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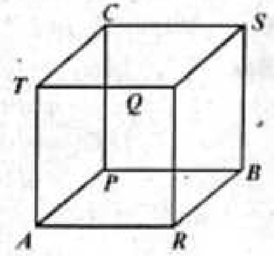
Chapter 8 ANALYTICAL GEOMETRY OF THREE DIMENSIONS

Exercise Set 8.1 (Page 347)

P and Q are the opposite vertices of a parallelepiped having its faces parallel to the coordinate planes. Find the coordinates of the other vertices and sketch the parallelepiped (Problems 1 – 3):

1. $P(-1, 1, 2)$, $Q(2, 3, 5)$

Sol. Complete the parallelepiped with faces parallel to the coordinates planes and PQ as a diagonal. Coordinates of the other vertices are (as in Definition 7.1)



$A(2, 1, 2), B(-1, 3, 2)$
$C(-1, 1, 5), R(2, 3, 2)$
$S(-1, 3, 5), T(2, 1, 5)$

2. $P(2, -1, -3)$, $Q(4, 0, -1)$

Sol. Please refer to the figure in **Problem 1**.
Coordinates of the other vertices are
$A(4, -1, -3), B(2, 0, -3), C(2, -1, -1)$
$R(4, 0, -3), S(2, 0, -1), T(4, -1, -1)$

3. $P(2, 5, -3)$, $Q(-4, 2, 1)$

Sol. Referring to the figure in **Problem 1**, coordinates of the remaining vertices are
$A(-4, 5, -3), B(2, 2, -3), C(2, 5, 1)$
$R(-4, 2, -3), S(2, 2, 1), T(-4, 5, 10)$.

Show that the three given points are the vertices of a right triangle, or the vertices of an isosceles triangle, or both. (Problems 4 – 7):

4. $A(1, 5, 0)$, $B(6, 6, 4)$, $C(0, 9, 5)$

Sol.
$$|AB| = \sqrt{(6-1)^2 + (6-5)^2 + (4-0)^2} = \sqrt{25 + 1 + 16} = \sqrt{42}$$
$$|BC| = \sqrt{(0-6)^2 + (9-6)^2 + (5-4)^2}$$

$$= \sqrt{36 + 9 + 1} = \sqrt{46}$$

and $|AC| = \sqrt{(1-0)^2 + (5-9)^2 + (0-5)^2}$
$= \sqrt{1 + 16 + 25} = \sqrt{42}$

Since $|AB| = |AC|$, therefore the triangle is isosceles.

5. $A(4, 9, 4)$, $B(0, 11, 2)$, $C(1, 0, 1)$

Sol. $|AB| = \sqrt{(0-4)^2 + (11-9)^2 + (2-4)^2}$
$= \sqrt{16 + 4 + 4} = \sqrt{24} = 2\sqrt{6}$
$|BC| = \sqrt{(1-0)^2 + (0-11)^2 + (1-2)^2}$
$= \sqrt{1 + 121 + 1} = \sqrt{123}$
and $|CA| = \sqrt{(1-4)^2 + (0-9)^2 + (1-4)^2}$
$= \sqrt{9 + 81 + 9} = \sqrt{99} = 3\sqrt{11}$

Clearly $|AB|^2 + |CA|^2 = 24 + 99 = 123 = |BC|^2$.
Thus ABC is a right triangle with right angle at A.

6. $A(1, 0, 2)$, $B(4, 3, 2)$, $C(0, 7, 6)$

Sol. $|AB| = \sqrt{(4-1)^2 + (3-0)^2 + (2-2)^2}$
$= \sqrt{9 + 9 + 0} = \sqrt{18} = 3\sqrt{2}$
$|BC| = \sqrt{(0-4)^2 + (7-3)^2 + (6-2)^2}$
$= \sqrt{16 + 16 + 16} = \sqrt{48} = 4\sqrt{3}$
and $|CA| = \sqrt{(1-0)^2 + (0-7)^2 + (2-6)^2}$
$= \sqrt{1 + 49 + 16} = \sqrt{66}$

Clearly $|AB|^2 + |BC|^2 = |AC|^2$
Hence, ABC is a right triangle with right angle at B.

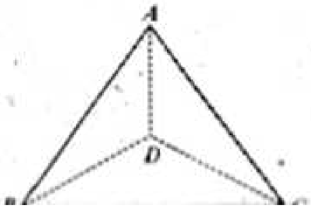
7. $A(2, 3, 4)$, $B(8, -1, 2)$, $C(-4, 1, 0)$

Sol. $|AB| = \sqrt{(8-2)^2 + (-1-3)^2 + (2-4)^2}$
$= \sqrt{36 + 16 + 4} = \sqrt{56}$
$|BC| = \sqrt{(-4-8)^2 + (1+1)^2 + (0-2)^2}$
$= \sqrt{144 + 4 + 4} = \sqrt{152}$
and $|CA| = \sqrt{(2+4)^2 + (3-1)^2 + (4-0)^2}$
$= \sqrt{36 + 4 + 16} = \sqrt{56}$

Since $|AB| = |CA|$, the triangle ABC is an isosceles one.

8. Show that the points $(1, 6, 1)$, $(1, 3, 4)$, $(4, 3, 1)$ and $(0, 2, 0)$ are the vertices of regular tetrahedron.

Sol. Let $A\ (1, 6, 1)$, $B(1, 3, 4)$
$C\ (4, 3, 1)$, $D(0, 2, 0)$
They will form the vertices of a regular tetrahedron provided
$|AB| = |AC| = |BC|$
$= |AD| = |CD| = |BD|$

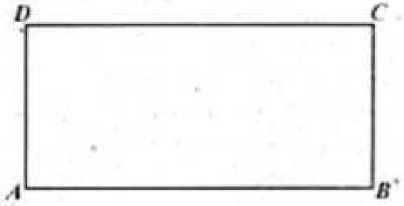


Now, $|AB| = \sqrt{(1-1)^2 + (3-6)^2 + (4-1)^2}$
$= \sqrt{0 + 9 + 9} = \sqrt{18} = 3\sqrt{2}$
$|AC| = \sqrt{(4-1)^2 + (3-6)^2 + (1-1)^2}$
$= \sqrt{9 + 9 + 0} = 3\sqrt{2}$
$|BC| = \sqrt{(4-1)^2 + (3-3)^2 + (1-4)^2}$
$= \sqrt{9 + 0 + 9} = 3\sqrt{2}$

Similarly, $|AD| = |CD| = |BD| = 3\sqrt{2}$
Hence the given points are the vertices of a regular tetrahedron.

9. Show that the points $(3, -1, 3)$, $(1, -1, 2)$, $(2, 1, 0)$ and $(4, 1, 1,)$ are the vertices of a rectangle.

Sol. Let $A = (3, -1, 3)$, $B = (1, -1, 2)$, $C = (2, 1, 0)$ and $D = (4, 1, 1)$
They will form a rectangle if $|AB| = |CD|$ and $|AC| = |BD|$ with $\angle A = 90°$.



Now, $|AB| = \sqrt{(1-3)^2 + (-1+1)^2 + (2-3)^2}$
$= \sqrt{4 + 0 + 1} = \sqrt{5}$
$|CD| = \sqrt{(4-2)^2 + (1-1)^2 + (1-0)^2}$
$= \sqrt{4 + 0 + 1} = \sqrt{5}$

Thus $|AB| = |CD|$.

Again $|AC| = \sqrt{(2-3)^2 + (1+1)^2 + (0-3)^2}$
$= \sqrt{1 + 4 + 9} = \sqrt{14}$
$|BD| = \sqrt{(4-1)^2 + (1+1)^2 + (1-2)^2}$
$= \sqrt{9 + 4 + 1} = \sqrt{14}$

Therefore, $|AC| = |BD|$

To prove that angle A is a right angle, we note that
$$|AD|^2 + |AB|^2 = [(4-3)^2 + (1+1)^2 + (1-3)^2] + 5$$
$$= 1 + 4 + 4 + 5$$
$$= 14 = |BD|^2$$

Hence $\angle A$ is a right angle and so $ABCD$ is a rectangle.

10. Under what conditions on x, y and z is the point $P(x, y, z)$ equidistant from the points $(3, -1, 4)$ and $(-1, 5, 0)$?

Sol. Let the given points be A and B respectively. Then since $|PA| = |PB|$, we have $(PA)^2 = (PB)^2$

or $(x-3)^2 + (y+1)^2 + (z-4)^2 = (x+1)^2 + (y-5)^2 + (z-0)^2$

or $x^2 - 6x + 9 + y^2 + 2y + 1 + z^2 + 16 - 8z$
$$= (x^2 + 2x + 1) + y^2 - 10y + 25 + z^2$$

or $-8x + 12y - 8z = 0$ or $2x - 3y + 2z = 0$

which is the required condition.

11. Find the coordinates of the point dividing the join of $A(-3, 1, 4)$ and $B(5, -1, 6)$ in the ration 3 : 5.

Sol. Coordinates of the point dividing the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m : n$ are

$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$

Hence the required point dividing the join of given points in the ratio 3 : 5 is

$$= \left(\frac{3.5 + 5(-3)}{3+5}, \frac{3(-1) + 5(1)}{3+5}, \frac{3(6) + 5(4)}{3+5}\right) = \left(0, 1, \frac{19}{4}\right)$$

12. Find the ratio in which the yz-plane divides the segment joining the points $A(-2, 4, 7)$ and $B(3, -5, 9)$.

Sol. Suppose the yz-plane divides the join of the given points in the ratio $m : n$. The x-coordinate of the point which divides the join of the given points is

$$x = \frac{m(3) + n(-2)}{m+n}$$

It must be zero as it lies on the yz-plane.

Therefore, $3m - 2n = 0$

or $\frac{m}{n} = \frac{2}{3}$ i.e., $m : n = 2 : 3$

13. Show that the centroid of the triangle whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3$; is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

Sol. Let $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ be the vertices of the triangle.

Mid point of D of BC is $\left(\frac{x_1 + x_2}{3}, \frac{y_1 + y_2}{3}, \frac{z_1 + z_2}{3}\right)$

Coordinates of the point dividing AD in the ratio 2 : 1 are

$$\left(\frac{x_1 + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{y_1 + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2}, \frac{z_1 + 2\left(\frac{z_2 + z_3}{2}\right)}{1+2}\right)$$

i.e., $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$

By symmetry, this point also lies on the two other medians.

Hence coordinates of the centroid of the triangle are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right)$$

14. Find the centroid of the tetrahedron whose vertices are (x_i, y_i, z_i), $i = 1, 2, 3, 4$.

Sol. Let $A = (x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ be the vertices of the tetrahedron.

The centroid E of the face BCD is

$$E = \left(\frac{x_2 + x_3 + x_4}{4}, \frac{y_2 + y_3 + y_4}{4}, \frac{z_2 + z_3 + z_4}{4}\right)$$

Coordinates of the point dividing AE in the ratio of 3 : 1 are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right)$$

which also lie on lines joining vertices to the centroids of the opposite faces. Thus the coordinates of the centroid of the tetrahedron are

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4}\right).$$

Exercise Set 8.2 (Page 354)

In each of Problems 1 – 4, find parametric equations, direction ratios, direction cosines and measures of the direction angles of the straight line through P and Q:

1. $P(1, -2, 0)$, $Q(5, -10, 1)$

Sol. Here $P = (1, -2, 0)$, $Q = (5, -10, 1)$,. The line is parallel to the vector $\mathbf{d} = [5-1, -10+2, 1-0] = [4, -8, 1]$

Parametric equations of the line are

$$x = 1 + 4t, y = -2 + 8t, z = 0 + t$$

Direction ratios of the line PQ are 4, –8, 1.

Also $|PQ| = \sqrt{(4)^2 + (-8)^2 + (1)^2} = \sqrt{16 + 64 + 1} = 9$

Direction cosines of PQ are $\frac{4}{9}, \frac{-8}{9}, \frac{1}{9}$ and the direction angles are

$$\arccos\frac{4}{9}, \arccos\frac{-8}{9} \text{ and } \arccos\frac{1}{9}$$

i.e., 63°37′, 152°44′, 83°37′.

2. $P(6, 5, -3)$, $Q(4, 1, 1)$

Sol. A direction vector of PQ is

$$\boldsymbol{d} = [6-4, 5-1, -3-1] = [2, 4, -4] = 2[1, 2, -2]$$

Equations of the line through $P(6, 5, -3)$ and parallel to $\boldsymbol{d}$ are

$$x = 6 + t, y = 5 + 2t, z = -3 - 2t$$

Direction ratios of PQ are 1, 2, −2 and its direction cosines are

$$\frac{1}{\sqrt{9}}, \frac{2}{\sqrt{9}}, \frac{-2}{\sqrt{9}} \quad i.e., \quad \frac{1}{3}, \frac{2}{3}, \frac{-2}{3}$$

Direction angles of PQ are

$$\alpha = \arccos\frac{1}{3} = 70°32' \quad \beta = \arccos\frac{2}{3} = 48°11'$$

$$\gamma = \arccos\frac{-2}{3} = 131°49'$$

3. $P(1, -5, 1)$, $Q(4, -5, 4)$

Sol. Here direction vector of the line is parallel to

$$\boldsymbol{d} = [3, 0, 3] = 3[1, 0, 1]$$

Equations of PQ are

$$x = 1 + t, y = -5 + 0t, z = 1 + t$$

Direction ratios of PQ are 1, 0, 1

Its direction cosines are $= \frac{1}{\sqrt{1+1}}, \frac{0}{\sqrt{2}}, \frac{3}{\sqrt{2}} = \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$

Direction angles of the line are

$$\arccos\frac{1}{\sqrt{2}}, \arccos 0, \arccos\frac{1}{\sqrt{2}} \quad i.e., \quad \frac{\pi}{4}, \frac{\pi}{2}, \frac{\pi}{4}.$$

4. $P(3, 5, 7)$, $Q(6, -8, 10)$

Sol. Here $\boldsymbol{d} = [6-3, -8-5, 10-7] = [3, -13, 3]$

$$|\boldsymbol{d}| = \sqrt{9 + 169 + 9} = \sqrt{187}$$

Parametric equation of the line through the point P having direction vector $\boldsymbol{d}$ are

$$x = 3 + 3t, y = 5 - 13t, z = 7 + 3t$$

Direction ratios of the line PQ are 3, −13, 3

Direction cosines of PQ are $\frac{3}{\sqrt{187}}, \frac{-13}{\sqrt{187}}, \frac{3}{\sqrt{187}}$

Measures of the direction angles are

$$\alpha = \arccos\left(\frac{3}{\sqrt{187}}\right) = 77°19'38''$$

$$\beta = \arccos\left(\frac{-13}{\sqrt{187}}\right) = 159°19'02''$$

$$\gamma = \arccos\left(\frac{3}{\sqrt{187}}\right) = 77°19'38''$$

5. Find the direction cosines of the coordinates axes.

Sol. Any two points on the x-axis are $(0, 0, 0)$ and $(a, 0, 0)$. Direction ratios of the x-axis are $a - 0, 0, 0 = a, 0, 0$ and hence the direction cosines are

$$= \frac{a}{\sqrt{a^2 + 0, 0}}, \frac{0}{\sqrt{a^2 + 0, 0}}, \frac{0}{\sqrt{a^2 + 0, 0}} = 1, 0, 0$$

Proceeding as above, we find that the direction cosines of the y-axis are 0, 1, 0 and the direction cosines of the z-axis are 0, 0, 1.

6. Prove that if measures of the direction angles of a straight line are α, β and γ, then $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$.

Sol. We know that if α, β, γ are the direction angles of a line, then

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

i.e., $(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = 1$

i.e., $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

7. If measures of two of the direction angles of a straight line are 45° and 60°, find measure of the third direction angle.

Sol. Here $\alpha = 45°, \beta = 60°$. We need to find γ

Now $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

or $\cos^2 45 + \cos^2 60 + \cos^2\gamma = 1$

or $\frac{1}{2} + \frac{1}{4} + \cos^2\gamma = 1$

or $\cos^2\gamma = 1 - \frac{1}{2} = \frac{1}{4}$

i.e., $\cos\gamma = \pm\frac{1}{2}$ or $\gamma = 60°$

8. The direction cosines l, m, n of two straight lines are given by the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$. Find measure of the angle between them.

Sol. We know that if l, m, n are the direction cosines of a line then

$$l^2 + m^2 + n^2 = 1 \qquad (1)$$

Also $l^2 + m^2 - n^2 = 0 \qquad (2)$

Subtracting (2) form (1), we get

$$n^2 = \frac{1}{2} \quad \text{or} \quad n = \frac{1}{\sqrt{2}} \quad \text{or} \quad -\frac{1}{\sqrt{2}} \qquad (3)$$

Again, adding (1) and (2), we have

$$2(l^2 + m^2) = 1 \quad \text{or} \quad l^2 + m^2 = \frac{1}{2} \qquad (4)$$

Now $l + m + n = 0$ (given)

or $\quad l = -\left(m - \frac{1}{\sqrt{2}}\right)$

Putting this values of l into (4), we have

$$\left[-m - \frac{1}{\sqrt{2}}\right]^2 + m^2 = \frac{1}{2}$$

or $\quad m^2 + \frac{2}{\sqrt{2}}m + \frac{1}{2} + m^2 = \frac{1}{2}$

i.e., $\quad 2m\left(m + \frac{1}{\sqrt{2}}\right) = 0 \Rightarrow m = 0$ or $\frac{-1}{\sqrt{2}}$

Thus $l = -\left(m + \frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}$ or 0

Values of l, m, n are $-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}$ or $0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$

Similarly, other values of l, m, n can be found by taking $n = -\frac{1}{\sqrt{2}}$.

Thus direction cosines of the two lines are

$$-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \text{ and } 0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$$

$$\cos\theta = \frac{1}{2} \quad \text{or} \quad \theta = \frac{\pi}{3}$$

is the angle between the lines.

9. The direction cosines l, m, n of two straight lines are given by the equations $l + m + n = 0$ and $2lm + 2ln - mn = 0$. Find measure of the angle between them.

Sol.
$$l + m + n = 0 \qquad (1)$$
$$2lm + 2ln - mn = 0 \qquad (2)$$

From (1), $n = -(l + m)$

Substituting into (2), we have

$$2lm - 2l(l + m) + m(l + m) = 0$$

or $\quad 2lm - 2l^2 - 2lm + lm + m^2 = 0$

or $\quad 2l^2 - lm - m^2 = 0$

or $\quad (l - m)(2l + m) = 0$

$\Rightarrow \quad l = m \quad$ or $\quad 2l = -m$

Solving

$l + m + n = 0$,

and $l - m = 0$, we get

$$\frac{l}{1} = \frac{m}{1} = \frac{n}{-1-1}$$

$l + m + n = 0$

$2l + m = 0$, we have

$$\frac{l}{-1} = \frac{m}{2} = \frac{n}{1-2}$$

Direction cosines of the lines are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}$ and $\frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, \frac{-1}{\sqrt{6}}$

If θ is measure of the angle between the two lines, then

$$\cos\theta = \frac{1(-1) + 1 \times 2 + (-2)(-1)}{\sqrt{6}.\sqrt{6}} = \frac{3}{6} = \frac{1}{2}$$

Thus $\quad \theta = \frac{\pi}{3}$

Find equations of the straight lines L and M in symmetric forms. Determine whether the pairs of lines intersect. Find the point of intersection if it exists. (Problems 10 – 12):

10. L : through $A(2, 1, 3)$, $B = (-1, 2, 4)$

M : through $P(5, 1, -2)$, $Q(0, 4, 3)$

Sol. Parametric equations of the lines are

$$L: \begin{cases} x = 2 - 3t \\ y = 1 + t \\ z = 3 + t \end{cases} \quad \text{and} \quad M: \begin{cases} x = 5 - 5s \\ y = 1 + 3s \\ z = -2 + 5s \end{cases}$$

The lines intersect if the system of equations

$$2 - 3t = 5 - 5s \qquad (1)$$
$$1 + t = 1 + 3s \qquad (2)$$
$$3 + t = -2 + 5s \qquad (3)$$

has a solution.

Solving (1) and (2) we obtain $t = \frac{-9}{14}$ and $s = \frac{3}{14}$. But these values of t and s do not satisfy (3). Hence the lines do not intersect. The lines are also non-parallel.

11. L : $\mathbf{r} = (3\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + t(6\mathbf{i} - 4\mathbf{j} - 3\mathbf{k})$

M : $\mathbf{r} = (5\mathbf{i} + 4\mathbf{j} - 7\mathbf{k}) + ss(14\mathbf{i} - 6\mathbf{j} + 2\mathbf{k})$

Sol. Equations of the lines are

$$L: \begin{cases} x = 3 + 6t \\ y = 2 \cdot 4t \\ z = -1 - 3t \end{cases} \quad \text{and} \quad M: \begin{cases} x = 5 + 14s \\ y = 4 - 6s \\ z = 7 + 2s \end{cases}$$

The lines intersect if the system of equations

$$3 + 6t = 5 + 14s \qquad (1)$$
$$2 - 4t = 4 - 6s \qquad (2)$$
$$-1 - 3t = 7 + 2s \qquad (3)$$

has a solution.

Solving (1) and (2) we obtain $t = -2$, $s = -1$. These values of t and s satisfy (3). Hence the lines intersect. Substituting $t = -2$ into the equation for L, the point of intersection is $(-9, 10, 5)$.

12. L : through $A(2, -1, 0)$ and parallel to $\mathbf{b} = [4, 3, -2]$

M : through $P(-1, 3, 5)$ and parallel to $\mathbf{c} = [1, 7, 3]$.

Sol. Equations of the lines are

$$L: \begin{bmatrix} x = 2 + 4t \\ y = -1 + 3t \\ z = 0 - 2t \end{bmatrix} \quad \text{and} \quad M: \begin{bmatrix} x = -1 + s \\ y = 3 + 7s \\ z = 5 + 3s \end{bmatrix}$$

The lines intersect if the system of equations

$$2 + 4t = -1 + s \qquad (1)$$
$$-1 + 3t = 3 + 7s \qquad (2)$$
$$-2t = 5 + 3s \qquad (3)$$

has a solution.

Solving (1) and (3), we get

$$t = -1 \quad \text{and} \quad s = -1$$

These values of t and s satisfy (2). Hence the lines intersect in a point. Substituting $s = -1$ in the equations for M, the point of intersection is $(-2, -4, 2)$.

Find the distance of the given point P from the given line L. (Problems 13 – 14):

13. $P = (3, -2, 1)$, L : $\begin{cases} x = 1 + t \\ y = 3 - 2t \\ z = -2 + 2t \end{cases}$

Sol. A point on L is $A = (1, 3, -2)$

$$\overrightarrow{AP} = [2, -5, 3]$$

Direction vector of L is $\mathbf{b} = [1, -2, 2]$

Required distance

$$d = \frac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|} = \frac{1}{3}\left|\text{Det}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 3 \\ 1 & -2 & 2 \end{bmatrix}\right|$$

$$= \frac{1}{3}|-4\mathbf{i} - \mathbf{j} + \mathbf{k}| = \frac{\sqrt{18}}{3} = \sqrt{2}.$$

14. $P = (0, -2, 1)$, L : $\dfrac{x-1}{4} = \dfrac{y+3}{-2} = \dfrac{z+1}{5}$.

Sol. A point on L is $A = (1, -3, -1)$

$$\overrightarrow{AP} = [-1, 1, 2,]$$

Direction vector of L is $\mathbf{b} = [4, -2, 5]$

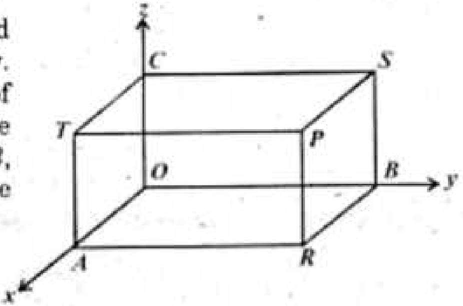
$$d = \frac{|\overrightarrow{AP} \times \mathbf{b}|}{|\mathbf{b}|} = \frac{1}{45}\left|\text{Det}\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ 4 & -2 & 5 \end{bmatrix}\right|$$

$$= \frac{1}{\sqrt{45}}|9\mathbf{i} + 13\mathbf{j} - \mathbf{k}| = \sqrt{\frac{254}{45}}$$

15. If the edges of a rectangular parallelepiped are a, b, c; show that angles between the four diagonals are given by

$$\arccos\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

Sol. Lengths the edges OA, OB and OC are a, b, c respectively. Therefore, the coordinates of the vertices of the parallelopiped (with OA, OB, OC as coordinate axes) are



$O = (0, 0, 0)$.
$A = (a, 0, 0)$, $B = (0, b, 0)$
$C = (0, 0, c)$, $R = (a, b, 0)$
$S = (0, b, c)$, $T = (a, 0, c)$
and $P = (a, b, c)$

The four diagonals are OP, AS, CR and BT.

Direction ratios of OP are a, b, c

direction cosines of OP are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Direction ratios of AS are $-a$, b, c

direction cosines of AS are

$$\frac{-a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Therefore, if α is the angle between OP and AS, then

$$\cos\alpha = \frac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}, \text{ [using } \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2]$$

i.e., $\alpha = \arccos\left(\dfrac{-a^2 + b^2 + c^2}{a^2 + b^2 + c^2}\right)$.

Similarly, the angles between the remaining diagonals are found to be one of the angles

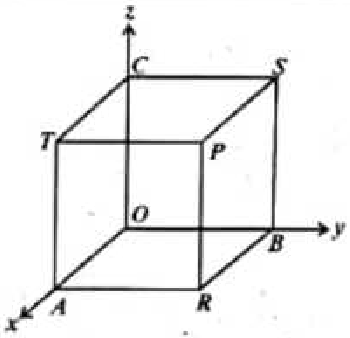
$$\arccos\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2 + b^2 + c^2}\right).$$

16. A straight line makes angles of measures α, β, γ, δ with the four diagonals of a cube. Prove that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Sol. Suppose the edge of the cube is a. Then as in the previous question the vertices of the cube are $O = (0, 0, 0)$

$A = (a, 0, 0), B = (0, a, 0)$
$C = (0, 0, a), R = (a, a, 0)$
$S = (0, a, a), T = (a, 0, a),$
and $P = (a, a, a)$
The diagonals are OP, AS, BT and CR



The direction cosines of OP are

$$\frac{a}{\sqrt{a^2+a^2+a^2}}, \frac{b}{\sqrt{a^2+a^2+a^2}}, \frac{c}{\sqrt{a^2+a^2+a^2}} = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad (1)$$

Similarly, direction cosines of AS are

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad (2)$$

direction cosines of BT are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \qquad (3)$$

and of CR are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$ (4)

Let the direction cosines of the line which makes angles α, β, γ and δ with $OP, AS,, BT$ and CR respectively be l, m, n. Then

$$\cos\alpha = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \qquad (5)$$

$$\cos\beta = \frac{-l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \qquad (6)$$

$$\cos\gamma = \frac{l}{\sqrt{3}} + \frac{-m}{\sqrt{3}} + \frac{n}{\sqrt{3}} \qquad (7)$$

and $$\cos\delta = \frac{l}{\sqrt{3}} + \frac{m}{\sqrt{3}} + \frac{-n}{\sqrt{3}} \qquad (8)$$

Squaring (5), (6), (7) and (8) and adding the results, we get

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}(l^2 + m^2 + n^2)$$

$$= \frac{4}{3}, \text{ since } l, m, n, \text{ are the direction cosines.}$$

17. Find equation of the straight line passing through the point $P(0, -3, 2)$ and parallel to the straight line joining the points $A(3, 4, 7)$ and $B(2, 7, 5)$.

Sol. The line joining the points $A(3, 4, 7)$ and $B(2, 7, 5)$ has direction ratios

$$2-3, 7-4, 5-7 \qquad \text{or} \qquad -1, 3, -2$$

As the required line is parallel to AB, it must have the direction ratios $-1, 3, -2$

Therefore, the line through the point $(0, -3, 2)$ with the direction ratios $-1, 3, -1$ is

$$\frac{x-0}{-1} = \frac{y+3}{3} = \frac{z+1}{-2} \quad \text{or} \quad \frac{x}{1} = -y + \frac{3}{3} = \frac{z+2}{2}$$

18. Find equations of the straight line passing through the point $P(2, 0, -2)$ and perpendicular to each of the straight lines

$$\frac{x-3}{2} = \frac{y}{2} = \frac{z+1}{2} \quad \text{and} \quad \frac{x}{3} = \frac{y+1}{-1} = \frac{z+2}{2}.$$

Sol. Direction ratios of the given lines are 2, 2, 2 and 3 - 1, 2. If the required line has direction ratios c_1, c_2, c_3 then by the condition of perpendicularity, we have

$$2c_1 + 2c_2 + 2c_3 = 0 \qquad \text{and} \qquad 3c_1 - c_2 + 2c_3 = 0$$

Therefore, $$\frac{c_1}{4+2} = \frac{c_2}{6-4} = \frac{c_3}{-2-6}$$

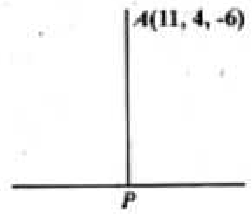
i.e., $\frac{c_1}{6} = \frac{c_2}{2} = \frac{c_3}{-8}$ *i.e.*, $c_1 : c_2 : c_3 = 3 : 1 : -4$

The required line through the point $P(2, 0, -2)$ with d.r.'s c_1, c_2, c_3 is $\frac{x-2}{3} = \frac{y}{1} = \frac{z+2}{-4}$

Find equations of the straight line through the given point A and intersecting at right angle the given straight line (Problems 19 – 20):

19. $A = (11, 4, -6)$ and $x = 4 - t, y = 7 + 2t, z = -1 + t$

Sol. Suppose the perpendicular from A meets the given line in P. The given line, in the symmetric form, has equations.



$$\frac{x-4}{-1} = \frac{y-7}{2} = \frac{z+1}{1} = t$$

We shall find t such that $(x, y, z) = (4 - t, 7 + 2t, -1 + t)$ are the coordinates of P.

Direction ratios of AP are
$4 - t - 11, 7 + 2t - 4, -1 + t + 6$
i.e., $-t - 7, 2t + 3, t + 5$ (1)

Direction ratios of the given line are $-1, 2, 1$

Since AP is perpendicular to the given line, so

$$(-1)(-t-7) + 2(2t+3) + 1(t+5) = 0$$

i.e., $t + 4t + t + 7 + 6 + 5 = 0$ or $t = -3$

Hence direction ratios of AP are $-4, -3, 2$

Equations of the required line AP are

$$\frac{x-11}{-4} = \frac{y-4}{-3} = \frac{z+6}{2}$$

20. $A = (5, -4, 4)$ and $\dfrac{x}{-1} = \dfrac{y-1}{1} = \dfrac{z}{-2} = t$ (1)

Sol. Let $P(x, y, z)$ be the point on the given line such that AP is perpendicular to (1). Direction ratios of the line AP are

$$x-5, y+4, z-4.$$

Since AP is perpendicular to (1), we have

$$-(x-5) + (y+4) - 2(z-4) = 0$$

or $\quad -(-t-5) + (t+1+4) - 2(-2t-4) = 0$ giving $t = -3$

Thus $P = (3, -2, 6)$

Direction ratios of AP are $1, -1, -1$.

Equations of the required line AP are

$$x = 5 + t, y = -4 - t, z = 4 - t$$

21. Find the length of the perpendicular from the point $P(x_1, y_1, z_1)$ to the straight line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}, \text{ where } l^2 + m^2 + n^2 = 1.$$

Sol. Let $P = (x_1, y_1, z_1)$. The point $A = (\alpha, \beta, \gamma)$ lies on the given line and its direction vector is $\boldsymbol{b} = [l, m, n]$. The required perpendicular distance from P to the given line is

$$d = \frac{|\overrightarrow{AP} \times \boldsymbol{b}|}{|\boldsymbol{b}|} = |[x_1 - \alpha, y_1 - \beta, z_1 - \gamma] \times [l, m, n]|$$

$$= \left| \text{Det} \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 - \alpha & y_1 - \beta & z_1 - \gamma \\ l & m & n \end{pmatrix} \right| = \left(\Sigma \begin{vmatrix} x_1 - \alpha & y_1 - \beta \\ l & m \end{vmatrix}^2 \right)^{1/2}$$

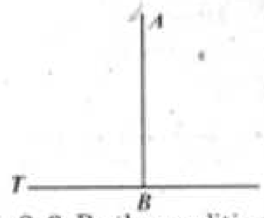
22. Find equations of the perpendicular from the point $P(1, 6, 3)$ to the straight line

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}.$$

Also obtain its length and coordinates of the foot of the perpendicular.

Sol. Let $A = (1, 6, 3)$ and P be the foot of the perpendicular on

$$\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3} = t$$



If $\quad x = t, y = 1 + 2t, z = 3t + 2$ are coordinates of P, then direction ratios of AP are

$$t - 1, 1 + 2t - 6, 3t + 2 - 3$$

or $\quad t - 1,\ 3t - 5,\ 3t - 5,\ 3t - 1$

Also direction ratios of the given line are 1, 2, 3. By the condition of perpendicularity, we have

$$(t-1)(1) + (2t-5)^2 + 3(3t-1) = 0$$

or $\quad 14t = 14$ *i.e.*, $t = 1$

Hence coordinates of the foot P of the perpendicular are $(1, 3, 5)$

Length of the perpendicular $= |AP|$

$$= \sqrt{(1-1)^2 + (3-6)^2 + (5-3)^2}$$

$$= \sqrt{0 + 9 + 4} = \sqrt{13}$$

Equation of the perpendicular AP are

$$\frac{x-1}{1-1} = \frac{y-6}{3-6} = \frac{z-3}{5-3} \text{ or } \frac{x-1}{0} = \frac{y-6}{-3} = \frac{z-3}{2}$$

23. Find a necessary and sufficient condition that the points $P(x_1, y_1, y_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ are collinear.

Sol. Equations of a line through $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ are

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} = t$$

Thus $x = x_1 + (x_2 - x_1)t, y = y_1 + (y_2 - y_1)t$; $z = z_1 + (z_2 - z_1)t$

If $\quad (x_3, y_3, z_3)$ lies on this line; then

$$x_3 = x_1 + (x_2 - x_1)t = (1-t)x_1 + t x_2$$

$$y_3 = y_1 + (y_2 - y_1)t = (1-t)y_1 + t y_2$$

$$z_3 = z_1 + (z_2 - z_1)t = (1-t)z_1 + z_2$$

Thus we have

$$(1-t)x_1 + t x_2 - x_3 = 0$$

$$(1-t)y_1 + t y_2 - y_3 = 0$$

$$(1-t)z_1 + t z_2 - z_3 = 0$$

Eliminating t from the last three equations, we get

$$\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} x_1 & y_2 & z_3 \\ x_1 & y_2 & z_3 \\ x_1 & y_2 & z_3 \end{vmatrix} = 0$$

which is a necessary condition for the three points to be collinear. Working algebra backward, we find that this condition is also sufficient.

24. If l_1, m_1, n_1; l_2, m_2, n_2, and l_3, m_3, n_3 are direction cosines of three mutually perpendicular lines, prove that the line whose direction-cosines are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ makes congruent angles with them.

Sol. We are given that

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \qquad (1)$$

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \qquad (2)$$

$$l_3 l_1 + m_3 m_1 + n_3 n_1 = 0 \qquad (3)$$

Also $l_i^2 + m_i^2 + n_i^2 = 1 \;,\; i = 1, 2, 3 \qquad (4)$

Suppose θ_1 is measure of the angle between the lines with d.c.'s l_1, m_1, n_1 and $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$. Then

$$\cos\theta_1 = \frac{l_1(l_1 + l_2 + l_3) + m_1(m_1 + m_2 + m_3) + n_1(n_1 + n_2 + n_3)}{\sqrt{(l_i^2+m_i^2+n_i^2)}\sqrt{(l_1+l_2+l_3)^2+(m_1+m_2+m_3)^2+(n_1+n_2+n_3)^2}}$$

$$= \frac{l_1^2+m_1^2+n_1^2+(l_1l_2+m_1m_2+n_1n_2)+(l_1l_3+m_1m_3+n_1n_3)}{\Sigma(l_1+l_2+l_3)^2}$$

$$= \frac{1}{\sqrt{\Sigma(l_1 + l_2 + l_3)}}$$

Similarly, measure of the angle θ_2, between lines with d.c.'s l_2, m_2, n_2 and $l_1 + l_2 + l_3, m_1 + m_2 + m_3, n_1 + n_2 + n_3$ is given by

$$\cos\theta_2 = \frac{1}{\sqrt{\Sigma(l_1 + l_2 + l_3)}}$$

In a similar manner, $\cos\theta_3 = \dfrac{1}{\sqrt{\Sigma(l_1 + l_2 + l_3)}}$

Thus $\theta_1 = \theta_2 = \theta_3$

25. A variable line in two adjacent positions has direction cosines l, m, n and $l + l\delta, m + m\delta, n + \delta n$. Show that measure of the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Sol. Since l, m, n and $l + \delta l, m + \delta m, n + \delta n$ are direction cosines of a line in two adjacent positions, we have

$$l^2 + m^2 + n^2 = 1 \qquad (1)$$

and $(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1 \qquad (2)$

Using (1), we get from (2)

$$2(\delta l + m\,\delta m + n\,\delta n) + (\delta l)^2 + (\delta m)^2 + (\delta m)^2 = 0 \qquad (3)$$

The small angle $\delta\theta$ between the two adjacent positions of the line is given by

$$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$

$$= 1 + l\,\delta l + m\,\delta m + n\,\delta n$$

or $\quad -1 + \cos\delta\theta = -2\sin^2\left(\dfrac{\delta\theta}{2}\right) = l\,\delta l + m\,\delta m + n\,\delta n$

i.e., $\quad 4\sin^2\left(\dfrac{\delta\theta}{2}\right) = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$, (using (3)) $\qquad (4)$

Since $\delta\theta$ is small, so is $\dfrac{\delta\theta}{2}$

Therefore, $\dfrac{\sin\left(\dfrac{\delta\theta}{2}\right)}{\delta\theta/2} \longrightarrow 1$

i.e., $\quad \sin\left(\dfrac{\delta\theta}{2}\right) \longrightarrow \dfrac{\delta\theta}{2}$

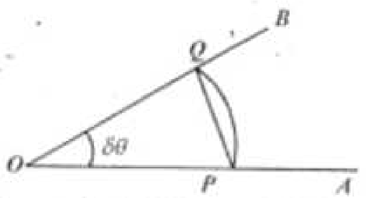
or $\quad \sin^2\left(\dfrac{\delta\theta}{2}\right) \longrightarrow \dfrac{(\delta\theta)^2}{4} \qquad (5)$

Hence from (4) and (5), we have

$$(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$$

Alternative Method:

Let OA, OB be the two adjacent positions of the line. Let PQ be the arc of the circle with centre O and radius 1. Then coordinates of P and Q are (l, m, n) and $(l + \delta l, m + \delta m, n + \delta n)$ respectively.



Now $\delta\theta$ = arc $PQ \longrightarrow$ chord PQ, since $\delta\theta$ is small

$$= [(\delta l)^2 + (\delta m)^2 + (\delta n)^2]^{1/2}$$

Therefore, $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Exercise Set 8.3 (Page 363)

Find an equation of the plane through the three given points (Problems 1 – 3):

1. (2, 1, 1), (6, 3, 1), (–2, 1, 2).

Sol. ***1st Method:***

Suppose equation of the required plane is

$$ax + by + cz + 1 = 0 \quad (1)$$

Since it passes through the points (2, 1, 1); (6, 3, 1); (–2, 1, 2) they satisfy (1). Therefore, we have

$$2a + b + c + 1 = 0$$
$$6a + 3b + c + 1 = 0$$
$$-2a + b + 2c + 1 = 0$$

Solving these equations, we get

$$a = -\frac{1}{4}, b = \frac{2}{4}, c = -1$$

Equation of the required plane is

$$-\frac{1}{4}x + \frac{2}{4}y - z + 1 = 0 \quad \text{or} \quad x - 2y + 4z - 4 = 0$$

2nd Method:

Let $P = (2, 1, 1), Q = (6, 3, 1), R = (-2, 1, 2)$

$$\overrightarrow{PQ} = [4, 2, 0], \overrightarrow{PR} = [-4, 0, 1]$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 4 & 2 & 0 \\ -4 & 0 & 1 \end{vmatrix} = 2i - 4j + 8k \quad (1)$$

(1) is a normal vector to the required plane. Equation of the plane through $R(-2, 1, 2)$ having (1) as a normal vector is

$$2(x + 2) - 4(y - 1) + 8(z - 2) = 0$$

or $x - 2y + 4z - 4 = 0$

2. (1, –1, 2), (–3, –2, 6), (6, 0, 1).

Sol. Let an equation of the plane be $ax + by + cz = 1$

Since it passes through the points (1, –1, 2), (–3, –2, 6), (6, 0, 1), we have

$$a - b + 2c = 1$$
$$-3a - 2b + 6c = 1$$
$$6a + 0b + c = 1$$

Solving these equations, we have

$$a = \frac{3}{17}, b = \frac{-16}{17}, c = \frac{-1}{17}$$

Hence an equation of the plane is $3x - 16y - z = 17$

3. (–1, 1, 1), (5, –8, –2), (4, 1, 0).

Sol. Let an equation of the plane be $ax + by + cz = 1$

Since it passes through the points (–1, 1, 1), (5, –8, –2) and (4, 1, 0), we have

$$-a + b + c = 1$$
$$5a - 8b - 2c = 1$$
$$4a + b = 1$$

Solving these equation, we obtain $a = \frac{1}{3}, b = \frac{1}{3}, c = \frac{5}{3}$

Hence an equation of the plane is $x - y + 5z = 3$

4. Find equations of the planes bisecting the angles between the planes

$$3x + 2y - 6z + 1 = 0 \text{ and } 2x + y - 5 = 0.$$

Sol. Points on the planes bisecting the angles between the given planes are equidistant from them. Let (x, y, z) be a point on the planes bisecting the angles between the given planes. Equations of the required planes are

$$\frac{2x + y + 2z - 5}{3} = \pm\frac{3x + 2y - 6z + 1}{7}$$

or $14x + 7y + 14z - 35 = \pm(9x + 6y - 18z + 3)$

i.e., $5x + y + 32z - 38 = 0$ and $23x + 13y - 4z - 32 = 0$

5. Transform the equations of the planes $3x - 4y + 5z = 0$ and $2x - y - 2z = 5$ to normal forms and hence find measure of the angle between them.

Sol. $3x - 4y + 5z = 0 \quad (1)$

Its normal form is

$$\frac{3}{5\sqrt{2}}x - \frac{4}{5\sqrt{2}}y + \frac{5}{5\sqrt{2}}z = 0$$

The plane $2x - y - 2z = 5$ has normal form as

$$\frac{2}{3}x - \frac{1}{3}y - \frac{2}{3}z = \frac{5}{3} \quad (2)$$

Measure of the angle between the planes is given by

$$\cos\theta = \frac{3}{5\sqrt{5}} \times \frac{2}{3} + \left(-\frac{4}{5\sqrt{2}}\right)\left(-\frac{1}{3}\right) + \left(\frac{5}{5\sqrt{2}}\right)\left(-\frac{2}{3}\right)$$
$$= \frac{2}{5\sqrt{2}} + \frac{4}{15\sqrt{2}} - \frac{2}{3\sqrt{2}} = \frac{6 + 4 - 10}{15\sqrt{2}} = 0$$

Hence $\theta = \frac{\pi}{2}$

6. Find equations of the planes through the points (4, –5, 3) and (2, 3, 1) and parallel to the coördinate exes.

Sol. Let a plane parallel to the x-axis be $ax + by + cz = 1$ where a, b, c are direction ratios of a normal to the plane. The direction cosines of the x-axis are (1, 0, 0). Normal to the plane is perpendicular to the x-axis.

Hence $a.\,1 + b.\,0 + c.\,0 = 0$ or $a = 0$.

Therefore, an equation of the plane parallel to the x-axis is

$$by + cz = 1.$$

Since the points (4, –5, 3) and (2, 3, 1) lie on this plane, we have

$$\left.\begin{aligned} -5b + 3c &= 1 \\ \text{and} \quad 3b + c &= 1 \end{aligned}\right\}$$

These equations give $b = \frac{1}{7}$, $c = \frac{4}{7}$

The plane parallel to the x-axis and passing through the given points is $\frac{1}{7}y + \frac{4}{7}z = 1$ or $y + 4z = 7$

Similarly, a plane parallel to the y-axis is $ax + cz = 1$. On substituting the given points into this equations, we have

$4a + 3c = 1$ and $2a + c = 1$ which give $a = 1, c = -1$.

The plane parallel to the y-axis and passing through the given points is $x - z = 1$.

The plane parallel to the z-axis can be found similarly as $4x + y - 11 = 0$.

7. Find an equation of the plane through the points (1, 0, 1) and (2, 2, 1) and perpendicular to the plane $x - y - z + 4 = 0$.

Sol. Let the required plane be $ax + by + cz = 1$. As this plane is to be perpendicular to $x - y - z + 4 = 0$, we have

$$a.\,1 + b(-1) + c(-1) = 0$$

$$i.e., \quad a - b - c = 0 \qquad (1)$$

Also the points (1, 0, 1) and (2, 2, 1) lie on the required plane

Therefore, $\quad a + c = 1 \qquad (2)$

and $\quad 2a + 2b + c = 1 \qquad (3)$

Solving (1), (2) and (3) simultaneously, we have

$$a = \frac{2}{5}, b = \frac{-1}{5}, c = \frac{3}{5}$$

Hence the required plane is

$$\frac{2}{5}x - \frac{1}{5}y + \frac{3}{5}z = 1 \text{ or } 2x - y + 3z = 5$$

8. Find an equation of the plane which is perpendicular bisector of the line segment joining the points (3, 4, –1) and (5, 2, 7).

Sol. Direction ratios of the line joining the given points are

$$5 - 3, 2 - 4, 7 - (-1) \qquad i.e., \quad 2, -2, 8$$

These are direction ratios of a normal to the required plane.

Also the mid-point of the line segment joining the given points is

$$\left(\frac{3+5}{2}, \frac{2+4}{2}, \frac{7-1}{2}\right) = (4, 3, 3).$$

Hence equation of the required plane is

$$2(x - 4) + (-2)(y - 3) + 8(z - 3) = 0$$

or $\quad 2x - 2y = 8z = 26 \qquad$ or $\qquad x - y + 4z = 13$

9. Show that the join of (0, –1, 0) and (2, 4, –1) intersects the joins of (1, 1, 1) and (3, 3, 9).

Sol. We first show that the two joins are coplanar.

Equation of the plane through (0, –1, 0), (2, 4, –1) and (1, 1, 1) is

$$\begin{vmatrix} x & y & z & 1 \\ 0 & -1 & 0 & 1 \\ 2 & 4 & -1 & 1 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0 \text{ or } 7x - 3y - z - 3 = 0$$

The fourth point (3, 3, 9) also satisfies this equation. Hence the two joins are coplanar.

Direction ratios of the join of

(0, –1, 0) and (2, 4, –1) are 2, 5, –1 $\qquad (1)$

and the direction ratios of the join of

(1, 1, 1) and (3, 3, 9) are 2, 2, 8 $\qquad (2)$

Since (1) and (2) are not proportional, the two joins are not parallel and hence being coplanar, they intersect.

10. The vertices of a tetrahedron are (0, 0, 0), (3, 0, 0), (0, –4, 0) and (0, 0, 5). Find equations of the planes of its faces.

Sol. Let the vertices be denoted by $A = (0, 0, 0)$, $B = (3, 0, 0)$, $C = (0, -4, 0)$ and $D = (0, 0, 5)$.

Equation of any plane through the point A is $ax + by + cz = 0$. If it passes through B and C, then $a = 0$ and $b = 0$. Thus the plane through A, B, C is $cz = 0$ or $z = 0$.

Similarly, plane through A, B and D is $y = 0$ and plane through A, C and D is $x = 0$.

Now we find the plane through B, C and D. This can be written in the intercept form as

$$\frac{x}{3} + \frac{y}{-4} + \frac{z}{5} = 1.$$

Equations of the planes of the four faces are

$$x = 0, y = 0, z = 0, \frac{x}{3} - \frac{y}{4} + \frac{z}{5} = 1$$

11. Find an equation of the plane through (5, – 1, 4) and perpendicular to each of the planes $x + y - 2z - 3 = 0$ and $2x - 3y + z = 0$

Sol. A plane through (5, –1, 4) is $a(x - 5) + b(y + 1) + c(z - 4) = 0$

Since it is perpendicular to each of the given planes, we have

$$a + b - 2c = 0 \quad \text{and} \quad 2a - 3b + c = 0$$

or $\dfrac{a}{1-6} = \dfrac{b}{-4-1} = \dfrac{c}{-3-2}$ or $\dfrac{a}{1} = \dfrac{b}{1} = \dfrac{c}{1}$ *i.e.,* $a : b : c = 1 : 1 : 1$

Hence equation of the required plane is

$$x - 5 + y + 1 + z - 4 = 0 \text{ or } x + y + z - 8 = 0$$

12. Find an equation of the plane each of whose point is equidistant from the points $A(2, -1, 1)$ and $B(3, 1, 5)$.

Sol. $\overrightarrow{AB} = [1, 2, 4]$

This vector is a normal of the required plane since each points of the plane is equidistant from A, B. Equation of a plane with $\overrightarrow{AB}$ as a normal vector is $x + 2y + 4z + d = 0$.

The mid-point $\left(\frac{5}{2}, 0, 3\right)$ of the line segment $\overrightarrow{AB}$ also lies on this plane. Hence $\frac{5}{2} + 12 + d = 0$ or $d = -\frac{29}{2}$

Equation of the required plane is

$$x + 2y + 4z - \frac{29}{2} = 0 \quad i.e., \quad 2x + 4y + 8z - 29 = 0$$

13. Find an equation of the plane through the point (3, –2, 5) and perpendicular to the line $x = 2 + 3t, y = 1 - 6t, z = -2 + 2t$.

Sol. Direction ratios of the line are 3, –6, 2

Since the plane is perpendicular to the given line, direction ratios of the line are direction ratios of a normal to the plane.

Equation of such a plane is $3x - 6y + 2z + d = 0$

Since it passes through (3, –2, 5), we have

$$9 + 12 + 10 + d = 0 \quad \text{or} \quad d = -31.$$

Equation of the required plane is $3x - 6y + 2z - 31 = 0$

14. Find parametric equations of the line containing the point (2, 4, - and perpendicular to the plane $3x + 3y - 7z = 9$.

Sol. Since the line is to be perpendicular to the given plane, direction ratios of the line are 3, 3, –7

Equations of the line through (2, 4, –3) with these direction ratios are

$$\frac{x-2}{3} = \frac{y-4}{3} = \frac{z+3}{-7} = t$$

i.e., $x = 2 + 3t, y = 4 + 3t, z = -3 - 7t$

are the required parametric equations of the line.

15. Write equation of the family of all planes whose distance from the origin is 7. Find those members of the family which are parallel to the plane $x + y + z + 5 = 0$.

Sol. An equation of the family of all required planes in normal form is

$$lx = my + nz = 7$$

where l, m, n are direction cosines of normals to the planes. Equation of the plane $x + y + z + 5 = 0$ in the normal from is

$$\frac{x}{-\sqrt{3}} + \frac{y}{-\sqrt{3}} + \frac{z}{-\sqrt{3}} - \frac{5}{-\sqrt{3}} = 0 \qquad (1)$$

A plane parallel to (1) has a normal vector with direction cosines

$$\left(\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}\right) \text{ or } \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Thus, there are two members of the family parallel to (1). They are

$$-\frac{1}{\sqrt{3}}x - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}z - 7 = 0$$

and $\dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}y + \dfrac{1}{\sqrt{3}}z - 7 = 0$

16. Fin an equation of the family of the plane which passes through the point (3, 4, 5), has an x-intercept equal to –5 and is perpendicular to the plane $2x + 3y - z = 8$.

Sol. Equation of a plane with intercepts –5, b and c on the x-axis, y-axis and z-axis respectively is

$$\frac{x}{-5} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (1)$$

As this plane is perpendicular to $2x + 3y - z = 8$, we have

$$2\left(-\frac{1}{5}\right) + 3\left(\frac{1}{b}\right) - \frac{1}{c} = 0 \quad \text{or} \quad \frac{3}{b} - \frac{1}{c} = \frac{2}{5} \qquad (2)$$

Also the plane (1) passes through (3, 4, 5), Therefore

$$\frac{3}{-5} + \frac{4}{b} + \frac{5}{c} = 1 \quad \text{or} \quad \frac{4}{b} + \frac{5}{c} = 1 + \frac{3}{5} = \frac{8}{5} \qquad (3)$$

From (2) and (3), we have $b = \dfrac{95}{18}$, $c = \dfrac{95}{16}$

Equation of the required plane is

$$\frac{x}{-5} + \frac{18y}{95} + \frac{16z}{95} = 1 \text{ or } 19x - 18y - 16z + 95 = 0$$

17. Show that the distance of the point $P(3, -4, 5)$ from the plane $2x + 5y - 6z = 16$ measured parallel to the line

$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$ is $\frac{60}{7}$.

Sol. Equations of the line through $P(3, -4, 5)$ and parallel to the given line are

$$\frac{x-3}{2} = \frac{x+4}{1} = \frac{z-5}{-2} \qquad (1)$$

Any point on this line is $Q(2t+3, t-4, 2t+5)$

If Q also lies on the given plane, then

$$2(2t+3) + 5(t-4) - 6(-2t+5) = 16 \text{ or } t = \frac{20}{7}$$

Thus $Q\left(\frac{61}{7}, \frac{-8}{7}, \frac{-5}{7}\right)$ is the point where the line (1) meets the given plane.

Required distance $= |PQ| = \left[\left(\frac{40}{7}\right)^2 + \left(\frac{20}{7}\right)^2 + \left(\frac{40}{7}\right)^2\right]^{1/2} = \frac{60}{7}$

18. Show that the lines

$L : x = 3 + 2t, \quad y = 2 + t, \quad z = -2 - 3t$

$M : x = -3 + 4s, \quad y = 5 - 4s, \quad z = 6 - 5s$

intersect. Find an equation of the plane containing these lines.

Sol. The lines intersect if the equations

$$3 + 2t = -3 + 4s$$
$$2 + t = 5 - 4s$$
$$-2 - 3t = 6 - 5s$$

have simultaneous solution. Solving the first two equations, we find $t = -1$ and $s = 1$. Last equation is satisfied by these values of t and s. Hence the lines intersect.

Equations of the lines in symmetric forms are

$$L: \frac{x-3}{2} = \frac{y-2}{1} = \frac{z+2}{-3} \qquad (1)$$

$$M: \frac{x+3}{4} = \frac{y-5}{-4} = \frac{z-6}{-5} \qquad (2)$$

A point on the line (1) is $(3, 2, -2)$.

Since the required plane is to contain both the lines, it will contain every point of both lines.

Any plane through $(3, 2, -2)$ is $a(x-3) + b(y-2)(+ c(z+2) = 0$

If this plane contains both (1) and (2), then normal vector of the plane is perpendicular to each of the two lines. Therefore,

$$2a + b - 3c = 0$$
$$4a - 4b - 5c = 0$$

i.e. $\frac{a}{-17} = \frac{b}{-2} = \frac{c}{-12}$

Hence equation of the desired plane is

$$17(x-3) + 2(y-2) + 12(z+2) = 0$$

i.e., $17x + 2y + 12z - 31 = 0$.

19. If a, b, c are the intercepts of a plane on the coordinate axes and r is the distance of the origin from the plane, prove that

$$\frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Sol. Equation of the plane, with a, b, c as intercepts on the x-axis, y-axis and z-axis respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Distance r of the plane from $(0, 0, 0)$ is

$$r = \frac{|-1|}{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)} \quad i.e., \quad \frac{1}{r^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

20. Find equations of two planes whose distances from the origin are 3 units each and which are perpendicular to the line through the points $A(7, 3, 1)$ and $B(6, 4, -1)$.

Sol. The line AB is normal to both the required planes. Direction ratios of AB are $1, -1, 2$. Direction cosines of the line AB are

$$\frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$$

Equations of the planes at distance 3 unit each from the origin are

$$lx + my + nz = \pm 3$$

where l, m, n are direction cosines of normals to the two planes.

Since AB is normal to the two planes, we have

$$\frac{1}{\sqrt{6}}x - \frac{1}{\sqrt{6}}y + \frac{2}{\sqrt{6}}z = \pm 3$$

or $x - y + 2z = \pm 3\sqrt{6}$ are equations of the desired planes.

Exercise Set 8.4 (Page 366)

1. Prove that the planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ intersect and equations of their line of intersection can be written in the form

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}.$$

Sol. Direction ratios of normals to the two planes are $4, 4, -5$ and $8, 12, -13$ which are not proportional. Thus the planes are not parallel and so they intersect.

To find equations of their line of intersection in a symmetric form, we choose $z = 0$. Then the two equations are

$$4x + 4y = 12 \qquad \text{and} \qquad 8x + 12y = 32$$

which give $x = 1, y = 2$. Hence $(1, 2, 0)$ is a point on the line.

Similarly, putting $x = 0$, we find

$$4y - 5z = 12 \qquad \text{and} \qquad 12y - 13z = 32$$

which give $z = -2$, $y = \frac{1}{2}$. Therefore another point on the line is

$$\left(0, \frac{1}{2}, -2\right)$$

The required line passes through $(1, 2, 0)$ and $\left(0, \frac{1}{2}, -2\right)$.

Its equations are

$$\frac{x-1}{0-1} = \frac{y-2}{\frac{1}{2}-2} = \frac{z-0}{-2-0} \quad \text{or} \quad \frac{x-2}{2} = \frac{y-2}{3} = \frac{z}{4}$$

2. Find a symmetric form for the line $x + y + z + 1 = 0 = 4x + y - 2z + 2$.

Sol. Suppose $z = 0$. Then the given equations are

$$x + y + 1 = 0 \quad \text{and} \quad 4x + y + 2 = 0$$

The equations give $x = -\frac{1}{3}, y = \frac{-2}{3}$.

Hence $A\left(\frac{-1}{3}, \frac{-2}{3}, 0\right)$ is a point lying on the line.

Again, we let $x = 0$ and have $-y + z + 1 = 0$ and $y - 2z + z = 0$

which give $z = \frac{1}{3}$ and $y = \frac{-4}{3}$

Therefore, another point on the given line is $B\left(0, \frac{-3}{4}, \frac{1}{3}\right)$

Equations of the line through A and B in a symmetric form are

$$\frac{x+\frac{1}{3}}{0+\frac{1}{3}} = \frac{y+\frac{2}{3}}{\frac{-4}{3}+\frac{2}{3}} = \frac{z-0}{\frac{1}{3}-0} \quad \text{or} \quad \frac{x+\frac{1}{3}}{1} = \frac{y+\frac{2}{3}}{-2} = \frac{z}{1}$$

3. Show that the lines

L : $x + 2y - z - 7 = 0 = y + z - 2x - 6$

M : $3x + 6y - 3z - 8 = 0 = 2x - y - z$ are parallel.

Sol. If l_1, m_1, n_1 are direction ratios of the line L then since normals of the two planes are also normal to the line we have

$$1 . l_1 + 2m_1 - n_1 = 0 \qquad \text{and} \qquad -2l_1 + m_1 + n_1 = 0$$

Therefore, $\frac{l_1}{3} = \frac{m_1}{1} = \frac{n_1}{5}$ (1)

Again if l_2 , m_2, n_2 are direction ratios of the line M we have, as before

$$3l_2 + 6m_2 - 3n_2 = 0 \qquad \text{and} \qquad 2l_2 - m_2 - n_2 = 0$$

Thus $\frac{l_2}{-9} = \frac{m_2}{-3} = \frac{n_2}{-15}$ (2)

From (1) and (2), we find that

$$\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} = \frac{-1}{3}$$

Therefore, the two given lines are parallel.

4. Show that the lines

L : $x + 2y - 1 = 0 = 2y - z - 1$

M : $x - y - 1 = 0 = x - 2y - z$

are perpendicular.

Sol. Let direction ratios of the line L be l_1, m_1, n_1. Then we have

$$1 . l_1 + 2m_1 + 0n_1 = 0 \qquad \text{and} \qquad 0 l_1 + 2m_1 - n_1 = 0$$

Therefore, $\frac{l_1}{-2} = \frac{m_1}{1} = \frac{n_1}{2} = k_1$ (say)

or $\quad l_1 = -2k_1, m_1 = k_1, n_1 = 2k_1$ (1)

Now, if l_2, m_2, n_2 are direction ratios of the line M, then we have

$$1 . l_2 - 1. m_2 + 0n_1 = 0 \qquad \text{and} \qquad 1 . l_2 - 0m_2 - 2n_2 = 0$$

Hence $\frac{l_2}{2} = \frac{m_2}{2} = \frac{n_2}{1} = k_2$ (say)

or $\quad l_2 = 2k_2, m_2 = 2k_2, n_2 = k_2$ (2)

From (1) and (2), we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = -4k_1k_2 + 2k_1k_2 + 2k_1k_2 = 0$$

Hence the given lines are perpendicular to each other.

5. Find equations of the straight line through the point $(1, 2, 3)$ and parallel to the line $x - y + 2z - 5 = 0 = 3x + y + z + 6$. (1)

Sol. Suppose direction ratios of the required line are l, m, n. Since the required line is parallel to the line (1), it is perpendicular to the normals of each of the planes constituting the line (1). Therefore,

$$1 . l - 1 . m + 2n = 0 \qquad \text{and} \qquad 3l + m + n = 0$$

From these equations, we get

$$\frac{l}{-3} = \frac{m}{5} = \frac{n}{4}$$

Equations of the required line are

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

6. Find equations of the planes containing the line $x - y - z = 0 = 2x - y + 3z - 5$ and perpendicular to the coordinate planes.

Sol. Any plane through the given line is

$$x + y - z + k(2x - y + 3z - 5) = 0$$

or $\quad (1 + 2k)x + (1 - k)y + (3k - 1)z - 5k = 0 \qquad (1)$

A normal vector to this plane is

$$[1 + 2k, 1 - k, 3k - 1]$$

The plane (1) is perpendicular to the yz-plane whose equation is $x = 0$. Hence $1(2 + 2k) = 0$ which gives $k = -\frac{1}{2}$

Putting this value of k into (1), we have

$$\frac{3}{2}y - \frac{5}{2}z + \frac{5}{2} = 0 \qquad \text{or} \qquad 3y - 5z + 5 = 0$$

as an equation of the plane perpendicular the to the yz-plane.

Equations of planes perpendicular to the zx and xy-planes can be found in a similar manner.

7. Fin an equation of the plane containing the line $x = 2t, y = 3t, z = 4t$ and intersection of the planes $x + y + z = 0$ and $2y - z = 0$.

Sol. A plane through the intersection of $x + y + z = 0$ and $2y - z = 0$ is

$$x + y + z + k(2y - z) = 0$$

i.e., $\quad x + (1 + 2k)y + (1 - k)z = 0 \qquad (1)$

Since this plane contains the line $x = 2t, y = 3t, z = 4t$, a normal of the plane is perpendicular to the line. Therefore,

$2 + 3(1 + 2k) + 4(1 - k) = 0,$ (since 2, 3, 4, are direction ratios of the line)

or $\quad 2 + 3 + 6k + 4 - 4k = 0$ giving $k = -\frac{9}{2}$.

Substituting for k into (1), equation of the required plane is

$$x + (1 - 9)y + \left(1 + \frac{9}{2}\right)z = 0 \quad \text{or} \quad 2x - 16y + 11z = 0.$$

8. Write an equation of the family of planes having x-intercept 5, y-intercept 2 and a nonzero z-intercept. Find the member of the family which is perpendicular to the plane

$$3x - 2y + z - 4 = 0. \qquad (1)$$

Sol. Let nonzero z-intercept be c.

Equation of the required family of planes is

$$\frac{x}{5} + \frac{y}{2} + \frac{z}{c} = 1, \quad \text{where } c \text{ is a parameter.}$$

If a member of this family is perpendicular to (1), then we have

$$\frac{3}{5} - \frac{2}{2} + \frac{1}{c} = 0, \quad i.e., \quad c = \frac{5}{2}$$

The required plane is

$$\frac{x}{5} + \frac{2}{2} + \frac{z}{5/2} = 1 \quad i.e., \quad \frac{x}{5} + \frac{y}{2} + \frac{2z}{5} = 1.$$

9. Find an equation of the plane passing through the point (2, −3, 1) and containing the line $x - 3 = 2y = 3z - 1$.

Sol. The given line can be written as

$$x - 2y - 3 = 0 = 2y - 3z + 1$$

Therefore, any plane through this line is

$$(x - 2y - 3) + k(2y - 3z + 1) = 0 \qquad (1)$$

If it passes through (2, −3, 1), then

$$(2 + 6 - 3) + k(-6 - 3 + 1) = 0 \text{ or } k = \frac{5}{8}$$

Putting this values of k into (1), we have

$$(x - 2y - 3) + \frac{5}{8}(2y - 3z + 1) = 0 \text{ or } 8x - 6y - 15z = 19$$

is an equation of the required plane.

10. Find an equation of the plane passing through the line of intersection of the planes $2x - y + 2z = 0$ and $x + 2y - 2z - 3 = 0$ and at unit distance from the origin.

Sol. Any plane through the intersection of the given planes is

$$2x - y + 3z + k(x + 2y - 2z - 3) = 0$$

or $\quad (2 + k)x + (2k - 1)y + (3 - 2k)z = 3k \qquad (1)$

Now the perpendicular distance of this plane from the origin is 1. Therefore,

$$\frac{|-3k|}{\sqrt{(2 + k)^2 + (2k - 1)^2 + (3 - 2k)^2}} = 1$$

or $\quad 9k^2 = 9k^2 - 12k + 14$ or $k = \frac{7}{6}$

Putting this value of k into (1), we get

$$19x + 8y + 4z = 21$$

as an equation of the required plane.

11. Find equations of the perpendicular from the origin to the line $x + 2y + 3z + 4 = 0 = 2x + 3y + 4z + 5$. Also find the coordinates of the foot of the perpendicular.

Sol. We first convert equations of the given line in a symmetric form. Let a, b, c be direction ratios of the given line. Since the line is perpendicular to the normal of each of the planes constituting the line, we have

$a + 2b + 3c = 0$

and $2a + 3b + 4c = 0$

or $\dfrac{a}{-1} = \dfrac{b}{2} = \dfrac{c}{-1}$

Therefore, 1, −2, 1 are direction ratios of the line.

Now, we find a point on the given line. Let a point be such that $z = 0$

Then $x + 2y + 4 = 0$ and $2x + 3y + 5 = 0$

Solving these equations, we have $x = 2, y = -3$. Thus a point on the line is $(2, -3, 0)$.

A symmetric form of the given line is

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z}{1} = t \quad \text{(say)} \qquad (1)$$

Then, $A(2 + t, -3 - 2t, t)$ is any point on the line. Let A be the foot of the perpendicular from $O(0, 0, 0)$ to the line (1). Direction ratios of OA are

$2 + t, -3 - 2t, t$

Since OA is perpendicular to (1), we have

$1(2 + t) - 2(-3 - 2t) + t = 0$

i.e., $6t + 8 = 0$ giving $t = -\dfrac{4}{3}$

Thus foot of the perpendicular is

$$A\left(\frac{2}{3}, \frac{-1}{3}, \frac{-4}{3}\right)$$

Perpendicular distance $= |OA| = \sqrt{\dfrac{4}{9} + \dfrac{1}{9} + \dfrac{16}{9}} = \sqrt{\dfrac{7}{3}}$

Equations of the perpendicular OA are

$$\frac{x-0}{\frac{2}{3}} = \frac{y-0}{-\frac{1}{3}} = \frac{z-0}{-\frac{4}{3}} \text{ or } \frac{x}{2} = \frac{y}{-1} = \frac{z}{-4}$$

12. A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the coordinate planes. Show that the locus of their point of intersection is given by $x^{-2} + y^{-2} + z^{-2} = p^{-2}$.

Sol. Let an equation of the variable plane be

$$lx + my + nz = p \text{ where } l^2 + m^2 + n^2 = 1 \qquad (1)$$

or $\dfrac{x}{p/l} + \dfrac{y}{p/m} + \dfrac{z}{p/n} = 1$

Thus coordinates of A, B, C are respectively

$$\left(\frac{p}{l}, 0, 0\right), \left(0, \frac{p}{m}, 0\right), \left(0, 2, \frac{p}{n}\right)$$

Equation of the plane through $A\left(\dfrac{p}{l}, 0, 0\right)$ and parallel to yz-plane is $x = \dfrac{p}{l}$

Similarly, equation of the plane through $B\left(0, \dfrac{p}{m}, 0\right)$ and parallel to xz-plane is $y = \dfrac{p}{m}$ and equation of the plane through $C\left(0, 0, \dfrac{p}{n}\right)$ and parallel to xy-plane is $z = \dfrac{p}{n}$.

Thus $l = \dfrac{p}{x}, m = \dfrac{p}{y}, n = \dfrac{p}{z}$.

Since $l^2 + m^2 + n^2 = 1$, we have

$$\frac{p^2}{x^2} + \frac{p^2}{y^2} + \frac{p^2}{z^2} = 1$$

or $x^{-2} + y^{-2} + z^{-2} = p^{-2}$ as the required locus.

13. Let $A, B\ C$ be the points as in **Problems 12**. Prove that the locus of the centroid of the tetrahedron $OABC$ is

$$x^{-2} + y^{-2} + z^{-2} = 16p^{-2}, O \text{ being the origin.}$$

Sol. From **Problem 12**, we have

$$A\left(\frac{p}{l}, 0, 0\right), B\left(0, \frac{p}{m}, 0\right), C\left(0, 2, \frac{p}{n}\right), O(0, 0, 0)$$

Coordinates of the centroid of the tetrahedron $OABC$ are

$$\left(\frac{p}{4l}, \frac{p}{4m}, \frac{p}{4n}\right)$$

i.e., $x = \dfrac{p}{4l}, y = \dfrac{p}{4m}, z = \dfrac{p}{4n}$

We eliminate l, m, n from these equation to get the required locus.

We have $l = \dfrac{p}{4x}, m = \dfrac{p}{4y}, n = \dfrac{p}{4z}$

Squaring these equations and adding the results, we have

$$1 = l^2 + m^2 + n^2 = \frac{p^2}{16x^2} + \frac{p^2}{16y^2} + \frac{p^2}{16z^2}$$

or $16 = p^2(x^{-2} + y^{-2} + z^{-2})$ *i.e.,* $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

Exercise Set 8.5 (Page 370)

1. Show that the straight line $\frac{x+3}{2} = \frac{y-4}{-7} = \frac{z}{3}$ is parallel to the plane $4x + 2y + 2z = 9$.

Sol. Direction ratios of a normal to the given line are [2, −7, 3] and direction ratios of a normal to the given plane are [4, 2, −2]. Since (2) (4) + (−7) (2) + (3) (−2) = 0, normal to the plane is perpendicular to the given line. Thus the line is parallel to the given plane.

2. Show that the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is perpendicular to the plane $4x + 8y + 12z + 19 = 0$.

Sol. Direction ratios of given line are 1, 2, 3,. And direction ratios of a normal to the plane are 4, 8, 12.

Since $\frac{1}{4} = \frac{2}{8} = \frac{3}{12}$, the given line is parallel to a normal to the given plane

i.e., the given line is perpendicular to the plane.

3. Find the conditions that the straight line $x = mz + a, y = nz + b$ may lie in the plane $Ax + By + Cz + D = 0$.

Sol. The given line in a symmetric form is

$$\frac{x-a}{m} = \frac{y-b}{n} = \frac{z}{1} = t \qquad \text{(say)}$$

Therefore, any point on the line is (x, y, z) where

$$x = a + mt, \qquad y = b + nt \qquad \text{and} \qquad z = 1$$

This point lies on the given plane if

$$A(a + mt) + B(b + nt) + Ct + D = 0$$

or $\quad (Am + Bn + C)t + Aa + Bb + D = 0$

which must be satisfied for every value of t. This implies that

$$Aa + Bb + D = 0 \qquad \text{and} \qquad Am + Bn + C = 0$$

which are required conditions for the given line to lie in the given plane.

4. Determine the point, if any, common to the straight line $x = 1 + t$, $y = t, z = -1 + t$ and the plane $x + y + z = 3$.

Sol. We find t such that $x = 1 + t, y = t$ and $z = -1 + t$ lie on the plane $x + y + z = 3$. This requires

$$(1 + t) + t + (-1 + t) = 3 \qquad \text{or} \qquad t = 1$$

Hence the common point is (2, 1, 0).

5. Find an equation of the plane through the point (x_1, y_1, z_1) and through the straight line $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$.

Sol. Any plane through the given line is

$$A(x-a) + B(y-b) + C(z-c) = 0 \qquad (1)$$

where $\quad Al + Bm + Cn = 0 \qquad (2)$

Since (x_1, y_1, z_1) lies on (1), we have

$$A(x_1 - a) + B(y_1 - b) + C(z_1 - c) = 0 \qquad (3)$$

Eliminating A, B, C from (1), (2) and (3), we get

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ x_1-a & y_1-b & z_1-c \end{vmatrix} = 0$$

or $\quad \Sigma(x-a)\,[m(z_1-c) - n(y_1-b)] = 0$

which is an equation of the required plane.

6. Find an equation of the plane passing through the straight line $x + 2z = 4, y - z = 8$ and parallel to the straight line

$$\frac{x-3}{2} = \frac{y+4}{3} = \frac{z-7}{4}. \qquad (1)$$

Sol. The straight line $x + 2z = 4, y - z = 8$ in a symmetric form is

$$\frac{x-4}{-2} = \frac{y-8}{1} = \frac{z}{1}$$

Any plane through this line is

$$a(x-4) + b(y-8) + cz = 0 \qquad (2)$$

where $\quad a(-2) + b(1) + c(1) = 0 \qquad (3)$

As the plane (2) is parallel to the line (1), we have

$$a(2) + b(3) + c(4) = 0 \qquad (4)$$

Eliminating a, b, c from (2), (3) and (4), we obtain

$$\begin{vmatrix} x-4 & y-8 & z \\ -2 & 1 & 1 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

or $\quad (x-4) + 10(y-8) + (-8)(z) = 0$ *i.e.*, $x + 10y - 8z = 84$

is an equation of the required plane.

7. Find an equation of the plane passing through the point (α, β, γ) and parallel to each of the straight lines

$$\frac{x-a}{l_1} = \frac{y-b}{m_1} = \frac{z-c}{n_1} \quad \text{and} \quad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}.$$

Sol. Suppose that the plane through the point (α, β, γ) is

$$a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0 \qquad (1)$$

If (1) is parallel to the given straight lines, we have

$$al_1 + bm_1 + cn_1 = 0 \quad (2)$$

and $$al_2 + bm_2 + cn_2 = 0 \quad (3)$$

Eliminating a, b, c from (1), (2) and (3), we get

$$\begin{vmatrix} x-\alpha & y-\beta & z-\gamma \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

i.e., $\Sigma(x-\alpha)(m_1n_2 - m_2n_1) = 0$ which is the required plane.

8. Find an equation of the plane through the straight line

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$$

and parallel to the straight line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ (1)

Sol. Any plane through the given line is

$$ax + by + cz + d + k(a'x + b'y + c'z + d') = 0$$

i.e., $(a + ka')x + (b + kb')y + (c + kc')d + ka' = 0$ (2)

Since this plane is parallel to the line (1), we have

$$(a + ka')l + (b + kb')m + (c + kc')n = 0$$

This gives $k = -\dfrac{al + bm + cn}{a'l + b'm + c'n}$

Putting this value of k into (2), we have equation of the required plane as

$$ax + by + cz + d = -\frac{al + bm + cn}{a'l + b'm + c'n} \times (a'x + b'y + c'z + d') = 0$$

or $(a'l + b'm + c'n)(ax + by + cz + d)$
$= (al + bm + cn)(a'x + b'y + c'z + d')$.

9. Prove that the straight lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-3}{3} = \frac{y-3}{4} = \frac{z-4}{5} \text{ are coplanar.}$$

Sol. We know that if the straight lines

$$\frac{x-\alpha_1}{l_1} = \frac{y-\beta_1}{m_1} = \frac{z-\gamma_1}{n_1} \text{ and } \frac{x-\alpha_2}{l_2} = \frac{y-\beta_2}{m_2} = \frac{z-\gamma_2}{n_2}$$

are coplanar, then

$$\begin{vmatrix} \alpha_2-\alpha_1 & \beta_2-\beta_1 & \gamma_2-\gamma_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

So the given straight lines are coplanar if

$$\begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

which is, of course, zero since the subtraction of the second row from the third row makes the first and second rows identical. Hence the given straight lines are coplanar.

10. Prove that the straight lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8} \text{ and } \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

intersect. Also find the point of intersection and the plane through them.

Sol. Let $\dfrac{x-1}{2} = \dfrac{y+1}{-3} = \dfrac{z+10}{8} = t$

This gives $x = 1 + 2t, y = -1 - 3t, z = 8t - 10$ (1)

Similarly, from the second line, we have

$$x = 4 + s, y = -3 - 4s, z = -1 + 7s \quad (2)$$

The lines intersect if the equations

$$1 + 2t = 4 - s, -1 - 3t = -3 - 4s \text{ and } 8t - 10 = -1 + 7s$$

have a simultaneous solution.

Solving the first two equations simultaneously, we have

$t = 2$ and $s = 1$.

The third of these equations is also satisfied by these values of t and s

Putting either $t = 2$ in (1) or $s = 1$ in (2), we have

$$x = 5, y = -7, z = 6$$

which is the required point of intersection of the given lines.

A plane through these lines must contain the point of intersection $(5, -7, 6)$. Suppose the plane is

$$a(x-5) + b(y+7) + c(z-6) = 0 \quad (3)$$

Since this plane is to contain both the given lines, we have

$$a(2) + b(-3) + c(8) = 0 \quad (4)$$

and $$a(1) + b(-4) + c(7) = 0 \quad (5)$$

Eliminating a, b, c from (3), (4) and (5), we have

$$\begin{vmatrix} x-5 & y+7 & z-6 \\ 2 & -3 & 8 \\ 1 & -4 & 7 \end{vmatrix} = 0$$

or $(x-5)(11) + (y+7)(-6) + (z-6)(-5) = 0$

i.e., $11x - 6y - 5z = 67$ is an equation of the required plane.

Exercise Set 8.6 (Page 375)

1. Show that the shortest between the lines $x + a = 2y = -12z$ and $x = y + 2a = 6(z - a)$ is $2a$.

Sol. The given lines are

$$\frac{x+a}{1} = \frac{y}{1/2} = \frac{z}{-1/12} \qquad (1)$$

and $x = y + 2a, x = 6(z - a)$ (2)

Any plane through (2) in $(x - y - 2a) + k(x - 6z + 6a) = 0$

or $(1 + k)x - y - 6kz - 2a + 6ka = 0$ (3)

This is parallel to (1) if

$$(1 + k) - \frac{1}{2} - 6k\left(-\frac{1}{12}\right) = 0$$

or $1 + k - \frac{1}{2} + \frac{k}{2} = 0$ or $\frac{3k}{2} = -\frac{1}{2}$ *i.e.,* $k = -\frac{1}{3}$

Substituting this value of k into (3), we get

$$\frac{2}{3}x - y + 2z - 4a = 0 \qquad (4)$$

A point on (1) is $(-a, 0, 0)$. Perpendicular distance of this point from (4) is

$$\frac{\left|-\frac{2}{3}a - 4a\right|}{\sqrt{\frac{4}{9} + 1 + 4}} = \frac{\frac{14a}{3}}{\frac{7}{3}} = 2a$$

which is the shortest distance between the given lines.

2. Find the shortest distance between the x-axis and the straight line $ax + by + cz + d = 0 = a'x + b'y + c'y + c'z + d'$.

Sol. Equations of the x-axis are

$$y = 0 = z \text{ or } \frac{x}{1} = \frac{y}{0} = \frac{z}{0} \qquad (1)$$

The other line is

$$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \qquad (2)$$

Any plane containing the line (2) is

$$(ax + by + cz + d) + k(a'x + b'y + c'z + d') = 0 \qquad (3)$$

or $(a + ka')x + (b + kb')y + (c + kc')z + d + kd' = 0$

This will be parallel to the x-axis if

$$a + ka' = 0 \quad \text{or} \quad k = -\frac{a}{a'}$$

Putting this value of k into (3), we get the plane parallel to (1) as

$$(ax + by + cz + d) - \frac{a}{a'}(a'x + b'y + c'z + d') = 0$$

or $(a'b - ab')y + (a'c - ac')z + (a'd - ad') = 0$ (4)

∴ Shortest distance between (1) and (2)

= Perpendicular distance of the plane (4) from any point, say, (1, 0, 0) on the x-axis.

$$= \frac{a'd - ad'}{\sqrt{(a'b - ab')^2 + (a'c - ac')^2}}$$

3. Show that the shortest distance between the straight lines

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4} \text{ and } \frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$$

is $\frac{1}{\sqrt{6}}$ and equations of the straight line perpendicular to both are

$$11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7.$$

Sol. If l, m, n are direction cosines of the line of shortest distance then, it being perpendicular to the given line, we have

$$2l + 3m + 4n = 0 \quad \text{and} \quad 3l + 4m + 5n = 0$$

Thus $\frac{l}{15-16} = \frac{m}{12-10} = \frac{n}{8-9}$ or $\frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$

i.e., $l = \frac{-1}{\sqrt{6}}, m = \frac{2}{\sqrt{6}}, n = \frac{-1}{\sqrt{6}}$

Length of the shortest distance

$$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$$

$$= \frac{-1}{\sqrt{6}}(2-1) + \frac{2}{\sqrt{6}}(4-2) + \left(\frac{-1}{\sqrt{6}}\right)(5-3) = \frac{1}{\sqrt{6}}$$

Equations of the shortest distance are, by the standard formula

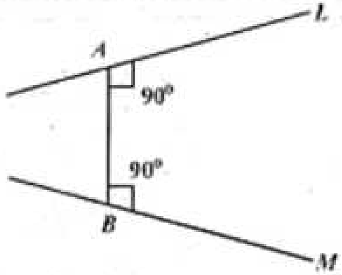
$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ -1 & 2 & -1 \end{vmatrix} = 0 = \begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ -1 & 2 & -1 \end{vmatrix}$$

i.e., $11x + 2y - 7z + 6 = 0 = 7x + y - 5z + 7$ as required.

4. Find the shortest between the lines

$$\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} \text{ and } \frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$$

Find equations of the straight line perpendicular to both the given straight lines and also its points of intersections with the given straight lines.



Sol. Let the shortest distance AB have direction cosines l, m, n. Then AB being perpendicular to both the given lines, we have

$$1 . l - 2m + n = 0$$

and $7l - 6m + n = 0$

i.e., $\dfrac{l}{4} = \dfrac{m}{6} = \dfrac{n}{8}$ or $\dfrac{l}{2} = \dfrac{m}{3} = \dfrac{n}{4}$

or $l = \dfrac{2}{\sqrt{29}}, m = \dfrac{3}{\sqrt{29}}, n = \dfrac{4}{\sqrt{29}}$

Shortest distance $= |(l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)|$

$$= \left|\frac{2}{\sqrt{29}}(-1-3) + \frac{3}{29}(-1-5) + \frac{4}{29}(-1-7)\right|$$

$$= \left|-\frac{(8+18+32)}{\sqrt{29}}\right| = \left|\frac{-58}{\sqrt{29}}\right|$$

$$= |-2\sqrt{29}| = 2\sqrt{29}$$

In order to find the coordinates of the points A and B, we note that A lies on

$$L : \frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1} = t \quad \text{(say)}$$

Then $x = t + 3, y = 5 - 2t, z = 7 - t$ are coordinates of A.

Similarly, $x = 7s - 1, y = -6s - 1, z = -1 + s$ are the coordinates of B.

Therefore, the direction ratios of AB are

$$7s - 1 - t - 3, -6s - 1 - 5 + 2t, -1 + s - 7 + t$$

or $7s - t - 4, -6s + 2t - 6, s + t - 8$.

Since AB is perpendicular to both the lines, we have

$$(7s - t - 4)1 + (-6 + 2t - 6)(-2) + (s + t - 8)(1) = 0$$

and $(7s - t - 4)(7) + (-6s + 2t - 6)(-6) + (s + t - 8)(1) = 0$

i.e., $20s - 4t = 0$ (1)

and $86s - 12t = 0$ (2)

(1) and (2) give $t = 0 = s$

Hence $A = (3, 5, 7)$ and $B = (-1, -1, -1)$

Also direction ratios of AB are $-4, -6, -8$ *i.e.*, $2, 3, 4$.

Equation of the shortest distance AB are $\dfrac{x-3}{2} = \dfrac{y-5}{3} = \dfrac{z-7}{4}$

5. Find the coordinates of the point on the join of $(-3, 7, -13)$ and $(-6, 1, -10)$ which is nearest to the intersection of the planes

$$2x - y - 3z + 32 = 0 \text{ and } 3x + 2y - 15z - 8 = 0.$$

Sol. Equations of the line through the points $(-3, 7, -13)$ and $(-6, 1, -10)$ are

$$\frac{x+3}{3} = \frac{y-7}{-6} = \frac{z+13}{3} \text{ or } \frac{x+3}{1} = \frac{y-7}{2} = \frac{z+13}{-1} = t \quad (1)$$

A point on (1) is $P(-3 + t, 7 + 2t, -13 - t)$.

We transform the equations

$$\left.\begin{aligned} 2x - y - 3z + 32 &= 0 \\ \text{and} \quad 3x + 2y - 15z - 0 &= 8 \end{aligned}\right\} \quad (2)$$

of the second straight line into symmetric form. Let $[a, b, c]$ be a direction vector of (2). Then

$$2a - b - 3c = 0$$

and $3a + 2b - 15c = 0$

Therefore, $\dfrac{a}{21} = \dfrac{b}{21} = \dfrac{c}{7}$ or $[a, b, c] = [3, 3, 1]$

Taking $z = 0$, equations (2) become

$$2x - y + 32 = 0 \quad \text{and} \quad 3x + 2y - 8 = 0$$

or $\dfrac{x}{56} = \dfrac{y}{112} = \dfrac{1}{7}$ or $x = -8, y = 16, z = 0$ is a point on (2).

Hence a symmetric form of equations (2) is

$$\frac{x+8}{3} = \frac{y-16}{3} = \frac{z}{1} = s \quad (3)$$

A point on (3) is

$$Q(-8 + 3s, 16 + 3s, s)$$

$$\overrightarrow{PQ} = [3s - t - 5, 3s - 2t + 9, s + t + 13]$$

Let $\overrightarrow{PQ}$ be normal to both (1) and (3). Then

$$3s - t - 5 + 6s - 4t + 18 - s - t - 13 = 0$$

and $9s - 3t - 15 + 9s - 6t + 27 + s + t + 13 = 0$

i.e., $8s - 6t = 0$ and $19s - 8t + 25 = 0$

Solving the last equations for s and t, we have

$$s = -3, t = -4$$

Substituting $t = -4$ in the coordinates for P, we get

$P(-7, -1, -9)$ as the required point.

6. Find the length and equations of the common perpendicular of the lines

L : $6x + 8y + 3z - 13 = 0$, $x + 2y + z - 3 = 0$

M : $3x - 9y + 5z = 0$, $x + y - z = 0$

Sol. We first transform the equations of L and M into symmetric forms. Putting $z = 0$ in the equations for L, we have

$$6x + 8y - 13 = 0 \quad \text{and} \quad x + 2y - 3 = 0$$

Therefore, $\dfrac{x}{2} = \dfrac{y}{2} = \dfrac{1}{4}$ or $x = \dfrac{1}{2}, y = \dfrac{5}{4}, z = 0$ is a point on L.

Let $[a, b, c]$ be a direction vector of L. Since L is perpendicular to normal of each plane constituting it, we have

$$6a + 8b + 3c = 0 \qquad \text{and} \qquad a + 2b + c = 0$$

so $\frac{a}{2} = \frac{b}{-3} = \frac{c}{4}$ or $[a, b, c] = [2, -3, 4]$

A symmetric form of equations of L is

$$\frac{x - \frac{1}{2}}{2} = \frac{y - \frac{5}{4}}{-3} = \frac{z}{4} \qquad (1)$$

Similarly, we can write M as

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \qquad (3)$$

Let $\boldsymbol{n} = [A, B, C]$ be the normal vector of the common perpendicular of both (1) and (2). Then

$$2A - 3B + 4C \qquad \text{and} \qquad A + 2B + 3C = 0$$

or $\frac{A}{-17} = \frac{B}{-2} = \frac{C}{7}$

Hence $\boldsymbol{n} = [A, B, C] = [-17, -2, 7]$

The points $P\left(\frac{1}{2}, \frac{5}{4}, 0\right)$ and $Q(0, 0, 0)$ lie on (1) and (2) respectively

$$\overrightarrow{PQ} = \left[\frac{1}{2}, \frac{5}{4}, 0\right]$$

Length of the common perpendicular to the two lines is the orthogonal projection of $\overrightarrow{PQ}$ on $\boldsymbol{n}$. Therefore, length of the common perpendicular

$$= \frac{|\overrightarrow{PQ} \cdot \boldsymbol{n}|}{|\boldsymbol{n}|} = \frac{\frac{17}{2} + \frac{5}{2}}{\sqrt{342}} = \frac{11}{\sqrt{342}}$$

To find equations of the common perpendicular, we proceed as follows:

A plane through L is $6x + 8y + 3z - 13 + k(x + 2y + z - 3) = 0$

i.e., $(6 + k)x + (8 + 2k)y + (3 + k)z - 3k - 13 = 0$

This plane contains the common perpendicular if

$$-17(6 + k) - 2(8 + 2k) + 7(3 + k) = 0$$

or $-102 - 17k - 16 - 4k + 21 + 7k = 0$

or $-14k - 97 = 0$ or $k = \frac{-97}{14}$

Hence the plane is $13x + 82y + 55z - 109 = 0$ (3)

Similarly, a plane containing M is

$$3x - 9y + 5z + m(x + y - z) = 0$$

or $(3 + m)x + (-9 + m)y + (5 - m)z = 0$

It contains the common perpendicular if

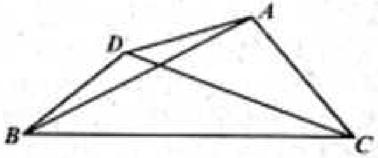
$$-17(3 + m) - 2(-9 + m) + 7(5 - m) = 0 \text{ or } -26m + 2 = 0$$

or $m = \frac{1}{13}$. The plane is $10x - 29y + 16z = 0$ (4)

(3) and (4) are the required equations of the common perpendicular.

7. Show that the shortest distance between any two opposite edges of the tetrahedron formed by the planes $y + z = 0, z + x = 0, x + y = 0, x + y + z = a$ is $\frac{2a}{\sqrt{6}}$ and that the three straight lines of the shortest distances intersect at the point $(-a, -a, -a)$.

Sol. Let the planes $y + z = 0, z + x = 0, x + y = 0$ and $x + y + z$ be ABC, ACD, ADB and BCD respectively.



Equations of line AC (being the intersection of ABC and ACD) are $y + z = 0 = z + x$.

or $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ (1)

The edge opposite to AC is BD which is intersection of the planes ADB and BCD. Its equations are

$$x + y = 0 \qquad \text{and} \qquad x + y + z = a$$

Any point on this line is $(0, 0, a)$. If l, m, n are direction ratios of this line then

$$1 . l + 1 . m + 0 . n = 0 \qquad \text{and} \qquad 1 . l + 1 . m + 1 . n = 0$$

or $\frac{l}{1} = \frac{m}{-1} = \frac{n}{0}$

Equation of BD in a symmetric form are $\frac{x - 0}{1} = \frac{y - 0}{-1} = \frac{z - a}{0}$ (2)

Now, if L, M, N are direction cosines of the shortest distance between (1) and (2) then

$$L . 1 + M . 1 + N(-1) = 0$$

and $L . 1 + M(-1) + N(0) = 0$

or $\frac{L}{-1} = \frac{M}{-1} = \frac{N}{-2}$ which give

i.e., $L = \frac{1}{\sqrt{6}}, M = \frac{1}{\sqrt{6}}, N = \frac{2}{\sqrt{6}}$

Shortest distance between the opposite edge AC and BD

$$= |L(x_2 - x_1) + M(y_2 - y_1) + N(z_2 - z_1)|$$

$$= \left|\frac{1}{\sqrt{6}}(0) + \frac{1}{\sqrt{6}}(0) + \frac{2}{\sqrt{6}}(a - 0)\right| = \frac{2a}{\sqrt{6}}$$

Similarly, the distances between the opposite edges AB, CD and BC, AD can be shown each equal to $\frac{2a}{\sqrt{6}}$.

Also equations of the line of shortest distance between AC and BD are

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} x & y & z-a \\ 1 & -1 & -1 \\ -1 & -1 & 2 \end{vmatrix} = 0$$

i.e., $x - y = 0$ (1)

and $-x + y + z + a = 0$ (2)

Now $(-a, -a, -a)$ satisfies both (1) and (2). Thus this point lies on the line of shortest distance between AC and BD. Similarly, $(-a, -a, -a)$ lies on the other two lines of shortest distances. Hence it lies on the intersection of all three lines of shortest distances.

8. Find the shortest distance between the straight line joining the points $A(3, 2, -4)$ and $B(1, 6, -6)$ and the straight line joining the points $C(-1, 1, -2)$ and $D(-3, 1, -6)$. Also find equations of the shortest distance and coordinates of the feet of the common perpendicular.

Sol. Equations of the line through $A(3, 2, -4)$ and $B(1, 6, -6)$ are

$$\frac{x-3}{1-3} = \frac{y-2}{6-2} = \frac{z+4}{-6+a} \text{ or } \frac{x-3}{-2} = \frac{y-2}{4} = \frac{z+4}{-2} = t \quad (1)$$

Equations of the lines through $C(-1, 1, -2)$ and $D(-3, 1, -6)$ are

$$\frac{x+1}{-3+1} = \frac{y-1}{1-1} = \frac{z+2}{-6+2} = s \text{ or } \frac{x+1}{-2} = \frac{y-1}{0} = \frac{z+2}{-4} = s \quad (2)$$

Suppose P and Q are the feet of the common perpendicular.

Coordinates of P are $(3 - 2t, 2 + 4t, -4 - 2t)$.

Coordinates of Q are $(-1 - 2s, 1, -2 - 4s)$.

$$\overrightarrow{PQ} = [-1 - 2s - 3 + 2t, 1 - 2 - 4t, -2 - 4s + 4 + 2t]$$
$$= [-2s + 2t - 4, -4t - 1, -4s + 2t + 2]$$

Since $\overrightarrow{PQ}$ is perpendicular to (1), we have

$$-2(-2s + 2t - 4) + 4(-4t - 1) - 2(-4s + 2t + 2) = 0$$

$\Rightarrow$ $12s - 24t = 0$ *i.e.*, $s = 2t$ (3)

Also $\overrightarrow{PQ}$ is perpendicular to (2). Therefore,

$$-2(-2s + 2t - 4) - 4(-4s + 2t + 2) = 0$$

or $4s - 4t + 8 + 16s - 8t - 8 = 0$

or $20s - 12t = 0$

Thus $5s - 3t = 0$ *i.e.*, $s = \frac{3}{5}t$ (4)

From (3) and (4), we get $s = 0, t = 0$

Thus $P = (3, 2, -4)$ and $Q = (-1, 1, -2)$

$$|PQ| = \sqrt{(-1-3)^2 + (1-2)^2 + (-2+4)^2}$$
$$= \sqrt{16 + 1 + 4} = \sqrt{21}$$

Equations of PQ are $\frac{x-3}{-4} = \frac{y-2}{-1} = \frac{z+4}{2}$.

Exercise Set 8.7 (Page 379)

1. Find the cylindrical coordinates of the point whose rectangular coordinates are:

(*a*) $(2\sqrt{3}, 2, -2)$ (*b*) $\left(\frac{16}{5}, \frac{12}{5}, 3\right)$

Sol.

(*a*) $(2\sqrt{3}, 2, -2)$

We have, $2\sqrt{3} = r\cos\theta$ and $2 = r\sin\theta$

Squaring these equations and adding the results, we have

$16 = r^2$ or $r = 4$

and $\tan\theta = \frac{1}{\sqrt{3}}$ giving $\theta = \frac{\pi}{6}$

Cylindrical coordinates are $\left(4, \frac{\pi}{6}, -2\right)$

(*b*) Here, $\frac{16}{5} = r\cos\theta$ and $\frac{12}{5} = r\sin\theta$

$r^2 = \frac{400}{25} = 16$ or $r = 4$

$\tan\theta = \frac{3}{4}$ or $\theta = \arctan\left(\frac{3}{4}\right)$

Cylindrical coordinates $\left[4, \arctan\left(\frac{3}{4}\right), 3\right]$

2. Change the following from cylindrical coordinates to rectangular coordinates.

(*a*) $\left(5, \frac{\pi}{6}, 3\right)$ (*b*) $\left(6, \frac{\pi}{3}, -5\right)$

Sol.

(*a*) $\left(5, \frac{\pi}{6}, 3\right)$

$r = 5, \theta = \frac{\pi}{6}$; $x = r\cos\theta = 5\cos\frac{\pi}{6} = \frac{5\sqrt{3}}{2}$

$y = r\sin\theta = 5\sin\frac{\pi}{6} = \frac{5}{\sqrt{2}}$

Rectangular coordinates are $\left(\frac{5\sqrt{3}}{2}, \frac{5}{\sqrt{2}}, 3\right)$

(b) Here $x = 6\cos\frac{\pi}{3} = 3$, $y = 6\sin\frac{\pi}{3} = 3\sqrt{3}$

Rectangular coordinates are $(3, 3\sqrt{3}, -5)$.

3. Find the spherical coordinates of the point whose rectangular coordinates are

(a) $(1, 1, \sqrt{6})$ (b) $(-2, 2\sqrt{3}, 4)$

(c) $(-\sqrt{3}, 1, -2)$ (d) $(4, -4\sqrt{3}, 6)$

Sol.

(a) $\rho = \sqrt{1+1+6} = \sqrt{8}$

$\tan\theta = \frac{y}{x} = 1$ or $\theta = \frac{\pi}{4}$

$z = \rho\cos\phi$ gives $\sqrt{6} = \sqrt{8}\cos\phi$,

$\cos\phi = \frac{\sqrt{3}}{2}$ or $\phi = \frac{\pi}{6}$

Spherical coordinates are $\left(\sqrt{8}, \frac{\pi}{4}, \frac{\pi}{6}\right)$.

(b) $\rho = \sqrt{4+12+16} = 4\sqrt{2}$

$\rho\cos\phi = z \Rightarrow 4\sqrt{2}\cos\phi = 4$

or $\cos\phi = \frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{\pi}{4}$

$\tan\theta = \frac{2\sqrt{3}}{-2} = -\sqrt{3} \Rightarrow \theta = \frac{2\pi}{3}$

Spherical coordinates are $\left(4\sqrt{2}, \frac{2\pi}{3}, \frac{\pi}{4}\right)$.

(c) $\rho = \sqrt{3+1+4} = 2\sqrt{2}$

$x = -\sqrt{3} = r\cos\theta$; $y = 1 = r\sin\theta$

$r = 2$ and $\tan\theta = \frac{-1}{\sqrt{3}}$ or $\theta = \frac{5\pi}{6}$

$z = \rho\cos\phi$ gives $-2 = 2\sqrt{2}\cos\phi$

or $\cos\phi = \frac{-1}{\sqrt{2}}$ and so $\phi = \frac{3\pi}{4}$

Required coordinates $= \left(2\sqrt{2}, \frac{5\pi}{6}, \frac{3\pi}{4}\right)$

(d) $\rho = \sqrt{16+48+36} = 10$

$4 = r\cos\theta$ and $-4\sqrt{3} = r\sin\theta$

Therefore, $r = \sqrt{16+48} = 8$

$\tan\theta = -\sqrt{3}$ and $\theta = \frac{5\pi}{3}$.

$6 = \rho\cos\phi = 10\cos\phi$

or $\cos\phi = \frac{3}{5}$ or $\phi = \arccos\frac{3}{5}$

Required coordinates $= \left[10, \frac{5\pi}{3}, \text{arcco}\left(\frac{3}{5}\right)\right]$

4. Find the rectangular coordinates of the point whose spherical coordinates are

(a) $\left(5, \frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left(4, \frac{\pi}{3}, \frac{2\pi}{3}\right)$

(c) $\left(0, \frac{\pi}{11}, \frac{\pi}{5}\right)$ (d) $\left(2, \frac{5\pi}{3}, \frac{3\pi}{4}\right)$

Sol.

(a) If (ρ, θ, ϕ) are the spherical polar coordinates of a point then the rectangular coordinates (x, y, z) are given by

$x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\phi$

Therefore, $x = 5 . \sin\frac{\pi}{2}\cos\frac{\pi}{2} = 0$

$y = 5 . \sin\frac{\pi}{2} . \sin\frac{\pi}{2} = 5$

and $z = 5 . \cos\frac{\pi}{2} = 0 = 0$

Hence $(x, y, z) = (0, 5, 0)$

(b) $x = 4 . \sin\frac{2\pi}{3} . \cos\frac{\pi}{3} = 4 . \frac{\sqrt{3}}{2} . \frac{1}{2} = \sqrt{3}$

$y = 4 . \sin\frac{2\pi}{3} . \sin\frac{\pi}{3} = 4. \frac{\sqrt{3}}{2} . \frac{\sqrt{3}}{2} = 3$

$z = 4 . \cos\frac{2\pi}{3} = 4\left(-\frac{1}{2}\right) = -2$

Hence $(x, y, z) = (\sqrt{3}, 3, -2)$

(c) $x = \rho\sin\phi\cos\theta = 0 . \sin\frac{\pi}{5}\cos\frac{\pi}{11} = 0$

$y = \rho\sin\phi\sin\theta = 0$; $z = \rho\cos\phi = 0$

Rectangular coordinates are $(0, 0, 0)$.

(d) $\rho = 2, \theta = \frac{5\pi}{3}, \phi = \frac{3\pi}{4}$

$x = \rho\sin\phi\cos\theta = 2\sin\left(\frac{3\pi}{4}\right)\cos\left(\frac{5\pi}{3}\right) = 2 . \frac{1}{\sqrt{2}} . \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$

$$y = \rho \sin \phi \sin \theta = 2 \cdot \frac{1}{\sqrt{2}}\left(\frac{-\sqrt{3}}{2}\right) = -\sqrt{\frac{3}{2}}$$

$$z = \rho \cos \phi = 2 \cos\left(\frac{3\pi}{4}\right) = 2 \cdot \left(\frac{-1}{\sqrt{2}}\right) = -\sqrt{2}.$$

Rectangular coordinates $\left(\frac{1}{\sqrt{2}}, -\sqrt{\frac{3}{2}}, -\sqrt{2}\right)$.

In (Problems 5 - 10), express the given equation in rectangular coordinates:

5. $\rho \cos \phi = 2$

Sol. Since $z = \rho \cos \phi$, the given equation in rectangular coordinates is $z = 2$.

6. $\rho = 2 \cos \theta \sin \phi$

Sol. $\rho^2 = 2\rho \cos \theta \sin \phi$ or $x^2 + y^2 + z^2 = 2x$

7. $\rho = 7 \sin \theta \sin \phi$

Sol. $\rho^2 = 7\rho \sin \phi \sin \theta$ or $x^2 + y^2 + z^2 = 7y$

8. $\rho^2 \cos 2\theta = a^2$

Sol. $\rho^2(2 \cos^2 \theta - 1) = a^2$

or $(x^2 + y^2 + z^2)\left(2 \cdot \frac{x^2}{x^2 + y^2} - 1\right) = a^2$

or $(x^2 + y^2 + z^2)(x^2 - y^2) = a^2(x^2 + y^2)$

9. $z = r^2 \cos 2\theta$.

Sol. $z = r^2(\cos^2 \theta - \sin^2 \theta) = x^2 - y^2$

10. $z = 1 + \sin \theta$

Sol. $z - 1 = \sin \theta$

or $r(z - 1) = r \sin \theta$

or $r^2(z - 1)^2 = r^2 \sin^2 \theta$

or $(x^2 + y^2)(z - 1)^2 = y^2$

In (Problems 11 - 14) express the given equation in cylindrical and spherical coordinates:

11. $(x + y)^2 - z^2 + 4 = 0$

Sol. For spherical coordinates, we set

$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$

in the given equation. It becomes

$(\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta)^2 - (\rho \cos \phi)^2 + 4 = 0$

or $\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta + 2 \sin \theta \cdot \cos \theta) - \rho^2 \cos^2 \phi + 4 = 0$

i.e., $\rho^2 \sin^2 \phi (1 + \sin 2\theta) - \rho^2 \cos^2 \phi + 4 = 0$

or $\rho^2(\sin^2 \phi - \cos^2 \phi) + \rho^2 \sin^2 \phi \cdot \sin 2\theta + 4 = 0$

i.e. $\rho^2(-\cos 2\phi) + \rho^2 \sin^2 \phi \cdot \sin 2\theta + 4 = 0$

i.e., $\rho^2[\sin^2 \phi \cdot \sin 2\theta - \cos 2\phi] + 4 = 0$

For cylindrical coordinates, we put

$x = r \cos \theta, y = r \sin \theta, z = z$, and the given equation becomes

$(r \cos \theta + r \sin \theta)^2 - z^2 + 4 = 0$

or $r^2(1 + \sin^2 \theta) - z^2 + 4 = 0$

12. $x^2 + y^2 + 2z = 6$

Sol. In spherical coordinates, the given equation is

$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \theta \sin^2 \phi + 2\rho \cos \phi = 6$

or $\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + 2\rho \cos \phi = 6$

or $\rho^2 \sin^2 \phi + 2\rho \cos \phi = 6$

In cylinderical coordinates, the given equation is

$r^2 \cos^2 \theta + r^2 \sin^2 \theta + 2z = 6$

or $r^2 + 2z = 6$

13. $x^2 - y^2 - z^2 = 1$

Sol. For cylindrical coordinates, we have $x = r \cos \theta$ and $y = r \sin \theta$. The given equation becomes

$r^2 \cos^2 \theta - r^2 \sin^2 \theta - z^2 = 1$

or $r^2 \cos 2\theta - z^2 = 1$

For spherical coordinates, we have

$x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi$

and $x^2 - y^2 - z^2 = 1$ becomes

$\rho^2 \sin^2 \phi \cos^2 \theta - \rho^2 \sin^2 \phi \sin^2 \theta - \rho^2 \cos^2 \phi = 1$

or $\rho^2(\sin^2 \phi \cos 2\theta - \cos^2 \phi) = 1$

14. $3x + y - 4z = 12$

Sol. In cylindrical coordinates, the given equation is

$3r \cos \theta + r \sin \theta - 4z = 12$

In spherical coordinates, the equation becomes

$3\rho \sin \phi \cos \theta + \rho \sin \phi \sin \theta - 4\rho \cos \phi = 12$

or $\rho(3 \sin \phi \cos \theta + \sin \phi \sin \theta - 4 \cos \phi) = 12$

15. Write the equation of the surface defined by

$\frac{(z-1)^2}{4} - \frac{(y+2)^2}{1} = 4(x - 4)$ relative to a new set of parallel axes with origin at $(4, -2, 1)$

Sol. Let the new origin be O' $(4, -2, 1)$. If any point P has coordinates (x, y, z) referred to the original axes and P has coordinates (x', y', z') referred to the new set of axes through O', then

$x = x' + 4, y = y' - 2, z = z' + 1$

Putting these values of x, y, z in the given equation, we get

$$\frac{(z'+1-1)^2}{4}-\frac{(y-2+2)^2}{1}=4(x'+4-4)$$

or $\frac{z'^2}{4}-\frac{y'^2}{2}=4x'$ as the required equation.

16. Write the equation $x^2-9y^2-4z^2-6x+18y+16z+20=0$ referred to new set of parallel axes with the origin at (3, 1, 2).

Sol. Put $x=x'+3, y=y'+1, z=z'+2$ in the given equation.
Therefore, the equation becomes
$(x'+3)^2-9(y'+1)^2-4(z+2)^2-6(x'+3)+18(y'+1)+16(z+2)+20=0$
or $x'^2-9y'^2-4z'^2+36=0$ or $9y'^2+4z'^2-x'^2=36$
or $\frac{y'^2}{4}+\frac{z'^2}{9}-\frac{x'^2}{36}=1$ is the required equation.

Exercise Set 8.8 (Page 382)

Find an equation of the trace of the given surface in the specified coordinate plane. Identify the trace (Problem 1–4):

1. $x^2+y^2+z^2-2xz+5z-4=0$; xy-plane

Sol. For the trace in xy-plane we put $z=0$ in the equation and get
$$x^2+y^2-4=0 \quad \text{or} \quad x^2+y^2=4$$
which is a circle with radius 2 units and centre at the origin.

2. $xy+yz+zx+zx=1$; xz-plane

Sol. For the trace in xz-plane, put $y=0$ in the given equation.
Therefore, $zx=1$ in the required trace which is *a* hyperbola.

3. $x^2+4y^2+z^2+4xy-2xz-2x-4y+z+1=0$; xy-plane

Sol. The trace in the xy-plane is found by putting $z=0$ in the given equation. Required trace is
$$x^2+4y^2+4xy-2x-4y+1=0$$
which is a pair of coincident lines.

4. $x^2+xy-3xz-2=0$; yz-plane

Sol. Here the trace in yz-plane is found by putting $x=0$ in the given equation. This implies that $-2=0$ which is absurd. Hence there is no trace in the yz-plane.

Find the intercepts of the given surface on the coordinates axes (Problems 5 – 6):

5. $x^2+4y^2+5xz-2x+y-3=0$

Sol. The intercepts on the x-axis, y-axis and z-axis are found by putting $y=z=0$, $z=x=0$ and $x=y=0$ respectively in the given equation.
Therefore, $x^2-2x-3=0$ gives $x=3,-1$

Intercepts on the x-axis are 3 and -1.
Intercepts on the y-axis are given by
$$4y^2+y-3=0$$
or $y=\frac{-1\pm\sqrt{1+48}}{8}=\frac{-1\pm7}{8}=-1,\frac{3}{4}$

Intercepts on the y-axis are -1 and $\frac{3}{4}$

There are no intercepts on the z-axis.

6. $2x^2-z^2-xy-8yz+y-z-2=0$ (1)

Sol. Here, we find intercepts on the x-axis by putting $y=0=z$ in (1).
i.e., $2x^2-2=0$ or $x=-1,1$
Therefore, -1, 1 are the intercepts on the x-axis
Putting $x=0=z$ in (1), we have
$y-2=0$ *i.e.,* 2 is the intercept on y-axis.
Again, setting $x=0=y$ in (1), we get $z^2+z+2=0$, which gives imaginary roots.
Hence z-intercepts are imaginary.

Exercise Set 8.9 (Page 383)

Write an equation of the surface obtained by revolving the given curve about the specified coordinate axis (Problems 1 – 5):

1. $x^2+2y^2=8, z=0$; (*a*) x-axis (*b*) y-axis

Sol.

(*a*) Since the curve is in xy-plane, for surface of revolution about x-axis we replace y^2 by y^2+z^2. Therefore, the required surface is $x^2+2(y^2+z^2)=8$

(*b*) For surface of revolution about the y-axis, we replace x^2 by x^2+z^2 in the given equation. Hence the required surface is
$$x^2+z^2+2y^2=8$$

2. $4x^2-9z^2=5, y=0$; (*a*) y-axis (*b*) z-axis

Sol.

(*a*) The curve is in the xz-plane. Therefore, for surface of revolution we replace z^2 by y^2+z^2 and get $4x^2-9(y^2+z^2)=5$.

(*b*) Here we replace x^2 by x^2+y^2 and have $4(x^2+y^2)-9z^2=5$ as the required surface of the revolution about z-axis.

3. $6y^2+6z^2=7, y=0$; (*a*) y-axis (*b*) z-axis

Sol.

(a) This curve in the yz-plane. For surface of revolution about y-axis, we replace z^2 by $x^2 + z^2$. The required surface is

$$6y^2 + 6(x^2 + z^2) = 7$$

(b) For surface of revolution about z-axis, replace y^2 by $x^2 + y^2$ and get $6(x^2 + y^2) + 6z^2 = 7$ as the required surface.

4. $2x + 3y = 6, z = 0;$ **(a)** x-axis **(b)** y-axis

Sol.

(a) We replace y^2 by $y^2 + z^2$ *i.e.,* y by $\sqrt{y^2 + z^2}$

Hence the surface of revolution about x-axis is

$$2x + 3\sqrt{y^2 + z^2} = 6$$

or $9(y^2 + z^2) = (6 - 2x)^2$

i.e., $4x^2 - 19(y^2 + z^2) - 24x + 36 = 0$

(b) Similarly, the surface of revolution about y-axis is

$$2\sqrt{x^2 + z^2} + 3y = 6$$

or $4(x^2 + z^2) = (6 - 3y)^2$

or $4x^2 - 9y^2 + 4z^2 + 36y - 36 = 0$

5. $y = 2,$ **(a)** y-axis **(b)** z-axis

Sol.

(a) The curve is in yz-plane. The surface of revolution about y-axis is $y = 2$ (Since the curve does not contain terms in z).

(b) The surface of revolution about z-axis is

$$\sqrt{x^2 + y^2} = 2 \quad \text{or} \quad x^2 + y^2 = 4$$

State which coordinate axis is the axis of revolution for the given surface and write equations for a generatix in the specified coordinate plane (Problems 6 – 9):

6. $x^2 + y^2 + z = 2;$ xz-plane

Sol. The given surface contains $x^2 + y^2$ which has been replaced for x^2 in the xz-plane. Therefore, the axis of revolution is the z-axis and hence curve is

$$x^2 + z = 2, y = 0 \quad \text{which is a parabola.}$$

7. $x^2 - 4y^2 - 4z^2 = 8$; xy-plane

Sol. We note that $y^2 + z^2$ has been substituted for y^2. Therefore, the axis of revolution is x-axis and the required curve is $x^2 - 4y^2 = 8, z = 0$ which is a hyperbola.

8. $x^2 - 4y^2 - 4z^2 = 0;$ xz-plane

Sol. Here the curve being in the xz-plane, z^2 has been replaced by $y^2 + z^2$. Therefore, the axis of revolution is the x-axis and the curve is

$$x^2 - 4z^2 = 0, y = 0.$$

9. $x^2y^2 + y^2z^2 = 1;$ xz-plane

Sol. The equation may be written as

$$y^2(x^2 + z^2) = 1 \quad \text{or} \quad x^2 + z^2 = \left(\frac{1}{y}\right)^2$$

This is of the form $x^2 + z^2 = [f(y)]^2$. Hence, y-axis is the axis of revolution and the curve is

$$z^2 = \frac{1}{y^2} \quad \text{or} \quad z = \frac{1}{y}, x = 0.$$

10. Find an equation of the **torus** obtained by revolving about y-axis the circle in the xy-plane with centre at $(a, 0, 0)$ and radius b, where $0 < b < a$.

Sol. Equation of the given circle is $(x - a)^2 + y^2 = b^2$. For the required surface, the axis of revolution being y-axis, we replace x^2 by $x^2 + z^2$ or x by $\sqrt{x^2 + z^2}$.

Hence equation of the torus is

$$\left(\sqrt{x^2 + z^2} - a\right)^2 + y^2 = b^2$$

or $x^2 + z^2 + a^2 - 2a\sqrt{x^2 + z^2} + y^2 - b^2 = 0$

or $(x^2 + y^2 + z^2 + a^2 - b^2)^2 = 4a^2(x^2 + z^2).$

Exercise Set 8.10 (Page 387)

Derive an equation of the cylinder from definition with given directrix and elements parallel to the given vector (Problems 1 – 3):

1. $x^2 + y^2 = 9,$ $\boldsymbol{n} = [1, -2, 1].$

Sol. The line L through $(x_1, y_1, 0)$ and parallel $\boldsymbol{n}$ is

$$L: \begin{cases} x = x_1 + t \\ y = y_1 - 2t \\ z = t \end{cases}$$

Thus, $x_1 = x - z, y_1 = y + 2z$. Equation of the directrix is $f(x, y) = x^2 + y^2 - 9 = 0$ and (x_1, y_1) lies on it. Hence $x_1^2 + y_1^2 - 9 = 0$. An equation of the cylinder is

$$(x - z)^2 + (y + 2z)^2 - 9 = 0$$

i.e., $x^2 + y^2 + 5z^2 - 2zx + 4yz - 9 = 0.$

2. $x + y = 4,$ $\boldsymbol{n} = [0, 2, -1].$

Sol. The line l through $(x_1, 0, z_1)$ and parallel to $\boldsymbol{n}$ is

$$L:\begin{cases} x = x_1 \\ y = 2t \\ z = z_1 - t \end{cases}$$

Here $t = \frac{y}{2}$. Hence $x_1 = x$, $z_1 = z + \frac{y}{2}$. Since $(x_1, 0, z_1)$ lies on $x + z = 4$, $x_1 + z_1 = 4$. Therefore, equation of the cylinder is

$$x + z + \frac{y}{2} = 4 \quad \text{or} \quad 2x + y + 2z = 8$$

3. $\frac{z^2}{4} + \frac{y^2}{9} = 1$, $\quad \boldsymbol{n} = [1, 1, 1]$

Sol. A line parallel to $\boldsymbol{n}$ and through the point $(0, y_1, z_1)$ is

$$L:\begin{cases} x = t \\ y = y_1 + t \\ z = z_1 + t \end{cases}$$

Therefore, $y_1 = y - x$, $z_1 = z - x$. The point $(0, y_1, z_1)$ lies on the directrix. Hence $\frac{z_1^2}{4} + \frac{y_1^2}{9} = 1$.

Equation of the cylinder is $\frac{(z-x)^2}{4} + \frac{(y-x)^2}{9} = 1$.

i.e., $13x^2 + 4y^2 + 9z^2 - 8xy - 18zx - 36 = 0$.

Discuss the given surfaces (Problems 4 – 11):

4. $\frac{x^2}{2} + \frac{y^2}{4} = 1$

Sol. The equation represents an ellipse in the xy-plane. The surface is a right elliptic cylinder parallel to the z-axis.

5. $yz = 2$ (1)

Sol. (1) represents a hyperbola in the yz-plane. Hence it is a right hyperbolic cylinder parallel to the x-axis.

6. $z = k$ (1)

Sol. (1) represent a plane parallel to the xy-plane. Thus it is a right cylinder made up of planes parallel to the xy-plane.

7. $9x^2 + 4z^2 - 18x - 16z - 11 = 0$ (1)

Sol. $(9x^2 - 18x) + (4z^2 - 16z) - 11 = 0$

or $\quad (9x^2 - 18x + 9) + (4z^2 - 16z + 16) - 9 - 16 - 11 = 0$

or $\quad 9(x^2 - 2x + 1) + 4(z^2 - 4z + 4) - 36 = 0$

i.e., $\quad 9(x-1)^2 + 4(z-2)^2 - 36 = 0$

or $\quad \frac{(x-1)^2}{4} + \frac{(z-2)^2}{9} = 1$

It is an ellipse in the xz-plane with centre at $(1, 0, 2)$. Major axis: 6; minor axis; 4. Hence (1) represents an elliptic cylinder parallel to the y-axis.

8. $z = \sqrt{y-1}$ (1)

Sol. (1) can be written as

$z^2 = y - 1$ which is a parabola in the yz-plane. Hence it represents a right parabolic cylinder parallel to the x-axis.

9. $x^2 + y^2 - 4x + 6y + 11 = 0$ (1)

Sol. (1) may be written as

$$(x^2 - 4x + 4) + (y^2 + 6y + 9) - 4 - 9 + 11 = 0$$

or $\quad (x-2)^2 + (y+3)^2 = 2$

which is a circle with centre at $(2, -3, 0)$. Thus (1) is represents a right circular cylinder parallel to the z-axis.

10. $z = \sin x$ (1)

Sol. (1) represent the sinusoidal cylinder parallel to the y-axis extending above and below the zx-plane.

11. $r = r_0$ (cylindrical coordinates)

Sol. $r = r_0$ in the rectangular coordinates is $x^2 + y^2 = r_0^2$ which represents a right circular cylinder parallel to the z-axis.

12. Write an equation of the right circular cylinder with radius 2 and centre at $(3, 0, 5)$.

Sol. The right circular cylinder is parallel to the y-axis. Equation of the circle in xz-plane with radius 2 and centre at $(3, 0, 5)$ is

$$(x-3)^2 + (z-5)^2 = 4$$

or $\quad x^2 + z^2 - 6x - 10z + 30 = 0$

This is the required equation of the right circular cylinder.

13. Write an equation of the right elliptic cylinder whose directrix is in the yz-plane with foci $(0, \pm 3, 0)$ and major axis 8.

Sol. Semi-major axis a of the ellipse $= 4$. Also $c = 3$

Therefore, $c^2 = a^2 - b^2$ $\quad$ *i.e.*, $\quad 9 = 16 - b^2$

or $\quad b = \sqrt{7}$, which is semi-minor axis of the ellipse.

Equation of the directrix of the cylinder is

$$\frac{y^2}{16} + \frac{z^2}{7} = 1$$

or $\quad 7y^2 + 16z^2 - 112 = 0$.

This is the required equation of the right elliptic cylinder.

Find an equation of the cone whose directrix and vertex are given (Problems 14 – 16):

14. Directrix: $y^2 + z^2 = 1$, $x = 2$; $\quad$ Vertex $A = (0, 0, 0)$

Sol. Let $P(x, y, z)$ be a point on the cone and let the element AP meet the directrix at $Q(x_1, y_1, z_1)$. Then $y_1^2 + z_1^2 = 1, x_1 = 2$ (1)

Equations of the line AQ are $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1}$

Since $x_1 = 2$, we have $y_1 = \dfrac{2y}{x}, z_1 = \dfrac{2z}{x}$

Substituting these values of y_1, z_1 into the first equation of (1), we have

$$\left(\frac{2y}{x}\right)^2 + \left(\frac{2z^2}{x}\right)^2 = 1$$

or $4y^2 + 4z^2 - x^2 = 0$, is the required equation of the cone.

15. Directrix: $4x^2 + (y-2)^2 = 4, z = 3$; Vertex $A = (0, 0, 0)$

Sol. Let $P(x, y, z)$ be a point on the cone and let $Q = (x_1, y_1, z_1)$ be the point on the directrix where the element AP meets the cone. Then

$$4x_1^2 + (y^1 - 2)^2 = 4, z_1 = 3$$

Equations of the line PQ are $\dfrac{x}{x_1} = \dfrac{y}{y_1} = \dfrac{z}{z_1}$

or $x_1 = \dfrac{xz_1}{z} = \dfrac{3x}{z}$, since $z_1 = 3$

$y_1 = \dfrac{yz_1}{z} = \dfrac{3y}{z}$

Equation of the cone is $4\left(\dfrac{3x}{z}\right)^2 + \left(\dfrac{3y}{z} - 2\right)^2 = 4$

i.e., $\dfrac{36x^2}{z^2} + \dfrac{9y^2 + 4z^2 - 12yz}{z^2} = 4$

or $36x^2 + 9y^2 + 4z^2 - 12yz = 4z^2$

or $12x^2 + 3y^2 - 4yz = 0$.

16. Directrix: $x^2 + 4y^2 - 2x + 8y - 4 = 0, z = 3$; Vertex $A = (-1, 2, 1)$

Sol. Let $P(x, y, z)$ be a point on the cone and suppose that the element AP meets the directrix at $Q(x_1, y_1, z_1)$. Then

$$x_1^2 + y_1^2 - 2x_1 + 8y_1 - 4 = 0, z_1 = 3 \quad (1)$$

Equation of the line PQ are

$$\frac{x+1}{x_1+1} = \frac{y-2}{y_1-2} = \frac{z-1}{z_1-1} = \frac{z-1}{3-1} = \frac{z-1}{2}, \text{ since } z_1 = 3$$

Therefore, $x_1 + 1 = \dfrac{2(x+1)}{z-1}$ and $y_1 - 2 = \dfrac{2(y-2)}{z-1}$

or $x_1 = \dfrac{2(x+1)}{z-1} - 1 = \dfrac{2x - z + 3}{z-1}$.

$$y_1 = \frac{2(y-2)}{z-1} + 2 = \frac{2y + 2z - 6}{z-1}$$

Substituting these values of x_1, y_1 into the first equation of (1), we have

$$\left(\frac{2x-z+3}{z-1}\right)^2 + 4\left(\frac{2y+2z-6}{z-1}\right)^2 - 2\left(\frac{2x-z+3}{z-1}\right) + 8\left(\frac{2y+2z-6}{z-1}\right) - 4 = 0$$

or $(2x - z + 3)^2 + 16(y + z - 3)^2 - 2(2x - z + 3)(z - 1)$
$\quad + 16(y + z - 3)(z - 1) - 4(z - 1)^2 = 0$

or $4x^2 + z^2 + 9 - 4xz + 12x - 6z + 16(y^2 + z^2 + 9 + 2yz - 6y - 6z)$
$\quad -2(2xz - 2x - z^2 + 4z - 3) + 16(yz - y + z^2 - 4z + 3)$
$\quad -4(z^2 - 2z + 1) = 0$

or $4x^2 + 16y^2 + 31z^2 - 8xz + 48yz + 16x - 112y - 166z + 203 = 0$

is the required equation of the cone.

17. Show that an equation of the cone with vertex at (3, 1, 2) and directrix $2x^2 + 3y^2 = 1, z = 0$ is $2x^2 + 3y^2 + 5z^2 - 3yz - 6zx + z - 1 = 0$.

Sol. Any line through (3, 1, 2) is

$$\frac{x-3}{l} = \frac{y-1}{m} = \frac{z-2}{n} = t \quad (1)$$

A point on this line is $(3 + lt, 1 + mt, 2 + nt)$

This lies on $2x^2 + 3y^2 = 1, z = 0$ if

$$2(3 + lt)^2 + 3(1 + mt)^2 = 1, 2 + nt = 0$$

or $2(9 + l^2t^2 + 6lt) + 3(1 + m^2t^2 + 2mt) = 1, \quad t = -\dfrac{2}{n}$

or $18 + 2l^2t^2 + 12lt + 3 + 3m^2t^2 + 6mt = 1, \quad t = -\dfrac{2}{n}$

or $18 + 2l^2 \times \dfrac{4}{n^2} + 12l\left(-\dfrac{2}{n}\right) + 3 + 3m^2\left(\dfrac{4}{n^2}\right) + 6m\left(-\dfrac{2}{n}\right) = 1$

or $20 + \dfrac{8l^2}{n^2} - \dfrac{24l}{n} + \dfrac{12m^2}{n^2} - \dfrac{12m}{n} = 0$

i.e., $20n^2 + 8l^2 - 24ln + 12m^2 - 12mn = 0$ (2)

Eliminating l, m, n from (1) and (2), we get

$$20(z-2)^2 + 8(x-3)^2 - 24(x-3)(x-2) + 12(y-1)^2 - 12(y-1)(z-2) = 0$$

or $5(z-2)^2 + 2(x-3)^2 - 6(x-3)(z-2) + 3(y-1)^2 - 3(y-1)(z-2) = 0$

or $2x^2 + 3y^2 + 5z^2 - 3yz - 6zx + z - 1 = 0$.

18. Show that an equation of the cylinder whose generators intersect the curve $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, z = 0$ and are parallel to $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$ is

$a(nx - lz)^2 + 2h(nx - lz)(ny - mz) + b(ny - mz)^2 + 2gn(nx - lz) + 2nf(ny - mz) + n^2c = 0$.

Sol. Let $P(x_1, y_1, z_1)$ be any point on the cylinder.

Equation of the generator through P are

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad (1)$$

Coordinates of any point on (1) are

$(x_1 + lt, y_1 + mt, z_1 + nt)$

This lies on the curve if

$$z_1 + nt = 0 \qquad (2)$$

and $a(x_1 + lt)^2 + 2h(x_1 + lt)(y^1 + mt) + b(y_1 + mt)^2$

$$+2g(x_1 + lt) + 2f(y_1 + mt) + c = 0 \qquad (3)$$

Eliminating t from (2) and (3), we get

$$a\left(x_1 - \frac{lz_1}{n}\right)^2 + 2h\left(x_1 - \frac{lz_1}{n}\right)\left(y_1 - \frac{mz_1}{n}\right) + b\left(y_1 - \frac{mz_1}{n}\right)2$$

$$+2g\left(x_1 - \frac{lz_1}{n}\right) + 2f\left(y1 - \frac{mz_1}{n}\right) + c = 0$$

On simplification, the locus of (x_1, y_1, z_1) is

$$a(nx - nl)^2 + 2h(nx - lz)(ny - mz) + b(ny - nz)^2$$
$$+ 2gn(nx - lz) + 2nf(ny - mz) + n^2c = 0$$

which is the required equation of the cylinder.

19. Show that an equation of the cylinder whose generators are parallel to the z-axis and which passes through the curve $x^2 + y^2 + z^2 = 1, x + y + z = 1$ is $x^2 + y^2 + xy - x - y = 0$

Sol. Let (x_1, y_1, z_1) be any point on the cylinder. Direction cosines of the z-axis are 0, 0, 1.

Equations of the generator through $P(x_1, y_1, z_1)$ are

$$\frac{x - x_1}{0} = \frac{y - y_1}{0} = \frac{z - z_1}{1} \qquad (1)$$

Any point on (1) is $(x_1, y_1, z_1 + t)$.

If this point lies on the curve, then

$$x_1^2 + y_1^2 + (z_1 + t)^2 = 1, \qquad (2)$$
$$x_1 + y_1 + z_1 + t = 1 \qquad (3)$$

Eliminating t from (2) and (3), we get

$$x_1^2 + y_1^2 + (1 - x_1 - y_1)^2 = 0$$

or $\quad 2x_1^2 + 2y_1^2 + 2x_1y_1 - 2x_1 - 2y_1 = 0$

The locus of (x_1, y_1, z_1) is $x^2 + y^2 + xy - x - y = 0$

which is an equation of the required cylinder.

20. Show that an equation of the right circular cylinder of radius 3 and whose axis is the line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z-5}{-1}$ is

$$5x^2 + 5y^2 + 8z^2 - 8xy + 4xy + 4yz - 6x - 42y - 96z + 225 = 0.$$

Sol. Let $P(x, y, z)$ be any point on the cylinder. $A(1, 3, 5)$ is a point on its axis. Let L be the foot of the perpendicular from P on its axis

Then AL = Projection of AP on its axis

$$= \frac{2(x-1) + 2(y-3) - 1(z-5)}{\sqrt{2^2 + 2^2 + (-1)^2}} = \frac{2x + 2y - z - 3}{3}$$

Now $AP^2 = AL^2 + LP^2$, where $LP = 3$

Thus $(x-1)^2 + (y-3)^2 + (z-5)^2 = \left[\frac{2x + 2y - z - 3}{3}\right]^2 + 3^2$

or $\quad 5x^2 + 5y^2 + 8z^2 - 8xy + 4yz + 4yz - 6x - 96z + 225 = 0$

is the required equation.

21. Show that an equation to the right circular cone with vertex at O, axis Oz and semi-vertical angle α is $x^2 + y^2 = z^2 \tan^2 \alpha$.

Sol. Let $P(x, y, z)$ be any point on the cone. Direction cosines of OP are proportional to x, y, z. Direction cosines of Oz are 0, 0, 1.

Hence $\quad \cos \alpha = \frac{x(0) + y(0) + z(1)}{\sqrt{x^2 + y^2 + z^2}}$

i.e., $\quad x^2 + y^2 + z^2 = z^2 \sec^2 \alpha$

$\quad x^2 + y^2 + z^2 = z^2(\tan^2 \alpha + 1) = z^2 \tan^2 \alpha + z^2$

or $\quad x^2 + y^2 = z^2 \tan^2 \alpha$ is the required equation.

22. Show that the general equation to a cone of second degree which passes through the coordinates axes is $fyz + gzx + hxy = 0$ **Sol.**

General equation to a cone of the second degree is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

This is satisfied by direction cosines of the coordinates axes

i.e., by $\quad$ 1, 0, 0; 0, 1, 0; 0, 0, 1

Thus $\quad a = 0, \; b = 0, \; c = 0$

Hence an equation of the cone is

$$2fxz + 2gzx + 2hxy = 0$$

i.e., $\quad fyz + gzx + hxy = 0$

23. Prove that an equation to the cone whose vertex is at the origin and which passes through the curve of intersection of the plane $lx + my + nz = p$ and the surface $ax^2 + by^2 + cz^2 = 1$ is

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2.$$

Sol. Any line through the origin is

$$\frac{x}{A} = \frac{y}{B} = \frac{z}{C} = t \quad (1)$$

A point on (1) is (At, Bt, Ct)

This lies on the curve if

$$lAt + mBt + nCt = p \quad (2)$$

and $a(A^2t^2) + b(B^2t^2) + c(C^2t^2) = 1 \quad (3)$

From (2), $t = \dfrac{p}{Al + Bm + Cn}$

Substituting this value of t into (3), we get

$$(aA^2 + bB^2 + cC^2)\frac{p^2}{(Al + Bm + Cn)^2} = 1$$

or $aA^2 + bB^2 + cC^2 = \left(\dfrac{Al + Bm + Cn}{p}\right)^2 \quad (4)$

Eliminating A, B, C from (1) and (4), we get

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)$$

as the required equation.

24. The plane $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = 1$ meets the coordinates axes in A, B, C. Prove that an equation to the cone generated by lines drawn from the origin to meet the circle ABC is

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + zy\left(\frac{a}{b} + \frac{b}{a}\right) = 0.$$

Sol. Coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$. Equations of the circle through A, B, C are given by

$$x^2 + y^2 + z^2 - ax - by - cz = 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad (1)$$

Any line through the origin is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = t \quad (2)$$

A point on this line is (lt, mt, nt).

This lies on (1) if

$$\frac{lt}{a} + \frac{mt}{b} + \frac{nt}{c} = 1 \quad (3)$$

and $(lt)^2 + (mt)^2 + (nt)^2 - a(lt) - b(mt) - c(nt) = 0$

i.e, $(l^2 + m^2 + n^2)t^2 - t(al + bm + cn) = 0 \quad (4)$

From (3), $t = \dfrac{1}{\frac{l}{a} + \frac{m}{b} + \frac{n}{c}}$

Setting this values of t in (4), we have

$$(l^2 + m^2 + n^2)\cdot\frac{1}{\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)^2} - \frac{al + bm + cn}{\left(\frac{l}{a} + \frac{m}{b} + \frac{n}{c}\right)} = 0$$

$\Rightarrow \quad (l^2 + m^2 + n^2) - (al + bm + cn)\left(\dfrac{l}{a} + \dfrac{m}{b} + \dfrac{n}{c}\right) = 0 \quad (5)$

Eliminating l, m, n from (2) and (5), we get

$$(x^2 + y^2 + z^2) - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

or $x^2 + y^2 + z^2 - \left(x^2 + y^2 + z^2 + \dfrac{a}{b}xy + \dfrac{a}{c}xz + \dfrac{b}{a}xy + \dfrac{b}{c}yz + \dfrac{c}{a}xz + \dfrac{c}{b}yz\right) = 0$

or $yz\left(\dfrac{b}{c} + \dfrac{c}{b}\right) + zx\left(\dfrac{c}{a} + \dfrac{a}{c}\right) + xy\left(\dfrac{a}{b} + \dfrac{b}{a}\right) = 0$

as the required equation.

25. Find an equation to the cone whose vertex is the origin and directrix is the circle $x = a, y^2 + z^2 = b^2$. Show that the trace of the cone in a plane parallel to the xy-plane is a hyperbola.

Sol. Any line through the origin is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} = t \quad (1)$$

A point on (1) is (lt, mt, nt)

This lies on $x = a, y^2 + z^2 = b^2$ if

$$lt = a, m^2t^2 + n^2t^2 = b^2$$

Eliminating t from the last two equations, we get

$$m^2\frac{a^2}{l^2} + n^2\frac{a^2}{l^2} = b^2$$

or $a^2(m^2 + n^2) = b^2l^2 \quad (2)$

Eliminating l, m, n from (1) and (2), we have

$$a^2(y^2 + z^2) = b^2x^2$$

which is the required equation of the cone.

A plane parallel to the xy-plane is $z = k$. Trace of the cone in this plane is $a^2(y^2 + k^2) = b^2x^2$

or $b^2x^2 - a^2y^2 = a^2k^2$, which is a hyperbola.

Exercise Set 8.11 (Page 391)

1. Show that $\rho = c$ is an equation of a sphere **(in spherical coordinates)** of radius c and centre at (0, 0, 0).

Sol. $\rho = c$ implies $\sqrt{x^2 + y^2 + z^2} = c$
or $x^2 + y^2 + z^2 = c^2$ which is an equation of a sphere with centre at the origin and radius c.

2. Find an equation of the sphere whose centre is on the y-axis and which passes through the points (0, 2, 2) and (4, 0, 0).

Sol. Let the centre of the sphere be $(0, b, 0)$ and its radius be r. Then equation of the sphere is $x^2 + (y - b)^2 + z^2 = r^2$.
Since the given points lie on it, we have

$$(0)^2 + (2 - b)^2 + (2)^2 = r^2 \qquad (1)$$

and $$(4)^2 + b^2 + (0)^2 = r^2 \qquad (2)$$

Subtracting (1) from (2), we get

$$-16 - 4b + 8 = 0 \qquad i.e., \qquad b = -2$$

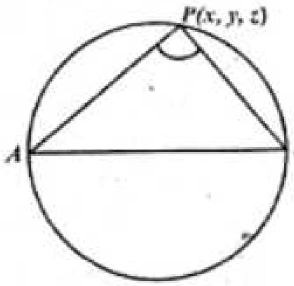
Putting this value of b into (1), we have $r^2 = 20$
Hence equation of the required sphere is

$$(x - 0)^2 + (y + 2)^2 + z^2 = 20$$

or $$x^2 + y^2 + z^2 + 4y - 16 = 0$$

3. Show that an equation of the sphere having the straight line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) as a diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$.

Sol. Let the given points forming a diameter of the sphere be A and B. If P is any point with coordinates (x, y, z) on the sphere, then APB is a right angle. This implies that AP is perpendicular to BP. Now the direction ratios of AP are $x - x_1$, $y - y_1$, $z - z_1$ and those of BP are $x - x_2, y - y_2, z - z_2$.



Hence $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$
which is the required equation of the sphere.

4. Find an equation of the sphere which passes through the points $A(-3, 6, 0)$, $B(-2, -5, -1)$ and $C(1, 4, 2)$ and whose centre lies on the hypotenuse of the right-angled triangle ABC.

Sol. $A(-3, 6, 0)$, $B(-2, -5, -1)$ and $C(1, 4, 2)$ are the given points.
Now $AB^2 = (-2 + 3)^2 = (-5 - 6)^2 + (-1 - 0)^2$

$$= 1 + 121 + 1 = 123$$

$$BC^2 = (1 + 2)^2 + (4 + 5)^2 + (2 + 1)^2$$
$$= 9 + 81 + 9 = 99$$
$$CA^2 = (1 + 3)^2 + (4 - 6)^2 + (2 - 0)^2$$
$$= 16 + 4 + 4 = 24$$

Thus $AB^2 = BC^2 + CA^2$ and so AB is the hypotenuse of the right triangle ABC.
Since centre of the sphere lies on AB, AB is a diameter for the sphere. Therefore, equation of the sphere is

$$(x + 3)(x + 2) + (y - 6)(y + 5) + (z - 0)(z + 1) = 0$$

or $$x^2 + 5x + 6 + y^2 - y - 30 + z^2 + z = 0$$

or $$x^2 + y^2 + z^2 + 5x - y + z - 24 = 0$$

The point (1, 4, 2) lies on this equation.

5. Prove that each of the following equation represents a sphere. Find the centre and radius of each:

(a) $x^2 + y^2 + z^2 - 6x + 4z = 0$
(b) $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$
(c) $4x^2 + 4y^2 + 4z^2 - 4x + 8y + 24z + 1 = 0$ (1)

Sol.

(a) The given equation written in the standard form is

$$(x - 3)^2 + y^2 + (z + 2)^2 = 13.$$

It is a sphere with centre at (3, 0, –2) and radius $\sqrt{13}$.

(b) The equation may be written as

$$(x + 1)^2 + (y - 2) + (z - 3)^2 = -5 + 1 + 4 + 9 = 9,$$

which represents a sphere with centre at (–1, 2, 3) and radius 3.

(c) Dividing through by 4 and re-arranging the terms in (1), we have

$$x^2 - x + y^2 + 2y + z^2 + 6z = -\frac{1}{4}$$

or $$\left(x - \frac{1}{2}\right)^2 + (y + 1)^2 + (z + 3)^2 = -\frac{1}{4} + \frac{1}{4} + 1 + 9 = 10$$

which is a sphere with centre at $\left(\frac{1}{2}, -1, -3\right)$ and radius $\sqrt{10}$.

6. Find an equation of the sphere through the points (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3). Also find its centre and radius.

Sol. Let an equation of the sphere be

$$x^2 + y^2 + z^2 + 2fx + 2gy + 2hz + c = 0 \qquad (1)$$

Since the given point lie on the sphere, they must satisfy this equation. Therefore, substituting these points into (1), we get

$$c = 0,$$
$$1 + 1 + 2g - 2h = 0, \quad \text{or} \quad g - h + 1 = 0$$
$$1 + 4 - 2f + 4g = 0 \quad \text{or} \quad 4g - 2f + 5 = 0$$

$$1 + 4 + 9 + 2f + 4g + 6h = 0 \quad \text{or} \quad f + 2g + 3h + 7 = 0$$

Solving these equations simultaneously, we have

$$g = \frac{-25}{14}, h = \frac{-11}{14} \quad \text{and} \quad f = \frac{-15}{14}$$

Hence an equation of the sphere is

$$x^2 + y^2 + z^2 + 2\left(\frac{-15}{14}\right)x + 2\left(\frac{-25}{14}\right)y + 2\left(\frac{-11}{14}\right)z = 0$$

or $\quad x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0$

or $\quad 7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$

Centre of the sphere is $= \left(\frac{15}{14}, \frac{25}{14}, \frac{11}{14}\right)$ and radius

$$= \sqrt{\left(\frac{15}{14}\right)^2 + \left(\frac{25}{14}\right)^2 + \left(\frac{11}{14}\right)^2} = \frac{\sqrt{971}}{14}$$

7. Find an equation of the sphere passing through the point $(0, -2, -4,)$, $(2, -1, -1)$ and having its centre on the straight line $2x - 3y = 0 = 5y + 2z$.

Sol. Let an equation of the sphere be

$$x^2 + y^2 + z^2 + 2gx + 2fy + 2hx + c = 0$$

Its centre is $(-g, -f, -h)$. Since it lies on the given line, we have

$-2g + 3f = 0 \qquad (1)$

and $\; -5f - 2k = 0 \qquad (2)$

Also the given points lie on the sphere. Therefore,

$4 + 16 - 4f - 8h + c = 0 \qquad (3)$

and $\; 4 + 1 + 1 + 4g - 2f - 2h + c = 0 \qquad (4)$

Solving equations (1) to (4) simultaneously, we get

$$f = -2, g = -3, h = 5 \quad \text{and} \quad c = 12$$

Hence the required equation of the sphere is

$$x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$$

8. Find an equation of the sphere which passes through the circle $x^2 + y^2 + z^2 = 9$, $2x + 3y + 4z = 5$ and the point $(1, 2, 3)$.

Sol. A sphere through the given circle is

$x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0 \qquad (1)$

It passes through $(1, 2, 3)$, then

$$1 + 4 + 9 - 9 + k(2 + 6 + 12 - 5) = 0 \quad i.e., \quad k = -\frac{1}{3}$$

Putting this value of k into (1), we have

$$x^2 + y^2 + z^2 - 9 - \frac{1}{3}(2x + 3y + 4z - 5) = 0$$

or $\quad 3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$

is the required equation.

9. Find an equation of the sphere through the circle $x^2 + y^2 + z^2 = 1$, $2x + 4y + 5z - 6 = 0$ and touching the plane $z = 0$.

Sol. Any sphere through the given circle is

$$x^2 + y^2 + z^2 - 1 + k(2x + 4y + 5z - 6) = 0$$

or $\quad x^2 + y^2 + z^2 + 2kx + 4ky + 5kz - 1 - 6k = 0 \qquad (1)$

Its centre is $\left(-k, -2k, -\frac{5k}{2}\right)$

and radius $= \sqrt{k^2 + 4k^2 + \frac{25k^2}{4} + 1 + 6k}$

As $z = 0$ is a tangent plane to the sphere, its distance from the centre of the sphere equals the radius of the sphere.

Thus $\quad \frac{\left|-\frac{5k}{2}\right|}{1} = \sqrt{\frac{45k^2}{4} + 6k + 1}$

or $\quad \frac{25k^2}{4} = \frac{45k^2}{4} + 6k + 1 \quad \text{or} \quad 5k^2 + 6k + 1 = 0$

i.e., $\quad k = \frac{-6 \pm \sqrt{36 - 20}}{10} = -1, \frac{-1}{5}$

Putting these values of k into (1), equations of the required spheres are $x^2 + y^2 + z^2 - 2x - 4y - 5z + 5 = 0$

and $\quad x^2 + y^2 = z^2 - \frac{2}{5}x - \frac{4}{5}y - z + \frac{1}{5} = 0.$

10. Show that the two circles $x^2 + y^2 + z^2 = 9$, $x - 2y + 4z - 13 = 0$ and $x^2 + y^2 + z^2 + 6y - 6z + 21 = 0$ $x + y + z + 2 = 0$ lie on the same sphere. Also find its equation.

Sol. Any sphere through the first circle is

$$x^2 + y^2 + z^2 + k(x - 2y + 4z - 13) = 0$$

and a sphere through the second circle is

$x^2 + y^2 + z^2 + 6y - 6z + 21 + h(x + y + z + 2) = 0 \qquad (2)$

If the given circles lie on the same sphere, then (1) and (2) must be identical. This requires, (by equating the coefficients of x, y, z and constant terms).

$$k = h, -2k = 6 + h, 4k = -6 + h \quad \text{and} \quad -9 - 13k = 21 + 2h$$

All these equations yield $k = h = -2$. Putting the value of k into (1) or of h into (2), we have

$$x^2 + y^2 + z^2 - 2x + 4y - 8z + 17 = 0$$

as an equation of the sphere on which the two circles lie.

11. Find an equation of the sphere for which the circle $x^2 + y^2 + z^2 + 7y - 2z + 2 = 0$, $2x + 3y + 4z - 8 = 0$ is a great circle.

Sol. A sphere through the given circle is

$x^2 + y^2 + z^2 + 7y - 2z + 2 + k(2x + 3y + 4z - 8) = 0 \qquad (1)$

Its centre is

$$\left(-k, -\frac{7+3k}{2}, 1-2k\right).$$

If the given circle is a great circle of (1) then the centre of the sphere must lie on the plane $2x + 3y + 4z - 8 = 0$.

Therefore, $-2k - \dfrac{3(7+3k)}{2} + 4(1-2k) - 8 = 0$

or $\quad -29k = 29 \quad$ or $\quad k = -1$

Putting $k = -1$ into (1), we have

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 10 = 0$$

as an equation of the required sphere.

12. Find an equation of the sphere with centre $(2, -1, -1)$ and tangent to the plane $x - 2y + z + 7 = 0$.

Sol. Here radius r of the required sphere is

$$r = \frac{|2+2-1+7|}{\sqrt{1+4+1}} = \frac{10}{\sqrt{6}}$$

Therefore, equation of the sphere is

$$(x-2)^2 + (y-1)^2 + (z+1)^2 = \frac{100}{6} = \frac{50}{3}$$

or $\quad 3x^2 + 3y^2 + 3z^2 - 12x + 6y + 6z + 6z - 32 = 0.$

13. Fin an equation of the plane tangent to the sphere

$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$ at the point $(1, 2, 3)$.

Sol. Centre of the given sphere is

$$M = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right), P = (1, 2, 3) \;;\; \overrightarrow{PM} = \left[\frac{-2}{3}, \frac{-3}{2}, \frac{-7}{3}\right]$$

It is a normal vector of the required plane through (1, 2, 3,).

Equation of the plane tangent to the sphere is

$$-\frac{2}{3}(x-1) - \frac{3}{2}(y-2) - \frac{7}{3}(z-3) = 0$$

i.e., $\quad 4x + 9y + 14z - 64 = 0.$

14. Find an equation of the sphere with centre at the point $P(-2, 4, -6)$ and tangent to the

(a) xy-plane **(b)** yz-plane **(c)** zx-plane

Sol.

(a) Magnitude of the perpendicular from $(-2, 4, -6)$ to the plane $z = 0$ is 6. This is the radius of the required sphere. Equation of the sphere with radius 6 and centre at $P(-2, 4, -6)$ is

$$(x+2)^2 + (y-4)^2 + (z+6)^2 = 36$$

i.e., $\quad x^2 + y^2 + z^2 + 4x - 8y + 12z + 20 = 0$

(b) Length of the perpendicular from $P(-2, 4, -6)$ to the plane $x = 0$ is 2 which is radius of the sphere. Equation of the sphere is

$$(x+2)^2 + (y-4)^2 + (z+6)^2 + 2^2$$

or $\quad x^2 + y^2 + z^2 + 4x - 8y + 12z + 52 = 0$

(c) Length of the perpendicular from $P(-2, 4, -6)$ to the plane $y = 0$ is 4. Equation of the sphere with centre at $T(-2, 4, -6)$ and radius is

$$(x+2)^2 + (y-4)^2 + (z+6)^2 = 4^2$$

or $\quad x^2 + y^2 + z^2 + 4x - 8y + 12z + 40 = 0.$

15. Find an equation of the surface whose points are equidistant from $P(7, 8, 2)$ and $Q(5, 2, -6)$.

Sol. Let $R(x, y, z)$ be a point on the surface such that

$$|RP| = |RQ|$$

i.e., $\quad (x-7)^2 + (y-8)^2 + (z-2)^2 = (x-5)^2 + (y-2)^2 + (z+6)^2$

or $\quad -14x - 16y - 4z + 117 = -10x - 4y + 12z + 65$

or $\quad 4x + 12y - 16z - 52 = 0$

i.e., $\quad x + 3y - 4z - 13 = 0$

is an equation of the required surface.

16. A point P moves such that the square of its distance from the origin is proportional to its distance from a fixed plane. Show that P always lies on a sphere.

Sol. Let the fixed plane be

$$lx + my + nz = p \quad \text{where } l^2 + m^2 + n^2 = 1$$

$P(x, y, z)$ be a point on the required locus. Therefore, from the hypothesis, we have

$OP^2 = k$(distance of P from the plane), where O is the origin (0, 0, 0).

or $\quad x^2 + y^2 + z^2 = k(lx + my + nz - p)$

i.e., $\quad x^2 + y^2 + z^2 - klx - kmy - knz - kp = 0$

Thus the locus of P is a sphere.

17. A sphere of radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2 + y^2 + z^2) = 4k^2$.

Sol. Any sphere passing through the origin is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$$

It meets the axes at $(-2u, 0, 0.), (0, -2v, 0)$ and $(0, 0, -2w)$.

Therefore,

$$A = (-2u, 0, 0), B = (0, -2v, 0), C = (0, 0, -2w).$$

Also the radius of this sphere is

$$\sqrt{u^2 + v^2 + w^2} = k$$

or $\quad u^2 + v^2 + w^2 = k \qquad (1)$

Now the centroid of the triangle ABC is

$$\left(\frac{-2u+0+0}{3}, \frac{0-2v+0}{3}, \frac{0+0-3w}{3}\right)$$

$$= \left(\frac{-2u}{3}, \frac{-2v}{3}, \frac{-2w}{3}\right) = (x_1, y_1, z_1) \text{ (say)}$$

Therefore, $x_1 = \frac{-2u}{3}, y_1 = \frac{-2v}{3}, z_1 = \frac{-3w}{3}$

or $\quad u = \frac{-3}{2}x_1, y = \frac{-3y_1}{2}, w = \frac{-3z_1}{2}$

Substituting these values into (1), we get

$$\frac{9}{4}x_1^2 + \frac{9}{4}y_1^2 + \frac{9}{4}z_1^2 = k^2$$

or $\quad 9(x_1^2 + y_1^2 + z_1^2) = 4k^2$

Hence the centroid of the ΔABC lies on the sphere

$$9(x^2 + y^2 + z^2) = 4k^2.$$

18. The plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ meets the axes in A, B, C. Find an equation of the circumcircle of the triangle ABC. Also find the coordinates of the centre of the circle.

Sol. The circle is the intersection of the given plane by a sphere through A, B, C. For convenience, we take the origin as a forth point on the sphere.

Coordinates of A, B, C, are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively.

The sphere through O, A, B, C is

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ 0 & 0 & 0 & 0 & 1 \\ a^2 & a & 0 & 0 & 1 \\ b^2 & 0 & b & 0 & 1 \\ c^2 & 0 & 0 & c & 1 \end{vmatrix} = 0$$

or $\quad x^2 + y^2 + z^2 - ax - by - cz = 0, \qquad (1)$

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \qquad (2)$$

are equations of the circumcircle of the triangle ABC.

Now the centre P of this circle is the foot of the perpendicular from the centre of the sphere (1) to the plane (2). Coordinates of G, the centre of the sphere, are $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$.

Direction ratios of PG, which is perpendicular to the plane (2), are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

Therefore, equations of PG are

$$\frac{x-\frac{a}{2}}{\frac{1}{a}} = \frac{y-\frac{b}{2}}{\frac{1}{b}} = \frac{z-\frac{c}{2}}{\frac{1}{c}} = t \text{ (say)}$$

or $\quad x = \frac{a}{2} + \frac{1}{a}t, \; y = \frac{b}{2} + \frac{1}{b}t, \; z = \frac{c}{2} + \frac{1}{c}t$

are coordinates of P. Since P lies on (2), we have

$$\frac{1}{2} + \frac{t}{a^2} + \frac{1}{2} + \frac{t}{b^2} + \frac{1}{2} + \frac{t}{c^2} = 1$$

or $\quad t = -\frac{1}{2}\,\frac{a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2}.$

Foot T of the perpendicular has coordinates $\left(\frac{a}{2} + \frac{t}{a}, \frac{b}{2} + \frac{t}{b}, \frac{c}{2} + \frac{t}{c}\right)$,

where $t = -\frac{1}{2}\,\frac{a^2b^2c^2}{b^2c^2 + c^2a^2 + a^2b^2}.$

19. A plane passes through a fixed point (a, b, c) and cuts the axes of coordinates in A, B, C. Find the locus of the centre of the sphere $OABC$ for different positions of the plane, O being the origin.

Sol. Let an equation of the plane be

$$\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1 \qquad (1)$$

As it passes through (a, b, c), we have

$$\frac{a}{\alpha} + \frac{b}{\beta} + \frac{c}{\gamma} = 1$$

Coordinates of A, B, C are

$(\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)$ respectively.

Suppose equation of the sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (2)$$

Since it passes through $(0, 0, 0), (\alpha, 0, 0), (0, \beta, 0), (0, 0, \gamma)$, we have

$d = 0$

$\alpha^2 + 2u\alpha = 0 \;\Rightarrow\; \alpha = -2u$

$\beta^2 + 2v\beta = 0 \;\Rightarrow\; \beta = -2v$

$\gamma^2 + 2w\gamma = 0 \;\Rightarrow\; \gamma = -2w$

Therefore, from(1), we have $\frac{a}{-2u} + \frac{b}{-2v} + \frac{c}{-2w} = 1$

or $\quad \frac{a}{-u} + \frac{b}{-v} + \frac{c}{-w} = 2$

Locus of the centre $(-u, -v, -w)$ of the sphere (2) is

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

or $\quad ax^{-1} + by^{-1} + cz^{-1} = 2.$

20. Find an equation of the sphere circumscribing the tetrahedron whose faces are $x = 0, y = 0, z = 0$ and $lx + my + nz + p = 0$.

Sol. Vertices of the tetrahedron are

$$\left(-\frac{p}{l}, 0, 0\right), \left(0, -\frac{p}{m}, 0\right), \left(0, 0, -\frac{p}{n}\right), (0, 0, 0)$$

Let an equation of the sphere circumscribing the tetrahedron be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad (1)$$

Since (1) passes through the vertices of the tetrahedron, we have

$$d = 0$$

$$\frac{p^2}{l^2} - \frac{2p}{l}u = 0 \quad \text{or} \quad u = \frac{p}{2l}$$

Similarly, $v = \frac{p}{2m}$ and $w = \frac{p}{2n}$

Equation of the sphere (1) becomes

$$x^2 + y^2 + z^2 + \frac{p}{l}x + \frac{p}{m}y + \frac{p}{n}z = 0.$$

Exercise Set 8.12 (Page 399)

Discuss and sketch the surface defined by each of the following equations:

1. $x^2 + y^2 + z^2 - 4x + 2y = 11$

Sol. The equation may be written as

$$(x - 2)^2 + (y + 1)^2 + z^2 = 16$$

This represents a sphere whose centre is at $(2, -1, 0)$ and radius 4.

2. $4x^2 + 4y^2 + 4z^2 - 4x + 16y + 12z + 1 = 0$

Sol. Dividing the given equation by 4, we get

$$x^2 + y^2 + z^2 - x + 4y + 3z + \frac{1}{4} = 0$$

i.e., $\left(x - \frac{1}{2}\right)^2 + (y + 2)^2 + \left(z + \frac{3}{2}\right)^2 = \frac{25}{4}$

which is an equation of a sphere with centre at $\left(\frac{1}{2}, -2, \frac{-3}{2}\right)$ and radius $\frac{5}{2}$.

3. $4x^2 + 9y^2 + 36z^2 = 36$

Sol. Dividing both sides of the given equation by 36, we have

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{1} = 1$$

This is an equation of an ellipsoid.

4. $x^2 + y^2 - 4z^2 = 2$

Sol. The given equation is

$$x^2 + y^2 = 4z^2$$

which is an equation of a right circular cone.

5. $x^2 + y^2 - z^2 - 4 = 0$

Sol. On division by 4, we get

$$\frac{x^2}{4} + \frac{y^2}{4} - \frac{z^2}{4} = 1$$

which is an equation of a hyperboloid of one sheet.

6. $x^2 + 9z^2 = 36 - 9z^2$

Sol. The given equation is

$$\frac{x^2}{36} + \frac{y^2}{4} + \frac{z^2}{4} = 1$$

which represents an ellipsoid of revolution.

7. $y^2 + z^2 = 4x$

Sol. The given equation is

$$y^2 + z^2 = 4x$$

which is an equation of (an elliptic) paraboloid of revolution.

8. $x^2 + 4y^2 = z^2 - 4$

Sol. The given equation can be written as

$$\frac{x^2}{4} + \frac{y^2}{1} - \frac{z^2}{4} = -1$$

and this represent a hyperboloid of two sheets.

9. $9x^2 - 4y = 9z^2$

Sol. The equation is

$$x^2 - z^2 = \frac{4}{9}y$$

which represents a hyperbolic paraboloid.

10. $x - y^2 - z^2 = 0$

Sol. The can be written as

$$x^2 + y^2 = x$$

which is a paraboloid of revolution.

11. $x^2 + y^2 = 2z - z^2$

Sol. The given equation is

$$(x - 1)^2 + y^2 + z^2 = 1$$

which is a sphere with centre $(1, 0, 0)$ and radius 1.

12. $x^2 + 4y^2 = 4 - z$

Sol. The given equation is

$$\frac{x^2}{4} + \frac{y^2}{1} = \frac{-1}{4}(z-1)$$

which is an elliptic paraboloid.

13. $x^2 + 4y^2 = 4x - 4z^2$

Sol. This equation can be written as

$$\frac{(x-2)^2}{4} + \frac{y^2}{1} + \frac{z^2}{1} = 1$$

This represents an ellipsoid with two of its semi-axes equal. It is ellipsoid of revolution with centre at (2, 0, 0).

14. $100x^2 + 25y^2 + 100 = 4z^2$

Sol. On division by 100, the equation is

$$\frac{x^2}{1} + \frac{y^2}{4} - \frac{z^2}{25} = -1$$

which is an equation of a hyperboloid of two sheets.

Exercise Set 8.13 (Page 406)

1. Prove that in a spherical triangle ABC

(*a*) $\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$

(*b*) $\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$

(*c*) $\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-c)}}$, where $2s = a + b + c$

State and prove similar results for $\frac{B}{2}$ and $\frac{C}{2}$.

Sol.

(*a*) $2\sin^2\frac{A}{2} = 1 - \cos A$

$$= 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}$$

$$= \frac{\cos b \cos c + \sin b \sin c - \cos a}{\sin b \sin c} = \frac{\cos(b-c) - \cos a}{\sin b \sin c}$$

$$= \frac{2\sin\frac{a+b-c}{2}\sin\frac{a-b+c}{2}}{\sin b \sin c} = \frac{2\sin(s-c)\sin(s-b)}{\sin b \sin c}$$

or $\sin^2\frac{A}{2} = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}$

or $\sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$

(*b*) $2\cos^2\frac{A}{2} = 1 + \cos A$

$$= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c}$$

$$= \frac{\cos a - \cos(b+c)}{\sin b \sin a} = \frac{2\sin\frac{a+b+c}{2}\sin\frac{b+c-a}{2}}{\sin b \sin c}$$

$$= \frac{2\sin s \sin(s-a)}{\sin b \sin c} \text{ or } \cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$$

(*c*) $\tan\frac{A}{2} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} = \sqrt{\frac{\sin(s-b)\sin(s-a)}{\sin s \sin(s-a)}}$, by Parts (*a*) and (*b*).

From **1**(*a*) and **1**(*b*), by symmetry, we have

$$\sin\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin a \sin c}}\ ; \cos\frac{B}{2} = \sqrt{\frac{\sin s \sin(s-b)}{\sin a \sin c}}$$

$$\sin\frac{C}{2} = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}}\ ; \cos\frac{C}{2} = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}}$$

$$\tan\frac{B}{2} = \frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s \sin(s-c)}}$$

2. Prove that in a spherical triangle ABC,

$$\frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

Sol. From **Problem 1 (*c*)**, we have

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$$

$$= \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s \sin^2(s-a)}} = \frac{r}{\sin(s-a)},$$

where $r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}$

Similarly, $\tan\frac{B}{2} = \frac{r}{\sin(s-b)}$ and $\tan\frac{C}{2} = \frac{r}{\sin(s-c)}$

Now $$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} = \frac{\sin(s-b)}{\sin(s-a)}$$

or $$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}} = \frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)} \quad (1)$$

Also, $$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}} = \frac{\sin\frac{A}{2}\cos\frac{B}{2}+\sin\frac{B}{2}\cos\frac{A}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}-\sin\frac{B}{2}\cos\frac{A}{2}}$$

$$= \frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}} \quad (2)$$

$$\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)} = \frac{2\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{2\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)}$$

$$= \frac{\sin\frac{1}{2}c\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}c\sin\frac{1}{2}(a-b)} \quad (3)$$

Therefore, from (2) and (3), we have

$$\frac{\sin\frac{A+B}{2}}{\sin\frac{A-B}{2}} = \frac{\sin\frac{1}{2}c\cos\frac{a-b}{2}}{\cos\frac{1}{2}c\sin\frac{a-b}{2}} \quad \text{or} \quad \frac{\left(\sin\frac{A+B}{2}\right)}{\dfrac{\sin\frac{A-B}{2}}{\sin\frac{a-b}{2}}\sin\frac{c}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

or $$\frac{\sin\frac{A+B}{2}}{\cos\frac{C}{2}} = \frac{\cos\frac{a-b}{2}}{\cos\frac{c}{2}}$$

$$\left(\text{since } \sin\frac{c}{2}\sin\frac{A-B}{2} = \cos\frac{C}{2}\sin\frac{a-b}{2}\right).$$

3. In any spherical triangle ABC, show that

$$2\cos\frac{a+b}{2}\cos\frac{a-b}{2}\tan\frac{c}{2} = \sin b\cos A + \sin a\cos B.$$

Sol. L.H.S. $= 2\cos\frac{a+b}{2}\cos\frac{a-b}{2}\tan\frac{c}{2}$

$= (\cos a + \cos b)\tan\frac{c}{2}$

R.H.S. $= \sin b\cos A + \sin a\cos B$

$$= \sin b \times \frac{\cos a - \cos b\cos c}{\sin b\sin c} + \sin a \times \frac{\cos b - \cos c\cos a}{\sin c\sin a}$$

$$= \frac{\cos a - \cos b\cos c}{\sin c} + \frac{\cos b - \cos c\cos a}{\sin c}$$

$$= \frac{\cos a - \cos b\cos c + \cos b - \cos c\cos a}{\sin c}$$

$$= \frac{(\cos a + \cos b) - \cos c(\cos b + \cos a)}{\sin c}$$

$$= \frac{(\cos a + \cos b)(1-\cos c)}{\sin c} = \frac{(\cos a + \cos b)\,.\,2\sin^2\frac{c}{2}}{2\sin\frac{c}{2}\cos\frac{c}{2}}$$

$$= (\cos a + \cos b)\frac{\sin\frac{c}{2}}{\cos\frac{c}{2}} = (\cos a + \cos b)\tan\frac{c}{2}.$$

4. In an equilateral triangle, show that

(a) $\sec A = 1 + \sec a$

(b) $\tan^2\frac{a}{2} = 1 - 2\cos A$

Sol.

(a) $\cos A = \dfrac{\cos a - \cos b\cos c}{\sin b\sin c}$

Since triangle is equilateral, $a = b = c$

$$\cos A = \frac{\cos a - \cos^2 a}{\sin^2 a} = \frac{\cos a(1-\cos a)}{1-\cos^2 a}$$

$$= \frac{\cos a}{1+\cos a} = \frac{1}{\sec a + 1}$$

or $1 + \sec a = \sec A$.

(b) $\cos A = \dfrac{\cos a - \cos b\cos c}{\sin b\sin c}$

As $a = b = c$

$$\cos A = \frac{\cos a - \cos^2 a}{\sin^2 a} = \frac{\cos a(1-\cos a)}{1-\cos^2 a} = \frac{\cos a}{1+\cos a}$$

$$1-2\cos A = 1-\frac{2\cos a}{1+\cos a} = \frac{1-\cos a}{1+\cos a} = \frac{2\sin^2\frac{a}{2}}{2\cos^2\frac{a}{2}} = \tan^2\frac{a}{2}$$

5. Find the direction of Qibla of each of the given places:

	Place	Latitude φ	Longitude λ
(a)	Islamabad	33° 40′ N	73° 8′ E
(b)	Karachi	24° 51.5′ N	67° 2′ E
(c)	Quetta	30° 15′ N	67° 0′ E
(d)	Peshawar	34° 1′ N	71° 40′ E
(e)	New York	40° 49′ N	74° 0′ W
(f)	Canberra	35° 17′ S	149° 8′ E

Sol.

(a) The classical longitude

$l = 73°8'E - 39°49.2'E = 33°18.8'$ (CE)

Now $p = \dfrac{\sin\phi}{\tan l} = \dfrac{\sin 33°40'}{\tan 33°18.8'}$

$q = \dfrac{\cos 33°40' \tan 21°25.2'}{\sin 33°18.8'}$

$\log p = \log\sin 33°40' - \log\tan 33°18.8'$

$= -0.25620 + 0.182295 = -0.0739$

$p = 0.8435$

$\log q = \log\cos 33°40' + \log\tan 21°25.2' - \log\sin 33°18.8'$

$= -0.079732 - 0.4064 - 0.26026 = -0.746392$

or $q = 0.1793$

Now, $\tan i = p - q = 0.8435 - 0.1793$

$= 0.6642$

or $i = 33°35.5'$ south of west.

(b) Here $l = 67°2'E - 39°49'.2E = 66°62' - 39°49.2$

$= 27°12.8'$ (CE)

$p = \dfrac{\sin 24°51.5'}{\tan 27°12.8'}$; $q = \dfrac{\cos 24°51.5' \tan 21°25.2'}{\sin 27°12.8'}$

$\log p = \log\sin 24°51.5' - \log\tan 27°12.8'$

$= -0.37636 + 0.28885 = -0.08751$

or $p = 0.8175$

$\log q = \log\cos 24°51.5' + \log\tan 21°25.2' - \log\sin 27°12.8'$

$= -0.04222 - 0.4064 + 0.33979$

$= -0.44862 + 0.33979 = -0.10833$

or $q = 0.77834$

$\tan i = p - q = 0.8175 - 0.77834 = 0.03916$

$i = 2°14.55'$ south of west.

(c) $l = 67°E - 39°49.2'E = 66°60' - 39°49.2' = 27°10.8'$

$p = \dfrac{\sin 30°15'}{\tan 27°10.8'}$, $q = \dfrac{\cos 30°15' \tan 21°15.5'}{\sin 27°10.8'}$

$\log p = \sin 30°15' - \log\tan 27°10.8'$

$= -0.29776 + 0.28946 = -0.0083$

$p = 0.9811$

$\log q = \log\cos 30°15' + \log\tan 21°25.2' - \log\sin 27°10.8'$

$= 0.06356 - 0.4064 + 0.34028 = -0.46996 + 0.34028$

$= -0.12968$

or $q = 0.74185$

$p - q = 0.9811 - 0.74185 = 0.23925$

$\tan i = 0.23925$

or $i = 13°27.3'$ south of west.

(d) $l = 71°40'E - 39°49.2'E = 70°100' - 39°49.2'$

$= 31°50.8'$

$p = \dfrac{\sin 34°1'}{\tan 31°50.8'}$, $q = \dfrac{\cos 34°1' \tan 21°25.2'}{\sin 31°50.8'}$

$\log p = \log\sin 34°1' - \log\tan 21°25.8'$

$= -0.25225 + 0.20680 = -0.04545$

or $p = 0.90063$

$\log q = \log\cos 34°1' + \log\tan 21°25.2' - \log\sin 31°50.8'$

$= -0.08151 - 0.4064 + 0.27765$

$= -0.48791 + 0.27765 = -0.21026$

or $q = 0.61622$

$p - q = 0.2844 = \tan i$

or $i = 15°52.55'$ south of west.

(e) $\phi_o = 21°25.2'N$

$\lambda_o = 39°49.2'N$

$l = 74° + 39°49.2'$ E $= 113°49.2'$ CW

$p = \dfrac{\sin\phi}{\tan l} = \dfrac{\sin 40°49'}{\tan 113°49.2'} = -0.288562$

$q = \dfrac{\cos\phi\tan\phi_o}{\sin l} = \dfrac{\cos 40°49' \tan 21°25.2'}{\sin 113°49.2'} = 0.324537$

$\tan i = p - q = -0.6131$

$i = -31.51°$

Since i is negative, the direction of Qibla is $31°30.6'$ north of east.

(f) $l = \lambda - \lambda_o$ CE $= 149°8' - 39°.49.2'$

$= 109°18.8'$ CE