{ds}{dt} = -\dfrac{1}{(t+1)^2} = v \qquad (1)$

$$v_0 = -\frac{1}{(0+1)^2} = -1 \; ; \; v_1 = -\frac{1}{(1+1)^2} = -\frac{1}{4} \; ; \; v_2 = \frac{1}{(2+1)^2} = -\frac{1}{9}$$

From (1), $a = \dfrac{d^2s}{dt^2} = \dfrac{2}{(t+1)^3}$

Let a_0, a_1, a_2 be the accelerations at $t = 0, 1, 2$ respectively. Then

$$a_0 = \frac{2}{1} = 2 \; ; \; a_1 = \frac{2}{(1+1)^3} = \frac{2}{8} = \frac{1}{4} \; ; \; a_2 = \frac{2}{(2+1)^3} = \frac{2}{27}$$

15. $s = t^2 + 2t + 5$

Sol. $v = \dfrac{ds}{dt} = 2t + 2 \quad ; \quad v_0 = 2$

$v_1 = 2(1) + 2 = 4 \quad ; \quad v_2 = 2(2) + 2 = 6$

$a = \dfrac{d^2s}{dt^2} = 2 \quad ; \quad a_0 = 2, a_1 = 2, a_2 = 2$

16. $s = t^2(t-1)$

Sol. $v = \dfrac{ds}{dt} = 3t^2 - 2t \quad ; \quad v_0 = 0$

$v_1 = 3 - 2 = 1 \quad ; \quad v_2 = 3(2)^2 - 2(2) = 12 - 4 = 8$

$a = \dfrac{d^2s}{dt^2} = 6t - 2 \quad ; \quad a_0 = -2$

$a_1 = 6(1) - 2 = 4 \quad ; \quad a_2 = 6(2) - 2 = 10$

17. A point moves in a straight line so that its distance s (in metres) after time t (in seconds) is $s = 4t^2 - 16t + 12$. Find

(i) the average velocity in the interval $[1, 1 + \Delta t]$

(ii) the velocity at $t = 1$.

Sol. The average velocity $\Delta s/\Delta t$ in the interval $[1, 1 + \Delta t]$ is

$$\frac{\Delta s}{\Delta t} = \frac{s(1+\Delta t) - s(1)}{1 + \Delta t - 1}$$

$$= \frac{4(1+\Delta t)^2 - 16(1+\Delta t) + 12 - 4 + 16 - 12}{\Delta t}$$

$$= \frac{4\Delta t(\Delta t - 2)}{\Delta t} = 4\Delta t - 8$$

The velocity at $t = 1$ is

$$\left.\frac{ds}{dt}\right]_{t=1} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0} (4\Delta t - 8) = -8$$

i.e., -8 m/sec.

18. The position of a body (in feet) at time t seconds is $s = t^3 - 6t^2 + 9t$. Find the body's acceleration each time its velocity is zero.

Sol. $s = t^3 - 6t^2 + 9t$

$\dfrac{ds}{dt} = 3t^2 - 12t + 9 \qquad (1)$

$\dfrac{d^2s}{dt^2} = 6t - 12 \qquad (2)$

Velocity of the body is zero if

$3t^2 - 12t + 9 = 0$ or $3(t-1)(t-3) = 0$

i.e., $t = 1, 3$

Acceleration of the body

(i) when $t = 1$ is $\left.\dfrac{d^2s}{dt^2}\right]_{t=1} = 6 - 12 = -6$

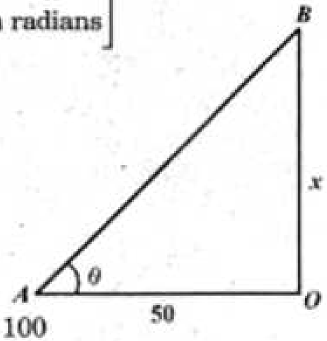
i.e., -6 ft/sec^2

(ii) when $t = 3$ is $\left.\dfrac{d^2s}{dt^2}\right]_{t=3} = 18 - 12 = 6$

i.e., 6 ft/sec^2

19. A ladder is placed 50 metres from a wall at an angle θ with the horizontal. Top of the ladder is x metres above the ground. If the bottom of the ladder is pushed toward the wall, find the rate of change of x with respect to θ when $\theta = 45°$.

$\left[\textbf{Hint: } \dfrac{d}{d\theta}(\tan\theta) = \sec^2\theta \qquad \text{if } \theta \text{ is in radians}\right]$



Sol. Let θ be in radians.

From the figure, we have

$\tan\theta = \dfrac{x}{50}$ or $x = 50\tan\theta$

Rate of change of x w.r.t. θ is

$\dfrac{dx}{d\theta} = 50\sec^2\theta$

$$\left.\frac{dx}{d\theta}\right|_{\theta = \frac{\pi}{4}} = 50\left(\sec\frac{\pi}{4}\right)^2 = 50 \times 2 = 100$$

i.e., 100 m/radian $= \left(\dfrac{100}{\frac{180}{\pi}}\right)$ m/deg $= \dfrac{100 \times \pi}{180}$ m/deg ≈ 1.75 m/deg

20. The number of litres of water in a tank, t minutes after the water starts draining out of the tank, is given by $f(t) = 200\,(300 - t)^2$

(i) What is the average rate at which the water flows out during the first 5 minutes?

(ii) How fast is the water running out at the end of 5 minutes?

Sol. Average rate at which the water flows out during the first five minutes

$$= \frac{f(5) - f(1)}{5 - 1} = \frac{200(30-5)^2 - 200(30-1)^2}{4}$$

$$= \frac{200(25^2 - 29^2)}{4} = -50 \times 216 = -10800$$

i.e., 10800 lit/min.

$$f'(t) = -400(30 - t)$$

The rate at which water runs out after 5 min

$$= f'(5) = -400(30-5) = -400 \times 25 = -10000$$

i.e., 10000 lit./min.

21. The heights (in feet) of a rocket t seconds after its launching, is given by $s = -t^3 + 96t^2 + 195t + 10, (t \geq 0)$.

Find

(i) the velocity of the rocket at any time t.

(ii) the velocity of the rocket when t = 0, 30, 70 seconds. Interpret the results

(iii) the maximum height attained by the rocket.

Sol.

(i) The velocity v of the rocket at any time t is

$$v = \frac{ds}{dt} = -3t^2 + 192t + 195$$

(ii) The velocity when $t = 0$ is 195 ft/sec.

When $t = 30$, $v = -3(30)^2 + 192.30 + 195 = 3255$

When $t = 50$, $v = -3(50)^2 + 192.50 + 195 = 2295$

When $t = 70$, $v = -3(70)^2 + 192.70 + 195 = -1065$

Interpretations:

The velocity with which rocket is launched = 195 ft/sec.

At $t = 30$ sec, it accelerates to 3255 ft/sec.

After 50 sec, the velocity is 2295 ft/sec. which is less than the velocity at $t = 30$. Thus the velocity is decreasing after some time.

The velocity is 0 if

$$-3t^2 + 192t + 195 = 0$$

i.e., $t^2 - 64t - 65 = 0$

or $t = 65, -1$

After 65 sec. of its fight, the velocity of the rocket is 0 and after 70 sec. it is -1065, which means it is coming back to the earth with a velocity of 1065 ft/sec. at this instant.

(iii) The maximum height is attained when the rocket has velocity 0, *i.e.,* at $t = 65$.

Maximum height attained

$$= -(65)^3 + 96(65)^2 + 195(65) + 10$$

$$= 143660 \quad \textit{i.e.,} \quad 143660 \text{ ft.}$$

22. The rupee cost $C(x)$ of producing x washing machines is given by

$$C(x) = 2000 + 100x - 0.1x^2$$

(i) Find the marginal cost at $x = 100$.

(ii) Show that the marginal cost at $x = 100$ is approximately the cost of producing the 101st washing machine.

Sol.

(i) $C(x) = 2000 + 100x - 0.1x^2$

$C'(x) = 100 - 0.2x$

Marginal cost at $x = 100$ is

$$C'(100) = 100 - 0.2(100) = 100 - 20 = 80$$

i.e., Rs. 80

(ii) Cost of producing the 101st washing machine is

$$C(101) - C(100)$$

$$= [2000 + 100 \times 101 - 0.1\,(101)^2] - [2000 + 100 \times 100 - 0.1\,(100)^2]$$

$$= (2000 + 10100 - 1020.10) - (2000 + 10000 - 1000.00)$$

$$= 1079.9 - 1000 = 79.90$$

i.e., Rs. 79.90 ≈ Rs. 80

Thus marginal cost at $x = 100$ is approximately the cost of producing the 101st washing machine.

23. The revenue $R(x)$ (in rupees) from selling x units of desks is given by

$$R(x) = 2000\left(1 - \frac{1}{x+2}\right).$$

(i) Find the marginal revenue when x number of desks are sold.

(ii) Use $R'(x)$ to estimate the increase in revenue that will result by selling the 9th desk.

Sol.

(i) Marginal revenue when x desks are sold

$$R'(x) = 2000\left(\frac{1}{(x+2)^2}\right) = \frac{2000}{(x+2)^2}$$

(ii) Approximate increase in revenue that will result by selling the 9th desk

$$= R'(8) = \frac{2000}{10^2} = 20 \quad i.e., \quad \text{Rs. } 20.$$

24. The cost $C(x)$ (in rupees) of producing x units of fans is

$$C(x) = 100x + 200000$$

and the revenue $R(x)$ (in rupees) of selling these x number of fans is $\quad R(x) = -0.02x^2 + 400x.$

Find the profit function $P(x)$ and the marginal profit at $x = 2000$. Calculate the actual profit realized from the sale of 2001 st fan.

Sol. Profit function $P(x)$ is given by

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= -0.02x^2 + 400x - 100x - 200000 \\ &= -0.02x^2 + 300x - 200000 \end{aligned}$$

Marginal profit is

$$P'(x) = -0.04x + 300$$

Marginal profit at $x = 2000$ is

$$\begin{aligned} P'(2000) &= -0.04\,(2000) + 300 \\ &= -80.00 + 300 = 220 \quad \Rightarrow \quad i.e., \ \text{Rs. } 220. \end{aligned}$$

Actual profit realized from the sale of 2001st fan

$$\begin{aligned} &= P(2001) - P(2000) \\ &= [-0.02(2001)^2 + 300(2001) - 200000] \\ &\quad - [-0.02(2000)^2 + 300(2000) - 200000] \\ &= -0.02[(2001)^2 - (2000)^2] + 300 \\ &= -0.02(4001) + 300 \\ &= -80.02 + 300 = 219.98 \quad i.e., \ \text{Rs. } 219.98 \end{aligned}$$

Thus the marginal profit at $x = 2000$ is approximately equal to the profit realized from the sale of 2001 st fan.

Exercise Set 2.2 (Page 66)

Differentiate with respect to x, (Problems 1 – 1[illegible]):

1. $\sqrt{a^2 + x^2}$

Sol. Let $\quad y = \sqrt{a^2 + x^2} = (a^2 + x^2)^{1/2}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(a^2 + x^2)^{-1/2} \frac{d}{dx}(a^2 + x^2) \\ &= \frac{1}{2}(a^2 + x^2)^{-1/2}(2x) = \frac{x}{\sqrt{a^2 + x^2}} \end{aligned}$$

2. $\sqrt[3]{x^2 + x + 1}$

Sol. Let $\quad y = (x^2 + x + 1)^{1/3}$

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{3}(x^2 + x + 1)^{-2/3} \frac{d}{dx}(x^2 + x + 1) \\ &= \frac{1}{3}(x^2 + x + 1)^{-2/3}(2x + 1) = \frac{2x + 1}{3(x^2 + x + 1)^{2/3}} \end{aligned}$$

3. $\dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$

Sol. Let $\quad y = \dfrac{\sqrt{a + x} - \sqrt{a - x}}{\sqrt{a + x} - \sqrt{a - x}}$

$$\begin{aligned} &= \frac{(\sqrt{a + x} - \sqrt{a - x})(\sqrt{a + x} - \sqrt{a - x})}{(\sqrt{a + x} - \sqrt{a - x})(\sqrt{a + x} - \sqrt{a - x})}, \ \text{(rationalizing)} \\ &= \frac{(a + x) + (a - x) - 2(\sqrt{a^2 - x^2})}{(a + x) - (a - x)} \\ &= \frac{2a - 2\sqrt{a^2 - x^2}}{2x} = \frac{a - \sqrt{a^2 - x^2}}{x} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{x\left[0 - \frac{1}{2}(a^2 - x^2)^{-1/2}(-2x)\right] - [a - \sqrt{a^2 - x^2}]\,.\,1}{x^2} \\ &= \frac{x\,.\,\dfrac{x}{\sqrt{a^2 - x^2}} - a + \sqrt{a^2 - x^2}}{x^2} \\ &= \frac{\dfrac{x^2}{\sqrt{a^2 - x^2}} + \sqrt{a^2 - x^2} - a}{x^2} = \frac{\dfrac{x^2 + a^2 - x^2}{\sqrt{a^2 - x^2}} - a}{x^2} \\ &= \frac{\dfrac{a^2}{\sqrt{a^2 - x^2}} - a}{x^2} = \frac{a^2 - a\sqrt{a^2 - x^2}}{x^2\sqrt{a^2 - x^2}} = \frac{a(a - \sqrt{a^2 - x^2})}{x^2\sqrt{a^2 - x^2}} \end{aligned}$$

4. $\dfrac{\sqrt{\sin x}}{\sin\sqrt{x}}$

Sol.
$$\begin{aligned} \frac{dy}{dx} &= \frac{\sin\sqrt{x}\left\{\frac{1}{2}(\sin x)^{-1/2}\cos x\right\} - \sqrt{\sin x}\left\{\cos\sqrt{x}\left(\frac{1}{2}\right)x^{-1/2}\right\}}{(\sin\sqrt{x})^2} \\ &= \frac{\sin\sqrt{x}\,.\,\dfrac{\cos x}{2\sqrt{\sin x}} - \dfrac{\sqrt{\sin x}\,.\,\cos\sqrt{x}}{2\sqrt{x}}}{(\sin\sqrt{x})^2} \end{aligned}$$

$$= \frac{\dfrac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}}}{(\sin\sqrt{x})^2}$$

$$= \frac{\sqrt{x}\sin\sqrt{x}\cos x - \sin x\cos\sqrt{x}}{2\sqrt{x}\sqrt{\sin x}\,\sin^2\sqrt{x}}$$

5. $\sqrt{\log_{10}(x^2+1)}$

Sol. Let $y = \sqrt{\log_{10}(x^2+1)} = \sqrt{\dfrac{\ln(x^2+1)}{\ln 10}}$

$$\frac{dy}{dx} = \frac{1}{\sqrt{\ln 10}}\cdot\frac{1}{2\sqrt{\ln(x^2+1)}}\cdot\frac{(2x)}{(x^2+1)}$$

$$= \frac{x}{\sqrt{\ln 10}\,(x^2+1)\sqrt{\ln(x^2+1)}}$$

6. $\tan(\sin x)$

Sol. $\dfrac{dy}{dx} = \sec^2(\sin x)\,.\,\dfrac{d}{dx}(\sin x) = \sec^2(\sin x)\,.\,\cos x$

$= \cos x\,.\,\sec^2(\sin x)$

7. $\arctan\left(\dfrac{x\sin\alpha}{1-x\cos\alpha}\right)$

Sol. $$\frac{dy}{dx} = \frac{\dfrac{d}{dx}\left(\dfrac{x\sin a}{1-x\cos a}\right)}{1+\left(\dfrac{x\sin a}{1-x\cos a}\right)^2}$$

$$= \frac{\dfrac{(1-x\cos a)\sin a - x\sin a(-\cos a)}{(1-x\cos a)^2}}{\dfrac{(1-x\cos a)^2+(x\sin a)^2}{(1-x\cos a)^2}}$$

$$= \frac{\sin a - x\sin a\cos a + x\sin a\cos a}{(1-x\cos a)^2+(x\sin a)^2}$$

$$= \frac{\sin a}{1-2x\cos a + x^2\cos^2 a + x^2\sin^2 a} = \frac{\sin a}{1-2x\cos a + x^2}$$

8. $\ln\left(\dfrac{x^2+x+1}{x^2-x+1}\right)$

Sol. $y = \ln(x^2+x+1) - \ln(x^2-x+1)$

$$\frac{dy}{dx} = \frac{1}{x^2+x+1}\frac{d}{dx}(x^2+x+1) - \frac{1}{x^2-x+1}\frac{d}{dx}(x^2-x+1)$$

$$= \frac{2x+1}{x^2+x+1} - \frac{2x-1}{x^2-x+1}$$

$$= \frac{(2x+1)(x^2-x+1)-(2x-1)(x^2+x+1)}{(x^2+x+1)(x^2-x+1)} = \frac{2-2x^2}{1+x^2+x^4}$$

9. x^{x^2}

Sol. $\ln y = x^2\ln x$

Differentiating both sides w.r.t. x, we have

$$\frac{1}{y}\frac{dy}{dx} = x^2\,.\,\frac{1}{x} + 2x\ln x$$

$$= x + 2x\ln x = x(1+2\ln x)$$

$$\frac{dy}{dx} = y\,.\,x(2\ln x+1)$$

$$= x^{x^2}.\,x(2\ln x+1) = x^{x^2+1}(2\ln x+1)$$

10. $\ln(x^2+x)$

Sol. Let $y = \ln(x^2+x)$

Differentiating w.r.t, we have

$$\frac{dy}{dx} = \frac{1}{x^2+x}\frac{d}{dx}(x^2+x) = \frac{2x+1}{x^2+x}$$

11. $(\arcsin x)^{x^{1/x}}$

Sol. Taking logarighm of both sides, we have

$\ln y = x^{1/x}\ln(\arcsin x)$

Differentiating both sides, we get

$$\frac{1}{y}\frac{dy}{dx} = x^{1/x}\,.\,\frac{\dfrac{1}{\sqrt{1-x^2}}}{\arcsin x} + \ln(\arcsin x)\frac{d}{dx}(x^{1/x}) \qquad (1)$$

Now, let $u = x^{1/x}$ or $\ln u = \dfrac{1}{x}\ln x$

$$\frac{1}{u}\frac{du}{dx} = \frac{1}{x}\cdot\frac{1}{x} + \ln x\left(-\frac{1}{x^2}\right) = \frac{1}{x^2} + \ln x\left(-\frac{1}{x^2}\right)$$

$$\frac{du}{dx} = u\left(\frac{1}{x^2} - \frac{1}{x^2}\ln x\right) = x^{1/x}\frac{1}{x^2}(1-\ln x)$$

$$= x^{\frac{1}{x}-2}(\ln e - \ln x) = x^{\frac{1}{x}-2}\ln\frac{e}{x}$$

Putting in (1)

$$\frac{1}{y}\frac{dy}{dx} = \frac{x^{1/x}}{\sqrt{1-x^2}\,\arcsin x} + x^{\frac{1}{x}-2}\ln\frac{e}{x}\ln(\arcsin x)$$

$$\frac{dy}{dx} = (\arcsin x)^{x^{1/x}} \times \left[x^{\frac{1}{x}-2}\ln\frac{e}{x}\ln(\arcsin x) + \frac{x^{1/x}}{\sqrt{1-x^2}\,(\arcsin x)}\right]$$

12. $|x^2 - 9|$

Sol.
$$\begin{aligned} y &= x^2 - 9, && \text{if } |x| \geq 3 \\ &= -x^2 + 9, && \text{if } |x| < 3 \end{aligned}$$

Therefore,
$$\begin{aligned} \frac{dy}{dx} &= 2x, && \text{if } |x| \geq 3 \\ &= -2x, && \text{if } |x| < 3 \end{aligned}$$

13. $\sqrt{x + \sqrt{x + \sqrt{x}}}$

Sol.
$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}\left(x + \sqrt{x + \sqrt{x}}\right)^{-1/2} \times \frac{d}{dx}\left(x + \sqrt{x + \sqrt{x}}\right) \\ &= \frac{1}{2\sqrt{x + \sqrt{x + \sqrt{x}}}} \times \left[1 + \frac{1}{2}(x + \sqrt{x})^{-1/2} \frac{d}{dx}(x + \sqrt{x})\right] \\ &= \frac{1}{2y}\left[1 + \frac{1}{2\sqrt{x + \sqrt{x}}}\left(1 + \frac{1}{2\sqrt{x}}\right)\right] \end{aligned}$$

14. $(x + |x|)^{1/2}$

Sol. Here
$$\begin{aligned} y &= (x + x)^{1/2} = (2x)^{1/2} && \text{if } x > 0 \\ &= (0 + 0) = 0 && \text{if } x = 0 \\ &= (x - x)^{1/2} = 0 && \text{if } x < 0 \end{aligned}$$

Thus
$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2}(2x)^{-1/2} \cdot 2 = \frac{1}{\sqrt{2x}} && \text{if } x > 0 \\ &= 0 && \text{if } x \leq 0 \end{aligned}$$

15. Differentiate $\arctan \dfrac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}$ with respect to $\arccos x^2$.

Sol. Let $u = \arccos x^2$

Then $x^2 = \cos u$

and
$$\begin{aligned} y &= \arctan\left(\frac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}\right) \\ &= \arctan\left(\frac{\sqrt{1 + \cos u} - \sqrt{1 - \cos u}}{\sqrt{1 + \cos u} + \sqrt{1 - \cos u}}\right) \\ &= \arctan\left(\frac{\sqrt{2\cos^2 \frac{u}{2}} - \sqrt{2\sin^2 \frac{u}{2}}}{\sqrt{2\cos^2 \frac{u}{2}} + \sqrt{2\sin^2 \frac{u}{2}}}\right) \\ &= \arctan\left(\frac{\cos \frac{u}{2} - \sin \frac{u}{2}}{\cos \frac{u}{2} + \sin \frac{u}{2}}\right) = \arctan\left(\frac{1 - \tan \frac{u}{2}}{1 + \tan \frac{u}{2}}\right) \end{aligned}$$

$$= \arctan\left(\tan\left(\frac{\pi}{4} - \frac{u}{2}\right)\right) = \frac{\pi}{4} - \frac{u}{2}$$

Thus $\dfrac{dy}{du} = -\dfrac{1}{2}$

Find $\dfrac{dy}{dx}$ (Problems 16 – 20):

16. $y = x^{\sin y}$

Sol. Taking logarithm of both sides, we have

$$\ln y = \sin y (\ln x)$$

Now differentiating w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{\sin y}{x} + (\ln x)\cos y \frac{dy}{dx} \quad \text{or} \quad \frac{dy}{dx}(x - xy\cos y \ln x) = y \sin y$$

i.e., $\dfrac{dy}{dx} = \dfrac{y \sin y}{x - xy(\cos y \ln x)}$

17. $x^y = e^{x-y}$

Sol. Taking logarithm of both sides, we have

$$\begin{aligned} y \ln x &= (x - y) \ln e \\ &= x - y \qquad (1) \end{aligned}$$

Differentiating both sides w.r.t. x, we have

$$y \cdot \frac{1}{x} + \ln x \cdot \frac{dy}{dx} = 1 - \frac{dy}{dx} \quad \text{or} \quad (\ln x + 1)\frac{dy}{dx} = 1 - \frac{y}{x} = \frac{x - y}{x}$$

or $\dfrac{dy}{dx} = \dfrac{x - y}{x(\ln x + 1)}$ (2)

From (1), we have

$$(1 + \ln x)y = x \quad \text{or} \quad y = \frac{x}{1 + \ln x}$$

Putting in (2), we get

$$\frac{dy}{dx} = \frac{x - \frac{x}{1 + \ln x}}{x(\ln x + 1)} = \frac{x + x\ln x - x}{x(\ln x + 1)^2} = \frac{\ln x}{(\ln x + 1)^2}$$

Alternative Method:

$$y \ln x = x - y \Rightarrow y(1 + \ln x) = x \quad \text{or} \quad y = \frac{x}{1 + \ln x}$$

Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = \frac{(1 + \ln x) \cdot 1 - x \cdot \frac{1}{x}}{(1 + \ln x)^2} = \frac{1 + \ln x - 1}{(1 + \ln x)^2} = \frac{\ln x}{(1 + \ln x)^2}$$

18. $y^x + x^y = c$

Sol. Let $u = y^x$ and $v = x^y$

Taking logarithm of both sides of the first equation

$$\ln u = x \ln y$$

Differentiating w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = x \cdot \frac{1}{y} + \ln y$$

$$\frac{du}{dx} = u\left(\frac{x}{y}\frac{dy}{dx} + \ln y\right) = y^x\left(\frac{x}{y}\frac{dy}{dx} + \ln y\right) \quad (1)$$

Now from $v = x^y$, taking logarithm, we get

$$\log v = y \ln x$$

Differentiating w.r.t. x, we obtain

$$\frac{1}{v}\frac{dv}{dx} = \frac{y}{x} + \ln x \frac{dy}{dx}$$

$$\frac{dv}{dx} = v\left[\frac{y}{x} + \ln x \cdot \frac{dy}{dx}\right] = x^y\left[\frac{y}{x} + \ln x \cdot \frac{dy}{dx}\right] \quad (2)$$

The given equation is $u + v = c$

Differentiating w.r.t. x, we get

$$\frac{du}{dx} + \frac{dv}{dx} = 0 \quad (3)$$

Putting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ from (1) and (2) into (3), we have

$$y^x\left[\frac{x}{y}\frac{dy}{dx} + \ln y\right] + x^y\left[\frac{y}{x} + \ln x \cdot \frac{dy}{dx}\right] = 0$$

or $\quad (xy^{x-1} + x^y \ln x)\frac{dy}{dx} = -\left[y^x \ln y + x^y \cdot \frac{y}{x}\right]$

or $\quad \frac{dy}{dx} = -\frac{y^2 \ln y + yx^{y-1}}{xy^{x-1} + x^y \ln x}$

19. $\frac{x+y}{x-y} = x^2 + y^2$

Sol. Differentiating w.r.t. x, we have

$$\frac{(x-y)(1+y') - (x+y)(1-y')}{(x-y)^2} = 2x + 2yy',$$

where $\quad y' = \frac{dy}{dx}$

or $\quad \frac{y'(x-y+x+y) + x - y - x - y}{(x-y)^2} = 2x + 2yy'$

or $\quad xy' - y = x(x-y)^2 + yy'(x-y)^2$

or $\quad y'[x - y(x-y)^2] = y + x(x-y)^2$

$$y' = \frac{dy}{dx} = \frac{y + x(x-y)^2}{x - y(x-y)^2}$$

20. $x + \arcsin y = xy$

Sol. $x + \arcsin y = xy \quad (1)$

Differentiating (1) w.r.t. x, we have $\quad 1 + \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = x\frac{dy}{dx} + y$

or $\quad \frac{dy}{dx}\left(\frac{1}{\sqrt{1-y^2}} - x\right) = y - 1 \quad$ or $\quad \frac{dy}{dx}\left(\frac{1 - x\sqrt{1-y^2}}{\sqrt{1-y^2}}\right) = y - 1$

Therefore, $\quad \frac{dy}{dx} = \frac{(y-1)\sqrt{1-y^2}}{1 - x\sqrt{1-y^2}}$

In Problems 21 – 30, find $f'(x)$ where:

21. $f(x) = x^2\sqrt{2ax - x^2}$

Sol. $f'(x) = x^2 \cdot \frac{1}{2}(2ax - x^2)^{-1/2}\frac{d}{dx}(2ax - x^2) + \sqrt{2ax - x^2} \cdot 2x$

$$= \frac{x^2}{2\sqrt{2ax - x^2}} \cdot (2a - 2x) + 2x\sqrt{2ax - x^2}$$

$$= \frac{x^2(a-x)}{\sqrt{2ax - x^2}} + 2x\sqrt{2ax - x^2}$$

$$= \frac{x^2(a-x) + 2x(2ax - x^2)}{\sqrt{2ax - x^2}} = \frac{5ax^2 - 3x^3}{\sqrt{2ax - x^2}}$$

22. $f(x) = \ln\left(\frac{e^x}{1+e^x}\right)$

Sol. $f(x) = \ln\frac{e^x}{1+e^x} = \ln e^x - \ln(1 + e^x) = x - \ln(1 + e^x)$

$$f'(x) = 1 - \frac{e^x}{1+e^x} = \frac{1 + e^x - e^x}{1 + e^x} = \frac{1}{1+e^x}$$

23. $f(x) = x^{\ln x}$

Sol. Taking ln of both sides, we get

$$\ln(f(x)) = \ln x \cdot \ln x = (\ln x)^2$$

Differentiating both sides, we have

$$\frac{f'(x)}{f(x)} = 2(\ln x) \cdot \frac{1}{x} = \frac{2\ln x}{x}$$

or $\quad f'(x) = f(x) \cdot \left(\frac{2\ln x}{x}\right) = x^{\ln x}\left(\frac{2\ln x}{x}\right) = \frac{2}{x}\ln x \cdot x^{\ln x}$

24. $f(x) = \ln\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$

Sol. $f(x) = \ln(1 + \sqrt{x}) - \ln(1 - \sqrt{x})$

$$f'(x) = \frac{\frac{1}{2\sqrt{x}}}{1+\sqrt{x}} - \frac{-\frac{1}{2\sqrt{x}}}{1-\sqrt{x}} = \frac{1}{2\sqrt{x}\,(1+\sqrt{x})} + \frac{1}{2\sqrt{x}\,(1-\sqrt{x})}$$

$$= \frac{1-\sqrt{x}+1+\sqrt{x}}{2\sqrt{x}\,(1+\sqrt{x})\,(1-\sqrt{x})} = \frac{2}{2\sqrt{x}\,(1-x)} = \frac{1}{\sqrt{x}\,(1-x)}$$

25. $f(x) = e^{ax}\cos(b\arctan x)$

Sol. $f(x) = e^{ax}\cos(b\arctan x)$

$$f'(x) = e^{ax}\cdot a\cos(b\arctan x) + e^{ax}\left[-\frac{b}{1+x^2}\sin(b\arctan x)\right]$$

$$= e^{ax}\left[a\cos(b\arctan x) - \frac{b\sin(b\arctan x)}{1+x^2}\right]$$

$$= \frac{e^{ax}}{1+x^2}\left[a(1+x^2)\cos(b\arctan x) - b\sin(b\arctan x)\right]$$

26. $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\ln\dfrac{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)}$

Sol. $f(x) = \dfrac{1}{\sqrt{b^2-a^2}}\left[\ln\left(\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)\right) - \ln\left(\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)\right)\right]$

$$f'(x) = \frac{1}{\sqrt{b^2-a^2}}\left(\frac{\frac{1}{2}\sqrt{b-a}\sec^2\left(\frac{x}{2}\right)}{\sqrt{b+a}+\sqrt{b-a}\tan\left(\frac{x}{2}\right)} - \frac{-\frac{1}{2}\sqrt{b-a}\sec^2\left(\frac{x}{2}\right)}{\sqrt{b+a}-\sqrt{b-a}\tan\left(\frac{x}{2}\right)}\right)$$

$$= \frac{\sqrt{b-a}\sec^2\left(\frac{x}{2}\right)}{2\sqrt{b^2-a^2}}\left[\frac{2\sqrt{b+a}}{(b+a)-(b-a)\tan^2\left(\frac{x}{2}\right)}\right]$$

$$= \frac{\cos^2\frac{x}{2}}{2\sqrt{b+a}\cos^2\frac{x}{2}}\left[\frac{2\sqrt{b+a}}{(b+a)\cos^2\left(\frac{x}{2}\right)-(b-a)\sin^2\frac{x}{2}}\right]$$

$$= \frac{1}{2\sqrt{b+a}}\,\frac{2\sqrt{b+a}}{b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)+a} = \frac{1}{a+b\cos x}$$

27. $f(x) = x\,a^x\sinh x$

Sol. $f'(x) = 1\,.\,a^x\sinh x + x\,.\,a^x\ln a\sinh x + xa^x\cosh x$

$= a^x\sinh x + xa^x\sinh x\ln a + xa^x\cosh x$

28. $f(x) = \dfrac{-\cos x}{2\sin^2 x} + \dfrac{1}{2}\ln\tan\left(\dfrac{x}{2}\right)$

Sol. $f'(x) = -\dfrac{1}{2}\left[\dfrac{\sin^2 x(-\sin x)-\cos x(2\sin x\cos x)}{\sin^4 x}\right] + \dfrac{1}{2}\,\dfrac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$

$$= -\frac{1}{2}\left[\frac{-\sin^2 x-2\cos^2 x}{\sin^3 x}\right] + \frac{1}{4}\cdot\frac{1}{\cos^2\frac{x}{2}}\cdot\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}$$

$$= \frac{1}{2}\left[\frac{\sin^2 x+2\cos^2 x}{\sin^3 x}\right] + \frac{1}{4\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \frac{\sin^2 x+2\cos^2 x}{2\sin^3 x} + \frac{1}{2\sin x}$$

$$= \frac{\sin^2 x+2\cos^2 x+\sin^2 x}{2\sin^3 x} = \frac{2}{2\sin^3 x} = \csc^3 x$$

29. $f(x) = \operatorname{arcsec}(\csc x+\sqrt{x})$

Sol. $f'(x) = \dfrac{1}{(\csc x+\sqrt{x})\sqrt{(\csc x+\sqrt{x})^2-1}}\times\dfrac{d}{dx}(\csc x+\sqrt{x})$

$$= \frac{1}{(\csc x+\sqrt{x})\sqrt{(\csc x+\sqrt{x})^2-1}}\times\left(-\csc x\cot x+\frac{1}{2\sqrt{x}}\right)$$

$$= \frac{1-2\sqrt{x}\csc x\cot x}{2\sqrt{x}\,(\csc x+\sqrt{x})\sqrt{\cot^2 x+x+2\sqrt{x}\csc x}}$$

30. $f(x) = \left(1+\dfrac{1}{x}\right)^{x^2}$

Sol. Taking ln of both sides, we have $\ln f(x) = x^2\ln\left(1+\dfrac{1}{x}\right)$

Differentiating w.r.t. x, we obtain

$$\frac{f'(x)}{f(x)} = 2x\,.\ln\left(1+\frac{1}{x}\right) + x^2\cdot\frac{1}{1+\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)$$

$$= 2x\cdot\ln\left(1+\frac{1}{x}\right) - \frac{1}{1+\frac{1}{x}}$$

$$f'(x) = f(x)\left[2x\ln\left(1+\frac{1}{x}\right) - \frac{x}{x+1}\right]$$

$$= \left(1+\frac{1}{x}\right)^{x^2}\left[2x\ln\left(1+\frac{1}{x}\right)-\frac{x}{x+1}\right]$$

Differentiate with respect to x each of the following (Problems 31 – 42).

31. $\arctan\left(\dfrac{1+2x}{2-x}\right)$

Sol. We have

$$\frac{d}{dx}\left[\arctan\left(\frac{1+2x}{2-x}\right)\right] = \frac{1}{1+\left(\dfrac{1+2x}{2-x}\right)^2}\,\frac{d}{dx}\left(\frac{1+2x}{2-x}\right)$$

$$= \frac{(2-x)^2}{(2-x)^2+(1+2x)^2}\cdot\frac{2(2-x)-(1+2x)(-1)}{(2-x)^2} = \frac{5}{5+5x^2} = \frac{1}{1+x^2}$$

32. $\ln(\arcsin e^x)$

Sol. Let $y = \ln(\arcsin e^x)$. Then

$$\frac{dy}{dx} = \frac{1}{\arcsin e^x}\cdot\frac{d}{dx}(\arcsin e^x)$$

$$= \frac{1}{\arcsin e^x}\cdot\frac{1}{\sqrt{1-(e^x)^2}}\cdot e^x = \frac{e^x}{\sqrt{1-(e^x)^2}\,\arcsin e^x}$$

33. $(\arcsin x^2)^\pi$

Sol. Let $u = \arcsin x^2$. Then

$$y = u^\pi$$

$$\frac{dy}{du} = \pi u^{\pi-1} = \pi(\arcsin x^2)^{\pi-1} \qquad (1)$$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^4}}\cdot 2x \qquad (2)$$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{2x\,\pi(\arcsin x^2)^{\pi-1}}{\sqrt{1-x^4}}$$

34. $f\left(\dfrac{x^2+1}{x^2-1}\right)$

Sol. Let $y = f\left(\dfrac{x^2+1}{x^2-1}\right)$

Set $u = \dfrac{x^2+1}{x^2-1}$ so that $y = f(u)$

$$\frac{dy}{du} = f'(u)$$

$$\frac{du}{dx} = \frac{(x^2-1)\,2x - 2x(x^2+1)}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}$$

$$\frac{dy}{dx} = \frac{dy}{du}\times\frac{du}{dx} = f'(u)\cdot\frac{-4x}{(x^2-1)^2} = f'\left(\frac{x^2+1}{x^2-1}\right)\cdot\frac{-4x}{(x^2-1)^2}$$

35. $\dfrac{1-\cosh x}{1+\cosh x}$

Sol. Using the identities $\cosh 2x = 2\sinh^2 x + 1$ and $\cosh 2x = 2\cosh^2 x - 1$, we have

$$y = \frac{-2\sinh^2\left(\frac{x}{2}\right)}{2\cosh^2\left(\frac{x}{2}\right)} = -\tanh^2\left(\frac{x}{2}\right)$$

Therefore, $\dfrac{dy}{dx} = -2\tanh\left(\dfrac{x}{2}\right)\operatorname{sech}^2\left(\dfrac{x}{2}\right)\cdot\dfrac{1}{2}$

$$= -\tanh\left(\frac{x}{2}\right)\operatorname{sech}^2\left(\frac{x}{2}\right)$$

36. $\ln(\tanh 2x)$

Sol. $\dfrac{dy}{dx} = \dfrac{1}{\tanh 2x}\cdot\dfrac{d}{dx}(\tan h\, 2x)$

$$= \frac{2\operatorname{sech}^2 2x}{\tanh 2x} = \frac{2}{\cosh^2 2x\cdot\dfrac{\sinh 2x}{\cosh 2x}} = \frac{4}{2\cosh 2x\sinh 2x}$$

$$= \frac{4}{\sinh 4x} = 4\operatorname{csch} 4x$$

37. $\log_{10}\left(\dfrac{x+1}{x}\right)$

Sol. $y = \log_{10}\left(\dfrac{x+1}{x}\right) = \dfrac{\ln\left(\dfrac{x+1}{x}\right)}{\ln 10} = \dfrac{1}{\ln 10}\,[\ln(x+1) - \ln x]$

$$y' = \frac{1}{\ln 10}\left(\frac{1}{x+1}-\frac{1}{x}\right) = \frac{1}{\ln 10}\cdot\frac{-1}{x(x+1)} = -\frac{1}{(\ln 10)\,x\,(x+1)}$$

38. $\arccos\left(\sqrt{1-x^2}\right)$

Sol. Let $y = \arccos\sqrt{1-x^2}$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(1-x^2)}}\,\frac{d}{dx}\left(\sqrt{1-x^2}\right) = \frac{-1}{\sqrt{x^2}}\cdot\frac{1}{2}\,\frac{-2x}{\sqrt{1-x^2}} = \frac{x}{|x|\,\sqrt{1-x^2}}$$

39. $\operatorname{arcsec}(\sinh x)$

Sol. Let $y = \operatorname{arcsec}(\sinh x)$

$$\frac{dy}{dx} = \frac{1}{\sinh x\sqrt{\sinh^2 x - 1}}\,\frac{d}{dx}(\sinh x) = \frac{\cosh x}{\sinh x\sqrt{\sinh^2 x - 1}}$$

40. arcsin(arccot ln x)

Sol. Let $y = \arcsin(\text{arccot} \ln x)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\text{arccot} \ln x)^2}} \frac{d}{dx} (\text{arccot} \ln x)$$

$$= \frac{1}{\sqrt{1-(\text{arccot} \ln x)^2}} \cdot \frac{-1}{1+(\ln x)^2} \cdot \frac{1}{x}$$

$$= \frac{-1}{x(1+\ln^2 x)\sqrt{1-(\text{arccot} \ln x)^2}}$$

41. $\cosh^{-1}(1+x^2)$

Sol. Let $y = \cosh^{-1}(1+x^2)$

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1+x^2)^2-1}} \frac{d}{dx}(1+x^2) = \frac{2x}{\sqrt{2x^2+x^4}} = \frac{2x}{|x|\sqrt{2+x^2}}$$

42. $\sinh^{-1}(\tanh x)$

Sol. $\frac{dy}{dx} = \frac{1}{\sqrt{\tanh^2 x + 1}} \cdot \frac{d}{dx} (\tanh x)$

$$= \frac{\text{sech}^2 x}{\sqrt{\tanh^2 x + 1}}$$

In Problems 43 – 54 find $\frac{dy}{dx}$:

43. $\sqrt{x} + \sqrt{y} = \sqrt{a}$

Sol. Differentiating, both sides w.r.t. x, we get

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

or $\quad x^{-1/2} + y^{-1/2}\frac{dy}{dx} = 0 \quad$ or $\quad y^{-1/2}\frac{dy}{dx} = -x^{-1/2}$

or $\quad \frac{dy}{dx} = -\frac{x^{-1/2}}{y^{-1/2}} = -\frac{y^{1/2}}{x^{1/2}} \quad$ or $\quad \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$

44. $xy^2 - 2xy + x = 1$

Sol. Differentiating both sides w.r.t x, we get

$$1 . y^2 + 2xy\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y + 1 = 0$$

or $\quad \frac{dy}{dx}(2xy - 2x) = -y^2 + 2y - 1$

or $\quad \frac{dy}{dx} = \frac{-y^2+2y-1}{2x(y-1)}$, provided $2x(y-1) \neq 0$

45. $x^3 + y^3 - 3axy = 0$

Sol. Differentiating w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} - 3a\left[x\frac{dy}{dx} + y . 1\right] = 0$$

or $\quad x^2 + y^2\frac{dy}{dx} - a\left[x\frac{dy}{dx} + y\right] = 0$

or $\quad (y^2 - ax)\frac{dy}{dx} = ay - x^2 \quad$ or $\quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$

46. $(x^2 + y^2)^3 = y$

Sol. Differentiating both sides w.r.t. x, we have

$$3(x^2+y^2)^2\frac{d}{dx}(x^2+y^2) = \frac{dy}{dx}$$

or $\quad 3(x^2+y^2)^2\left(2x + 2y\frac{dy}{dx}\right) = \frac{dy}{dx}$

or $\quad \frac{dy}{dx}(1 - 6y(x^2+y^2)^2) = 6x(x^2+y^2)^2$

$$\frac{dy}{dx} = \frac{6x(x^2+y^2)^2}{1-6y(x^2+y^2)^2} \quad \text{provided } 1 - 6y(x^2+y^2)^2 \neq 0$$

47. $\arctan\left(\frac{y}{x}\right) + yx^2 = 1$

Sol. Differentiating both sides w.r.t. x, we have

$$\frac{1}{1+\frac{y^2}{x^2}}\frac{d}{dx}\left(\frac{y}{x}\right) + x^2\frac{dy}{dx} + 2xy = 0$$

or $\quad \frac{x^2}{x^2+y^2} \times \frac{x\frac{dy}{dx} - y . 1}{x^2} + x^2\frac{dy}{dx} + 2xy = 0$

or $\quad \frac{dy}{dx}\left[\frac{x}{x^2+y^2} + x^2\right] = \frac{y}{x^2+y^2} - 2xy$

or $\quad \frac{dy}{dx}\left[\frac{x + x^2(x^2+y^2)}{x^2+y^2}\right] = \frac{y - 2xy(x^2+y^2)}{x^2+y^2}$

or $\quad \frac{dy}{dx} = \frac{y - 2xy(x^2+y^2)}{x + x^2(x^2+y^2)} = \frac{y(1-2x^3-2xy^2)}{x(1+x^3+xy^2)}$

48. $\arctan(x + y) = \arcsin(e^y + x)$.

Sol. Differentiating both the sides w.r.t. x implicitly, we have

$$\frac{1}{1+(x+y)^2}\left(1 + \frac{dy}{dx}\right) = \frac{1}{\sqrt{1-(e^y+x)^2}}\left(e^y\frac{dy}{dx} + 1\right)$$

or $\quad \dfrac{dy}{dx}\left[\dfrac{1}{1+(x+y)^2}-\dfrac{e^y}{\sqrt{1-(e^y+x)^2}}\right]$

$$= \frac{1}{\sqrt{1-(e^y+x)^2}} - \frac{1}{1+(x+y)^2}$$

or $\quad [\sqrt{1-(e^y+x)^2} - e^y(1+(x+y)^2]\dfrac{dy}{dx}$

$$= 1+(x+y)^2 - \sqrt{1-(e^y+x)^2}$$

or $\quad \dfrac{dy}{dx} = \dfrac{1+(x+y)^2-\sqrt{1-(e^y+x)^2}}{\sqrt{1-(e^y+x)^2}-e^y(1+(x+y)^2)}$

49. $y = \arcsin(\ln x) - \ln(\arctan x)$

Sol. Differentiating both sides w.r.t. x, we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(\ln x)^2}}\frac{d}{dx}(\ln x) - \frac{1}{\arctan x}\frac{d}{dx}(\arctan x)$$

$$= \frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{1}{x} - \frac{1}{\arctan x}\cdot\frac{1}{1+x^2}$$

$$= \frac{1}{x\sqrt{1-(\ln x)^2}} - \frac{1}{(1+x^2)\arctan x}$$

50. $y \arcsin x - x \arctan y = 1$

Sol. Differentiating both sides of the above equation, w.r.t. x, we get

$$\frac{dy}{dx}\cdot\arcsin x + y\cdot\frac{1}{\sqrt{1-x^2}} - \left(1.\arctan y + x\cdot\frac{1}{1+y^2}\frac{dy}{dx}\right) = 0$$

or $\quad \left(\arcsin x - \dfrac{x}{1+y^2}\right)\dfrac{dy}{dx} = \arctan y - \dfrac{y}{1-x^2}$

or $\quad \left(\dfrac{(1+y^2)\arcsin x - x}{1+y^2}\right)\dfrac{dy}{dx} = \dfrac{\sqrt{1-x^2}\arctan y - y}{\sqrt{1-x^2}}$

or $\quad \dfrac{dy}{dx} = \dfrac{(1+y^2)(\sqrt{1-x^2}\arctan y - y)}{\sqrt{1-x^2}((1+y^2)\arcsin x - x)}$

51. $\arcsin(\ln xy) = x + y^2$

Sol. Differentiating both sides of the above equation, w.r.t. x, we get

$$\frac{1}{\sqrt{1-(\ln xy)^2}}\cdot\frac{d}{dx}(\ln xy) = 1 + 2y\frac{dy}{dx}$$

or $\quad \dfrac{1}{\sqrt{1-(\ln xy)^2}}\cdot\dfrac{1}{xy}\left(1.y + x\dfrac{dy}{dx}\right) = 1 + 2y\dfrac{dy}{dx}$

or $\quad \dfrac{1}{\sqrt{1-(\ln xy)^2}}\cdot\dfrac{1}{y}\cdot\dfrac{dy}{dx} - 2y\dfrac{dy}{dx} = 1 - \dfrac{1}{\sqrt{1-(\ln xy)^2}}\cdot\dfrac{1}{x}$

or $\quad \left[\dfrac{1}{y\sqrt{1-(\ln xy)^2}} - 2y\right]\dfrac{dy}{dx} = \dfrac{x\sqrt{1-(\ln xy)^2}-1}{x\sqrt{1-(\ln xy)^2}}$

or $\quad \dfrac{1-2y^2\sqrt{1-(\ln xy)^2}}{y\cdot\sqrt{1-(\ln xy)^2}}\cdot\dfrac{dy}{dx} = \dfrac{x\sqrt{1-(\ln xy)^2}-1}{x\sqrt{1-(\ln xy)^2}}$

or $\quad \dfrac{dy}{dx} = \dfrac{y(x\sqrt{1-(\ln xy)^2}-1)}{x(1-2y^2\sqrt{1-(\ln xy)^2})}$

52. $\operatorname{arcsec}(x^2+y) - e^x = \dfrac{1}{x+y}$

Sol. $\operatorname{arcsec}(x^2+y) - e^x = (x+y)^{-1}$ (1)

Differentiating both sides of (1), w.r.t. x, we get

$$\frac{1}{(x^2+y)\sqrt{(x^2+y)^2-1}}\left(2x+\frac{dy}{dx}\right) - e^x = -(x+y)^{-2}\times\left(1+\frac{dy}{dx}\right)$$

or $\quad \left[\dfrac{1}{(x^2+y)\sqrt{(x^2+y)^2-1}} + \dfrac{1}{(x+y)^2}\right]\dfrac{dy}{dx}$

$$= e^x - \frac{1}{(x+y)^2} - \frac{2x}{(x^2+y)\sqrt{(x^2+y)^2-1}}$$

or $\quad \left(\dfrac{(x+y)^2+(x^2+y)\sqrt{(x^2+y)^2-1}}{(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}}\right)\dfrac{dy}{dx}$

$$= \frac{e^x.(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1} - (x^2+y)\sqrt{(x^2+y)^2-1} - 2x(x+y)^2}{(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1}}$$

or $\quad \dfrac{dy}{dx} = \dfrac{e^x(x+y)^2(x^2+y)\sqrt{(x^2+y)^2-1} - (x^2+y)\sqrt{(x^2+y)^2-1} - 2x(x+y)^2}{(x+y)^2+(x^2+y)\sqrt{(x^2+y)^2-1}}$

53. $x = a(t - \sin t), y = a(1 - \cos t)$

Sol. Differentiating, above equations w.r.t. t, we get

$$\frac{dx}{dt} = a(1-\cos t)$$

and $\quad \dfrac{dy}{dt} = a(0 - (-\sin t)) = a\sin t$

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{a\sin t}{a(1-\cos t)} = \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{2\sin^2\frac{t}{2}} = \frac{\cos\frac{t}{2}}{\sin\frac{t}{2}} = \cot\frac{t}{2}$$

54. $x = \dfrac{3at}{1+t^2}, y = \dfrac{3at^2}{1+t^2}$

Sol. $\dfrac{dx}{dt} = \dfrac{(1+t^2)\,3a - 3at\,(2t)}{(1+t^2)^2} = \dfrac{3a[1+t^2-2t^2]}{(1+t^2)^2} = \dfrac{3a(1-t^2)}{(1+t^2)^2}$

From $y = \dfrac{3at^2}{1+t^2}$, we have

$$\frac{dy}{dt} = 3a\,\frac{(1+t^2)\,2t - t^2\,(2t)}{(1+t^2)^2} = 3a\,\frac{2t}{(1+t^2)^2} = \frac{6at}{(1+t^2)^2}$$

Thus $\dfrac{dy}{dx} = \dfrac{dy}{dt}\cdot\dfrac{dt}{dx} = \dfrac{6at}{(1+t^2)^2}\times\dfrac{(1+t^2)^2}{3a(1-t^2)} = \dfrac{2t}{1-t^2}$

Alternative Method:

$$y = \frac{3at^2}{1+t^2} = t\cdot\frac{3at}{1+t^2} = tx \quad i.e.,\ y = tx \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{dy}{dx} = \frac{dt}{dx}\cdot x + t\,.\,1 = \frac{dt}{dx}\cdot x + t \qquad (2)$$

But from $x = \dfrac{3at}{1+t^2}$

$$\frac{dx}{dt} = 3a\,.\,\frac{(1+t^2)\,.\,1 - t\,.\,2t}{(1+t^2)^2} = \frac{3a(1-t^2)}{(1+t^2)^2}$$

Now (2) becomes $\dfrac{dy}{dx} = \dfrac{(1+t^2)^2}{3a(1-t^2)}\cdot\dfrac{3at}{1+t^2} + t$

$$= \frac{t(1+t^2)}{1-t^2} + t = \frac{t+t^3+t-t^3}{1-t^2} = \frac{2t}{1-t^2}$$

Differentiate with respect to x (problems 55–60)

55. $y = \sqrt[3]{\dfrac{x(x^2+1)}{(x-1)^2}}$

Sol. $y = \left[\dfrac{x(x^2+1)}{(x-1)^2}\right]^{1/3}$ (1)

Taking natural logrithms of both sides of (1), we get

$$\ln y = \frac{1}{3}\ln\left[\frac{x(x^2+1)}{(x-1)^2}\right]$$

$$= \frac{1}{3}[\ln x + \ln(x^2+1) - 2\ln(x-1)] \qquad (2)$$

Differentiating both sides of (2), w.r.t. x, we get

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{3}\left[\frac{1}{x} + \frac{1}{x^2+1}\cdot 2x - 2\cdot\frac{1}{x-1}\right]$$

$$= \frac{1}{3}\cdot\frac{(x^2+1)(x-1) + 2x\,.\,x(x-1) - 2x(x^2+1)}{x(x^2+1)(x-1)}$$

$$= \frac{x^3 - x^2 + x - 1 + 2x^3 - 2x^2 - 2x^3 - 2x}{3x(x^2+1)(x-1)}$$

or $\dfrac{dy}{dx} = y\cdot\dfrac{x^3 - 3x^2 - x - 1}{3x(x^2+1)(x-1)}$

$$= \frac{x^{1/3}(x^2+1)^{1/3}}{(x-1)^{2/3}}\cdot\frac{x^3-3x^2-x-1}{3x(x^2+1)(x-1)} = \frac{x^3-3x^2-x-1}{3x^{2/3}(x^2+1)^{2/3}(x-1)^{5/3}}$$

56. $y = \dfrac{\sqrt{x}\,(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}$

Sol. Taking natural logarithms of both sides of the above equation, we get

$$\ln y = \ln\frac{x^{1/2}(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}$$

or $\ln y = \dfrac{1}{2}\ln x + \dfrac{2}{3}\ln(1-2x) - \dfrac{3}{4}\ln(2-3x) - \dfrac{4}{3}\ln(3-4x)$ (1)

Differentiating both sides of (1), w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x} + \frac{2}{3}\frac{1}{1-2x}(-2) - \frac{3}{4}\frac{1}{2-3x}(-3) - \frac{4}{3}\frac{1}{3-4x}(-4)$$

$$= \frac{1}{2x} - \frac{4}{3(1-2x)} + \frac{9}{4(2-3x)} + \frac{16}{3(3-4x)}$$

$$= \left[\frac{1}{2x} + \frac{9}{4(2-3x)}\right] + \left[\frac{16}{3(3-4x)} - \frac{4}{3(1-2x)}\right]$$

$$= \frac{2(2-3x)+9x}{4x(2-3x)} + \frac{16(1-2x)-4(3-4x)}{3(3-4x)(1-2x)}$$

$$= \frac{4-6x+9x}{4x(2-3x)} + \frac{16-32x-12+16x}{3(3-4x)(1-2x)}$$

$$= \frac{4+3x}{4x(2-3x)} + \frac{4-16x}{3(3-4x)(1-2x)}$$

or $\dfrac{dy}{dx} = y\left[\dfrac{4+3x}{4x(2-3x)} + \dfrac{4(1-4x)}{3(1-2x)(3-4x)}\right]$

$$= \frac{\sqrt{x}\,(1-2x)^{2/3}}{(2-3x)^{3/4}(3-4x)^{4/3}}\left[\frac{4+3x}{4x(2-3x)} + \frac{4(1-4x)}{3(1-2x)(3-4x)}\right]$$

57. $y = (\tan x)^{\cot x} + (\cot x)^{\tan x}$

Sol. Let $u = (\tan x)^{\cot x}$ (1)

and $v = (\cot x)^{\tan x}$ (2)

Then $y = u + v$ (3)

Taking natural logarithms of both sides of (1), we get

$$\ln u = \cot x \ln \tan x \quad (4)$$

Differentiating (4) w.r.t. x, we have

$$\frac{1}{u}\frac{du}{dx} = (-\operatorname{cosec}^2 x) \ln \tan x + \cot x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$$= -\operatorname{cosec}^2 x \ln \tan x + \frac{1}{\tan^2 x} \cdot \frac{1}{\cos^2 x}$$

$$= -\operatorname{cosec}^2 x \ln \tan x + \operatorname{cosec}^2 x \left(\because \frac{1}{\tan x \cos x} = \frac{1}{\sin x}\right)$$

$$= \operatorname{cosec}^2 x (1 - \ln \tan x)$$

$$= \operatorname{cosec}^2 x (\ln e - \ln \tan x)$$

$$= \operatorname{cosec}^2 x \ln \frac{e}{\tan x} = \operatorname{cosec}^2 x \cdot \ln (e \cot x)$$

or $$\frac{du}{dx} = u \cdot \operatorname{cosec}^2 x \cdot \ln (e \cot x)$$

$$= (\tan x)^{\cot x} \cdot \operatorname{cosec}^2 x \cdot \ln (e \cot x)$$

Taking natural logarithms of both sides of (2), we have

$$\ln v = \tan x \ln \cot x \quad (5)$$

Differentiating both sides of (5), w.r.t. x, we get

$$\frac{1}{v}\frac{dv}{dx} = \sec^2 x \cdot \ln \cot x + \tan x \cdot \frac{1}{\cot x} \cdot (-\operatorname{cosec}^2 x)$$

$$= \sec^2 x \ln \cot x - \frac{1}{\cot^2 x} \cdot \frac{1}{\sin^2 x}$$

$$= \sec^2 x \ln \cot x - \sec^2 x \left(\because \frac{1}{\cot x} \cdot \frac{1}{\sin x} = \frac{1}{\cos x}\right)$$

$$= -\sec^2 x (1 - \ln \cot x)$$

$$= -\sec^2 x \ln \frac{e}{\cot x} \quad (\because 1 = \ln e)$$

$$= -\sec^2 x \ln (e \tan x)$$

or $$\frac{dv}{dx} = v(-\sec^2 x \ln (e \tan x))$$

$$= -(\cot x)^{\tan x} \sec^2 x \ln (e \tan x)$$

Differentiating bothsides of (3) w.r.t. x, we have

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= (\tan x)^{\cot x} \operatorname{cosec}^2 x \ln(e \cot x) - (\cot x)^{\tan x} \sec^2 x \ln (e \tan x)$$

58. $y = x^x e^x \sin (\ln x)$

Sol. Taking natural logarithms of both sides, we get

$$\ln y = \ln x^x + \ln e^x + \ln (\sin (\ln x))$$

or $$\ln y = x \ln x + x + \ln (\sin (\ln x))$$

Differentiating both sides of the above equation w.r.t x, we have

$$\frac{1}{y}\frac{dy}{dx} = \left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) + 1 + \frac{1}{\sin (\ln x)} \cdot \cos (\ln x) \cdot \frac{1}{x}$$

$$= \ln x + 1 + 1 + \frac{1}{x} \cdot \frac{\cos (\ln x)}{\sin (\ln x)}$$

or $$\frac{dy}{dx} = y\left(\ln x + 2 + \frac{1}{x} \cdot \frac{\cos (\ln x)}{\sin (\ln x)}\right)$$

$$= x^x e^x \sin (\ln x)\left(2 + \ln x + \frac{1}{x} \cot (\ln x)\right)$$

59. $y = \dfrac{(x+2)^2}{(x+1)(x^2+3)^3}$

Sol. Taking natural logarithms of both sides of the given equation, we get

$$\ln y = \ln (x+2)^2 - [\ln (x+1) + \ln (x^2+3)^3]$$

or $$\ln y = 2 \ln (x+2) - \ln (x+1) - 3 \ln (x^2+3)$$

Differentiating both sides of the above equation, we have

$$\frac{1}{y} \cdot \frac{dy}{dx} = 2 \cdot \frac{1}{x+2} - \frac{1}{x+1} - \frac{3}{x^2+3} \cdot 2x$$

$$= \frac{2(x+1)(x^2+3) - (x+2)(x^2+3) - 6x(x+2)(x+1)}{(x+2)(x+1)(x^2+3)}$$

$$= \frac{2(x^3+x^2+3x+3) - (x^3+2x^2+3x+6) - 6x(x^2+3x+2)}{(x+1)(x+2)(x^2+3)}$$

$$= \frac{2x^3 + 2x^2 + 6x + 6 - x^3 - 2x^2 - 3x - 6 - 6x^3 - 18x^2 - 12x}{(x+1)(x+2)(x^2+3)}$$

or $$\frac{dy}{dx} = y\left(\frac{-5x^3 - 18x^2 - 9x}{(x+1)(x+2)(x^2+3)}\right)$$

$$= -y \cdot \frac{5x^3 + 18x^2 + 9x}{(x+1)(x+2)(x^2+3)}$$

$$= -\frac{(x+2)^2}{(x+1)(x^2+3)^3} \cdot \frac{5x^3+18x^2+9x}{(x+1)(x+2)(x^2+3)}$$

$$= -\frac{(x+2)(5x^3+18x+9x)}{(x+1)^2(x^2+3)^4}$$

60. $y = \exp.\left(\operatorname{arccsc}\left(\frac{1}{x}\right)\right)$

Sol. Taking ln of both sides, we have

$$\ln y = \operatorname{arc\,csc}\left(\frac{1}{x}\right)$$

Therefore, $\frac{1}{y}\frac{dy}{dx} = \frac{-1}{\frac{1}{x}\sqrt{\frac{1}{x^2}-1}}\frac{d}{dx}\left(\frac{1}{x}\right), \ \frac{1}{|x|} > 1$

$$= \frac{1}{x^2} \cdot \frac{x^2}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = \frac{y}{\sqrt{1-x^2}}, \ |x| < 1$$

$$= \frac{\exp.\left(\operatorname{arccsc}\left(\frac{1}{x}\right)\right)}{\sqrt{1-x^2}}, \ |x| < 1$$

Exercise Set 2.3 (Page 75)

1. Find $\Delta y, dy, \Delta y - dy$ if

(*i*) $y = x^3 - 1, x = 1, \Delta x = -0.5$

(*ii*) $y = \sqrt{3x-2}, x = 2, \Delta x = 0.3$

Sol.

(*i*) Here $y = x^3 - 1$ (1)

$$\Delta y = (x+\Delta x)^3 - 1 - x^3 + 1$$

Setting $x = 1, \Delta x = -0.5$, we get

$$\Delta y = (1-0.5)^3 - 1^3$$
$$= 0.125 - 1 = -0.875$$

From (1), we have

$$\frac{dy}{dx} = 3x^2. \text{ Therefore, } dy = 3x^2 dx$$

When $x = 1, dx = \Delta x = -0.5$,

$$dy = 3(-0.5) = -1.5$$
$$\Delta y - dy = -0.875 + 1.5 = 0.625$$

(*ii*) $y = \sqrt{3x-2}$ (1)

From (1), we have $\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$.

Therefore, $dy = \frac{3}{2\sqrt{3x-2}}dx$

When $x = 2, dx = \Delta x = 0.3$,

$$dy = \frac{3}{2\times 2}(0.3) = 0.2250$$
$$\Delta y = \sqrt{3(x+\Delta x)-2} - \sqrt{3x-2}$$

Setting $x = 2, \Delta x = 0.3$, we get

$$\Delta y = \sqrt{3(2.3)-2} - \sqrt{3\times 2-2} = \sqrt{4.9} - 2 \approx 2.2136 - 2$$
$$\approx 0.2136$$
$$\Delta y - dy \approx 0.2136 - 0.2250 = -0.0114$$

2. Use differentials to approximate

(*i*) $\sqrt{26.2}$ (*ii*) $\sqrt{80.9}$ (*iii*) $\sqrt[3]{123}$

(*iv*) $\cos 61°$ (*v*) $(3.02)^4$ (*vi*) $\tan 29°$

(*i*) $\sqrt{26.2}$

Sol. We consider $y = f(x) = \sqrt{x}$ (1)

with $x = 25$ and $\Delta x = 1.2$

From (1), we have $dy = \frac{1}{2\sqrt{x}}dx$ (2)

Substituting $x = 25, dx = \Delta x = 1.2$ in (2), we get

$$dy = \frac{1}{10}(1.2) = 0.12$$

Now, $dy \approx \Delta y = y + \Delta y - y = \sqrt{x+\Delta x} - \sqrt{x}$

Therefore, $0.12 \approx \sqrt{26.2} - \sqrt{25} = \sqrt{26.2} - 5$

or $\sqrt{26.2} \approx 5 + 0.12 = 5.1200$

Using calculator, we find $\sqrt{26.2} \approx 5.1186$

Error in the approximation $= 0.0014$

(*ii*) $\sqrt{80.9}$

Sol. Let $y = f(x) = \sqrt{x}$ with $x = 81$

and $\Delta x = -0.1 = dx$

$$dy = \frac{1}{2\sqrt{x}}dx = \frac{1}{2\sqrt{81}} \times (-0.1)$$
$$= \frac{-0.1}{18} \approx -0.005556$$

Now $dy \approx \Delta y = \sqrt{x+\Delta x} - \sqrt{x} = \sqrt{80.9} - 9$.

Therefore, $-0.005556 \approx \sqrt{80.9} - 9$

or $\sqrt{80.9} \approx 9 - 0.005556 = 8.994444$

Using calculator, $\sqrt{80.9} \approx 8.994443$

Error in approximation = 0.000001

(iii) $\sqrt[3]{123}$

Sol. Let $y = f(x) = x^{1/3}$

with $x = 125$ and $\Delta x = -2$

$$dy = \frac{1}{3x^{2/3}}dx = \frac{1}{3(125)^{2/3}}\cdot(-2) = \frac{-2}{75} \approx -0.0267$$

$$dy \approx \Delta y = f(x+\Delta x) - f(x).$$

Therefore, $-0.0267 \approx \sqrt[3]{123} - \sqrt[3]{125}$

or $\sqrt[3]{123} \approx 5 - 0.0267 = 4.9733$

But $\sqrt[3]{123} = 4.9732$ by using calculator

Error in the approximation = 0.0001

(iv) $\cos 61^\circ$

Sol. Let $y = f(x) = \cos x$

with $x = 60^\circ = \frac{\pi}{3}$ and $\Delta x = 1^\circ = \frac{\pi}{180}$

$$dy = -\sin x\, dx \qquad (1)$$

Putting $x = \frac{\pi}{3}$, $dx = \frac{\pi}{180}$ in (1), we have

$$dy = \left(-\sin\frac{\pi}{3}\right)\frac{\pi}{180} = -\frac{\sqrt{3}}{2}\cdot\frac{\pi}{180}$$

$$dy \approx \Delta y = f(x+\Delta x) - f(x)$$

i.e., $\frac{-\sqrt{3}\,\pi}{360} \approx \cos 61^\circ - \cos 60^\circ$

or $\cos 61^\circ \approx \frac{1}{2} - \frac{\sqrt{3}\,\pi}{360} \approx 0.5 - 0.01512 = .48488$

From tables, $\cos 61^\circ \approx 0.48481$

Error in approximation = .00007

(v) $(3.02)^4$

Sol. Let $y = f(x) = x^4$ with $x = 3$ and $\Delta x = 0.02$

$$dy = 4x^3\,dx$$

$$dy\big]_{x=3} = 4\times 3^3\,(0.02) = 2.16.$$

Since $dy \approx \Delta y = (x+\Delta x)^4 - x^4$, therefore,

$$2.16 \approx (3.02)^4 - 3^4$$

or $(3.02)^4 \approx 81 + 2.16 = 83.16$

But $(3.02)^4 = 83.1817$

Error in approximation = -0.0217.

(vi) $\tan 29^\circ$

Sol. We let $f(x) = \tan x$, with $x = \frac{\pi}{6}$ and $\Delta x = \frac{-\pi}{180}$

$$dy = \sec^2 x\, dx$$

$$dy\big]_{x=\frac{\pi}{6}} = \sec^2\frac{\pi}{6}\left(-\frac{\pi}{180}\right) = \frac{4}{3}\times\frac{-\pi}{180} \approx -0.0233$$

Now $dy \approx \Delta y = f(x+\Delta x) - f(x)$

Therefore, $-0.0233 \approx \tan 29^\circ - \tan 30^\circ$

or $\tan 29^\circ \approx \frac{1}{\sqrt{3}} - 0.0233 \approx 0.5774 - 0.0233$

≈ 0.5541

But $\tan 29^\circ \approx 0.5543$

Error in approximation = -0.0002

3. The side of a cube is measured with a possible error of ±2%. Find the percentage error in the surface area of one face of the cube.

Sol. Let x be edge of the cube.

Area A of a face is

$$A = x^2$$

$$dA = 2x\,dx$$

$$\frac{dA}{A} = \frac{2x}{x^2}.dx = 2\frac{dx}{x}$$

But $\frac{dx}{x} = \pm 0.02$

Therefore, $\frac{dA}{A} = 2(\pm 0.02) = \pm 0.04$

The percentage error in the surface area is ±4%

4. A box with a square base has its height twice its width. If the width of the box is 8.5 inches with a possible error of ±0.3 inches, find the possible error in the volume of the box.

Sol. Let x in be the width of the box. Then its volume V is

$$V = 2x^3$$

$dV = 6x^2\,dx$. But $dx = \pm\frac{3}{10}$ (in)

Therefore change in volume

$$dV = 6(8.5)^2\left(\pm\frac{3}{10}\right) = \pm 130.05 \text{ (in}^3)$$

Thus the error in the volume of box is ± 130.05 cubic inches.

5. The radius x of a circle increases from $x = 10$ centimetres (cm) to $x + \Delta x = 10.1$ cm. Find the corresponding change in the area of the circle. Also find the percentage change in the area.

Sol. Let A be area of the circle of radius x. Then $A = \pi x^2$

$$dA = 2\pi x\, dx$$

Now, $x = 10$ cm and $\Delta x = 0.1$ cm

Change in the area of the circle is

$$\Delta A \approx dA = 2\pi(10)(0.1) = 2\pi\,(\text{cm}^2)$$

Relative change in the area $= \dfrac{2\pi}{\pi(10)^2} = \dfrac{2}{100} = 0.02$

Percentage change $= \dfrac{2}{100} \times 100 = 2\%$

6. The diameter of a plant was 8 inches. After one year the circumference of the plant increased by 2 inches. How much did

(i) the diameter of the plant increase?

(ii) the cross-sectional area of the plant change?

Sol. If x is the radius of the plant, then its circumference $C = 2\pi x$

Therefore, $dC = 2\pi dx$

Change in circumference is $dC = 2$

and so the change Δx in radius is given by

$$2 = 2\pi dx \quad \text{or} \quad dx = \frac{1}{\pi}$$

Thus the diameter increased by $\dfrac{2}{\pi}$ inches.

Area A of the cross-section is

$$A = \pi x^2$$
$$dA = 2\pi x\, dx$$

When $x = 4$, $dx = \dfrac{1}{\pi}$ and change in area is

$$\Delta A \approx dA = 2\pi(4)\frac{1}{\pi} = 8 \quad i.e., \quad 8 \text{ in}^2$$

7. Sand pouring from a chute forms a conical pile whose altitude is always equal to the radius. If the radius of the pile is 10cm, find the approximate change in radius when volume increases by 2 cm^3.

Sol. The volume of the conical pile of radius r and height r is

$$V = \frac{1}{3}\pi r^3$$
$$dV = \pi r^2 dr$$

Now $\Delta V \approx dV = 2$ cm^3, when $r = 10$

Change in radius $= \Delta r = dr$

Therefore, $2 \approx \pi(10)^2\, dr$ or $dr \approx \dfrac{1}{50\pi}$

i.e., change in radius $\approx \dfrac{1}{50\pi}$ cm.

8. A dome is in the shape of a hemisphere with radius 60 feet. The dome is to be painted with a layer of 0.01 inch thickness. Use differentials to estimate the amount of the paint required.

Sol. If V is volume of hemisphere with radius r, then

$$V = \frac{2\pi r^3}{3}$$
$$dV = 2\pi r^2\, dr \qquad (1)$$

We need change in the volume when change Δr in r is $\dfrac{1}{1200}$ ft.

Letting $r = 60$ and $dr = \dfrac{1}{1200}$ in (1), we have

$$dV = 2\pi \times 60 \times 60 \times \frac{1}{1200} = 6\pi\,(\text{ft}^3), \text{ so}$$

$\Delta V \approx 6\,\pi\,(\text{ft.}^3)$, that is,

the amount of paint $\approx 6\,\pi$ cubic ft.

9. The side of a building is in the shape of a square surmounted by an equilateral triangle. If the length of the base is 15 metres with an error of 1%, find the percentage error in the area of the side.

Sol. Let x m be the length of the base. Then area A of the side is given by

$$A = x^2 + \frac{\sqrt{3}}{4}x^2 = \left(1 + \frac{\sqrt{3}}{4}\right)x^2$$

$$dA = 2\left(1 + \frac{\sqrt{3}}{4}\right)x\,dx \qquad (1)$$

It is given that $\dfrac{dx}{x} = 0.01 \Rightarrow dx = x\left(\dfrac{1}{100}\right)$ and $x = 15$(m).

Now we need to find $\dfrac{dA}{A} \times 100$

From (1), we have

$$dA = 2\left(1 + \frac{\sqrt{3}}{4}\right)15 \times 15\left(\frac{1}{100}\right), \text{ so}$$

$$\frac{\Delta A}{A} \approx \frac{dA}{A} = \frac{2 \times 15^2\left(\frac{\sqrt{3}}{4} + 1\right) \times \frac{1}{100}}{\left(1 + \frac{\sqrt{3}}{4}\right)15^2} = \frac{2}{100}$$

Thus the percentage error in the area of the side is approximately equal to 2%.

10. A boy makes a paper cup in the shape of a right circular cone with height four times its radius. If the radius is changed from 2 cm to 1.5 cm but the height remains four times the radius, find the approximate decrease in the capacity of the cup.

Sol. If r is the radius of the base and h is height of the cup , then its volume V is given by

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi r^2 (4r) = \frac{4}{3}\pi r^3$$

$$dV = 4\pi r^2\, dr \qquad (1)$$

Now it is given that

$$r = 2 \text{ cm and } \Delta r = dr = -0.5$$

Then from (1), change in the capacity of the cup

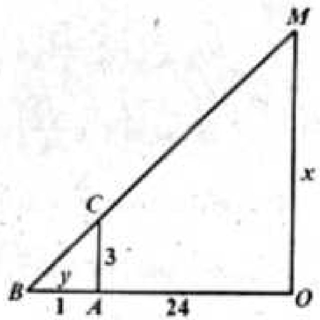
$$\Delta V \approx dV = 4\pi(2)^2\left(-\frac{1}{2}\right) = -8\pi$$

The –ve sign shows that there is decrease in the capacity of the cup, which is approximately equal to 8π cm^3.

11. To estimate the height of Minar-i-Pakistan, the shadow of a 3 metre pole placed 24 metres from the Minar is measured. If the length of the shadow is 1 metre with a percentage error of 1%, find the height of the Minar. Also find the percentage error in the height so found.

Sol. If x m is height of the Minar, then from the figure

$$\frac{x}{25} = \frac{3}{1}$$



Therefore, $x = 25 \times 3 = 75$.

Height of the Minar = 75 m.

If y is the actual length of the shadow of the pole, then

$$\frac{y + 24}{x} = \frac{y}{3}$$

or $\quad 3y + 72 = xy$

or $\quad 3dy = x\,dy + y\,dx$

or $\quad (3 - x)\,dy = y\,dx$

or $\quad (3 - x)\dfrac{dy}{y} = dx \qquad (1)$

Now $\dfrac{dy}{y} = \pm 0.01$. When $x = 75$, relative error in the height $= \dfrac{dx}{x}$.

From (1), we get

$$\frac{-72}{75}(\pm 0.01) = \frac{dx}{75} \quad \text{or} \quad \frac{dx}{75} = \pm\frac{24}{25} \times \frac{1}{100}$$

Percentage error

$$= \pm\frac{24}{25} \times \frac{1}{100} \times 100 = \pm 0.96\%.$$

12. Oil spilled from a tanker spreads in a circle whose radius increases at the rate of 2 ft/sec. How fast is the area increasing when the radius of the circle is 40 feet?

Sol. Let r be the radius of the circle at any instant t. Then area A of the circle is

$$A = \pi r^2$$

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt} \qquad (1)$$

We have to find $\dfrac{dA}{dt}$ when $\dfrac{dr}{dt} = 2$ and $r = 40$. Substituting into (1), we have

$$\frac{dA}{dt} = 2\pi \times 40 \times 2 = 160\pi$$

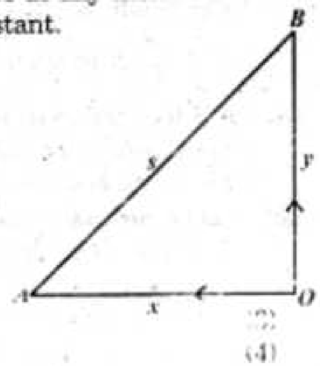
Thus area of the circle changes at the rate of $160\,\pi$ ft^2/sec.

13. From a point O, two cars leave at the same time. One car travels west and after t seconds its position is $x = t^2 + t$ feet. The other car travels north and it covers $y = t^2 + 3t$ feet in t seconds. At what rate is the distance between the two cars changing after 5 seconds.

Sol. Let A, B be the positions of the two cars at any instant t and let s be the distance between them at this instant.

$$s^2 = x^2 + y^2 \qquad (1)$$

$$s\frac{ds}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} \qquad (2)$$



We have to find $\dfrac{ds}{dt}$ at the instant when $t = 5$.

We have

$$x = t^2 + t \qquad (3)$$

$$y = t^2 + 3t \qquad (4)$$

Differentiating (3) and (4) w.r.t. t, we have

$$\frac{dx}{dt} = 2t + 1, \qquad \left.\frac{dx}{dt}\right|_{t = 5} = 11$$

$$\frac{dy}{dt} = 2t + 3, \qquad \left.\frac{dy}{dt}\right|_{t = 5} = 13$$

After 5 sec. the distances of the two cars from O are

$$x = 5^2 + 5 = 30$$
$$y = 5^2 + 15 = 40$$

and $s^2 = 30^2 + 40^2$ from (1)

i.e., $s = 50$

Substituting into (2), we get

$$50\frac{ds}{dt} = 30 \times 11 + 40 \times 13 = 330 + 520 = 850$$

or $$\frac{ds}{dt} = \frac{850}{50} = 17$$

Therefore, the distance between the two cars is changing at the rate of 17 ft./sec.

14. Sand falls from a container at the rate of 10 ft^3/min and forms a conical pile whose height is always double the radius of the base. How fast is the height increasing when the pile is 5 feet high?

Sol. Let h be the height of the pile at any instant t. Radius of the pile = $\frac{h}{2}$. Volume V of the pile is $V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

It is given that $\frac{dV}{dt} = 10$ and we need to find $\frac{dh}{dt}$ at the instant when $h = 5$. Therefore, from (1), we have

$$10 = \frac{\pi}{4} \cdot 5^2 \frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt} = \frac{10 \times 4}{25\,\pi} = \frac{8}{5\,\pi}$$

The height of the pile is changing at the rate of

$\frac{8}{5\pi}$ ft/min ≈ 0.51 ft./min.

15. A 6 feet tall man is walking toward a lamp post 16 feet high at a speed of 5 ft/sec. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing?

Sol. Let x be man's distance from the lamp post OP and z be the distance of the tip of his shadow from O.

i.e., $OM = x$, $OA = z$

From the similar triangles, we have

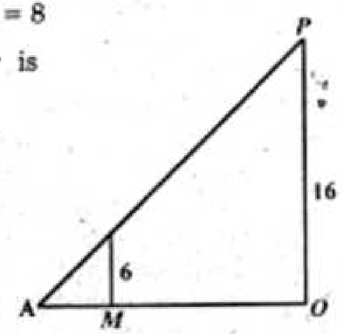
$$\frac{16}{z} = \frac{6}{z - x} \quad \text{or} \quad 16z - 6z = 16x$$

i.e., $5z = 8x$ (1)

Differentiating (1), w.r.t. t, we get

$$5\frac{dz}{dt} = 8\frac{dx}{dt}$$

We substitute $\frac{dx}{dt} = 5$ and find that $\frac{dz}{dt} = 8$



Therefore the tip of man's shadow is moving at the rate of 8 ft./sec.

If y is the length of the shadow, then $MA = y$. From the similar triangles we have

$$\frac{16}{x + y} = \frac{6}{y} \quad \text{or} \quad 16y - 6y = 6x$$

i.e., $5y = 3x$

Therefore, $5\frac{dy}{dt} = 3\frac{dx}{dt}$

Substituting $\frac{dx}{dt} = 5$, we have $5\frac{dy}{dt} = 3 \times 5 \Rightarrow \frac{dy}{dt} = 3$

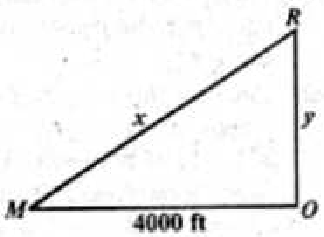
Thus the shadow is changing at the rate of 3 ft./sec.

16. At a distance of 4000 feet from a launching site, a man is observing a rocket being launched. If the rocket lifts off vertically and is rising at a speed of 600 ft/sec. when it is at an altitude of 3000 feet, how fast is the distance between the rocket and the man changing at this instant?

Sol. Let y be altitude of the rocket and x be the distance between the man and the rocket at any instant t.

We have

$$x^2 = y^2 + 4000^2 \quad (1)$$



When $y = 3000$ ft, we have from (1),

$$x^2 = 3000^2 + 4000^2$$
$$= 9000000 + 16000000$$
$$= 25000000$$
$$x = 5000$$

Differentiating (1) w.r.t. t, we get

$$x\frac{dx}{dt} = y\frac{dy}{dt} \quad (2)$$

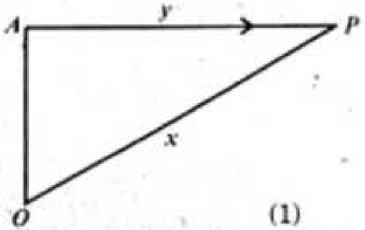
When $y = 3000$, $\frac{dy}{dt} = 600$ (given) and we have to find $\frac{dx}{dt}$ at this instant. From (2), we have

$$\frac{dx}{dt} = \frac{3000}{5000} \times 600 = 360$$

Thus the distance between the rocket and man is changing at the rate of 360 ft/sec.

17. An airplane flying horizontally at an altitude of 3 miles and a speed of 480 miles per hour passes directly above an observer on the ground. How fast is the distance of the observer to the airplane increasing after 30 seconds?

Sol. Let O be the observer on the ground and P be the airplane at some instant t.



Let $OP = x$, $AP = y$

It is given that $OA = 3$.

From the right triangle, we have

$$3^2 + y^2 = x^2 \quad (1)$$

The distance travelled by the plane 30 sec. after it has passed above the observer = 4 miles. Substituting $y = 4$ in (1) we get $x = 5$.

We have to find $\frac{dx}{dt}$ at the instant when $t = 30$ sec. and $\frac{dy}{dt} = 480$.

Implicit differentiation of (1) gives

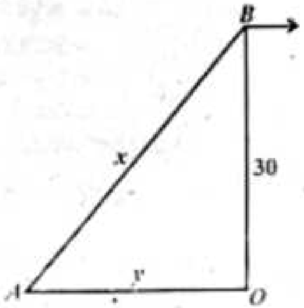
$$y\frac{dy}{dt} = x\frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt}$$

$$\left.\frac{dx}{dt}\right|_{t = 30 \text{ sec.}} = \frac{4}{5} \times 480 = 384$$

The rate of change of the distance of the plane from the observer = 384 miles/hr.

18. A boy flies a kite at an altitude of 30 metres. If the kite flies horizontally away from the boy at the rate of 2 m/sec, how fast is the string being let out when the length of the string released is 70 metres?

Sol. Let x be the length of the string let out at some instant t, K be the kite at an altitude of 30m and let $OA = y$. The kite flies horizontally away from the boy at the rate of 2 m/sec.



From ΔAOK, we have

$$x^2 = 30^2 + y^2 \quad (1)$$

Therefore, $$x\frac{dx}{dt} = y\frac{dy}{dt} \quad (2)$$

When $x = 70$, we have from (1),

$$y^2 = 70^2 - 30^2 = 4900 - 900 = 4000 - 20\sqrt{10}$$

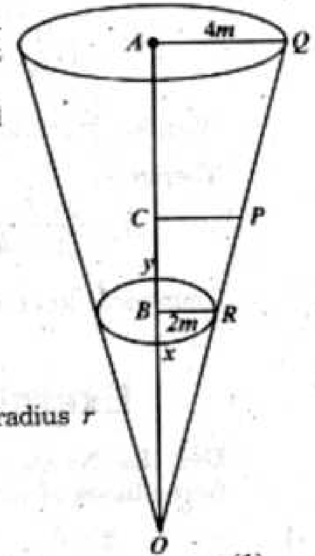
and $\frac{dy}{dt} = 2$ at this instant.

Substituting in (2), we get $\frac{dx}{dt} = \frac{20\sqrt{10} \times 2}{70} = \frac{4\sqrt{10}}{7}$

Thus the string is being let out at the rate of $\frac{4\sqrt{10}}{7}$ m/sec.

19. A water tank is in the shape of frustrum of a cone with height 6 metres and upper and lower radii 4 metres and 2 metres respectively. If water pours into the tank at the rate of 20 m^3/min, how fast is the water level rising when the water is half way up?

Sol. Extend the tank downward so as to form a cone. Let $BO = x$ m so that the height of the cone is $x + 6$.



Suppose that at some instant water level is at C where $BC = y$ and let $CP = r$.

From similar Δ's AOQ and BOR, we get

$$\frac{6 + x}{4} = \frac{x}{2}$$

i.e., $x = 6\ (= BO)$.

From Δ's COP and BOR, we have

$$\frac{y + 6}{r} = \frac{6}{2} \quad i.e., \quad r = \frac{y + 6}{3}$$

Volume V of the frustum with upper radius r and lower radius 2 is

$$V = \frac{1}{3}\pi r^2 (y + 6) - \frac{1}{3}\pi(2)^2 \times 6$$

$$= \frac{\pi}{9 \times 3}(y + 6)^3 - \frac{1}{3}\pi(2)^2 \times 6 \quad (1)$$

Differentiating (1) w.r.t. t, we obtain

$$\frac{dV}{dt} = \frac{\pi}{9}\cdot(y + 6)^2\frac{dy}{dt} \quad (2)$$

It is given that $\frac{dV}{dt} = 20$ and we need find $\frac{dy}{dt}$ when water is half way up *i.e.*, when $y = 3$. Therefore, from (2), we have

$$20 = \frac{\pi}{9}\cdot 9^2\frac{dy}{dt} \quad \text{or} \quad \frac{dy}{dt} = \frac{20}{9\pi}$$

Thus water level is rising at the rate of $\frac{20}{9\pi}$ m/min.

20. A 12 metre long water trough, with vertical cross-sections in the shape of equilateral triangles (one vertex down), is being filled at the rate of 4 m^3/min. How fast is the water level rising at the instant when the depth of the water is $1\frac{1}{2}$ metres?

Sol. When the water is x ft deep, a vertical cross section of water has area $= \frac{1}{2}x . x \csc 60^\circ = \frac{x^2}{\sqrt{3}}$.

Volume of water at this instant is

$$V = 12 \times \frac{x^2}{\sqrt{3}}$$

$$\frac{dV}{dt} = \left(\frac{24}{\sqrt{3}}\right) x \frac{dx}{dt}$$

We need $\frac{dx}{dt}$ at the instant when $x = \frac{3}{2}$ m and $\frac{dV}{dt} = 4$.

Therefore,

$$4 = \frac{24}{\sqrt{3}} \frac{3}{2} \frac{dx}{dt} \quad \text{or} \quad 4 = 12\sqrt{3}\, \frac{dx}{dt} \quad \text{or} \quad \frac{dx}{dt} = \frac{1}{3\sqrt{3}}$$

Thus water level is rising at the rate of $\frac{1}{3\sqrt{3}}$ m/min.

Exercise Set 2.4 (Page 80)

Use the Newton-Raphson method to approximate, up to four places of decimal, a root of each of the following:

1. $x^3 - 3x - 3 = 0$ with $x_0 = 2$

Sol. Let $f(x) = x^3 - 3x - 3$.

Then $f'(x) = 3x^2 - 3 = 3(x^2 - 1)$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we have, $x_{n+1} = x_n - \frac{x_n^3 - 3x_n - 3}{3x_n^2 - 3} = \frac{3x_n^3 - 3x_n - x_n^3 + 3x_n + 3}{3x_n^2 - 3}$

or $$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 - 3} \qquad (1)$$

For $n = 0$, the equation (1) becomes $x_1 = \frac{2x_0^3 + 3}{2x_0^2 - 3}$

Now putting $x_0 = 2$ in the above equation, we get

$$x_1 = \frac{2(2)^3 + 3}{3(2)^2 - 3} = \frac{2 \times 8 + 3}{3 \times 4 - 3} = \frac{16 + 3}{12 - 3} = \frac{19}{9}$$

For $n = 1$, the equation (1) becomes

$$x_2 = \frac{2x_1^3 + 3}{3x_1^2 - 3} = \frac{2\left(\frac{19}{9}\right)^3 + 3}{3\left(\frac{19}{9}\right)^2 - 3} \qquad \left(\because x_1 = \frac{19}{9}\right)$$

$$\approx \frac{2(9.4088) + 3}{3(4.4568) - 3} = \frac{18.8176 + 3}{13.3704 - 3} = \frac{21.8176}{10.3704}$$

$$\approx 2.1038$$

For n_2, the equation (1) becomes

$$x_3 = \frac{2x_2^3 + 3}{3(x_2)^2 - 3}$$

$$\approx \frac{2(2.1038)^3 + 3}{3(2.1038)^2 - 3} \qquad [\because x_2 \approx 2.1038]$$

$$\approx \frac{2(9.3114) + 3}{3(4.4260) - 3} = \frac{18.6228 + 3}{13.2780 - 3} = \frac{21.6228}{10.2780}$$

$$\approx 2.1038$$

Thus the required root is approximately equal to 2.1038.

2. $x^3 - 5x + 3 = 0$ with $x_0 = 0$

Sol. If $f(x) = x^3 - 5x + 3$, then $f'(x) = 3x^2 - 5$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we have $x_{n+1} = x_n - \frac{x_n^3 - 5x_n + 3}{3x_n^2 - 5} = \frac{3x_n^3 - 5x_n - x_n^3 + 5x_n - 3}{3x_n^2 - 5}$

or $$x_{n+1} = \frac{2x_n^3 - 3}{3x_n^2 - 5} \qquad (1)$$

For $n = 0$, the equation (1) becomes $x_1 = \frac{2x_0^3 - 3}{3x_0^2 - 5}$

Now putting $x_0 = 0$ in the above equation, we get

$$x_1 = \frac{2(0)^3 - 3}{3(0)^2 - 5} = \frac{-3}{-5} = \frac{3}{5} = 0.6$$

For $n = 1$, the equation (1) becomes

$$x_2 = \frac{2x_1^3 - 3}{3x_1^2 - 5} = \frac{2(.6)^3 - 3}{3(.6) - 5} \qquad [\because x_1 = 0.6]$$

$$= \frac{2(0.216) - 3}{3(0.36) - 5} = \frac{0.432 - 3}{1.08 - 5} = \frac{-2.568}{-3.92} = \frac{2.568}{3.92} \approx 0.6551$$

For $n = 2$, the equation (1) becomes

$$x_3 = \frac{2x_2^3 - 3}{3x_2^2 - 5} \approx \frac{2(0.6551)^3 - 3}{3(0.6551)^2 - 5} \qquad (\because\ x_2 \approx 0.6551)$$

$$\approx \frac{2(0.28114) - 3}{3(0.42916) - 5} = \frac{0.56228 - 3}{1.28748 - 5} = \frac{-2.43772}{-3.71252} \approx 0.6566$$

For $n = 3$, the equation (1) becomes

$$x_4 = \frac{2x_3^3 - 3}{3x_3^2 - 5} \approx \frac{2(0.6566)^3 - 3}{3(0.6566)^2 - 5} \qquad (\because\ x_3 \approx 0.6566)$$

$$\approx \frac{2(0.28308) - 3}{3(0.43112) - 5} = \frac{0.56616 - 3}{1.29336 - 5} = \frac{-2.43384}{-3.70664} \approx 0.6566$$

Thus the required root is approximately equal to 0.6566.

3. $e^{-x} - \sin x = 0$ with $x_0 = 0.5$

Sol. If $f(x) = e^{-x} - \sin x$, then

$f'(x) = -e^{-x} - \cos x = -(e^{-x} + \cos x)$

Using Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

we have $x_{n+1} = x_n - \dfrac{e^{-x_n} - \sin x_n}{-(e^{-x_n} + \cos x_n)}$

or $$x_{n+1} = x_n + \frac{e^{-x_n} - \sin x_n}{e^{-x_n} + \cos x_n} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 + \frac{e^{-x_0} - \sin x_0}{e^{-x_0} + \cos x_0}$$

Putting $x_0 = 0.5$, in the above equation, we get

$$x_1 = 0.5 + \frac{e^{-.5} - \sin(0.5)}{e^{-.5} + \cos(0.5)} \quad (e \approx 2.71828 \text{ and } 0.5 \text{ rad} \approx 28.6478^\circ)$$

$$\approx 0.5 + \frac{0.6065 - 0.4794}{0.6065 + 0.8776} = 0.5 + \frac{0.1271}{1.4841}$$

$$\approx 0.5 + 0.0856 = 0.5856$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 + \frac{e^{-x_1} - \sin(x_1)}{e^{-x_1} + \cos x_1}$$

$$\approx 0.5856 + \frac{e^{-0.5856} - \sin(0.5856)}{e^{-0.5856} + \cos(0.5856)} \qquad (\because\ x_1 \approx 0.5856)$$

$$\approx 0.5856 + \frac{0.5568 - 0.5527}{0.5568 + 0.8334} = 0.5856 + \frac{0.0041}{1.3902}$$

$$\approx 0.5856 + 0.0029 = 0.5885$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 + \frac{e^{-x_2} - \sin x_2}{e^{-x_2} + \cos x_2}$$

$$\approx 0.5885 + \frac{e^{-0.5885} - \sin(0.5885)}{e^{-0.5885} + \cos(0.5885)} \qquad (\because\ x_2 \approx 0.5885)$$

$$\approx 0.5885 + \frac{0.55515 - 0.55511}{0.55515 + 0.83177} = 0.5885 + \frac{0.00004}{1.38692}$$

$$\approx 0.5885 + 0.000029 = 0.588529 \approxeq 0.5885$$

Thus the required root is approximately equal to 0.5885.

4. $e^x - 3x = 0$ with $x_0 = 0$

Sol. If $f(x) = e^x - 3x$, then $f'(x) = e^x - 3$

Using Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

we have

$$x_{n+1} = x_n - \frac{e^{x_n} - 3x_n}{e^{x_n} - 3} = \frac{x_n e^{x_n} - 3x_n - e^{x_n} + 3x_n}{e^{x_n} - 3} = \frac{x_n e^{x_n} - e^{x_n}}{e^{x_n} - 3}$$

or $$x_{n+1} = \frac{e^{x_n}(x_n - 1)}{e^{x_n} - 3} \qquad (1)$$

For $n = 0$, the equation (1) becomes $x_1 = \dfrac{e^{x_0}(x_0 - 1)}{e^{x_0} - 3}$

Putting $x_0 = 0$, we get

$$x_1 = \frac{e^0(0 - 1)}{e^0 - 3} = \frac{1 \times (-1)}{1 - 3} = \frac{-1}{-2} = \frac{1}{2} = 0.5$$

For $n = 1$, the equation (1) becomes

$$x_2 = \frac{e^{x_1}(x_1 - 1)}{e^{x_1} - 3} = \frac{e^{0.5}(0.5 - 1)}{e^{0.5} - 3} \qquad (\because\ x_1 = 0.5)$$

$$\approx \frac{1.6487(0.5 - 1)}{1.6487 - 3} = \frac{1.6487 \times (-0.5)}{-1.3513} \approx \frac{-0.8244}{-1.3513}$$

$$\approx 0.6101$$

For $n = 2$, the equation (1) becomes

$$x_3 = \frac{e^{x_2}(x_2 - 1)}{e^{x_2} - 3} \approx \frac{e^{0.6101}(0.6101 - 1)}{e^{0.6101} - 3} \approx \frac{1.8406(-0.3899)}{1.8406 - 3}$$

$$= \frac{-0.7176}{-1.1594} \approx 0.6189$$

For $n = 3$, the equation (1) becomes

$$x_4 = \frac{e^{x_3}(x_3 - 1)}{e^{x_3} - 3} \approx \frac{e^{0.6189}(0.6189 - 1)}{e^{0.6189} - 3} \approx \frac{1.8569 \times (-0.3811)}{1.8569 - 3}$$

$$\approx \frac{-0.7077}{-1.1431} \approx 0.6191$$

For $n = 4$, the equation (1) becomes

$$x_5 = \frac{e^{x_4}(x_4 - 1)}{e^{x_4} - 3} \approx \frac{e^{0.6191}(0.6191 - 1)}{e^{0.6191} - 3} \qquad (\because\ x_4 \approx 0.6191)$$

$$\approx \frac{1.8573 \times (0.3809)}{1.8573 - 3} \approx \frac{-0.7074}{-1.1427} \approx 0.6191$$

Thus the required root is approximately equal to 0.6191.

5. $4 \sin x = e^x$ in the interval $]0, 0.5[$

Sol. If $f(x) = 4 \sin x - e^x$, then $f'(x) = 4 \cos x - e^x$

$$f(0) = 4 \sin 0 - e^0 = 4 \times 0 - 1 = -1$$

$$f(0.5) = 4 \sin (0.5) - e^{0.5} \quad (0.5 \text{ rad} \approx 28.6478^\circ)$$

$$\approx 4(0.4794) - 1.6487 = 1.9176 - 1.6487$$

$$\approx 0.2689$$

As $f(0)$ and $f(0.5)$ are of opposite signs, so there is a root between 0 and 0.5.

Let us take $x_0 = \frac{\pi}{12} \approx 0.2618.$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we get, $$x_{n+1} = x_n - \frac{4 \sin x_n - e^{x_n}}{4 \cos x_n - e^{x_n}} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 - \frac{4 \sin x_0 - e^{x_0}}{4 \cos x_0 - e^{x_0}} = \frac{\pi}{12} - \frac{4 \sin \frac{\pi}{12} - e^{\pi/12}}{4 \cos \frac{\pi}{12} - e^{\pi/12}}$$

$$\approx 0.2618 - \frac{4(0.2588) - 1.2993}{4(0.9659) - 1.2993}$$

$$\approx 0.2618 - \frac{1.0352 - 1.2993}{3.8636 - 1.2993} = 0.2618 - \frac{-0.2641}{2.5643}$$

$$\approx 0.2618 + 0.10299 = 0.36479 \approx 0.3648$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 - \frac{4 \sin x_1 - e^{x_1}}{4 \cos x_1 - e^{x_1}}$$

$$\approx 0.3648 - \frac{4\sin (0.3648) - e^{0.3648}}{4\cos (0.3648) - e^{0.3648}} \quad (0.3648 \text{ rad} \approx 20.9015^\circ)$$

$$\approx 0.3648 - \frac{4(0.3568) - 1.4402}{4(0.9342) - 1.4402}$$

$$\approx 0.3648 - \frac{1.4272 - 1.4402}{3.7368 - 1.4402} = 0.3648 - \frac{-0.0130}{2.2966}$$

$$\approx 0.3648 + 0.00566 = 0.37046 \approx 0.3705$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 - \frac{4 \sin x_2 - e^{x_2}}{4 \cos x_2 - e^{x_2}}$$

$$\approx 0.3705 - \frac{4 \sin (0.3705) - e^{0.3705}}{4 \cos (0.3705) - e^{0.3705}} \quad (\because\ x_2 \approx 0.3705)$$

$$\approx 0.3705 - \frac{4(0.3621) - 1.4485}{4(0.9321) - 1.4485} \quad (0.3705 \text{ rad.} \approx 21.2280^\circ)$$

$$\approx 0.3705 - \frac{1.4484 - 1.4485}{3.7286 - 1.4485} = 0.3705 - \frac{-0.0001}{2.2801}$$

$$\approx 0.3705 + 0.00004386 = 0.37054386$$

$$\approx 0.3705$$

Thus the required root is approximately equal to .3705.

6. $\sin x = 1 - x$ with $x_0 = 0$

Sol. If $f(x) = \sin x - 1 + x$, then

$$f'(x) = \cos x + 1$$

Using Newton-Raphson formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$,

we get, $$x_{n+1} = x_n - \frac{\sin x_n - 1 + x_n}{\cos x_n + 1} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 - \frac{\sin x_0 - 1 + x_0}{\cos x_0 + 1}$$

Putting $x = 0$ in the above equation, we have

$$x_1 = 0 - \frac{\sin 0 - 1 + 0}{\cos 0 + 1} = -\frac{0-1}{1+1} = \frac{1}{2} = 0.5$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 - \frac{\sin x_1 - 1 + x_1}{\cos x_1 + 1}$$

$$= 0.5 - \frac{\sin(0.5) - 1 + 0.5}{\cos(0.5) + 1} \quad (\because x_1 = 0.5)$$

$$\approx 0.5 - \frac{0.4794 - 1 + 0.5}{.8776 + 1} = 0.5 - \frac{-0.0206}{1.8776} = 0.5 + \frac{0.0206}{1.8776}$$

$$\approx 0.5 + 0.01097 = 0.51097 \approx 0.5110$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 - \frac{\sin x_2 - 1 + x_2}{\cos x_2 + 1}$$

$$\approx 0.5110 - \frac{\sin(0.5110) - 1 + 0.5110}{\cos(0.5110) + 1}$$

$$\approx 0.5110 - \frac{0.4890 - 1 + 0.5110}{0.8723 + 1} = 0.5110 - \frac{1-1}{1.8723} \approx 0.5110$$

Thus the required root is approximately equal to 0.5110.

Alternative Method:

Let $f(x) = 1 - x - \sin x$, then

$f'(x) = -1 - \cos x = -(1 + \cos x)$

Using Newton-Raphson formula $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$,

we get $$x_{n+1} = x_n - \frac{1 - x_n - \sin x_n}{-(1 + \cos x_n)}$$

$$= x_n + \frac{1 - x_n - \sin x_n}{1 + \cos x_n} \qquad (1)$$

For $n = 0$, the equation (1) becomes

$$x_1 = x_0 + \frac{1 - x_0 - \sin x_0}{1 + \cos x_0}$$

Putting $x_0 = 0$ in the above equation, we have

$$x_1 = 0 + \frac{1 - 0 - 0}{1 + 1} = \frac{1}{2} = 0.5$$

For $n = 1$, the equation (1) becomes

$$x_2 = x_1 + \frac{1 - x_1 - \sin x_1}{1 + \cos x_1}$$

$$= 0.5 + \frac{1 - 0.5 - \sin(0.5)}{1 + \cos(0.5)} \quad (0.5 \text{ rad.} \approx 28.6478^\circ)$$

$$\approx 0.5 + \frac{1 - 0.5 - 0.4794}{1 + 0.8776} = 0.5 + \frac{0.0206}{1.8776}$$

$$\approx 0.5 + 0.01097 = 51097 \approx 0.5110$$

For $n = 2$, the equation (1) becomes

$$x_3 = x_2 + \frac{1 - x_2 - \sin x_2}{1 + \cos x_2}$$

$$\approx 0.5110 + \frac{1 - 0.5110 - \sin(0.5110)}{1 + \cos(0.5110)} \quad (0.511 \text{ rad.} \approx 29.2781^\circ)$$

$$\approx 0.5110 + \frac{1 - 0.5110 - 0.4890}{1 + 0.8723} = 0.5110 + \frac{1-1}{1.8723}$$

$$\approx 0.5110$$

Thus the required root is approximately equal to 0.5110.

Exercise Set 2.5 (Page 86)

In each of Problems 1 – 4, find the nth order derivative:

1. $\dfrac{x}{x^2 - a^2}$

Sol. $\dfrac{x}{x^2 - a^2} = \dfrac{x}{(x-a)(x+a)}$

Let $\dfrac{x}{(x-a)(x+a)} = \dfrac{A}{x-a} + \dfrac{B}{x+a}$

or $x = A(x+a) + B(x-a)$

Putting $x = a$ and $x = -a$ in the above equation successively, we get

$$a = 2Aa \quad \text{or} \quad A = \frac{1}{2}$$

and $$-a = -2Ba \quad \text{or} \quad B = \frac{1}{2}$$

Thus $\dfrac{x}{x^2 - a^2} = \dfrac{1}{2}\left[\dfrac{1}{x-a} + \dfrac{1}{x+a}\right]$

Therefore,

$$\frac{d^n}{dx^n}\left(\frac{x}{x^2 - a^2}\right) = \frac{1}{2}\left[\frac{(-1)^n n!}{(x-a)^{n+1}} + \frac{(-1)^n n!}{(x+a)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[\frac{1}{(x-a)^{n+1}} + \frac{1}{(x+a)^{n+1}}\right]$$

2. $\dfrac{x^4}{(x-1)(x-2)}$

Sol. $\dfrac{x^4}{(x-1)(x-2)} = x^2 + 3x + 7 + \dfrac{15x-14}{(x-1)(x-2)}$

Now, $\dfrac{15x-14}{(x-1)(x-2)} = \dfrac{-1}{x-1} + \dfrac{16}{x-2}$, on resolving into partial fractions

Hence $y = \dfrac{x^4}{(x-1)(x-2)}$

$= x^2 + 3x + 7 + \dfrac{-1}{x-1} + \dfrac{16}{x-2}$

For $n > 2$, we have

$$y^{(n)} = 16\frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x-1)^{n+1}} = (-1)^n n!\left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$$

3. $e^{ax}\sin(bx+c)$

Sol. Differentiating w.r.t. x, we get

$$y' = e^{ax}\,.\,a\sin(bx+c) + e^{ax}b\cos(bx+c)$$
$$= e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$$

Now, putting $a = r\cos\theta$, $b = r\sin\theta$, we get

$$r = \sqrt{a^2+b^2}, \quad \text{and} \quad \theta = \arctan\frac{b}{a} \qquad (1)$$

Then $y' = e^{ax}[r\sin(bx+c)\cos\theta + r\cos(bx+c)\sin\theta]$

$= re^{ax}[\sin(bx+c)\cos\theta + \cos(bx+c)\sin\theta]$

$= re^{ax}\sin(bx+c+\theta)$

Similarly, we can get

$y'' = r^2e^{ax}\sin(bx+c+2\theta)$ and generalizing

$y^{(n)} = r^ne^{ax}\sin(bx+c+n\theta)$

$$= (a^2+b^2)^{n/2}e^{ax}\sin\left[bx+c+n\arctan\frac{b}{a}\right]$$

on putting the values of r and θ from (1).

4. $e^{ax}\cos^2 x\sin x$

Sol. $y = e^{ax}\cos^2 x\sin x$

$$= \frac{1}{2}e^{ax}[(1+\cos 2x)\sin x]$$
$$= \frac{1}{2}e^{ax}e^{ax}\sin x + \frac{1}{2}e^{ax}(\cos 2x\sin x)$$
$$= \frac{1}{2}e^{ax}\sin x + \frac{1}{4}e^{ax}[\sin 3x - \sin x]$$
$$= \frac{1}{4}e^{ax}\sin x + \frac{1}{4}e^{ax}\sin 3x$$

Hence $y^{(n)} = \dfrac{1}{4}(a^2+1)^{n/2}e^{ax}\sin\left(x + n\arctan\dfrac{1}{a}\right) +$

$$\frac{1}{4}(a^2+9)^{n/2}\sin\left(3x + n\arctan\frac{3}{a}\right)$$

5. If $x^y = e^{x-y}$, find $\dfrac{d^ny}{dx^n}$

Sol. $x^y = e^{x-y}$

or $y\ln x = x - y$ or $y(1+\ln x) = x$ (1)

Let $u = y$ and $v = 1 + \ln x$

Then $v^{(n)} = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$.

Differentiating (1) n times by Leibniz' Theorem, we have

$$y^{(n)}(1+\ln x) + n\,y^{(n-1)}\times\frac{1}{x} + \frac{n(n-1)}{2}y^{(n-2)}\left(\frac{-1}{x^2}\right)$$
$$+ \dots + ny'\frac{(-1)^{n-2}(n-2)!}{x^{n-1}} + y\frac{(-1)^{n-1}(n-1)!}{x^n} = 0,$$

which gives $y^{(n)}$ in terms of $y^{(n-1)}, y^{(n-2)}, \dots, y'$.

6. If $f(x) = \ln(1+\sqrt{1-x})$, prove that

$4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$

Sol. $f(x) = \ln(1+\sqrt{1-x})$

$$f'(x) = \frac{1}{1+\sqrt{1-x}}\left(\frac{-1}{2\sqrt{1-x}}\right)$$

or $$2\sqrt{1-x}\,f'(x) = \frac{-1}{1+\sqrt{1-x}} = \frac{-(1-\sqrt{1-x})}{(1+\sqrt{1-x})(1-\sqrt{1-x})}$$
$$= \frac{-(1-\sqrt{1-x})}{x}$$

or $2x\sqrt{1-x}\,f'(x) = -1 + \sqrt{1-x}$

Differentiating the above equation, we get

$$2x\sqrt{1-x}\,f''(x) + 2f'(x)\left[1\sqrt{1-x} - \frac{x}{2\sqrt{1-x}}\right] = -\frac{1}{2\sqrt{1-x}}$$

or $4x(1-x)f''(x) + 2f'(x)[2(1-x)-x] = -1$

or $4x(1-x)f''(x) + 2(2-3x)f'(x) + 1 = 0$

Differentiating n times by Leibniz' Theorem, we get

$$2x(1-x)f^{(n+2)}(x) + \{(2n+2)-(4n+3)x\}f^{(n+1)}(x)$$
$$-n(2n+1)f^{(n)}(x) = 0. \text{ (Verify!)}$$

7. If $y = \arctan x$, show that
$(1 + x^2) y^2 + 2xy = 0$
Hence find the value of $y^{(n)}$ when $x = 0$

Sol. Differentiating $y = \arctan x$, w.r.t. x, we get

$$y' = \frac{1}{1+x^2} \qquad y'(0) = 1$$

or $(1 + x^2) y' = 1$

Differentiating the above equation, we have

$$(1 + x^2) y'' + 2xy' = 0, y''(0) = 0$$

Differentiating n times by Leibniz' Theorem, we get

$$(1+x^2)y^{(n+2)}+n(2x)y^{(n+1)}+\frac{n(n-1)}{2!}(2)y^{(n)}+2xy^{(n+1)}+n(2)y^{(n)}=0$$

or $(1+x^2) y^{(n+2)} + 2(n + 1) y^{(n+1)} x + (n^2+n) y^{(n)} = 0$

or $(1+x^2) y^{(n+2)} + 2(n + 1) xy^{(n+1)} + n(n+1) y^{(n)} = 0$

Putting $x = 0, y^{(n+2)}(0) = -n(n + 1) y^{(n)}(0)$ (1)

Putting $n = 2, y^{(4)}(0) = -2.3\, y''(0) = (0)$

Putting $n = 4, y^{(6)}(0) = -6.7 . y^{(4)}(0) = (0)$ and so on.

Generalizing, we have $y^{(2n)}(0) = (0)$

Putting $n = 1$ in (1), we get $y'''(0) = -1.2 . y'(0)$

$$= -2.1 = (-1)^1 2!$$

Putting $n = 3, y^{(5)}(0) = -3.4\, y'''(0) = -3.4\,(-2.1)$

$$= (-1)^2 4!$$

Putting $n = 5, y^{(7)}(0) = -5.6 . y^{(5)}(0)$

$$= -6.5 . (-1)^2 4!$$

$$= (-1)^3 . 6 . 5 . 4! = (-1)^3 6!$$

and so on, generalizing, we get

$$y^{(2n+1)}(0) = (-1)^n (2n)!$$

8. If $y = \sin (a \arcsin x)$, prove that
$(1 - x^2) y^{(n+2)} = (2n + 1) xy^{(n+1)} + (n^2 - a^2) y^{(n)}$

Sol. $y = \sin (a \arcsin x)$

$$y' = \cos (a \arcsin x) \times \frac{a}{\sqrt{1-x^2}}$$

$$(1 - x^2)^{1/2} y' = a \cos (a \arcsin x)$$

Squaring both sides, we have

$$(1 - x^2) y'^2 = a^2 \cos^2 (a \arcsin x) = a^2 [1 - \sin^2 (a \arcsin x)]$$

or $(1 - x^2) y'^2 = a^2(1 - y^2)$

Differentiating again, we have

$$(1 - x^2)\, 2y' y'' - 2x\, y'^2 = a^2(-2yy')$$

$$(1 - x^2) y'' - xy' = -a^2y$$

$$(1 - x^2) y'' - xy' + a^2y = 0$$

Differentiating n times by Leibniz' Theorem, we get

$$(1 - x^2) y^{(n+2)} + n . (-2x) y^{(n+1)} + \frac{n(n-1)}{2} (-2) y^{(n)}$$
$$- xy^{(n+1)} - n . y^{(n)} + a^2 y^{(n)} = 0$$

or $(1-x^2) y^{(n+2)} - (2n+1) xy^{(n+1)} - (n^2-a^2) y(n) = 0$

or $(1 - x^2) y^{(n+2)} = (2n + 1) xy^{(n+1)} + (n^2 - a^2) y^{(n)}$

9. If $y = e^{m \arcsin x}$ show that
$(1 - x^2) y^{(n+2)} - (2n + 1) xy^{(n+1)} - (n^2 + m^2) y^{(n)} = 0$
Find the value of $y^{(n)}$ at $x = 0$

Sol. $$y' = e^{m \arcsin x} \frac{m}{\sqrt{1-x^2}} = \frac{my}{\sqrt{1-x^2}}$$

or $y'^2 (1 - x^2) = m^2 y^2$ (1)

Differentiating (1), we have

$$2y' y''(1 - x^2) + y'^2 (-2x) = m^2\, 2y\, y'$$

or $(1 - x^2) y'' - xy' = m^2y.$

Using Leibniz' Theorem, we get from the above equation

$$(1 - x^2) y^{(n+2)} + n(-2x) y^{(n+1)} + \frac{n(n-1)}{2!} (-2)y^{(n)} - xy^{(n+1)} - ny^{(n)} = m^2 y^{(n)}$$

or $(1 - x^2) y^{(n+2)} - (2n + 1) xy^{(n+1)} - (n^2 + m^2) y^{(n)} = 0$

when $x = 0$, $y(0) = 1$, $y'(0) = m, y''(0) = m^2$

$$y^{(n+2)}(0) = (n^2 + m^2) y^{(n)}(0)$$

$$y'''(0) = (1 + m^2) y'(0) = (1^2 + m^2)\, m$$

$$y^{(4)}(0) = (2^2 + m^2) y''(0) = (2^2 + m^2)m^2$$

$$y^{(5)}(0) = (3^2 + m^2) y'''(0) = (3^2 + m^2)(1^2 + m^2)\, m$$

$$y^{(6)}(0) = (4^2 + m^2) y^{(4)}(0) = (4^2 + m^2)(2^2 + m^2)\, m^2$$

Thus it follows that when n is even

$$y^{(n)}(0) = [(n-2)^2+m^2]\, [(n-4)^2+m] \ldots [4^2+m^2]\, [2^2+m^2]\, m^2$$

When n is odd

$$y^{(n)}(0) = [(n-2)^2+m^2]\,[(n-4)^2+m^2] \ldots [1^2+m^2]\, m$$

10. Find $y^{(n)}(0)$ if

(i) $y = \ln[x + \sqrt{1+x^2}]$

(ii) $y = (x + \sqrt{1+x^2})^m$

Sol.

(i) $y = \ln(x + \sqrt{1+x^2})$

Differentiating, we have

$$y' = \frac{1}{x+\sqrt{1+x^2}}\left[1 + \frac{x}{\sqrt{1+x^2}}\right] = \frac{1}{(x+\sqrt{1+x^2})} \cdot \frac{(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}$$

$$= \frac{1}{\sqrt{1+x^2}} \qquad (1)$$

or $(1+x^2)\,y'^2 = 1$

Differentiating, we get

$$(1+x^2)\,.\,2y'\,y'' + 2xy'2 = 0$$

or $(1+x^2)\,y'' + xy' = 0 \qquad (2)$

Differentiating n times by Leibniz' Theorem, we obtain

$$(1+x^2)\,y^{(n+2)} + n.2xy^{(n+1)} + \frac{n(n-1)}{2}(2)\,y^{(n)} + xy^{(n+1)} + n.1.\,y^{(n)} = 0$$

or $(1+x^2)\,y^{(n+2)} + (2n+1)\,xy^{(n+1)} + n^2 y^{(n)} = 0 \qquad (3)$

Putting $x = 0$ in (1), (2), (3), we get

$$y'(0) = 1\,,\; y''(0) = 0$$

$$y^{(n+2)}(0) = -n^2\,y^{(n)}(0)$$

Putting $n = 1\,,\; y'''(0) = -(1)^2\,y'(0) = -1 = -(1)^2$

$n = 2\,,\; y^{(4)}(0) = -(2)^2\,y''(0) = -(2)^2\,(0) = 0$

$n = 3\,,\; y^{(5)}(0) = -(3)^2\,y'''(0) = -(3)^2\,(-(1)^2)$

$n = 4\,,\; y^{(6)}(0) = -(4)^2\,y'(0) = 0$ and so and so on

Thus $y^{(2m)}(0) = 0.$

Putting $n = 5,$

$$y^{(7)}(0) = -5^2\,y^{(5)}(0)$$
$$= -5^2\,.\,-3^2\,.\,-(1)^2$$
$$= (-1)^3\,.\,1^2\,.\,3^2\,.\,5^2$$

Similarly, $y^{(9)}(0) = (-1)^4\,.\,1^2\,.\,3^2\,.\,5^2\,.\,7^2$ and generalizing

$$y^{(2n+1)}(0) = (-1)^n\,.\,1^2\,.\,3^2\,.\,5^2 \ldots (2n-1)^2$$

(ii) $y = (x + \sqrt{1+x^2})^m \qquad (1)$

$$y' = m(x+\sqrt{1+x^2})^{m-1}\left[1 + \frac{x}{\sqrt{1+x^2}}\right]$$

$$= \frac{m(x+\sqrt{1+x^2})^m}{\sqrt{1+x^2}} = \frac{m\,y}{\sqrt{1+x^2}}$$

(2)

or $y'^2 = \dfrac{m\,y^2}{1+x^2}$ or $(1+x^2)\,y'^2 = m^2\,y^2$

Differentiating it again, we have

$$(1+x^2)\,2y'\,y'' + 2xy'^2 = m^2\,.\,2yy'$$

$$(1+x^2)\,y'' + xy' - m^2\,y = 0 \qquad (3)$$

Differentiating it n times by Leibniz' Theorem, we get

$$(1+x^2)y^{(n+2)} + n.2xy^{(n+1)} + \frac{n(n-1)}{2\,!}2y^{(n)} + xy^{(n+1)} + ny^{(n)} - m^2 y^{(n)} = 0$$

or $(1+x^2)y^{(n+2)} + (2^n+1)xy^{(n+1)} + (n^2-m^2)y^{(n)} = 0 \qquad (4)$

Putting $x = 0$ successively in (1), (2), (3), (4), we obtain

$$y(0) = 1,\, y'(0) = m,\, y''(0) = m^2$$

and $y^{(n+2)}(0) = (m^2 - n^2)\,y^{(n)}(0) \qquad (5)$

Putting $n = 1, 3, 5, \ldots$ in (5), we get

$$y'''(0) = (m^2 - 1^2)\,y'(0) = (m^2 - 1^2)\,m$$

$$y^{(5)}(0) = (m^2 - 3^2)\,y'''(0) = (m^2 - 3^2)\,(m^2 - 1^2).m$$

$$y^{(7)}(0) = (m^2 - 5^2)\,(m^2 - 3^2)\,(m^2 - 1^2)\,m$$

and generalizing, we find

$$y^{(2n+1)}(0) = \{m^2-(2n-1)^2\}\,\{m^2-(2n-3)^2\} \ldots (m^2-5^2)\,(m^2-3^2)(m^2-1^2)m$$

Similarly, putting $n = 2, 4, 6, \ldots$ in (5) and generalizing, we have

$$y^{(2n)}(0) = \{m^2-(2n-2)^2\}\,\{m^2-(2n-4)^2\} \ldots \{m^2-4^2\}\,\{m^2-2^2\}\,m^2$$

11. If $y = a\cos(\ln x) + b\sin(\ln x)$, prove that

$$x^2\,y^{(n+2)} + (2n+1)\,xy^{(n+1)} + (n^2+1)\,y^{(n)} = 0$$

Sol. $y = a\cos(\ln x) + b\sin(\ln x)$

$$y' = -a\cos(\ln x)\,\frac{1}{x} + b\cos(\ln x)\,.\,\frac{1}{x}$$

or $xy' = b\cos(\ln x) - a\sin(\ln x)$

Differentiating it again, we get

$$xy'' + y' = -\frac{b\sin(\ln x)}{x} - \frac{a\cos(\ln x)}{x}$$

or $x^2y'' + xy' = -[a\cos(\ln x) + b\sin(\ln x)] = -y$

or $x^2y'' + xy' + y = 0$ (1)

Differentiating (1) n times by Leibniz' Theorem, we have

$$x^2y^{(n+2)} + n.2xy^{(n+1)} + \frac{n(n-1)}{2!}2.y^{(n)} + xy^{(n+1)} + ny^{(n)} + y^{(n)} = 0$$

or $x^2y^{(n+2)} + (2n+1)xy^{(n+1)} + (n^2+1)y^{(n)} = 0$

12. Show that

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{(-1)^n n!}{x^{n+1}}\left[\ln x - 1 - \frac{1}{2} - \frac{1}{3} \cdots - \frac{1}{n}\right]$$

Sol. Let $u = \frac{1}{x}, v = \ln x$

$$u^{(n)} = \frac{(-1)^n n!}{x^{n+1}}, v' = \frac{1}{x}; v'' = \frac{-1}{x^2}$$

$$u^{(n-1)} = \frac{(-1)^{n-1}(n-1)!}{x^n}, v^{(n)} = \frac{(-1)^{n-1}(n-1)!}{x^n}$$

Now, by Leibniz' Theorem, we have

$$\frac{d^n}{dx^n}\left(\frac{\ln x}{x}\right) = \frac{d^n}{dx^n}(u\ v)$$

$$= {}^nC_0 u^{(n)} v + {}^nC_1 u^{(n-1)} v' + \dots + {}^nC_{n-1} u' v^{(n-1)} + {}^nC_n u v^{(n)}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\ln x + n\frac{(-1)^{n-1}(n-1)!}{x^n}\left(\frac{1}{x}\right) + \frac{n(n-1)}{2}.$$

$$\frac{(-1)^{n-2}(n-2)!}{x^{n-1}}\left(-\frac{1}{x^2}\right) + \dots + \frac{1}{x}.\frac{(-1)^{n-1}(n-1)!}{x^n}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\ln x - \frac{(-1)^n n!}{x^{n+1}} - \frac{1}{2}\frac{(-1)^n n!}{x^{n+1}} - \dots + \frac{-1}{n}\frac{(-1)^n n!}{x^{n+1}}$$

$$= \frac{(-1)^n n!}{x^{n+1}}\left(\ln x - 1 - \frac{1}{2} - \dots - \frac{1}{n}\right)$$

Exercise Set 2.6 (Page 97)

Evaluate the given limits (Problems 1 – 5):

1. $\lim\limits_{(x,y)\to(0,0)} \frac{5-x^2}{4+x+y}$

Sol. $\lim\limits_{(x,y)\to(0,0)} \frac{5-x^2}{4+x+y} = \frac{\lim\limits_{(x,y)\to(0,0)}(5-x^2)}{\lim\limits_{(x,y)\to(0,0)}(4+x+y)} = \frac{5-0}{4+0+0} = \frac{5}{4}$

2. $\lim\limits_{(x,y)\to(1,-1)} e^{-xy}$

Sol. $\lim\limits_{(x,y)\to(1,-1)} e^{-xy} = e^{-(1)(-1)} = e$

3. $\lim\limits_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy}$

Sol. $\lim\limits_{(x,y)\to(0,0)} \frac{e^{xy}\sin xy}{xy} = \lim\limits_{(x,y)\to(0,0)} e^{xy}.\lim\limits_{(x,y)\to(0,0)} \frac{\sin xy}{xy} = 1$

4. $\lim\limits_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

Sol. Setting $x = r\cos\theta, y = r\sin\theta$, we have the given limit

$$= \lim_{r\to 0}\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}, \text{ since } r^2 = x^2+y^2$$

and $(x,y)\to(0,0)$ implies $r\to 0$

$$= \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

5. $\lim\limits_{(x,y)\to(2,2)} \frac{x^3-2xy+3x^2-2y}{x^2y+4y^2-6x^2+24y}$

Sol. $\lim\limits_{(x,y)\to(2,2)} \frac{x^3-2xy+3x^2-2y}{x^2y+4y^2-6x^2+24y}$

$$= \frac{\lim\limits_{(x,y)\to(2,2)}(x^3-2xy+3x^2-2y)}{\lim\limits_{(x,y)\to(2,2)}(x^2y+4y^2-6x^2+24y)}$$

$$= \frac{8-8+12-4}{8+16-24+48} = \frac{8}{48} = \frac{1}{6}$$

In each of Problems 6 – 10, show that the given limit doe not exist:

6. $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$

Sol. The limit does not exist if we show that $f(x, y) = \dfrac{xy}{x^2+y^2}$ approaches to different values as $(x, y) \to (0, 0)$ along different paths. Let (x, y) approach $(0, 0)$ along the line $y = mx$.

Then $f(x, y) = \dfrac{mx^2}{x^2(1+m^2)} = \dfrac{m}{1+m^2}$, since $x \neq 0$

Thus $f(x, y)$ will have different values for different values of m. Hence the given limit does not exist.

7. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$

Sol. Let $x = r\cos\theta, y = r\sin\theta$. Then $r^2 = x^2 + y^2$.

As $(x, y) \to (0, 0)$, $r \to 0$. Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}$$

$$= \lim_{r\to 0} (\cos^2\theta - \sin^2\theta) = \cos 2\theta$$

which is independent of r and can have any value. Thus the given limit does not exist.

8. $\lim\limits_{(x,y)\to(0,0)} \dfrac{ax^2+by}{cy^2+dx}$

Sol. Let $(x, y) \to (0, 0)$ along the line $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{ax^2+by}{cy^2+dx} = \lim_{(x,y)\to(0,0)} \frac{ax^2+bmx}{cm^2x^2+dx}$$

$$= \lim_{(x,y)\to(0,0)} \frac{ax+bm}{cm^2x+d}$$

$$= \frac{bm}{d}, \text{ which has different values for}$$

different m. Hence the given limit does not exist.

9. $\lim\limits_{(x,y)\to(0,0)} \dfrac{(x^2+y^2)^2}{x^4+y^4}$

Sol. Let $(x, y) \to (0, 0)$ along the line $ax + by = 0$. Then

$$\lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)^2}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{\left(x^2+\frac{a^2}{b^2}x^2\right)^2}{x^4+\frac{a^4}{b^4}x^4}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(a^2+b^2)^2}{a^4+b^4} = \frac{(a^2+b^2)^2}{a^4+b^4}$$

which depends on the values of a and b.

Hence the limit does not exist.

10. $\lim\limits_{(x,y)\to(0,0)} \left(\dfrac{xy^2}{x^2+y^4}\right)$

Sol. Let $(x, y) \to (0, 0)$ along $y = mx$. Then

$$f(x, y) = \frac{xy^2}{x^2+y^4} = \frac{m^2x^3}{x^2+m^4x^4} = \frac{m^2x}{1+m^4x^2}$$

$$\frac{m^2x}{1+m^4x^2} \to 0 \text{ as } x \to 0$$

Thus along every straight line through the origin,

$f(x, y) \to 0$ as $(x, y) \to (0, 0)$

Next, let $(x, y) \to (0, 0)$ along $x = y^2$. Then

$$f(x, y) = \frac{y^4}{y^4+y^4} = \frac{1}{2}, \text{ so that } f(x, y) \to \frac{1}{2} \text{ as } (x, y) \to (0, 0)$$

along the parabola $x = y^2$. Hence the limit does not exist.

11. Let $f(x, y) = \begin{cases} \dfrac{xy^2}{x^3+y^3} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Show that f is not continuous at the origin.

Sol. We evaluate $\lim\limits_{(x,y)\to(0,0)} f(x, y)$.

Let $(0, 0)$ be approached along the line $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{x^3+y^3} = \lim_{(x,y)\to(0,0)} \frac{m^2x^3}{x^3+m^3x^3}$$

$$= \frac{m^2}{1+m^3}, \text{ since } x \neq 0,$$

which will have different values for different m. Therefore, $\lim\limits_{(x,y)\to(0,0)} f(x, y)$ does not exist. Hence the function is not continuous at $(0, 0)$.

12. Find a such that the function

$$f(x, y) = \begin{cases} \dfrac{3xy}{\sqrt{x^2+y^2}} & \text{if } (x, y) \neq (0, 0) \\ a & \text{if } (x, y) = (0, 0) \end{cases}$$

is continuous at $(0, 0)$.

Sol. Let $x = r\cos\theta$, $y = r\sin\theta$. Then $r \to 0$ as $(x, y) \to (0, 0)$

$$f(x, y) = \frac{3r^2\sin\theta\cos\theta}{r} = 3r\sin\theta\cos\theta$$

$3r\sin\theta\cos\theta \to 0$ as $r \to 0$.

Thus $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$

The function will be continuous at the origin if $f(0, 0) = 0$ which requires $a = 0$.

13. Let $f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Examine the continuity at (0, 0). Do $f_x(0, 0)$ and $f_y(0, 0)$ exist?

Sol. Let $x = r = \cos\theta, y = r \sin\theta$. Then $r \to 0$ as $(x, y) \to (0, 0)$

$$f(x, y) = \frac{r^3(\cos^3\theta + \sin^3\theta)}{r^2} = r(\cos^3\theta + \sin^3\theta)$$

$$r(\cos^3\theta + \sin^3\theta) \to 0 \text{ as } r \to 0.$$

Therefore, f is continuous at (0, 0).

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{k - 0}{k} = 1$$

Thus $f_x(0, 0)$ and $f_y(0, 0)$ exist.

14. Let $f(x, y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Prove that f is not continuous at (0, 0). Do $f_x(0, 0)$ and $f(0, 0)$ exist?

Sol. Let (0, 0) be approached along the line $y = mx$

$$f(x, y) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2 + m^2} \to 0 \text{ as } x \to 0.$$

But if we approach (0, 0) along $y = x^2$, then

$$f(x, y) = \frac{x^4}{x^4 + x^4} = \frac{1}{2} \text{ and so } \lim_{(x, y) \to (0, 0)} f(x, y) = \frac{1}{2}$$

Thus $\lim\limits_{(x, y) \to (0, 0)} f(x, y)$ does not exist.

Hence f is not continuous at (0, 0).

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0$$

$$f_y(0, 0) = \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = 0$$

Thus $f_x(0, 0) = f_y(0, 0) = 0$.

Find the first order derivatives of the given functions (Problems 15 – 22):

15. $f(x, y) = x^{y^2}$

Sol. $f(x, y) = x^{y^2}$

$$f_x = y^2 . x^{y^2 - 1} \;;\; f_y = 2y . x^{y^2} \ln x, \left[\frac{d}{dx}(a^x) = a^x \ln a\right]$$

16. $f(x, y) = e^{x^2 + y^2}$

Sol. $f(x, y) = e^{x^2 + y^2}$

$$f_x = (e^{x^2 + y^2})\, 2x = 2xe^{x^2 + y^2} \;;\; f_y = (e^{x^2 + y^2})\, 2y = 2y\, e^{x^2 + y^2}$$

17. $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$

Sol. $f(x, y) = \arctan\left(\dfrac{y}{x}\right)$

$$f_x = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = \frac{1}{1 + \left(1 + \frac{y}{x}\right)} \cdot \left(-\frac{y}{x^2}\right)$$

$$= -\frac{1}{x^2\left\{1 + \left(\frac{y}{x}\right)^2\right\}} = -\frac{y}{x^2\left\{\left(\frac{x^2 + y^2}{x^2}\right)\right\}} = -\frac{y}{x^2 + y^2}$$

$$f_y = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = \frac{1}{\frac{x^2 + y^2}{x^2}} \cdot \frac{1}{x} = \frac{x^2}{x^2 + y^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

18. $f(x, y) = \arctan(x + y)$

Sol. Let $z = \arctan(x + y)$

Then $\dfrac{\partial z}{\partial x} = \dfrac{1}{1 + (x + y)^2} . 1 = \dfrac{1}{1 + (x + y)^2}$

and $\dfrac{\partial z}{\partial y} = \dfrac{1}{1 + (x + y)^2} . 1 = \dfrac{1}{1 + (x + y)^2}$

19. $f(x, y) = e^{ax} \sin by$

Sol. Let $z = e^{ax} \sin by$

Then $\dfrac{\partial z}{\partial x} = (e^{ax} . a) \sin by = a\, e^{ax} \sin by$

and $\dfrac{\partial z}{\partial y} = e^{ax} (\cos by)\, b = b\, e^{ax} \cos by$

20. $f(x, y) = \ln(x^2 + y^2)$

Sol. Let $z = \ln(x^2 + y^2)$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}\cdot 2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}\cdot 2y = \frac{2y}{x^2+y^2}$$

21. $f(x, y) = \ln\left[\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right]$

Sol. $f(x, y) = \ln\left\{\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right\}$

$$= \ln\{\sqrt{x^2+y^2}-x\} - \ln\{\sqrt{x^2+y^2}+x\}$$

$$f_x = \frac{1}{\sqrt{x^2+y^2}-x}\cdot\frac{\partial}{\partial x}(\sqrt{x^2+y^2}-x) - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\frac{\partial}{\partial x}(\sqrt{x^2+y^2}+x)$$

$$= \frac{1}{\sqrt{x^2+y^2}-x}\cdot\left\{\frac{1}{2}(x^2+y^2)^{-1/2}(2x)-1\right\} - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\left\{\frac{1}{2}(x^2+y^2)^{-1/2}(2x)+1\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2}-x}\cdot\left\{\frac{x}{(x^2+y^2)^{1/2}}-1\right\} - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\left\{\frac{x}{(x^2+y^2)^{1/2}(2x)}+1\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2}-x}\cdot\left\{\frac{x-(x^2+y^2)^{1/2}}{(x^2+y^2)^{1/2}}\right\} - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\left\{\frac{x+(x^2+y^2)^{1/2}}{(x^2+y^2)^{1/2}}\right\}$$

$$= \frac{-1}{(x^2+y^2)^{1/2}} - \frac{1}{(x^2+y^2)^{1/2}} = \frac{-2}{\sqrt{x^2+y^2}}$$

$$f_y = \frac{1}{\sqrt{x^2+y^2}-x}\cdot\frac{\partial}{\partial y}(\sqrt{x^2+y^2}-x) - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\frac{\partial}{\partial y}(\sqrt{x^2+y^2}+x)$$

$$= \frac{1}{\sqrt{x^2+y^2}-x}\cdot\left\{\frac{1}{2}(x^2+y^2)^{-1/2}.(2y)\right\} - \frac{1}{\sqrt{x^2+y^2}}\cdot\left\{\frac{1}{2}(x^2+y^2)^{-1/2}.(2y)\right\}$$

$$= \frac{1}{\sqrt{x^2+y^2}-x}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}\right) - \frac{1}{\sqrt{x^2+y^2}+x}\cdot\left(\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$= \frac{y}{\sqrt{x^2+y^2}}\cdot\left\{\frac{\sqrt{x^2+y^2}+x-\sqrt{x^2+y^2}+x}{(\sqrt{x^2+y^2}-x)(\sqrt{x^2+y^2}+x)}\right\}$$

$$= \frac{y}{\sqrt{x^2+y^2}}\cdot\left(\frac{2x}{x^2+y^2-x^2}\right)$$

$$= \frac{2xy}{y^2\sqrt{x^2+y^2}} = \frac{2x}{y\sqrt{x^2+y^2}}$$

22. $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$

Sol. $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = (x^2+y^2+z^2)^{-1/2}$

Now, $f_x = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial x}(x^2+y^2+z^2)$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}.(2x) = \frac{-x}{(x^2+y^2+z^2)^{3/2}}$$

$$f_y = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial y}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2y) = \frac{-y}{(x^2+y^2+z^2)^{3/2}}$$

$$f_z = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$= -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2z) = \frac{-z}{(x^2+y^2+z^2)^{3/2}}$$

Find all the four second order partial derivatives (Problems 23 – 26):

23. e^{x-y}

Sol. Let $z = e^{x-y}$ Then

$$\frac{\partial z}{\partial x} = e^{x-y}.1 = e^{x-y}$$

$$\frac{\partial^2 z}{\partial x^2} = e^{x-y}.1 = e^{x-y}$$

and $\frac{\partial z}{\partial y} = e^{x-1}.(-1) = -e^{x-y}$

$$\frac{\partial^2 z}{\partial y^2} = (-e^{x-y})(-1) = e^{x-y}$$

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(-e^{x-y}) = -e^{x-y}.1 = -e^{x-y}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)(e^{x-y}) = e^{x-y} \cdot -1 = -e^{x-y}$$

24. $\dfrac{x+y}{x-y}$

Sol. Let $z = \dfrac{x+y}{x-y}$

$$\frac{\partial z}{\partial x} = \frac{(x-y) \cdot 1 - (x+y) \cdot 1}{(x-y)^2} = \frac{-2y}{(x-y)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = (-2y)(-2) \cdot \frac{1}{(x-y)^3} \cdot 1 = \frac{4y}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(\frac{-2y}{(x-y)^2}\right)$$

$$= \frac{(x-y)^2(-2) - (-2y)(2)(x-y)(-1)}{(x-y)^4}$$

$$= \frac{-2(x-y)[x-y+2y]}{(x-y)^4} = \frac{-2(x+y)}{(x-y)^3}$$

Also $\dfrac{\partial z}{\partial y} = \dfrac{(x-y)(1) - (x+y)(-1)}{(x-y)^2} = \dfrac{2x}{(x-y)^2}$

$$\frac{\partial^2 z}{\partial y^2} = (2x)(x-y)^{-3}(-2)(-1) = \frac{4x}{(x-y)^3}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2x}{(x-y)^2}\right)$$

$$= \frac{(x-y)^2 \cdot 2 - 2x \cdot 2(x-y) \cdot 1}{(x-y)^4}$$

$$= \frac{2(x-y)[x-y-2x]}{(x-y)^4}$$

$$= \frac{2(-x-y)}{(x-y)^3} = \frac{-2(x+y)}{(x-y)^3}$$

25. e^{x^y}

Sol. Let $u = e^{x^y}$. Then

$$\frac{\partial u}{\partial x} = e^{x^y} \times yx^{y-1} = yx^{y-1} e^{x^y}$$

$$\frac{\partial u}{\partial y} = e^{x^y} \frac{\partial}{\partial y}(x^y) = e^{x^y} x^y \ln x$$

$$\frac{\partial^2 u}{\partial x^2} = y(y-1)x^{y-2} e^{x^y} + yx^{y-1} \cdot e^{x^y} \cdot (yx^{y-1})$$

$$= [y(y-1)\, x^{y-2} + y^2 x^{2y-2}]\, e^{x^y}$$

$$\frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial}{\partial y}(y\, x^{y-1} e^{x^y})$$

$$= x^{y-1} e^{x^y} \frac{\partial}{\partial x}(y) + yx^{y-1} \frac{\partial}{\partial x}(e^{x^y}) + ye^{x^y} \frac{\partial}{\partial x}(x^{y-1})$$

$$= x^{y-1} e^{x^y} + yx^{y-1} e^{x^y}(x^y \ln x) + ye^{x^y} x^{y-1} \ln x$$

$$= e^{x^y}(x^{y-1} + yx^{2y-1} \ln x + yx^{y-1} \ln x)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(e^{x^y} x^y \ln x) = e^{x^y} \frac{\partial}{\partial x}(x^y) \ln x + \frac{\partial}{\partial x}(e^{x^y})\, x^y \ln x$$

$$= e^{x^y} x^y (\ln x)^2 + e^{x^y} x^y \ln x\,(x^y \ln x)$$

$$= e^{x^y}[x^y(\ln x)^2 + x^{2y}(\ln x)^2]$$

$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}(e^{x^y} x^y \ln x)$$

$$= e^{x^y} x^y \frac{\partial}{\partial x}(\ln x) + e^{x^y} \ln x \frac{\partial}{\partial x}(x^y) + x^y \ln x \frac{\partial}{\partial x}(e^{x^y})$$

$$= e^{x^y} x^y \times \frac{1}{x} + e^{x^y} \ln x\, yx^{y-1} + x^y \ln x\, e^{x^y} yx^{y-1}$$

$$= e^{x^y} x^{y-1} + e^{x^y} y \ln x\, x^{y-1} + e^{x^y}(x^{2y-1} y \ln x)$$

$$= e^{x^y}[x^{y-1} + yx^{y-1} \ln x + x^{2y-1} y \ln x]$$

26. $\tan(\arctan x + \arctan y)$

Sol. Let $z = \tan(\arctan x + \arctan y)$

$$= \frac{\tan \arctan x + \tan \arctan y}{1 - \tan(\arctan x)\tan(\arctan y)} = \frac{x+y}{1-xy}$$

$$\frac{\partial z}{\partial x} = \frac{(1-xy) \cdot 1 - (x+y)(-y)}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(1-xy) \cdot 1 - (x+y)(-x)}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{\partial^2 z}{\partial x^2} = (1+y^2)[-2(1-xy)^{-3}(-y)] = \frac{2y(1+y^2)}{(1-xy)^3}$$

$$\frac{\partial^2 z}{\partial y^2} = (1+x^2).[-2(1-2y)^{-3} \times (-x)] = \frac{2x(1+x^2)}{(1-xy)^3}$$

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left[\frac{1+y^2}{(1-xy)^2}\right]$$

$$= \frac{(1-xy)^2 \cdot 2y - (1+y^2) \cdot 2(1-xy)(-x)}{(1-xy)^4}$$

$$= \frac{2y(1-xy)^2 + 2x(1+y^2)(1-xy)}{(1-xy)^4}$$

$$= \frac{2y(1-xy) + 2x(1+y^2)}{(1-xy)^3} = \frac{2(x+y)}{(1-xy)^3}$$

Similarly, or by symmetry (students are advised to workout themselves)

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{2(y+x)}{(1-xy)^3}$$

In Problems 27 – 32 verify that $f_{xy} = f_{yx}$:

27. $f(x, y) = e^{xy}\cos(bx + c)$

Sol. $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(e^{xy})\cos(bx+c) + e^{xy}\frac{\partial}{\partial x}\cos(bx+c)$

$$= ye^{xy}\cos(bx+c) - be^{xy}\sin(bx+c)$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \cos(bx+c)\left\{y\frac{\partial}{\partial y}(e^{xy}) + e^{xy}\frac{\partial}{\partial y}(y)\right\}$$

$$-b\sin(bx+c)\frac{\partial}{\partial y}(e^{xy})$$

$$= \cos(bx+c)\{xy\,e^{xy} + e^{xy}\} - b\sin(bx+c)\,xe^{xy}$$

or $\quad f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = xy\,e^{xy}\cos(bx+c) + e^{xy}\cos(bx+c)$

$$-bx\,e^{xy}\sin(bx+c) \qquad (1)$$

$$f_y = \frac{\partial f}{\partial y} = x\,e^{xy}\cos(bx+c)$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = x\,e^{xy}\frac{\partial}{\partial x}\cos(bx+c) + x\cos(bx+c)\frac{\partial}{\partial x}(e^{xy})$$

$$+ e^{xy}\cos(bx+c)\frac{\partial}{\partial x}(x)$$

$$= -bx\,e^{xy}\sin(bx+c) + xy\cos(bx+c)\,e^{xy} + e^{xy}\cos(bx+c)$$

$$= xy\,e^{xy}\cos(bx+c) + e^{xy}\cos(bx+c) - bx\,e^{xy}\sin(bx+c) \qquad (2)$$

From (1) and (2), we see that $f_{xy} = f_{yx}$

28. $f(x, y) = \ln(e^x + e^y)$

Sol. $f(x, y) = \ln(e^x + e^y)$

$$f_y = \frac{\partial f}{\partial y} = \frac{1}{e^x + e^y}\frac{\partial}{\partial x}(e^x + e^y)$$

$$= \frac{1}{e^x + e^y}\cdot e^y = \frac{e^y}{e^x + e^y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{(e^x + e^y)\,.\,0 - e^y(e^x)}{(e^x + e^y)^2} = \frac{e^x e^y}{(e^x + e^y)^2}$$

or $\quad f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = -\frac{e^{x+y}}{(e^x + e^y)^2} \qquad (1)$

Now $f_x = \frac{\partial f}{\partial x} = \frac{1}{(e^x + e^y)}\cdot\frac{\partial}{\partial x}(e^x + e^y) = \frac{1}{e^x + e^y} = \frac{e^x}{e^x + e^y}$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{(e^x + e^y)\,.\,0 - e^x\,.\,e^y}{(e^x + e^y)^2} = -\frac{e^x e^y}{(e^x + e^y)^2}$$

or $\quad f_{xy} = \frac{\partial^2 f}{\partial x\,\partial y} = -\frac{e^{x+y}}{(e^x + e^x)^2} \qquad (2)$

Hence (1) and (2), we get $f_{xy} = f_{yx}$

29. $f(x, y) = \ln\left(\frac{x^2 + y^2}{xy}\right)$

Sol. $f(x, y) = \ln\left(\frac{x^2 + y^2}{xy}\right) = \ln(x^2 + y^2) - \ln(xy)$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 + y^2}\cdot\frac{\partial}{\partial x}(x^2 + y^2) - \frac{1}{xy}\cdot\frac{\partial}{\partial x}(xy)$$

$$= \frac{1}{x^2 + y^2}(2y) - \frac{1}{xy}\,.\,(x) = \frac{2y}{x^2 + y^2} - \frac{x}{xy} = \frac{2y}{x^2 + y^2} - \frac{1}{y}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{(x^2 + y^2)\frac{\partial}{\partial x}(2y) - (2y)\frac{\partial}{\partial x}(x^2 + y^2)}{(x^2 + y^2)^2} - \frac{\partial}{\partial x}\left(\frac{1}{y}\right)$$

$$= \frac{(x^2 + y^2)\,(0) - 2y\,(2x)}{(x^2 + y^2)^2} - 0 = -\frac{4xy}{(x^2 + y^2)^2}$$

or $\quad f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y} = -\frac{4xy}{(x^2 + y^2)^2} \qquad (1)$

$$f_x = \frac{1}{(x^2 + y^2)}\frac{\partial}{\partial x}(x^2 + y^2) - \frac{1}{(xy)}\frac{\partial}{\partial x}(xy)$$

$$= \frac{1}{x^2 + y^2}\cdot(2x) - \frac{1}{(xy)}\cdot y = \frac{2x}{x^2 + y^2} - \frac{1}{x}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{(x^2 + y^2)\frac{\partial}{\partial y}(2x) - (2x)\frac{\partial}{\partial y}(x^2 + y^2)}{(x^2 + y^2)^2} - \frac{\partial}{\partial y}\left(\frac{1}{x}\right)$$

$$= \frac{(x^2 + y^2)0 - (2x)\,(0 + 2y)}{(x^2 + y^2)^2} - 0 = -\frac{4xy}{(x^2 + y^2)^2}$$

or $\quad f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x} = -\frac{4xy}{(x^2 + y^2)^2} \qquad (2)$

From (1) and (2), we get $f_{xy} = f_{yx}$

30. $f(x, y) = x^y + y^x$

Sol. $f(x, y) = x^y + y^x$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^y) + \frac{\partial}{\partial y}(y^x) = x^y\ln x + xy^{x-1}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = x^y\frac{\partial}{\partial x}(\ln x) + \ln x\frac{\partial}{\partial x}(x^y) + x\frac{\partial}{\partial x}(y^{x-1}) + y^{x-1}\frac{\partial}{\partial x}(x)$$

$$= x^{y-1} + yx^{y-1}\ln x + xy^{x-1}\ln y + y^{x-1}$$

or $$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = x^{y-1}(1 + y\ln x) + y^{x-1}(1 + x\ln y) \qquad (1)$$

Now $f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^y) + \frac{\partial}{\partial x}(y^x) = yx^{y-1} + y^x \ln y \,.\, 1$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = y \,.\, \frac{y}{\partial y}(x^{y-1}) + (x^{y-1})\frac{\partial}{\partial y}(y) + y^x \frac{\partial}{\partial y}(\ln y) + \ln y \frac{\partial}{\partial y}(y^x)$$

$$= y \,.\, x^{y-1} \,.\, \ln x + x^{y-1} + y^x \,.\, \frac{1}{y} + \ln y (x \,.\, y^{x-1})$$

$$= yx^{y-1}\ln x + x^{y-1} + y^{x-1} + xy^{x-1}\ln y$$

$$= x^{y-1}(y\ln x + 1) + y^{x-1}(1 + x\ln y) \qquad (2)$$

From (1) and (2), we get $f_{xy} = f_{yx}$

31. $f(x, y) = x\sin xy + y\cos xy$

Sol. $f_x = \sin xy + xy\cos xy - y^2 \sin xy$

$$f_{xy} = x\cos xy + x\cos xy - x^2y\sin xy - 2y\sin xy - xy^2\cos xy$$

$$= (2x - xy^2)\cos xy - (x^2y + 2y)\sin xy \qquad (1)$$

$$f_y = x^2\cos xy + \cos xy - xy\sin xy$$

$$f_{yx} = 2x\cos xy - x^2y\sin xy - y\sin xy - y\sin xy - xy^2\cos xy$$

$$= (2x - xy^2)\cos xy - (x^2y + 2y)\sin xy \qquad (2)$$

It is clear from (1) and (2) that $f_{xy} = f_{yx}$

32. $f(x, y) = \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$

Sol. $f(x, y) = \dfrac{xy}{\sqrt{1 + x^2 + y^2}}$

$$f_y = \frac{\partial f}{\partial y} = x \cdot \frac{\sqrt{1 + x^2 + y^2} \,.\, 1 - y \,.\, \frac{1}{2}\frac{2y}{\sqrt{1 + x^2 + y^2}}}{(\sqrt{1 + x^2 + y^2})^2}$$

$$= \frac{x}{1 + x^2 + y^2}\left[\frac{1 + x^2 + y^2 - y^2}{\sqrt{1 + x^2 + y^2}}\right] = \frac{x(1 + x^2)}{(1 + x^2 + y^2)^{3/2}}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{x(1 + x^2)}{(1 + x^2 + y^2)^{3/2}}\right]$$

$$= \frac{(1 + x^2 + y^2)^{3/2}(1 + 3x^2) - x(1 + x^2) \,.\, {}^3/_2 (1 + x^2 + y^2)^{1/2} \,.\, 2x}{(1 + x^2 + y^2)^3}$$

$$= (1 + x^2 + y^2)^{1/2}\left[\frac{(1 + x^2 + y^2)(1 + 3x^2) - (x + x^3) \,.\, 3x}{(1 + x^2 + y^2)^3}\right]$$

$$= \frac{1}{(1 + x^2 + y^2)^{5/2}}[1 + x^2 + y^2 + 3x^2 + 3x^4 + 3x^2y^2 - 3x^2 - 3x^4]$$

$$= \frac{1 + x^2 + y^2 + 3x^2y^2}{(1 + x^2 + y^2)^{5/2}} \qquad (1)$$

Also

$$f_x = \frac{\partial f}{\partial x} = y \cdot \frac{\sqrt{1 + x^2 + y^2} \,.\, 1 - x \,.\, \frac{1}{2}\frac{2x}{\sqrt{1 + x^2 + y^2}}}{(1 + x^2 + y^2)}$$

$$= \frac{y}{1 + x^2 + y^2} \cdot \frac{1 + y^2}{\sqrt{1 + x^2 + y^2}} = \frac{y(1 + y^2)}{(1 + x^2 + y^2)^{3/2}}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{y(1 + y^2)}{(1 + x^2 + y^2)^{3/2}}\right]$$

$$= \frac{(1 + x^2 + y^2)^{3/2}(1 + 3y^2) - y(1 + y^2)\frac{3}{2}(1 + x^2 + y^2)^{1/2} \,.\, 2y}{(1 + x^2 + y^2)^3}$$

$$= (1 + x^2 + y^2)^{1/2}\frac{[(1 + x^2 + y^2)(1 + 3y^2) - 3y^2(1 + y^2)]}{(1 + x^2 + y^2)^3}$$

$$= \frac{[1 + x^2 + y^2 + 3x^2y^2]}{(1 + x^2 + y^2)^{5/2}} \qquad [1 + x^2 + y^2 + 3x^2y^2] \qquad (2)$$

From (1) and (2), we see that

$$f_{xy} = f_{yx}$$

Show that each of the following functions satisfies Laplace's equation $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$. (Problems 33 – 36):

33. $f(x, y) = \sin x \sinh y$

Sol. $\dfrac{\partial f}{\partial x} = \cos x \sinh y$ and $\dfrac{\partial^2 f}{\partial x^2} = -\sin x \sinh y \qquad (1)$

$\dfrac{\partial f}{\partial y} = \sin x \cosh y$ and $\dfrac{\partial^2 f}{\partial y^2} = \sin x \sinh y \qquad (2)$

Adding (1) and (2), we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -\sin x \sinh y + \sin x \sinh y = 0$$

34. $f(x, y) = e^{-x}\cos y$

Sol. $\dfrac{\partial f}{\partial x} = -e^{-x}\cos y$ and $\dfrac{\partial^2 f}{\partial x^2} = e^{-x}\cos y \qquad (1)$

$\dfrac{\partial f}{\partial y} = e^{-x}(-\sin y) = -e^{-x}\sin y$ and $\dfrac{\partial^2 f}{\partial y^2} = -e^{-x}\cos y \qquad (2)$

Adding (1) and (2), we have

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = e^{-x}\cos y - e^{-x}\cos y = 0$$

35. $f(x, y) = \ln\sqrt{x^2 + y^2}$

Q9

Sol. $f(x, y) = \ln\sqrt{x^2 + y^2} = \frac{1}{2}\ln(x^2 + y^2)$

$$\frac{\partial f}{\partial x} = \frac{1}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2x = \frac{x}{x^2 + y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2 + y^2).1 - x.2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} \qquad (1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}\cdot\frac{1}{x^2 + y^2}\cdot 2y = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{(x^2 + y^2).1 - y.2y}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2} \qquad (2)$$

Adding (1) and (2), we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0$$

36. $f(x, y) = \arctan\left(\frac{2xy}{x^2 - y^2}\right)$

Sol. $f(x, y) = \arctan\left(\frac{2xy}{x^2 - y^2}\right)$

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \frac{4x^2y^2}{(x^2 - y^2)^2}} \times \frac{(x^2 - y^2)\,2y - 2xy\,.\,2x}{(x^2 - y^2)^2}$$

$$= \frac{(x^2 - y^2)^2}{(x^2 + y^2)^2} \times \frac{-2y(x^2 + y^2)}{(x^2 - y^2)^2} = \frac{-2y}{x^2 + y^2}$$

$$\frac{\partial^2 f}{\partial x^2} = (-2y)(-1)(x^2 + y^2)^{-2}\cdot 2x = \frac{4xy}{(x^2 + y^2)^2} \qquad (1)$$

$$\frac{\partial f}{\partial y} = \frac{1}{1 + \frac{4x^2y^2}{(x^2 - y^2)^2}} \times \frac{(x^2 - y^2)\,2x - 2xy\,.\,(-2y)}{(x^2 - y^2)^2}$$

$$= \frac{(x^2 - y^2)^2}{(x^2 + y^2)^2} \times \frac{2x\,(x^2 + y^2)}{(x^2 - y^2)^2} = \frac{2x}{(x^2 + y^2)}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x\,(-1)\,(x^2 + y^2)^{-2} \times 2y\,\frac{-4xy}{(x^2 + y^2)^2} \qquad (2)$$

Adding (1) and (2), we see that

$$\frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

37. $f(x, y) = x^2\arctan\left(\frac{y}{x}\right) - y^2\arctan\left(\frac{x}{y}\right)$, show that $\frac{\partial^2 f}{\partial x\,\partial y} = \frac{x^2 - y^2}{x^2 + y^2}$.

Sol. $f(x, y) = x^2\arctan\frac{y}{x} - y^2\,\text{arc}\tan\frac{x}{y}$

$$\frac{\partial f}{\partial y} = x^2\,.\,\frac{\partial}{\partial y}\left(\arctan\frac{y}{x}\right) + \arctan\frac{y}{x}\cdot\frac{\partial}{\partial y}\cdot(x^2)$$

$$-y^2\,.\,\frac{\partial}{\partial y}\left(\arctan\frac{x}{y}\right) + \text{arc}\tan\frac{x}{y}\cdot\frac{\partial}{\partial y}(y^2)$$

$$= \frac{x^2\left(\frac{1}{x}\right)}{1 + \left(\frac{y}{x}\right)^2} + \left(\arctan\frac{y}{x}\right)\cdot 0 - y^2\,\frac{1\left(-\frac{x}{y^2}\right)}{1 + \left(\frac{x}{y}\right)^2} - 2y\arctan\frac{x}{y}$$

$$= \frac{x}{\frac{x^2 + y^2}{x^2}} + \frac{x}{\frac{x^2 + y^2}{y^2}} - 2y\arctan\frac{x}{y}$$

$$= \frac{x^3}{x^2 + y^2} + \frac{xy^2}{x^2 + y^2} - 2y\arctan\frac{x}{y}$$

$$= \frac{x(x^2 + y^2)}{x^2 + y^2} - 2y\arctan\frac{x}{y} = x - 2y\arctan\frac{x}{y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x - 2y\arctan\frac{x}{y}\right) = 1 - 2y\,\frac{1}{1 + \left(\frac{x}{y}\right)^2} \times \frac{1}{y}$$

or $$\frac{\partial^2 f}{\partial x\partial y} = 1 - \frac{2}{1 + \frac{x^2}{y^2}} = 1 - \frac{2y^2}{x^2 + y^2} = \frac{x^2 + y^2 - 2y^2}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2}$$

38. If $f(x, y) = \frac{x^2 + y^2}{x + y}$, that $(f_x - f_y)^2 = 4(1 - f_x - f_y)$

Sol. $f(x, y) = \frac{x^2 + y^2}{x + y}$

$$f_x = \frac{(x + y)\frac{\partial}{\partial x}(x^2 + y^2) - (x^2 + y^2)\left(\frac{\partial}{\partial x}\right)(x + y)}{(x + y)^2}$$

$$= \frac{(x + y)\,(2x) - (x^2 + y^2)\,.\,1}{(x + y)^2} = \frac{x^2 + 2xy - y^2}{(x + y)^2}$$

$$f_y = \frac{(x + y)\,2y - (x^2 + y^2)\,.\,1}{(x + y)^2} = \frac{2xy + y^2 - x^2}{(x + y)^2}$$

$$f_x - f_y = \frac{x^2 + 2xy - y^2 - (2xy + y^2 - x^2)}{(x + y)^2}$$

$$= \frac{2x^2 - 2y^2}{(x+y)^2} = \frac{2(x^2 - y^2)}{(x+y)^2} = \frac{2(x-y)}{x+y}$$

$$(f_x - f_y)^2 = \frac{4(x-y)^2}{(x+y)^2}$$

Again, $1 - f_x - f_y = 1 - \frac{x^2 + 2xy - y^2}{(x+y)^2} - \frac{2xy + y^2 - x^2}{(x+y)^2}$

$$= 1 - \left[\frac{2xy + 2xy}{(x+y)^2}\right] = 1 - \frac{4xy}{(x+y)^2}$$

$$= \frac{(x+y)^2 - 4xy}{(x+y)^2} = \frac{(x-y)^2}{(x+y)^2}$$

Therefore, $4(1 - f_x - f_y) = \frac{4(x-y)^2}{(x+y)^2}$

Hence $(f_x - f_y)^2 = 4(1 - f_x - f_y)$

39. Show that the function $f(x, y) = \sin xy$ satisfies the differential equation $x^2 f_{xx} - y^2 f_{yy} = 0$.

Sol. $f(x, y) = \sin xy$

$f_x = y\cos(xy)$ and $f_{xx} = -y^2 \sin(xy)$

$f_y = x\cos(xy)$ and $f_{yy} = -x^2 \sin(xy)$

$x^2 f_{xx} - y^2 f_{yy} = -x^2y^2 \sin(xy) + y^2x^2 \sin(xy) = 0$ as required

40. Let $f(x, y) = \begin{cases} x^2 \arctan\left(\frac{y}{x}\right) - y^2 \arctan\left(\frac{x}{y}\right) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

Sol. $f_x(x, y) = x^2 \frac{1}{1 + \frac{y^2}{x^2}} \cdot \left(\frac{-y}{x^2}\right) + 2x \arctan\frac{y}{x} - y^2 \cdot \frac{1}{\left(1 + \frac{x^2}{y^2}\right)} \cdot \frac{1}{y}$

$$= \frac{-x^2y}{x^2 + y^2} + 2x \arctan\frac{y}{x} - \frac{y^3}{x^2 + y^2}$$

$$= \frac{-y(x^2 + y^2)}{x^2 + y^2} + 2x \tan\frac{y}{x}$$

$$= 2x \arctan\frac{y}{x} - y$$

Hence $f_x(0, k) = -k$.

Similarly, $f_y(x, y) = x - 2y \arctan\frac{x}{y}$

and so $f_y(h, 0) = h$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

$$f_{xy}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k} = \lim_{k \to 0} \frac{-k}{k} = -1$$

And $f_{yx}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h} = \lim_{h \to 0} \frac{h}{h} = 1$

Thus $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

41. (*i*) Let $f(x, y, z) = x^3 + 3yz + \sin xyz$. Prove that $f_{xyz} = f_{zxy}$

(*ii*) If $f(x, y, z, w) = \frac{xy}{z + w}$, show that $f_{xyzw} = \frac{2}{(x + w)^3}$.

Sol.

(*i*) $f(x, y, z) = x^3 + 3yz + \sin xyz$

$f_x = 3x^2 + yz\cos xyz$

$f_{xy} = z\cos xyz + yz(xz)(-\sin xyz) = z\cos xyz - xyz^2 \sin xyz$

$f_{xyz} = \cos xyz - xyz\sin xyz - 2xyz(\sin xyz) - x^2y^2z^2(\cos xyz)$ (1)

$f_z = 3y + xy\cos xyz$

$f_{zx} = y\cos xyz - xy^2z\sin xyz$

$f_{zxy} = \cos xyz - xyz\sin xyz - 2xyz\sin xyz - x^2y^2z^2\cos xyz$ (2)

The result follows from (1) and (2).

(*ii*) If $f(x, y, z, w) = \frac{xy}{z + w}$, show that $f_{xyzw} = \frac{2}{(x + w)^3}$.

Sol. $f(x, y, z, w) = \frac{xy}{x + w}$

$f_x = \frac{y}{z + w}$ and $f_{xy} = \frac{1}{z + w}$

$$f_{xyz} = \frac{-1}{(z + w)^2}$$

$f_{xyzw} = \frac{0 - (-1) \cdot 2(z + w)}{(z + w)^4} = \frac{2}{(z + w)^3}$ as required.

In Problems 42 – 45, find $\frac{dy}{dx}$ by using partial derivatives:

42. $y^2 + x^2y + ax^4 = 0$

Sol. Let $f(x, y) = y^2 + x^2y + ax^4 = 0$

Now $f_x = 2xy + 4ax^3$