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Solutions Manual For

CALCULUS WITH ANALYTIC GEOMETRY

By

A BOARD OF EXPERIENCED PROFESSORS

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Chapter 1 REAL NUMBERS, LIMITS AND CONTINUITY

Exercise Set 1.1 (Page 17)

1. If $a, b \in \boldsymbol{R}$ and $a + b = 0$, prove that $a = -b$

Sol. Since $b \in \boldsymbol{R}$, there is an element $-b \in \boldsymbol{R}$ such that

$$b + (-b) = 0 \qquad (1)$$

By hypothesis, $a + b = 0$ (2)

Adding $-b$ to both sides of (2) and applying the above property of additive inverse, we get

$$a + b + (-b) = -b \quad \text{or} \quad a + (b + (-b)) = -b$$

(Associative property of addition)

or $a + 0 = -b$ by (1)

i.e., $a = -b$ as required.

2. Prove that $(-a)(-b) = ab$ for all $a, b \in \boldsymbol{R}$.

Sol. We have, $ab + a(-b) + (-a)(-b) = ab + [a(-b) + (-a)(-b)]$,

(Associative property of addition)

or $a[b + (-b)] + (-a)(-b) = ab + [a + (-a)](-b)$

(Distributive property)

or $a.0 + (-a)(-b) = ab + 0 . (-b)$

i.e., $(-a)(-b) = ab$, since $a . 0 = 0 = 0 . (-b)$.

3. Prove that $\big||a| - |b|\big| \le |a - b|$ for every $a, b \in \boldsymbol{R}$.

Sol. By Theorem 1.5 (v), we have

$$|a + b| \le |a| + |b| \qquad (1)$$

Replacing b by $-b$, (1) becomes

$$|a - b| \le |a| + |-b| = |a| + |b|, \qquad (2)$$

since $|-b| = |b|$

Replace a by $b - a$ in (2) to get

$$|-a| \le |b - a| + |b|$$

or $|a| - |b| \le |b - a| = |a - b|$ (3)

Again, in $|b - a| \le |a| + |b|$, replace b by $a - b$ to have

$$|-b| \le |a| + |a - b| \text{ or } |b| - |a| \le |a - b|$$

Multiplying both sides of the inequality by -1, we get

$$|a| - |b| \ge -|a - b|$$

or $-|a - b| \le |a| - |b|$ (4)

Combining (3) and (4), we have $-|a - b| \le |a| - |b| \le |a - b|$

or $\big||a| - |b|\big| \le |a - b|$, by Theorem 1.5 (*iv*).

4. Express $3 < x < 7$ in modulus notation

Sol. We know that $|x - a| < l$ implies $a - l < x < a + l$

Now $3 < x < 7$

Therefore, by comparison,

$$a - l = 3 \qquad (1)$$

$$a + l = 7 \qquad (2)$$

Adding (1) and (2), we get $2a = 10$ or $a = 5$

Subtracting (1) from (2), we have $2l = 4$ or $l = 2$

Hence the given inequality can be expressed in the modulus notation as $|x - 5| < 2$

5. Let $\delta > 0$ and $a \in \mathbf{R}$. Show that $a - \delta < x < a + \delta$ if and only if $|x - a| < \delta$.

Sol. Suppose $a - \delta < x < a + \delta$. These inequalities can be written as

$$a - \delta < x \qquad (1)$$

and $$x < a + \delta \qquad (2)$$

From (1) and (2), we have respectively

$$-\delta < x - a \qquad (3)$$

and $$x - a < \delta \qquad (4)$$

Combining (3) and (4), we get

$$-\delta < x - a < \delta \text{ or } |x - a| < \delta \text{ by Theorem 1.5 (iv)}$$

Conversely, let $|x - a| < \delta$. By Theorem 1.5 (*iv*), we have

$$-\delta < x - a < \delta \text{ or } a - \delta < x < a + \delta \text{ as desired}$$

6. Give an example of a set of rational numbers which is bounded above but does not have a rational supremum.

Sol. Consider the set S of rational numbers defined by

$$S = \{x \in Q : x^2 < 2\}$$

The supremum of S is $\sqrt{2}$ which is not a rational number.

Solve each of the following inequalities (Problems 7 – 15)

7. $|2x + 5| > |2 - 5x|$

Sol. Associated equation is $|2x + 5| = |2 - 5x|$

This is equivalent to

$$2x + 5 = 2 - 5x \qquad (1)$$

or $$2x + 5 = -2 + 5x \qquad (2)$$

From (1), we get $x = -\frac{3}{7}$ and from (2), we have $x = \frac{7}{3}$

These are the boundary numbers for the given inequality. The number line is divided by the boundary numbers into regions as shown:

Region A, test $x = -1$:	$\lvert -2 + 5\rvert > \lvert 2 + 5\rvert$	False
Region B, test $x = 0$:	$\lvert 5\rvert > \lvert 2\rvert$	True
Region C, test $x = 3$:	$\lvert 6 + 5\rvert > \lvert 2 - 15\rvert$	False

Thus the solution set is

$$\left\{x : \frac{-3}{7} < x < \frac{7}{3}\right\} = \left]-\frac{3}{7}, \frac{7}{3}\right[$$

8. $$\left|\frac{x + 8}{12}\right| < \frac{x - 1}{10} \qquad (1)$$

Sol. (1) is equivalent to the compound inequality

$$-\frac{x - 1}{10} < \frac{x + 8}{12} < \frac{x - 1}{10} \text{ or } -6x + 6 < 5x + 40 < 6x - 6$$

This is equivalent to $-11x < 34$ and $46 < x$

i.e., $-\frac{34}{11} < x$ and $46 < x$

The solution set is

$$\left\{x : -\frac{34}{11} < x\right\} \cap \{x : 46 < x\} = \{x : 46 < x\} =]46, \infty[$$

Alternative Method:

Associated equation is $\frac{x + 8}{12} = \pm \frac{x - 1}{10}$

i.e., $5x + 40 = \pm 6(x - 1)$

i.e., $5x + 40 = 6x - 6$ and $5x + 40 = -6x + 6$

or $x = 46$ and $x = -\frac{34}{11}$

These boundary numbers divide the number line as shown:

Region A, test $x = -4$:	$\left\lvert\frac{-4 + 8}{12}\right\rvert < \frac{-4 - 1}{10}$	False
Region B, test $x = 45$:	$\left\lvert\frac{45 + 8}{12}\right\rvert < \frac{45 - 1}{10}$	False
Region C, test $x = 47$:	$\left\lvert\frac{47 - 8}{12}\right\rvert < \frac{47 - 1}{10}$	True

The solution set is $\{x : x > 46\} =]46, \infty[$.

9. $|x| + |x - 1| > 1$

Sol. The associated equation is

$$|x| + |x - 1| = 1 \text{ or } \pm x \pm (x - 1) = 1$$

This is equivalent to

$$x + x - 1 = 1 \qquad (1)$$

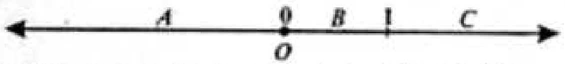
$x + x - 1 = -1$ (2)

$x - (x - 1) = 1$ (3)

$-x + x - 1 = 1$ (4)

From (1) and (2), we find $x = 1, 0$.

These boundary numbers divide the number line as shown:



Region A, test $x = -1$: $|-1| + |-1-1| > 1$ True

Region B, test $x = \frac{1}{2}$: $\left|-\frac{1}{2}\right| + \left|\frac{1}{2} - 1\right| > 1$ False

Region C, test $x = 2$: $|2| + |2-1| > 1$ True

The solution set is $\{x : x < 0\} \cup \{x : x > 1\} =]-\infty, 0[\cup]1, \infty[$.

10. $2x^2 - 25x + 12 > 0$

Sol. The associated equation is

$$12x^2 - 25x + 12 = 0$$

$$x = \frac{25 \pm \sqrt{625 - 576}}{24} = \frac{25 \pm 7}{24} = \frac{4}{3}, \frac{3}{4}$$

These boundary numbers divide the number line as shown.

Region A, test $x = 0$: $12 > 0$ True

Region B, test $x = 1$: $12 - 25 + 12 > 0$ False

Region C, test $x = 2$: $48 - 50 + 12 > 0$ True

The solution set is $\left\{x : x < \frac{3}{4}\right\} \cup \left\{x : x > \frac{4}{3}\right\}$

$$= \left]-\infty, \frac{3}{4}\right[\cup \left]\frac{4}{3}, \infty\right[$$

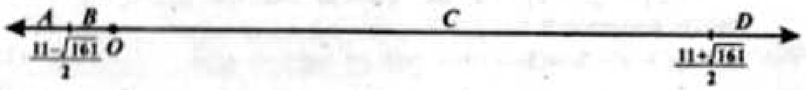
11. $\frac{x-1}{2} - \frac{1}{x} > \frac{4}{x} + 5$

Sol. The associated equation is $\frac{x-1}{2} - \frac{1}{x} = \frac{4}{x} + 5$

or $x^2 - x - 2 = 8 + 10x$ or $x^2 - 11x - 10 = 0$

$$x = \frac{11 \pm \sqrt{121 + 40}}{2} = \frac{11 \pm \sqrt{161}}{2} = \frac{11 + \sqrt{161}}{2}, \frac{11 - \sqrt{161}}{2}$$

The point $x = 0$ is a free boundary number. These boundary numbers divide the number line as shown:



Region A, test $x = -1$: $-1 + 1 > -4 + 5$ False

Region B, text $x = -5.0$: $-\frac{1.5}{2} - \frac{1}{-0.5} > \frac{4}{-0.5} + 5$

i.e., $-0.75 + 2 > -8 + 5$ True

Region C, test $x = 5$: $\frac{5-1}{2} - \frac{1}{5} > \frac{4}{5} + 5$ False

Region D, test $x = 13$: $\frac{13-1}{2} - \frac{1}{13} > \frac{4}{13} + 5$ True

The solution set is $\left]\frac{11 - \sqrt{161}}{2}, 0\right[\cup \left]\frac{11 + \sqrt{161}}{2}, \infty\right[$

12. $|x^2 - x + 1| > 1$ (1)

Sol. $|x^2 - x + 1| > 1$ is equivalent to

$x^2 - x + 1 > 1$ or $x^2 - x + 1 < 1$

For the inequality $x^2 - x + 1 > 1$, the associated equation

$x^2 - x + 1 = 1$

gives $x(x - 1) = 0$

or $x = 0, 1$ are the boundary numbers for $x^2 - x + 1 > 0$

The number line is divided by 0 and 1 as shown:

Region A, test $x = -1$: $1 + 1 + 1 > 1$ True

Region B, text $x = \frac{1}{2}$: $\frac{1}{4} - \frac{1}{2} + 1 > 1$ False

Region C, test $x = 2$: $4 - 2 + 1 > 1$ True

The solution set of $x^2 - x + 1 > 1$ is

$\{x : x < 0\} \cup \{x : x > 1\} =]-\infty, 0[\cup]1, \infty[$.

For $x^2 - x + 1 < -1$, one must have $x^2 - x + 2 < 0$

i.e., $\left(x - \frac{1}{2}\right)^2 + \frac{7}{4} < 0$ which is impossible for real x.

Thus the solution set of (1) is $]-\infty, 0[\cup]1, \infty[$.

13. $x^{-2} - 4x^{-1} + 4 > 0$

Sol. The given inequality is equivalent to

$$\frac{1}{x^2} - \frac{4}{x} + 4 > 0 \text{ or } \left(\frac{1 - 2x}{x}\right)^2 > 0$$

or $x = \frac{1}{2}$ is a boundary number. $x = 0$ is a free boundary number since x occurs in the denominator of the inequality. The boundary numbers divide the number line into regions as shown:

Region A, test $x = -1$: $\left(\dfrac{1+2}{-1}\right)^2 > 0$ True

Region B, test $x = \frac{1}{4}$: $\left(\dfrac{1-\frac{1}{2}}{\frac{1}{4}}\right)^2 > 0$ True

Region C, test $x = 1$: $\left(\dfrac{1-2}{1}\right)^2 > 0$ True

The solution set is $\{x : x < 0\} \cup \left\{x : 0 < x < \frac{1}{2}\right\} \cup \left\{x : x > \frac{1}{2}\right\}$

$$= \left]-\infty, 0\right[\;\cup\; \left]0, \tfrac{1}{2}\right[\;\cup\; \left]\tfrac{1}{2}, \infty\right[$$

14. $\dfrac{2x}{x+2} \geq \dfrac{x}{x-2}$ (1)

Sol. $x = -2, 2$ are free boundary numbers for (1). The associated equation

$\dfrac{2x}{x+2} = \dfrac{x}{x-2}$ is equivalent to

$2x^2 - 4x = x^2 + 2x$ or $x^2 - 6x = 0$

i.e., $x = 0, 6$ are boundary numbers for (1).

The boundary numbers divide the number line into region as shown:

Region A, test $x = -3$: $\dfrac{-6}{-3+2} \geq \dfrac{-3}{-3-2}$ True

Region B, test $x = -1$: $\dfrac{-2}{-1+2} \geq \dfrac{-1}{-1-2}$ False

Region C, test $x = 1$: $\dfrac{2}{1+2} \geq \dfrac{2}{1-2}$ True

Region D, test $x = 3$: $\dfrac{6}{3+2} \geq \dfrac{3}{3-2}$ False

Region E, test $x = 7$: $\dfrac{14}{7+2} \geq \dfrac{7}{7-2}$ True

Since equality sign occurs in the inequality, the boundary numbers 0, 6 are in the solution set.

The solution set is

$\{x : x < -2\} \cup \{x : 0 \leq x < 2\} \cup \{x : x \geq 6\}$

$= \left]-\infty, -2\right[\;\cup\; [0, 2[\;\cup\; [6, \infty[$

...lems 23 – 30):

15. $x^4 - 5x^3 - 4x^2 + 20x \leq 0$

Sol. The associated equation is

$x^4 - 5x^3 - 4x^2 + 20x = 0$ (1)

i.e., $x(x^3 - 5x^2 - 4x + 20) = 0$

or $x[x^2(x-5) - 4(x-5)] = 0$ or $x(x-2)(x+2)(x-5) = 0$

$x = 0, -2, 2, 5$ are the boundary numbers for (1).

Locate the boundary numbers on a number line and check eac region whether it belongs to the solution set or not.

Region A, test $x = -3$: $-3(-3-2)(-3+2)(-3-5) \leq 0$ False

Region B, test $x = -1$: $-1(-1-2)(-1+2)(-1-5) \leq 0$ True

Region C, test $x = 1$: $1(1-2)(1+2)(1-5) \leq 0$ False

Region D, test $x = 3$: $3(3-2)(3+2)(3-5) \leq 0$ True

Region E, test $x = 6$: $6(6-2)(6+2)(6-5) \leq 0$ False

The solution set consists of regions B and D. The bounda numbers are in these regions and since equality occurs in (1), th belong to the solution sets. The solution set is

$\{x : -2 \leq x \leq 0\} \cup \{x : 2 \leq x \leq 5\} = [-2, 0] \cup [2, 5]$

16. The cost function $C(x)$ and the revenue function $R(x)$ for produci x units of certain product are given by

$C(x) = 5x + 350$; $R(x) = 50 - x^2$

Find the values of x that yield a profit.

Sol. A profit is produced if revenue exceeds cost. Therefore for a profit

$R(x) > C(x)$

i.e., $50x - x^2 > 5x + 350$

or $x^2 - 45x + 350 < 0$ (1)

The associated equation of (1) is

$x^2 - 45x + 350 = 0$ or $(x-10)(x-35) = 0$

which gives $x = 10, 35$ as the boundary points. Locate these poi on a number line and check which regions belong to the solution.

Region A, test $x = 0$: $(-10)(-35) < 0$ False

Region B, test $x = 15$: $(15-10)(15-35) < 0$ True

Region C, test $x = 40$: $(40-10)(40-35) < 0$ False

Thus the solution set for a profit is $\{x : 10 < x < 35\}$

Function f from R to R is defined by the given formula. Determine the domain of the function (Problems 17 – 22)

$f(x) = \sqrt{1 - x^2}$

As soon as the numerical value of x exceeds 1, $f(x)$ becomes imaginary.

Hence the domain of definition of this function is $|x| \le 1$.

$f(x) = \dfrac{a + x}{a - x}$

Here $f(x)$ becomes infinite when $x = a$ and for every other real value of x, we get the corresponding real value of $f(x)$. Hence the domain of this function is the set of all real numbers except $x = a$.

$f(x) = \dfrac{1}{\sqrt{(1 - x)(2 - x)}}$

Here $f(x)$ becomes infinite when $x = 1$ or $x = 2$. Also when $x \in]1, 2[$, the value of $f(x)$ becomes imaginary *i.e.*, $f(x)$ is not defined for any value of x where $1 < x < 2$. Therefore, the domain of definition of this function is the set of all real numbers x except when $x \in [1, 2]$.

$f(x) = \sqrt{3 + x} + \sqrt{7 - x}$

Here when x exceeds 7, the value of $f(x)$ does not remain real. Similarly, when $x < -3$, $f(x)$ does not remain real. For every other real value of x, $f(x)$ is defined in the set of real numbers. Hence the required domain is the closed interval $[-3, 7]$.

$$f(x) \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ \sqrt{x - 1} & \text{if } x > 2 \end{cases}$$

The function is defined by two rules for all real numbers. Hence the domain of f is R.

$f(x) = \sqrt{\dfrac{x - 4}{x + 1}}$

The function is not defined when $x = -1$. For $-1 < x < 4$, the numerator is negative while the denominator is positive and so the value of the function is imaginary. Hence Dom $f = R - [-1, 4[$

Draw the graph of the following functions (Problems 23 – 30):

23. $f(x) = |x| + |x - 1|$ for all $x \in R$

Sol. This can be rewritten as

$$y = f(x) = \begin{cases} -x + 1 - x = 1 - 2x, & \text{when } x < 0 \\ x + 1 - x = 1, & \text{when } 0 \le x \le 1 \\ x + x - 1 = 2x - 1, & \text{when } x > 1 \end{cases}$$

So, the graph of the function will consist of three parts.

$y = 1 - 2x,$ when $x < 0$ (1)

$y = 1,$ when $0 \le x \le 1$ (2)

$y = 2x - 1,$ when $x > 1$ (3)

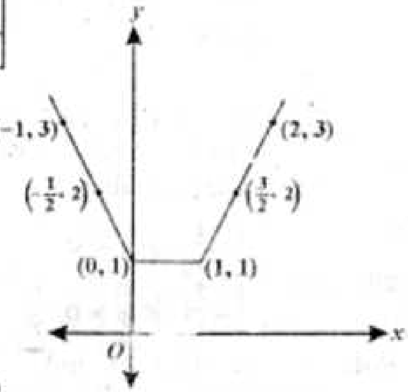
For (1), we have the following table of values

x	$-\frac{1}{2}$	-1	-2
y	2	3	5

For (2), its graph is a line segment in the interval $0 \le x \le 1$, parallel to the x-axis at a unit distance.

For (3), we have the following table of values to be plotted

x	$\frac{3}{2}$	2	3
y	2	3	5



Thus we have the graph as shown

24. $f(x) = [x] + [x + 1]$ for all $x \in R$

Sol. This can be rewritten as (if $y = f(x)$)

$y = 1$, if $0 \le x < 1$

$y = 3$, if $1 \le x < 2$

$y = 5$, if $2 \le x < 3$

$y = 7$, if $3 \le x < 4$

... ...

and $y = -1$, if $-1 \le x < 0$

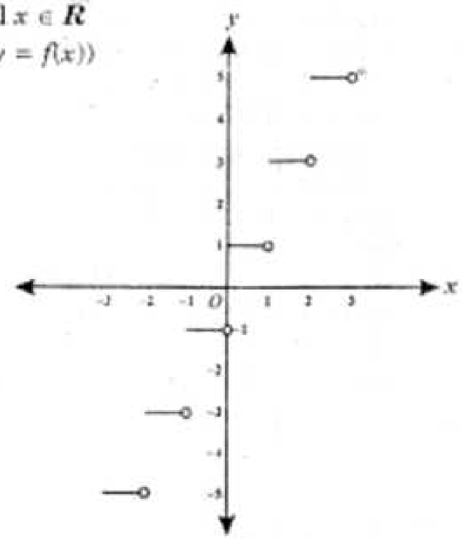
$y = -3$, if $-2 \le x < -1$

$y = -5$, if $-3 \le x < -2$

$y = -7$, if $-4 \le x < -3$

... ...

The graph is as shown



25. $f(x) = x - [x]$ for all $x \in [-3, 3]$ *(Saw-tooth function)*

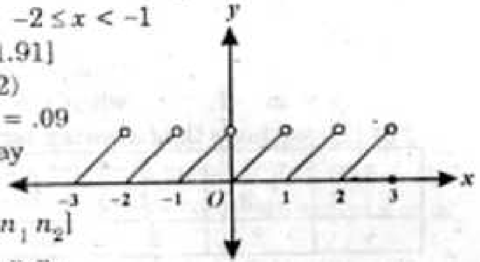
Sol. When x is an integer, (whether positive or negative), then $f(x) = 0$

If x is a negative fraction, say $x = -n \,.\, n_1 n_2$, where n, n_1, n_2 are positive integers, then

$$f(x) = -n \,.\, n_1 n_2 - [-n \,.\, n_1 n_2]$$
$$= -n \,.\, n_1 n_2 + n + 1 = 1 - .n_1 n_2$$

For -1.91 in the interval $-2 \le x < -1$

$$f(-1.91) = -1.91 - [-1.91]$$
$$= -1.91 - (-2)$$
$$= -1.91 + 2 = .09$$



If x is a positive fraction, say $x = n \,.\, n_1 n_2$, then

$$f(x) = n \,.\, n_1 n_2 - [n \,.\, n_1 n_2]$$
$$= n \,.\, n_1 n_2 - n = .n_1 n_2$$

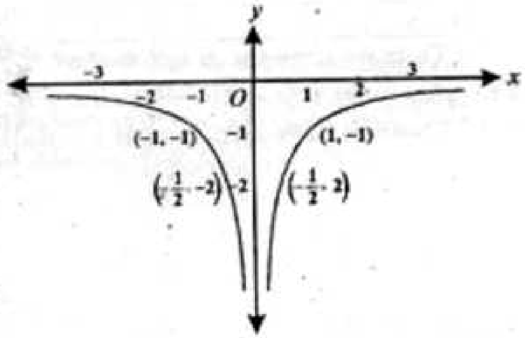
For 1.62 in the interval $1 \le x < 2$

$$f(1.62) = 1.62 - [1.62] = 1.62 - 1 = .62$$

The graph is as shown

26. $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x < 0 \\ -\dfrac{1}{x} & \text{if } x > 0 \end{cases}$

Sol. The function is not defined at $x = 0$. If x is negative, $f(x)$ is negative, and the value of $f(x)$ increases numerically as x decreases numerically so that $f(x)$ is $-\infty$ as x is near to zero from the left.



When x is near to zero and positive, then $f(x)$ is $-\infty$ and $f(x)$ decreases numerically as x increases so that $f(x)$ is near to 0 as x becomes very large. The graph is as shown.

27. $f(x) = x^2 + 2x - 1$ for all $x \in \mathbf{R}$ (1)

Sol. We rewrite (1) as

$$f(x) = y = x^2 + 2x - 1, \text{ that is,}$$
$$y = (x + 1)^2 - 2$$

or $(x + 1)^2 = y + 2$ or $X^2 = Y$ (2)

where $X = x + 1$ and $Y = y + 2$

Now (2) is a parabola which is symmetric about the Y-axis and has its vertex at

$$X = 0, Y = 0$$

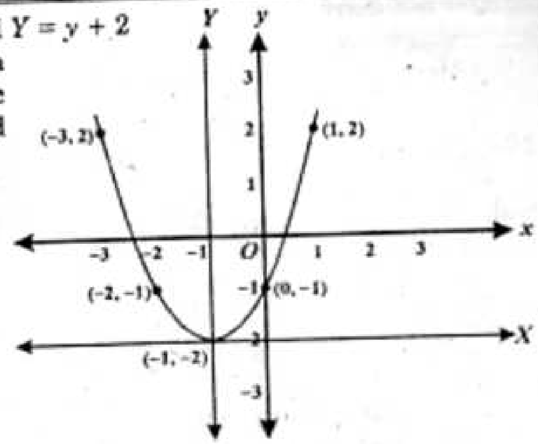
i.e., $x + 1 = 0$

and $y + 2 = 0$

or $x = -1$

and $y = -2$

It has the graph as shown.



28. $f(x) = \dfrac{1}{x^2}$, $x \ne 0$

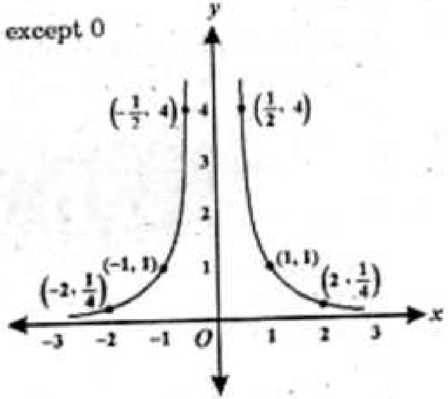
Sol. $f(x) = y = \dfrac{1}{x^2}$

y is defined for all values of x except 0

y is always positive, therefore the graph lies entirely above the x-axis.

$f(x_2) > f(x_1)$ if $x_2 > x_1$ for negative values of x that is, f is increasing in the interval $(-\infty, 0)$

$f(x_2) < f(x_1)$ if $x_2 > x_1$ for positive values of x that is, f is decreasing in the interval $(0, \infty)$.



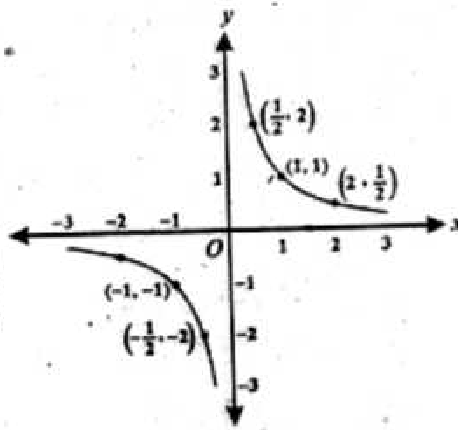
Hence we have the graph as shown in the figure.

29. $f(x) = \dfrac{1}{x}$, $x \ne 0$

Sol. $f(x) = y = \dfrac{1}{x}$

Here y is defined for all values of x except $x = 0$

When x is +ve, y is also +ve and when x is −ve, y is also −ve, therefore the graph lies in the first and third quadrants.



y is a decreasing function of x *i.e.*, as x increases, y decreases and vice versa.

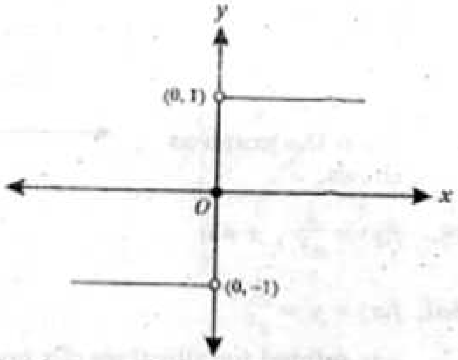
Hence the graph is a rectangular hyperbola as shown in the figure.

30. $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$ This is know as **singum (sgn)** ***function***

Sol. For $x > 0$, the graph is the straight line $y = 1$ (parallel to x-axis).

Similarly, for $x < 0$, the graph is the straight line $y = -1$.

The origin is also part of the graph since $f(0) = 0$. The points $(0, 1)$ and $(0, -1)$ are not on the graph.



Find the supremum and infimum (if they exist) of each of the given sets (Problems 31 – 34)

31. $\left\{(-1)^n\left(1-\dfrac{1}{n}\right), n = 1, 2, 3, \ldots\right\}$

Sol. The given set is $\left\{0, \dfrac{1}{2}, \dfrac{-2}{3}, \dfrac{3}{4}, \dfrac{-4}{5}, \dfrac{5}{6}, \dfrac{-6}{7}, \ldots\right\}$

It is clear that ..., -3 -2, -1 are lower bounds of the set. Since any real number greater than -1 is not a lower bound, we infer that -1 is the Inf of the set.

Again, 1, 2, 3, ... are upper bounds of the set. But any real smaller than 1 is not an upper bound. Thus 1 is the Sup.

32. The set of all nonnegative integers

Sol. Since this set starts from 0 and extends to $+\infty$,

Inf = 0 and Sup does not exist.

33. The set $A = \{x \in \mathbf{R} : 0 < x \le 3\}$

Sol. Inf $A = 0$ and Sup $A = 3$

34. The set $B = \{x \in \mathbf{R} : x^2 - 2x - 3 < 0\}$

Sol. $x^2 - 2x - 3 < 0$

Implies $(x - 3)(x + 1) < 0$

Two cases arise:

Case I: $x - 3 > 0$ and $x + 1 < 0$

i.e., $x > 3$ and $x < -1$.

Since there is no real number which is greater than 3 and less than -1, so this is not possible.

Case II: $x - 3 < 0$ and $x + 1 > 0$

$\Rightarrow$ $x < 3$ and $x > -1$. Thus $-1 < x < 3$

$\Rightarrow$ Inf $= -1$ and Sup $= 3$

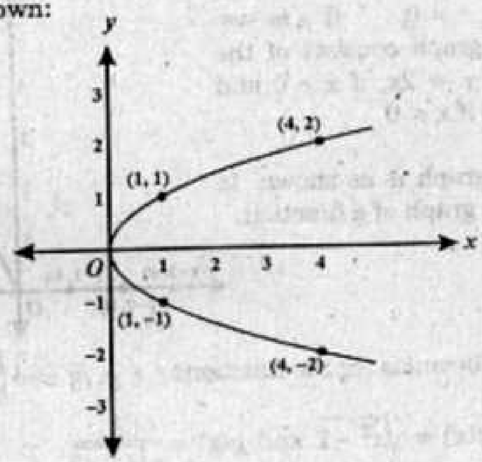
Sketch the graph of the given equation. Also determine which one is the graph of a function (Problems 35 – 38)

35. $y^2 = x$

Sol. It is clear that x is always positive. The graph passes through the origin. As y increases (numerically) x increases and is positive. We have the following table of some particular values:

x	0	1	4	9	16
y	0	±1	±2	±3	±4

The graph is as shown:



By the vertical line test, the graph is not of a function.

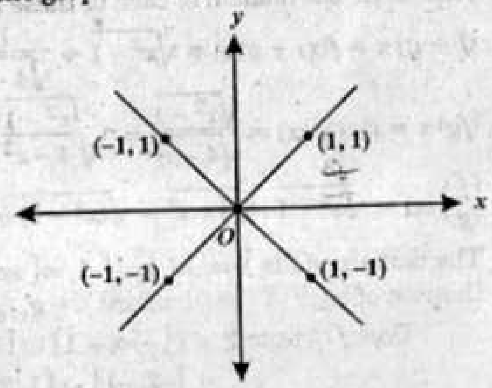
36. $|x| = |y|$

Sol. Here, we have

$x = \pm y$

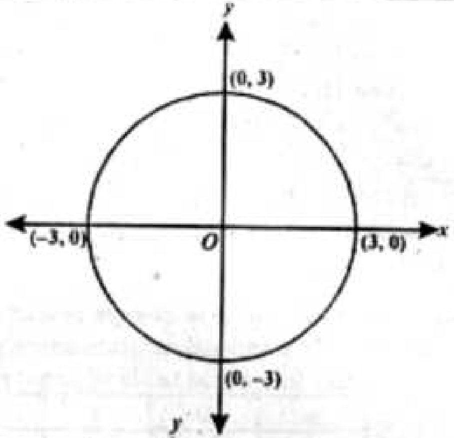
which is a pair of straight lines through the origin.

It is not the graph of a function.



37. $x^2 + y^2 = 9$

Sol. It is a circle with centre at the origin and having radius 3.
The graph is not a function.



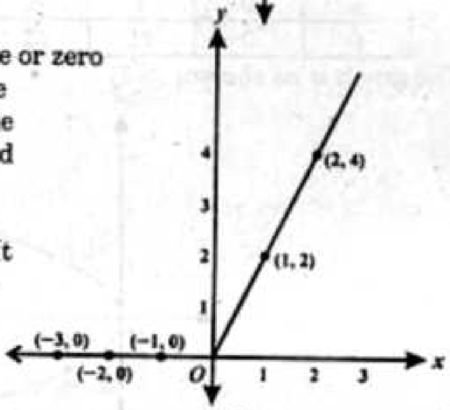
38. $y = |x| + x$

Sol. We have

$$y = x + x, \text{ if } x \text{ is +ve or zero}$$
$$= 0, \quad \text{if } x \text{ is −ve}$$

The graph consists of the lines $y = 2x$, if $x \geq 0$ and $y = 0$ if $x < 0$

The graph is as shown. It is the graph of a function.



39. Find formulas for the functions $f + g$, fg and $\frac{f}{g}$, where

$$f(x) = \sqrt{x^2 - 1} \text{ and } g(x) = \frac{1}{\sqrt{4 - x^2}}$$

Also write the domain of each of these functions.

Sol. $(f + g)\,x = f(x) + g(x) = \sqrt{x^2 - 1} + \frac{1}{\sqrt{4 - x^2}}$

$$(fg)\,x = f(x)\,g(x) = \frac{\sqrt{x^2 - 1}}{\sqrt{4 - x^2}} = \sqrt{\frac{x^2 - 1}{4 - x^2}}$$

$$\left(\frac{f}{g}\right)(x) = \sqrt{x^2 - 1}\,.\,\sqrt{4 - x^2} = \sqrt{(x^2 - 1)(4 - x^2)}$$

The domain of f is $]-\infty, -1\,] \cup [\,1, \infty[$ and the domain of g is $]-2, 2[$

Domain of each of the functions $f + g$, fg and f/g is

$$\text{Dom } f \cap \text{Dom } g = (]-\infty, -1] \cup [1, \infty[) \cap\,]-2, 2[$$
$$= \,]-2, -1] \cup [1, 2[$$

40. Find formulas for fog and gof, where

$$f(x) = \sqrt{x^3 - 3} \text{ and } g(x) = x^2 + 3$$

Sol. We know that $(fog)(x) = f(g(x))$

$$= f(x^2 + 3), \text{ by defining rule of } g$$
$$= \sqrt{(x^2 + 3)^2 - 3} \text{ by defining rule of } f$$
$$= \sqrt{x^4 + 6x^2 + 6}$$

Again $(gof) = g(f(x))$

$$= g(\sqrt{x^2 - 3}) \text{ by defining rule of } f$$
$$= x^2 - 3 + 3 = x^2$$

Exercise Set 1.2 (Page 35)

Evaluate the indicated limits (Problems 1 – 30):

1. $\lim\limits_{x \to 2} \frac{x - 2}{\sqrt{2 + x}}$

Sol. $\lim\limits_{x \to 2} \frac{x - 2}{\sqrt{2 + x}} = \frac{\lim\limits_{x \to 2} (x - 2)}{\lim\limits_{x \to 2} \sqrt{2 + x}} = \frac{0}{2} = 0$

2. $\lim\limits_{x \to 1} \frac{x^3 - 1}{x - 1}$

Sol. $\lim\limits_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim\limits_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1}$

$$= \lim_{x \to 1} (x^2 + x + 1) = 3$$

3. $\lim\limits_{x \to 1} \left(\frac{1}{1 - x} - \frac{3}{1 - x^3}\right)$

Sol. The given limit

$$= \lim_{x \to 1} \frac{1 + x + x^2 - 3}{1 - x^3} = \lim_{x \to 1} \frac{x^2 + x - 2}{1 - x^3}$$
$$= \lim_{x \to 1} \frac{(x + 2)(x - 1)}{-(x - 1)(x^2 + x + 1)} = \lim_{x \to 1} \frac{x + 2}{-(x^2 + x + 1)}$$
$$= \frac{3}{-3} = -1$$

4. If $P_n(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$, prove that

$$\lim_{x \to a} P_n(x) = P_n(a)$$

Sol. $\lim\limits_{x \to a} P_n(x)$

$$= \lim_{x \to a} \{a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n\}$$

$$= \lim_{x \to a} a_0 x^n + \lim_{x \to a} a_1 x^{n-1} + \ldots + \lim_{x \to a} a_{n-1} x + \lim_{x \to a} a_n,$$

by Theorem 1.26 (*i*)

$$= a_0 a^n + a_1 a^{n-1} + \ldots + a_{n-1} a + a_n \quad \left\{\begin{array}{l}\text{by Theorem 1.26 } (ii) \\ \text{and Theorem 1.24}\end{array}\right.$$

$$= P_n(a)$$

5. $\lim\limits_{x \to 0} \dfrac{\csc x - \cot x}{x}$ ... (1)

Sol. (1) may be written as

$$\lim_{x \to 0} \frac{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}{x} = \lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$$

$$= \lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x \sin x (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{1 - \cos^2 x}{x \sin x (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin^2 x}{x \,.\, \sin x (1 + \cos x)}$$

$$= \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{1 + \cos x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right) \cdot \lim_{x \to 0} \left(\frac{1}{1 + \cos x}\right)$$

$$= 1 \,.\, \frac{1}{2} = \frac{1}{2}$$

6. $\lim\limits_{x \to 0} \dfrac{\sin ax}{\sin bx}$

Sol. The given limit is

$$\lim_{x \to 0} \left[\left(\frac{\sin ax}{ax}\right)\left(\frac{bx}{\sin bx}\right)\left(\frac{ax}{bx}\right)\right]$$

$$= \lim_{x \to 0} \left(\frac{\sin ax}{ax}\right) . \lim_{x \to 0} \frac{bx}{\sin bx} . \lim_{x \to 0} \frac{ax}{bx}$$

$$= 1.1 \,.\, \frac{a}{b} = \frac{a}{b}$$

7. $\lim\limits_{x \to 0} \dfrac{1 - \cos x}{x^2}$ (1)

Sol. (1) may be written as

$$\lim_{x \to 0} \frac{(1 - \cos x)(1 + \cos x)}{x^2 (1 + \cos x)} = \lim_{x \to 0} \left(\frac{1 - \cos^2 x}{x^2} \cdot \frac{1}{1 + \cos x}\right)$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)^2 \cdot \lim_{x \to 0} \frac{1}{1 + \cos x}$$

$$= (1)^2 \,.\, \frac{1}{2} = \frac{1}{2}$$

8. $\lim\limits_{y \to x} \dfrac{y^{\frac{2}{3}} - x^{\frac{2}{3}}}{y - x}$

Sol. $$\lim_{y \to x} \frac{y^{\frac{2}{3}} - x^{\frac{2}{3}}}{y - x} = \lim_{y \to x} \frac{\left(y^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left(y^{\frac{1}{3}} + x^{\frac{1}{3}}\right)}{\left(y^{\frac{1}{3}} - x^{\frac{1}{3}}\right)\left(y^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + x^{\frac{2}{3}}\right)}$$

$$= \lim_{y \to x} \frac{y^{\frac{1}{3}} + x^{\frac{1}{3}}}{y^{\frac{2}{3}} + x^{\frac{1}{3}} y^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{2x^{\frac{1}{3}}}{3x^{\frac{2}{3}}} = \frac{2}{3} x^{\frac{1}{3}}$$

9. $\lim\limits_{x \to \pi} \dfrac{\tan(\sin x)}{\sin x}$

Sol. Let $\sin x = \theta$

When $x \longrightarrow \pi$, $\theta \longrightarrow 0$. Therefore,

$$\lim_{x \to \pi} \frac{\tan(\sin x)}{\sin x} = \lim_{\theta \to 0} \frac{\tan \theta}{\theta} = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \times \frac{1}{\cos \theta}$$

$$= \lim_{\theta \to \theta} \frac{\sin \theta}{\theta} \,.\, \lim_{\theta \to 0} \frac{1}{\cos \theta} = 1 \times 1 = 1$$

10. $\lim\limits_{x \to 0} x \sin\left(\dfrac{1}{x}\right)$

Sol. Here $|f(x) - 0|$

$$= \left|x \sin \frac{1}{x} - 0\right| = \left|x \sin \frac{1}{x}\right|$$

$$= |x| \left|\sin \frac{1}{x}\right| \le |x|, \left(\text{since } \left|\sin \frac{1}{x}\right| \le 1\right)$$

If we take $\varepsilon = \delta$, then $|f(x) - 0| < \varepsilon$

whenever $|x| < \delta$. Hence $\lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0$

Alternative Method:

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = \lim_{x\to 0} x \,.\, \lim_{x\to 0} \sin\left(\frac{1}{x}\right)$$
$$= 0.\ \text{some number in } [-1, 1] = 0$$

11. $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$

Sol. Here we find $\lim_{x\to 0^-} \sin\left(\frac{1}{x}\right)$

Since $\left|\sin\left(\frac{1}{x}\right)\right| \le 1$, then limit repeatedly takes the values 1 and −1 or any value between −1 and 1. Hence this limit cannot exist.

Similarly, $\lim_{x\to 0^+} \sin\left(\frac{1}{x}\right)$ does not exist.

Thus $\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

12. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x+1}$

Sol. The given limit is $\lim_{x\to\infty} \frac{x\sqrt{1+\frac{1}{x^2}}}{x(1+\frac{1}{x})} = \lim_{x\to\infty} \frac{\sqrt{1+\frac{1}{x^2}}}{1+\frac{1}{x}} = \frac{1}{1} = 1$

13. $\lim_{x\to\infty} \frac{4x^3-2x^2+1}{3x^3-5}$

Sol. Dividing both the numerator and denominator by x^3, we get the given limit

$$= \lim_{x\to\infty} \frac{4-\frac{2}{x^2}+\frac{1}{x^3}}{3-\frac{5}{x^3}} = \frac{4}{3}$$

14. $\lim_{x\to\infty} \left(1+\frac{2}{x}\right)^x$

Sol. $\lim_{x\to\infty} \left[\left(1+\frac{2}{x}\right)^{x/2}\right]^2 = e^2, \left(\text{since} \lim_{x\to\infty}\left(1+\frac{2}{x}\right)^{x/2} = e\right)$

15. $\lim_{x\to\infty} \left(1-\frac{1}{x}\right)^x$

Sol. $\lim_{x\to\infty} \left[\left(1-\frac{1}{x}\right)^{-x}\right]^{-1} = e^{-1} = \frac{1}{e}$

16. $\lim_{x\to\infty} \left(\frac{x}{1+x}\right)^x$

Sol. The given limit is

$$\lim_{x\to\infty} \left(\frac{1+x}{x}\right)^{-x} = \lim_{x\to\infty} \left(1+\frac{1}{x}\right)^{-x} = \lim_{x\to\infty} \left[\left(1+\frac{1}{x}\right)^x\right]^{-1} = e^{-1} = \frac{1}{e}$$

17. $\lim_{x\to\infty} \frac{a^x-1}{x}, (a > 1)$

Sol. Let $a^x - 1 = z$

or $a^x = 1 + z$ or $x = \log_a (1+z)$

If $x \longrightarrow \infty$, then $z \longrightarrow \infty$. Therefore,

$$\lim_{x\to\infty} \frac{a^x-1}{x} = \lim_{z\to\infty} \frac{z}{\log_a (1+z)} = \lim_{z\to\infty} \frac{1}{\log_a (1+z)^{1/z}}$$
$$= \frac{1}{0} = \infty.\ \text{(We assume } \infty^0 = 1)$$

18. $\lim_{x\to\infty} \frac{x^4-2x^2+6}{x^2+7}$ (1)

Sol. Dividing both the numerator and denominator by x^4, (1) becomes

$$= \lim_{x\to\infty} \frac{1-\frac{2}{x^2}+\frac{6}{x^4}}{\frac{1}{x^2}+\frac{7}{x^4}} = \frac{\lim_{x\to\infty}\left(1-\frac{2}{x^2}+\frac{6}{x^2}\right)}{\lim_{x\to\infty}\left(\frac{1}{x^2}+\frac{7}{x^4}\right)} = \frac{1}{0} = \infty$$

19. $\lim_{x\to\pm\infty} \left[\frac{x^2}{x+1} - \frac{x^2}{x+3}\right]$

Sol. The given limit is

$$\lim_{x\to\pm\infty} \left[\frac{x^3+3x^2-x^3-x^2}{(x+1)(x+3)}\right] = \lim_{x\to\pm\infty} \left[\frac{2x^2}{x^2+4x+3}\right]$$
$$= \lim_{x\to\pm\infty} \frac{2}{1+\frac{4}{x}+\frac{3}{x^2}} = 2$$

20. $\lim\limits_{x \to \infty} (x - \sqrt{x^2 - a^2})$

Sol. The given limit is

$$\lim_{x \to \infty} \frac{(x - \sqrt{x^2 - a^2})\,(x + \sqrt{x^2 - a^2})}{x + \sqrt{x^2 - a^2}} = \lim_{x \to \infty} \frac{x^2 - x^2 + a^2}{x + \sqrt{x^2 - a^2}} = 0$$

21. $\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{x^{3/2}}$

Sol. $\lim\limits_{x \to \infty} \dfrac{x^2 + 1}{x^{3/2}} = \lim\limits_{x \to \infty} \dfrac{1 + \dfrac{1}{x^2}}{\dfrac{1}{x^{1/2}}} = \infty$

22. $\lim\limits_{x \to \infty} \dfrac{5x^3 + 3x^2 - 1}{x - 4x^4}$

Sol. Since the limit of a quotient of polynomials as $x \longrightarrow \pm\infty$ is the same as the limit of the quotient of the highest power terms in the numerator and denominator, we have

$$\lim_{x \to \pm\infty} \frac{5x^3 + 2x^2 - 1}{x - 4x^4} = \lim_{x \to \pm\infty} \frac{5}{4x} = \lim_{x \to \pm\infty} \frac{5}{4x} = 0$$

23. $\lim\limits_{x \to \infty} \dfrac{3 - 2x^4}{1 + x}$

Sol. The given limit equals

$$\lim_{x \to \infty} \frac{-2x^4}{x} = \lim_{x \to \infty} -2x^3 = -\infty$$

24. $\lim\limits_{x \to -1} \dfrac{x^{1/3} + 1}{x + 1}$

Sol. $\lim\limits_{x \to -1} \dfrac{x^{1/3} + 1}{x + 1} = \lim\limits_{h \to 0} \dfrac{(-1 + h)^{1/3} + 1}{h} = \lim\limits_{h \to 0} \dfrac{-(1 - h)^{1/3} + 1}{h}$

$$= \lim_{h \to 0} \frac{-\left[1 + \frac{1}{3}(-h) + \dfrac{\frac{1}{3}\left(\frac{1}{3} - 1\right)}{2!}(-h)^2 + \ldots\right] + 1}{h}$$

$$= \lim_{h \to 0} \frac{\left(-1 + \frac{1}{3}h + \frac{1}{9}h^2 + \ldots\right) + 1}{h}$$

$$= \lim_{h \to 0} \left(\frac{1}{3} + \frac{1}{9}h + \ldots\right) = \frac{1}{3}$$

25. $\lim\limits_{x \to 3^-} \left(\dfrac{1}{x - 3} - \dfrac{1}{|x - 3|}\right)$

Sol. $\lim\limits_{x \to 3^-} \left[\dfrac{1}{x - 3} - \dfrac{1}{|x - 3|}\right] = \lim\limits_{x \to 3^-} \left[\dfrac{1}{x - 3} - \dfrac{1}{3 - x}\right]$

$$= \lim_{x \to 3^-} \left[\frac{1}{x - 3} + \frac{1}{x - 3}\right]$$

$$= \lim_{x \to 3^-} \left[\frac{2}{x - 3}\right] = -\infty$$

26. $\lim\limits_{x \to -2^-} \dfrac{x^2 + 2x - 8}{x^2 - 4}$

Sol. $\lim\limits_{x \to -2^-} \dfrac{x^2 + 2x - 8}{x^2 - 4} = \lim\limits_{x \to -2^-} (x^2 + 2x - 8) \times \lim\limits_{x \to -2^-} \dfrac{1}{x^2 - 4}$

$$= (4 - 4 - 8) \cdot \frac{1}{(-2)^2 - 4} = -\infty$$

27. $\lim\limits_{x \to 1^-} \dfrac{\sqrt{1 - x^2}}{1 - x}$ (1)

Sol. We write (1) as

$$\lim_{x \to 1^-} \frac{\sqrt{(1 - x)\,(1 + x)}}{1 - x} = \lim_{x \to 1^-} \sqrt{\frac{1 + x}{1 - x}} = \frac{\sqrt{2}}{0} = \infty$$

28. $\lim\limits_{x \to 1^+} \dfrac{x - 1}{\sqrt{x^2 - 1}}$

Sol. $\lim\limits_{x \to 1^+} \dfrac{x - 1}{\sqrt{x^2 - 1}} = \lim\limits_{x \to 1^+} \sqrt{\dfrac{x - 1}{x + 1}} = \sqrt{\dfrac{0}{2}} = 0$

29. $\lim\limits_{x \to 2^-} \dfrac{\sqrt{4 - x^2}}{\sqrt{6 - 5x + x^2}}$ (1)

Sol. (1) may be written as

$$\lim_{x \to 2^-} \sqrt{\frac{(2 - x)\,(2 + x)}{(2 - x)\,(3 - x)}} = \lim_{x \to 2^-} \sqrt{\frac{2 + x}{3 - x}} = \frac{\sqrt{4}}{1} = 2$$

30. $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x}$

Sol. $\lim\limits_{x \to \infty} \dfrac{x + \sin x}{x} = \lim\limits_{x \to \infty} \dfrac{x}{x} + \lim\limits_{x \to \infty} \dfrac{\sin x}{x}$

$$= \lim_{x\to\infty} 1 + \lim_{x\to\infty} \frac{\sin x}{x} = 1 + 0,$$

$$= 1, \begin{cases}\text{since } \sin x \text{ remains}\\ \text{bounded for all values of } x\end{cases}$$

31. Let $f(x) = \begin{cases} x^2 + 3 & \text{if } x \le 1 \\ x + 1 & \text{if } x > 1 \end{cases}$

Find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$

Sol. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x + 1) = 2$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2 + 3) = 4$

32. $f(x) = \begin{cases} 3 & \text{if } x \le -2 \\ -\frac{1}{2}x^2 & \text{if } -2 < x < 2 \\ 3 & \text{if } x \ge 2 \end{cases}$

Find $\lim_{x\to \pm 2^+} f(x)$ and $\lim_{x\to \pm 2^-} f(x)$

Sol. $\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} (3) = 3$

$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} \left(-\frac{1}{2}x^2\right) = -2$

$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \left(-\frac{1}{2}x^2\right) = -2$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (3) = 3$

33. Let $f(x) = \begin{cases} x^2 - 1 & \text{if } x \le 2 \\ \sqrt{x + 7} & \text{if } x > 2 \end{cases}$

Find $\lim f(x)$ as $x \to 2$.

Sol. $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} (x^2 - 1) = 3$

$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \sqrt{x + 7} = \sqrt{2 + 7} = 3$

Thus $\lim_{x\to 2^-} = \lim_{x\to 2^+} f(x) = 3 = \lim_{x\to 2} f(x)$

34. Let $f(x) = \begin{cases} \cos x & \text{if } x \le 0 \\ 1 - x & \text{if } x > 0 \end{cases}$

Find $\lim_{x\to 0} f(x)$.

Sol. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \cos x = 1$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (1 - x) = 1$

Thus, $\lim_{x\to 0} f(x) = 1$

35. Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x^3 & \text{if } x > 1 \end{cases}$

Show that $\lim_{x\to 1} f(x) = 1$

Sol. $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (x^2) = 1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x^3) = 1$

Hence, $\lim_{x\to 1} f(x) = 1$

36. Let $f(x) = \begin{cases} x + 2 & \text{if } x \le -1 \\ ax^2 & \text{if } x > -1 \end{cases}$

Find a so that $\lim_{x\to -1} f(x)$ exists.

Sol. $\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} (x + 2) = 1$

$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} ax^2 = a$

If $\lim_{x\to -1} f(x)$ exists, we must have

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^+} f(x)$$

Therefore, $1 = a$.

37. Evaluate $\lim_{x\to 3^-} \frac{3 - x}{|x - 3|}$

Sol. As $x \to 3^-$, so $|x - 3| = -(x - 3) = 3 - x$,

Therefore, $\lim_{x\to 3^-} \frac{3 - x}{|x - 3|} = \lim_{x\to 3^-} \frac{3 - x}{3 - x} = 1$

38. Evaluate $\lim_{x\to 0^-} \frac{x}{x - |x|}$

Sol. $\lim_{x\to 0^-} \frac{x}{x - |x|} = \lim_{x\to 0^-} \frac{x}{x + x}$ $\quad (|x| = -x \text{ as } x \to 0^-)$

$$= \lim_{x\to 0^-} \frac{x}{2x} = \frac{1}{2}$$

39. Find $\lim_{h \to 0} \frac{|-1+h|-1}{h}$

Sol. $\lim_{h \to 0} \frac{|-1+h|-1}{h} = \lim_{h \to 0} \frac{|-(1-h)|-1}{h} = \lim_{h \to 0} \frac{1-h-1}{h} = -1$

40. Evaluate, [...] being the bracket function:

(*i*) $\lim_{x \to 1} [2x](x-1)$ (*ii*) $\lim_{x \to 0} [x][x+1]$

(*iii*) $\lim_{x \to 0} x\left[\frac{1}{x}\right]$ (*iv*) $\lim_{x \to 0} x^3\left[\frac{1}{x}\right]$

Sol.

(*i*) $\lim_{x \to 1^-} [2x](x-1) = \lim_{x \to 1^-} [2x] \cdot \lim_{x \to 1^-} (x-1) = (1)(0) = 0$

$\lim_{x \to 1^+} [2x](x-1) = \lim_{x \to 1^+} [2x] \cdot \lim_{x \to 1^+} (x-1) = 2 \cdot 0 = 0$

Hence $\lim_{x \to 1} [2x](x-1) = 0$

(*ii*) $\lim_{x \to 0^-} [x][x+1] = \lim_{x \to 0^-} [x] \cdot \lim_{x \to 0^-} [x+1] = (-1)(0) = 0$

$\lim_{x \to 0^-} [x][x+1] = \lim_{x \to 0^+} [x] \cdot \lim_{x \to 0^+} [x+1] = (0)(1) = 0$

Thus $\lim_{x \to 0} [x][x+1] = 0$

(*iii*) $\lim_{x \to 0} x\left[\frac{1}{x}\right]$

In general, $\frac{1}{x} - 1 \le \left[\frac{1}{x}\right] \le \frac{1}{x}$, by definition of the bracket function.

For $x < 0$, $1 = x\left(\frac{1}{x}\right) \le x\left[\frac{1}{x}\right] \le x\left(\frac{1}{x} - 1\right) = 1 - x$

So $\lim_{x \to 0^-} x\left[\frac{1}{x}\right] = 1$ by Theorem 1.32 (*v*)

For $x > 0$, $1 - x = x\left(\frac{1}{x} - 1\right) \le x\left[\frac{1}{x}\right] \le x\left(\frac{1}{x}\right) = 1$

$\lim_{x \to 0^+} x\left[\frac{1}{x}\right] = 1$

It follows that $\lim_{x \to 0} x\left[\frac{1}{x}\right] = 1$

(*iv*) $\lim_{x \to 0} x^3\left[\frac{1}{x}\right]$

In general, $\frac{1}{x} - 1 \le \left[\frac{1}{x}\right] \le \frac{1}{x}$

For $x < 0$, $x^2 = x^3\left(\frac{1}{x}\right) \le x^3\left[\frac{1}{x}\right] < x^3\left(\frac{1}{x} - 1\right) = x^2 - x^3$

and so $\lim_{x \to 0^-} x^3\left[\frac{1}{x}\right] = 0$

For $x > 0$

$x^2 - x^3 = x^3\left(\frac{1}{x} - 1\right) \le x3\left[\frac{1}{x}\right] \le x^3\left(\frac{1}{x}\right) = x^2$

and thus $\lim_{x \to 0^+} x^3\left[\frac{1}{x}\right] = 0$

It follows that $\lim_{x \to 0} x^3\left[\frac{1}{x}\right] = 0$

Exercise Set 1.3 (Page 42)

Discuss the continuity of the following functions at the indicated points/set (Problems 1 – 7):

1. $f(x) = |x-3|$ at $x = 3$

Sol. $f(3) = |3-3| = 0$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} |x-3| = \lim_{h \to 0} |3-h-3|$, putting $x = 3-h$

$= \lim_{h \to 0} |-h| = 0$

$\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} |x-3| = \lim_{h \to 0} |3+h-3|$, putting $x = 3+h$

$= \lim_{h \to 0} |h| = 0$

Thus $\lim_{x \to 3^+} f(x)(= \lim_{x \to 3^-} f(x)(= f(3)$

Hence f is continuous at $x = 3$

2. $f(x) = \begin{cases} \frac{x^2-9}{x-3} & \text{if } x \ne 3 \\ 0 & \text{if } x = 3 \end{cases}$ at $x = 3$

Sol. $\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} (x+3) = 6$

But $f(3) = 0$ (given)

Therefore, $\lim_{x \to 3} f(x) \ne f(3)$

Hence f is discontinuous at $x = 3$

3. $f(x) = \begin{cases} x - 4 & \text{if } -1 < x \le 2 \\ x^2 - 6 & \text{if } 2 < x < 5 \end{cases}$ at $x = 2$

Sol. $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (x - 4) = -2$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x^2 - 6) = -2$

When $x = 2, f(x) = x - 4$

Thus $f(2) = 2 - 4 = -2$

Therefore, $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = f(2)$

Hence the function is continuous at $x = 2$.

4. $f(x) = \begin{cases} \dfrac{x^3 - 27}{x^2 - 9} & \text{if } x \ne 3 \\ 6 & \text{if } x = 3 \end{cases}$ at $x = 3$

Sol. $\lim\limits_{x \to 3} f(x) = \lim\limits_{x \to 3} \dfrac{x^3 - 27}{x^2 - 9} = \lim\limits_{x \to 3} \dfrac{(x-3)(x^2 + 3x + 9)}{(x-3)(x+3)}$

$= \lim\limits_{x \to 3} \dfrac{x^2 + 3x + 9}{x + 3} = \dfrac{27}{6} = \dfrac{9}{2}$

But $f(3) = 6$ (given)

Thus $\lim\limits_{x \to 3} f(x) \ne f(3)$

Hence the function is discontinuous at $x = 3$.

5. $f(x) = \begin{cases} \sin\left(\dfrac{1}{x}\right) & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ at $x = 0$

Sol. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \sin \dfrac{1}{x} = \sin(-\infty)$ which is not a definite number.

It can have any value in $[-1, 1]$

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} \sin \dfrac{1}{x} = \sin \infty$, which is also not a definite number.

Hence the limit of $f(x)$ does not exist as $x \to 0$, so the function is discontinuous at $x = 0$.

6. $f(x) = \sin x$ for all $x \in \mathbf{R}$.

Sol. Let θ be any arbitrary real number.

$\lim\limits_{x \to \theta^-} f(x) = \lim\limits_{x \to \theta^-} \sin x = \sin \theta$

$\lim\limits_{x \to \theta^+} f(x) = \lim\limits_{x \to \theta^+} \sin x = \sin \theta$

Also $f(\theta) = \sin \theta$

Thus $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^+} = f(\theta)$

and so $f(x)$ is continuous at $x = \theta$

Since θ is any real number, $\sin x$ is continuous at every $x \in \mathbf{R}$.

7. $f(x) = \begin{cases} \dfrac{x^2}{a} - a & \text{if } 0 < x < a \\ 0 & \text{if } x = a \\ a - \dfrac{a^2}{x} & \text{if } x > a \end{cases}$ at $x = a$

Sol. $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^-} \left(\dfrac{x^2}{a} - a\right) = 0$

$\lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} \left(a - \dfrac{a^2}{x}\right) = 0$

Therefore $\lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^+} = f(a) = 0$

Hence $f(x)$ is continuous at $x = a$.

8. Determine the points of continuity of the function $f(x) = x - [x]$ for all $x \in \mathbf{R}$.

Sol. Here $[x]$ is the bracket function and stands for the greatest integer not greater than x.

So, when $0 \le x < 1, [x] = 0; [x] = -1,$ if $-1 \le x < 0$

when $1 \le x < 2, [x] = 1; [x] = -2,$ if $-2 \le x - 1$

when $2 \le x < 3, [x] = 2; [x] = -3,$ if $-3 \le x - 2$ etc.

Hence the given function may be defined as

$f(x) = x$ in $0 \le x < 1$ (1)

$= x - 1$ in $1 \le x < 2$ (2)

$= x - 2$ in $2 \le x < 3$ (3)

etc.

Also $f(x) = x + 1$ in $-1 \le x < 0$ (4)

$= x + 2$ in $-2 \le x < -1$ (5)

At $x = 1;$ $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} x = 1$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} (x - 1) = 0$

Therefore, the limit of $f(x)$ at $x = 1$ does not exist.

Hence, f is not continuous at $x = 1$.

At $x = 0,$ $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (x + 1) = 1$, from (4)

$\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} x = 0$, from (1)

Thus $\lim\limits_{x \to 0} f(x)$ does not exist.

So, the function is discontinuous at $x = 0$

Similarly, it is discontinuous for integral values of x, both positive and negative but it is continuous at every other real value of x.

9. Discuss the continuity of $x - |x|$ at $x = 1$.

Sol. Let $f(x) = x - |x|$

$$\lim_{x \to 1^-} x - |x| = 0$$

$$\lim_{x \to 1^+} x - |x| = 0$$

$$f(1) = 1 - 1 = 0$$

Thus $\lim\limits_{x \to 1} f(x) = f(1)$ and so the function is continuous at $x = 1$.

10. Show that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is irrational} \\ 1 - x & \text{if } x \text{ is rational} \end{cases}$$

is continuous at $x = \dfrac{1}{2}$

Sol. $f\left(\dfrac{1}{2}\right) = 1 - \dfrac{1}{2} = \dfrac{1}{2}$

$$\lim_{x \to 1/2} f(x) = \lim_{x \to 1/2} (1 - x) = \frac{1}{2}$$

Hence the limit of $f(x)$ exists at $x = \dfrac{1}{2}$ and is equal to the value of

$f(x)$ at $x = \dfrac{1}{2}$

Thus $f(x)$ is continuous at $x = \dfrac{1}{2}$

11. Show that the function $f :]\,0, 1\,] \to \boldsymbol{R}$ defined by $f(x) = \dfrac{1}{x}$ is continuous on $]\,0, 1]$. Is $f(x)$ bounded on this interval? Explain.

Sol. $f(x)$ is defined for all real values of x such that $0 < x \leq 1$ and its limit exists at each such x and equals to its value there, so it is continuous on $]0, 1]$.

When $x = 1$, $f(x) = 1$ which is its lower bound.

So it is bounded below. $f(x)$ increases indefinitely as x becomes small. Thus $f(x)$ is not bounded above. Hence $f(x)$ is not bounded on $]0, 1]$.

12. Let $f(x) = \begin{cases} \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Is f continuous at $x = 0$?

Sol. $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} \cos\left(\dfrac{1}{x}\right)$,

which may be any value in $[-1, 1]$. Thus the limit does not exist. Similarly, $\lim\limits_{x \to 0^+} f(x)$ does not exist.

Hence $\lim\limits_{x \to 0} f(x)$ does not exist and the function cannot be continuous at $x = 0$.

13. Let $f(x) = \begin{cases} (x - a) \sin\left(\dfrac{1}{x - a}\right) & \text{if } x \neq a \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = a$.

Sol. Here $|f(x) - f(a)| = \left|(x - a) \sin \dfrac{1}{x - a} - 0\right|$

$$= \left|(x - a) \sin \frac{1}{x - a}\right| = |x - a| \left|\sin \frac{1}{x - a}\right|$$

$$\leq |x - a|, \left[\text{since } \left|\sin \frac{1}{x - a}\right| \leq 1\right]$$

$$< \varepsilon \quad \text{if } |x - a| < \delta = \varepsilon$$

Thus by definition, $f(x)$ is continuous at $x = a$.

14. Let $f(x) = \begin{cases} x \cos\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Show that f is continuous at $x = 0$

Sol. Here $|f(x) - f(0)| = \left|x \cos \dfrac{1}{x} - 0\right| = \left|x \cos \dfrac{1}{x}\right|$

$$= |x| \left|\cos \frac{1}{x}\right| \leq |x|, \left(\text{since } \left|\cos \frac{1}{x}\right| \leq 1\right)$$

$$< \varepsilon \text{ if } |x| < \delta = \varepsilon$$

Hence $f(x)$ is continuous at $x = 0$ by definition.

15. Let $f(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

Sol. $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} x^2 \lim\limits_{x \to 0} \sin\left(\dfrac{1}{x}\right)$

$= 0$ (some number between -1 and 1) $= 0$

$f(0) = 0$

Thus $f(x)$ is continuous at $x = 0$.

16. Let $f(x) = \begin{cases} x\sin\left(\frac{|x|}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

Discuss the continuity of f at $x = 0$

Sol. Now $\lim\limits_{x \to 0^-} x \sin\frac{|x|}{x} = \lim\limits_{x \to 0^-} x \lim\limits_{x \to 0^-} \sin\frac{-x}{x}$

$= 0 . \sin(-1) = 0$

$\lim\limits_{x \to 0^+} x \sin\frac{|x|}{x} = \lim\limits_{x \to 0^+} x \lim\limits_{x \to 0^+} \sin\frac{x}{x} = 0 . \sin 1 = 0$

Thus $\lim\limits_{x \to 0} f(x) = 0$

Also $f(0) = 0$

Hence f is continuous at $x = 0$

17. Find c such that the function $f(x) = \begin{cases} \frac{1-\sqrt{x}}{x-1} & \text{if } 0 \le x < 1 \\ c & \text{if } x = 1 \end{cases}$

is continuous for all $x \in [0, 1]$.

Sol. Let a be any point of $[0, 1[$

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{1-\sqrt{x}}{x-1} = \frac{1-\sqrt{a}}{a-1} = f(a)$$

Thus f is continuous at a and since a is an arbitrary point of $[0, 1[$, f is continuous on $[0, 1[$. In order that f be continuous at the point $x = 1$, we must have $\lim\limits_{x \to 1} f(x) = f(1) = c$

Now $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \frac{1-\sqrt{x}}{x-1} = \lim\limits_{x \to 1^-} \frac{-(\sqrt{x}-1)}{(\sqrt{x}-1)(\sqrt{x}+1)} = -\frac{1}{2}$

Thus f is continuous at $x = 1$ if $c = -\frac{1}{2}$.

Hence f is continuous on $[0, 1]$ for $c = -\frac{1}{2}$.

In Problems 18 – 20, find the points of discontinuity of the given functions.

18. $f(x) = \begin{cases} x + 4 & \text{if } -6 \le x < -2 \\ x & \text{if } -2 \le x < 2 \\ x - 4 & \text{if } 2 \le x < 6 \end{cases}$

Sol. $\lim\limits_{x \to -2^-} f(x) = \lim\limits_{x \to -2^-} (x + 4) = 2$

$\lim\limits_{x \to -2^+} f(x) = \lim\limits_{x \to -2^+} (x) = -2$

Thus $\lim\limits_{x \to -2^-} f(x) \neq f(x) \lim\limits_{x \to -2^+}$ and so the function is discontinuous at $= -2$.

Again $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x = 2$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x - 4) = -2$

Thus $\lim\limits_{x \to 2^-} f(x) \neq \lim\limits_{x \to 2^+} f(x)$ and therefore the function is discontinuous at $x = 2$.

The points of discontinuity of f are $x = -2, 2$.

19. $g(x) = \begin{cases} x^3 & \text{if } x < 1 \\ -4 - x^2 & \text{if } 1 \le x \le 10 \\ 6x^2 + 46 & \text{if } x > 10 \end{cases}$

Sol. $\lim\limits_{x \to 1^-} g(x) = \lim\limits_{x \to 1^-} x^3 = 1$

$\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} (-4 - x^2) = -5$

Thus $\lim\limits_{x \to 1} g(x)$ does not exist and so g is discontinuous at $x = 1$

Next, we find $\lim g(x)$ as $x \to 10$

$\lim\limits_{x \to 10^-} g(x) = \lim\limits_{x \to 10^-} (-4 - x^2) = -104$

$\lim\limits_{x \to 10^+} g(x) = \lim\limits_{x \to 10^+} (6x^2 + 46) = 646$

Hence $\lim\limits_{x \to 10} g(x)$ does not exist and so the function is discontinuous at $x = 10$

20. $f(x) = \begin{cases} x + 2 & \text{if } 0 \le x < 1 \\ x & \text{if } 1 \le x < 2 \\ x + 5 & \text{if } 2 \le x < 3 \end{cases}$

Sol. We check the continuity of f at $x = 1$ and $x = 2$

$\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} (x + 2) = 3$

$\lim\limits_{x \to 1^+} f(x) = \lim\limits_{x \to 1^+} x = 1$

Thus $\lim\limits_{x \to 1} f(x)$ does not exist and therefore the function is discontinuous at $x = 1$.

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} x = 2$

$\lim\limits_{x \to 2^+} f(x) = \lim\limits_{x \to 2^+} (x + 5) = 7$

Therefore, $\lim_{x \to 2} f(x)$ does not exist.

The function is also discontinuous at $x = 2$.

21. Find constants a and b such that the function f defined by

$$f(x) = \begin{cases} x^3 & \text{if } 0 < -1 \\ ax + b & \text{if } -1 \le x < 1 \\ x^2 + 2 & \text{if } x \ge 1 \end{cases}$$ is continuous for all x.

Sol. It is easy to see that the given function is continuous for all x possibly except at $x = -1$ and 1.

Now $\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} x^3 = -1$

$\lim_{x \to -1^+} f(x) = \lim_{x \to -1^+} (ax + b) = -a + b$

If the function is continuous at $x = -1$, we must have

$$-a + b = -1 \qquad (1)$$

Again $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (ax + b) = a + b$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + 2) = 3$

For continuity of f at $x = 1$, we must have

$$a + b = 3 \qquad (2)$$

Solving (1) and (2) simultaneously, we obtain $a = 2, b = 1$.

Find the interval on which the given function is continuous. Also find points where it is discontinuous. (Problems 22 – 26).

22. $f(x) = \dfrac{x^2 - 5}{x - 1}$

Sol. The function $f(x) = \dfrac{x^2 - 5}{x - 1}$ is not defined at $x = 1$. Thus $f(x)$ is discontinuous at $x = 1$.

The numerator $x^2 - 5$ is continuous at every point of $\boldsymbol{R}$ and so is the denominator $x - 1$. Hence $f(x)$ is continuous at every point of $\boldsymbol{R} - \{1\}$.

23. $f(x) = \dfrac{x}{|x|}$

Sol. $f(x)$ is not defined at $x = 0$ and so it is discontinuous at $x = 0$. The function is continuous at every other point of $\boldsymbol{R}$.

24. $f(x) = \dfrac{\sin x}{x}$

Sol. The function $\dfrac{\sin x}{x}$ is not defined at $x = 0$. Hence it is discontinuous at $x = 0$. The function is continuous at every other point of $\boldsymbol{R}$ since $\sin x$ and x are continuous on $\boldsymbol{R}$.

25. $f(x) = \tan x$

Sol. $f(x) = \dfrac{\sin x}{\cos x}$

The function is not defined at $x = (2n + 1)\dfrac{\pi}{2}$, where n is an integer. Thus, $f(x)$ is discontinuous at these points, $f(x)$ is continuous on all other points of $\boldsymbol{R}$.

26. $f(x) = \begin{cases} \sin x & \text{if } x \le \pi/4 \\ \cos x & \text{if } x > \pi/4 \end{cases}$

Sol. $\lim_{x \to \frac{\pi}{4}^-} f(x) = \lim_{x \to \frac{\pi}{4}^-} \sin x = \dfrac{1}{\sqrt{2}}$

$\lim_{x \to \frac{\pi}{4}^+} f(x) = \lim_{x \to \frac{\pi}{4}^+} \cos x = \dfrac{1}{\sqrt{2}}$

More-over, $f\left(\dfrac{\pi}{4}\right) = \sin\dfrac{\pi}{4} = \dfrac{1}{\sqrt{2}}$

The function is continuous at $x = \dfrac{\pi}{4}$. We also know that $\sin x$ and $\cos x$ are continuous at every point of $\boldsymbol{R}$. Hence $f(x)$ is continuous at every point of $\boldsymbol{R}$.

In Problems 27 – 34, examine whether the given function is continuous at $x = 0$

27. $f(x) = \begin{cases} (1 + 3x)^{1/x} & \text{if } x \ne 0 \\ e^2 & \text{if } x = 0 \end{cases}$

Sol. $f(x) = (1 + 3x)^{1/x}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} [(1 + 3x)^{1/3x}]^3 = e^3$

$f(0) = e^2$

Since $e^3 \ne e^2$, $f(x)$ is discontinuous at $x = 0$

28. $f(x) = \begin{cases} (1 + x)^{1/x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1 + x)^{1/x} = e$ but $f(0) = 1$

Since $e \neq 1$, $f(x)$ is discontinuous $x = 0$

29. $f(x) = \begin{cases} (1+2x)^{1/x} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} (1+2x)^{1/x} = [\lim_{x \to 0} (1+2x)^{1/2x}]^2 = e^2$

And $f(0) = e^2$

Thus $f(x)$ is continuous at $x = 0$

30. $f(x) = \begin{cases} (e^{-1/x}) & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} e^{-1/x^2} = \lim_{x \to 0} \frac{1}{e^{1/x^2}} = 0$

But $f(0) = 1$, so $f(x)$ is discontinuous at $x = 0$

31. $f(x) = \begin{cases} \dfrac{e^{1/x}}{1+e^{1/x}} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. When $x \to 0^-$, $\frac{1}{x} \to -\infty$ and so $e^{1/x} \to e^{-\infty} = 0$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{e^{1/x}}{1+e^{1/x}} = \frac{0}{1+0} = 0$$

As $x \to 0^+$, $\frac{1}{x} \to \infty$ and so $e^{1/x} \to e^{\infty} = \infty$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{e^{1/x}}{1+e^{1/x}} = \frac{\infty}{\infty}$$

Thus $\lim_{x \to 0} f(x)$ does not exist.

Hence $f(x)$ is discontinuous at $x = 0$.

32. $f(x) = \begin{cases} \dfrac{e^{1/x^2}}{e^{1/x^2}-1} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{e^{1/x^2}}{e^{1/x^2}-1} = \lim_{x \to 0} \frac{1}{1 - \frac{1}{e^{1/x^2}}} = 1$

$f(0) = 1$

Thus $\lim_{x \to 0} f(x) = f(0)$ and so $f(x)$ is continuous at $x = 0$

33. $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{2x} . 2 = 1.2 = 2$

$f(0) = 1$

Since $\lim_{x \to 0} f(x) \neq f(0)$, $f(x)$ is discontinuous at $x = 0$

34. $f(x) = \begin{cases} \dfrac{\sin 3x}{\sin 2x} & \text{if } x \neq 0 \\ \dfrac{2}{3} & \text{if } x = 0 \end{cases}$

Sol. $\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot \frac{2x}{\sin 2x} \times \frac{3}{2}$

$= 1 \times \frac{3}{2} = \frac{3}{2}$

$f(0) = \frac{2}{3}$

Thus $\lim_{x \to 0} f(x) \neq f(0)$ and so $f(x)$ is discontinuous at $x = 0$.

35. Let $f(x) = x^2$ and $g(x) = \begin{cases} -4 & \text{if } x \leq 0 \\ |x-4| & \text{if } x > 0 \end{cases}$

Determine whether fog and gof are continuous at $x = 0$

Sol. $(fog)(x) = f(g(x))$

$= f(-4)$, if $x \leq 0$

$= f(|x-4|)$, if $x > 0$

Thus $(fog)(x) = 16$, if $x \leq 0$

$= (x-4)^2$, if $x > 0$

Now $\lim_{x \to 0^-} (fog)(x) = 16$

$\lim_{x \to 0^+} (fog)(x) = \lim_{x \to 0^+} (x-4)^2 = 16$

$(fog)(0) = 16$

Thus fog is continuous at $x = 0$

Again, $(gof)\,x = g(f(x)) = g(x^2)$

$= -4$, if $x^2 \leq 0$

$= |x^2 - 4|$, if $x^2 > 0$

$\lim_{x \to 0^-} (gof)(x) = -4$

$\lim_{x \to 0^+} (gof)(x) = \lim_{x \to 0^+} |x^2 - 4| = 4$

Thus $\lim_{x \to 0} (gof)(x)$ does not exist and so gof is discontinuous at $x=0$.