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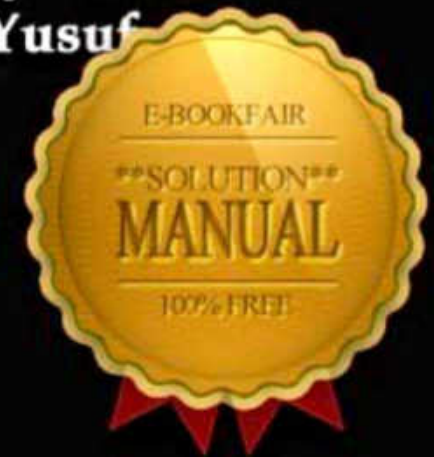
Group of **Jg** Network

# Calculus With Analytic Geometry

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## Calculus With Analytic Geometry

By  
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10. The upper half of the area inside the cardioid  $r = 2a(1 + \cos \theta)$  and outside the parabola  $\frac{2a}{r} = 1 + \cos \theta$  is revolved about the initial line. Show that the volume of the solid generated is  $18\pi a^3$ .

**Sol.** The curves intersect at the points where  $2a(1 + \cos \theta) = \frac{2a}{1 + \cos \theta}$

$$\text{or } (1 + \cos \theta)^2 = 1$$

$$\text{or } \cos \theta (\cos \theta + 2) = 0$$

$$\text{Therefore, } \cos \theta = 0$$

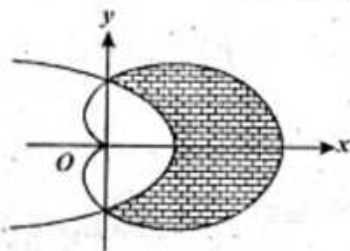
$$\text{or } \cos \theta = -2$$

But  $\cos \theta = -2$  is not possible. So

$$\cos \theta = 0 \text{ i.e., } \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

Required volume

$$\begin{aligned} &= \frac{2}{3} \pi \int_0^{\pi/2} \left\{ (2a(1 + \cos \theta))^3 - \left( \frac{2a}{1 + \cos \theta} \right)^3 \right\} \sin \theta d\theta \\ &= \frac{-16\pi a^3}{3} \int_0^{\pi/2} (1 + \cos \theta)^3 d(-\cos \theta) + \frac{-16\pi a^3}{3} \int_0^{\pi/2} (1 + \cos \theta)^{-3} d(-\cos \theta) \\ &= \frac{-4\pi a^3}{3} [(1 + \cos \theta)^4]_0^{\pi/2} - \frac{8\pi a^3}{3} [(1 + \cos \theta)^{-2}]_0^{\pi/2} \\ &= \frac{-4\pi a^3}{3} (1 - 16) - \frac{8\pi a^3}{3} \left( 1 - \frac{1}{4} \right) = 20\pi a^3 - 2\pi a^3 = 18\pi a^3. \end{aligned}$$



## Exercise Set 10.1 (Page 463)

Evaluate (Problems 1 – 19):

1.  $\int_0^1 \int_1^2 dx dy$

**Sol.**  $\int_0^1 \int_1^2 dx dy = \int_0^1 [x]_1^2 dy = \int_0^1 [2 - 1] dy$   
 $= \int_0^1 dy = [y]_0^1 = |1 - 0| = 1.$

2.  $\int_1^2 \int_0^3 (x + y) dx dy$

**Sol.**  $\int_1^2 \int_0^3 (x + y) dx dy = \int_1^2 \left[ \left( \frac{x^2}{2} \right)_0^3 + [xy]_0^3 \right] dy$   
 $= \int_1^2 \left[ \left( \frac{9}{2} - \frac{0}{2} \right) + (3y - 0y) \right] dy$   
 $= \int_1^2 \left[ \frac{9}{2} + 3y \right] dy = \frac{9}{2} [y]_1^2 + 3 \left[ \frac{y^2}{2} \right]_1^2$   
 $= \frac{9}{2} (2 - 1) + \frac{3}{2} (4 - 1) = 9.$

3.  $\int_2^4 \int_1^2 (x^2 + y^2) dy dx$

$$\begin{aligned}
 \text{Sol. } \int_2^4 \int_1^2 (x^2 + y^2) dy dx &= \int_2^4 \left[ x^2 y + \frac{y^3}{3} \right]_1^2 dx \\
 &= \int_2^4 \left[ x^2(2-1) + \frac{1}{3}(8-1) \right] dx \\
 &= \int_2^4 \left[ x^2 + \frac{7}{3} \right] dx = \left[ \frac{x^3}{3} + \frac{7}{3}x \right]_2^4 \\
 &= \frac{1}{3}(64-8) + \frac{7}{3}(4-2) = \frac{70}{3}.
 \end{aligned}$$

$$4. \int_0^1 \int_{x^2}^x xy^2 dy dx$$

$$\begin{aligned}
 \text{Sol. } \int_0^1 \int_{x^2}^x xy^2 dy dx &= \int_0^1 \left( x \left[ \frac{y^3}{3} \right]_{x^2}^x \right) dx = \int_0^1 \left[ x \left( \frac{x^3}{3} - \frac{x^6}{3} \right) \right] dx \\
 &= \int_0^1 \left( \frac{x^4}{3} - \frac{x^7}{3} \right) dx = \left[ \frac{x^5}{15} - \frac{x^8}{24} \right]_0^1 \\
 &= \left( \frac{1}{15} - \frac{0}{15} \right) - \left( \frac{1}{24} - \frac{0}{24} \right) = \frac{1}{40}.
 \end{aligned}$$

$$5. \int_1^2 \int_0^{y^{3/2}} \frac{x}{y^2} dx dy$$

$$\begin{aligned}
 \text{Sol. } \int_1^2 \int_0^{y^{3/2}} \frac{x}{y^2} dx dy &= \int_1^2 \frac{1}{y^2} \left[ \frac{x^2}{2} \right]_0^{y^{3/2}} dy = \int_1^2 \frac{1}{y^2} \left( \frac{y^3}{2} - \frac{0}{2} \right) dy \\
 &= \frac{1}{2} \int_1^2 y dy = \frac{1}{2} \left[ \frac{y^2}{2} \right]_1^2 = \frac{1}{4} \left( 4 - \frac{1}{2} \right) = \frac{3}{4}.
 \end{aligned}$$

$$6. \int_0^1 \int_x^{\sqrt{x}} (y + y^3) dy dx$$

$$\begin{aligned}
 \text{Sol. } \int_0^1 \int_x^{\sqrt{x}} (y + y^3) dy dx &= \int_0^1 \left[ \frac{y^2}{2} + \frac{y^4}{4} \right]_x^{\sqrt{x}} dx \\
 &= \frac{1}{2} \int_0^1 \left[ x - x^2 + \frac{x^2}{2} - \frac{x^4}{2} \right] dx \\
 &= \frac{1}{2} \int_0^1 \left[ x - \frac{x^2}{2} - \frac{x^4}{2} \right] dx \\
 &= \frac{1}{2} \left[ \frac{x^2}{2} - \frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{2} \left( \frac{1}{2} - \frac{1}{6} - \frac{1}{10} \right) = \frac{7}{60}.
 \end{aligned}$$

$$7. \int_0^1 \int_0^{x^2} xe^y dy dx$$

$$\begin{aligned}
 \text{Sol. } \int_0^1 \int_0^{x^2} xe^y dy dx &= \int_0^1 \left( x [e^y]_0^{x^2} \right) dx = \int_0^1 [x(e^{x^2} - e^0)] dx \\
 &= \int_0^1 (xe^{x^2} - x) dx = \left[ \frac{1}{2}e^{x^2} - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}e - \frac{1}{2} - \frac{1}{2}e^0 \\
 &= \frac{e}{2} - 1.
 \end{aligned}$$

$$8. \int_2^4 \int_y^{8-y} y dx dy$$

$$\begin{aligned}
 \text{Sol. } \int_2^4 \int_y^{8-y} y dx dy &= \int_2^4 [yx]_y^{8-y} dy = \int_2^4 [y(8-y-y)] dy \\
 &= \int_2^4 (8y - 2y^2) dy = \left[ 4y^2 - \frac{2}{3}y^3 \right]_2^4 \\
 &= 4 \times 16 - \frac{2}{3} \times 64 - 4 \times 4 + \frac{2}{3} \times 8 = 48 - \frac{112}{3} = \frac{32}{3}.
 \end{aligned}$$

$$9. \int_0^4 \int_{y/2}^2 e^{x^2} dx dy$$

**Sol.** The region of integration is bounded by  $0 \leq y \leq 4$ ,  $x = \frac{y}{2}$  and  $x = 2$ .

This region is also enclosed by  $0 \leq x \leq 2$ ,  $y = 0$  and  $y = 2x$ . The given integral is

$$\begin{aligned} &= \int_0^2 \int_0^{2x} e^{x^2} dy dx = \int_0^2 e^{x^2} [y]_0^{2x} dx \\ &= \int_0^2 2xe^{x^2} dx = [e^{x^2}]_0^2 = e^4 - 1. \end{aligned}$$

$$10. \int_0^2 \int_{y^2}^4 y \cos x^2 dx dy$$

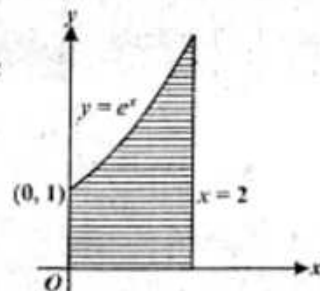
**Sol.** We change the order of integration. The region of integration is  $0 \leq y \leq 2$ ,  $y^2 \leq x \leq 4$ . This is equivalent to  $0 \leq x \leq 4$ ,  $0 \leq y \leq \sqrt{x}$ . The given integral equals

$$\begin{aligned} \int_0^4 \int_0^{\sqrt{x}} y \cos x^2 dy dx &= \frac{1}{2} \int_0^4 [y^2]_0^{\sqrt{x}} \cos x^2 dx = \frac{1}{2} \int_0^4 x \cos x^2 dx \\ &= \frac{1}{4} [\sin x^2]_0^4 = \frac{1}{4} \sin 16. \end{aligned}$$

11.  $\int_D \int dy dx$  and  $\int_D \int dx dy$ , where  $D$  is the region bounded by the  $y$ -axis, the line  $x = 2$  and the curve  $y = e^x$ .

$$\begin{aligned} \text{Sol. } \int_D \int dy dx &= \int_0^2 \int_1^{e^x} dy dx = \int_0^2 (e^x - 1) dx \\ &= [(e^x - x)]_0^2 = e^2 - 3 \end{aligned}$$

$$\int_D \int dx dy = \int_1^{e^2} \int_{\ln y}^2 dx dy$$



$$= \int_1^{e^2} (2 - \ln y) dy = [2y - y \ln y + y]_1^{e^2} = e^2 - 3.$$

$$12. \int_0^{\pi/2} \int_0^2 r^2 \cos \theta dr d\theta$$

$$\begin{aligned} \text{Sol. } \int_0^{\pi/2} \int_0^2 r^2 \cos \theta dr d\theta &= \int_0^{\pi/2} \cos \theta \left[ \frac{r^3}{3} \right]_0^2 d\theta = \int_0^{\pi/2} \frac{8}{3} \cos \theta d\theta \\ &= \frac{8}{3} [\sin \theta]_0^{\pi/2} = \frac{8}{3} (1 - 0) = \frac{8}{3}. \end{aligned}$$

$$13. \int_0^{2\pi} \int_0^{1-\cos \theta} r^3 \cos^2 \theta dr d\theta$$

$$\begin{aligned} \text{Sol. } \int_0^{2\pi} \int_0^{1-\cos \theta} r^3 \cos^2 \theta dr d\theta &= \int_0^{2\pi} \cos^2 \theta \left[ \frac{r^4}{4} \right]_0^{1-\cos \theta} d\theta \\ &= \int_0^{2\pi} \frac{1}{4} [\cos^2 \theta (1 - \cos \theta)^4] d\theta \\ &= \frac{1}{4} \int_0^{2\pi} \cos^2 \theta [1 - 4 \cos \theta + 6 \cos^2 \theta - 4 \cos^3 \theta + \cos^4 \theta] d\theta \\ &= \frac{1}{4} \int_0^{2\pi} [\cos^2 \theta - 4 \cos^3 \theta + 6 \cos^4 \theta - 4 \cos^5 \theta + \cos^6 \theta] d\theta \\ &= \frac{1}{4} \cdot 2 \int_0^{\pi} [\cos^2 \theta - 4 \cos^3 \theta + 6 \cos^4 \theta - 4 \cos^5 \theta + \cos^6 \theta] d\theta \\ &= \frac{1}{2} \int_0^{\pi} \cos^2 \theta d\theta - 2 \int_0^{\pi} \cos^3 \theta d\theta + 3 \int_0^{\pi} \cos^4 \theta d\theta - 2 \int_0^{\pi} \cos^5 \theta d\theta + \frac{1}{2} \int_0^{\pi} \cos^6 \theta d\theta \\ &= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \cos^2 \theta d\theta - 2(0) + 3 \cdot 2 \int_0^{\pi/2} \cos^4 \theta d\theta - 2(0) + \frac{1}{2} \cdot 2 \int_0^{\pi/2} \cos^6 \theta d\theta \end{aligned}$$

$$\begin{aligned}
 &= \int_0^{\pi/2} \cos^2 \theta d\theta + 6 \int_0^{\pi/2} \cos^4 \theta d\theta + \int_0^{\pi/2} \cos^6 \theta d\theta \\
 &= \frac{1}{2} \frac{\pi}{2} + 6 \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2} + \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \frac{\pi}{2}, \text{ by Wallis formula} \\
 &= \left( \frac{1}{2} + \frac{9}{4} + \frac{15}{48} \right) \frac{\pi}{2} = \left( \frac{24 + 108 + 15}{48} \right) \frac{\pi}{2} = \frac{49}{32} \pi.
 \end{aligned}$$

14.  $\iint_D e^{-(x^2+y^2)} dx dy$ , where  $D$  is the region in the first quadrant bounded by the circles  $x^2 + y^2 = 1$  and  $x^2 + y^2 = 4$ .

**Sol.** Changing into polar coordinates, the given integral is

$$\begin{aligned}
 &= \int_0^{\pi/2} \int_1^2 e^{-r^2} r dr d\theta = \int_0^{\pi/2} \left[ -\frac{1}{2} e^{-r^2} \right]_1^2 d\theta \\
 &= \frac{1}{2} (e^{-1} - e^{-4}) \left( \frac{\pi}{2} \right) = \frac{\pi}{4} (e^{-1} - e^{-4}).
 \end{aligned}$$

15.  $\iint_D \frac{dx dy}{1+x^2+y^2}$ , where  $D$  is the closed disc of radius  $a$  with centre at the origin.

**Sol.** Changing into polar coordinates, the given integral is

$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^a \frac{r dr d\theta}{1+r^2} = \int_0^{2\pi} \left[ \frac{1}{2} \ln(1+r^2) \right]_0^a d\theta \\
 &= \frac{1}{2} \ln(1+a^2) \int_0^{2\pi} d\theta = \pi \ln(1+a^2).
 \end{aligned}$$

16.  $\iint_D \frac{x^2}{(x^2+y^2)^2} dA$ , where  $D$  is the region in the first quadrant bounded by the circles  $x^2 + y^2 = a^2$ ,  $x^2 + y^2 = b^2$ ,  $0 < a < b$ .

**Sol.** Changing into polar coordinates, the given integral is

$$\int_0^{2\pi} \int_a^b \frac{r^2 \cos^2 \theta}{r^4} r dr d\theta$$

$$\int_0^{2\pi} \cos^2 \theta \left[ \ln r \right]_a^b d\theta = \ln \frac{b}{a} \int_0^{2\pi} \cos^2 \theta d\theta = \frac{\pi}{4} \ln \frac{b}{a}.$$

$$17. \int_{-a}^a \int_0^{\sqrt{a^2-x^2}} (x^2+y^2)^{3/2} dy dx.$$

**Sol.** Changing into polar coordinates, the given integral is

$$\int_0^{\pi} \int_0^a r^3 \cdot r dr d\theta = \int_0^{\pi} \left[ \frac{r^5}{5} \right]_0^a d\theta = \frac{a^5}{5} \int_0^{\pi} d\theta = \frac{\pi a^5}{5}.$$

$$18. \int_0^1 \int_0^{\sqrt{1-y^2}} \sin(x^2+y^2) dx dy$$

**Sol.** Changing into polar coordinates, we have the integral

$$\begin{aligned}
 &= \int_0^{\pi/2} \int_0^1 (\sin r^2) r dr d\theta = \int_0^{\pi/2} \left[ -\frac{1}{2} \cos r^2 \right]_0^1 d\theta \\
 &= \frac{1}{2} (1 - \cos 1) \int_0^{\pi/2} d\theta = \frac{\pi}{4} (1 - \cos 1).
 \end{aligned}$$

$$19. \int_0^4 \int_0^{\sqrt{4y-y^2}} (x^2+y^2) dx dy$$

**Sol.** The region of integration is bounded by

$$0 \leq x \leq \sqrt{4y-y^2} \text{ and } 0 \leq y \leq 4$$

Now  $x = \sqrt{4y-y^2}$  is the circle  $x^2 + y^2 - 4y = 0$

or  $x^2 + y^2 = 4y$ . In polar coordinates this takes the form

$$r^2 = 4r \sin \theta, \text{ or } r = 4 \sin \theta$$

On changing into polar coordinates, the given integral is

$$\begin{aligned}
 &= \int_0^{2\pi} \int_0^{4 \sin \theta} r^2 \cdot r dr d\theta = \int_0^{2\pi} 64 \sin^4 \theta d\theta \\
 &= 64 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ (using Wallis formula)} = 12\pi.
 \end{aligned}$$

20. (a). Let  $D_a$  be the region bounded by the circle  $x^2 + y^2 = a^2$ .

$$\text{Define } \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy = \lim_{a \rightarrow \infty} \int_{D_a} e^{-(x^2+y^2)} dx dy.$$

Evaluate this improper integral.

**Sol.(a).** Changing into polar coordinates, the given integral

$$\begin{aligned} I &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \int_0^a e^{-r^2} r dr d\theta = \lim_{a \rightarrow \infty} \int_0^{2\pi} \left[ -\frac{1}{2} e^{-r^2} \right]_0^a d\theta \\ &= \lim_{a \rightarrow \infty} \int_0^{2\pi} \frac{1}{2} (1 - e^{-a^2}) d\theta = \lim_{a \rightarrow \infty} \left[ \frac{1}{2} \theta - \frac{1}{2} e^{-a^2} \theta \right]_0^{2\pi} \\ &= \pi - \lim_{a \rightarrow \infty} \frac{\pi}{e^{a^2}} = \pi. \end{aligned}$$

20. (b). Use part (a) to prove that  $\int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}$

$$\begin{aligned} \text{Sol. } \int_{-a}^a \int_{-a}^a e^{-(x^2+y^2)} dy dx &= \int_{-a}^a e^{-y^2} \left( \int_{-a}^a e^{-x^2} dx \right) dy \\ &= \left( \int_{-a}^a e^{-x^2} dx \right) \left( \int_{-a}^a e^{-y^2} dy \right) = \left( \int_{-a}^a e^{-x^2} dx \right)^2 = 4 \left( \int_0^a e^{-x^2} dx \right)^2 \end{aligned}$$

Letting  $a \rightarrow \infty$ , we have

$$\begin{aligned} &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\ &= 4 \left( \int_0^{\infty} e^{-x^2} dx \right)^2 = \pi \quad \text{from Part (a) above.} \end{aligned}$$

$$\text{Therefore, } \int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \sqrt{\pi}.$$

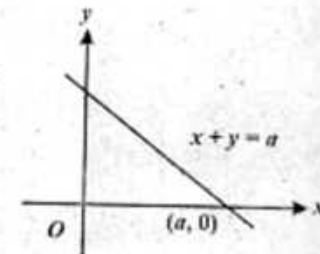
## Exercise Set 10.2 (Page 468)

1. By means of double integration, find the area of the region bounded by

(a). the coordinate axes and the straight line  $x + y = a$

**Sol.(a).** The required area

$$\begin{aligned} A &= \int_0^a \int_{a-x}^0 dy dx = \int_0^a [y]_{a-x}^0 dx \\ &= \int_0^a (x-a) dx = \left[ \frac{x^2}{2} \right]_0^a = -\frac{a^2}{2}. \end{aligned}$$

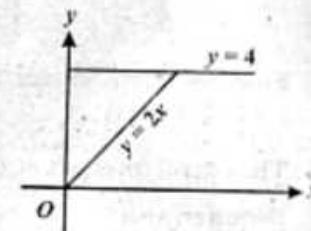


1(b). the y-axis, the straight line  $y = 2x$ , and the straight line  $y = 4$

**Sol.**

(b). Required area

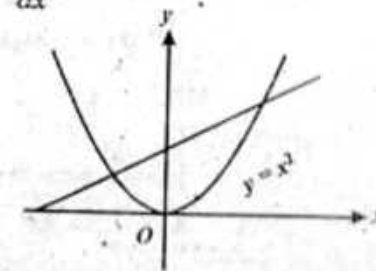
$$\begin{aligned} A &= \int_0^2 \int_{2x}^4 dy dx = \int_0^2 [y]_{2x}^4 dx \\ &= \int_0^2 (4-2x) dx \\ &= [4x - x^2]_0^2 = 8 - 4 = 4. \end{aligned}$$



2. Find the area bounded by the parabola  $y = x^2$  and the straight line  $y = 2x + 3$

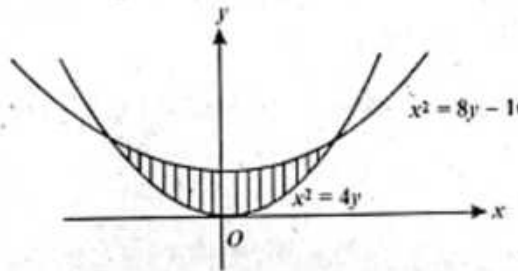
**Sol.** Solving  $y = x^2$  and  $y = 2x + 3$  simultaneously, we get the limits of integration for  $x$  as  $-1, 3$ . The required area is

$$\begin{aligned} &= \int_{-1}^3 \int_{x^2}^{2x+3} dy dx = \int_{-1}^3 [y]_{x^2}^{2x+3} dx \\ &= \int_{-1}^3 (2x+3-x^2) dx \\ &= \left[ x^2 + 3x - \frac{x^3}{3} \right]_{-1}^3 \\ &= 9 + 9 - 9 - \left( 1 - 3 + \frac{1}{3} \right) = \frac{32}{3}. \end{aligned}$$



3. Find the area bounded by  $x^2 = 4y$  and  $8y = x^2 + 16$

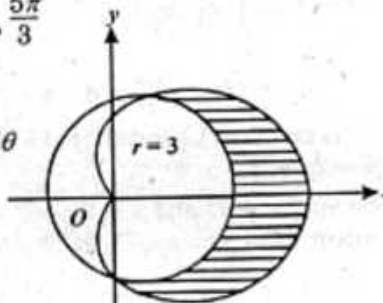
Sol. Solving the equation  $x^2 = 4y$  and  $8y = x^2 + 16$ , we get  $x = \pm 4$ . The required area is

$$\begin{aligned}
 &= \int_{-4}^4 \int_{\frac{x^2}{4}}^{\frac{x^2}{8} + 2} dy dx \\
 &= \int_{-4}^4 \left[ y \right]_{\frac{x^2}{4}}^{\frac{x^2}{8} + 2} dx \\
 &= \int_{-4}^4 \left[ \frac{x^2}{8} + 2 - \frac{x^2}{4} \right] dx = \int_{-4}^4 \left[ 2 - \frac{x^2}{8} \right] dx \\
 &= \left[ 2x - \frac{1}{8} \frac{x^3}{3} \right]_{-4}^4 = 8 - \frac{8}{3} - \left( -8 + \frac{8}{3} \right) = 16 - \frac{16}{3} = \frac{32}{3}.
 \end{aligned}$$


4. Find the area outside the circle  $r = 3$  and inside the cardioid  $r = 2(1 + \cos \theta)$ .

Sol. The curves intersect at  $\theta = \frac{\pi}{3}, \frac{5\pi}{3}$

Required area

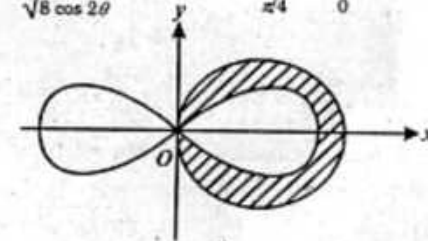
$$\begin{aligned}
 &= 2 \int_0^{\pi/3} \int_3^{2(1+\cos \theta)} r dr d\theta \\
 &= 2 \int_0^{\pi/3} \left[ \frac{r^2}{2} \right]_3^{2(1+\cos \theta)} d\theta \\
 &= 2 \int_0^{\pi/3} \left[ \frac{4(1+2\cos \theta + \cos^2 \theta)}{2} - \frac{9}{2} \right] d\theta \\
 &= \int_0^{\pi/3} (-5 + 8\cos \theta + 4\cos^2 \theta) d\theta
 \end{aligned}$$


$$\begin{aligned}
 &= \int_0^{\pi/3} (-5 + 8\cos \theta + 2(1 + \cos 2\theta)) d\theta \\
 &= [-3\theta + 8\sin \theta + \sin 2\theta]_0^{\pi/3} \\
 &= -3 \cdot \frac{\pi}{3} + 8 \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2} - \pi.
 \end{aligned}$$

5. Find the area inside the circle  $r = 4 \sin \theta$  and outside the lemniscate  $r^2 = 8 \cos 2\theta$ .

Sol. Since the two curves intersect at  $\theta = \frac{\pi}{6}, \frac{5\pi}{6}$  and  $r^2 = 8 \cos 2\theta = 0$  at  $\theta = \frac{\pi}{4}$ , the desired area

$$= 2 \int_{\pi/6}^{\pi/4} \int_{\sqrt{8 \cos 2\theta}}^{4 \sin \theta} r dr d\theta + 2 \int_{\pi/4}^{\pi/2} \int_0^{4 \sin \theta} r dr d\theta$$



$$\begin{aligned}
 &= 2 \int_{\pi/6}^{\pi/4} \left[ \frac{r^2}{2} \right]_{\sqrt{8 \cos 2\theta}}^{4 \sin \theta} d\theta + 2 \int_{\pi/4}^{\pi/2} \left[ \frac{r^2}{2} \right]_0^{4 \sin \theta} d\theta \\
 &= 2 \int_{\pi/6}^{\pi/4} (8 \sin^2 \theta - 4 \cos 2\theta) d\theta + 2 \int_{\pi/4}^{\pi/2} 8 \sin^2 \theta d\theta \\
 &= 8 \int_{\pi/6}^{\pi/4} (1 - 2 \cos 2\theta) d\theta + 8 \int_{\pi/4}^{\pi/2} (1 - \cos 2\theta) d\theta \\
 &= 8 \left[ \theta - \sin 2\theta \right]_{\pi/6}^{\pi/4} + 8 \left[ \theta - \frac{\sin 2\theta}{2} \right]_{\pi/4}^{\pi/2} \\
 &= 8 \left[ \frac{\pi}{4} - 1 - \frac{\pi}{6} + \frac{\sqrt{3}}{2} \right] + 8 \left[ \frac{\pi}{2} - \frac{\pi}{4} + \frac{1}{2} \right] \\
 &= 8 \left[ \frac{3\pi - 12 - 2\pi + 6\sqrt{3}}{12} \right] + 8 \left[ \frac{\pi}{4} + \frac{1}{2} \right]
 \end{aligned}$$

$$= \frac{2\pi}{3} - 4\sqrt{3} - 8 + 2\pi + 4 = \frac{8\pi}{3} + 4\sqrt{3} - 4.$$

6. Find the volume in the first octant between the planes  $z = 0$ ,  $z = x + y + 2$  and inside the cylinder  $x^2 + y^2 = 16$ .

$$\begin{aligned} \text{Sol. } V &= \int_0^4 \int_0^{\sqrt{16-x^2}} (x+y+2) dy dx \\ &= \int_0^4 \left[ xy + \frac{y^2}{2} + 2y \right]_0^{\sqrt{16-x^2}} dx \\ &= \int_0^4 \left[ x\sqrt{16-x^2} + \frac{16-x^2}{2} + 2\sqrt{16-x^2} \right] dx \\ &= -\frac{1}{3} \left[ (16-x^2)^{3/2} \right]_0^4 + \left[ 8x - \frac{x^3}{6} \right]_0^4 + 2 \left[ \frac{x\sqrt{16-x^2}}{2} + 8 \arcsin \frac{x}{4} \right]_0^4 \\ &= \frac{64}{3} + \left( 32 - \frac{64}{6} \right) + 2 \left[ 8 \left( \frac{\pi}{2} \right) \right] = \frac{64}{3} + \frac{64}{3} + 8\pi = \frac{128}{3} + 8\pi. \end{aligned}$$

7. Find the volume bounded by the cylinder  $x^2 + y^2 = 4$  and the planes  $y + z = 4$  and  $z = 0$ .

$$\begin{aligned} \text{Sol. } V &= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y) dy dx = 2 \int_{-2}^2 \int_0^{\sqrt{4-x^2}} 4 dy dx \\ &= 8 \int_{-2}^2 \sqrt{4-x^2} dx = 16 \int_0^2 \sqrt{4-x^2} dx \\ &= 16 \left[ x \frac{\sqrt{4-x^2}}{2} + 2 \arcsin \frac{x}{2} \right]_0^2 = 16 \left( 2 \cdot \frac{\pi}{2} \right) = 16\pi. \end{aligned}$$

8. Find the volume of the solid in the first octant bounded by the coordinate planes and the plane  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ,  $a$ ,  $b$  and  $c$  being positive.

$$\text{Sol. From } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1, \text{ we have } z = c \left( 1 - \frac{x}{a} - \frac{y}{b} \right).$$

Volume of the solid

$$\begin{aligned} &= \int_0^a \int_0^{b(1-\frac{x}{a})} c \left( 1 - \frac{x}{a} - \frac{y}{b} \right) dy dx \\ &= c \int_0^a \left[ y - \frac{xy}{a} - \frac{y^2}{2b} \right]_0^{b(1-\frac{x}{a})} dx \\ &= c \int_0^a \left[ b \left( 1 - \frac{x}{a} \right) - \frac{x}{a} b \left( 1 - \frac{x}{a} \right) - \frac{1}{2b} \cdot b^2 \left( 1 - \frac{x}{a} \right)^2 \right] dx \\ &= c \int_0^a \left[ b - \frac{bx}{a} - \frac{bx}{a} + b \frac{x^2}{a^2} - \frac{b}{2} \left( 1 - \frac{2x}{a} + \frac{x^2}{a^2} \right) \right] dx \\ &= c \int_0^a \left( \frac{b}{2} - \frac{bx}{a} + \frac{bx^2}{2a^2} \right) dx = c \left[ \frac{bx}{2} - \frac{b}{a} \frac{x^2}{2} + \frac{b}{2} \frac{x^3}{3a^2} \right]_0^a \\ &= c \left[ \frac{ba}{2} - \frac{ba}{2} + \frac{ba}{6} \right] = \frac{abc}{6}. \end{aligned}$$

9. Find the volume of the solid bounded by the paraboloid  $z = 4 - x^2 - y^2$  and the  $xy$ -plane.

Sol. The region  $D$  in the  $xy$ -plane is bounded by  $x^2 + y^2 = 4$ .

$$\begin{aligned} \text{Volume} &= \iint_D (4 - x^2 - y^2) dx dy = \int_0^{2\pi} \int_0^2 (4 - r^2) r dr d\theta \\ &= \int_0^{2\pi} \left[ 2r^2 - \frac{r^4}{4} \right]_0^2 d\theta = 4 \int_0^{2\pi} d\theta = 8\pi. \end{aligned}$$

10. Find the volume of the solid bounded by the graphs of  $x^2 + y^2 = 4$ ,  $z = \sqrt{16 - x^2 - y^2}$ ,  $z = 0$ .

Sol. Required volume

$$V = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{16-x^2-y^2} dy dx$$

We change the integral into polar coordinates. From  $x = r \cos \theta$ ,  $y = r \sin \theta$ , we have  $x^2 + y^2 = r^2$ , so that

$z = \sqrt{16 - r^2}$ . The bounds become  $0 \leq r \leq 2$  and  $0 \leq \theta \leq 2\pi$

$$\begin{aligned} V &= \int_0^{2\pi} \int_0^2 \sqrt{16 - r^2} r dr d\theta = \frac{1}{3} \int_0^{2\pi} [(16 - r^2)^{3/2}]_0^2 d\theta \\ &= -\frac{1}{3} \int_0^{2\pi} [(12)^{3/2} - (16)^{3/2}] d\theta = -\frac{1}{3} \int_0^{2\pi} (24\sqrt{3} - 64) d\theta \\ &= -\frac{1}{3} (24\sqrt{3} - 64) \theta \Big|_0^{2\pi} = -\frac{1}{3} (24\sqrt{3} - 64) 2\pi \\ &= \frac{-16\pi}{3} (3\sqrt{3} - 8) = \frac{16\pi}{3} (8 - 3\sqrt{3}). \end{aligned}$$

11. Find the centre of gravity of the plane area of uniform density bounded by

$x^2 = 4y$  and  $8y = x^2 + 16$  in the first quadrant.

**Sol.** The parabolas intersect at the points where

$$\frac{x^2}{4} = \frac{x^2 + 16}{8}, \quad \text{or} \quad x = \pm 4$$

Points of intersection are  $(4, 4)$ ,  $(-4, 4)$ . Let  $(\bar{x}, \bar{y})$  be coordinates of the centre of gravity of the area lying in the first quadrant. Then

$$\bar{x} = \frac{\int \int x dA}{\int \int dA}; \quad \bar{y} = \frac{\int \int y dA}{\int \int dA}$$

$$\begin{aligned} \text{Now, } \int \int dA &= \int_0^4 \int_{x^2/4}^{(x^2+16)/8} dy dx = \int_0^4 \left( \frac{x^2+16}{8} - \frac{x^2}{4} \right) dx \\ &= \frac{1}{8} \int_0^4 (16 - x^2) dx = \frac{1}{8} \left[ 16x - \frac{x^3}{3} \right]_0^4 = \frac{16}{3}. \\ \int \int x dA &= \int_0^4 \int_{x^2/4}^{(x^2+16)/8} x dy dx = \frac{1}{8} \int_0^4 (16x - x^3) dx \\ &= \frac{1}{8} \left[ 8x^2 - \frac{x^4}{4} \right]_0^4 = 8 \end{aligned}$$

$$\text{Hence } \bar{x} = \frac{8}{16/3} = \frac{3}{2}$$

$$\begin{aligned} \int \int y dA &= \int_0^4 \int_{x^2/4}^{(x^2+16)/8} y dy dx \\ &= \frac{1}{2} \int_0^4 \left[ \left( \frac{x^2+16}{8} \right)^2 - \left( \frac{x^2}{4} \right)^2 \right] dx \\ &= \frac{1}{2} \cdot \frac{1}{64} \int_0^4 (256 + 32x^2 - 3x^4) dx \\ &= \frac{1}{128} \left[ 256x + \frac{32x^3}{3} - \frac{3x^5}{5} \right]_0^4 \\ &= \frac{1}{128} \cdot 256 \times 4 \left[ 1 + \frac{2}{3} - \frac{3}{5} \right] = \frac{8 \times 16}{15}. \\ \bar{y} &= \frac{128/15}{16/3} = \frac{8}{5} \end{aligned}$$

$$\text{Centre of gravity } (\bar{x}, \bar{y}) = \left( \frac{3}{2}, \frac{8}{5} \right).$$

12. Find the mass of semicircular wire whose density varies as the distance from the diameter joining the ends.

**Sol.**

Let  $O$  be the centre of the semi-circular wire of radius  $r$ . Let  $PQ$  be a small strip of the wire so that  $m \angle POQ = d\theta$ . Then

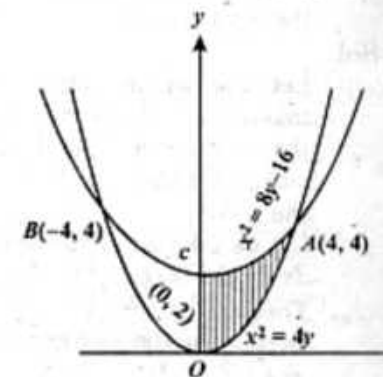
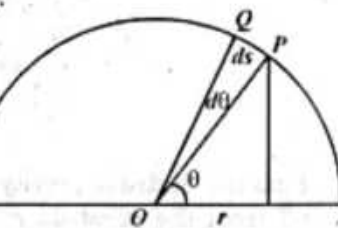
$$\frac{ds}{r} = d\theta, \quad ds = r d\theta$$

Height of  $PQ$  above the diameter  $= r \sin \theta$ .

Density of the strip  $PQ = k(r \sin \theta)$ .

Required mass

$$\begin{aligned} &= \int_0^\pi r d\theta k r \sin \theta \\ &= kr^2 \int_0^\pi \sin \theta d\theta \\ &= kr^2 [1 - \cos \theta]_0^\pi = 2kr^2. \end{aligned}$$



13. Find the mass of the square plate of side  $a$  if the density varies as the square of the distance from a vertex.

**Sol.**

Let the square plate be so taken that its one vertex is at the origin, one side along the  $x$ -axis and the other one along the  $y$ -axis.

Let us take an elemental area  $dx dy$  at  $P(x, y)$ .

Then  $OP^2 = x^2 + y^2$

Density of the elemental area  $= k(x^2 + y^2)$ .

Thus the required mass

$$\begin{aligned} &= \int_0^a \int_0^a k(x^2 + y^2) dx dy = \int_0^a \left[ \frac{x^3}{3} + xy^2 \right]_0^a dy \\ &= k \int_0^a \left( \frac{a^3}{3} + ay^2 \right) dy = k \left[ \frac{a^3}{3} y + \frac{ay^3}{3} \right]_0^a = k \left[ \frac{a^4}{3} + \frac{a^4}{3} \right] = k \frac{2a^4}{3}. \end{aligned}$$

14. Find the mass of a circular plate of radius  $a$  if the density varies as the square of the distance from a point on the circumference to the centre of the circle.

**Sol.** Let an equation of the circular plate be  $x^2 + y^2 = a^2$ .

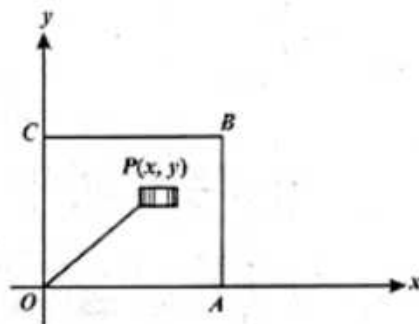
Density  $= k(x^2 + y^2)$ .

Mass of the plate  $= \iint k(x^2 + y^2) dA$

$$\begin{aligned} &= \int_0^{2\pi} \int_0^a kr^2 r dr d\theta = k \int_0^{2\pi} \left[ \frac{r^4}{4} \right]_0^a d\theta \\ &= \frac{ka^4}{4} \int_0^{2\pi} d\theta = \frac{ka^4}{4} \cdot 2\pi = \frac{k\pi a^4}{2}. \end{aligned}$$

15. Find the centre of gravity of a plate in the form of the segment cut off from the parabola  $y^2 = 8x$  by its latus rectum  $x = 2$ , if the density varies as the distance from the latus rectum.

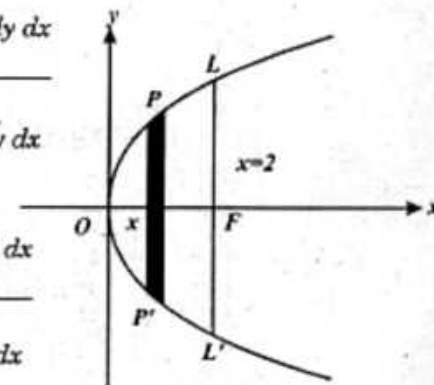
**Sol.** Cut off a small strip  $PP'$  of breadth  $dx$  at a distance of  $x$  from the vertex and a distance of  $2 - x$  from the latus rectum.



The density of the strip  $= k(2 - x)$ , where  $k$  is any constant.

Let  $(\bar{x}, \bar{y})$  be the centre of gravity of the centre plate. Then by symmetry  $\bar{y} = 0$

$$\begin{aligned} \bar{x} &= \frac{\int_0^2 \int_{-\sqrt{8x}}^{\sqrt{8x}} k(2-x)x dy dx}{\int_0^2 \int_{-\sqrt{8x}}^{\sqrt{8x}} k(2-x) dy dx} \\ &= \frac{\int_0^2 x(2-x) 2\sqrt{2} x^{1/2} dx}{\int_0^2 (2-x) 2\sqrt{2} x^{1/2} dx} \\ &= \frac{\int_0^2 (2x^{3/2} - x^{5/2}) dx}{\int_0^2 (2x^{1/2} - x^{3/2}) dx} = \frac{2 \cdot \frac{2}{5} [x^{5/2}]_0^2 - \frac{2}{7} [x^{7/2}]_0^2}{\frac{4}{3} \cdot 2\sqrt{2} - \frac{2}{5} \cdot 8\sqrt{2}} = \frac{\frac{16\sqrt{2}}{5} - \frac{16\sqrt{2}}{7}}{\frac{8\sqrt{2}}{3} - \frac{8\sqrt{2}}{5}} = \frac{\frac{2}{5} - \frac{2}{7}}{\frac{1}{3} - \frac{1}{5}} \\ &= \frac{4}{35} \times \frac{15}{2} = \frac{6}{7} \end{aligned}$$



Hence the centre of gravity is  $\left(\frac{6}{7}, 0\right)$ .

16. Find the centre of gravity of a plate in the form of the upper half of the cardioid  $r = a(1 + \cos \theta)$  if the density varies as the distance from the pole.

**Sol.** Here  $r = a(1 + \cos \theta)$ . Let us take any strip at a point  $P$  of the plate so that its area is  $r d\theta dr$ . Its density  $= kr$ .

If  $(\bar{x}, \bar{y})$  is the centre of gravity of the upper half then

$$\begin{aligned}
 \bar{x} &= \frac{\int_0^\pi \int_0^{a(1+\cos\theta)} (r \, d\theta \, dr) \, kr \cdot r \cos\theta}{\int_0^\pi \int_0^{a(1+\cos\theta)} (r \, d\theta \, dr) \, kr} \\
 &= \frac{\int_0^\pi \left[ \int_0^{a(1+\cos\theta)} r^3 \, dr \right] \cos\theta \, d\theta}{\int_0^\pi \left[ \int_0^{a(1+\cos\theta)} r^2 \, dr \right] d\theta} \\
 &= \frac{\int_0^\pi \frac{a^4(1+\cos\theta)^4}{4} \cos\theta \, d\theta}{\int_0^\pi \frac{a^3(1+\cos\theta)^3}{3} d\theta} \\
 &= \frac{3a}{4} \frac{\int_0^\pi (1+\cos\theta)^4 \cos\theta \, d\theta}{\int_0^\pi (1+\cos\theta)^3 d\theta} \\
 &= \frac{3a}{4} \frac{\int_0^\pi 16 \cos^3 \frac{\theta}{2} \left( 2 \cos^2 \frac{\theta}{2} - 1 \right) d\theta}{\int_0^\pi 8 \cos^6 \frac{\theta}{2} d\theta}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{3a}{2} \frac{\int_0^\pi \left( 2 \cos^{10} \frac{\theta}{2} - \cos^8 \frac{\theta}{2} \right) d\theta}{\int_0^\pi \cos^6 \frac{\theta}{2} d\theta} \quad \text{Put } t = \frac{\theta}{2} \text{ or } d\theta = 2 \, dt \\
 &= \frac{3a}{2} \frac{\int_0^{\pi/2} (2 \cos^{10} t - \cos^8 t) dt}{\int_0^{\pi/2} \cos^6 t \, dt} \\
 &= \frac{3a}{2} \frac{\left( 2 \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} - \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \right) \frac{\pi}{2}}{\left( \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \right) \frac{\pi}{2}} \\
 &= \frac{3a}{2} \frac{\frac{63}{128} - \frac{35}{128}}{\frac{5}{16}} = \frac{3a}{2} \left( \frac{28}{128} \times \frac{16}{5} \right) = \frac{21a}{20}
 \end{aligned}$$

$$\begin{aligned}
 \bar{y} &= \frac{\int_0^\pi \left[ \int_0^{a(1+\cos\theta)} r^3 \, dr \right] \sin\theta \, d\theta}{\int_0^\pi \left[ \int_0^{a(1+\cos\theta)} r^2 \, dr \right] d\theta} \\
 &= \frac{3a \int_0^\pi (1+\cos\theta)^4 \sin\theta \, d\theta}{4 \int_0^\pi (1+\cos\theta)^3 d\theta}
 \end{aligned}$$

Put  $1 + \cos\theta = t$  or  $-\sin\theta \, d\theta = dt$ .

$$\int_0^{\pi} (1 + \cos \theta)^4 \sin \theta d\theta = -\int_2^0 t^4 dt = \int_0^2 t^4 dt = \left| \frac{t^5}{5} \right|_0^2 = \frac{32}{5}$$

$$\begin{aligned} \int_0^{\pi} (1 + \cos \theta)^3 d\theta &= \int_0^{\pi} (1 + 3 \cos \theta + 3 \cos^2 \theta + \cos^3 \theta) d\theta \\ &= \int_0^{\pi} 1 \cdot d\theta + 3 \int_0^{\pi} \cos^2 \theta d\theta \\ &\left[ \text{Applying } \int_0^{2a} f(x) dx = \int_0^a f(x) dx + \int_0^a f(2a-x) dx \right] \\ &= \pi + 3.2 \int_0^{\pi/2} \cos^2 \theta d\theta = \pi + 6 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi + \frac{3\pi}{2} = \frac{5\pi}{2} \end{aligned}$$

$$\text{Hence } \bar{y} = \frac{3a}{4} \cdot \frac{5}{5\pi} = \frac{3a}{4} \cdot \frac{64}{25\pi} = \frac{48a}{25\pi}$$

$$\text{Centre of gravity } (\bar{x}, \bar{y}) = \left( \frac{21a}{20}, \frac{48a}{25\pi} \right)$$

17. Find  $I_x, I_y, I_0$  for the area enclosed by the loop of  $y^2 = x^2(2-x)$ .

Sol. Let  $A$  be the area of the loop. Then

$$\begin{aligned} A &= 2 \int_0^2 y dx = 2 \int_0^2 x \sqrt{2-x} dx \\ &= 2 \int_0^{\pi/2} (2 \sin^2 \theta) \sqrt{2} \cos \theta \cdot 4 \sin \theta \cos \theta d\theta \end{aligned}$$

(Putting  $x = 2 \sin^2 \theta$  or  $dx = 4 \sin \theta \cos \theta d\theta$ )

$$\begin{aligned} A &= 16\sqrt{2} \int_0^{\pi/2} \sin^3 \theta \cos^2 \theta d\theta \\ &= 16\sqrt{2} \frac{2 \cdot 1}{5 \cdot 3 \cdot 1} = \frac{32\sqrt{2}}{15} \end{aligned}$$

$$I_x = \int_0^2 \int_{-x\sqrt{2-x}}^{x\sqrt{2-x}} y^2 dy dx = 2 \int_0^2 \int_0^{x\sqrt{2-x}} y^2 dy dx \quad (1)$$

$$= \frac{2}{3} \int_0^2 [y^3]_0^{x\sqrt{2-x}} dx = \frac{2}{3} \int_0^2 x^3(2-x)^{3/2} dx$$

Put  $x = 2 \sin^2 \theta \Rightarrow dx = 4 \sin \theta \cos \theta d\theta$ . Then

$$\begin{aligned} I_x &= \frac{2}{3} \int_0^{\pi/2} (2 \sin^2 \theta)^3 2^{3/2} \cos^3 \theta \cdot 4 \sin \theta \cos \theta d\theta \\ &= \frac{128\sqrt{2}}{3} \int_0^{\pi/2} \sin^7 \theta \cos^4 \theta d\theta \\ &= \frac{128\sqrt{2}}{3} \frac{6 \cdot 4 \cdot 2 \cdot 3}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} \\ &= 60A \left( \frac{16}{11 \cdot 9 \cdot 7 \cdot 5} \right) \quad \text{from (1)} \\ &= \frac{64}{231} A \end{aligned}$$

$$\begin{aligned} I_y &= \int_0^2 \int_{-x\sqrt{2-x}}^{x\sqrt{2-x}} x^2 dy dx = 2 \int_0^2 \left[ \int_0^{x\sqrt{2-x}} dy \right] x^2 dx \\ &= 2 \int_0^2 x \sqrt{2-x} x^2 dx = 2 \int_0^2 x^3 \sqrt{2-x} x^2 dx \\ &= 2 \int_0^{\pi/2} 8 \sin^6 \theta \sqrt{2} \cos \theta \cdot 4 \sin \theta \cos \theta d\theta \end{aligned}$$

(on putting  $x = 2 \sin^2 \theta$  or  $dx = 4 \sin \theta \cos \theta d\theta$ )

$$\begin{aligned} I_y &= 64\sqrt{2} \int_0^{\pi/2} \sin^7 \theta \cos^2 \theta d\theta \\ &= 64\sqrt{2} \frac{6 \cdot 4 \cdot 2 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} = (30A) \left( \frac{16}{9 \cdot 7 \cdot 5} \right) = \frac{32}{21} A \end{aligned}$$

$$I_0 = \int_0^2 \int_{-x\sqrt{2-x}}^{x\sqrt{2-x}} (x^2 + y^2) dy dx = 2 \int_0^2 \left[ \int_0^{x\sqrt{2-x}} (x^2 + y^2) dy \right] dx$$

$$\begin{aligned}
&= 2 \int_0^2 \left[ x^2 y + \frac{y^3}{3} \right]_{-x\sqrt{2-x}}^{x\sqrt{2-x}} dx \\
&= 2 \int_0^2 x^2 \cdot x \sqrt{2-x} dx + \frac{2}{3} \int_0^2 x^3 (2-x)^{3/2} dx \\
&= 2 \int_0^2 x^3 \sqrt{2-x} dx + \frac{2}{3} \int_0^2 x^3 (2-x)^{3/2} dx \\
&= 2 \int_0^2 x^3 \sqrt{2-x} \left( 1 + \frac{2-x}{3} \right) dx = 2 \int_0^2 x^3 \sqrt{2-x} \left( \frac{5-x}{3} \right) dx \\
&= \frac{10}{3} \int_0^2 x^3 \sqrt{2-x} dx - \frac{2}{3} \int_0^2 x^4 \sqrt{2-x} dx \\
&= \frac{10}{3} \int_0^{\pi/2} 8 \sin^6 \theta \cdot \sqrt{2} \cos \theta \cdot 4 \sin \theta \cos \theta d\theta \\
&\quad - \frac{2}{3} \int_0^{\pi/2} 16 \sin^8 \theta \cdot \sqrt{2} \cos \theta \cdot 4 \sin \theta \cos \theta d\theta \\
&\quad \left( \text{Putting } x = 2 \sin^2 \theta \right. \\
&\quad \left. \text{or } dx = 4 \sin \theta \cos \theta d\theta \right) \\
&= \frac{320\sqrt{2}}{3} \int_0^{\pi/2} \sin^7 \theta \cos^2 \theta d\theta - \frac{128\sqrt{2}}{3} \int_0^{\pi/2} \sin^9 \theta \cos^2 \theta d\theta \\
&= \frac{320\sqrt{2}}{3} \cdot \frac{6 \cdot 4 \cdot 2 \cdot 1}{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} - \frac{128\sqrt{2}}{3} \cdot \frac{8 \cdot 6 \cdot 4 \cdot 2 \cdot 1}{11 \cdot 9 \cdot 7 \cdot 5 \cdot 3 \cdot 1} \\
&= \frac{304}{3} \cdot \frac{16}{63} - \frac{604}{3} \cdot \frac{8 \times 16}{99 + 35} = \frac{1604}{63} - \frac{5124}{693} \\
&= \left( \frac{1760 - 512}{693} \right) A = \frac{1248}{693} A = \frac{416}{231} A.
\end{aligned}$$

18. Find  $I_x$  and  $I_y$  of the area of

- (a) the circle  $r = 2(\sin \theta + \cos \theta)$   
 (b) one loop of  $r^2 = \cos 2\theta$ .

Sol.

(a) The given equation can be written as

$$r^2 = 2(r \sin \theta + r \cos \theta)$$

$$\Rightarrow x^2 + y^2 = 2(y + x) \Rightarrow (x-1)^2 + (y-1)^2 = 2$$

$$\text{Total area of the circle} = \pi(\sqrt{2})^2 = 2\pi$$

$$\text{M.I. about } AA' = \frac{2\pi \cdot 2}{4} = \pi$$

$$\text{M.I. about } Ox = I_x$$

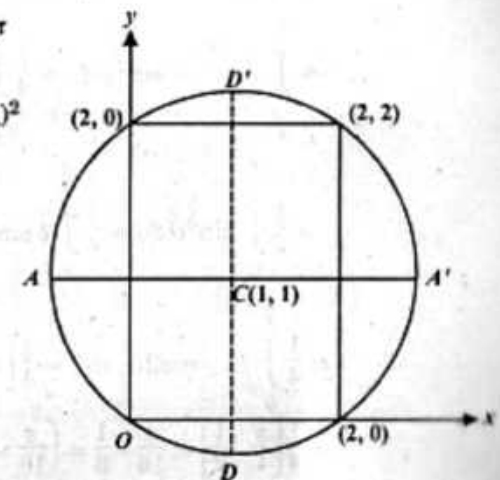
$$= \pi + 2\pi(1)^2$$

$$= \pi + 2\pi$$

$$= 3\pi$$

$$\text{Similarly, } I_y = 3\pi$$

$$\text{Hence } I_x = I_y = 3\pi$$

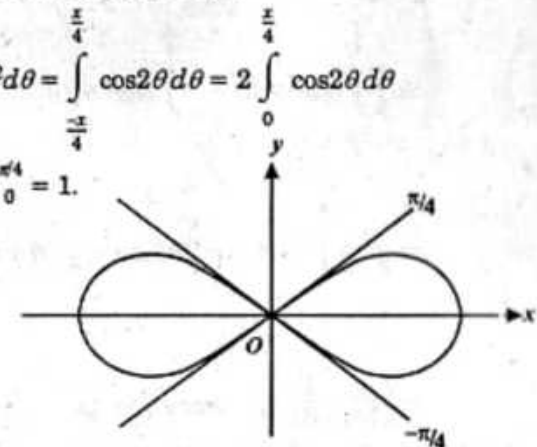


But  $A = 2\pi$ , where  $A$  is the area of the circle. Therefore,

$$I_x = I_y = \frac{3A}{2}.$$

(b) Let  $A$  be the area of the loop. Then

$$\begin{aligned}
A &= 2 \int_{-\pi/4}^{\pi/4} r^2 d\theta = \int_{-\pi/4}^{\pi/4} \cos 2\theta d\theta = 2 \int_0^{\pi/4} \cos 2\theta d\theta \\
&= [\sin 2\theta]_0^{\pi/4} = 1.
\end{aligned}$$



$$I_x = \int_{-\pi/4}^{\pi/4} \int_0^1 r^2 \sin^2 \theta r d\theta dr$$

$$= \int_{-\pi/4}^{\pi/4} \int_0^1 r^3 dr \sin^2 \theta d\theta = \frac{1}{4} \int_{-\pi/4}^{\pi/4} \sin^2 \theta d\theta = \frac{1}{2} \int_0^{\pi/4} \sin^2 \theta d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \sin^2 \theta d\theta = \frac{1}{4} \int_0^{\pi/4} 2 \sin^2 \theta d\theta$$

$$= \frac{1}{4} \int_0^{\pi/4} (1 - \cos 2\theta) d\theta = \frac{1}{4} \left[ \theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{4} \left[ \frac{\pi}{4} - \frac{1}{2} \right] = \frac{\pi}{16} - \frac{1}{8} = \left( \frac{\pi}{16} - \frac{1}{8} \right) A, \text{ as } A = 1$$

$$I_y = \int_{-\pi/4}^{\pi/4} \int_0^1 r^2 \cos^2 \theta r d\theta dr = \int_{-\pi/4}^{\pi/4} \int_0^1 r^3 dr \cos^2 \theta d\theta$$

$$= \frac{1}{4} \int_{-\pi/4}^{\pi/4} \cos^2 \theta d\theta = \frac{1}{2} \int_0^{\pi/4} \cos^2 \theta d\theta$$

$$= \frac{1}{2 \cdot 2} \int_0^{\pi/4} (1 + \cos 2\theta) d\theta = \frac{1}{4} \left[ \theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{4} \left[ \frac{\pi}{4} + \frac{1}{2} \right] = \frac{\pi}{16} + \frac{1}{8}$$

$$= \left( \frac{\pi}{16} + \frac{1}{8} \right) A, \text{ (since } A = 1).$$

19. Find the moment of inertia with respect to the x-axis of a plate having for its edges one arch of the curve  $y = \sin x$  and the x-axis if its density varies as the distance from the x-axis.

Sol. Here  $A = \int_0^{\pi/2} \int_0^{\sin x} ky dy dx = \frac{k}{2} \int_0^{\pi/2} \sin^2 x dx = \frac{k}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{k\pi}{8}$

or  $k = \frac{8A}{\pi}$

$$I_x = \int_0^{\pi/2} \int_0^{\sin x} y^2 \cdot ky dy dx = \frac{k}{4} \int_0^{\pi/2} \sin^4 x dx$$

$$= \frac{k}{4} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi k}{64} = \frac{3\pi}{64} \left( \frac{8A}{\pi} \right) = \frac{3}{8} A.$$

### Exercise Set 10.3 (Page 477)

1. Evaluate in SIX different ways

$$I = \int \int \int_S (x + 2y + 4z) dx dy dz, \text{ where } S \text{ is defined by}$$

$$1 \leq x \leq 2, -1 \leq y \leq 0, 0 \leq z \leq 3$$

Sol.  $I = \int_1^2 \int_{-1}^0 \int_0^3 (x + 2y + 4z) dz dy dx$

$$= \int_1^2 \int_{-1}^0 [(x + 2y)z + 2z^2]_0^3 dy dx$$

$$= \int_1^2 \int_{-1}^0 (18 + 3x + 6y) dy dx = \int_1^2 [18y + 3xy + 3y^2]_{-1}^0 dx$$

$$= \int_1^2 (15 + 3x) dx = \left[ 15x + \frac{3x^2}{2} \right]_1^2 = \frac{39}{2}.$$

The other five orders of integration are similar.

2. Evaluate  $I = \int_0^4 \int_0^{4-x} \int_0^{4-x-y} dz dy dx$ . Also change the order of integration so that z-integration is performed last and find its value.

$$\begin{aligned}\text{Sol. } I &= \int_0^4 \int_0^{4-x} (4-x-y) dy dx = \int_0^4 \left[ 4y - xy - \frac{y^2}{2} \right]_0^{4-x} dx \\ &= \int_0^4 \left[ 4(4-x) - x(4-x) - \frac{(4-x)^2}{2} \right] dx \\ &= \int_0^4 \left( 8 - 4x + \frac{x^2}{2} \right) dx = \left[ 8x - 2x^2 + \frac{x^3}{6} \right]_0^4 = \frac{32}{3}.\end{aligned}$$

If  $z$ -integration is performed last, then

$$\begin{aligned}I &= \int_0^4 \int_0^{4-z} \int_0^{4-y-z} dx dy dz \quad \text{and} \quad I = \int_0^4 \int_0^{4-z} \int_0^{4-x-z} dy dx dz \\ \int_0^4 \int_0^{4-z} \int_0^{4-y-z} dx dy dz &= \int_0^4 \int_0^{4-z} (4-y-z) dy dz \\ &= \int_0^4 \left[ 4y - \frac{y^2}{2} - yz \right]_0^{4-z} dz = \int_0^4 \left[ 4(4-z) - \frac{1}{2}(4-z)^2 - z(4-z) \right] dz \\ &= \int_0^4 \left( 8 - 4z + \frac{z^2}{2} \right) dz = 8z - 4\frac{z^2}{2} + \frac{z^3}{6} \Big|_0^4 = 32 - 32 + \frac{64}{6} = \frac{32}{3}\end{aligned}$$

Similarly,

$$\int_0^4 \int_0^{4-z} \int_0^{4-x-z} dy dx dz = \frac{32}{3}.$$

**Evaluate (Problems 3 - 10):**

$$3. \int_0^2 \int_0^1 \int_0^1 xyz \sqrt{2-x^2-y^2} dx dy dz$$

$$\begin{aligned}\text{Sol. } I &= \int_0^2 \int_0^1 yz \left[ -\frac{1}{3}(2-x^2-y^2)^{3/2} \right]_0^1 dy dz \\ &= \int_0^2 \int_0^1 yz \left[ -\frac{1}{3}(1-y^2)^{3/2} + \frac{1}{3}(2-y^2)^{3/2} \right] dy dz\end{aligned}$$

$$\begin{aligned}&= \int_0^2 z \left[ \frac{1}{15}(1-y^2)^{5/2} - \frac{1}{15}(2-y^2)^{5/2} \right]_0^1 dz \\ &= \int_0^2 z \left( \frac{4\sqrt{2}}{15} - \frac{2}{15} \right) dz = \left[ \left( \frac{4\sqrt{2}}{15} - \frac{2}{15} \right) \frac{z^2}{2} \right]_0^2 \\ &= 2 \left( \frac{4\sqrt{2}}{15} - \frac{2}{15} \right) = \frac{4}{15} (2\sqrt{2} - 1).\end{aligned}$$

$$4. \int_0^a \int_0^{\sqrt{a^2-y^2}} \int_0^{\sqrt{a^2-x^2-y^2}} x dz dx dy$$

$$\begin{aligned}\text{Sol. } I &= \int_0^a \int_0^{\sqrt{a^2-y^2}} x [z]_0^{\sqrt{a^2-x^2-y^2}} dx dy \\ &= \int_0^a \int_0^{\sqrt{a^2-y^2}} x \sqrt{a^2-x^2-y^2} dx dy \\ &= \int_0^a \left[ -\frac{1}{3}(a^2-x^2-y^2)^{3/2} \right]_0^{\sqrt{a^2-y^2}} dy \\ &= \int_0^a -\frac{1}{3} [0 - (a^2-y^2)^{3/2}] dy = \frac{1}{3} \int_0^a (a^2-y^2)^{3/2} dy\end{aligned}$$

Put  $y = a \sin \theta$  so that  $dy = a \cos \theta d\theta$  and

$$I = \frac{1}{3} \int_0^{\pi/2} a^4 \cos^4 \theta d\theta = \frac{a^4}{3} \left[ \frac{3}{4} \cdot \frac{\pi}{2} \right] = \frac{\pi a^4}{16}.$$

$$5. \int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{4-y^2-x^2} dx dy dz$$

$$\text{Sol. } I = \int_0^2 \int_0^{\sqrt{4-x^2}} [x]_{y^2+x^2-4}^{4-y^2-x^2} dy dx$$

$$= 2 \int_0^2 \int_0^{\sqrt{4-z^2}} (4-z^2-y^2) dy dz = 2 \int_0^2 \left[ (4-z^2)y - \frac{y^3}{3} \right]_0^{\sqrt{4-z^2}} dz$$

$$= \frac{4}{3} \int_0^2 (4-z^2)^{3/2} dz$$

Put  $z = 2 \sin \theta$  so that  $dz = 2 \cos \theta d\theta$  and

$$= I = \frac{4}{3} \int_0^{\pi/2} 8.2 \cos^4 \theta d\theta = \frac{64}{3} \cdot \frac{3}{4} \cdot \frac{\pi}{2} = 4\pi.$$

6.  $\int \int \int_S z dx dy dz$ ,  $S$  bounded by

$$z = \sqrt{x^2 + y^2}, z = 0, x = \pm 1, y = \pm 1.$$

Sol.  $I = \int_{-1}^1 \int_{-1}^1 \int_0^{\sqrt{x^2+y^2}} z dz dy dx = \int_{-1}^1 \int_{-1}^1 \left[ \frac{z^2}{2} \right]_0^{\sqrt{x^2+y^2}} dy dx$

$$= \frac{1}{2} \int_{-1}^1 \int_{-1}^1 (x^2 + y^2) dy dx = \frac{1}{2} \int_{-1}^1 \left[ x^2 y + \frac{y^3}{3} \right]_{-1}^1 dx$$

$$= \frac{1}{2} \int_{-1}^1 \left( 2x^2 + \frac{2}{3} \right) dx = \frac{1}{2} \left[ \frac{2x^3}{3} + \frac{2}{3}x \right]_{-1}^1 = \frac{1}{2} \cdot \frac{8}{3} = \frac{4}{3}.$$

7.  $\int \int \int_S 15x^2 z^2 dx dy dz$ ,  $S$  bounded by

$$x^2 + y^2 = 1, x^2 + z^2 = 1.$$

Sol.  $I = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 15x^2 z^2 dz dy dx$

$$= 15 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[ x^2 \frac{z^3}{3} \right]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy dx$$

$$= 15 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{3} x^2 (1-x^2)^{3/2} dy dx$$

$$= 10 \int_{-1}^1 [x^2 (1-x^2)^{3/2} y]_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 10 \int_{-1}^1 2x^2 (1-x^2)^{3/2} \sqrt{1-x^2} dx$$

$$= 20 \int_{-1}^1 x^2 (1-x^2)^2 dx = 20 \int_{-1}^1 (x^2 - 2x^4 + x^6) dx$$

$$= 20 \left[ \frac{x^3}{3} - \frac{2x^5}{5} + \frac{x^7}{7} \right]_{-1}^1$$

$$= 20 \left[ \frac{1}{3} - \frac{2}{5} + \frac{1}{7} - \left( -\frac{1}{3} + \frac{2}{5} - \frac{1}{7} \right) \right] = 20 \times 2 \left[ \frac{8}{3 \times 5 \times 7} \right] = \frac{64}{21}.$$

8.  $\int \int \int_S x^2 y^2 z dx dy dz$ ,  $S$  defined by

$$0 \leq z \leq x^2 - y^2, 0 \leq x \leq 1, 0 \leq y \leq 1$$

Sol.  $I = \int_0^1 \int_0^1 \int_0^{x^2-y^2} x^2 y^2 z dz dy dx$

$$= \int_0^1 \int_0^1 \left[ x^2 y^2 \frac{z^2}{2} \right]_0^{x^2-y^2} dy dx = \frac{1}{2} \int_0^1 \int_0^1 x^2 y^2 (x^4 - 2x^2 y^2 + y^4) dy dx$$

$$= \frac{1}{2} \int_0^1 \left[ x^6 \frac{y^3}{3} - 2x^4 \frac{y^5}{5} + x^2 \frac{y^7}{7} \right]_0^1 dx = \frac{1}{2} \int_0^1 \left( \frac{x^6}{3} - \frac{2x^4}{5} + \frac{x^2}{7} \right) dx$$

$$= \frac{1}{2} \left[ \frac{x^7}{21} - \frac{2x^5}{25} + \frac{x^3}{21} \right]_0^1 = \frac{1}{2} \left[ \frac{1}{21} - \frac{2}{25} + \frac{1}{21} \right] = \frac{4}{525}.$$

9.  $\int \int \int_S (x+1) dx dy dz$ ,  $S$  defined by

$$y = 0, y = x \text{ for } 0 \leq x \leq 1 \text{ and } -y^2 \leq z \leq x^2.$$

$$\begin{aligned}
 \text{Sol. } I &= \int_0^1 \int_0^x \int_{-y^2}^{x^2} (x+1) dz dy dx = \int_0^1 \int_0^x [(x+1)z]_{-y^2}^{x^2} dy dx \\
 &= \int_0^1 \int_0^x (x+1)(x^2+y^2) dy dx = \int_0^1 \left[ (x+1)x^2y + (x+1)\frac{y^3}{3} \right]_0^x dx \\
 &= \int_0^1 \left[ (x+1)x^3 + (x+1)\frac{x^3}{3} \right] dx = \frac{4}{3} \int_0^1 (x^4 + x^3) dx \\
 &= \frac{4}{3} \left[ \frac{x^5}{5} + \frac{x^4}{4} \right]_0^1 = \frac{4}{3} \left[ \frac{1}{5} + \frac{1}{4} \right] = \frac{3}{5}.
 \end{aligned}$$

10.  $\iiint_S yz dx dy dz$  :  $S$  in the first octant bounded above by  $z = 1$  and below by  $z = \sqrt{x^2 + y^2}$

**Sol.** Since  $S$  is in the first octant, the region  $D$  in the  $xy$ -plane is also in the first quadrant. The surfaces  $z = 1$  and  $z = \sqrt{x^2 + y^2}$  intersect in the curve  $1 = \sqrt{x^2 + y^2}$  in the  $xy$ -plane.

$$\begin{aligned}
 I &= \int_0^1 \int_0^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^1 yz dz dx dy \\
 &= \int_0^1 \int_0^{\sqrt{1-y^2}} \left[ y \frac{z^2}{2} \right]_{\sqrt{x^2+y^2}}^1 dx dy \\
 &= \int_0^1 \int_0^{\sqrt{1-y^2}} \frac{y}{2} (1 - x^2 - y^2) dx dy \\
 &= \int_0^1 \frac{y}{2} \left[ x - \frac{x^3}{3} - y^2x \right]_0^{\sqrt{1-y^2}} dy \\
 &= \int_0^1 \frac{y}{2} \left[ \sqrt{1-y^2} (1-y^2) - \frac{(1-y^2)^{3/2}}{3} \right] dy
 \end{aligned}$$

$$= \frac{1}{3} \int_0^1 y (1-y^2)^{3/2} dy = \frac{1}{3} \left[ \frac{(1-y^2)^{5/2}}{-5} \right]_0^1 = \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{15}.$$

**Find the volume of the given solid (Problems 11 – 13):**

11. Bounded by the coordinate planes and  $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1$ .

**Sol.** Required volume

$$\begin{aligned}
 &= \int_0^a \int_0^{b(1-\sqrt{x/a})^2} \int_0^{c(1-\sqrt{x/a}-\sqrt{y/b})^2} dz dy dx \\
 &= \int_0^a \int_0^{b(1-\sqrt{x/a})^2} c \left( 1 - \sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} \right)^2 dy dx \\
 &= \int_0^a \int_0^{b(1-\sqrt{x/a})^2} c \left[ \left( 1 - \sqrt{\frac{x}{a}} \right)^2 - 2 \left( 1 - \sqrt{\frac{x}{a}} \right) \sqrt{\frac{y}{b}} + \frac{y}{b} \right] dy dx \\
 &= \int_0^a c \left[ \left( 1 - \sqrt{\frac{x}{a}} \right)^2 y - 2 \left( 1 - \sqrt{\frac{x}{a}} \right) \frac{y^{3/2}}{3} + \frac{y^2}{2b} \right]_0^{b(1-\sqrt{x/a})^2} dx \\
 &= c \int_0^a \left[ b \left( 1 - \sqrt{\frac{x}{a}} \right)^4 - \frac{4b^{3/2}}{3\sqrt{b}} \left( 1 - \sqrt{\frac{x}{a}} \right)^4 + \frac{b^2}{2b} \left( 1 - \sqrt{\frac{x}{a}} \right)^4 \right] dx \\
 &= c \int_0^a \left( 1 - \sqrt{\frac{x}{a}} \right)^4 \left( b - \frac{4}{3}b + \frac{b}{2} \right) dx \\
 &= \frac{1}{6} bc \int_0^a \left[ 1 - 4\sqrt{\frac{x}{a}} + 6\frac{x}{a} - 4\left(\frac{x}{a}\right)^{3/2} + \left(\frac{x}{a}\right)^2 \right] dx \\
 &= \frac{1}{6} bc \left[ x - \frac{4}{\sqrt{a}} \frac{x^{3/2}}{3/2} + 6\frac{x^2}{2a} - \frac{4}{a^{3/2}} \frac{x^{5/2}}{5/2} + \frac{x^3}{3a^2} \right]_0^a \\
 &= \frac{1}{6} bc \left[ a - \frac{8}{3}a + 3a - \frac{8}{5}a + \frac{1}{3}a \right] = \frac{1}{6} abc \left[ 4 - \frac{8}{3} - \frac{8}{5} + \frac{8}{5} + \frac{1}{3} \right] = \frac{abc}{90}
 \end{aligned}$$

12. Bounded above by  $z = 4 - x^2 - y^2$  and below by  $z = 4 - 2x$ .

**Sol.** The region  $D$  in the  $xy$ -plane is the curve of intersection of the surfaces  $z = 4 - x^2 - y^2$  and  $z = 4 - 2x$

$$\text{i.e., } 4 - 2x = 4 - x^2 - y^2 \text{ or } x^2 + y^2 - 2x = 0.$$

$$\text{Thus } -\sqrt{2x - x^2} \leq y \leq \sqrt{2x - x^2}; 0 \leq x \leq 2.$$

Required volume

$$\begin{aligned} V &= \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} \int_{4-2x}^{4-x^2-y^2} dz \, dy \, dx \\ &= \int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (4 - x^2 - y^2 - 4 + 2x) \, dy \, dx \\ &= \int_0^2 \left[ (2x - x^2)y - \frac{y^3}{3} \right]_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} dx \\ &= \int_0^2 \left[ 2(2x - x^2)^{3/2} - \frac{2}{3}(2x - x^2)^{3/2} \right] dx = \frac{4}{3} \int_0^2 (2x - x^2)^{3/2} dx \end{aligned}$$

Put  $x - 1 = X$  so that  $dx = dX$

$$V = \frac{4}{3} \int_{-1}^1 (1 - X^2)^{3/2} dX$$

Now put  $X = \sin \theta$ ;  $dX = \cos \theta d\theta$  so that

$$V = \frac{4}{3} \int_0^{\pi/2} \cos^4 \theta d\theta = \frac{8}{3} \int_0^{\pi/2} \cos^4 \theta d\theta = \frac{8}{3} \cdot \frac{3}{4} \cdot \frac{\pi}{2} = \frac{\pi}{2}.$$

### 13. Bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

**Sol.** Required volume

$$\begin{aligned} &= 8 \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \int_0^{c\sqrt{1-x^2/a^2-y^2/b^2}} dz \, dy \, dx \\ &= 8c \int_0^a \int_0^{b\sqrt{1-x^2/a^2}} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} \, dy \, dx \end{aligned}$$

$$\begin{aligned} &= 8c \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \left[ \frac{b^2}{a^2} \left( \frac{a^2-x^2}{b^2} \right) - \frac{y^2}{b^2} \right]^{1/2} dy \, dx \\ &= 8c \int_0^a \int_0^{b/a\sqrt{a^2-x^2}} \frac{1}{b} \sqrt{\frac{b^2}{a^2} (a^2-x^2) - y^2} \, dy \, dx \\ &= \frac{8c}{b} \int_0^a \left[ y \sqrt{\frac{b^2}{a^2} (a^2-x^2) - y^2} + \right. \\ &\quad \left. \frac{b^2(a^2-x^2)}{2a^2} \arcsin \left( \frac{y}{\frac{b}{a}\sqrt{a^2-x^2}} \right) \right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx \\ &= \frac{8c}{b} \int_0^a \left( 0 + \frac{b^2(a^2-x^2)}{2a^2} \cdot \frac{\pi}{2} \right) dx = \frac{2\pi bc}{a^2} \left[ a^2x - \frac{x^3}{3} \right]_0^a = \frac{4\pi abc}{3}. \end{aligned}$$

$$14. \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^1 z(x^2+y^2) \, dz \, dy \, dx$$

by changing to cylindrical coordinates.

**Sol.** The solid is bounded below by  $z = x^2 + y^2$  and above  $z = 1$ . The region  $D$  in the  $xy$ -plane is  $x^2 + y^2 = 1$ . Changing into cylindrical coordinates, we have

$$\begin{aligned} I &= \int_0^{2\pi} \int_0^1 \int_{r^2}^1 (zr^2) r \, dz \, dr \, d\theta = \int_0^{2\pi} \int_0^1 \left[ r^3 \cdot \frac{z^2}{2} \right]_{r^2}^1 dr \, d\theta \\ &= \int_0^{2\pi} \int_0^1 r^3 \left( \frac{1}{2} - \frac{r^2}{2} \right) dr \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[ \frac{r^4}{4} - \frac{r^6}{6} \right]_0^1 d\theta = \frac{1}{24} \int_0^{2\pi} d\theta = \frac{\pi}{12} \end{aligned}$$

### 15. Use cylindrical coordinates to evaluate

$$I = \int \int \int_S z \sqrt{x^2 + y^2} \, dV, \text{ where } S \text{ is the hemisphere}$$

$$x^2 + y^2 + z^2 \leq 4, z \geq 0.$$

$$\begin{aligned}
 \text{Sol. } I &= \int_0^2 \int_0^{2\pi} \int_0^{\sqrt{4-r^2}} (zr) r \, dz \, d\theta \, dr \\
 &= \int_0^2 \int_0^{2\pi} \left[ r^2 \cdot \frac{z^2}{2} \right]_0^{\sqrt{4-r^2}} d\theta \, dr = \int_0^2 \int_0^{2\pi} \left( 2r^2 - \frac{1}{2}r^4 \right) d\theta \, dr \\
 &= \int_0^2 \left[ \left( 2r^2 - \frac{1}{2}r^4 \right) \theta \right]_0^{2\pi} dr = \pi \int_0^2 (4r^2 - r^4) dr = \pi \left[ \frac{4}{3}r^3 - \frac{r^5}{5} \right]_0^2 \\
 &= \pi \left[ \frac{32}{3} - \frac{32}{5} \right] = \frac{64}{15}.
 \end{aligned}$$

16. Evaluate  $I = \int \int_S \sqrt{x^2 + y^2} \, dV$ , where  $S$  is bounded above by the plane  $y + z = 4$ , below by  $z = 0$  and on the sides by  $x^2 + y^2 = 16$ .

**Sol.** Changing into cylindrical coordinates, we note that

$$\begin{aligned}
 z &= 4 - y = 4 - r \cos \theta; \quad 0 \leq r \leq 4 \\
 I &= \int_0^{2\pi} \int_0^4 \int_0^{4-r \cos \theta} r \cdot r \, dz \, dr \, d\theta \\
 &= \int_0^{2\pi} \int_0^4 r^2 (4 - r \cos \theta) dr \, d\theta = \int_0^{2\pi} \left[ 4 \frac{r^3}{3} - \frac{r^4}{4} \cos \theta \right]_0^4 d\theta \\
 &= \int_0^{2\pi} \left[ \frac{256}{3} - 64 \cos \theta \right] d\theta = \left[ \frac{256}{3} \theta - 64 \sin \theta \right]_0^{2\pi} = \frac{512}{3} \pi.
 \end{aligned}$$

17. Use spherical coordinates to evaluate

$$I = \int \int \int_S z \sqrt{x^2 + y^2 + z^2} \, dx \, dy \, dz, \text{ where } S \text{ is defined by}$$

$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - x^2 - y^2}.$$

**Sol.**  $S$  is bounded below by the cone  $z^2 = x^2 + y^2$  and above by the hemisphere  $z^2 = 1 - x^2 - y^2$ . In spherical coordinates equation of the cone becomes

$$\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = \rho^2 \sin^2 \phi$$

or  $\tan^2 \phi = 1$  giving  $\phi = \frac{\pi}{4}$ .

The hemisphere has the equation  $\rho = 1$

$$\begin{aligned}
 I &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 (\rho \cos \phi) \rho (\rho^2 \sin \phi) d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^{\pi/4} \left[ \frac{\rho^5}{5} \right]_0^1 \cos \phi \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^{\pi/4} \cos \phi \sin \phi \, d\phi \, d\theta \\
 &= \frac{1}{5} \int_0^{2\pi} \left[ \frac{\sin^2 \phi}{2} \right]_0^{\pi/4} d\theta = \frac{1}{5} \cdot \frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{10}.
 \end{aligned}$$

18. Evaluate  $I = \int \int \int_S \frac{dx \, dy \, dz}{x^2 + y^2 + z^2}$ , where  $S$  is the region above  $z = 0$  bounded by the cone  $z = \sqrt{3x^2 + 3y^2}$  and the spheres  $x^2 + y^2 + z^2 = 9$  and  $x^2 + y^2 + z^2 = 25$ .

**Sol.** Equations of the spheres in spherical coordinates are  $\rho = 3$  and  $\rho = 5$ . Equation of the cone is  $z^2 = 3(x^2 + y^2)$  i.e.,  $\rho^2 \cos^2 \phi = 3(\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta)$

or  $\tan^2 \phi = \frac{1}{3}$ , i.e.,  $\phi = \frac{\pi}{6}$

$$\begin{aligned}
 I &= \int_0^{2\pi} \int_0^{\pi/6} \int_3^5 \frac{\rho^2 \sin \phi \, d\rho \, d\phi \, d\theta}{\rho^2} = 2 \int_0^{2\pi} \int_0^{\pi/6} \sin \phi \, d\phi \, d\theta \\
 &= 2 \int_0^{2\pi} [-\cos \phi]_0^{\pi/6} d\theta = 2 \int_0^{2\pi} \left( 1 - \frac{\sqrt{3}}{2} \right) d\theta = 2\pi(2 - \sqrt{3}).
 \end{aligned}$$

19. Evaluate  $I = \int \int \int_S \sqrt{z} \, dx \, dy \, dz$ , where  $S$  is in first octant bounded by  $x^2 + y^2 + z^2 = 16$  and the planes  $z = 0$ ,  $x = \sqrt{3}y$ ,  $x = y$ .

**Sol.** Changing into spherical coordinates, we have

$$I = \int_0^{\pi/2} \int_0^{\pi/4} \int_0^4 \sqrt{\rho \cos \phi} (\rho^2 \sin \phi) d\rho \, d\theta \, d\phi$$

$$\begin{aligned}
&= \frac{2}{7} \int_0^{\pi/2} \int_{\pi/6}^{\pi/4} [\rho^{7/2}]_0^4 \sqrt{\cos \phi} \sin \phi d\theta d\phi \\
&= \frac{2}{7} (4)^{7/2} \int_0^{\pi/2} \left( \frac{\pi}{4} - \frac{\pi}{6} \right) \sqrt{\cos \phi} \sin \phi d\phi \\
&= \frac{2}{7} \cdot 128 \cdot \frac{\pi}{12} \left[ -\frac{(\cos \phi)^{3/2}}{3/2} \right]_0^{\pi/2} = \frac{64\pi}{21} \times \frac{2}{3} = \frac{128\pi}{63}
\end{aligned}$$

**Alternative method:** Changing into cylindrical coordinates, we get

$$\begin{aligned}
I &= \int_{\pi/6}^{\pi/4} \int_0^4 \int_0^{\sqrt{16-r^2}} \sqrt{z} r dz dr d\theta \\
&= \int_{\pi/6}^{\pi/4} \int_0^4 \frac{2}{3} (\sqrt{16-r^2})^{3/2} r dr d\theta = \int_{\pi/6}^{\pi/4} \left[ \frac{1}{3} \left( -\frac{(16-r^2)^{7/4}}{7/4} \right) \right]_0^4 d\theta \\
&= \frac{1}{3} \times \frac{4}{7} \cdot 2^7 \int_{\pi/6}^{\pi/4} d\theta = \frac{4 \times 128}{21} \times \frac{\pi}{12} = \frac{128\pi}{63}
\end{aligned}$$

20. Find the volume bounded by the torus  $\rho = 3 \sin \phi$

**Sol.** Required volume

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^\pi \int_0^{3 \sin \phi} \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\pi \left[ \frac{\rho^3}{3} \right]_0^{3 \sin \phi} \sin \phi d\phi d\theta \\
&= 9 \int_0^{2\pi} \int_0^\pi \sin^4 \phi d\phi d\theta = 9 \int_0^{2\pi} \int_0^\pi \left( \frac{1 - \cos 2\phi}{2} \right)^2 d\phi d\theta \\
&= 9 \int_0^{2\pi} \int_0^\pi \left[ \frac{3 - 4 \cos 2\phi + \cos 4\phi}{8} \right] d\phi d\theta \\
&= \frac{9}{8} \int_0^{2\pi} \left[ 3\phi - \frac{4 \sin 2\phi}{2} + \frac{\sin 4\phi}{4} \right]_0^\pi d\theta = \frac{9}{8} \int_0^{2\pi} 3\pi d\theta = \frac{27}{4} \pi
\end{aligned}$$

## Exercise Set 10.4 (Page 480)

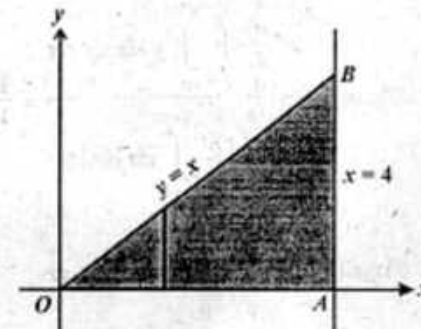
1. Find the centroid of each of the following volumes:

(a) Under  $z^2 = xy$  and above the triangle  $y = x, y = 0, x = 4$ .

**Sol.** The base is the triangle  $OAB$ . This can be swept if we vary  $x$  from 0 to 4 and  $y$  from 0 to  $x$  (i.e., from  $x$ -axis to the line  $y = x$ ).

The variation of  $z$  from the triangle  $OAB$  to the surface  $z^2 = xy$  implies  $z$  varies from 0 to  $\sqrt{xy}$ .

Let  $G(\bar{x}, \bar{y}, \bar{z})$  be the required centroid.



$$\begin{aligned}
&\int_0^4 \int_0^x \int_0^{\sqrt{xy}} dz dy dx \\
&= \int_0^4 \int_0^x [z]_0^{\sqrt{xy}} dy dx \\
&= \int_0^4 \int_0^x \sqrt{x} \sqrt{y} dy dx
\end{aligned}$$

$$= \int_0^4 \sqrt{x} \left[ \frac{y^{3/2}}{3/2} \right]_0^x dx = \frac{2}{3} \int_0^4 x^2 dx = \frac{2}{3} \left[ \frac{x^3}{3} \right]_0^4 = \frac{128}{9}$$

$$\begin{aligned}
\text{Now, } \int_0^4 \int_0^x \int_0^{\sqrt{xy}} x dz dy dx &= \int_0^4 \int_0^x x [z]_0^{\sqrt{xy}} dy dx = \int_0^4 \int_0^x x^{3/2} y^{1/2} dy dx \\
&= \int_0^4 x^{3/2} \left[ \frac{y^{3/2}}{3/2} \right]_0^x dx = \frac{2}{3} \int_0^4 x^3 dx = \frac{2}{3} \left[ \frac{x^4}{4} \right]_0^4 = \frac{128}{3}
\end{aligned}$$

$$\begin{aligned}
\text{Hence } \bar{x} &= \frac{\int_0^4 \int_0^x \int_0^{\sqrt{xy}} x dx dy dx}{\int_0^4 \int_0^x \int_0^{\sqrt{xy}} dz dy dx} = \frac{128/3}{128/9} = 3
\end{aligned}$$

$$\begin{aligned}
 \text{Again, } \int_0^4 \int_0^x \int_0^{\sqrt{xy}} y \, dz \, dy \, dx &= \int_0^4 \int_0^x y [z]_0^{\sqrt{xy}} \, dy \, dx = \int_0^4 \int_0^x \sqrt{x} y^{3/2} \, dy \, dx \\
 &= \int_0^4 \sqrt{x} \left[ \frac{y^{5/2}}{5/2} \right]_0^x \, dx = \frac{2}{3} \int_0^4 x^3 \, dx = \frac{2}{5} \left[ \frac{x^4}{4} \right]_0^4 = \frac{128}{5}
 \end{aligned}$$

$$\text{Thus } \bar{y} = \frac{\int_0^4 \int_0^x \int_0^{\sqrt{xy}} y \, dz \, dy \, dx}{\int_0^4 \int_0^x \int_0^{\sqrt{xy}} dz \, dy \, dx} = \frac{128/5}{128/9} = \frac{9}{5}$$

$$\begin{aligned}
 \text{Finally, } \int_0^4 \int_0^x \int_0^{\sqrt{xy}} z \, dz \, dy \, dx &= \int_0^4 \int_0^x \left[ \frac{z^2}{2} \right]_0^{\sqrt{xy}} \, dy \, dx = \frac{1}{2} \int_0^4 \int_0^x xy \, dy \, dx \\
 &= \frac{1}{2} \int_0^4 x \left[ \frac{y^2}{2} \right]_0^x \, dx = \frac{1}{2} \int_0^4 x^3 \, dx = \frac{1}{4} \left[ \frac{x^4}{4} \right]_0^4 = 16
 \end{aligned}$$

$$\text{So, } \bar{z} = \frac{\int_0^4 \int_0^x \int_0^{\sqrt{xy}} z \, dz \, dy \, dx}{\int_0^4 \int_0^x \int_0^{\sqrt{xy}} dz \, dy \, dx} = \frac{16}{128/9} = \frac{144}{128} = \frac{9}{8}$$

$$\text{Thus } G(\bar{x}, \bar{y}, \bar{z}) = G\left(3, \frac{9}{5}, \frac{9}{8}\right)$$

1. (b) within the cylinder  $r = 2 \cos \theta$ , bounded above by the paraboloid  $z = r^2$  and below by the plane  $z = 0$ .

Sol. Equation of the cylinder is given in the cylindrical polar coordinates as  $r = 2 \cos \theta$ . (1)

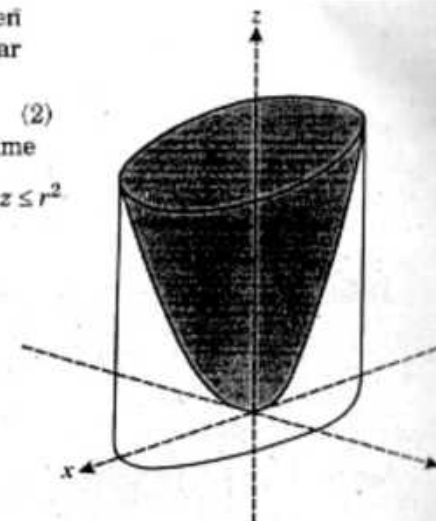
Equation of the given paraboloid in cylindrical polar coordinates is

$$z = r^2 \quad (2)$$

Clearly, for the required volume

$$0 \leq r \leq 2 \cos \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq z \leq r^2$$

Let  $G(\bar{x}, \bar{y}, \bar{z})$  be the centroid of the volume bounded by the cylinder (1), under the paraboloid (2) and above the plane  $z = 0$  (i.e., the  $xy$ -plane)



$$\begin{aligned}
 \text{Now, } \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \int_0^{r^2} r \, dz \, dr \, d\theta &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r [z]_0^{r^2} \, dr \, d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^3 \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{r^4}{4} \right]_0^{2 \cos \theta} \, d\theta \\
 &= 4 \int_{-\pi/2}^{\pi/2} \cos^4 \theta \, d\theta = 4 \cdot 2 \int_0^{\pi/2} \cos^4 \theta \, d\theta = 8 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} \int_0^{r^2} r \cos \theta \, r \, dz \, dr \, d\theta &= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^2 \cos \theta [z]_0^{r^2} \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r^4 \cos \theta \, dr \, d\theta \\
 &= \int_{-\pi/2}^{\pi/2} \left[ \frac{r^5}{5} \right]_0^{2 \cos \theta} \cos \theta \, d\theta
 \end{aligned}$$

$$= \frac{32}{5} \int_{-\pi/2}^{\pi/2} \cos^6 \theta d\theta = \frac{32}{5} \cdot 2 \int_0^{\pi/2} \cos^6 \theta d\theta$$

$$= \frac{64}{5} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{3\pi}{2}$$

$$\text{Thus, } \bar{x} = \frac{\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r r \cos\theta r dz dr d\theta}{\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r r dz dr d\theta} = \frac{2\pi}{3\pi/2} = \frac{4}{3}$$

$$\text{Again, } \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r r \sin\theta r dz dr d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \sin\theta \left| \frac{z^2}{2} \right|_0^r dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^4 \sin\theta dr d\theta = \int_{-\pi/2}^{\pi/2} \left| \frac{r^5}{5} \right|_0^{2\cos\theta} \sin\theta d\theta$$

$$= \frac{32}{5} \int_{-\pi/2}^{\pi/2} \cos^5 \theta \sin\theta d\theta = 0, \text{ as } \cos^5 \theta \sin\theta \text{ is odd function of } \theta$$

$$\text{Hence } \bar{y} = \frac{\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r r \sin\theta r dz dr d\theta}{\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r r dz dr d\theta} = 0$$

$$\text{Finally, } \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r z r dz dr d\theta = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r \left| \frac{z^2}{2} \right|_0^r dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \frac{r^5}{2} dr d\theta = \frac{1}{2} \int_{-\pi/2}^{\pi/2} \left| \frac{r^6}{6} \right|_0^{2\cos\theta} d\theta = \frac{32}{3} \int_0^{\pi/2} \cos^6 \theta d\theta$$

$$= \frac{32}{5} \cdot 2 \int_0^{\pi/2} \cos^6 \theta d\theta = \frac{32}{3} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} = \frac{\pi}{2} = \frac{5\pi}{3}$$

$$\text{Thus, } \bar{z} = \frac{\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r z r dz dr d\theta}{\int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^r r dz dr d\theta} = \frac{5\pi/3}{3\pi/2} = \frac{10}{9}$$

$$\text{Thus } G(\bar{x}, \bar{y}, \bar{z}) = G\left(\frac{4}{3}, 0, \frac{10}{9}\right)$$

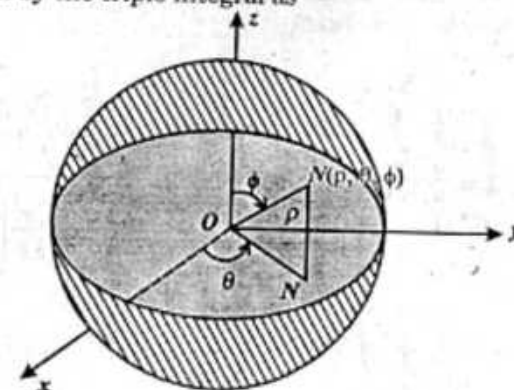
2. Find the mass of a sphere of radius  $r$  if the density varies inversely as the square of the distance from the centre.

**Sol.** Equation of a sphere with centre at the origin and radius  $a$  in spherical polar coordinates is  $\rho = a$

The volume element  $\rho^2 \sin \phi d\phi d\theta d\rho$  at  $P(\rho, \theta, \phi)$  has density  $\frac{k}{\rho^2}$ ,

as the distance  $|OP|^2 = \rho^2 = x^2 + y^2 + z^2$ .

Clearly,  $0 \leq \rho \leq a$ ,  $0 \leq \theta \leq 2\pi$  and  $0 \leq \phi \leq \pi$ . The mass  $M$  of the sphere is given by the triple integral as



$$M = \int_0^a \int_0^{2\pi} \int_0^\pi \frac{k}{\rho^2} \rho^2 \sin \phi d\phi d\theta d\rho$$

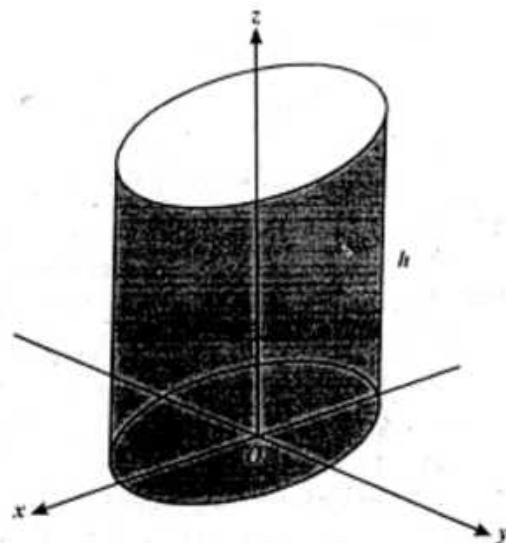
$$= \int_0^a \int_0^{2\pi} \left| -k \cos \phi \right|_0^\pi d\theta d\rho = \int_0^a \int_0^{2\pi} 2k d\theta d\rho$$

$$= \int_0^a 2k |\theta|_0^{2\pi} d\rho = \int_0^a 4k\pi d\rho = 4k\pi \rho|_0^a = 4k\pi a.$$

3. Find the centre of gravity of a right circular cylinder of radius  $r$  and height  $h$  if the density varies as the distance from the base.

**Sol.** Equation of a cylinder in cylindrical polar coordinates is  $r = a$  and axis of the cylinder is the  $z$ -axis. Now  $x$  varies from  $-a$  to  $a$  and  $\theta$  varies from  $0$  to  $2\pi$ . The variation of  $z$  is from  $0$  to  $h$ .

By symmetry the c. g. lies on the  $z$ -axis and let it be  $G(0, 0, \bar{z})$ . The density of the mass element  $r dz d\theta dr$  at  $P(r, \theta, z)$  of height  $z$  from the base i.e., the  $xy$ -plane) is  $\lambda z$ , where  $\lambda$  is constant.



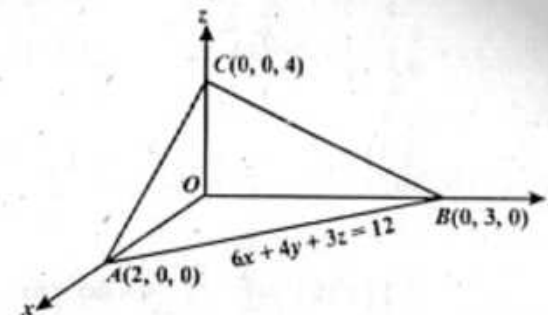
$$\begin{aligned} \bar{z} &= \frac{\int_0^a \int_0^{2\pi} \int_0^h \lambda z^2 r dz d\theta dr}{\int_0^a \int_0^{2\pi} \int_0^h \lambda z r dz d\theta dr} = \frac{\int_0^a \int_0^{2\pi} r \left| \frac{z^3}{3} \right|_0^h d\theta dr}{\int_0^a \int_0^{2\pi} r \left| \frac{z^2}{2} \right|_0^h d\theta dr} \\ &= \frac{\frac{h^3}{3} \int_0^a \int_0^{2\pi} r d\theta dr}{\frac{h^2}{2} \int_0^a \int_0^{2\pi} r d\theta dr} = \frac{2h}{3} \end{aligned}$$

Hence centre of gravity is  $(0, 0, \frac{2h}{3})$ .

4. Find the moments of inertia  $I_x, I_y, I_z$  of the following volumes:

(a) bounded by the coordinate planes and  $6x + 4y + 3z = 12$ .

**Sol.** The volume is as shown in the figure



Clearly,  $0 \leq x \leq 2, 0 \leq y \leq \frac{6-3x}{2}, 0 \leq z \leq \frac{12-6x-4y}{3}$

$V =$  volume of the tetrahedron  $OABC$

$$\begin{aligned} &= \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} dz dy dx = \int_0^2 \int_0^{\frac{6-3x}{2}} |z|_0^{\frac{12-6x-4y}{3}} dy dx \\ &= \int_0^2 \int_0^{\frac{6-3x}{2}} \frac{12-6x-4y}{3} dy dx \\ &= \frac{1}{3} \int_0^2 \left[ (12-6x) |y|_0^{\frac{6-3x}{2}} - 2 |y^2|_0^{\frac{6-3x}{2}} \right] dx \\ &= \frac{1}{3} \int_0^2 \left[ (6-3x)^2 - \frac{1}{2} (6-3x)^2 \right] dx \\ &= \frac{1}{6} \int_0^2 (6-3x)^2 dx = \frac{1}{6} \left| \frac{(6-3x)^3}{-9} \right|_0^2 = \frac{1}{6} \left[ 0 + \frac{63}{9} \right] = 4. \end{aligned}$$

$$\begin{aligned} &= \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} x^2 dz dy dx \\ &= \int_0^2 \int_0^{\frac{6-3x}{2}} x^2 |z|_0^{\frac{12-6x-4y}{3}} dy dx = \int_0^2 \int_0^{\frac{6-3x}{2}} x^2 \left( \frac{12-6x-4y}{3} \right) dy dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{3} \int_0^2 \int_0^{\frac{6-3x}{2}} [x^2(12-6x) - 4x^2y] dy dx \\
&= \frac{1}{3} \int_0^2 \left[ x^2(12-6x) \left| y \right|_0^{\frac{6-3x}{2}} - 2x^2 \left| y^2 \right|_0^{\frac{6-3x}{2}} \right] dx \\
&= \frac{1}{3} \int_0^2 \left[ x^2(6-3x)^2 - \frac{2x^2}{4} (6-3x)^2 \right] dx \\
&= \frac{1}{6} \int_0^2 x^2(6-3x)^2 dx = \frac{1}{6} \int_0^2 x^2[36 + 9x^2 - 36x] dx \\
&= \frac{1}{6} \int_0^2 (36x^2 + 9x^4 - 36x^3) dx = \frac{1}{6} \left[ 12x^3 + \frac{9}{5}x^5 - 9x^4 \right]_0^2 \\
&= \frac{1}{6} \left[ 96 + \frac{288}{5} - 144 \right] = \frac{1}{6} \left[ \frac{288}{5} - 48 \right] \\
&= \frac{1}{6} \left[ \frac{288 - 240}{5} \right] = \frac{48}{6 \times 5} = \frac{8}{5} = \frac{2}{5} V.
\end{aligned}$$

Again, 
$$\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} y^2 dz dy dx = \int_0^2 \int_0^{\frac{6-3x}{2}} y^2 \left| z \right|_0^{\frac{12-6x-4y}{3}} dy dx$$

$$\begin{aligned}
&= \int_0^2 \int_0^{\frac{6-3x}{2}} y^2 \left( \frac{12-6x-4y}{3} \right) dy dx \\
&= \int_0^2 \int_0^{\frac{6-3x}{2}} \left[ y^2 \frac{(12-6x)}{3} - \frac{4}{3} y^3 \right] dy dx \\
&= \int_0^2 \left[ \frac{2y^3}{9} (6-3x) - \frac{y^4}{3} \right]_0^{\frac{6-3x}{2}} dx
\end{aligned}$$

$$\begin{aligned}
&= \int_0^2 \left[ \frac{2}{9} \left( \frac{(6-3x)^3}{8} (6-3x) - \frac{((6-3x)^4)}{3} \right) \right] dx \\
&= \int_0^2 \left[ \frac{1}{36} (6-3x)^4 - \frac{(6-3x)^4}{48} \right] dx = \frac{1}{144} \int_0^2 (6-3x)^4 dx \\
&= \frac{-1}{432} \int_0^2 (6-3x)^4 (-3) dx = \frac{-1}{432} \left[ \frac{(6-3x)^5}{5} \right]_0^2 = \frac{-1}{432} \left[ -\frac{6^5}{5} \right] \\
&= \frac{6 \cdot 6 \cdot 6 \cdot 6 \cdot 6}{6 \cdot 6 \cdot 6 \cdot 2 \cdot 5} = \frac{18}{5} = \frac{9}{10} V
\end{aligned}$$

Finally, 
$$\int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} z^2 dz dy dx = \int_0^2 \int_0^{\frac{6-3x}{2}} \left[ \frac{z^3}{3} \right]_0^{\frac{12-6x-4y}{3}} dy dx$$

$$\begin{aligned}
&= \frac{1}{3^4} \int_0^2 \int_0^{\frac{6-3x}{2}} (12-6x-4y)^3 dy dx \\
&= \frac{1}{3^4} \int_0^2 \left[ \frac{(12-6x-4y)^4}{-16} \right]_0^{\frac{6-3x}{2}} dx \\
&= -\frac{1}{16 \cdot 3^4} \int_0^2 [0 - (12-6x)^4] dx \\
&= -\frac{1}{16 \cdot 3^4} \left[ \frac{(12-6x)^5}{-30} \right]_0^2 \\
&= -\frac{1}{16 \cdot 3^4} \left[ \frac{(12)^5}{30} \right] = \frac{32}{5} = \frac{8}{5} V
\end{aligned}$$

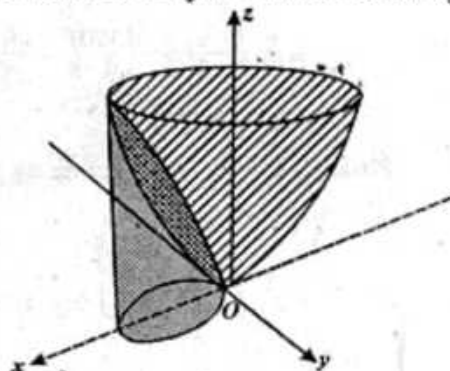
$$\begin{aligned}
I_x &= \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} (y^2 + z^2) dz dy dx \\
&= \frac{9}{10} V + \frac{8}{5} V = \frac{125}{10} V = \frac{5}{2} V.
\end{aligned}$$

$$I_y = \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} (x^2 + y^2) dz dy dx = \frac{2}{5}V + \frac{8}{5}V = 2V$$

$$I_z = \int_0^2 \int_0^{\frac{6-3x}{2}} \int_0^{\frac{12-6x-4y}{3}} (x^2 + y^2) dz dy dx \\ = \frac{2}{5}V + \frac{9}{10}V = \frac{13}{10}V.$$

4. (b) inside  $x^2 + y^2 = 4x$ , bounded above by  $z = 0$  and below by  $x^2 + y^2 = 4z$ .

Sol. Let  $V$  be the volume inside the cylinder  $x^2 + y^2 = 4x$  and below by the paraboloid  $x^2 + y^2 = 4z$  and above the plane  $z = 0$ . In cylindrical polar coordinates equation of the cylinder becomes  $r^2 = 4r \cos \theta$  and that of the paraboloid  $r^2 = 4z$ .



$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}, 0 \leq r \leq 4 \cos \theta, 0 \leq z \leq \frac{r^2}{4}$$

$$V = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \int_0^{r^2/4} r dz dr d\theta \\ = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} r \left| z \right|_0^{r^2/4} dr d\theta \\ = \frac{2}{4} \int_0^{\pi/2} \int_0^{4 \cos \theta} r^3 dr d\theta = \frac{1}{2} \int_0^{\pi/2} \left[ \frac{r^4}{4} \right]_0^{4 \cos \theta} d\theta \\ = \frac{4^4}{8} \int_0^{\pi/2} \cos^4 \theta d\theta = \frac{256}{8} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 6\pi.$$

$$2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \int_0^{r^2/4} x^2 r dz dr d\theta = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \int_0^{r^2/4} r^2 \cos^2 \theta r dz dr d\theta \\ = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} r^3 \cos^2 \theta \left| z \right|_0^{r^2/4} dr d\theta \\ = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \frac{r^5}{4} \cos^2 \theta dr d\theta \\ = \frac{1}{2} \int_0^{\pi/2} \left| \frac{r^6}{6} \right|_0^{4 \cos \theta} \cos^2 \theta d\theta \\ = \frac{4^6}{12} \int_0^{\pi/2} \cos^8 \theta d\theta \\ = \frac{4^6}{12} \cdot \frac{7}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{140V}{18}.$$

$$2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \int_0^{r^2/4} y^2 r dz dr d\theta = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \int_0^{r^2/4} r^2 \sin^2 \theta r dz dr d\theta \\ = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \int_0^{r^2/4} r^3 \sin^2 \theta \left| z \right|_0^{r^2/4} dr d\theta \\ = 2 \int_0^{\pi/2} \int_0^{4 \cos \theta} \frac{r^5}{4} \sin^2 \theta dr d\theta \\ = \frac{1}{2} \int_0^{\pi/2} \left| \frac{r^6}{6} \right|_0^{4 \cos \theta} \sin^2 \theta d\theta \\ = \frac{4^6}{12} \int_0^{\pi/2} \sin^6 \theta d\theta \\ = \frac{4^6}{12} \cdot \frac{1}{8} \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{20V}{18}.$$

$$\begin{aligned}
 2 \int_0^{2\pi} \int_0^{4 \cos \theta} \int_0^{r^2/4} z^2 r \, dz \, dr \, d\theta &= 2 \int_0^{2\pi} \int_0^{4 \cos \theta} \int_0^{r^2/4} \frac{r^5}{16} \, dz \, dr \, d\theta \\
 &= \frac{1}{8} \int_0^{2\pi} \int_0^{4 \cos \theta} r^5 \left| z \right|_0^{r^2/4} \, dr \, d\theta \\
 &= \frac{1}{32} \int_0^{2\pi} \int_0^{4 \cos \theta} r^7 \, dr \, d\theta \\
 &= \frac{1}{32} \int_0^{2\pi} \left| \frac{r^8}{8} \right|_0^{4 \cos \theta} \, d\theta \\
 &= \frac{4^8}{32} \times \frac{1}{8} \int_0^{2\pi} \sin^8 \theta \, d\theta \\
 &= \frac{4^8}{32 \times 8} \cdot \frac{7 \cdot 5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} \\
 &= \frac{105V}{32 \times 144} = \frac{35V}{18}.
 \end{aligned}$$

$$\begin{aligned}
 \text{Thus, } I_x &= 2 \int_0^{2\pi} \int_0^{4 \cos \theta} \int_0^{r^2/4} (y^2 + z^2) r \, dz \, dr \, d\theta \\
 &= \frac{20V}{18} + \frac{35V}{18} = \frac{55V}{18}.
 \end{aligned}$$

$$\begin{aligned}
 I_y &= 2 \int_0^{2\pi} \int_0^{4 \cos \theta} \int_0^{r^2/4} (x^2 + z^2) r \, dz \, dr \, d\theta \\
 &= \frac{140V}{18} + \frac{35V}{18} = \frac{175V}{18}.
 \end{aligned}$$

$$\begin{aligned}
 I_z &= 2 \int_0^{2\pi} \int_0^{4 \cos \theta} \int_0^{r^2/4} (x^2 + y^2) r \, dz \, dr \, d\theta \\
 &= \frac{140V}{18} + \frac{20V}{18} = \frac{80V}{9}.
 \end{aligned}$$

5. For the right circular cone of radius  $r$  and height  $h$ , find the moment of inertia with respect to:

Sol. (a) its axis

Let us take a right circular cone with vertex at  $O(0, 0, 0)$ , its axis along the  $z$ -axis as shown in figure.

Use spherical polar coordinates  $\rho, \theta, \phi$ .

$$x = \rho \sin \phi \cos \theta,$$

$$y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

Now the respective variations of coordinates for the points of the cone are

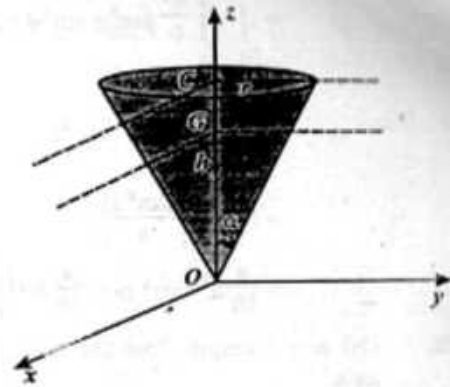
$$0 \leq \rho \leq h \sec \phi, \quad 0 \leq \phi \leq \alpha, \quad 0 \leq \theta \leq 2\pi$$

The volume  $V$  of the cone is given by the following triple integral

$$\begin{aligned}
 V &= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\alpha \left| \frac{\rho^3}{3} \right|_0^{h \sec \phi} \sin \phi \, d\phi \, d\theta = \int_0^{2\pi} \int_0^\alpha \frac{h^3}{3} \sec^3 \phi \sin \phi \, d\phi \, d\theta \\
 &= \frac{h^3}{3} \int_0^{2\pi} \int_0^\alpha \tan \phi \sec^2 \phi \, d\phi \, d\theta = \frac{h^3}{3} \int_0^{2\pi} \left| \frac{\tan^2 \phi}{2} \right|_0^\alpha \, d\theta \\
 &= \frac{h^3}{3} \tan^2 \alpha \Big|_0^{2\pi} = \frac{2\pi h^3}{6} \tan^2 \alpha = \frac{\pi h^3}{3} \cdot \frac{r^2}{h^2} = \frac{1}{3} \pi r^2 h.
 \end{aligned}$$

Now  $x^2 + y^2 = \rho^2 \sin^2 \phi$ . Hence

$$\begin{aligned}
 I_x &= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} (x^2 + y^2) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \rho^4 \sin^3 \phi \, d\rho \, d\phi \, d\theta \\
 &= \int_0^{2\pi} \int_0^\alpha \left| \frac{\rho^5}{5} \right|_0^{h \sec \phi} \sin^3 \phi \, d\phi \, d\theta
 \end{aligned}$$



$$\begin{aligned}
&= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \frac{h^5}{5} \sec^5 \phi \sin^3 \phi d\phi d\theta \\
&= \frac{h^5}{5} \int_0^{2\pi} d\theta \int_0^\alpha \tan^3 \phi \sec^2 \phi d\phi \\
&= \frac{2\pi h^5}{5} \left| \frac{\tan^4 \phi}{4} \right|_0^\alpha \\
&= \frac{\pi}{10} h^5 \tan^4 \alpha = \frac{\pi}{10} h^5 \left( \frac{r^4}{h^4} \right) = \frac{3}{10} r^2 \left( \frac{\pi r^2 h}{3} \right) = \frac{3r^2}{10} V.
\end{aligned}$$

5. (b) any straight line through its vertex and perpendicular to its axis.

**Sol.** A line perpendicular to the axis ( $z$ -axis) through the vertex can be taken as the  $x$ -axis

$$\begin{aligned}
\text{Now, } &\int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} y^2 \rho^2 \sin \phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \rho^4 \sin^3 \phi \sin^2 \theta d\rho d\phi d\theta \\
&= \int_0^{2\pi} \int_0^\alpha \left| \frac{\rho^5}{5} \right|_0^{h \sec \phi} \sin^3 \phi \sin^2 \theta d\phi d\theta \\
&= \frac{h^5}{5} \int_0^{2\pi} \int_0^\alpha \sec^5 \phi \sin^3 \phi \cos^2 \theta d\phi \\
&= \frac{4h^5}{5} \int_0^{2\pi} \cos^2 \theta d\theta \int_0^\alpha \sec^2 \phi \tan^3 \phi d\phi \\
&= \frac{4h^5}{5} \cdot \frac{1}{2} \cdot \frac{\pi}{2} \left| \frac{\tan^4 \phi}{4} \right|_0^\alpha \\
&= \frac{\pi h^5}{20} \tan^4 \alpha = \frac{\pi h^5}{20} \cdot \frac{r^4}{h^4} = \frac{\pi h r^4}{20}.
\end{aligned}$$

$$\text{Also, } \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} z^2 \rho^2 \sin \theta d\rho d\phi d\theta$$

$$\begin{aligned}
&= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \rho^4 \cos^2 \phi \sin \phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} d\theta \int_0^\alpha \left| \frac{\rho^5}{5} \right|_0^{h \sec \phi} \cos^2 \phi \sin \phi d\phi \\
&= \frac{2\pi h^5}{5} \int_0^\alpha \sec^5 \phi \cos^2 \phi \sin \phi d\phi \\
&= \frac{2\pi h^5}{5} \int_0^\alpha \tan \phi \sec^2 \phi d\phi = \frac{2\pi h^5}{5} \left| \frac{\tan^2 \phi}{2} \right|_0^\alpha \\
&= \frac{\pi h^5}{5} \cdot \tan^2 \alpha = \frac{\pi h^5}{5} \cdot \frac{r^2}{h^2} = \frac{\pi h^3 r^2}{5}.
\end{aligned}$$

$$\begin{aligned}
\text{Hence, } I_x &= \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} (y^2 + z^2) \rho^2 \sin \phi d\rho d\phi d\theta \\
&= \frac{\pi h r^4}{20} + \frac{\pi h^3 r^2}{5} = \frac{\pi h r^2}{3} \cdot \frac{3}{5} \left[ h^3 + \frac{r^2}{4} \right] = \frac{3}{5} \left( h^2 + \frac{r^2}{4} \right) V.
\end{aligned}$$

5. (c) any line through its centre of gravity and perpendicular to its axis.

**Sol.** The symmetry of the cone shows that its c.g. lies on the  $z$ -axis.

Let  $G(0, 0, \bar{z})$  be the c.g. of the cone.

$$\begin{aligned}
&\int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} z \rho^2 \sin \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \rho^3 \sin \phi \cos \phi d\rho d\phi d\theta \\
&= \int_0^{2\pi} d\theta \int_0^\alpha \left| \frac{\rho^4}{4} \right|_0^{h \sec \phi} \sin \phi \cos \phi d\phi \\
&= \frac{2\pi h^4}{4} \int_0^\alpha \sec^4 \phi \cos \phi d\phi \\
&= \frac{\pi h^4}{2} \int_0^\alpha \tan \phi \sin \phi \cos \phi d\phi
\end{aligned}$$

$$\begin{aligned}
 &= \frac{\pi h^4}{2} \left| \frac{\tan^2 \phi}{2} \right|_0^\alpha \\
 &= \frac{\pi h^2 r^2}{4} \tan^2 \alpha = \frac{\pi h^4}{4} \cdot \frac{r^2}{h^2} = \frac{\pi h^2 r^2}{4}
 \end{aligned}$$

$$\text{Thus, } \bar{z} = \frac{\int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} z \rho^2 \sin \phi d\rho d\phi d\theta}{\int_0^{2\pi} \int_0^\alpha \int_0^{h \sec \phi} \rho^2 \sin \phi d\rho d\phi d\theta} = \frac{\pi h^2 r^2 / 4}{\pi r^2 h / 3} = \frac{3h}{4}$$

Hence  $G\left(0, 0, \frac{3h}{4}\right)$  is the c.g. of the cone.

By the principle of parallel axes of moments of inertia, we have  
M.I. about the  $x$ -axis = M.I. about an axis through  $G$  and parallel to the  $x$ -axis + M.I. of mass  $V$  placed at  $G$  about the  $x$ -axis.

or  $\frac{3}{5}\left(h^2 + \frac{r^2}{4}\right)V = \text{M.I. about an axis through } G \text{ and parallel to the } x\text{-axis} + V\left(\frac{3h}{4}\right)^2$ , by 5(b)

Thus M.I. about an axis through  $G$  and parallel to the  $x$ -axis

$$\begin{aligned}
 &= \frac{3}{5}\left(h^2 + \frac{r^2}{4}\right)V - \frac{9h^2}{16}V \\
 &= \frac{3}{80}(16h^2 + 4r^2 - 15h^2)V = \frac{3}{80}(h^2 + 4r^2)V.
 \end{aligned}$$

5. (d) any diameter of its base.

Sol. By the principle of parallel axes of moments inertia, we have:

M.I. about a diameter (parallel to the  $x$ -axis) through  $C$   
= M.I. about an axis through  $G$  (parallel to the  $x$ -axis) + M.I. of mass  $V$  placed at  $G$  about an axis through  $C$  (parallel to the  $x$ -axis)

$$\begin{aligned}
 &= \frac{3}{80}(h^2 + 4r^2)V + \left(\frac{h}{4}\right)V \\
 &= \frac{V}{80}[3h^2 + 12r^2 + 5h^2] = \frac{V}{80}(8h^2 + 12r^2) \\
 &= \frac{1}{20}(2h^2 + 3r^2)V.
 \end{aligned}$$