Calculus With Analytic Geometry

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and $f_y = 2y + x^2$

Therefore, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{2xy + 4ax^3}{2y + x^2}$

43. $3x^2 - y^2 + x^3 = 0$

Sol. Let $f(x, y) = 3x^2 - y^2 + x^3 = 0$

Now $f_x = 6x + 3x^2$ and $f_y = -2y$

Therefore, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{6x + 3x^2}{-2y} = \dfrac{6x + 3x^2}{2y}$

44. $x^2 + xy + y^2 + ax + by = 0$

Sol. Let $f(x, y) = x^2 + xy + y^2 + ax + by = 0$

Now $f_x = 2x + y + a$

and $f_y = x + 2y + b$

Thus $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{2x + y + a}{x + 2y + b}$

45. $x^3 + x^2 + xy^2 + \sin y = 0.$

Sol. Let $f(x, y) = x^3 + x^2 + xy^2 + \sin y = 0$

Now, $f_x = 3x^2 + 2x + y^2$

$f_y = 2xy + \cos y$

Therefore, $\dfrac{dy}{dx} = -\dfrac{f_x}{f_y} = -\dfrac{3x^2 + 2x + y^2}{2xy + \cos y}$

Chapter 3 GENERAL THEOREMS, INDETERMINATE FORMS

Exercise Set 3.1 (Page 107)

Discuss the validity of Rolle's Theorem. Find c (wherever possible) such that $f'(c) = 0$ (Problems 1 – 6):

1. $f(x) = x^2 - 3x + 2$ on $[1, 2]$

Sol. $f(1) = (1)^2 - 3(1) + 2 = 1 - 3 + 2 = 0$

and $f(2) = (2)^2 - 3(2) + 2 = 4 - 6 + 2 = 0$

Thus $f(1) = f(2)$

Moreover, $f(x)$ is continuous on $[1, 2]$ and differentiable on $]1, 2[$

Hence all the conditions of Rolle's Theorem are satisfied.

Therefore, there must exist a point c in $]1, 2[$ such that $f'(c) = 0$

Now $f'(x) = 2x - 3$ and so $f'(c) = 2c - 3$

$f'(c) = 0 \Rightarrow 2c - 3 = 0$ or $c = \dfrac{3}{2}$

Hence Rolle's Theorem is valid and $c = \dfrac{3}{2}$

2. $f(x) = \sin^2 x$ on $[0, \pi]$

Sol. $f(0) = \sin^2(0) = 0$ and $f(\pi) = \sin^2(\pi) = 0$

Thus $f(0) = 0 = f(\pi).$

Moreover, $f(x)$ is continuous on $[0, \pi]$ and differentiable on $]0, \pi[$.

Hence all the conditions of Rolle's Theorem are satisfied.

There must exist a point c in the interval $]0, \pi[$ such that $f'(c) = 0$

Now $f'(c) = 0$

Now $f'(x) = 2 \sin x \cos x$

$f'(c) = 2 \sin c \cos c = \sin 2c$

$f'(c) = 0 \Rightarrow \sin 2c = 0$ or $2c = 0, \pi \Rightarrow c = 0, \dfrac{\pi}{2}$

Thus $c = \dfrac{\pi}{2}$, since $0 \notin]0, \pi[$

Hence Rolle's Theorem is valid and $c = \dfrac{\pi}{2}$

3. $f(x) = 1 - x^{3/4}$ on $[-1, 1]$

Sol. $f(-1) = 1 - (-1)^{3/4} = 1 - \left(\dfrac{\pm 1 \pm i}{\sqrt{2}}\right) \neq 0$

$$f(1) = 1 - (1)^{3/4} = 1 - 1 = 0$$

Thus $f(-1) \neq f(1)$ and one of the conditions of Rolle's Theorem is not satisfied. The Rolle's Theorem is not valid and we cannot calculate the value of c.

4. $f(x) = \dfrac{1 - x^2}{1 + x^2}$ on $[-1, 1]$

Sol. $f(-1) = 0 = f(1)$. $f(x)$ is continuous on $[-1, 1]$ and differentiable on $]-1, 1[$.

$$f'(x) = \frac{(1 + x^2)(-2x) - (1 - x^2)\cdot 2x}{(1 + x^2)^2} = \frac{-4x}{(1 + x^2)^2}$$

and $f'(0) = \dfrac{-4(0)}{(1 + 0)^2} = 0$

Thus Rolle's Theorem holds for the given function and $c = 0$.

5. $f(x) = x(x + 3)e^{-\frac{1}{2}x}$ on $[-3, 0]$

Sol. $f(-3) = -3(-3 + 3)e^{-\frac{1}{2}(-3)} = -3(0)\,e^{\frac{3}{2}} = 0$

$f(0) = 0\,(0 + 3)e^{-\frac{1}{2}(0)} = 0$

Thus $f(-3) = 0 = f(0)$

Moreover, $f(x)$ is continuous on $[-3, 0]$ and differentiable on $]-3, 0[$. By Rolle's Theorem, there must exist a point c in the interval $]-3, 0[$ such that $f'(c) = 0$

Now, $f'(x) = (x^2 + 3x)\left(-\frac{1}{2}\cdot e^{-\frac{1}{2}x}\right) + e^{-\frac{1}{2}x}.\,(2x + 3)$

$$= e^{-\frac{x}{2}}\left[\frac{-x^2 - 3x}{2} + 2x + 3\right] = e^{-\frac{x}{2}}\left[\frac{-x^2 - 3x + 4x + 6}{2}\right]$$

$$= -\frac{1}{2}\cdot e^{-\frac{x}{2}}\,(x^2 - x - 6)$$

$f'(c) = 0$ gives $c^2 - c - 6 = 0$

$\Rightarrow$ $(c - 3)(c + 2) = 0$ $\Rightarrow$ $c = 3, -2$

But only $c = -2$ lies in the interval $]-3, 0[$.

Hence the Rolle's Theorem is valid and $c = -2$.

6. $f(x) = 2 + (x - 1)^{3/2}$ on $[0, 2]$

Sol. $f(0) = 2 + (0 - 1)^{3/2} = 2 + (-1)^{3/2} = 2 - i$ or $2 + i$ $\quad (i = \sqrt{-1})$

$f(2) = 2 + (2 - 1)^{3/2} = 2 + (1)^{3/2} = 3$

Thus $f(0) \neq f(2)$

One of the conditions of Rolle's Theorems is not satisfied and so Rolle's Theorem is not valid.

Find c (wherever possible) of the Mean Value Theorem (Problems 7 – 10):

7. $f(x) = x^3 - 3x - 1$ on $\left[\frac{-11}{7}, \frac{13}{7}\right]$

Sol. $f\left(-\frac{11}{7}\right) = \left(-\frac{11}{7}\right)^3 - 3\left(-\frac{11}{7}\right) - 1 = \frac{-1331}{343} + \frac{33}{7} - 1 = \frac{-57}{343}$

$f\left(\frac{13}{7}\right) = \left(\frac{13}{7}\right)^3 - 3\left(\frac{13}{7}\right) - 1 = \frac{2197}{343} - \frac{39}{7} - 1 = \frac{-57}{343}$

Now, $f'(x) = 3x^2 - 3$; $f'(c) = 3c^2 - 3$

By the Mean Value Theorem,

$f(b) - f(a) = (b - a)f'(c)$ implies

$$\frac{-57}{343} - \left(\frac{-57}{343}\right) = \left[\left(\frac{13}{7}\right) - \left(\frac{-11}{7}\right)\right](3c^2 - 3)$$

or $\dfrac{-57}{343} + \dfrac{57}{343} = \left(\dfrac{13}{7} + \dfrac{11}{7}\right)(3c^2 - 3)$ or $0 = \left(\dfrac{13}{7} + \dfrac{11}{7}\right)(3c^2 - 3)$

$\Rightarrow$ $3c^2 - 3 = 0$ or $c = \pm 1$.

8. $f(x) = \sqrt{x - 2}$ on $[2, 4]$

Sol. $f(2) = 0$, $\quad f(4) = \sqrt{4 - 2} = \sqrt{2}$

$$f'(x) = \frac{1}{2\sqrt{x - 2}}$$

By the Mean Value Theorem, we have

$\dfrac{f(4) - f(2)}{4 - 2} = f'(c)$ *i.e.,* $\dfrac{\sqrt{2}}{2} = \dfrac{1}{2\sqrt{c - 2}}$

or $\sqrt{2}\sqrt{c - 2} = 1$ or $2c - 4 = 1$ or $c = \dfrac{5}{2}$

9. $f(x) = x^3 - 5x^2 + 4x - 2$ on $[1, 3]$

Sol. $f(1) = 1 - 5 + 4 - 2 = -2$

$f(3) = 27 - 45 + 12 - 2 = -8$

$f'(x) = 3x^2 - 10x + 4$

By the Mean Value Theorem, we have

$$\frac{f(3) - f(1)}{3 - 1} = f'(c)$$

$\dfrac{-8 + 2}{2} = 3c^2 - 10c + 4$ or $3c^2 - 10c + 7 = 0$

$c = \dfrac{10 \pm \sqrt{100 - 84}}{6} = \dfrac{10 \pm 4}{6} = \dfrac{7}{3}, 1.$ Thus $c = \dfrac{7}{3}$

10. $f(x) = x^{2/3}$ on $[-1, 1]$.

Sol. Here $f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$

$f'(0)$, is not defined at $x = 0$.

Thus $f(x)$ is not differentiable on $]-1, 1[$ and the M.V.T. fails.

In Problems 11–15, use the Mean Value Theorem to show that:

11. $|\sin x - \sin y| \le |x - y|$ for any real numbers x, y.

Sol. Let $f(t) = \sin t$. Then, $f(t)$ is continuous and differentiable for every real t, we apply the M.V.T. to $f(t) = \sin t$ in the interval $[x, y]$ where x, y are any real numbers. Therefore,

$$\frac{\sin y - \sin x}{y - x} = \cos z, z \in]x, y[\qquad (f'(t) = \cos t)$$

Taking modulus of both sides, we get

$$\left|\frac{\sin y - \sin x}{y - x}\right| = |\cos z|$$

or $|\sin x - \sin y| = |x - y|\ |\cos z| \le |x - y|$,

since $|\cos z| \le 1$.

12. $\left|\frac{\cos ax - \cos bx}{x}\right| \le |b - a|$, if $x \ne 0$.

Sol. Let $f(t) = \cos t$. $f(t)$ is continuous and differentiable for all real t. We apply the M.V.T. to $f(t)$ in the interval $[ax, bx]$ where $x \ne 0$. Therefore,

$$\frac{\cos bx - \cos ax}{bx - ax} = -\sin z, z \in]ax, bx[\qquad (f'(t) = -\sin t)$$

or $\left|\frac{\cos bx - \cos ax}{x}\right| = |b - a|\ |\sin(-z)| \le |b - a|$,

since $|\sin(-z)| \le 1$.

13. $|\tan x + \tan y| \ge |x + y|$ for all real numbers x and y in the interval $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$

Sol. The function $f(t) = \tan t$ is continuous on $[-x, y] \subset \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[$ and differentiable in $]-x, y[$. Thus M.V.T. applies and we have

$$\frac{\tan y - \tan(-x)}{y + x} = \sec^2 z, z \in]-x, y[\qquad (f'(t) = \sec^2 t)$$

or $\left|\frac{\tan x + \tan y}{y + x}\right| = |\sec^2 z|$

i.e., $|\tan x + \tan y| = |x+y|\ |\sec^2 z| \ge |x+y|$, since $|\sec^2 z| \ge 1$.

14. $(1 + x)^a > 1 + ax$, where $a > 1$ and $x > 0$. [Bernoulli's Inequality]

Sol. Let $f(x) = (1 + x)^a - (1 + ax)$. Then $f(0) = 0$ and f satisfies the conditions of the M.V.T. on $[0, x]$

By applying the M.V.T. to f on $[0, x]$, we have

$$\frac{f(x) - f(0)}{x - 0} = f'(c), \qquad c \in]0, x[$$

i.e., $(1 + x)^a - (1 + ax) = xf'(c) = x[a(1 + c)^{a-1} - a]$

$= ax[(1 + c)^{a-1} - 1] > 0$,

since $x > 0, a > 1, (1 + c)^{a-1} > 1$

Therefore, $(1 + x)^a > 1 + ax$.

15. $\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$. Also approximate $\sqrt{168}$ by using the Mean Value Theorem.

Sol. Consider the function f defined by $f(x) = \sqrt{x}$ on the interval $[25, 27]$. f satisfies the hypothesis of the M.V.T. on $[25, 27]$. Therefore,

$$\frac{\sqrt{27} - \sqrt{25}}{27 - 25} = f'(c) = \frac{1}{2\sqrt{c}}, \qquad (1)$$

$c \in]25, 27[$ *i.e.*, $25 < c < 27$, or $\sqrt{25} < \sqrt{c} < \sqrt{27}$

i.e., $\frac{1}{\sqrt{27}} < \frac{1}{\sqrt{c}} < \frac{1}{\sqrt{25}} = \frac{1}{5}$

Now, $\frac{1}{6} = \frac{1}{\sqrt{36}} < \frac{1}{\sqrt{27}}$ and so $\frac{1}{6} < \frac{1}{\sqrt{c}} < \frac{1}{5}$ (2)

From (1), we have $\sqrt{27} - 5 = \frac{1}{\sqrt{c}}$. Substitution in (2) yields

$\frac{1}{6} < \sqrt{27} - 5 < \frac{1}{5}$ as desired.

Consider $f(x) = \sqrt{x}$ on $[168, 169]$

By the MVT, $f(169) - f(168) = f'(c)(169 - 168)$,

where $168 < c < 169$.

Therefore, $\sqrt{169} - \sqrt{168} = \frac{1}{2\sqrt{c}}$

The exact value of $\sqrt{c}$ is not known but it is near 13 and so

$$13 - \sqrt{168} \approx \frac{1}{26}$$

or $\sqrt{168} \approx 13 - \frac{1}{26} = 12\frac{25}{26} \approx 12.9615$.

16. Let a function f be continuous on $[a, b]$ and $f'(x) = 0$ for all $x \in]a, b[$. Prove that f is constant on $[a, b]$. Use this to show that $\sin^2 x + \cos^2 x = 1$ for all real numbers x.

Sol. Let $f(x)$ be the differentiable in the interval $]a, b[$ and let x_1, x_2, be any two points belonging to this interval such that $x_2 > x_1$.
Applying the Mean Value Theorem on the interval $[x_1, x_2]$, we see that there exists a number c between x_1 and x_2 such that

$$f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$$

But $f'(c) = 0$

Hence $f(x_2) - f(x_1) = 0 \Rightarrow f(x_2) = f(x_1)$

Thus f assumes the same value at any two points in $[a, b]$ and so it is constant on $[a, b]$.

Set $f(x) = \sin^2 x + \cos^2 x$

$$f'(x) = 2\sin x\cos x - 2\cos x\sin x$$
$$= 0, \text{ for all real } x.$$

Hence $f(x)$ is a constant.

Let $f(x) = \sin^2 x + \cos^2 x = c$. (1)

where c is an arbitrary constant. Since (1) holds for all real x, it holds, in particular, for $x = 0$,

i.e., $\sin^2 0 + \cos^2 0 = c$ or $c = 1$

Thus, $\sin^2 x + \cos^2 x = 1$.

17. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x > 1. \end{cases}$

Does the Mean Value Theorem hold for f on $\left[\frac{1}{2}, 2\right]$?

Sol. It is easy to see that f is continuous on $\left[\frac{1}{2}, 2\right]$.

We check whether f is differentiable on $\left[\frac{1}{2}, 2\right]$. To this end, we check whether $f'(1)$ exists.

$$Lf'(1) = \lim_{x\to 1^-} \frac{f(x) - f(1)}{x - 1} = \lim_{x\to 1^-} \frac{x^2 - 1}{x - 1}$$
$$= \lim_{x\to 1^-} (x + 1) = 2$$

$$Rf'(1) = \lim_{x\to 1^+} \frac{f(x) - (1)}{x - 1} = \lim_{x\to 1^+} \frac{x - 1}{x - 1} = 1$$

Thus, $Lf'(1) \ne Rf'(1)$.

Therefore, $f'(1)$ does not exist and the M.V.T. does not hold on $\left[\frac{1}{2}, 2\right]$

18. Let n be a positive integer. Apply Rolle's Theorem to the function

$$F(x) = \begin{vmatrix} f(x) & f(a) & f(b) \\ x^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix}$$

to obtain a result that generalizes the Mean Value Theorem. Does the result hold if $n < 0$?

Sol. $F(a) = \begin{vmatrix} f(a) & f(a) & f(b) \\ a^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix} = 0$ and $F(b) = \begin{vmatrix} f(b) & f(a) & f(b) \\ b^n & a^n & b^n \\ 1 & 1 & 1 \end{vmatrix} = 0$

Thus $F(a) = F(b)$

By Rolle's Theorem, there exists $c \in]a, b[$ such that $F'(c) = 0$.

Now $F'(x) = \begin{vmatrix} f'(x) & f(a) & f(b) \\ n\cdot x^{n-1} & a^n & b^n \\ 0 & 1 & 1 \end{vmatrix}$

$$= f'(x)(a^n - b^n) - nx^{n-1}(f(a) - f(b))$$

$$F'(c) = 0$$

$$\Rightarrow \quad f'(c)(a^n - b^n) = nc^{n-1}(f(a) - f(b))$$

or $$\frac{f(b) - f(a)}{b^n - a^n} = \frac{f'(c)}{nc^{n-1}}$$

which is the required generalization of the Mean Value Theorem.

If $n < 0$, the theorem holds if $0 \notin [a, b]$.

19. Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points on the graph of the parabola $y = f(x) = ax^2 + bx + c$. By the Mean Value Theorem, there is a point (x_3, y_3) on the curve where tangent line is parallel to the chord AB. Show that $x_3 = \dfrac{x_1 + x_2}{2}$.

Sol. $f(x) = ax^2 + bx + c \quad ; \quad f'(x) = 2ax + b$

By the M.V.T., we have

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_3)$$

or $$\frac{ax_2^2 + bx_2 + c - ax_1^2 - bx_1 - c}{x_2 - x_1} = 2ax_3 + b$$

i.e., $$\frac{a(x_2 - x_1)(x_2 + x_1) + b(x_2 - x_1)}{x_2 - x_1} = 2ax_3 + b$$

or $$a(x_2 + x_1) + b = 2ax_3 + b$$

i.e., $x_3 = \dfrac{x_1 + x_2}{2}$ as required.

20. Show that $f(x) = x^3 - 3x^2 + 2$ is monotonically increasing on every interval.

Sol. $f(x) = x^3 - 3x^2 + 3x + 2$ and
$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x-1)^2$
It is obvious that $3(x-1)^2 = 0$ if $x = 1$. But $3(x-1)^2 > 0$ if $x \in]-\infty, 1[$ and $x \in]1, \infty[$. That is $f'(x) > 0$ for $x \in]-\infty, 1[$ and $x \in]1, \infty[$
So we conclude that $f(x)$ is increasing on $]-\infty, 1[$ and $]1, \infty[$.
For $x = 1$, $f'(x) = 0$
We can verify that $f(x)$ is increasing on every interval containing 1.
Thus $f(x)$ is monotonically increasing on every interval.

21. Prove that $f(x) = 2x - \arctan x - \ln(x + \sqrt{x^2+1})$ is increasing on $[0, \infty[$.

Sol. $f(x) = 2x - \text{arc}\tan x - \ln(x + \sqrt{x^2+1})$ and

$$f'(x) = 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}}\left(1 + \frac{1}{2}\cdot\frac{1}{\sqrt{x^2+1}}\cdot 2x\right)$$
$$= 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}}\cdot\left(1 + \frac{x}{\sqrt{x^2+1}}\right)$$
$$= 2 - \frac{1}{1+x^2} - \frac{1}{x+\sqrt{x^2+1}}\cdot\frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}}$$
$$= 2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} = 2 - \frac{1}{\sqrt{1+x^2}} - \frac{1}{1+x^2}$$
$$= \frac{2(1+x^2) - \sqrt{1+x^2} - 1}{1+x^2}$$
$$= \frac{(2\sqrt{1+x^2}+1)(\sqrt{1+x^2}-1)}{1+x^2}$$ [If $\sqrt{1+x^2} = t$, then the numerator $= 2t^2 - t - 1 = (2t+1)(t-1)$]

Now $2\sqrt{1+x^2} + 1 > 0$ for all $x \in \boldsymbol{R}$ and $\sqrt{1+x^2} - 1 > 0$ for $x \in \boldsymbol{R} - \{0\}$. That is $f'(x) > 0$ if $x \in \boldsymbol{R} - \{0\}$
For $x = 0$, $\sqrt{1+x^2} - 1 = 0$, that is, $f'(x) = 0$ for $x = 0$
It is obvious that $f'(x)$ is positive for $x \in]0, \infty[$ and $f'(x) = 0$ for $x = 0$, therefore, $f(x)$ is increasing on $[0, \infty[$.

22. Show that $\dfrac{\tan x}{x}$ is an increasing function for $0 < x < \dfrac{\pi}{2}$

Sol. Let $f(x) = \dfrac{\tan x}{x}$; $f'(x) = \dfrac{x\sec^2 x - \tan x}{x^2}$ (1)

Let $\phi(x) = x\sec^2 x - \tan x$
$\phi'(x) = \sec^2 x + x(2\sec^2 x \tan x) - \sec^2 x$
$= 2x\sec^2 x \tan x$

When $0 < x < \frac{\pi}{2}$, $\phi'(x) > 0$. Thus $\phi(x)$ is an increasing function for $0 < x < \frac{\pi}{2}$.
Since $\phi(0) = 0$, $\phi(x) > \phi(0)$ for $x > 0$.
i.e., $x\sec^2 x - \tan x > 0$
or $x\sec^2 x > \tan x$ when $0 < x < \frac{\pi}{2}$
From (1), we have $f'(x) > 0$ when $0 < x < \frac{\pi}{2}$
i.e., $f(x)$ is an increasing function for $0 < x < \frac{\pi}{2}$

23. Determine the interval on which
$$f(x) = 2x^3 - 15x + 36x + 1$$
is increasing or decreasing.

Sol. $f(x) = 2x^3 - 15x^2 + 36x + 1$; $f'(x) = 6x^2 - 30x + 36$
For $f(x)$ to be an increasing function
$f'(x) > 0$
i.e., $6x^2 - 30x + 36 > 0$ or $x^2 - 5x + 6 > 0$
or $(x-2)(x-3) > 0$
Now two cases arise.
Case I. $(x-2) > 0$ and $(x-3) > 0$
$\Rightarrow$ $x > 2$ and $x > 3$
Combining the two statements, we can say that $f(x)$ is an increasing function for all $x > 3$ *i.e.,* for all $x \in]3, \infty[$.
Case II. $x - 2 < 0$ and $x - 3 < 0$ $\Rightarrow$ $x < 2$ and $x < 3$
Combining the two statements it is evident that $f(x)$ is an increasing function for all $x < 2$ *i.e.*, for all $x \in]-\infty, 2[$.
Now for $f(x)$ to be a decreasing function
$f'(x) < 0$
i.e., $6x^2 - 30x + 36 < 0$ $\Rightarrow$ $(x-2)(x-3) < 0$
Two cases arise:
Case I. $(x-2) > 0$ and $(x-3) < 0$
$\Rightarrow$ $x > 2$ and $x < 3$
$\Rightarrow$ $x \in]2, 3[$. Thus f is decreasing in $]2, 3[$.
Case II. $(x-2) < 0$ and $(x-3) > 0$
$\Rightarrow$ $x < 2$ and $x > 3$
which is an absurdity because there is no such number which is less than 2 and greater than 3.

24. If $x > 0$, prove that:

$$x - \ln(1+x) > \frac{x^2}{2(1+x)}$$

Sol. Let $f(x) = x - \ln(1+x) - \frac{x^2}{2(1+x)}$

$$= x - \ln(1+x) - \frac{1}{2}\left[x - 1 + \frac{1}{1+x}\right]$$

$$= x - \ln(1+x) - \frac{1}{2}x + \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{1+x}$$

$$f'(x) = 1 - \frac{1}{1+x} - \frac{1}{2} + \frac{1}{2(1+x)^2}$$

$$= \frac{1}{2} + \frac{1}{2(1+x)^2} - \frac{1}{1+x}$$

$$= \frac{(1+x)^2 + 1 - 2(1+x)}{2(1+x)^2}$$

$$= \frac{1 + x^2 + 1 - 2}{2(1+x)^2} = \frac{x^2}{2(1+x)^2}$$

which is positive for every $x > 0$.

Thus $f(x)$ is an increasing function for $x > 0$.

But $f(0) = 0$

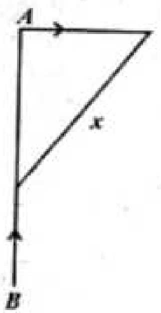
Hence $f(x) > 0$ for $x > 0$

i.e., $x - \ln(1+x) - \frac{x^2}{2(1+x)} > 0$ *i.e.*, $x - \ln(1+x) > \frac{x^2}{2(1+x)}$

5. A ship sails east from port A at 10 nautical miles pet hour. At the same time, another ship leaves port B, which is 100 nautical miles due south of port A, and sails north at 25 nautical miles per hour. For how long is the distance between the ships decreasing?

ol. Let x be the distance between the ships after t hours. Then

$$x^2 = (10t)^2 + (100 - 25t)^2$$
$$= 100t^2 + 10000 + 625t^2 - 5000t$$
$$= 725t^2 - 5000t + 10000$$



Differentiating w.r.t. t, we have

$$2x\frac{dx}{dt} = 1450t - 5000$$

or $x\frac{dx}{dt} = 725t - 2500$

or $\frac{dx}{dt} < 0$

whenever $725t - 2500 < 0$ (since x is positive)

or $t < \frac{2500}{725} = \frac{100}{29}$

Thus $\frac{dx}{dt} < 0$ for $0 < t < \frac{100}{29}$ and the distance x between the ships decreases for $\frac{100}{29}$ hours.

Exercise Set 3.2 (Page 114)

1. Write the Maclaurin's Formula for the function $f(x) = \sqrt{1+x}$ with remainder after two terms.

Sol. Here $f(x) = (1+x)^{1/2}, f(0) = 1$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} = \frac{1}{2\sqrt{1+x}}, f'(0) = \frac{1}{2}$$

$$f''(x) = -\frac{1}{4}(1+x)^{-3/2} = -\frac{1}{4}\cdot\frac{1}{(1+x)^{3/2}},$$

$$f''(\theta x) = -\frac{1}{4(1+\theta x)^{3/2}}$$

By Maclaurin's Theorem with remainder after two terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x), \quad 0 < \theta < 1$$

or $\sqrt{1+x} = 1 + \frac{1}{2}x + \frac{1}{2!}x^2\left\{-\frac{1}{4(1+\theta x)^{3/2}}\right\}$

$$= 1 + \frac{1}{2}\cdot x - \frac{1}{8}\cdot\frac{x^2}{(1+\theta x)^{3/2}}$$

2. Find, by Maclaurin's Formula, the first four terms of the expansion of $f(x) = e^{ax}\cos bx$ and write the remainder after n terms.

Sol. Here $f(x) = e^{ax}\cos bx$

$$f^{(n)}(x) = (a^2+b^2)^{n/2}e^{ax}\cos(bx + n\phi) \qquad \phi = \arctan\frac{b}{a}$$

Taking $n = 1, 2, 3$

$$f'(x) = (a^2+b^2)^{1/2}e^{ax}\cos(bx+\phi)$$
$$f''(x) = (a^2+b^2)e^{ax}\cos(bx+2\phi)$$
$$f'''(x) = (a^2+b^2)^{3/2}\cos(bx+3\phi)$$

Now $f(0) = 1$

$$f'(0) = \sqrt{a^2+b^2}\cos(\phi) = \sqrt{a^2+b^2}\cdot\frac{a}{\sqrt{a^2+b^2}} = a$$

$$f''(0) = (a^2+b^2)\cos 2\phi = (a^2+b^2)(\cos^2\phi - \sin^2\phi)$$

$$= (a^2+b^2)\left(\frac{a^2-b^2}{a^2+b^2}\right) = a^2 - b^2$$

$$f'''(0) = (a^2 + b^2)^{3/2} \cos 3\phi$$
$$= (a^2 + b^2)^{3/2} [4 \cos^3 \phi - 3 \cos \phi]$$
$$= (a^2 + b^2)^{3/2} \left[4 \frac{a^3}{(a^2 + b^2)^{3/2}} - \frac{3a}{(a^2 + b^2)^{1/2}}\right]$$
$$= (a^2 + b^2)^{3/2} \left[\frac{4a^3 - 3a(a^2 + b^2)}{(a^2 + b^2)^{3/2}}\right]$$
$$= a^3 - 3ab^2 = a(a^2 - 3b^2)$$
$$f^{(n)}(\theta x) = (a^2 + b^2)^{n/2} e^{a\theta x} \cos\left(b\theta x + n \arctan \frac{b}{a}\right), \quad 0 < \theta < 1$$

By Maclaurin's Theorem with remainder after n terms, we have

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(\theta x)$$

Therefore, $e^{ax} \cos bx = 1 + ax + \dfrac{a^2 - b^2}{2!} x^2 + \dfrac{a(a^2 - 3b^2)}{3!} x^3$

$$+ \ldots + \frac{x^n}{n!} (a^2 + b^2)^{n/2} e^{a\theta x} \cos\left(b\theta x + n \arctan \frac{b}{a}\right).$$

Find the Maclaurin's series of the given functions (Problems 3 – 9):

3. $f(x) = \sin x$

Sol.

$$\begin{aligned} f(x) &= \sin x, & f(0) &= 0 \\ f'(x) &= \cos x, & f'(0) &= 1 \\ f''(x) &= -\sin x, & f''(0) &= 0 \\ f'''(x) &= -\cos x, & f'''(0) &= -1 \\ f^{(4)}(x) &= \sin x, & f^{(4)}(0) &= 0 \end{aligned}$$

Generalizing, we get

$$f^{(r)}_{(x)} = \sin\left(x + r \cdot \frac{\pi}{2}\right), \quad f^{(r)}_{(0)} = \sin r \cdot \frac{\pi}{2}$$
$$f^{(2n-1)}_{(x)} = \sin\left[x + (2n-1)\frac{\pi}{2}\right], f^{(2n-1)}_{(0)} = (-1)^{n-1}$$

$R_n = \dfrac{x^n}{n!} f^{(n)}(\theta x)$, where R_n is the remainder after n terms and $0 < \theta < 1$.

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right) \longrightarrow 0$$

Thus $f(x) = \sin x$ can be expanded into an infinite series. Hence by Maclaurin's Theorem, for all values of x, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^{r-1}}{(r-1)!} f^{(r-1)}_{(0)} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + (-1)^{n-1} \cdot \frac{x^{2n-1}}{(2n-1)!} + \ldots$$

4. $f(x) = \cos x$

Sol.

$$\begin{aligned} f(x) &= \cos x, & f(0) &= 1 \\ f'(x) &= -\sin x, & f'(0) &= 0 \\ f''(x) &= -\cos x, & f''(0) &= -1 \\ f'''(x) &= \sin x, & f'''(0) &= 0 \\ f^{(4)}(x) &= \cos x, & f^{(4)}(0) &= 1 \end{aligned}$$

$$f^{(n)}(x) = \cos\left(x + \frac{n\pi}{2}\right), \; f^{(n)}(\theta x) = \cos\left(\theta x + \frac{n\pi}{2}\right)$$

$$f^{(n)}(0) = \cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}, & \text{if } n \text{ is even} \end{cases}$$

Now $R_n = \dfrac{x^n}{n!} f^{(n)}(\theta x), \; 0 < \theta < 1.$

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \frac{x^n}{n!} \cos\left(\theta x + \frac{n\pi}{2}\right)$$
$$= \lim_{n \to \infty} \frac{x^n}{n!} \lim_{n \to \infty} \cos\left(\theta x + \frac{n\pi}{2}\right) = 0$$

Hence by Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{(-1)^n}{(2n)!} x^{2n} + \ldots$$

5. $f(x) = \tan x$

Sol. $f(x) = \tan x, \quad f(0) = 0$

$$f'(x) = \sec^2 x = 1 + \tan^2 x, \; f'(0) = 1$$
$$f''(x) = 2 \tan x \sec^2 x$$
$$= 2 \tan x (1 + \tan^2 x)$$
$$= 2 \tan x + 2 \tan^3 x, \; f''(0) = 0$$
$$f'''(x) = 2 \sec^2 x + 6 \tan^2 x \sec^2 x$$
$$= 2(1 + \tan^2 x) + 6 \tan^2 x (1 + \tan^2 x)$$
$$= 2 + 2 \tan^2 x + 6 \tan^2 x + 6 \tan^4 x$$
$$= 2 + 8 \tan^2 x + 6 \tan^4 x, \; f'''(0) = 2$$
$$f^{(4)}(x) = 16 \tan x \sec^2 x + 24 \tan^3 x \sec^2 x$$
$$= 16 \tan x (1 + \tan^2 x) + 24 \tan^3 x (1 + \tan^2 x)$$
$$= 16 \tan x + 16 \tan^3 x + 24 \tan^3 x + 24 \tan^5 x$$
$$= 16 \tan x + 40 \tan^3 x + 24 \tan^5 x, \; f^{(4)}(0) = 0$$

$f^{(5)}(x) = 16\sec^2 x + 120\tan^2 x\sec^2 x + 120\tan^4 x\sec^2 x,\ f^{(5)}(0) = 16$

$\vdots \qquad \vdots \qquad \vdots$

By Maclaurin's Theorem,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \frac{x^5}{5!}f^{(5)}(0) + \ldots$$

Substituting the values, we have

$$\tan x = 0 + x \times 1 + \frac{x^2}{2!}(0) + \frac{x^3}{3\times2\times1}(2) + \frac{x^4}{4!}(0) + \frac{x^5}{5\times4\times3\times2} \times 16 + \ldots$$

$$= x + \frac{x^3}{3} + \frac{2}{15}x^5 + \ldots$$

6. $f(x) = \sec x$

Sol. $y = \sec x, y(0) = \sec 0 = 1$

$y_1 = \sec x\tan x, \quad y_1(0) = 0$

$y_2 = 2\sec^3 x - \sec x = 2y^3 - y, \quad y_2(0) = 1$

$y_3 = 6y^2 y_1 - y_1, \ y_3(0) = 0$

$y_4 = 6y^2 y_2 + 12yy_1^2 - y_2, \ y_4(0) = 5$

$y_5 = 6y^2 y_3 + 12yy_1y_2 + 12y_1^3 + 24yy_1y_2 - y_3$

$\quad = 6y^2 y_3 + 36yy_1y_2 + 12y_1^3 - y_3, \ \ y_5(0) = 0$

$y_6 = 6y^2 y_4 + 12yy_1y_3 + 36(y_1^2y_2 + yy_2^2 + yy_1y_3) + 36y_1^2y_2 - y_4$

$y_6(0) = 6 \,.\, 1 \,.\, 5 + 36(0 + 1 + 0) + 36 \cdot 0 \cdot 1 - 5 = 61$

Now by Maclaurin's Theorem,

$$y = y(0) + xy_1(0) + \frac{x^2}{2!}y_2(0) + \frac{x^3}{3!}y_3(0) + \ldots$$

or $\quad \sec x = 1 + x \,.\, 0 + \frac{x^2}{2!} \cdot 1 + \frac{x^3}{3!} \,.\, 0 + \frac{x^4}{4!} \,.\, 5 + \frac{x^5}{5!} \,.\, 0 + \frac{x^6}{6!} \,.\, 61 + \ldots$

$$= 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots$$

7. $f(x) = \ln(1 - x)$

Sol. $f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1-x)^n} \cdot (-1)^n = \frac{(-1)^{2n-1}(n-1)!}{(1-x)^n} = -\frac{(n-1)!}{(1-x)^n}$

$f^{(n)}(\theta x) = -\frac{(n-1)!}{(1-\theta x)^n}, \ 0 < \theta < 1$

$f'(0) = -1, f''(0) = -1, f'''(0) = -2, \ldots f^{(4)}(0) = -(3!)$

$$f(x) = f(x) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!} \cdot f'''(0) + \frac{x^4}{4!}f^{(4)}(0) + \ldots$$

$$\ln(1-x) = 0 + x(-1) + \frac{x^2}{2\times1}(-1) + \frac{x^3}{3\times2\times1} \times (-2) + \frac{x^4}{4!}(-(3!)) + \ldots$$

Therefore, $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots$

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8. $f(x) = e^{\sin x}$

Sol. $f(x) = e^{\sin x}, \ f(0) = e^0 = 1$

$f'(x) = e^{\sin x}\cos x, \ f'(0) = 1$

$f''(x) = e^{\sin x}\cos^2 x - e^{\sin x}\sin x, \ f''(0) = 1$

$f'''(x) = e^{\sin x}\cos^3 x - 2e^{\sin x}\cos x\sin x - e^{\sin} x\sin x\cos x - e^{\sin x}\cos x$

$\quad = e^{\sin x}\cos^3 x - 3e^{\sin x}\sin x\cos x - e^{\sin x}\cos x$

$\quad = e^{\sin x}\cos^3 x - \frac{3}{2}e^{\sin x}\sin 2x - e^{\sin x}\cos x, \ f'''(0) = 0$

$f^{(4)}(x) = e^{\sin x}\cos^4 x - 3e^{\sin x}\cos^2 x\sin x - \frac{3}{2}e^{\sin x}\sin 2x\cos x$

$\quad -\frac{3}{2}e^{\sin x}\, 2\cos 2x - e^{\sin x}\cos^2 x + e^{\sin x}\sin x,$

$f^{(4)}(0) = 1 - 3 - 1 = -3$

Hence $\quad e^{\sin x} = 1 + x \,.\, 1 + \frac{x^2}{2!} \,.\, 1 + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(-3) + \cdots$

$$= 1 + x + \frac{x^2}{2!} - \frac{x^4}{8} - \cdots$$

9. $f(x) = a^x$

Sol. $f(x) = a^x, \qquad f(0) = 1$

$f'(x) = a^x \ln a, \qquad f'(0) = \ln a$

$f''(x) = a^x(\ln a)^2, \qquad f''(0) = (\ln a)^2$

$f'''(x) = a^x(\ln a)^3, \qquad f'''(0) = \ln(a)^3$

$\vdots \qquad \vdots \qquad \vdots$

$f^{(n)}(x) = a^x(\ln a)^n, \qquad f^{(n)}(\theta x) = a^{\theta x}(\ln a)^n$

Therefore,

$$a^x = 1 + x(\ln a) + \frac{x^2}{2!}(\ln a)^2 + \frac{x^3}{3!}(\ln a)^3 + \ldots + \frac{x^{n-1}}{(n-1)!}(\ln a)^{n-1} + R_n,$$

where $\ R_n = \frac{x^n}{n!}a^{\theta x}(\ln a)^n = \frac{(x\ln a)^n}{n!}a^{\theta x}, \ 0 < \theta < 1.$

To show that $\lim_{n\to\infty} R_n = 0$, we use the Ratio Test

$$\lim_{n\to\infty}\frac{R_{n+1}}{R_n} = \lim_{n\to\infty}\frac{(x\ln a)^{n+1}}{(n+1)!} \times \frac{n!}{(x\ln a)^n}\frac{a^{\theta x}}{a^{\theta x}}$$

$$= \lim_{n\to\infty}\frac{x\ln a}{n+1} = 0 < 1,$$

if x and a are fixed numbers. The series is convergent.

Hence $\lim_{n \to \infty} R_n = 0$

$$a^x = 1 + x \ln a + \frac{(x \ln a)^2}{2!} + \frac{(x \ln a)^3}{2!} + \dots + \frac{(x \ln a)^n}{n!} + \dots \infty$$

10. Apply Taylor's Theorem to prove that

$$(a+b)^m = a^m + \frac{m}{1!} a^{m-1} b + a^{m-2} b^2 + \dots$$

for all real $m, a > 0, -a < b < a$.

Sol. Consider $f(x+b) = (x+b)^m$

$$f(x) = x^m \ ; \ f'(x) = m\, x^{m-1}$$
$$f''(x) = m(m-1)x^{m-2} \ ; \ f'''(x) = m(m-1)(m-2)x^{m-3}$$

By Taylor's Theorem,

$$f(x+b) = f(x) + bf'(x) + \frac{b^2}{2!} f''(x) + \frac{b^3}{3!} f'''(x) + \dots$$
$$= x^m + bm\, x^{m-1} + \frac{b^2}{2!} m(m-1) x^{m-2} + \frac{b^3}{3!} m(m-1)(m-2) x^{m-3} + \dots$$

Therefore,

$$f(a+b) = (a+b)^m = a^m + \frac{m}{1!} a^{m-1} b + \frac{m(m-1)}{2!} a^{m-2} b^2 + \dots$$

Here $R_n = \dfrac{b^n}{(n-1)!} (1-\theta)^{n-1} f^{(n)}(a + \theta b), \quad 0 < \theta < 1.$

$$f^{(n)}(x) = m(m-1)(m-2)(m-n+1)\, x^{m-n}$$
$$f^{(n)}(a+\theta b) = m(m-1)(m-2)\dots(m-n+1)(a+\theta b)^{m-}$$
$$R_n = \frac{b^n}{(n-1)!}(1-\theta)^{n-1}\, m(m-1)(m-1)(m-2)(m-n+1)(a+\theta b)^{m-n}$$
$$= \frac{b^n (1-\theta)^{n-1} n!}{(m-n)!\,(n-1)!} \cdot (a+\theta b)^{m-n}$$

$R_n \to 0$ as $n \to \infty$ for all real $m, a > 0, \ -a < b < a$

Hence $(a+b)^m = a^m + \dfrac{m}{1!} a^{m-1} b + \dfrac{m(m-1)}{2!} a^{m-2} b^2 + \dots$

11. Prove that:

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a + (x-a)\theta)$$

stating the conditions under which it holds.

Sol. $f(x) = f(a) + (x-a) f'(a) + \dfrac{(x-a)^2}{2!} f''(a) + \dots + \dfrac{(x-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \dfrac{(x-a)^n}{n!} f^{(n)}(a + \overline{x-a}\,\theta)$

The conditions for which the above theorem holds are as follows:
The function f is such that

(i) $f, f'', f''', \dots, f^{(n-1)}$ are continuous on $[a, x]$

(ii) $f^{(n)}$ exists in $]a, x[$.

then there exists a real number θ, $0 < \theta < 1$, such that

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(x-1)!} f^{(n-1)}(a) + \frac{(x-a)^n}{n!} f^{(n)}(a + \overline{x-a}\,\theta)$$

Here $f(x) = f(a + \overline{x-a})$

Expanding by Taylor's Theorem with Lagrange's form of remainder, we have

$$f(x) = f(a + \overline{x-a})$$
$$= f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^{n-1}}{(x-1)!} f^{n-1}(a) + \frac{(x-a)^n}{n!} f^n (a + \overline{x-a}\,\theta)$$

12. Use Taylor's Theorem to prove that

$$\ln \sin(x+h) = \ln \sin x + h \cot x - \frac{1}{2} h^2 \csc^2 x + \frac{1}{3} h^3 \cot x \csc^2 x + \dots$$

Sol. Let $f(x) = \ln \sin(x+h)$

Then $f(x) = \ln \sin x \ ; \ f'(x) = \dfrac{1}{\sin x} \cdot \cos x = \cot x$

$$f''(x) = -\csc^2 x \ ; \ f'''(x) = -2 \csc x(-\csc x \cot x)$$
$$= 2 \csc^2 x \cot x$$

By Taylor's Theorem, we get

$$\ln \sin(x+h) = \ln \sin x + h \cot x + \frac{h^2}{2!}(-\csc^2 x) + \frac{h^3}{3!}(2 \csc^2 x \cot x) + \dots$$
$$= \ln \sin x + h \cot x - \frac{h^2}{2} \csc^2 x + \frac{h^3}{3} \csc^2 x \cot x + \dots$$

13. Show that, under certain conditions to be stated, $f(a+h) = f(a) + hf'(a+\theta h)$ where $0 < \theta < 1$. Prove also the limiting value of θ, when h decreases indefinitely, is $\frac{1}{2}$.

Sol. $f(a+h) = f(a) + hf'(a+\theta h)$, holds by Taylor's Theorem with remainder after one term.

Also by Taylor's theorem with Lagrange's form of remainder after two terms,

$$f(a+h) = f(a)+hf'(a) + \frac{h^2}{2!}f''(a+\theta' h), \text{ where } 0 < \theta' < 1$$

The two equations give

$$f(a) + hf'(a+\theta h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta' h)$$

or $$hf'(a+\theta h) = hf'(a) + \frac{h^2}{2!}f''(a+\theta' h)$$

or $$f'(a+\theta h) = f'(a) + \frac{h}{2}f''(a+\theta' h)$$

or $$f'(a+\theta h) - f'(a) = \frac{h}{2}f''(a+\theta' h)$$

or $$\theta h f''(a+\theta\theta'' h) = \frac{h}{2}f''(a+\theta' h), \quad 0 < \theta'' < 1$$

or $$\theta f''(a+\theta\theta'' h) = \frac{1}{2}f''(a+\theta' h)$$

Letting $h \longrightarrow 0$, we have

$$\theta f''(a) = \frac{1}{2}f''(a) \quad \text{or} \quad \theta = \frac{1}{2}$$

14. If the functions f, ϕ and Ψ are continuous on $[a, b]$ and differentiable in $]a, b[$, show that there exists a point $\xi \in]a, b[$ such that

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0$$

Hence deduce Lagrange's and Cauchy's mean value theorems.

Sol. We form a new function $F(x) = f(x) + A\,\phi(x) + B\,\psi(x)$

where A and B are constants to be chosen such that

$$f(a) + A\,\phi(a) + B\,\psi(a) = 0 \qquad (1)$$
$$f(b) + A\,\phi(b) + B\,\psi(b) = 0 \qquad (2)$$

The function, $F(x)$ satisfies the conditions of Rolle's Theorem. Hence there exists a point $\xi \in [a, b]$ such that

$$f'(\xi) + A\,\phi'(\xi) + B\,\psi'(\xi) = 0 \qquad (3)$$

Eliminating A and B from (1), (2) and (3), we obtain

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0 \qquad (4)$$

Deduction. Take $\psi(x) = k$, where k is some constant. Then

$$\psi(a) = \psi(b) = k$$

and $\psi'(\xi) = 0$.

Hence from (4), we have

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a) & \phi(a) & k \\ f(b) & \phi(b) & k \end{vmatrix} = 0$$

or $$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & k \end{vmatrix} = 0 \text{ by } R_2 - R_3$$

or $$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & 1 \end{vmatrix} = 0$$

or $$f'(\xi)[\phi(a)-\phi(b)] - \phi'(\xi)[f(a)-f(b)] = 0$$

or $$[\phi(b)-\phi(a)]f'(\xi) = [f(b)-f(a)]\,\phi'(\xi)$$

$$\Rightarrow \frac{f(b)-f(a)}{\phi(b)-\phi(a)} = \frac{f'(\xi)}{\phi'(\xi)} \qquad (5)$$

which is Cauchy's Mean Value Theorem.

Now take $\phi(x) = x$ so that $\phi(a) = a$, $\phi(b) = b$ and $\phi'(\xi) = 1$

Putting these values in (5), we get

$$\frac{f(b)-(a)}{b-a} = \frac{f'(\xi)}{1}$$

which is Lagrang's Mean Value Theorem.

15. Assuming f'' to be continuous on $[a, b]$ and differentiable on $]a, b[$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where both c and ξ belong to $]a, b[$.

Sol. Consider the function

$$\phi(x) = f(x) + Ax + Bx^2 \qquad (1)$$

where A and B are constants to be determined such that

$$\phi(a) = \phi(b) = \phi(c)$$

Thus we have

$$f(a) + Aa + Ba^2 = f(b) + Ab + Bb^2 = f(c) + Ac + Bc^2$$

or $$(a-b)A + (a^2-b^2)B + [f(a)-f(b)] = 0$$

and $$(b-c)A + (b^2-c^2)B + [f(b)-f(c)] = 0$$

13. Show that, under certain conditions to be stated, $f(a+h) = f(a) + h f'(a+\theta h)$ where $0 < \theta < 1$. Prove also the limiting value of θ, when h decreases indefinitely, is $\frac{1}{2}$.

Sol. $f(a+h) = f(a) + hf'(a+\theta h)$, holds by Taylor's Theorem with remainder after one term.

Also by Taylor's theorem with Lagrange's form of remainder after two terms,

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta' h), \text{ where } 0 < \theta' < 1$$

The two equations give

$$f(a) + hf'(a+\theta h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta' h)$$

or $$hf'(a+\theta h) = hf'(a) + \frac{h^2}{2!} f''(a+\theta' h)$$

or $$f'(a+\theta h) = f'(a) + \frac{h}{2} f''(a+\theta' h)$$

or $$f'(a+\theta h) - f'(a) = \frac{h}{2} f''(a+\theta' h)$$

or $$\theta h f''(a+\theta\theta'' h) = \frac{h}{2} f''(a+\theta' h), \qquad 0 < \theta'' < 1$$

or $$\theta f''(a+\theta\theta'' h) = \frac{1}{2} f''(a+\theta' h)$$

Letting $h \longrightarrow 0$, we have

$$\theta f''(a) = \frac{1}{2} f''(a) \qquad \text{or} \qquad \theta = \frac{1}{2}$$

14. If the functions f, ϕ and Ψ are continuous on $[a, b]$ and differentiable in $]a, b[$, show that there exists a point $\xi \in]a, b[$ such that

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0$$

Hence deduce Lagrange's and Cauchy's mean value theorems.

Sol. We form a new function $F(x) = f(x) + A\,\phi(x) + B\,\psi(x)$ where A and B are constants to be chosen such that

$$f(a) + A\,\phi(a) + B\,\psi(a) = 0 \tag{1}$$
$$f(b) + A\,\phi(b) + B\,\psi(b) = 0 \tag{2}$$

The function, $F(x)$ satisfies the conditions of Rolle's Theorem. Hence there exists a point $\xi \in [a, b]$ such that

$$f'(\xi) + A\,\phi'(\xi) + B\,\psi'(\xi) = 0 \tag{3}$$

Eliminating A and B from (1), (2) and (3), we obtain

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & \psi'(\xi) \\ f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \end{vmatrix} = 0 \tag{4}$$

Deduction. Take $\psi(x) = k$, where k is some constant. Then

$$\psi(a) = \psi(b) = k$$

and $\psi'(\xi) = 0$.

Hence from (4), we have

$$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a) & \phi(a) & k \\ f(b) & \phi(b) & k \end{vmatrix} = 0$$

or $$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & k \end{vmatrix} = 0 \quad \text{by } R_2 - R_3$$

or $$\begin{vmatrix} f'(\xi) & \phi'(\xi) & 0 \\ f(a)-f(b) & \phi(a)-\phi(b) & 0 \\ f(b) & \phi(b) & 1 \end{vmatrix} = 0$$

or $$f'(\xi)\,[\phi(a) - \phi(b)] - \phi'(\xi)\,[f(a) - f(b)] = 0$$

or $$[\phi(b) - \phi(a)]\,f'(\xi) = [f(b) - f(a)]\,\phi'(\xi)$$

$$\Rightarrow \quad \frac{f(b) - f(a)}{\phi(b) - \phi(a)} = \frac{f'(\xi)}{\phi'(\xi)} \tag{5}$$

which is Cauchy's Mean Value Theorem.

Now take $\phi(x) = x$ so that $\phi(a) = a$, $\phi(b) = b$ and $\phi'(\xi) = 1$

Putting these values in (5), we get

$$\frac{f(b) - (a)}{b - a} = \frac{f'(\xi)}{1}$$

which is Lagrang's Mean Value Theorem.

15. Assuming f'' to be continuous on $[a, b]$ and differentiable on $]a, b[$, show that

$$f(c) - f(a)\frac{b-c}{b-a} - f(b)\frac{c-a}{b-a} = \frac{1}{2}(c-a)(c-b)f''(\xi)$$

where both c and ξ belong to $]a, b[$.

Sol. Consider the function

$$\phi(x) = f(x) + Ax + Bx^2 \tag{1}$$

where A and B are constants to be determined such that

$$\phi(a) = \phi(b) = \phi(c)$$

Thus we have

$$f(a) + Aa + Ba^2 = f(b) + Ab + Bb^2 = f(c) + Ac + Bc^2$$

or $$(a-b)A + (a^2 - b^2)B + [f(a) - f(b)] = 0$$

and $$(b-c)A + (b^2 - c^2)B + [f(b) - f(c)] = 0$$

Solving these equation simultaneously by cross-multiplication, we have

$$\frac{A}{(a^2-b^2)[f(b)-f(c)]-(b^2-c^2)]f(a)-f(b)]}$$
$$=\frac{B}{(b-c)[f(a)-f(b)]-(a-b)[f(b)-(c)]}$$
$$=\frac{1}{(a-b)(b^2-c^2)-(a^2-b^2)(b-c)}$$

i.e., $A=\dfrac{(a^2-b^2)[f(b)-f(c)]-(b^2-c^2)[f(a)-f(b)]}{(a-b)(b-c)(c-a)}$ (2)

$B=\dfrac{(b-c)[f(a)-f(b)]-(a-b)[f(b)-f(c)]}{(a-b)(b-c)(c-a)}$ (3)

Now $f''(x)$ exists in $[a, b]$. Therefore, $f(x)$, $f'(x)$ exist and are differentiable in $[a, b]$.

Also, $Ax + Bx^2$ is differentiable in $[a, b]$ and so

$\phi(x)$ is differentiable in $[a, b]$

Now, consider the interval $[a, c]$

Since this interval is included in $[a, b]$, we see that

$\phi(x)$ is differentiable in $[a, c]$.

Also $\phi(a)=\phi(c)$ (4)

Hence by Rolle's Theorem, there is a value α in $[a, c]$, such that

$\phi'(\alpha)=0$ (5)

Similarly applying Rolle's Theorem to the interval $[c, b]$ we conclude that there is a value β in $[c, b]$, such that

$\phi'(\beta)=0$ (6)

Now, consider the function

$F(x)=\phi'(x)=f'(x)+A+2Bx$

We have $F'(x)=f''(x)+2B$ (7)

which exists in $[a, b]$. Therefore,

$F(x)$ is differentiable in $[a, b[$ and hence also in $]\alpha, \beta[$ which is contained in $[a, b]$.

Also $F(\alpha)=\phi'(\alpha)=0$, by (5)

and $F(\beta)=\phi'(\beta)=0$, by (6)

Thus $F(\alpha)=F(\beta)$.

Hence by Rolle's Theorem there is a number ξ in $]\alpha, \beta[$ and so also in $]a, b[$ such that

$F'(\xi)=0$

i.e., $f''(\xi)+2B=0$, by (7)

or $f''(\xi)+2\dfrac{(b-c)[f(a)-f(b)]-(a-b)[f(b)-f(c)]}{(a-b)(b-c)(c-a)}=0$

$$\Rightarrow \quad \frac{1}{2}(c-a)(c-b)f''(\xi)$$
$$=\frac{(b-c)[f(a)-f(b)]-(a-b)[f(b)-f(c)]}{(a-b)}$$

or $\dfrac{1}{2}(c-a)(c-b)f''(\xi)$

$$=\frac{b-c}{a-b}f(a)-\left(\frac{b-c+a-b}{a-b}\right)f(b)+f(c)$$
$$=-\frac{b-c}{b-a}f(a)-\frac{c-a}{b-a}f(b)+f(c)$$
$$=f(c)-f(a)\frac{b-c}{b-a}-f(b)\frac{c-a}{b-a}$$

Hence $f(c)-f(a)\dfrac{b-c}{b-a}-f(b)\dfrac{c-a}{b-a}=\dfrac{1}{2}(c-a)(c-b)f''(\xi)$

16. Show that the number θ which occurs in the Taylor's Theorem with Lagrange's form of remainder after n terms approaches the limit $\dfrac{1}{1+n}$ as $h\to 0$ provided that $f^{(n+1)}(x)$ is continuous and different from zero at $x=a$.

Sol. Applying Taylor's Theorem with remainder after n terms and $(n+1)$ terms successively, we obtain

$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^{n-1}}{(n-1)!}f^{n-1}(a)+\frac{h^n}{n!}f^{(n)}(a+\theta h)$$
$$f(a+h)=f(a)+hf'(a)+\ldots+\frac{h^n}{n!}f^n(a)+\frac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta' h)$$

These gives $\dfrac{h^n}{n!}f^{(n)}(a+\theta h)=\dfrac{h^n}{n!}f^{(n)}(a)+\dfrac{h^{n+1}}{(n+1)!}f^{(n+1)}(a+\theta' h)$

or $f^{(n)}(a+\theta h)-f^{(n)}(a)=\dfrac{h}{n+1}f^{(n+1)}(a+\theta' h)$

Applying Lagrange's Mean Value Theorem to the left-hand side, we have

$$\theta h f^{(n+1)}(a+\theta''\theta h)=\frac{h}{n+1}f^{(n+1)}(a+\theta' h), 0<\theta''<1$$

or $\theta=\dfrac{1}{n+1}\dfrac{f^{(n+1)}(a+\theta' h)}{f^{(n+1)}(a+\theta''\theta h)}$ or $\lim\limits_{h\to 0}\theta=\dfrac{1}{n+1}$

Exercise Set 3.3 (Page 126)

Evaluate the given limits (Problems 1 – 48):

1. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x}$ $\left(\frac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x}}{\sin x} = \lim\limits_{x \to 0} \dfrac{e^x + e^{-x}}{\cos x} = 2$ $\left(\frac{0}{0}\right)$

2. $\lim\limits_{x \to 0} \dfrac{e^{x^2} - 1}{\cos x - 1}$ $\left(\frac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{e^{x^2} - 1}{\cos x - 1} = \lim\limits_{x \to 0} \dfrac{2xe^{x^2}}{-\sin x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2[x\,(2x)\,e^{x^2} + e^{x^2}]}{-\cos x} = \lim_{x \to 0} \frac{2\,e^{x^2}\,(2x^2 + 1)}{-\cos x}$$

$$= \frac{2\,e^0\,(0 + 1)}{-1} = -2$$

3. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x - \sin x}$ $\left(\frac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x - \sin x} = \lim\limits_{x \to 0} \dfrac{1 - \sec^2 x}{1 - \cos x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{-2\sec^2 x \tan x}{\sin x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-2\sec^4 x - 4\sec^2 x \tan^2 x}{\cos x} = -2$$

4. $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{\cos 3x + 1}$ $\left(\frac{0}{0}\right)$

Sol. $\lim\limits_{x \to \pi} \dfrac{\sin^2 x}{\cos 3x + 1} = \lim\limits_{x \to \pi} \dfrac{2\sin x \cos x}{-3\sin 3x} = \lim\limits_{x \to \pi} \dfrac{\sin 2x}{-3\sin 3x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to \pi} \frac{2\cos 2x}{-9\cos 3x} = \frac{2}{9}$$

5. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1 - \cos x}}$ $\left(\frac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{1 - \cos x}} = \lim\limits_{x \to 0} \dfrac{\sin x}{\left(2\sin^2 \frac{x}{2}\right)^{1/2}} = \lim\limits_{x \to 0} \dfrac{\sin x}{\sqrt{2}\sin \frac{x}{2}}$

$$= \lim_{x \to 0} \frac{2\sin \frac{x}{2} \cos \frac{x}{2}}{\sqrt{2}\sin \frac{x}{2}} = \lim_{x \to 0} \sqrt{2}\cos \frac{x}{2} = \sqrt{2}$$

6. $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^2 \tan x}$

Sol. We have $\dfrac{\tan x - \sin x}{x^2 \tan x} = \dfrac{\frac{\sin x}{\cos x} - \sin x}{x^2 \cdot \frac{\sin x}{\cos x}} = \dfrac{\sin x\,(1 - \cos x)}{x^2 \sin x} = \dfrac{1 - \cos x}{x^2}$

Now, $\lim\limits_{x \to 0} \dfrac{\tan x - \sin x}{x^2 \tan x} = \lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

7. $\lim\limits_{x \to 1} \dfrac{nx^{n+1} - (n+1)\,x^n + 1}{(x-1)^2}$

Sol. $\lim\limits_{x \to 1} \dfrac{nx^{n+1} - (n+1)\,x^n + 1}{(x-1)^2}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 1} \frac{n(n+1)x^n - n(n+1)\,x^{n-1}}{2(x-1)} \quad \left(\frac{0}{0}\right)$$

$$= \frac{n(n+1)}{2} \lim_{x \to 1} \frac{x^n - x^{n-1}}{x - 1} = \frac{n\,(n+1)}{2} \lim_{x \to 1} \frac{x^{n-1}\,(x-1)}{x-1}$$

$$= \frac{n(n+1)}{2} \lim_{x \to 1} (x^{n-1}) = \frac{n\,(n+1)}{2} \cdot 1 = \frac{n(n+1)}{2}$$

8. $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$ $\left(\frac{0}{0}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x} = \lim\limits_{x \to 0} \dfrac{e^x + 2\sin x - e^{-x}}{x\cos x + \sin x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{-x\sin x + \cos x + \cos x}$$

$$= \lim_{x \to 0} \frac{e^x + 2\cos x + e^{-x}}{-x\sin x + 2\cos x} = \frac{4}{2} = 2$$

9. $\lim\limits_{x \to 0} \dfrac{\ln(1 - x^2)}{\ln \cos x}$ $\left(\frac{0}{0}\right)$

Sol. $$\lim_{x \to 0} \frac{\ln(1-x^2)}{\ln \cos x} = \lim_{x \to 0} \frac{\frac{1}{(1-x^2)}(-2x)}{\frac{1}{\cos x}(-\sin x)}$$

$$= \lim_{x \to 0} \frac{\frac{-2x}{(1-x^2)}}{\frac{-\sin x}{\cos x}} = \lim_{x \to 0} \frac{2x \cos x}{(1-x^2)\sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2[-x \sin x + \cos x]}{(1-x^2)\cos x + \sin x\,(-2x)}$$

$$= \lim_{x \to 0} \frac{-2x \sin x + 2 \cos x}{(1-x^2)\cos x - 2x \sin x} = \frac{2}{1} = 2$$

10. $$\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x} \qquad \left(\frac{0}{0}\right)$$

Sol. $$\lim_{x \to 0} \frac{\cosh x - \cos x}{x \sin x} = \lim_{x \to 0} \frac{\sinh x + \sin x}{x \cos x + \sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \frac{\cosh x + \cos x}{-x \sin x + \cos x + \cos x} = \frac{2}{2} = 1$$

11. $$\lim_{x \to 1} \frac{1 - x + \ln x}{1 - \sqrt{2x - x^2}} \qquad \left(\frac{0}{0}\right)$$

Sol. $$\lim_{x \to 1} \frac{1 - x + \ln x}{1 - \sqrt{2x - x^2}} = \lim_{x \to 1} \frac{-1 + \frac{1}{x}}{-\frac{1}{2}(2x - x^2)^{-1/2}(2 - 2x)}$$

$$= \lim_{x \to 1} -\frac{\frac{(1-x)}{x}}{(2x - x^2)^{-1/2}(1-x)} = \lim_{x \to 1} -\frac{(1-x)(2x-x^2)^{1/2}}{x(1-x)}$$

$$= \lim_{x \to 1} -\frac{(2x-x^2)^{1/2}}{x} = -\frac{(2-1)^{1/2}}{1} = -1$$

12. $$\lim_{x \to 0} \frac{x \cos x - \ln(1+x)}{x^2}$$

Sol. $$\lim_{x \to 0} \frac{x \cos x - \ln(1+x)}{x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-x \sin x + \cos x - \frac{1}{1+x}}{2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\{x \cos x + \sin x\} - \sin x + (1+x)^{-2}}{2} = \frac{1}{2}$$

13. $$\lim_{x \to 0} \frac{\sin x - \ln(e^x \cos x)}{x \sin x}$$

Sol. $$\lim_{x \to 0} \frac{\sin x - \ln(e^x \cos x)}{x \sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x - \frac{1}{e^x \cos x}(-e^x \sin x + \cos x\, e^x)}{x \cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x + \frac{e^x(\sin x - \cos x)}{e^x \cos x}}{x \cos x + \sin x}$$

$$= \lim_{x \to 0} \frac{\cos x + \tan x - 1}{x \cos x + \sin x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\sin x + \sec^2 x}{-x \sin x + \cos x + \cos x} = \frac{1}{2}$$

14. $$\lim_{x \to 0} \frac{1 + \sin x - \cos x + \ln(1-x)}{x \tan^2 x}$$

Sol. $$\lim_{x \to 0} \frac{1 + \sin x - \cos x + \ln(1-x)}{x \tan^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{x \,.\, 2 \tan x \sec^2 x + \tan^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{(1-x)}}{2x \tan x (1 + \tan^2 x) + \tan^2 x}$$

$$= \lim_{x \to 0} \frac{\cos x + \sin x - \frac{1}{1-x}}{2x \tan x + 2x \tan^3 x + \tan^2 x}$$

$$= \lim_{x \to 0} \frac{-\sin x + \cos x - \frac{1}{(1-x)^2}}{2x \sec^2 x + 2 \tan x + 6x \tan^2 x \sec^2 x + 2 \tan^3 x + 2 \tan x \sec^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{-\cos x - \sin x - \frac{2}{(1-x)^3}}{2 \sec^2 x + 4x \sec^2 x \tan x + 2 \sec^2 x + 6 \tan^2 x \sec^2 x}$$

$$+\ 12x \sec^2 x \tan^3 x + 12x \tan x \sec^4 x$$
$$+\ 6 \tan^2 x \sec^2 x + 4 \sec^2 x \tan^2 x + 2 \sec^4 x$$
$$= \frac{-1-2}{2+2+2} = \frac{-3}{6} = \frac{-1}{2}$$

15. If $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ exists, find the values of a and the limit.

Sol. $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ $\quad \left(\dfrac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{2 \cos 2x + a \cos x}{3x^2} \qquad (1)$$

The denominator of (1) $\to 0$ as $x \to 0$
and since the given limit exists and is finite, the numerator
$2 \cos 2x + a \cos x$ of (1) must $\to 0$ as $x \to 0$
i.e., $2 + a = 0$ or $a = -2$
With this value of a, (1) becomes

$$\lim_{x \to 0} \frac{2 \cos 2x - 2 \cos x}{3x^2} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{-4 \sin 2x + 2 \sin x}{6x} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{-8 \cos 2x + 2 \cos x}{6} = \frac{-8+2}{6} = -1$$

16. $\lim\limits_{x \to 0} \dfrac{\ln (\sin 3x)}{\ln (\sin x)}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

Sol. $\lim\limits_{x \to 0} \dfrac{\ln (\sin 3x)}{\ln (\sin x)} = \lim\limits_{x \to 0} \dfrac{\frac{1}{\sin 3x} 3 \cos 3x}{\frac{1}{\sin x} \cos x}$

$$= \lim_{x \to 0} \frac{3 \sin x \cos 3x}{\sin 3x \cos x} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{3[\sin x\,(-3 \sin 3x) + \cos 3x \cos x]}{-\sin 3x \sin x + \cos x\,(3 \cos 3x)}$$
$$= \lim_{x \to 0} \frac{-9 \sin x \sin 3x + 3 \cos x \cos 3x}{-\sin x \sin 3x + 3 \cos x \cos 3x} = \frac{3}{3} = 1$$

17. $\lim\limits_{x \to 0} \left(\dfrac{1}{x \arcsin x} - \dfrac{1}{x^2}\right)$

Sol. The given limit $= \lim\limits_{x \to 0} \left(\dfrac{x - \arcsin x}{x^2 \arcsin x}\right)$ $\quad \left(\dfrac{0}{0}\right)$

Put $\arcsin x = y$ so that $x = \sin y$ and $y \to 0$ as $x \to 0$.

The limit becomes $\lim\limits_{y \to 0} \left(\dfrac{\sin y - y}{y \sin^2 y}\right)$ $\quad \left(\dfrac{0}{0}\right)$

$$= \lim_{y \to 0} \left(\frac{\cos y - 1}{2y \sin y \cos y + \sin^2 y}\right) = \lim_{y \to 0} \frac{\cos y - 1}{y \sin 2y + \sin^2 y} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{y \to 0} \left(\frac{-\sin y}{2y \cos 2y + \sin 2y + \sin 2y}\right)$$
$$= \lim_{y \to 0} \left(\frac{-\sin y}{2y \cos 2y + 2 \sin 2y}\right) \quad \left(\frac{0}{0}\right)$$
$$= \lim_{y \to 0} \left(\frac{-\cos y}{2 \cos 2y - 4y \sin 2y + 4 \cos 2y}\right) = \frac{-1}{2+4} = -\frac{1}{6}$$

18. $\lim\limits_{x \to a} \dfrac{\ln (x-a)}{\ln (e^x - e^a)}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

Sol. $\lim\limits_{x \to a} \dfrac{\ln (x-a)}{\ln (e^x - e^a)} = \lim\limits_{x \to a} \dfrac{\frac{1}{(x-a)}}{\frac{1}{e^x - e^a} \cdot e^x}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x \to a} \frac{e^x - e^a}{e^x (x-a)} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to a} \frac{e^x}{e^x + (x-a) e^x} = \frac{e^a}{e^a} = 1$$

19. $\lim\limits_{x \to 0} \dfrac{\ln (\tan x)}{\ln x}$

Sol. $\lim\limits_{x \to 0} \dfrac{\ln (\tan x)}{\ln x}$ $\quad \left(\dfrac{\infty}{\infty}\right)$

$$= \lim_{x \to 0} \frac{\left(\frac{1}{\tan x}\right) \sec^2 x}{\frac{1}{x}} = \lim_{x \to 0} \frac{x}{\sin x \cos x} \quad \left(\frac{0}{0}\right)$$
$$= \lim_{x \to 0} \frac{2x}{2 \sin x \cos x} = \lim_{x \to 0} \frac{2x}{\sin 2x} = 1, \text{ since } \lim_{\theta \to 0} \frac{\theta}{\sin \theta} = 1$$

20. $\lim\limits_{x \to 0} \log_{\tan x} (\tan 2x)$

Sol. $\lim\limits_{x \to 0} \log_{\tan x} (\tan 2x)$

$$= \lim_{x \to 0} \frac{\ln \tan 2x}{\ln \tan x} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x}\cdot 2\sec^2 2x}{\frac{1}{\tan x}\cdot \sec^2 x} = \lim_{x\to 0}\frac{\frac{\cos 2x}{\sin 2x}\cdot\frac{2}{\cos^2 2x}}{\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}}$$

$$= \lim_{x\to 0}\frac{\frac{2}{\sin 2x\cos 2x}}{\frac{1}{\sin x\cos x}} = \lim_{x\to 0}\frac{2\sin x\cos x}{\sin 2x\cos 2x}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{\sin 2x\cos 2x} = \lim_{x\to 0}\frac{1}{\cos 2x} = \frac{1}{1} = 1$$

21. $\lim\limits_{x\to a}(x-a)\csc\left(\frac{\pi x}{a}\right)$

Sol. $\lim\limits_{x\to a}(x-a)\csc\left(\frac{\pi x}{a}\right)$ $\qquad (0\times\infty)$

$$= \lim_{x\to a}\frac{x-a}{\sin\frac{\pi x}{a}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to a}\frac{1}{\frac{\pi}{a}\cos\frac{\pi x}{a}} = \frac{a}{\pi}\lim_{x\to a}\frac{1}{\cos\frac{\pi x}{a}} = \frac{a}{\pi}\cdot\frac{1}{-1} = -\frac{a}{\pi}$$

22. $\lim\limits_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$

Sol. $\lim\limits_{x\to 1}(1-x)\tan\left(\frac{\pi x}{2}\right)$ $\qquad (0\times\infty)$

$$= \lim_{x\to 1}\frac{(1-x)}{\cot\frac{\pi x}{2}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\frac{-1}{\left(-\csc^2\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} = \lim_{x\to 1}\frac{\sin^2\left(\frac{\pi x}{2}\right)}{\frac{\pi}{2}} = \frac{2}{\pi}$$

23. $\lim\limits_{x\to 0} x\ln(\tan x)$

Sol. $\lim\limits_{x\to 0} x\ln(\tan x) = \lim\limits_{x\to 0}\dfrac{\ln\tan x}{\frac{1}{x}}$ $\qquad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0}\frac{\frac{1}{\tan x}\cdot\sec^2 x}{-\frac{1}{x^2}} = \lim_{x\to 0}\frac{\frac{\cos x}{\sin x}\cdot\frac{1}{\cos^2 x}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to 0} -\frac{x^2}{\sin x\cos x} = \lim_{x\to 0} -\frac{2x^2}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} -\frac{4x}{2\cos 2x} = 0$$

24. $\lim\limits_{x\to 0} x\tan\left(\frac{\pi}{2}-x\right)$ $\qquad (0\times\infty)$

Sol. $\lim\limits_{x\to 0} x\tan\left(\frac{\pi}{2}-x\right) = \lim\limits_{x\to 0}\dfrac{\tan\left(\frac{\pi}{2}-x\right)}{\frac{1}{x}}$ $\qquad \left(\frac{\infty}{\infty}\right)$

$$= \lim_{x\to 0}\frac{-\sec^2\left(\frac{\pi}{2}-x\right)}{-x^{-2}} = \lim_{x\to 0}\frac{x^2}{\cos^2\left(\frac{\pi}{2}-x\right)} = \lim_{x\to 0}\frac{x^2}{\sin^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{2x}{2\sin x\cos x} = \lim_{x\to 0}\frac{2x}{\sin 2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{2}{2\cos 2x} = \frac{2}{2} = 1$$

25. $\lim\limits_{x\to \pi/2}\tan x\ln(\sin x)$ $\qquad (\infty\times 0)$

Sol. $\lim\limits_{x\to \pi/2}\tan x\ln(\sin x) = \lim\limits_{x\to \pi/2}\dfrac{\ln\sin x}{\cot x}$ $\qquad \left(\frac{0}{0}\right)$

$$= \lim_{x\to \pi/2}\frac{\frac{1}{\sin x}\cdot\cos x}{-\csc^2 x} = \lim_{x\to \pi/2} -\frac{\cos x\sin^2 x}{\sin x}$$

$$= \lim_{x\to \pi/2} -\cos x\sin x = 0$$

26. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)$ $\qquad (\infty-\infty)$

Sol. $\lim\limits_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right) = \lim\limits_{x\to 0}\dfrac{e^x-1-x}{x(e^x-1)} = \lim\limits_{x\to 0}\dfrac{e^x-1-x}{xe^x-x}$ $\qquad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\frac{e^x-1}{xe^x+e^x-1} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{e^x}{xe^x + e^x + e^x} = \lim_{x \to 0} \frac{e^x}{e^x(x+2)} = \lim_{x \to 0} \frac{1}{x+2} = \frac{1}{2}$$

27. $\lim\limits_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right]$

Sol. $\lim\limits_{x \to 0} \left[\frac{a}{x} - \cot\left(\frac{x}{a}\right)\right]$ $(\infty - \infty)$

$$= \lim_{x \to 0} \left(\frac{a}{x} - \frac{\cos\frac{x}{a}}{\sin\frac{x}{a}}\right) = \lim_{x \to 0} \frac{a\sin\frac{x}{a} - x\cos\frac{x}{a}}{x\sin\frac{x}{a}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cos\frac{x}{a} - \left\{-\frac{x}{a}\sin\frac{x}{a} + \cos\frac{x}{a}\right\}}{\frac{x}{a}\cos\frac{x}{a} + \sin\frac{x}{a}} = \lim_{x \to 0} \frac{\frac{x}{a}\cdot\sin\frac{x}{a}}{\frac{x}{a}\cdot\cos\frac{x}{a} + \sin\frac{x}{a}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{a}\cdot\frac{x}{a}\cos\frac{x}{a} + \frac{1}{a}\cdot\sin\frac{x}{a}}{-\frac{x}{a^2}\sin\frac{x}{a} + \frac{1}{a}\cdot\cos\frac{x}{a} + \frac{1}{a}\cdot\cos\frac{x}{a}} = \frac{0+0}{-0+\frac{1}{a}+\frac{1}{a}} = 0$$

28. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

Sol. $\lim\limits_{x \to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$ $(\infty - \infty)$

$$= \lim_{x \to 1} \frac{x\ln x - x + 1}{(x-1)\ln x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{x\cdot\frac{1}{x} + \ln x - 1}{(x-1)\frac{1}{x} + \ln x} = \lim_{x \to 1} \frac{\ln x}{1 - \frac{1}{x} + \ln x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{\frac{1}{x}}{\frac{1}{x^2} + \frac{1}{x}} = \frac{1}{1+1} = \frac{1}{2}$$

29. $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$

Sol. $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$ $(\infty - \infty)$

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x \to \pi/2} \left(\frac{1 - \sin x}{\cos x}\right) \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = 0$$

30. $\lim\limits_{x \to 1} \left(\frac{2}{x^2-1} - \frac{1}{x-1}\right)$

Sol. $\lim\limits_{x \to 1} \left(\frac{2}{x^2-1} - \frac{1}{x-1}\right) = \lim\limits_{x \to 1} \left\{\frac{2}{(x-1)(x+1)} - \frac{1}{(x-1)}\right\}$ $(\infty - \infty)$

$$= \lim_{x \to 1} \frac{2 - (x+1)}{(x-1)(x+1)} = \lim_{x \to 1} \frac{1-x}{x^2-1} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 1} \frac{-1}{2x} = -\frac{1}{2}$$

31. $\lim\limits_{x \to \infty} (\sqrt{x^2 + 5x} - x)$

Sol. This is of the form $\infty - \infty$. But application of L. Hospital's rule to this limit results in complicated derivatives. We proceed as under:

$$\lim_{x \to \infty} (\sqrt{x^2 + 5x} - x)$$

$$= \lim_{x \to \infty} \frac{(\sqrt{x^2 + 5x} - x)(\sqrt{x^2 + 5x} + x)}{\sqrt{x^2 + 5x} + x}$$

$$= \lim_{x \to \infty} \frac{x^2 + 5x - x^2}{\sqrt{x^2 + 5x} + x} = \lim_{x \to \infty} \frac{5x}{\sqrt{x^2 + 5x} + x}$$

$$= \lim_{x \to \infty} \frac{5}{\sqrt{1 + \frac{5}{x}} + 1}$$ **(dividing the numerator and the denominator by x)**

$$= \frac{5}{2}$$

Alternative Method:

As $\sqrt{x^2 + 5x} - x = \sqrt{x^2\left(1 + \frac{5}{x}\right)} - x = x\left(\sqrt{1 + \frac{5}{x}} - 1\right)$, so

$$\lim_{x \to \infty} \sqrt{x^2 + 5x} - x = \lim_{x \to \infty} \frac{\sqrt{1 + \frac{5}{x}} - 1}{\frac{1}{x}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{2}\cdot\frac{-\frac{5}{x^2}}{\sqrt{1 + \frac{5}{x}}}}{-\frac{1}{x^2}} = \lim_{x \to \infty} \frac{5}{2}\cdot\frac{1}{\sqrt{1 + \frac{5}{x}}} = \frac{5}{2}\cdot\frac{1}{1} = \frac{5}{2}$$

32. $\lim\limits_{x \to \infty} (e^x + e^{-x})^{2/x}$

Sol. Let $y = (e^x + e^{-x})^{2/x}$. Then $\ln y = \dfrac{2}{x} \ln (e^x + e^{-x})$ and

$$\lim_{x \to \infty} \ln y = \lim_{x \to \infty} \frac{2 \ln (e^x + e^{-x})}{x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \infty} \frac{2 \cdot \dfrac{1}{e^x + e^{-x}} \cdot (e^x - e^{-x})}{1} = \lim_{x \to \infty} \frac{2e^x (1 - e^{-2x})}{e^x (1 + e^{-2x})}$$

$$= \lim_{x \to \infty} \frac{2(1 - e^{-2x})}{1 + e^{-2x}} = \frac{2(1-0)}{1+0} = 2$$

$$\lim_{x \to \infty} \ln y = 2 \Rightarrow \lim_{x \to \infty} y = e^2$$

Thus $\lim\limits_{x \to \infty} (e^x + e^{-x})^{2/x} = e^2$

33. $\lim\limits_{x \to 0} \left(\dfrac{1}{x}\right)^{\tan x}$ $\qquad \infty^0$

Sol. Let $y = \left(\dfrac{1}{x}\right)^{\tan x}$ or $\ln y = \tan x \ln \left(\dfrac{1}{x}\right)$

$$= -\tan x \ln x = -\frac{\ln x}{\cot x}. \text{ Then}$$

$$\lim_{x \to 0} (\ln y) = -\lim_{x \to 0} \frac{\ln x}{\cot x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= -\lim_{x \to 0} \frac{\dfrac{1}{x}}{-\csc^2 x} = \lim_{x \to 0} \frac{\sin^2 x}{x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{2 \sin x \cos x}{1} = \frac{0}{1} = 0$$

Hence $\lim\limits_{x \to 0} y = e^0 = 1$

34. $\lim\limits_{x \to \pi/2} (\cos x)^{-x + \pi/2}$

Sol. Let $y = (\cos x)^{-x + \frac{\pi}{2}}$ or $\ln y = \left(\dfrac{\pi}{2} - x\right) \ln \cos x$

and $$\lim_{x \to \pi/2} (\ln y) = \lim_{x \to \pi/2} \frac{\ln \cos x}{\left(\dfrac{1}{\frac{\pi}{2} - x}\right)} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \pi/2} \frac{\dfrac{1}{\cos x} \cdot (-\sin x)}{\left(\frac{\pi}{2} - x\right)^{-2}} = \lim_{x \to \pi/2} -\frac{\tan x}{\left(\frac{\pi}{2} - x\right)^{-2}} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x \to \pi/2} -\frac{\left(\frac{\pi}{2} - x\right)^2}{\cot x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \pi/2} -\frac{2 \cdot \left(\frac{\pi}{2} - x\right)(-1)}{-\csc^2 x} = \lim_{x \to \pi/2} \frac{2\left(\frac{\pi}{2} - x\right)}{-\csc^2 x} = \frac{0}{-1} = 0$$

Hence $\lim\limits_{x \to \pi/2} y = e^0 = 1$ or $\lim\limits_{x \to \pi/2} (\cos x)^{\pi/2 - x} = 1$

35. $\lim\limits_{x \to \infty} \left(\dfrac{x+a}{x-a}\right)^x$

Sol. Let $y = \left(\dfrac{x+a}{x-a}\right)^x$. Then $\ln y = x \ln \left(\dfrac{x+a}{x-a}\right)$ and $\lim\limits_{x \to \infty} x \ln \left(\dfrac{x+a}{x-a}\right)$

$$= \lim_{z \to 0} \frac{1}{z} \ln \frac{1 + az}{1 - az}, \text{ on setting } z = \frac{1}{x}$$

$$= \lim_{z \to 0} \frac{\ln \dfrac{1+az}{1-az}}{z} = \lim_{z \to 0} \frac{\ln (1 + az) - \ln (1 - az)}{z} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{z \to 0} \frac{\dfrac{a}{1+az} + \dfrac{a}{1-az}}{1} = \lim_{z \to 0} \frac{2a}{1 - a^2 z^2} = 2a$$

Thus $\lim\limits_{x \to \infty} \ln y = 2a$ or $\lim\limits_{x \to \infty} y = e^{2a}$

36. $\lim\limits_{x \to 0} \left(\dfrac{\sinh x}{x}\right)^{1/x^2}$

Sol. Let $y = \left(\dfrac{\sinh x}{x}\right)^{1/x^2}$ or $\ln y = \dfrac{1}{x^2} \ln \left(\dfrac{\sinh x}{x}\right)$

Now $\dfrac{\ln \dfrac{\sinh x}{x}}{x^2}$ is of the form $\dfrac{0}{0}$ as $x \to 0$, since $\dfrac{\sinh x}{x} \to 1$ as $x \to 0$.

Therefore, $$\lim_{x \to 0} \ln y = \lim_{x \to 0} \frac{\ln \left(\dfrac{\sinh x}{x}\right)}{x^2} = \lim_{x \to 0} \frac{\dfrac{\cosh x}{\sinh x} - \dfrac{1}{x}}{2x}$$

$$= \lim_{x\to 0} \frac{x\cosh x - \sinh x}{2x^2 \sinh x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cosh x + x\sinh x - \cosh x}{4x\sinh x + 2x^2\cosh x}$$

$$= \lim_{x\to 0} \frac{\sinh x}{4\sinh x + 2x\cosh x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{\cosh x}{4\cosh x + 2\cosh x + 2x\sinh x} = \frac{1}{6}$$

Thus $\lim_{x\to 0} y = e^{1/6}$.

37. $\lim_{x\to 0} (\tan x)^{\sin 2x}$

Sol. Let $y = (\tan x)^{\sin 2x}$ or $\ln y = \sin 2x \ln \tan x = \dfrac{\ln \tan x}{\csc 2x}$

$$\lim_{x\to 0} (\ln y) = \lim_{x\to 0} \frac{\ln \tan x}{\csc 2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0} \frac{\frac{\sec^2 x}{\tan x}}{-2\csc 2x \cot 2x}$$

$$= \lim_{x\to 0} \frac{\frac{1}{\sin x \cos x}}{\frac{-2\cos 2x}{\sin^2 2x}} = \lim_{x\to 0} -\frac{\sin^2 2x}{\cos 2x\,(2\sin x\cos x)}$$

$$= \lim_{x\to 0} -\frac{\sin 2x}{\cos 2x} = -\frac{0}{1} = 0$$

Hence $\lim_{x\to 0} y = e^0 = 1$

38. $\lim_{x\to 0} (1 + \sin x)^{\cot x}$

Sol. Let $y = (1 + \sin x)^{\cot x}$

or $\ln y = \cot x \ln(1 + \sin x) = \dfrac{\ln(1+\sin x)}{\tan x} \qquad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{\frac{1}{1+\sin x}\cdot \cos x}{\sec^2 x} = \lim_{x\to 0} \frac{\cos^3 x}{1+\sin x} = 1$$

Thus $\lim_{x\to 0} y = e^1 = e$

39. $\lim_{x\to \pi/2} (\sec x)^{\cot x}$

Sol. Let $y = (\sec x)^{\cot x}$ or $\ln y = \cot x \ln \sec x = \dfrac{\ln \sec x}{\tan x}$

$$\lim_{x\to \pi/2} (\ln y) = \lim_{x\to \pi/2} \frac{\ln \sec x}{\tan x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to \pi/2} \frac{\frac{1}{\sec x}\sec x\tan x}{\sec^2 x}$$

$$= \lim_{x\to \pi/2} \frac{\tan x}{\sec^2 x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to \pi/2} \frac{\sec^2 x}{2\tan x\sec^2 x}$$

$$= \lim_{x\to \pi/2} \frac{1}{2\tan x} = \lim_{x\to \pi/2} \frac{1}{2}\cot x = 0$$

Therefore, $\lim_{x\to \pi/2} y = e^0 = 1$

40. $\lim_{x\to 1} (1 - x^2)^{1/\ln(1-x)}$

Sol. Let $y = (1 - x^2)^{1/\ln(1-x)}$

or $\ln y = \dfrac{1}{\ln(1-x)} \ln(1 - x^2) = \dfrac{\ln(1-x^2)}{\ln(1-x)}$

$$\lim_{x\to 1} (\ln y) = \lim_{x\to 1} \frac{\ln(1-x^2)}{\ln(1-x)} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 1} \frac{\frac{1}{(1-x^2)}(-2x)}{\frac{1}{(1-x)}(-1)} = \lim_{x\to 1} \frac{2x(1-x)}{(1-x^2)}$$

$$= \lim_{x\to 1} \frac{2x}{(1+x)} = 1$$

Thus $\lim_{x\to 1} y = e^1 = e$

41. $\lim_{x\to 1} \left[\tan\left(\frac{x\pi}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)}$

Sol. Let $y = \left[\tan\left(\frac{x\pi}{4}\right)\right]^{\tan\left(\frac{\pi x}{2}\right)}$

$$\ln y = \tan\frac{\pi x}{2}\ln\tan\frac{\pi x}{4} = \frac{\ln\tan\frac{\pi x}{4}}{\cot\frac{\pi x}{2}}$$

$$\lim_{x\to 1}\ln y = \lim_{x\to 1}\frac{\ln\tan\frac{\pi x}{4}}{\cot\frac{\pi x}{2}} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 1}\frac{\frac{1}{\tan\frac{\pi x}{4}}\cdot\left(\sec^2\frac{\pi x}{4}\right)\cdot\frac{\pi}{4}}{\frac{\pi}{2}\cdot\left(-\csc^2\frac{\pi x}{2}\right)}$$

$$= \lim_{x\to 1} -\frac{\sin^2\frac{\pi x}{2}}{2\sin\frac{\pi x}{4}\cos\frac{\pi x}{4}} = \lim_{x\to 1} -\frac{\sin^2\frac{\pi x}{2}}{\sin\frac{\pi x}{2}}$$

$$= \lim_{x\to 1}\left(-\sin\frac{\pi x}{2}\right) = -1$$

Thus $\lim\limits_{x\to 1} y = e^{-1} = \frac{1}{e}$

42. $\lim\limits_{x\to \pi/2}(1-\sin x)^{\cos x}$

Sol. Let $y = (1-\sin x)^{\cos x}$

or $\ln y = \cos x \ln(1-\sin x) = \dfrac{\ln(1-\sin x)}{\sec x}$

$$\lim_{x\to \pi/2}\ln y = \lim_{x\to \pi/2}\frac{\ln(1-\sin x)}{\sec x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to \pi/2}\frac{\frac{-\cos x}{1-\sin x}}{\sec x\tan x}$$

$$= \lim_{x\to \pi/2}\left[\frac{-\cos x}{1-\sin x}\times\frac{\cos x\times\cos x}{\sin x}\right]$$

$$= \lim_{x\to \pi/2}\frac{-\cos^3 x}{\sin x-\sin^2 x} \qquad \left(\frac{0}{0}\right)$$

$$= \frac{-3\cos^2 x(-\sin x)}{\cos x - 2\sin x\cos x} = \lim_{x\to \pi/2}\frac{3\sin x\cos x}{1-2\sin x}$$

$$= \lim_{x\to \pi/2}\frac{3}{2}\cdot\frac{\sin 2x}{1-2\sin x} = 0$$

Therefore, $\lim\limits_{x\to \pi/2} y = e^0 = 1$

Alternative Method:

$$\lim_{x\to \pi/2}\frac{\frac{-\cos x}{1-\sin x}}{\sec x\tan x} = \lim_{x\to \pi/2}\frac{\frac{-\cos^2 x}{1-\sin \pi}}{\tan x} = \lim_{x\to \pi/2}\frac{-\cos x(1-\sin^2 x)}{\sin x(1-\sin x)}$$

$$= \lim_{x\to \pi/2}\frac{\cos x(1+\sin x)}{\sin x} = -\frac{0(2)}{1} = 0$$

Thus $\lim\limits_{x\to \pi/2} y = e^0 = 1$

43. $\lim\limits_{x\to 0}(\cot x)^{\sin 2x}$

Sol. Let $y = (\cot x)^{\sin 2x}$ or $\ln y = \sin 2x \ln(\cot x)$

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln\cot x}{\csc 2x} \qquad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{\frac{-\csc^2 x}{\cot x}}{-2\csc 2x\cot 2x} = \lim_{x\to 0}\frac{\frac{1}{\sin x\cos x}}{2\frac{\cos 2x}{\sin^2 2x}}$$

$$= \lim_{x\to 0}\frac{\sin 2x}{\cos 2x} = 0$$

$\ln(\lim\limits_{x\to 0} y) = 0$ or $\lim\limits_{x\to 0} y = e^0 = 1$

44. $\lim\limits_{x\to 0}\dfrac{\tanh x-\sinh x}{x^2}$

Sol.
$$\lim_{x\to 0}\frac{\tanh x-\sinh x}{x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\operatorname{sech}^2 x-\cosh x}{2x} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-2\operatorname{sech}^2 x\tanh x-\sinh x}{2} = 0$$

45. $\lim\limits_{x\to 0}\dfrac{\sqrt{x}-\sqrt{\sin x}}{x^{5/2}}$

Sol.
$$\lim_{x\to 0}\frac{\sqrt{x}-\sqrt{\sin x}}{x^{5/2}} = \lim_{x\to 0}\frac{\sqrt{x}-\left\{x-\frac{x^3}{3!}+\frac{x^5}{5!}\cdots\right\}^{1/2}}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x}\left\{1 - \frac{x^2}{6} + \frac{x^4}{120} - \dots\right\}^{1/2}}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x}\left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right)\right]^{1/2}}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\sqrt{x} - \sqrt{x}\left[1 - \frac{1}{2}\left(\frac{x^2}{6} - \frac{x^4}{120} + \dots\right) + \frac{\frac{1}{2}\left(\frac{1}{2} - 1\right)}{2!} \cdot \left(-\left(\frac{x^2}{6} - \frac{x^4}{1200} + \dots\right)\right)^2 + \dots\right]}{x^{5/2}}$$

$$= \lim_{x \to 0} \frac{\frac{1}{2} \cdot \frac{x^2}{6} + \text{higher powers of } x}{x^2} = \frac{1}{12}$$

46. $\lim_{x \to 0} \frac{\sinh x - \sin x}{\sin^3 x}$

Sol. $\lim_{x \to 0} \frac{\sinh x - \sin x}{\sin^3 x}$ $\left(\frac{0}{0}\right)$

$$= \lim_{x \to 0} \frac{\cosh x - \cos x}{3 \sin^2 x \cos x}, \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\sinh x + \sin x}{6 \sin^2 x \cos x - 3 \sin^3 x}, \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\cosh x + \cos x}{6 \cos^3 x - 12 \sin^2 x \cos x - 9 \sin^2 x \cos x} = \frac{2}{6} = \frac{1}{3}$$

47. $\lim_{x \to 0} \frac{(1 + x)^{1/x} - e + \frac{e x}{2}}{x^2}$

Sol. To expand $(1 + x)^{1/x}$ into an infinite series, we let

$$y = (1 + x)^{1/x} \quad \text{or} \quad \ln y = \frac{1}{x} \ln (1 + x)$$

$$= \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \dots\right) = 1 - \frac{x}{2} + \frac{x^2}{3} - \dots$$

Therefore, $y = \exp\left(1 - \frac{x}{2} + \frac{x^2}{3} - \dots\right) = e \cdot e^{-\frac{x}{2} + \frac{x^2}{3} - \dots}$

$$= e\left[1 + \left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right) + \frac{1}{2}\left(-\frac{x}{2} + \frac{x^2}{3} - \dots\right)^2 + \dots\right]$$

$$= e\left(1 - \frac{x}{2} + \frac{11x^2}{24} - \dots\right)$$

Hence, $\lim_{x \to 0} \frac{(1 + x)^{1/x} - e + \frac{e x}{2}}{x^2}$

$$= \lim_{x \to 0} \frac{e\left(1 - \frac{x}{2} + \frac{11x^2}{24} - \dots\right) - e + \frac{e x}{2}}{x^2}$$

$$= \lim_{x \to 0} \left(\frac{11}{24} e + \text{powers of } x\right) = \frac{11}{24} e$$

48. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

Sol. $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

$$= \lim_{x \to 0} \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) - \left\{1 + x + \frac{x^2}{2} - \frac{x^4}{8} + \dots\right\}}{x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)}$$

$$= \lim_{x \to 0} \frac{\frac{x^3}{3!} + \text{higher powers in } x}{\frac{x^3}{6} + \text{higher powers in } x} = \frac{\frac{1}{6}}{\frac{1}{6}} = 1.$$

49. Use L' Hospital's Rule to prove that

$$\lim_{x \to \infty} \left[\frac{a^{1/x} + b^{1/x}}{2}\right]^x = \sqrt{ab}, \quad a > 0, b > 0.$$

Sol. The given limit is of the from 1^∞.

Let $y = \left[\frac{a^{1/x} + b^{1/x}}{2}\right]^x$ or $\ln y = x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right)$

Now, limit $x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right)$ is of the form $\infty \times 0$ as $x \to \infty$

$$\lim_{x \to \infty} x \ln\left(\frac{a^{1/x} + b^{1/x}}{2}\right) = \lim_{x \to \infty} \frac{\ln (a^{1/x} + b^{1/x}) - \ln 2}{\frac{1}{x}} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \infty} \frac{\frac{1}{a^{1/x} + b^{1/x}} \cdot \left[-\frac{1}{x^2} (a^{1/x} \ln a + b^{1/x} \ln b)\right]}{-\frac{1}{x^2}}$$

$$= \lim_{x \to \infty} \frac{a^{1/x} \ln a + b^{1/x} \ln b}{a^{1/x} + b^{1/x}} = \frac{\ln a + \ln b}{2} = \ln (ab)^{1/2}$$

Thus $\lim\limits_{x\to\infty} \ln y = \ln\sqrt{ab}$ or $\lim\limits_{x\to\infty} y = \sqrt{ab}$ as required.

50. If f is a thrice differentiable function, prove, by using L'Hospital's Rule, that

(i) $$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} = f'(x)$$

(ii) $$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} = f''(x)$$

(iii) $$\lim_{h\to 0}\frac{f(x+h)-f(x)-hf'(x)-\frac{h^2}{2}f''(x)}{h^3} = \frac{f'''(x)}{6}$$

Sol.

(i) $$\lim_{h\to 0}\frac{f(x+h)-f(x-h)}{2h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{f'(x+h)+f'(x-h)}{2}$$

$$= \frac{2f'(x)}{2} = f'(x)$$

(ii) $$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{f'(x+h)-f'(x-h)}{2h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{f''(x+h)+f''(x-h)}{2} = \frac{2f''(x)}{2} = f''(x)$$

(iii) $$\lim_{h\to 0}\frac{f(x+h)-f(x)-hf'(x)-\frac{h^2}{2}f''(x)}{h^3} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{f'(x+h)-f'(x)-hf''(x)}{3h^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{f''(x+h)-f''(x)}{6h} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{h\to 0}\frac{f'''(x+h)}{6} = \frac{f'''(x)}{6}$$

51. Determine a, b, c, d and e such that

$$\lim_{x\to 0}\frac{\cos ax + bx^3 + cx^2 + dx + e}{x^4} = \frac{2}{3}$$

Sol. If the limit is to be of the indeterminate form $\frac{0}{0}$, then

$\cos 0 + e = 0$ *i.e.,* $e = -1$.

Now $$\lim_{x\to 0}\frac{\cos ax + bx^3 + cx^2 + dx - 1}{x^4} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{-a\sin ax + 3bx^2 + 2cx + d}{4x^3}$$

It will be of the form $\frac{0}{0}$ if $d = 0$

$$\lim_{x\to 0}\frac{-a\sin ax + 3bx^2 + 2cx}{4x^3} = \lim_{x\to 0}\frac{-a^2\cos ax + 6bx + 2c}{12x^2}$$

If this limit is of the form $\frac{0}{0}$, then $-a^2 + 2c = 0$

$$\lim_{x\to 0}\frac{-a^2\cos ax + 6bx + 2c}{12x^2} \qquad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{a^3\sin ax + 6b}{24x}$$

It is of the form $\frac{0}{0}$ if $b = 0$.

Therefore, $\lim\limits_{x\to 0}\frac{a^3\sin ax}{24x} = \frac{2}{3}$ or $\lim\limits_{x\to 0}\frac{a^4\cos ax}{24} = \frac{2}{3}$

i.e., $\frac{a^4}{24} = \frac{2}{3}$ or $a^4 = 16 \Rightarrow a^2 = \pm 4 \Rightarrow a = \pm 2, \pm 2i$

We take real values of a

From above $a^2 = 2c$ yields $c = 2$.

Thus $a = \pm 2$, $b = 0$, $c = 2$, $d = 0$, $e = -1$.