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Calculus With Analytic Geometry

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Calculus
With
Analytic Geometry

By
S.M Yusuf

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Chapter 7 PLANE CURVES II

Exercise Set 7.1 (Page 284)

Find equations of the asymptotes of the following curves:

1. $y = \dfrac{(x-2)^2}{x^2}$

Sol. The equation can be written as

$$x^2 y = (x-2)^2$$

or $\quad x^2(y-1) = -4x + 4 \qquad (1)$

We find the asymptotes parallel to coordinate axes.

The highest power of y in (1) is y and its coefficient is x^2.

Asymptotes parallel to the y-axis are

$$x^2 = 0$$

i.e., two coincident asymptotes $x = 0$.

Coefficient of highest power of x, *i.e.*, of x^2 in (1) is $y - 1$. Therefore, $y - 1 = 0$ is an asymptotes parallel to the x-axis.

Hence the required asymptotes are

$$x = 0 \quad \text{and} \quad y = 1.$$

2. $x^2 y^2 = 12(x-3)$

Sol. Asymptotes parallel to the y-axis are

$$x^2 = 0 \quad i.e., \quad x = 0$$

Asymptotes parallel to the x-axis are

$$y^2 = 0 \quad \text{or} \quad y = 0$$

3. $2xy = x^2 + 3$

Sol. Asymptote parallel to the y-axis is

$$x = 0.$$

There is no asymptote parallel to the x-axis.

For oblique asymptotes, equation of the curve can be written as

$$2xy - x^2 - 3 = 0$$

Here, putting $y = m$ and $x = 1$, we have

$$\phi_2(m) = 2m - 1 = 0 \text{ gives } m = \frac{1}{2}$$

$$\phi'_2(m) = 2$$

$$\phi_1(m) = 0$$

To find c we apply the formula

$$c\phi'_2(m) + \phi_1(m) = 0$$

or $2c = 0$ or $c = 0$

Hence an oblique asymptote is $y = \frac{1}{2}x$.

4. $x^2(x-y)^2 + a^2(x^2 - y^2) = a^2xy$ (1)

Sol. The equation is

$$x^4 + x^2y^2 + 2x^3 + a^2x^2 - a^2y^2 - a^2xy = 0$$

Coefficient of highest power of y *i.e.*, of y^2 in (1) is $x^2 - a^2$

Hence $x^2 - a^2 = 0$ or $x = \pm a$

are asymptote parallel to the y-axis.

There is no asymptote parallel to the x-axis.

Now, for oblique asymptotes,

$\phi_4(m) = 1 + m^2 - 2m = 0$ gives $m = 1, 1$

$\phi'_4(m) = 2m - 2$

$\phi''_4(m) = 2$

$\phi_3(m) = 0$

$\phi'_3(m) = 0$

$\phi_2(m) = a^2(1 - m^2) - a^2m$

To find c, we apply the formula

$$\frac{c^2}{2!}\phi''_4(m) + c\,\phi'_3(m) + \phi_2(m) = 0$$

or $c^2 + a^2(1 - m^2) - a^2m = 0$

Putting $m = 1$, $c^2 = a^2$ or $c = \pm a$

Hence the oblique asymptotes are $y = x \pm a$.

5. $(x-y)^2(x^2+y^2) - 10(x-y)x^2 + 12y^2 + 2x + y = 0$ (1)

Sol. Here coefficient of x^4 is 1 and that of y^4 is 1 and so (1) has no asymptotes parallel to the coordinate axes.

For oblique asymptotes, we have

$\phi_4(m) = (1-m)^2(1+m^2) = 0$ gives $m = 1, 1$

and the other two values of m are imaginary.

$\phi_4(m) = (1 - 2m + m^2)(1 + m^2)$

$= m^4 - 2m^3 + 2m^2 - 2m + 1$

$\phi'_4(m) = 4m^3 - 6m^2 + 4m - 2$

$\phi''_4(m) = 12m^3 - 12m + 4$

$\phi_3(m) = -10(1-m) = 10m - 10$

$\phi'_3(m) = 10$

$\phi_2(m) = 12m^2$

To find c, we use

$$\frac{c^2}{2!}\phi''_4(m) + c\phi'_3(m) + \phi_2(m) = 0$$

or $c^2(6m^2 - 6m + 2) + 10c + 12m^2 = 0$

Putting $m = 1$, the above equation becomes

$2c^2 + 10c + 12 = 0$

or $c^2 + 5c + 6 = 0$

or $(c+2)(c+3) = 0$

or $c = -2, -3$

Hence the asymptotes are

$y = x - 2, y = x - 3$

6. $x^2y + xy^2 + xy + y^2 + 3x = 0$ (1)

Sol. Coefficient of the highest power of y in (1) is $x + 1$. Therefore,

$x + 1 = 0$

is an asymptote parallel to y-axis

For oblique asymptotes, we have

$\phi_3(m) = m + m^2 = 0$ gives $m = 0, -1$

$\phi'_3(m) = 1 + 2m$

$\phi_2(m) = m + m^2$

To find c, we apply the formula

$c\phi'_3(m) + \phi_2(m) = 0$ or $c(1 + 2m) + m + m^2 = 0$

(2)

Putting $m = 0$ in (2), we have $c = 0$.

Thus $y = 0$ is an asymptote parallel to the x-axis.

Putting $m = -1$ in (2), we get

$c(-1) - 1 + 1 = 0$ or $c = 0$

Therefore, $y = -x$ is an oblique asymptote.

7. $(x - y + 1)(x - y - 2)(x + y) = 8x - 1$

Sol. The equation can be written as

$[(x-y)^2 - (x-y) - 2](x+y) = 8x - 1$

or $(x-y)^2(x+y) - (x-y)(x+y) - 2(x+y) - 8x + 1 = 0$

or $(x-y)^2(x+y) - (x-y^2) - 10x - 2y + 1 = 0$

There are no asymptotes parallel to the coordinate axes.

Here $\phi_3(m) = (1-m)^2(1+m) = 0$ gives $m = 1, 1, -1$

$\phi_3(m) = (1 - 2m + m^2)(1 + m) = m^3 - m^2 - m + 1$

$\phi'_3(m) = 3m^2 - 2m - 1$

$\phi''_3(m) = 6m - 2$

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$\phi_2(m) = -1 + m^2$

$\phi_1(m) = -10 - 2m$

To find c when $m = -1$, we use

$c\phi'_3(m) + \phi_2(m) = 0$

or $c(3m^2 - 2m - 1) + (-1 + m^2) = 0$ (1)

Putting $m = -1$ in (1), we get $c(3 + 2 - 1) + 0 = 0$

or $4c = 0$ or $c = 0$

Therefore, $y = -x$ is an asymptote

To find c for $m = 1, 1$, we apply the formula

$$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$$

or $\frac{c^2}{2}(6m - 2) + c(2m) - 10 - 2m = 0$

Putting $m = 1$, the above equation becomes

$$\frac{c^2}{2}(4) + 2c - 12 = 0$$

or $c^2 + c - 6 = 0$ or $(c + 3)(c - 2) = 0$

Thus $c = 2, -2$

Hence $y = x + 2$ and $y = x - 3$ are asymptotes.

8. $y^3 + x^2y + 2xy - y + 1 = 0$

Sol. Here $\phi_3(m) = m^3 + m + 2m^2$

$\phi_3(m) = 0$ gives

$m^3 + 2m^2 + m = 0$

or $m(m^2 + 2m + 1)$

i.e., $m = 0, \ -1, -1$

$\phi'_3(m) = 3m^2 + 1 + 4m$

$\phi''_3(m) = 6m + 4$

$\phi_2(m) = 0, \phi'_2(m) = 0, \phi_1(m) = -m$

To find c, when m has two equal values, we use

$$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$$

or $\frac{c^2}{2}(6m + 4) + 0 - m = 0$ or $c^2(3m + 2) - m = 0$ (1)

Putting $m = -1$ in (1), we have $-c^2 + 1 = 0$

or $c^2 = 1$ or $c = \pm 1$

Asymptotes parallel to each other are

$y = -x + 1$ and $y = -x - 1$

To find c when $m = 0$, we apply the formula

$c\phi'_3(m) + \phi_2(m) = 0$

or $c(3m^2 + 1 + 4m) + 0 = 0$ *i.e.*, $c = 0$

Therefore $y = 0$ is an asymptote.

Hence the required asymptotes are

$y = 0, y + x = \pm 1$

9. $y(x - y)^2 = x + y$

Sol. The given equation can be written as

$y(x - y)^2 - x - y = 0$

Here $\phi_3(m) = m(1 - m)^2$

$\phi_3(m) = 0$ gives $m = 1, 1, 0$

Again, $\phi_3(m) = m(m^2 - 2m + 1) = 3m^2 - 2m^2 + m$

$\phi'_3(m) = 3m^2 - 4m + 1$

$\phi''_3(m) = 6m - 4$

$\phi_2(m) = 0, \phi'_2(m) = 0$

$\phi_1(m) = -1 - m$

To find c, when $m = 1, 1$, we apply the formula

$$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$$

or $\frac{c^2}{2}(6m - 4) + 0 - 1 - m = 0$

or $c^2(3m - 2) - 1 - m = 0$ (1)

Putting $m = 1$ in (1), we have $c^2 - 2 = 0$

or $c = \pm\sqrt{2}$

The corresponding asymptotes are

$y = x \pm \sqrt{2}$

To find c when $m = 0$, we use

$c\phi'_3(m) + \phi_2(m) = 0$

or $c(3m^2 - 4m + 1) = 0$ *i.e.*, $c = 0$

The corresponding asymptote is $y = 0$

10. $x^2y^2(x^2 - y^2)^2 = (x^2 + y^2)^3$ (1)

Sol. Coefficient of the highest power of x *i.e.*, of x^6 in (1) is

$y^2 - 1$

and coefficient of the highest power of y *i.e.*, of y^6 in (1) is

$x^2 - 1$

Thus asymptotes parallel to the coordinate axes are

$x = \pm 1, \quad y = \pm 1$

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For oblique asymptotes, we have

$\phi_8(m) = m^2(1-m^2)^2$

$\phi_8(m) = 0$, gives $m = 0, 0, 1, 1, -1, -1$.

Again, $\phi_8(m) = m^2(m^4 - 2m^2 + 1)$

$= m^6 - 2m^4 + m^2$

$\phi'_8(m) = 6m^5 - 8m^3 + 2m$

$\phi''_8(m) = 30m^4 - 2m + 2$

$\phi_7(m) = 0$

$\phi'_7(m) = 0$

$\phi_6(m) = -(1+m^2)^3$

To find c for equal values of m, we apply the formula.

$$\frac{c^2}{2!}\phi''_8(m) + c\phi'_7(m) + \phi_6(m) = 0$$

or $\frac{c^2}{2!}(30m^4 - 24m^2 + 2) - (1+m^2)^3 = 2$ (2)

(i) When $m = 0, 0$, (2) gives $c^2 - 1 = 0$ or $c = \pm 1$

Hence $y = \pm 1$ and are the asymptotes which have already been determined.

(ii) When $m = 1, 1$, (2) gives

$\frac{c^2}{2}(30 - 24 + 2) - (1+1)^3 = 0$ or $4c^2 = 8$

or $c = \pm\sqrt{2}$

Thus $y = x \pm \sqrt{2}$ are asymptotes.

(iii) When $m = -1, -1$, (2) gives

$\frac{c^2}{2!}(30 - 24 + 2) - 8 = 0$ or $c = \pm\sqrt{2}$

Therefore, the asymptotes are $y = -x \pm \sqrt{2}$

11. $xy^2 = (x+y)^2$

Sol. The given equation can be written as

$xy^2 - (x + 2xy + y^2) = 0$ (1)

Coefficient of the highest power of y *i.e.*, of y^2 in (1) is $x - 1$.

Thus $x - 1 = 0$ is an asymptote parallel to the y-axis.

Now $\phi_3(m) = m^2 = 0$ gives $m = 0, 0$

$\phi'_3(m) = 2m$

$\phi_2(m) = -(1 + 2m + m^2)$

For $m = 0$, $\phi'_3(m) = 0$ but $\phi_2(m) = -1$.

Thus $c\phi'_3(m) + \phi_2(m) = 0$ is not an identity an so there is no asymptotes corresponding to $m = 0$.

12. $xy^2 - x^2y - 3x^2 - 2xy + y^2 + x - 2y + 1 = 0$ (1)

Sol. Highest power of y in (1) is y^2 and its coefficient is

$x + 1$

An asymptote parallel to the y-axis is $x + 1 = 0$

For oblique asymptotes, we have

$\phi_3(m) - m = 0$ gives

$m(m-1) = 0$ or $m = 0, m = 1$

$\phi'_3(m) = 2m - 1$

$\phi_2(m) = -3 - 2m + m^2$

To find c we apply the formula

$c\phi'_3(m) + \phi_2(m) = 0$

i.e., $c(2m-1) - 3 - 2m + m^2 = 0$ (2)

(i) When $m = 0$, from (2), we get $-c - 3 = 0$ or $c = -3$

Thus $y = -3$ is an asymptote.

(ii) When $m = 1$, (2) gives $c(1) - 3 - 2 + 1 = 0$ or $c = 4$.

$y = x + 4$

13. $r = \frac{a}{\theta}$

Sol. When $r = \infty$, $\theta = 0$. There can be only one asymptote to the curve. Differentiating (1) w.r.t. r, we have

$$1 = -\frac{a}{\theta^2}\frac{d\theta}{dr} \quad \text{or} \quad \frac{d\theta}{dr} = -\frac{\theta^2}{a}$$

Therefore, $\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{a^2}{\theta^2}\cdot\frac{-\theta^2}{a} = -a\ (= p)$.

Equation of an asymptote is

$p = r\sin(\alpha - \theta)$

i.e., $-a = r\sin(0 - \theta)$ or $a = r\sin\theta$

14. $r = \frac{a}{\sqrt{\theta}}$ (1)

Sol. Here $\theta = 0$ when $r = \infty$. Differentiating (1) w.r.t. r, we get

$$1 = \frac{-1}{2}a\theta^{-3/2}\frac{d\theta}{dr} \quad \text{or} \quad \frac{d\theta}{dr} = \frac{-2}{a}\theta^{3/2}$$

or $r^2\frac{d\theta}{dr} = \frac{a^2}{\theta}\left(\frac{-2}{a}\theta^{3/2}\right) = -2a\sqrt{\theta}$

$$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}(-2a\sqrt{\theta}) = 0\ (= p)$$

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Equation of an asymptote is

$$p = r\sin(\alpha - \theta) \quad i.e., \quad 0 = r\sin(0 - \theta)$$
$$\sin\theta = 0 \quad i.e., \quad \theta = 0$$

15. $r = a\cos\theta + b$ (1)

Sol. When $r = \infty, \theta = 0, \ \pi, 2\pi, 3\pi ...$

Differentiating (1) w.r.t. r, we have

$$1 = -a\csc\theta\cot\theta\frac{d\theta}{dr} = -\frac{a\cos\theta}{\sin^2\theta}\frac{d\theta}{dr}$$

or $$\frac{d\theta}{dr} = -\frac{\sin^2\theta}{a\cos\theta}$$

or $$r^2\frac{d\theta}{dr} = -\left(\frac{a + b\sin\theta}{\sin\theta}\right)^2 \cdot \frac{\sin^2\theta}{a\cos\theta} = -\frac{(a + b\sin\theta)^2}{a\cos\theta}$$

$$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{-(a + b\sin\theta)^2}{a\cos\theta} = -a$$

And $$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{-(a + b\sin\theta)^2}{a\cos\theta} = a.$$

Thus the limiting value of $r^2\frac{d\theta}{dr}$ is $-a$ when $\theta \to 0, 2\pi, 4\pi$ etc. and its limiting value is a when $\theta \to \pi, 3\pi, 5\pi$ etc. Therefore, there are only two asymptotes to the curve whose equations are

$$-a = r\sin(0 - \theta) \quad \text{and} \quad a = r\sin(\pi - \theta).$$

But both of these equations yield the single equation

$$r\sin\theta = a$$

Hence there is only one asymptotes of the curve.

16. $r = 2a\sin\theta\tan\theta$ (1)

Sol. When $r = \infty, \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$

Differentiating (1) w.r.t. r, we have

$$1 = [2a\sin\theta\sec^2\theta + 2a\cos\theta\tan\theta]\frac{d\theta}{dr}$$

$$= \left[\frac{2a\sin\theta}{\cos^2\theta} + 2a\sin\theta\right]\frac{d\theta}{dr} = \frac{2a\sin\theta(1 + \cos^2\theta)}{\cos^2\theta}\frac{d\theta}{dr}$$

or $$\frac{d\theta}{dr} = \frac{\cos^2\theta}{2a\sin\theta(1 + \cos^2\theta)}$$

or $$r^2\frac{d\theta}{dr} = \frac{4a^2\sin^4\theta}{\cos^2\theta} \cdot \frac{\cos^2\theta}{2a\sin\theta(1 + \cos^2\theta)} = \frac{2a\sin^2\theta}{\cos^2\theta}$$

$$\lim_{\theta\to\pi/2} r^2\frac{d\theta}{dr} = \lim_{\theta\to\pi/2}\frac{2a\sin^3\theta}{1 + \cos^2\theta} = 2a$$

And $$\lim_{\theta\to 3\pi/2} r^2\frac{d\theta}{dr} = \lim_{\theta\to 3\pi/2}\frac{2a\sin^2\theta}{1 + \cos^2\theta} = -2a$$

Thus $\lim r^2\frac{d\theta}{dr} = 2a$ when $\theta \to \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}$ etc.

$\lim r^2\frac{d\theta}{dr} = -2a$ when $\theta \to \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}$ etc.

Equations of the asymptotes are

$$2a = r\sin\left(\frac{\pi}{2} - \theta\right) \quad \text{and} \quad -2a = r\sin\left(\frac{3\pi}{2} - \theta\right)$$

i.e., $2a = r\cos\theta$ and $-2a = -r\cos\theta$.

Thus there is only one asymptote whose equation is $r\cos\theta = 2a$.

17. $r\sin 2\theta = a\cos 3\theta$

Sol. Here $$r = \frac{a\cos 3\theta}{\sin 2\theta}$$ (1)

When $r = \infty, 2\theta = 0, \pi, 2\pi, 3\pi, ...$

i.e., $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi, ...$

Differentiating (1) w.r.t. r, we get

$$1 = \frac{\sin 2\theta(-3a\sin 3\theta) - a\cos 3\theta(2\cos 2\theta)}{\sin^2 2\theta}\frac{d\theta}{dr}$$

Therefore, $$\frac{d\theta}{dr} = \frac{-\sin^2 2\theta}{3a\sin 2\theta\sin 3\theta + 2a\cos 2\theta\cos 3\theta}$$

$$r^2\frac{d\theta}{dr} = \frac{-a^2\cos^2 3\theta}{3a\sin 2\theta\sin 3\theta + 2a\cos 2\theta\cos 3\theta}$$

$\lim r^2 . \frac{d\theta}{dr} = 0,$ when $\theta \to \frac{\pi}{2}, \frac{3\pi}{2}$ etc.

$\lim r^2\frac{d\theta}{dr} = \frac{-a^2}{2a}$ when $\theta \to 0, 2\pi, 4\pi$ etc.

$$= \frac{-a}{2}$$

$$\lim_{\theta\to\pi, 3\pi, \text{etc}} r^2\frac{d\theta}{dr} = \frac{-a^2}{-2a} = \frac{a}{2}$$

The asymptotes are

(i) $0 = r\sin\left(\frac{\pi}{2} - \theta\right)$

(ii) $\frac{-a}{2} = r\sin(0 - \theta)$

(iii) $\frac{a}{2} = r\sin\sin(\pi - \theta).$

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On simplification, we have

(*i*) as $r\cos\theta = 0$ *i.e.*, $\theta = \frac{\pi}{2}$

(*ii*) as $\frac{a}{2} = r\sin\theta$ and

(*iii*) as $\frac{a}{2} = r\sin\theta$

Thus, there are only two distinct asymptotes viz:

$$\theta = \frac{\pi}{2} \quad \text{and} \quad \frac{a}{2} = r\sin\theta.$$

18. $r = \dfrac{a}{1-\cos\theta}$ (1)

Sol. When $r = \infty$, $\theta = 0, 2\pi, 4\pi$ etc.

Differentiating (1) w.r.t. r, we have

$$1 = \frac{-a\sin\theta}{(1-\cos\theta)^2}\cdot\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-(1-\cos\theta)^2}{a\sin\theta}$

$$r^2\frac{d\theta}{dr} = \frac{a^2}{(1-\cos\theta)^2}\cdot\frac{-(1-\cos\theta)^2}{a\sin\theta} = \frac{-a}{\sin\theta}$$

$$\lim_{\theta\to 0,\,2\pi,\,\text{etc.}} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0,\,2\pi,\,\text{etc.}}\left(\frac{-a}{\sin\theta}\right)$$

which diverges to $-\infty$.

Hence, there is no asymptote of the curve.

19. $r\sin n\theta = a$

Sol. $r = \dfrac{a}{\sin n\theta}$ (1)

When $r = \infty$, $n\theta = k\pi$, where k is an integer.

Differentiating (1) w.r.t. r, we get

$$1 = \frac{-na\cos n\theta}{\sin^2 n\theta}\cdot\frac{d\theta}{dr} \quad \text{or} \quad \frac{d\theta}{dr} = \frac{-\sin^2 n\theta}{na\cos n\theta}$$

$$r^2\frac{d\theta}{dr} = \frac{a^2}{\sin^2 n\theta}\left(\frac{-\sin^2 n\theta}{na\cos n\theta}\right) = \frac{-a}{n\cos n\theta}$$

$$\lim_{\theta\to k\pi/n} r^2\frac{d\theta}{dr} = \lim_{\theta\to k\pi/n}\frac{-a}{n\cos n\theta} = \frac{-a}{n}\sec k\pi,$$

Hence equation of the asymptote is

$$-\frac{a}{n}\sec k\pi = r\sin\left(\frac{k\pi}{n}-\theta\right) = -r\sin\left(\theta-\frac{k\pi}{n}\right)$$

i.e., $\dfrac{a}{n}\sec k\pi = r\sin\left(\theta-\dfrac{k\pi}{n}\right)$.

20. $r(e^\theta - 1) = a(e^\theta + 1)$

Sol. $r = \dfrac{a(e^\theta+1)}{e^\theta-1}$ (1)

Differentiating (1) w.r.t. r, we get

$$1 = \frac{a(e^\theta-1)e^\theta - a(e^\theta+1)e^\theta}{(e^\theta-1)^2}\cdot\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-(e^\theta-1)^2}{2ae^\theta}$

$$r^2\frac{d\theta}{dr} = \frac{a^2(e^\theta+1)^2}{(e^\theta-1)^2}\cdot\frac{-(e^\theta-1)^2}{2ae^\theta} = \frac{-a(e^\theta+1)^2}{2e^\theta}$$

$$\lim_{\theta\to 0} r^2\frac{d\theta}{dr} = \lim_{\theta\to 0}\frac{-a(e^\theta+1)^2}{2e^\theta}, \text{ since } r\to\infty \text{ when } \theta\to 0$$

$$= \frac{-4a}{2} = -2a$$

Equation of an asymptote is

$$-2a = r\sin(0-\theta) = -r\sin\theta$$

i.e., $r\sin\theta = 2a$,

21. $r^n\sin n\theta = a^n$

Sol. $r^n = \dfrac{a^n}{\sin n\theta}$ (1)

Here, if $r = \infty$ then $n\theta = k\pi$, where k is any integer.

Differentiating (1) w.r.t. r, we get

$$nr^{n-1} = \frac{-na^n\cos n\theta}{\sin^2 n\theta}\cdot\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-r^{n-1}\sin^2 n\theta}{a^n\cos n\theta}$

$$r^2\frac{d\theta}{dr} = \frac{-r^{n+1}\sin^2 n\theta}{a^n\cos n\theta} = \frac{-a^n}{\sin n\theta}\cdot\frac{a}{\sin^{1/n} n\theta}\cdot\frac{\sin^2 n\theta}{a^n\cos n\theta}$$

$$= \frac{-a\sin^{(n-1)/n} n\theta}{\cos n\theta}$$

$$\lim_{\theta\to k\pi/n} r^2\frac{d\theta}{dr} = \lim_{\theta\to k\pi/n}\frac{-a\sin^{(n-1)/n} n\theta}{\cos n\theta} = 0.$$

Equation of an asymptote is $0 = r\sin\left(\dfrac{k\pi}{n}-\theta\right)$

i.e., $\dfrac{k\pi}{n}-\theta = 0$ or $\theta = \dfrac{k\pi}{n}$, where k is any integer.

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22. $r^2 \sin\theta = a^2 \cos 2\theta$

Sol. $\dfrac{r^2}{a^2} = \dfrac{\cos 2\theta}{\sin\theta}$ (1)

When $r = \infty$, $\theta = 0, \pi, 2\pi$, etc.

Differentiating (1) w.r.t. r, we get

$$\frac{2r}{a^2} = \frac{\sin\theta(-2\sin 2\theta) - \cos 2\theta\cos\theta}{\sin^2\theta}\frac{d\theta}{dr}$$

Therefore, $\dfrac{d\theta}{dr} = \dfrac{-2r\sin^2\theta}{a^2(2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)}$

$$r^2\frac{d\theta}{dr} = \frac{a^2\cos 2\theta}{\sin\theta}\times\frac{-2r\sin^2\theta}{a^2(2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta)}$$

$$= \pm\frac{2\cos 2\theta\sin\theta}{2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta}\times\sqrt{\frac{a^2\cos 2\theta}{\sin\theta}}$$

$$= \pm\frac{2a\cos 2\theta\sqrt{\sin\theta}\sqrt{\cos 2\theta}}{2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta}$$

Taking limits as $\theta \to 0, \pi, 2\pi$ etc.

$$\lim_{\theta\to 0\text{ etc.}} r^2\frac{d\theta}{dr} = \pm\lim_{\theta\to 0\text{ etc.}}\frac{2a(\cos 2\theta)^{3/2}\sqrt{\sin\theta}}{2\sin\theta\sin 2\theta + \cos\theta\cos 2\theta} = 0$$

Equation of an asymptote is

$0 = r\sin(0 - \theta)$ *i.e.*, $\theta = 0$.

Exercise Set 7.2 (Page 293)

Locate the points of relative extreme of each of the following curves (Problems 1 - 10):

1. $f(x) = 2x^3 - 15x^2 + 36x + 10$

Sol. Here $f(x) = 2x^3 - 15x^2 + 36x + 10$

$f'(x) = 6x2 - 30x + 36$

For points of maxima and minima

$f'(x) = 0$ gives $6(x^2 - 5x + 6) = 0$

or $(x-2)(x-3) = 0$ *i.e.*, $x = 2, x = 3$

$f''(x) = 12x - 30$

When $x = 2, f''(x) = 24 - 30 = -6 < 0$.

Hence f has relative maximum at $x = 2$.

When $x = 3, f''(x) = 36 - 30 = 6 > 0$

Therefore, f has relative minimum at $x = 3$.

2. $f(x) = 3x^4 - 4x^3 + 5$

Sol. $f'(x) = 12x^3 - 12x^2$

For extreme values,

$f'(x) = 0$ gives $12x^3 - 12x^2 = 0$

or $12x^2(x-1) = 0$ or $x = 0, x = 1$

$f''(x) = 36x^2 - 24x$

When $x = 1, f''(x) = 36 - 24 = 12 > 0$

Thus f has relative minimum at $x = 1$

When $x = 0, f''(x) = 0$. The second derivative test fails.

We apply **Theorem, 7.7**

For $x \in]-h, h[$, $f'(x)$ does not change sign, h being small. Thus $f'(x)$ has no relative extrema at $x = 0$.

3. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$

Sol. $f(x) = 12x^5 - 45x^4 + 40x^3 + 6$

$f'(x) = 60x^4 - 180x^3 + 120x^2$

For relative maximum and minimum

$f'(x) = 0 \Rightarrow 60x^4 - 180x^3 + 120x^2 = 0$

or $60x^2(x^2 - 3x + 2) = 0$ or $60x^2(x-2)(x-1) = 0$

i.e., $x = 0, 2, 1$

$f''(x) = 240x^3 - 540x^2 + 240x$

$= 60x(4x^2 - 9x + 4)$

When $x = 2, f''(x) = 240 > 0$.

Therefore, f has relative minimum at $x = 2$

When $x = 1, f''(x) = -60 < 0$.

Thus f has relative maximum at $x = 1$.

At $x = 0, f''(x) = 0$.

The second derivative test fails. We apply **Theorem 7.7.**

For $x \in]-h, h[$, $f(x)$ does not change sign.

Thus $f(x)$ has no extreme at $x = 0$.

4. $f(x) = (x-1)(x-2)(x-3)$

Sol. $f(x) = (x-1)(x^2 - 5x + 6) = x^3 - 6x^2 + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(x) = 0$ gives $3x^2 - 12x + 11 = 0$

or $x = \dfrac{12 \pm \sqrt{144 - 132}}{6} = \dfrac{12 \pm \sqrt{12}}{6} = \dfrac{6 \pm \sqrt{3}}{3} = 2 \pm \dfrac{1}{\sqrt{3}}$

$f''(x) = 6x - 12$

When $x = 2 + \dfrac{1}{\sqrt{3}}, f''(x) = 12 + 2\sqrt{3} - 12 = 2\sqrt{3} > 0$

Therefore, f has relative minimum at $x = 2 + \dfrac{1}{\sqrt{3}}$

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When $x = 2 - \frac{1}{\sqrt{3}}$, $f''(x) = 12 - 2\sqrt{3} - 12 = -2\sqrt{3} < 0$

Hence f has relative maximum at $x = 2 - \frac{1}{\sqrt{3}}$

5. $f(x) = \sin x \cos 2x$

Sol. $f(x) = \sin x \cos 2x$
$= \sin x\,(1 - 2\sin^2 x)$
$= \sin x - 2\sin^3 x$
$f'(x) = \cos x - 6\sin^2 x \cos x$
$= \cos x\,(1 - 6\sin^2 x)$

For maxima and minima, $f'(x) = 0$

$\Rightarrow \quad \cos x = 0 \quad$ or $\quad 1 - 6\sin^2 x = 0$

$\Rightarrow \quad x = \pm\frac{\pi}{2} \quad$ or $\quad 6\sin^2 x = 1 \quad$ or $\quad \sin x = \frac{1}{6}$

or $\quad \sin x = \pm\frac{1}{\sqrt{6}}$

$$f''(x) = -\sin x\,(1 - 6\sin^2 x) + \cos x\,(-12\sin x \cos x)$$

When $x = \frac{\pi}{2}$, $f''(x) = 5 > 0$ and so f has relative minimum at $x = \frac{\pi}{2}$

When $x = \frac{-\pi}{2}$, $f''(x) = -5 < 0$ and so f has relative maximum at

$$x = \frac{-\pi}{2}.$$

When $\sin x = \frac{1}{\sqrt{6}}$

$$f''(x) = \frac{1}{\sqrt{6}}(1-1) + \cos x\left(-12\frac{1}{\sqrt{6}}\right)(\cos x) = \frac{-12}{\sqrt{6}}\cos^2 x < 0$$

Thus f has relative maximum at $x = \arcsin\left(\frac{1}{\sqrt{6}}\right)$

When $\sin x = -\frac{1}{\sqrt{6}}$, we have

$$f''(x) = \cos x\left[-12\cos x.\left(-\frac{1}{\sqrt{6}}\right)\right] = \frac{12}{\sqrt{6}}\cos^2 x > 0$$

Therefore, f has relative minimum at $x = \arcsin\left(-\frac{1}{\sqrt{6}}\right)$.

6. $f(x) = a\sec x + b\csc x$, $(0 < a < b)$

Sol. $f(x) = a\sec x + b\csc x$, $(0 < a < b)$ (1)
$f'(x) = a\sec x\tan x - b\csc x\cot x$

$$= \frac{a}{\cos x} \times \frac{\sin x}{\cos x} - \frac{b}{\sin x}\cdot\frac{\cos x}{\sin x} = \frac{a\sin x}{\cos^2 x} - \frac{b\cos x}{\sin^2 x}$$

$$= \frac{a\sin^3 x - b\sin^3 x}{\sin^2 x\cos^2 x} \qquad (2)$$

For extreme values,

$f'(x) = 0 \quad$ gives $\quad a\sin^3 x - b\cos^3 x = 0$

or $\quad a\sin^3 x = b \quad \Rightarrow \quad \frac{\sin^3 x}{\cos^3 x} = \frac{b}{a}$

$\Rightarrow \quad \tan^3 x = \frac{b}{a} \quad \Rightarrow \quad \tan x = \left(\frac{b}{a}\right)^{1/3}$

$\tan x = \frac{b^{1/3}}{a^{1/3}} \quad$ *i.e.*, $\quad \sin x = \pm\frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$

and $\quad \cos x = \pm\frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$

Since $\tan x$ is +ve, both $\sin x$ and $\cos x$ have the same sign.
Differentiating (2), we get

$$f''(x) = \frac{3a\sin^2 x\cos x + 3b\cos^2 x\sin x}{\sin^2 x\cos^2 x}$$
$$+ \text{ term involving } (a\sin^3 x - b\cos^3 x)$$
$$= \frac{3\,(a\sin x + b\cos x)}{\sin x\cos x}$$
$$+ \text{ term involving } a\,(\sin^3 x - b\cos^3 x)$$

When $\sin x$ and $\cos x$ are positive
$f''(x) > 0$

Thus f has relative minimum when $\sin x = \frac{b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$

and $\cos x = \frac{a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$.

When both $\sin x$ and $\cos x$ are negative, $f''(x) < 0$
Thus f has relative at the point x where f maximum

$$\sin x = \frac{-b^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}} \quad \text{and} \quad \cos x = \frac{-a^{1/3}}{\sqrt{a^{2/3} + b^{2/3}}}$$

7. $f(x) = \sin x\cos^2 x$

Sol. $f'(x) = \cos^3 x - 2\sin^2 x\cos x$
$= \cos x\,(\cos^2 x - 2\sin^2 x)$
$= \cos x\,(1 - 3\sin^2 x)$

$f'(x) = 0 \quad$ gives $\quad \cos x = 0, \sin^2 x = \frac{1}{3}$

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$\Rightarrow \quad x = \pm\frac{\pi}{2}, \sin x = \pm\frac{1}{\sqrt{3}}$

$f''(x) = -\sin x\,(1 - 3\sin^2 x) + \cos x\,(-6 \sin x \cos x)$

When $x = \frac{\pi}{2}, f''(x) = 2 > 0$

Thus f has relative minimum at $x = \frac{\pi}{2}$.

When $x = \frac{-\pi}{2}$,

$f''(x) = -2 < 0$ and so f has relative maximum at $x = \frac{-\pi}{2}$.

When $\sin x = \frac{1}{\sqrt{3}}, f''(x) < 0$

and so f has relative maximum at $x = \arcsin\frac{1}{\sqrt{3}}$

When $\sin x = -\frac{1}{\sqrt{3}}, f''(x) > 0$

and so f has relative minimum at $x = \arcsin\left(\frac{-1}{\sqrt{3}}\right)$.

8. $f(x) = e^x \cos(x - a)$

Sol. $f'(x) = re^x \cos(x - a + \theta)$,

Where $r = \sqrt{2}, \theta = \frac{\pi}{4}$ *i.e.,* $f'(x) = \sqrt{2}\, e^x \cos\left(x - a + \frac{\pi}{4}\right)$

For extreme values, $\cos\left(x - a + \frac{\pi}{4}\right) = 0$ gives $x - a + \frac{\pi}{4} = \pm\frac{\pi}{2}$

or $\quad x = a + \frac{\pi}{4}, \qquad a - \frac{3\pi}{4}$

$f''(x) = 2e^x \cos\left(x - a + \frac{\pi}{2}\right)$

Now, when $x - a = \frac{\pi}{4}, f''(x) = -2e^{a+\frac{\pi}{4}}\frac{1}{\sqrt{2}} = -\sqrt{2}\, e^{a+\frac{\pi}{4}} < 0$

Thus f has relative maximum at $x = \frac{\pi}{4} + a$.

When $x - a = -\frac{3\pi}{4}, f''(x) = \sqrt{2}\, e^{a-\frac{3\pi}{4}} > 0$.

Therefore, f has relative minimum at $x = a - \frac{3\pi}{4}$

9. $f(x) = x^x$

Sol. Let $y = f(x) = x^x \qquad$ *i.e.,* $\quad y = x^x$

Taking logarithm, we have

$\ln y = x \ln x$

Differentiating, $\frac{1}{y}\frac{dy}{dx} = x \cdot \frac{1}{x} + \ln x \cdot 1 = 1 + \ln x$

or $\quad \frac{dy}{dx} = y\,(1 + \ln x) = x^x\,(1 + \ln x) \qquad (1)$

$\frac{dy}{dx} = 0 \quad$ gives $\quad x^x\,(1 + \ln x) = 0$

Since x^x cannot be zero, we have

$1 + \ln x = 0$

or $\quad \ln x = -1 \qquad$ or $\qquad x = e^{-1} = \frac{1}{e}$

Differentiating (1), we get

$\frac{d^2y}{dx^2} = x^x \cdot \frac{1}{x}$ + terms involving $(1 + \ln x)$

Putting $x = \frac{1}{e}, \frac{d^2y}{dx^2} > 0$, as the term involving $(1 + \ln x)$ will vanish at $x = \frac{1}{e}$.

Hence $f(x) = x^x$ has a relative minimum at $x = e^{-1}$.

10. $f(x) = \frac{\ln x}{x}, 0 < x < \infty$.

Sol. Let $\quad f(x) = y = \frac{\ln x}{x}$

$\frac{dy}{dx} = \frac{x \cdot \frac{1}{x} - \ln x \cdot 1}{x^2} = \frac{1 - \ln x}{x^2} \qquad (1)$

For maxima and minima, $\frac{dy}{dx} = 0 \quad$ gives $\quad 1 - \ln x = 0$

$\Rightarrow \quad \ln x = 1 \;\Rightarrow\; x = e$

Differentiating (1), we have

$\frac{d^2y}{dx^2} = \frac{1}{x^2}\left(-\frac{1}{x}\right) - \frac{2}{x^3}(1 - \ln x) = -\frac{1}{x^3} - \frac{2}{x^3}(1 - \ln x)$

When $x = e, \frac{d^2y}{dx^2} < 0$. Therefore, $x = e$ is a point of relative maximum.

11. Find the relative extreme of y if $r = 1 - \sin\theta$

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Sol. By **(6.29)**, we have

$$\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\dfrac{dr}{d\theta}}{-r\sin\theta + \cos\theta\dfrac{dr}{d\theta}} = \frac{(1-\sin\theta)\cos\theta - \sin\theta\cos\theta}{-\sin\theta(1-\sin\theta)-\cos^2\theta}$$

$$= \frac{\cos\theta(1-2\sin\theta)}{-\sin\theta(1-\sin\theta)-\cos^2\theta}$$

For extrema, $\frac{dy}{dx} = 0$ gives

$$\cos\theta(1-2\sin\theta) = 0$$

$$\Rightarrow \quad \cos\theta = 0 \text{ or } 1 - 2\sin\theta = 0 \quad \Rightarrow \quad \theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

Critical points are $\left(0, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{2}\right), \left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$

We use the first derivative test to check the nature of these critical points.

At $\theta = 91°, \frac{dy}{dx} = \frac{-0.01(1-2\times0.99)}{-0.99(1-0.99)-(-0.01)^2} = \frac{+\text{ive}}{-\text{ive}} = -\text{ive}$

$\theta = 89°, \frac{dy}{dx} = \frac{0.01(1-2\times0.99)}{-0.99(1-0.99)-(0.01)^2} = \frac{-\text{ive}}{-\text{ive}} = +\text{ive}$

Relative minimum at $\left(0, \frac{\pi}{2}\right)$.

At $\theta = 269°, \frac{dy}{dx} = \frac{-0.01(1-2\times(-0.99))}{0.99(1+0.99)-(0.01)^2} = \frac{-\text{ive}}{+\text{ive}} = -\text{ive}$

$\theta = 271°, \frac{dy}{dx} = \frac{0.01(1-2(-0.99))}{0.99(1+0.99)-(0.01)^2} = \frac{+\text{ive}}{+\text{ive}} = +\text{ive}$

Relative minimum at $\left(2, \frac{3\pi}{2}\right)$.

At $\theta = 151°, \frac{dy}{dx} = \frac{-0.87(1-2\times0.48)}{-0.48(1-0.48)-(-0.87)^2} = \frac{-\text{ive}}{-\text{ive}} = +\text{ive}$

$\theta = 149°, \frac{dy}{dx} = \frac{-0.85(1-2\times0.51)}{-0.51(1-0.51)-(-0.85)^2} = \frac{+\text{ive}}{-\text{ive}} = -\text{ive}$

Relative maximum at $\left(\frac{1}{2}, \frac{5\pi}{6}\right)$.

At, $\theta = 31°, \frac{dy}{dx} = \frac{0.85(1-2\times0.51)}{-0.51(1-0.51)-(0.85)^2} = \frac{-\text{ive}}{-\text{ive}} = +\text{ive}$

$\theta = 29°, \frac{dy}{dx} = \frac{0.87(1-2\times0.48)}{-0.48(1-0.48)-(0.87)^2} = \frac{+\text{ive}}{-\text{ive}} = -\text{ive}$

Relation maximum at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$.

Thus y has relative maximum at $\left(\frac{1}{2}, \frac{\pi}{6}\right), \left(\frac{1}{2}, \frac{5\pi}{6}\right)$ and relative minimum at $\left(0, \frac{\pi}{2}\right), \left(2, \frac{3\pi}{2}\right)$.

12. Find the point on the straight line $2x - 7y + 5 = 0$ that is closest to the origin.

Sol. Let $P(a, b)$ be a point on the line

$$2x - 7y + 5 = 0 \qquad (1)$$

The distance p of $P(a, b)$ from the origin is

$$p = \sqrt{a^2 + b^2} \qquad (2)$$

Since $P(a, b)$ lies on (1), we have

$$2a - 7b + 5 = 0 \qquad (3)$$

The point $P(a, b)$ is closest to the origin if the distance p is minimum.

From (3), we get $b = \frac{(2a+5)}{7}$

Substituting this value of b into (2) and after squaring, we obtain

$$49p^2 = 53a^2 + 20a + 25$$

Differentiating w.r.t. a, we have

$$98p\frac{dp}{da} = 106a + 20 \text{ or } \frac{dp}{da} = \frac{1}{98p}(106a + 20)$$

$$\frac{dp}{da} = 0 \qquad \text{implies } a = \frac{-10}{53}$$

$$\frac{d^2p}{da^2} = -\frac{1}{98p^2}\frac{dp}{da}(106a + 20) + \frac{1}{98p} \times 106$$

p is minimum for $a = -\frac{10}{53}$, since $\frac{d^2p}{da^2}$ is positive at this point.

When $a = -\frac{10}{53}, b = \frac{245}{371} = \frac{35}{53}$

Hence $P\left(\frac{-10}{53}, \frac{35}{53}\right)$ is the required point.

13. Find the extrema of the radii vectors of the curve

$$\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta}; a > 0, b > 0.$$

Sol. $\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta} \qquad (1)$

Differentiating both sides w.r.t. θ, we get

$$-\frac{2c^4}{r^3}\frac{dr}{d\theta} = -\frac{2a^2}{\sin^3\theta}\cos\theta + \frac{2b^2\sin\theta}{\cos^3\theta}$$

$$\frac{dr}{d\theta} = 0 \text{ gives } -\frac{2a^2\cos\theta}{\sin^3\theta} + \frac{2b^2\sin\theta}{\cos^3\theta} = 0$$

or $\quad a^2\cos^4\theta = b^2\sin^4\theta$ or $\dfrac{\sin^4\theta}{\cos^4\theta} = \dfrac{a^2}{b^2}$ or $\tan^2\theta = \dfrac{a}{b}$

$$\sin^2\theta = \frac{a}{a+b} \text{ and } \cos^2\theta = \frac{b}{a+b}$$

Now $\dfrac{dr}{d\theta} = \dfrac{r^3}{c^4}\left(\dfrac{a^2\cos\theta}{\sin^3\theta} - \dfrac{b^2\sin\theta}{\cos^3\theta}\right) = \dfrac{r^3}{c^4}\,\dfrac{(a^2\cos^4\theta - b^2\sin^4\theta)}{\sin^3\theta\cos^3\theta}$

$$= \frac{r^3}{c^4\sin^3\theta\cos^3\theta}(a^2\cos^4\theta - b^2\sin^4\theta)$$

Differentiating it again, we have

$$\frac{d^2r}{d\theta^2} = \frac{r^4}{c^4\sin^3\theta\cos^3\theta}\begin{pmatrix}(-4a^2\cos^3\theta\sin\theta - 4b^2\sin^3\theta\cos\theta)\\ +\text{ terms involving } (a^4\cos^4\theta - b^2\sin^4\theta)\end{pmatrix}$$

$$= \frac{-4r^4}{c^4\sin^2\theta\cos^2\theta}(a^2\cos^2\theta + b^2\sin^2\theta) + \ldots$$

$$= \frac{-4c^4\sin^2\theta\cos^2\theta}{a^2\cos^2\theta + b^2\sin^2\theta} + \ldots, \qquad \text{using (1)}$$

$$= -\text{ive, when } \sin^2\theta = \frac{a}{a+b},\ \cos^2\theta = \frac{b}{a+b}$$

Thus r is maximum for this value of θ.

The maximum value of r is given by $\dfrac{c^4}{r^2} = \dfrac{a^2}{\dfrac{a}{a+b}} + \dfrac{b^2}{\dfrac{b}{a+b}}$

or $\quad \dfrac{c^4}{r^2} = a(a+b) + b(a+b) = (a+b) + b(a+b) = (a+b)^2$

or $\quad \dfrac{r^2}{c^4} = \dfrac{1}{(a+b)^2}$ or $r = \dfrac{c^2}{a+b}$.

Find the points of inflection of each of the following curves (Problems 14 – 17):

14. $y = \dfrac{x^3 - x}{3x^2 + 1}$

Sol. $y = \dfrac{x^3 - x}{3x^2 + 1} \qquad (1)$

$$= \frac{1}{3}x - \frac{\frac{4}{3}x}{3x^2+1} = \frac{1}{3}x - \frac{4}{3}\,\frac{x}{3x^2+1}$$

$$\frac{dy}{dx} = \frac{1}{3} - \frac{4}{3}\,\frac{3x^2 + 1 - x(6x)}{(3x^2+1)^2} = \frac{1}{3} - \frac{4}{3}\,\frac{(1-3x^2)}{(3x^2+1)^2}$$

$$\frac{d^2y}{dx^2} = -\frac{4}{3}\left[\frac{(3x^2+1)^2(-6x) - (1-3x^2).\,2(3x^2+1)\,6x}{(3x^2+1)^4}\right]$$

$$= -\frac{4}{3}\left[\frac{-6x(3x^2+1) - 12x(1-3x^2)}{(3x^2+1)^2}\right]$$

$$= -\frac{4}{3}\left[\frac{-18x^3 - 6x - 12x + 36x^3}{(3x^2+1)^3}\right]$$

$$= -\frac{4}{3}\left[\frac{18x^3 - 18x}{(3x^2+1)^2}\right] = -24\,\frac{(x^3 - x)}{(3x^2+1)^2}$$

$$= -\frac{24x(x^2-1)}{(3x^2+1)^2} \qquad (2)$$

From (1) and (2), the possible points of inflection are $(0, 0), (1, 0), (-1, 0)$

If $\quad x < 0, \dfrac{d^2y}{dx^2} < 0$ and if $x > 0, \dfrac{d^2y}{dx^2} > 0$, ($x$ is small).

Thus $(0, 0)$ is a point of inflection.

Similarly, we find that $(1, 0)$ and $(-1, 0)$ are points of inflection.

15. $x = (y-1)(y-2)(y-3)$

Sol. $\dfrac{dx}{dy} = (y-2)(y-3) + (y-1)(y-3) + (y-1)(y-2)$

$$= (y^2 - 5y + 6) + (y^2 - 4y + 3) + (y^2 - 3 + 2)$$

$$= 3y^2 - 12y + 11$$

$$\frac{d^2x}{dy^2} = 6y - 12$$

$\dfrac{d^2x}{dy^2} = 0$ gives $y = 2$

When $y = 2, x = 0$

Thus $(0, 2)$ is a possible point of inflection

If $y < 2, \dfrac{d^2x}{dy^2} < 0$ and if $y > 2, \dfrac{d^2x}{dy^2} > 0$. Thus $(0, 2)$ is a point if inflection.

16. $y^2 = x(x+1)^2 \qquad (1)$

Sol. Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = (x+1)^2 + x\,.\,2(x+1)$$

$$= (x+1)(x+1+2x) = (x+1)(3x+1)$$

$$\frac{dy}{dx} = \frac{(x+1)(3x+1)}{2y} = \frac{(x+1)(3x+1)}{2\sqrt{x}\,(x+1)}$$

$$= \frac{1}{2}\left(\frac{3x+1}{\sqrt{x}}\right) = \frac{1}{2}\left(3\sqrt{x} + x^{-1/2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left[\frac{3}{2}x^{-1/2} - \frac{1}{2}x^{-3/2}\right] = \frac{1}{4}x^{-1/2}\left(3 - x\,x^{-1}\right)$$

$$\frac{d^2y}{dx^2} = 0 \qquad \text{gives} \quad x = \frac{1}{3}$$

Putting $x = \frac{1}{3}$ in (1), we have

$$y^2 = \frac{1}{3}\left(\frac{4}{3}\right)^2 = \frac{16}{27}\ ; \quad y = \pm\frac{4}{3\sqrt{3}}$$

Thus possible points of inflection are

$$\left(\frac{1}{3}, \frac{4}{3\sqrt{3}}\right) \text{ and } \left(\frac{1}{3}, -\frac{4}{3\sqrt{3}}\right)$$

If $\quad x < \frac{1}{3}, \frac{d^2y}{dx^2} < 0 \quad$ and if $\ x > \frac{1}{3}, \frac{d^2y}{dx^2} > 0.$

Thus $x = \frac{1}{3}$ gives point of inflection.

Hence $\left(\frac{1}{3}, \pm\frac{4}{3\sqrt{3}}\right)$ are points if inflection.

17. $a^2y^2 = x^2(a^2 - x^2)$ (1)

Sol. Differentiating (1) w.r.t. x, we get

$$2a^2y\frac{dy}{dx} = 2x(a^2 - x^2) - 2x^3$$

$$= 2a^2x - 2x^3 - 2x^3 = 2x(a^2 - 2x^2)$$

or $$\frac{dy}{dx} = \frac{2x(a^2 - 2x^2)}{2a^2 \cdot y} = \frac{x(a^2 - 2x^2)}{a \cdot x\sqrt{a^2 - x^2}} = \frac{a^2 - 2x^2}{a\sqrt{a^2 - x^2}}$$

$$a\frac{d^2y}{dx^2} = \frac{\sqrt{a^2 - x^2}(-4x) - (a^2 - 2x^2) \times -x\sqrt{(a^2 - x^2)}}{(a^2 - x^2)}$$

$$= \frac{(a^2 - x^2)(-4x) - x(a^2 - 2x^2)}{(a^2 - x^2)^{3/2}}$$

$$= \frac{-4a^2x + 4x^3 - a^2x + 2x^3}{(a^2 - x^2)^{3/2}}$$

$$= \frac{6x^3 - 5a^2x}{(a^2 - x^2)^{3/2}} = \frac{x(6x^2 - 5a^2)}{(a^2 - x^2)^{3/2}}$$

$\frac{d^2y}{dx^2} = 0$ gives $x = 0$ and so $y = 0$.

Therefore, the possible point of inflection is (0, 0).

For $\ x < 0$ and $x > 0$, $\frac{d^2y}{dx^2}$ changes signs. Therefore, (0, 0) is a point of inflection.

18. Find a and b so that the function f given by $f(x) = ax^3 + bx^2$ has (1, 6) as a point of inflection.

Sol. $f(x) = ax^3 + bx^2$

$f'(x) = 3ax^2 + 2bx$

$f''(x) = 6ax + 2b$

Since (1, 6) is a point of inflection

$f''(1) = 6a + 2b = 0$

or $\quad b = -a$ (1)

Moreover, $6 = f(1) = a + b$ (2)

Solving (1) and (2) simultaneously, we get

$a = -3 \quad$ and $\quad b = 9.$

19. Find the intervals in which the curves $y = 3x^5 - 40x^3 + 3x - 20$ faces (*i*) upward (*ii*) downward. Also find the points of inflection.

Sol. $y = 3x^5 - 40x^3 + 3x - 20$

$$\frac{dy}{dx} = 15x^4 - 120x^2 + 3$$

$$\frac{d^2y}{dx^2} = 60x^3 - 240x^2 = 60x(x^2 - 4)$$

Now $\frac{d^2y}{dx^2} = 0$, when $x = 0, 2, -2$

Hence we consider the intervals

$]-\infty, -2[$, $]-2, 0[$, $]0, 2[$ and $[2, \infty[$

In the interval $]-\infty, -2[$ *i.e.*, when $x < -2$

$\frac{d^2y}{dx^2} < 0$ and so the curve is concave downward

In the interval $]-2, 0[$, *i.e.*, for $-2 < x < 0$

$\frac{d^2y}{dx^2} > 0$ and the curve is concave upward.

In the interval $]0, 2[$, $\frac{d^2y}{dx^2} < 0$ and so the curve is concave downward. In the interval $]2, \infty[$ *i.e.*, for $2 < x < \infty$.

$\frac{d^2y}{dx^2} > 0$, and so the curve is concave upward.

The possible points of inflection are

(–2, 198), (0, –20), (2, 238).

From the concavity of the curve, we find that these are all points of inflection.

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20. Find the intervals in which the curve $y = (x^2 + 4x + 5)\,e^{-x}$ faces upward or downward. Also find its points of inflection.

Sol. $y = (x^2 + 4x + 5)\,e^{-x}$

$$\frac{dy}{dx} = -(x^2 + 4x + 5)e^{-x} + e^{-x}(2x + 4)$$
$$= -(x^2 + 2x + 1)e^{-x}$$
$$\frac{d^2y}{dx^2} = (x^2 + 2x + 1)e^{-x} - e^{-x}(2x + 2)$$
$$= (x^2 - 1)e^{-x}$$

Now $\frac{d^2y}{dx^2} = 0$ if $x = 1, -1$.

$\frac{d^2y}{dx^2} > 0$ for $x > 1$ and $x < -1$

Thus the curve is concave up in $]1, \infty[$ and $]-\infty, -1[$.

For $-1 < x < 1$, $\frac{d^2y}{dx^2} < 0$. Hence the curve faces down in $]-1, 1[$. The possible points of inflection are at $x = -1, 1$.

i.e., the possible points of inflection are $\left(1, \frac{10}{e}\right)$ and $(-1, 2e)$.

For $x < 1$ $\frac{d^2y}{dx^2} < 0$ and for $x > 1$, $\frac{d^2y}{dx^2} > 0$

Thus $\frac{d^2y}{dx^2}$ changes sign when passing though the point $\left(1, \frac{10}{e}\right)$ and so this is a point of inflection.

Similarly, for $x < -1$, $\frac{d^2y}{dx^2} > 0$ and for $x > -1$, $\frac{d^2y}{dx^2} < 0$.

Therefore, $\frac{d^2y}{dx^2}$ changes sign when passing though the point $(-1, 2e)$ and this is also a point of inflection.

21. Use calculus to show that $5x^2 - 20x + 81 > 0$ for all x.

Sol. $f(x) = 5x^2 - 20x + 81$

$f'(x) = 10x - 20$

$f''(x) = 10$

$f(x)$ has a stationary point at $x = 2$, $f(2) = 61$ *i.e.*, $(2, 61)$ is a critical point of $f(x)$. Since $f''(x) = 10 > 0$, $f(x)$ has absolute minimum at the point $(2, 61)$. The curve is concave up and so $f(x) > 0$ for all x.

22. Show that $x^4 - 4x^3 + 12x^2 + 40 > 0$ for all x.

Sol. $f(x) = x^4 - 4x^3 + 12x^2 + 40$

$f'(x) = 4x^3 - 12x^2 + 24x$

$= 4x(x^2 - 3x + 6)$

$f'(x) = 0 \Rightarrow x = 0$ or $x^2 - 3x + 6 = 0$

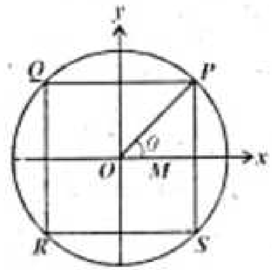
The equation $x^2 - 3x + 6 = 0$ has imaginary roots. Thus $f(x)$ has an extreme value at $x = 0$.

$f''(x) = 12x^2 - 24x + 24$

$f''(0) = 24 > 0$

Thus $f(x)$ has absolute minimum at $x = 0$. The curve is concave up and so $f(x)$ is always positive.

23. Find the dimensions of the rectangle of maximum area that can be inscribed in a circled of radius r.



Sol. Let the length PQ and breath QR of the inscribed rectangle be $2x$ and $2y$ respectively. Suppose O is the centre of the circle and OP makes an angle θ with the line Ox parallel to PQ.

Then $|OM| = x$ and $|PM| = y$.

From the right triangle OMP, we have

$x = OP\cos\theta = r\cos\theta$,

$y = OP\sin\theta = r\sin\theta$,

where $OP = r$ is the radius of the circle. Area A of the rectangle $PQRS$ is

$A = 4xy = 4r^2\sin\theta\cos\theta$.

This area is a function of θ and we need to maximize it.

$$\frac{dA}{d\theta} = 4r^2(\cos^2\theta - \sin^2\theta)$$
$$= 4r^2\cos 2\theta$$

For extreme values, $\frac{dA}{d\theta} = 0$ implies

$\cos 2\theta = 0$ or $\theta = \frac{\pi}{4}$

$$\frac{d^2A}{d\theta^2} = -8r^2\sin 2\theta$$
$$\frac{d^2A}{d\theta^2} = -8r^2 < 0 \text{ if } \theta = \frac{\pi}{4}$$

Thus, $\theta = \frac{\pi}{4}$ gives the rectangle of maximum area. The dimensions of the required rectangle are

$2x = 2r\cos\frac{\pi}{4} = \sqrt{2}\,r$

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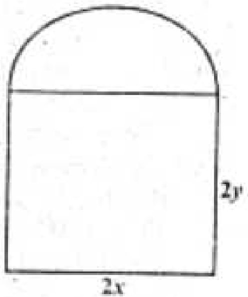
$$2y = 2r \sin \frac{\pi}{4} = \sqrt{2}\, r$$

Thus, the inscribed rectangle of maximum area is a square of side $\sqrt{2}\, r$.

24. A window has the shape of a rectangle surmounted by a semi-circle. Find the dimensions that maximize the area of the window if its perimeter is m metres.

Sol. Let the dimensions of the window be as shown in the figure. Area enclosed is

$$A = 4xy + \frac{1}{2}\pi x^2$$



The perimeter m of the window is

$$m = 4x + 4y + \pi x$$

or $\quad y = \frac{1}{4}(m - 4x - \pi x)$

Therefore, $A = 4x \cdot \frac{1}{4}(m - 4x - \pi x) + \frac{1}{2}\pi x^2$

$$= mx - 4x^2 - \frac{1}{2}\pi x^2$$

$$\frac{dA}{dx} = m - 8x - \pi x$$

For extreme values, $\frac{dA}{dx} = 0$ gives $m - 8x - \pi x = 0$ *i.e.*, $x = \frac{m}{\pi + 8}$

$$\frac{d^2A}{dx^2} = -8 - \pi = -(\pi + 8) < 0$$

Thus, the area is maximum when $x = \frac{m}{\pi + 8}$

$$y = \frac{1}{4}(m - 4x - \pi x) = \frac{m}{\pi + 8}$$

The required dimensions are

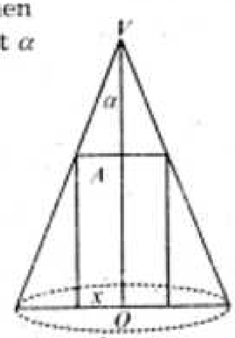
Breadth $= \frac{2m}{\pi + 8}$ meters. Height $= \frac{2m}{\pi + 8}$ meters.

25. Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.

Sol. Let h be the height and α the vertical angle of the cone. Also, let x be the radius of the base of the cylinder.

Let V be the vertex of the cone and O, the centre of the base. Join V to O, cutting the top surface of the cylinder in A. Then height of the cylinder

$$= OA = OV - VA = h - x \cot \alpha$$



Let S be the curved surface of the cylinder. Then

$$S = 2\pi x(h - x\cot \alpha) = 2\pi h x - 2\pi x^2 \cot \alpha$$

$$\frac{dS}{dx} = 2\pi h - 4\pi x \cot$$

$$\frac{dS}{dx} = 0 \text{ gives } 2\pi h - 4\pi x \cot \alpha = 0$$

or $\quad h = 2x \cot \alpha \quad$ or $\quad x = \frac{h}{2} \tan \alpha$

$$\frac{d^2S}{dx^2} = -4\pi \cot \alpha < 0$$

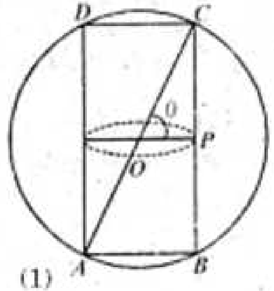
Hence S is maximum when $x = \frac{h}{2} \tan \alpha$

But, radius of the base of the cone $= h \tan \alpha$

Hence the required result.

26. Find the surface of the right cylinder of greatest surface which can be inscribed in a sphere of radius r.

Sol. Let O be the centre of the sphere of radius r and $ABCD$ be any cylinder inscribed in the sphere.



Let $\angle POC = \theta$.

Height of the cylinder $= |BC| = 2r \sin \theta$

Radius of its base $= |OP| = r \cos \theta$

Total curved surface S of the cylinder is

$$S = 2\pi(r \cos \theta)^2 + 2\pi(r \cos \theta)(2r \sin \theta)$$

$$S = 2\pi(r \cos \theta)^2 + 2\pi(r \cos \theta)(2r \sin \theta)$$

$$= 2\pi r^2(\cos^2 \theta + 2 \sin \theta \cos \theta)$$

$$= 2\pi r^2(\cos^2 \theta + \sin 2\theta) \quad (1)$$

$$\frac{dS}{d\theta} = 2\pi r^2[-2 \sin \theta \cos \theta + 2 \cos 2\theta]$$

$$= 2\pi r^2[2 \cos 2\theta - \sin 2\theta]$$

$$\frac{dS}{d\theta} = 0 \text{ gives } \sin 2\theta = 2 \cos 2\theta$$

$\Rightarrow \quad \tan 2\theta = 2 \quad \Rightarrow \quad 2\theta =$ the acute angle tanarc 2

$$\frac{d^2S}{d\theta^2} = 2\pi r^2[-4 \sin 2\theta - 2 \cos 2\theta]$$

$$= 2\pi r^2(-2)[2 \sin 2\theta - 2 \cos 2\theta]$$

$$= -4\pi a^2(2 \sin 2\theta + \cos 2\theta)$$

$$= -4\pi a^2\left[2 \cdot \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{5}}\right] = -4\pi a^2\left(\frac{5}{\sqrt{5}}\right)$$

$$= -4\pi\sqrt{5}\, a^2 < 0$$

Thus S is maximum when $\tan 2\theta = 2$

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32. A local train has 1200 passengers at a fare of Rs. 2 each. For every paisa the fare is reduced, 10 more passengers ride the train. What fare should be charged to maximize the revenue?

Sol. Suppose the fare is lowered by x paisa. Then $10x$ more passengers ride the train.

Total number of passengers at the new fare = $1200 + 10x$.

Total revenue R earned at new fare of Rs. $\left(2 - \frac{x}{100}\right)$ per passenger is

$$R = (1200 + 10x)\left(2 - \frac{x}{100}\right) = 2400 + 8x - \frac{x^2}{100}$$

$$\frac{dR}{dx} = 8 - \frac{x}{5}$$

For extreme values, $\frac{dR}{dx} = 0$ gives $8 - \frac{x}{5} = 0$

or $x = 40$.

R is maximum for this value of x since $\frac{d^2R}{dx^2} < 0$. Thus the fare should be lowered by 40 paisa to earn a maximum revenue.

33. A merchant has 200 quintals of cattle that he can sell at a profit of Rs. 500 per quintal. If the cattle gain 5 quintals per week, but the profit falls by Rs. 10 per quintal per week, when should the cattle be sold to obtain maximum profit?

Sol. Suppose the cattle are sold after x weeks to get maximum profit.

Weight grained by the cattle in x weeks = $5x$ quintals.

Total weight of cattle after x weeks = $(200 + 5x)$ quintals.

Rate of profit after x weeks = $(500 - 10x)$ Rs. per quintal.

The profit P when the cattle are sold is

$$P = (200 + 5x)(500 - 10x)$$
$$= 100{,}000 + 500x - 50x^2$$

$$\frac{dP}{dx} = 500 - 100x$$

For extreme values $\frac{dP}{dx} = 0$ gives $500 - 100x = 0$ or $x = 5$.

P is maximum for this value of x, since $\frac{d^2P}{dx^2}$ is negative.

Thus the cattle should be sold after 5 weeks to obtain maximum profit.

Exercise Set 7.3 (Page 301)

Determine the nature of the singular point (0, 0) (Problems 1 – 4):

1. $(x^2 + y^2)^2 = 4a^2xy$

Sol. Tangents at the origin are

$xy = 0$

i.e., $x = 0$ and $y = 0$

Since they are real and distinct, the origin is a node

2. $y^2(a^2 - x^2) = x^2(b - x)^2$

Sol. The given equation can be written as

$$a^2y^2 - x^2y^2 = x^2(b^2 - 2bx + x^2)$$

or $x^4 + x^2y^2 - 2bx^3 + b^2x^2 - a^2y^2 = 0$

Equating the lowest degree terms to zero, we get

$b^2x^2 - a^2y^2 = 0$ or $b^2x^2 = a^2y^2$

or $\pm bx = ay$ or $y = \pm\frac{b}{a}x$

Since these tangents at the origin are real and distinct, the origin is a node.

3. $(x^2 + y^2)(2a - x) = b^2x$

Sol. $(x^2 + y^2)x - 2a(x^2 + y^2) + b^2x = 0$

The lowest degree terms equated to zero give

$x = 0$

Therefore, the origin is a cusp.

4. $a^2(x^2 - y^2) = x^2y^2$

Sol. Tangents ate the origin are

$$a^2(x^2 - y^2) = 0$$

$\Rightarrow$ $(x^2 - y^2) = 0 \Rightarrow (x - y)(x + y) = 0$

Since these tangents are real and distinct, the origin is a node.

Find the position and nature of the multiple points on the given curves (Problems 5 – 10):

5. $x^2(x - y) + y^2 = 0$

Sol. Let $f(x, y) = x^3 - x^2y + y^2$ (1)

$f_x = 3x^2 - 2xy$ (2)

$f_y = -x^2 + 2y$ (3)

$f_{xx} = 6x - 2y$ (4)

$f_{xy} = -2x$ (5)

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i.e., $\sin 2\theta = \dfrac{2}{\sqrt{5}}$ and $\cos 2\theta = \dfrac{1}{\sqrt{5}}$

The maximum value of S

$$= 2\pi r^2\left[\frac{1+\cos 2\theta}{2} + \sin 2\theta\right], \text{ from (1)}$$

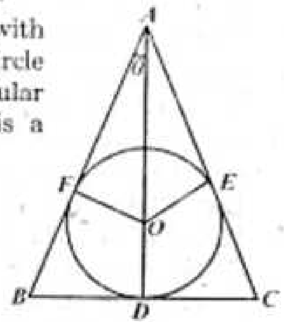
$$= 2\pi r^2 \frac{1}{2}[1 + \cos 2\theta + 2 \text{ in } 2\theta]$$

$$= 2\pi r^2 \frac{1}{2}\left[1 + \frac{1}{\sqrt{5}} + 2\cdot\frac{2}{\sqrt{5}}\right], \left(\begin{matrix}\text{Putting the values}\\ \text{of } \cos 2\theta \text{ and } \sin 2\theta\end{matrix}\right)$$

$$= \pi r^2 [1 + \sqrt{5}].$$

27. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

Sol. Let ABC be an isosceles triangle with $AB = AC$, O the centre of the inscribed circle and OD, OE, OF respectively perpendicular to BC, CA and AB. Evidently, AOD is a straight line.



Let $\angle OAF = \theta$, then

$AF = AE = r \cot \theta$

$OA = r \csc \theta$

Hence $AD = r + r \csc \theta$

so that $BD = DC = AD \tan \theta$

$P = (r + r \csc \theta) \tan \theta$

$= r(\tan \theta + \sec \theta) = BF = CE$

Perimeter of $\Delta ABC = 2AF + 4BD$

$$= 2r \cot \theta + 4r(\tan \theta + \sec \theta) \qquad (1)$$

$$\frac{dP}{d\theta} = 2r[-\csc^2 \theta] + 4r[\sec^2 \theta + \sec \theta \tan \theta]$$

$$= \frac{-2r}{\sin^2 \theta} + 4r\left[\frac{1}{\cos^2 \theta} + \frac{\sin \theta}{\cos^2 \theta}\right]$$

$$= -\frac{2r}{\sin^2 \theta} + \frac{4r(1 + \sin \theta)}{\cos^2 \theta}$$

$$= \frac{-2r \cos^2 \theta + 4r(\sin^2 \theta + \sin^3 \theta)}{\sin^2 \theta \cos^2 \theta}$$

$$= \frac{2r[2 \sin^3 \theta - \cos^2 \theta + 2 \sin^2 \theta]}{\sin^2 \theta \cos^2 \theta}$$

$$= \frac{2r[2\sin^3\theta + 3\sin^2\theta - 1]}{\sin^3\theta\cos^2\theta}$$

$$= \frac{2r}{\sin^2\theta\cos^2\theta}(2\sin \theta - 1)(\sin \theta + 1)^2$$

$\dfrac{dP}{d\theta} = 0$ gives $\sin \theta = \dfrac{1}{2}, -1$

But $\sin \theta = -1$ is inadmissible. So $\theta = \dfrac{\pi}{6}$

$\dfrac{d^2P}{d\theta^2} = \dfrac{2r}{\sin^2 \theta \cos^2 \theta}(2 \cos \theta)(\sin \theta + 1)^2$ + terms involving the factor $(2 \sin \theta - 1)$ which vanishes when $\theta = \dfrac{\pi}{6}$

When $\theta = \dfrac{\pi}{6}, \dfrac{d^2P}{d\theta^2} > 0$

Thus P is minimum when $\theta = \dfrac{\pi}{6}$

Minimum value of $P = 2r \cot \dfrac{\pi}{6} + 4r\left(\tan \dfrac{\pi}{6} + \sec \dfrac{\pi}{6}\right)$, from (1)

$$= 2r\sqrt{2} + 4r\left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}}\right)$$

$$= 2\sqrt{3}\,r + 4r(\sqrt{3}) = 6\sqrt{3}\,r.$$

28. A cone is circumscribed to a sphere of radius r. Show that when the volume of the cone is minimum, its altitude is $4r$ and its semi-vertical angle is $\arcsin\left(\dfrac{1}{3}\right)$.

Sol. Please refer to the figure of Problem 27 which represents the section of the sphere and the cone through the axis of the cone. Let $BD = x$ and $AD = y$. Then the volume of the cone is given by

$$V = \frac{1}{3}\pi x^2 y \qquad (1)$$

We now find a relation between x and y. We have

$\csc \theta = \dfrac{AO}{OF} = \dfrac{y - r}{r}$ and $\cot \theta = \dfrac{AD}{BD} = \dfrac{y}{x}$

Now $\csc^2 \theta - \cot^2 \theta = 1 \Rightarrow \dfrac{(y-r)^2}{r^2} - \dfrac{y^2}{x^2} = 1$

$\Rightarrow x^2(y - r)^2 - r^2y^2 = r^2x^2$

$\Rightarrow x^2(y^2 + r^2 - 2ry) - r^2y^2 = r^2x^2$

$\Rightarrow x^2y^2 + x^2r^2 - 2rx^2y - r^2y^2 = r^2x^2$

$\Rightarrow y(x^2y - 2rx^2 - r^2y) = 0$

$\Rightarrow (x^2 - r^2)y = 2rx^2$

$$\Rightarrow y = \frac{2rx^2}{x^2 - r^2} \qquad (2)$$

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Substituting the value of y from (2) into (1), we ge

$$V = \frac{1}{3}\pi x^2 \frac{2rx^2}{x^2 - r^2} = \frac{2}{3}\pi r\left(\frac{x^4}{x^2 - r^2}\right)$$

$$\frac{dV}{dx} = \frac{2}{3}\pi r \frac{(x^2 - r^2).4x^3 - x^4.2x}{(x^2 - r^2)^2} = \frac{2}{3}\pi r \frac{x^3(x^2 - 2r^2)}{(x^2 - r^2)^2}$$

$$\frac{dV}{dx} = 0 \quad \text{gives} \quad x^2 - 2r^2 = 0$$

$\Rightarrow$ $x = \sqrt{2}\, r$, other value of x being inadmissible

$$\frac{d^2V}{dx^2} = \frac{2\pi r}{3}\frac{x}{(x^2 - r^2)^2}(2x) + \text{ terms involving the factor } (x^2 - 2x^2)$$

When $x = \sqrt{2}\, r$, $\frac{d^2V}{dx^2} > 0$. Therefore,

V is minimum when $x = \sqrt{2}\, r$

From (2), we have, $y = \frac{2r\,.\,2r^2}{2r^2 - r^2} = \frac{4r^3}{r^2} = 4r.$

Also $\sin\theta = \frac{OF}{OA} = \frac{r}{y - r} = \frac{r}{4r - r} = \frac{1}{3}$

or $\theta = \arcsin\frac{1}{3}$ as required.

29. A farmer has 1000 metres of barbed wire with which he is to fences off three sides of a rectangular field, the fourth side being bounded by a straight canal. How can the farmer enclose the largest field?

Sol. Suppose the length of the side parallel to the canal is x m and that of the one perpendicular to it is y m. Then the area

$A = xy$ is to be maximized subject to

$$x + 2y = 1000 \tag{1}$$

or $x = 1000 - 2y$

Thus $A = 1000y - 2y^2$

$$\frac{dA}{dy} = 1000 - 4y$$

For extreme values, $\frac{dA}{dy} = 0$ gives $y = 250$. It is easy to see that A is maximum for this value of y. From (1), when $y = 250$, $x = 500$. Thus the dimensions of the largest field are 500m by 250 m.

30. A topless rectangular box with a square base is to have volume of 1926 cubic cm. The material for the base costs Rs. 3 per square cm and the material for the sides costs Rs. 2 per square cm. What dimensions should the box have to minimize its cost?

Sol. Let the length of the base be x cm and that of each side be y cm. Then $x^2y = 1296$. (1)

Cost of material for the base $= 3x^2$ Rs.
Cost of material for the four sides $= 8xy$ Rs.
Total cost

$$C = 3x^2 + 8xy \tag{2}$$

This is to be minimized subject to (1). Eliminating y between (1) and (2), we have

$$C = 3x^2 + 8x\,.\,\frac{1296}{x^2} = 3x^2 + \frac{10368}{x}$$

$$\frac{dC}{dx} = x - \frac{10368}{x^2}$$

$$\frac{dC}{dx} = 0 \quad \text{gives} \quad 6x^2 - 10368 = 0$$

or $x^3 = 1728$, $\Rightarrow$ $x = 12$.

C is minimum for this value of x. Putting $x = 12$ in (1), we get $y = 9$. Thus the required dimensions of the box are

12 cm, 12 cm, 9 cm.

31. An open rectangular box is to be made from a sheet of cardboard 8 dm by 5 dm by cutting equal squares from each corner and turning up sides. Find the edge of the square which makes the volume a maximum.

Sol. Let the side of the square cut from each corner be x dm. Then the edges of the box formed by bending the sides are $8 - 2x$, $5 - 2x$ and x decimeters.

Volume V of the box is

$$V = x(8 - 2x)(5 - 2x);\ 0 < x < \frac{5}{2}$$
$$= 4x^3 - 26x^2 + 40x$$

$$\frac{dV}{dx} = 12x^2 - 52x + 40$$

For extreme values, $\frac{dV}{dx} = 0$

Therefore, $12x^2 - 52x + 40 = 0$

or $3x^2 - 13x + 10 = 0$ *i.e.,* $x = 1, \frac{10}{3}$

But $x = \frac{10}{3}$ is inadmissible.

$$\frac{d^2V}{dx^2} = 6x - 13 < 0, \text{ for } x = 1$$

Thus V is maximum when $x = 1$ dm.

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$f_{yy} = 2$ (6)

Putting each of the equations (2) and (3) equal to zero, we get

$3x^2 - 2xy = 0$ (7)

and $-x^2 + 2y = 0$ (8)

From (8), we have $y = \frac{x^2}{2}$

Putting it into (7), we have

$3x^2 - x^3 = 0$

$x^2(3 - x) = 0$

or $x = 0$ and $x = 3$

When $x = 0$, $y = 0$

Thus (0, 0) may be a multiple point

When $x = 3$, $y = \frac{9}{2}$

Therefore, $\left(3, \frac{9}{2}\right)$ may be the another such point

But (0, 0) satisfies (1) and $\left(3, \frac{9}{2}\right)$ does not lie on (1).

Hence (0, 0) is the only multiple point.

At this point

$f_{xx} = 0$, $f_{yy} = 2$, $f_{xy} = 0$

Now $(f_{xy})^2 - f_{xx}f_{yy} = 0 - 0 = 0$

Hence (0, 0) is a cusp.

6. $y^3 = x^3 + ax^2$

Sol. $f(x, y) = x^3 - y^3 + ax^2 = 0$

$f_x = 3x^2 + 2ax$; $f_{xx} = 6x + 2a$

$f_y = -3y^2$; $f_{yy} = -6y$

$f_{xy} = 0$

$f_x = 0$ gives $x(3x + 2a) = 0$ *i.e.,* $x = 0, -\frac{2a}{3}$

$f_y = 0$ gives $3y^2 = 0$ *i.e.,* $y = 0$

Hence the possible multiple points are (0, 0), $\left(-\frac{2a}{3}, 0\right)$

Of these only (0, 0) satisfies the given equation.

Hence (0, 0) is the only multiple point.

At (0, 0);

$f_{xy} = 0$, $f_{xx} = 2a$, $f_{yy} = 0$

$(f_{xy})^2 - f_{xx}f_{yy} = 0$

Hence (0, 0) is a cusp.

7. $x^4 + y^3 - 2x^3 + 3y^2 = 0$

Sol. Let $f(x, y) = x^4 + y^3 - 2x^2 + 3y^2 = 0$

$f_x = 4x^3 - 6x^2$

$f_y = 3y^2 + 6y$

$f_{xy} = 0$

$f_{xx} = 12x^2 - 12x$

$f_{yy} = 6y + 6$

$f_x = 0$ gives $2x^2(2x - 3) = 0$ *i.e.,* $x = 0, \frac{3}{2}$

$f_y = 0$ gives $3y(y + 2) = 0$ *i.e.,* $y = 0, -2$

The possible multiple points are

(0, 0) (0, −2), $\left(\frac{3}{2}, 0\right)$, $\left(\frac{3}{2}, -2\right)$

Of these only (0, 0) satisfies the given equation of the curve and so (0, 0) is a multiple point.

At (0, 0)

$f_{xy} = 0$, $f_{xx} = 0$, $f_{yy} = 6$

$(f_{xy})^2 - f_{xx}f_{yy} = 0 - 0 = 0$

Thus (0, 0) is a cusp.

8. $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$

Sol. Let $f(x, y) = x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0$

$f_x = 3x^2 + 4x + 2y + 5$

$f_y = 2x - 2y - 2$

$f_{xy} = 2$

$f_{xx} = 6x + 4$

$f_{yy} = -2$

$f_x = 0$ gives $3x^2 + 4x + 2y + 5 = 0$ (1)

$f_y = 0$ gives $2x - 2y - 2 = 0$ or $x - y - 1 = 0$

or $y = x - 1$ (2)

Putting from (2) into (1), we have

$3x^2 + 4x + 2(x - 1) + 5 + 0$ or $3x^2 + 6x + 3 = 0$

or $x^2 + 2x + 1 = 0$ or $x = -1$

When $x = -1$, $y = -2$ from (1)

The possible multiple point is (−1, −2)

It satisfies the given equation, therefore,, it is multiple point.

At (−1, −2)

$f_{xy} = 2$, $f_{xx} = 2$, $f_{yy} = -2$

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$(f_{xy})^2 - f_{xx} f_{yy} = 4 - 4(-2) = 0$

Hence $(-1, -2)$ is a cusp.

9. $(2y + x + 1)^2 - 4(1 - x)^5 = 0$

Sol. Let $f(x, y) = (2y + x + 1)^2 - 4(1 - x)^5 = 0$

$f_x = 2(2y + x + 1) + 20(1 - x)^4$

$f_y = 4(2y + x + 1)$

$f_x = 0$ gives $2(2y + x + 1) = -(x - 1)^4$

or $2y + x + 1 = -10(x - 1)^4$

$f_y = 0$ gives $2y + x + 1 = 0$

Also $f_{xy} = 4$, $f_{xx} = 2 - 80(1 - x)^3$, $f_{yy} = 8$

Solving $f_x = 0$, $f_y = 0$ simultaneously, we get

$x = 1$ and $y = -1$

Therefore, $(1, -1)$ may be a multiple point.

Since it satisfies the given equation, it is a multiple point.

At $(1, -1)$

$(f_{xy})^2 - f_{xx} f_{yy} = 16 - (2)(8) = 0$

Thus $(1, -1)$ is a cusp.

10. $(y^2 - a^2)^3 + x^4(2x + 3a)^2 = 0$

Sol. $f(x, y) = (y^2 - a^2)^3 + x^4(2x + 3a)^2$

$f_x = 4x^3(2x + 3a)^2 + 4x^4(2x + 3a)$

$= 4x^3(2x + 3a)(2x + 3a + x) = 12x^3(2x + 3a)(x + c)$

$f_x = 0$ gives $x = 0, -\frac{3}{2}a, -a$

$f_y = 3 \cdot 2y(y^2 - a^2)^2 = 6y(y^2 - a^2)^2$

$f_y = 0$ gives $y = 0,\ y = a\ \ y = -a$

Thus the possible multiple points are

$(0, 0)$, $(0, a)$, $(0, -a)$

$\left(-\frac{3}{2}a, 0\right)$, $\left(-\frac{3}{2}a, a\right)$, $\left(-\frac{3}{2}a, -a\right)$

$(-a, 0)$, $(-a, a)$, $(-a, -a)$

Of these nine points the following satisfy the given equation

$(0, \pm a)$, $\left(-\frac{3}{2}a, \pm a\right)$, $(-a, 0)$

These are the multiple points.

$f_{xx} = 36x^2(2x + 3a)(x + a) + 24x^3(x + a) + 12x^2(2x + 3a)$

$f_{yy} = 6(y^2 - a^2)^2 + 24y^2(y^2 - a^2)$

$= 6(y^2 - a^2)[y^2 - a^2 + 4y^2]$

$= 6(y^2 - a^2)(5y^2 - a^2)$

$f_{xy} = 0$

At $(0, \pm a)$, $f_{xx} = 0$, $f_{yy} = 0$

Therefore, $(0, \pm a)$ are multiple points of order higher than two. It can be easily seem that $(-a, 0)$ is a node and $\left(\frac{-3}{2}a, \pm a\right)$ are cusps.

11. Show that the origin is a node, a cusp or an isolated point on the curve $y^2 = ax^2 + ax^3$ according as a is positive, zero, or negative respectively.

Sol. The lowest degree terms equated to zero give

$y^2 - ax^2 = 0$

When a is positive, we get

$y^2 = ax^2$ i.e., $y = \pm\sqrt{a}x$

which are real and distinct. Therefore, the origin is a node.

(i) When $a = 0$, tangents at the origin are $y^2 = 0$ which are real and coincident. Hence the origin is a cusp.

(ii) When a is negative, tangents at the origin are $y = \pm\sqrt{a}\,x$.

Since a is negative, $\sqrt{a}$ is imaginary.

Hence the tangent at the origin are imaginary and the origin is a conjugate point.

Find equations of the tangent at the multiple points of the given curves (Problems 12 – 13):

12. $x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4 = 0$

Sol. $f(x, y) = x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 + 3a^2y^2 - a^4$

$f_x = 4x^3 - 12ax^2 + 8a^2x$

$f_y = -6ay^2 + 6a^2y$

$f_x = 0$ gives

$4x^3 - 12ax^2 + 8a^2x = 0 \Rightarrow 4x(x^2 - 3ax + 2a^2) = 0$

$\Rightarrow 4x(x - 2a)(x - a) = 0 \Rightarrow x = 0, a, 2a$

$f_y = 0$ gives

$-6ay^2 + 6a^2y = 0$ or $-6ay(y - a) = 0$

$\Rightarrow y = 0,\ y = a$

The possible multiple points are

$(0, a)$, (a, a), $(2a, a)$

$(0, 0)$, $(a, 0)$, $(2a, 0)$

Of these $(0, a)$, $(a, 0)$, $(2a, a)$ satisfy the given equation of the curve.

Thus $(0, a)$, $(a, 0)$, $(2a, a)$ are the multiple points.

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(i) Shifting the origin to $(0, a)$, we get the new equation as

$$x^4 - 4ax^3 + 4a^2x^2 - 2a(y+a)^3 + 3a^2(y+a)^2 - a^4 = 0$$

$$\Rightarrow \quad x^4 - 4ax^3 + 4a^2x^2 - 2a(y^3 + 3y^2a + 3ya^2 + a^3) + 3a^2(y^2 + 2ay + a^2) - a^4 = 0$$

$$\Rightarrow x^4 - 4ax^3 - 2ay^3 + 4a^2x^2 - 6a^2y^2 + 3a^2y^2 + (-6a^3y + 6a^3y) = 0$$

$$\Rightarrow \quad x^4 - 4ax^3 - 2ay^3 + a^2(4x^2 - 3y^2) = 0$$

Tangents at the new origin are $4x^2 - 3y^2 = 0$

or $\quad 3y^2 = 4x^2 \qquad$ or $\qquad y = \pm\dfrac{2}{\sqrt{3}}x$

Hence tangents at $(0, a)$ are $y - a = \pm\dfrac{2}{\sqrt{3}}x$.

Shifting the origin to $(a, 0)$, we get the equation of the curve as

$$(x+a)^4 - 4a(x+a)^3 - 2ay^3 + 4a^2(x+a)^2 + 3a^2y^2 - a^4 = 0$$

$$\Rightarrow \quad x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4 - 4a(x^3 + 3x^2a + 3xa^2 + a^3) - 2ay^3 + 4a^2(x^2 + 2xa + a^2) + 3a^2y^2 - a^4 = 0$$

$$\Rightarrow \quad x^4 - 2a^2x^2 - 2ay^3 + 3a^2y^2 = 0$$

$$\Rightarrow \quad x^4 - 2ay^3 - 2a^2x^2 + 3a^2y^2 = 0$$

$$\Rightarrow \quad x^4 - 2ay^3 - a^2(2x^2 - 3y^2) = 0$$

Tangents at the new origin are

$$2x^2 - 3y^2 = 0$$

or $\quad 3y^2 = 2x^2 \qquad$ or $\qquad y = \pm\dfrac{\sqrt{2}}{\sqrt{3}}x$

Hence equations of the tangents at the multiple point $(a, 0)$ are

$$y = \pm\sqrt{\frac{2}{3}}(x - a)$$

Shifting the origin to $(2a, a)$, we get the equation of the curve as

$$(x + 2a)^4 - 4a(x + 2a)^3 - 2a(y + a)^3 + 4a^2(x + 2a)^2 + 3a^2(y + a)^2 - a^4 = 0$$

$$\Rightarrow \quad x^4 + 4x^3(2a) + 6x^2(2a)^2 + 4x(2a)^3 + (2a)^4 - 4a[x^3 + 3x^2(2a)^2 + 3x(2a)^2 + (2a)^3] - 2a[y^3 + 3y^2a + 3ya^2 + a^3] + 4a^2[x^2 + 2x(2a) + 4a^2] + 3a^2[y^2 + 2ya + a^2] - a^4 = 0$$

$$\Rightarrow \quad x^4 + 8ax^3 + 24a^2x^2 + 32a^3x + 16x^4 - 4ax^3 - 24a^2x^2 - 48a^3x - 32a^4 - 2ay^3 - 6a^2y^2 - 6a^3y - 2a^4 + 4a^2x^2 + 16a^2x + 16a^4 + 3a^2y^2 + 6a^2y + 3a^4 - a^4 = 0$$

$$\Rightarrow \quad x^4 + a(4x^3 - 2y^3) + a^2(4x^2 - 3y^2) = 0$$

Equating to zero the lowest degree terms, we get

$4x^2 - 3y^2 = 0$ as tangents at the new origin.

Equations of these tangents are

$$3y^2 = 4x^2, \; y = \pm\frac{2}{\sqrt{3}}x$$

So equations of the tangents at $(2a, a)$ are

$$y - a = \pm\frac{2}{\sqrt{3}}(x - 2a).$$

13. $(y - 2)^2 = x(x - 1)^2$

Sol. $f(x, y) = (y - 2)^2 - x(x - 1)^2$

$$f_x = -(x-1)^2 - 2x(x-1) = -[(x-1)^2 + 2x(x-1)]$$

$$= -(x-1)[x - 1 + 2x] = -(x-1)(3x-1)$$

$f_x = 0 \quad$ gives $\quad x = 1, \dfrac{1}{3}$

$f_y = 2(y - 2)$

$f_y = 0 \quad$ gives $\quad y = 2$

The possible multiple points are $(1, 2)$, $\left(\dfrac{1}{3}, 2\right)$

Of these only $(1, 2)$ satisfies the equation of the curve.

Shifting the origin to $(1, 2)$, we get the new equation as

$$y^2 - (x + 1)x^2 = 0$$

$$y^2 - x^2 - x^3 = 0$$

$$x^3 + x^2 - y^2 = 0$$

Tangents at the new origin are

$$x^2 - y^2 = 0$$

or $\quad y^2 = x^2 \qquad$ or $\qquad y = \pm x$

Hence equations of the tangents at $(1, 2)$ are

$y - 2 = \pm(x - 1)$	$y - 2 = -(x - 1)$
$y - 2 = x - 1$	$= -x + 1$
or $\quad x - y + 1 = 0$	or $\quad x + y = 3$

Tangents at the multiple point $(1, 2)$ are

$x - y + 1 = 0 \quad$ and $\quad x + y = 3$

Find the nature of the cusps on the given curves (Problems 14 – 17):

14. $x^2(x - y) + y^2 = 0$

Sol. The curve has coincident tangents $y^2 = 0$ at the origin. Hence the origin is a cusp and the branches of the curve through it are real. Equation of the curve can be written as

$$y^2 - x^2y + x^3 = 0$$

$$y = \frac{x^2 \pm \sqrt{x^4 - 4x^3}}{2} = \frac{x^2 \pm x\sqrt{x^4 - 4x}}{2} = \frac{x^2 \pm x\sqrt{x(x-4)}}{2}$$

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The values of y are real only for negative values of x near origin.
Hence the origin is cusp.
Also for any particular negative value of x, y has opposite signs *i.e.*, the curve exists on both sides of the x-axis, the cuspidal tangent.
The cusp is of the first species.
Hence origin is a single cusp of the first species.

15. $x^3 + y^3 - 2ay^2 = 0$

Sol. $y^3 = x^3 + ax^2$ (1)

Tangent at the origin are $x^2 = 0$
i.e., the curve has two coincident tangent $x^2 = 0$ at the origin.
From (1) $ax^2 = y^3$ (Neglecting x^3)

or $x^2 = \dfrac{y^3}{a}$ or $x = \pm y\sqrt{\dfrac{y}{a}}$

The values of x are real only for positive values of y.
Hence origin is a single cusp.
Also, for any particular positive value of y, x has opposite signs. *i.e.*, the curve exists on both sides of the y-axis, the cuspidal tangent.
The cusp is of the first species.
Hence the origin is a single cusp of the first species.

16. $x^6 - ayx^4 - a^3x^2y + a^4y^2 = 0$

Sol. $x^6 - ayx^4 - a^3x^2y + a^4y^2 = 0$ (1)

Tangents at the origin are $y^2 = 0$.
Hence origin is a cusp.
(1) can be written as

$$a^4y^2 - a(x^4 + a^2x^2)y + x^6 = 0$$

$$y = \frac{a(x^4 + a^2x^2) \pm \sqrt{a^2(x^4 + a^2x^2)^2 - 4a^4x^6}}{2a^4}$$

$$= \frac{ax^2(x^2 + a^2) \pm \sqrt{a^2\{a^8 + 2a^2x^6 + a^4x^4 - 4a^2x^6\}}}{2a^4}$$

$$= \frac{a^2x^2(x^2 + a^2) \pm \sqrt{a^2(x^4 - a^2x^2)^2}}{2a^4}$$

$$= \frac{ax^2(x^2 + a^2) \pm a(x^4 - a^2x^2)}{2a^4}$$

$$= \frac{ax^2(x^2 + a^2) \pm ax^2(x^2 - a^2)}{2a^4}$$

$$= \frac{x^2[(x^2 + a^2) \pm (x^2 - a^2)]}{2a^3} = \frac{x^4}{a^3}, \ \frac{x^2}{a}$$

The values of y are real and positive for both positive and negative values of x.
The curve exists on one side of the x-axis. This shows that (0, 0) is double cusp of the second species.

17. $y^3 = (x - a)^2(2x - a) = 0$

Sol. $y^3 = (x - a)^2(2x - a) = 0$ (1)

Shifting the origin to the point $(a, 0)$, equation (1) becomes

$$y^3 = x^2(2x + a) \quad (2)$$

Tangents to (2) at the new origin are $x^2 = 0$.
Since the tangents are coincident, the new origin is cusp
From (2), neglecting $2x^3$, we get

$$ax^2 = y^3 \quad \text{or} \quad x = \pm\sqrt{\frac{y^3}{a}}$$

The value of x are real only for positive values of y. Hence the new origin is a single cusp.
Also for any particular positive value of y, x has opposite signs *i.e.*, the curve exists on both sides of the new y-axis, the cuspidal tangent.
The cusp is of the first species.
Hence the point $(a, 0)$ is a single cusp of the first species.

Exercise Set 7.4 (Page 310)

Discuss and sketch each of the following curves:

1. $3ay^2 = x^2(x - a)$

Sol.

I. Since only even power of y occurs in the given equation, the curve is symmetric about the x-axis.

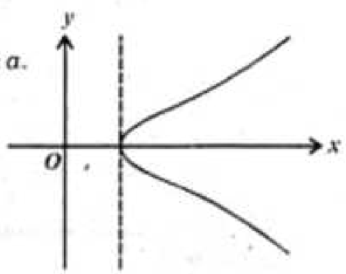
II. It passes through the origin and tangents at the origin are
$x^2 + 3y^2 = 0$ which are imaginary.

III. It has no asymptotes.

IV. It can be written as

$$y^2 = \frac{x^2(x - a)}{3a}$$

$$= \pm\frac{x\sqrt{x - a}}{\sqrt{3a}}, \ y \text{ is real only when } x \geq a.$$



V. It cuts the x-axis at $(a, 0)$ and $x = a$ is a tangent to the curve at $(a, 0)$.

VI. As x increases beyond a, y also increases.

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Hence following is the shape of the curve.

2. $ay^2 = x(a^2 - x^2)$

Sol.

I. It is symmetric about the x-axis.

II. It passes through the origin.
Tangent at the origin is $x = 0$

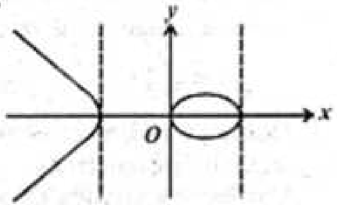
III. It has no asymptotes.

IV. It cuts the x-axis at $x = 0, \pm a$.
$x = \pm a$ are tangents to the curve at $(\pm a, 0)$.
It cuts the y-axis only at the origin.

V. Equation of the curve can be written as

$$y = \pm \frac{\sqrt{x}}{\sqrt{a}} \sqrt{a^2 - x^2}. \; x$$

cannot take any value larger than a.



However, when x is negative, it cannot take any value numerically les than a. The curve is as shown.

3. $y^2 = x^2(4 - x^2)$

Sol.

I. It is symmetric about the x-axis.

II. It is symmetric about the y-axis.

III. On changing x into $-x$ and y into $-y$, the equation of the curve remains unchanged. Therefore, if is symmetric in the opposite quadrants.

IV. The curve is passing through the origin and tangents at the origin are $^2 - 4x^2 = 0$ or $y = \pm 2x$
which are real and distinct.
So the origin is a node.

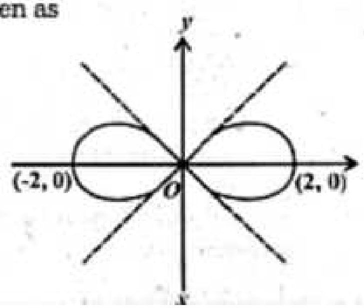
V. It has no asymptotes – oblique or parallel to the coordinate axes.

VI. It cuts the x-axis at $x = 0, x = \pm 2$
It cuts the y-axis at the origin only.

VII. Equation of the curve can be written as

$$y = x\sqrt{4 - x^2}$$

Hence no point of the curve lies beyond $x = 2$ and $x = -2$ because when x is greater than 2 numerically, y becomes imaginary.
We have the following shape of the curve.



4. $x(x^2 + y^2) = a(x^2 - y^2)$

Sol.

I. One changing y into $-y$, equation of the curve remains unchanged. Hence it is symmetric about the x-axis.

II. It is passing through the origin and tangents at the origin are
$x^2 - y^2 = 0$
i.e., $(x - y)(x + y) = 0$ *i.e.*, $y = \pm x$ which are real and distinct.
The origin is a node.

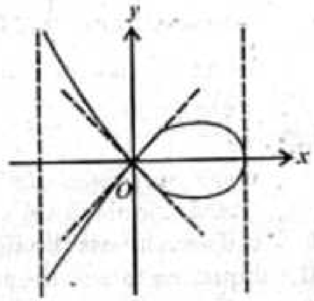
III. Coefficient of y^2 is $x + a$. Therefore,
$x + a = 0$ is an asymptotes parallel to the y-axis.

IV. It cuts the x-axis at $x = a$.
$x - a = 0$ is tangent to the curve at $(a, 0)$.
It cuts the y-axis only at the origin.

V. Equation of t e curve can be written as

$$y^2 = \frac{x^2(a - x)}{a + x}$$

which shows that no point of the curve lies beyond $x = a$ since y becomes imaginary for $x > a$.
The sketch of the curve is as shown.



5. $y(a^2 + x^2) = a^2x$

Sol.

I. On changing x into $-x$ and y into $-y$, equation of the curve remains unchanged. Therefore, it is symmetric in the opposite quadrants.

II. It passing through the origin. Tangent at the origin is
$y - x = 0$

III. Equating to zero the coefficient of the highest power of x, we get
$y = 0$ as an asymptote to the curve

IV. It cuts the coordinate axes only at the origin.

The equation can be written as $y = \dfrac{ax}{a^2 + x^2}$

When x is positive, y is positive. When x is negative, y is negative.

$$\frac{dy}{dx} = \frac{(a^2 + x^2)a - ax(2x)}{(a^2 = x^2)^2}$$

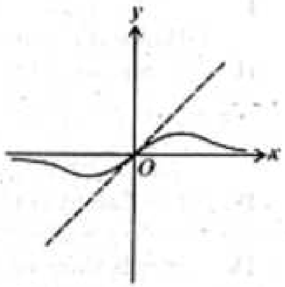
$$= \frac{a(a^2 - x^2)}{(a^2 + x^2)^2} \quad > 0 \text{ if } |x| < a$$

$$< 0 \text{ if } |x| > a$$

$$= 0 \text{ if } |x| = a$$

Thus the curve has tangents parallel to the x-axis at $\left(a, \frac{1}{2}\right)$ and $\left(-a, -\frac{1}{2}\right)$.

As x increases from 0 to a, y increases from 0 to $\frac{1}{2}$ and as x decreases from 0 to $-a$, y decreases from 0 to $-\frac{1}{2}$. As x increases from a to ∞, y decreases from $\frac{1}{2}$ to 0. As x decreases from $-a$ to $-\infty$, y increases from $-\frac{1}{2}$ to 0.



Hence we have the following sketch of the curve.

6. $y^2x = a\,(x^2 - a^2)$ (1)

Sol.

I. Since only even powers of y occur in the equation, the curve symmetric about the x-axis.

II It does not pass through the origin.

III. Equating to zero the coefficient of y^3, we get

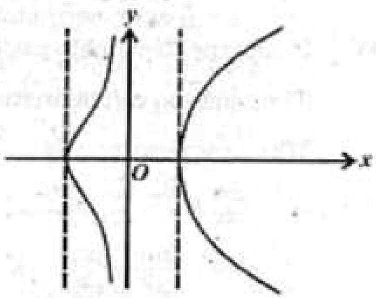
$x = 0$ as an asymptote.

Putting $y = 0$, we get $x = \pm a$. Therefore, the curve meets the x-axis at $x = a$ and $x = -a$.

(1) can be written as $y^2 = \dfrac{a\,(x^2 - a^2)}{x}$

If x is positive and less than a then y is imaginary. Hence no portion of the curve lies between (0, 0) and $(a, 0)$.

If x is negative and $x^2 < a^2$ then y^2 is positive. If x is negative and $x^2 > a^2$ then y^2 is negative so that y is imaginary. Hence no portion of the curve lies byond $(-a, 0)$. As x increases from a to ∞, y increases numerically from 0 to ∞.



As x decreases from 0 to $-a$, y increases numerically from 0 to ∞.
$x = \pm a$ are tangents to the curve at $(\pm a, 0)$.
Hence we have sketch of the curve as shown.

7. $x = (y-1)(y-2)(y-3)$

Sol.

I. The curve has no symmetry.

II. It does not pass through the origin.

III. It has no asymptotes.

IV. It cuts the y-axis at $y = 1, y = 2, y = 3$ and the x-axis at $x = -6$.
Equation of the curve may be written as

$$x = y^3 - 6y^2 + 11y - 6$$

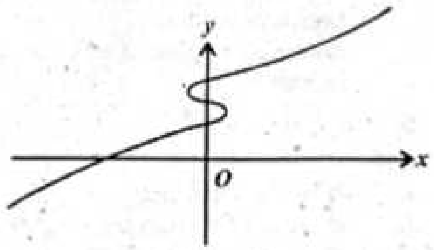
$$\frac{dx}{dy} = 3y^2 - 12y + 11$$

$$\frac{dx}{dy} = 0 \Rightarrow 3y^2 - 12y + 11 = 0$$

$$y = \frac{12 \pm \sqrt{144 - 132}}{6} = \frac{6 \pm \sqrt{3}}{3} = 1.427, 2.577$$

Thus tangents to the curve are parallel to the y-axis at the points where $y = 1.427, 2.577$. As y increase from 1 to 2, x is positive. This portion of the curve lies in the first quadrant. When y increases from 2 to 3, x is negative and this portion of the curve lies in the second quadrant. As y increases from 0 to 1, x remains negative and it increases from -6 to 0. This portion of the curve lies in the second quadrant.

As y increases beyond 3. x increases and remains positive. This portion of the curve lies in the first quadrant. When y is negative, x is also negative and as y increases numerically, x also increases numerically, This portion of the curve lies in the third quadrant.



Hence we have the following sketch.

8. $y^2(2x - 1) = x(x - 1)$ (1)

Sol.

I It is symmetric about the x-axis.

II. It is passing through the origin.
The tangent at the origin is $x = 0$ (y-axis)

III. An asymptote parallel to y-axis is

$$2x - 1 = 0 \qquad i.e., \qquad x = \frac{1}{2}$$

IV. Putting $y = 0$, we get $x = 0, x = 1$

Thus the curve cuts the x-axis at $x = 0, x = 1$.
It does not meet the y-axis.
To shift the origin to $(1, 0)$, we put

$$x - 1 = x' \quad \text{or} \quad x = x' + 1$$

Therefore, (1) becomes

$$y^2(2x' + 1) = (x' + 1)x'$$

Tangent at the new origin is $x' = 0$
i.e., $x - 1 = 0$ is tangent to the curve at $(1, 0)$.

V. Equation of the curve can be written as

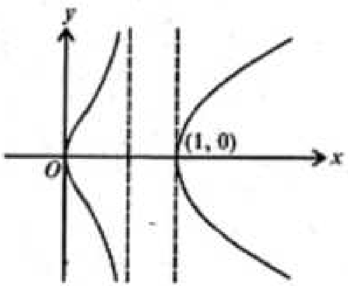
$$y^2 = \frac{x(x-1)}{2x-1}$$

When $0 < x < \frac{1}{2}$, y is real and takes both positive and negative values.

When $\frac{1}{2} < x < 1$, y becomes imaginary and so no part of the curve lies between $x = \frac{1}{2}$ and $x = 1$

Whew $x > 1$, y is real and assumes both positive and negative values.

Sketch of the curve is as shown.



9. $xy^2 = (x + y)^2$

Sol.

I. It has no symmetry

II. It passes through the origin.
Tangents at the origin are $(x + y)^2 = 0$
i.e., There are two coincident tangents

$$y = -x \text{ at the origin.}$$

III. Coefficient of y^2 is $x - 1$. Thus an asymptote parallel to y = axis is $x = 1$

IV. It cuts the coordinate axes only at the origin.
It cuts the asymptote $x = 1$ at $\left(1, -\frac{1}{2}\right)$

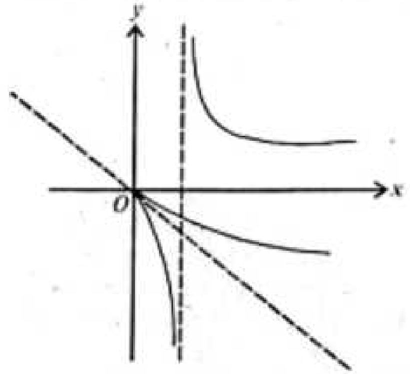
V. The equation of the curve can be written as

$$y^2(x-1) - 2xy - x^2 = 0$$

Therefore, $y = \dfrac{2x \pm \sqrt{4x^2 + 4x^2(x-1)}}{(2x-2)}$

$$= \frac{x(1 \pm \sqrt{x})}{x-1}$$

Thus x cannot take negative values.
When x is positive but < 1, y is negative
When $x > 1$, y decreases as x increases
Sketch of the curve is as shown.



10. $x^2(y + 1) = x(x - 1)$

Sol.

I. It has no symmetry.

II. It passes through the origin.
Tangents at the origin are

$$x^2 + 4y^2 = 0, \text{ which are imaginary.}$$

Therefore, the origin is an isolated point.

III. Asymptotes parallel to the x-axis is

$$y + 1 = 0 \quad \textit{i.e.,} \quad y = -1$$

Asymptotes parallel to y-axis is

$$x - 4 = 0, \quad x = 4$$

For oblique asymptotes, the equation of the curve can be written as

$$x^2y - xy^2 + x^2 + 4y^2 = 0$$

Now, $\phi_3(m) = m - m^2 = 0$ gives $m = 0, 1$

$$\phi'_3(m) = 1 - 2m, \qquad \phi_2(m) = 1 + 4m^2$$

To find c, we apply the formula

$$c\phi'_3(m) + \phi_2(m) = 0$$

i.e., $c(1 - 2m) + 1 + 4m^2 = 0$ (1)

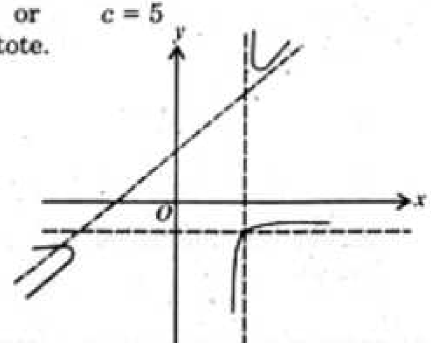
When $m = 0$, $c = 1$. Thus $y = -1$ is the asymptote already found above.

When $m = 1$, from (1), we have

$$c(1-2) + 1 + 4 = 0 \quad \text{or} \quad c = 5$$

Hence $y = x + 5$, is an asymptote.

IV. It cuts the coordinate axes at $(0, 0)$ which is an isolated point.
It cuts $y = -1$ at $x = 4$. Thus the curve cuts the asymptote $y = -1$ at $(4, -1)$.
It cuts the asymptote $x = 4$ at $y = -1$, *i.e.*, at $(4, -1)$



It cuts $y = x + 5$ at $\left(-\frac{25}{4}, -\frac{5}{4}\right)$

Sketch of the curve is as shown.

11. $y(x - y)^2 = x + y$

Sol.

I. On changing x into $-x$ and y into $-y$, equation of curve becomes

$$y(-x + y)^2 = -(x + y) \quad \text{or} \quad y(x - y)^2 = (x + y)$$

i.e., equation of the curve remains unchanged and so it is symmetric in the opposite quadrants.

II. It passes through the origin and tangent there at is $x + y = 0$

i.e., $y = -x$.

III. Coefficient of x^2 is y. Hence $y = 0$ is an asymptote.

For oblique asymptotes, equation of the curve can be written as

$$y(y^2 - 2xy + x^2) - (x + y) = 0$$

Here, $\phi_3(m) = m(m^2 - 2m + 1)$. $\phi_3(m) = 0$ gives $m(m - 1)^2 = 0$

$$\phi'_3(m) = 3m^2 - 4m,\ \phi''_3(m) = 6m - 4$$

$$\phi_2(m) = 0,\ \phi'_2(m) = 0$$

$$\phi_1(m) = -1 - m.$$

To find c when $m = 1$, we apply the formula

$$\frac{c^2}{2!}\phi''_3(m) + c\phi'_2(m) + \phi_1(m) = 0$$

i.e., $\frac{c^2}{2}(6m - 4) - (m + 1) = 0$

or $\quad c^2(3m - 2) = m + 1. \qquad (1)$

Putting $m = 1$ in (1), we get $c^2 = 2$ or $c = \pm\sqrt{2}$.

Hence equations of two asymptotes are

$$y = x \pm \sqrt{2}$$

To find c, when $m = 0$, we apply $c\phi'_3(m) + \phi_2(m) = 0$

i.e., $c(3m^2 - 4m) = 0$

or $\quad c = 0$

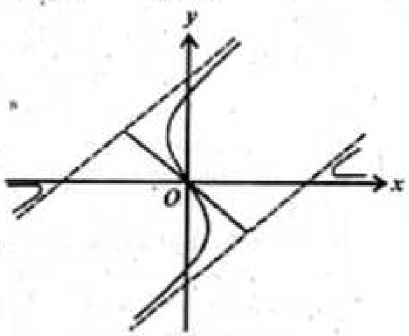
Thus $y = 0$ is an asymptote (already found)

IV. Putting $x = 0$, we find that the curve cuts y-axis at

$$y = 0,\quad y = 1,\quad y = -1$$

It cuts the x-axis only at the origin.

Sketch of the curve is as shown.



12. $y^4 - x^4 + xy = 0 \qquad (1)$

Sol.

I. It is symmetric in the opposite quadrants as on changing x into $-x$ and y into $-y$, equation of the curve remains unchanged.

II. It passes through the origin.

Tangents at the origin are $x = 0$, $\quad y = 0$

Thus the origin is a node.

III. There are no asymptotes parallel to the coordinate axes.

For oblique asymptotes, we have

$$\phi_4(m) = m^4 - 1,\ \phi_3(m) = 0$$

$$\phi_4(m) = 0 \quad \text{give} \quad m = \pm 1$$

$$\phi'_4(m) = 4m^3$$

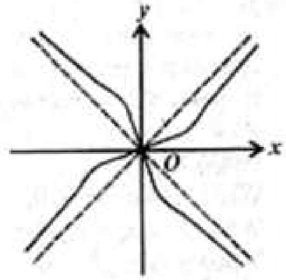
$$c\phi'_4(m) + \phi_3(m) = 0$$

gives $c = 0$.

Hence $y = \pm x$ are oblique asymptotes of (1).

IV. It cuts the coordinate axes at the origin only.

Sketch of the curve is as shown.



13. $x^3 + y^3 = 3ax^2$, $(a > 0)$

Sol.

I. There is no symmetry about any axis.

II. The curve passes through the origin and tangents at the origin are $x^2 = 0$. Thus the origin is a cusp.

III. There is no asymptote parallel to the axes.

For oblique asymptotes, we have

$$\phi_3(m) = m^3 + 1 = 0 \text{ gives } m = -1$$

$$\phi'_3(m) = 3m^2,\ \phi_2(m) = -3a$$

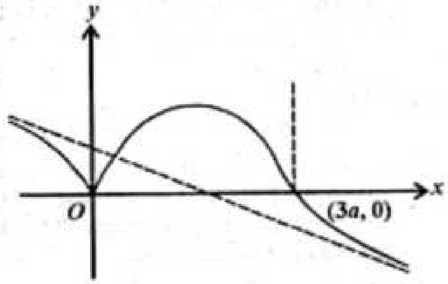
$$c\phi'_3(m) + \phi_2(m) = 0 \quad \text{gives} \quad c(3m^2) - 3a = 0$$

When $m = -1$, $3c = 3a$ $\quad$ or $\quad c = a$

Thus $x + y = a$ is an asymptote of curve.

IV. It cuts the x-axis at $(0, 0)$ and $(3a, 0)$

x and y cannot both be negative, for this would make L.H.S. negative and R.H.S. positive.



Hence no part of the curve exists in the third quadrant.
Sketch of the curve is as shown.

14. $y^2 = (x-a)(x-b)(x-c)$

Sol. We suppose that $a, b, c,$ are positive and consider the following possibilities.

Case I. $a < b < c$

Case II. $a = b < c$

Case III. $a < b = c$

Case IV. $a = b = c$

Case I. $a < b < c$ (I)

It is symmetric about the x-axis.
It does not pass through the origin.
It meets the x-axis at $(a, 0)$, $(b, 0)$ and $(c, 0)$ but it does not meet the y-axis. $x = a$, $x = b$, $x = c$ are tangents at $(a, 0)$, $(b, 0)$ and $(c, 0)$ respectively.

When $x < a, y^2 < 0$, *i.e.,* y is not real when $a < x < b$
When $b < x < c, y^2 < 0$, *i.e.,* y is not real.
When $x > c, y^2 > 0$

Hence no part of the curve lies to the left of the line $x = a$ and between the lines $x = b, x = c$.

As x increases beyond c, y^2 also increases. We have

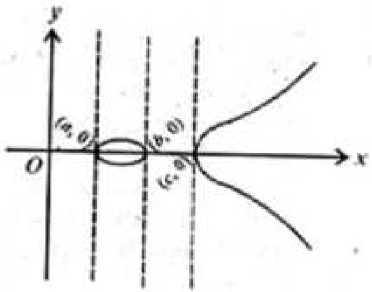
$$2y\frac{dy}{dx} = 3x^2 - 2(a+b+c)x + (ab+bc+ca)$$

or
$$\frac{dy}{dx} = \frac{3x^2 - 2(a+b+c)x + (ab+bc+ca)}{2\sqrt{(x-a)(x-b)(x-c)}}$$

$$= \frac{x^{1/2}\left[3 - 2(a+b+c)\frac{1}{x} + (ab+bc+ca)\left(\frac{1}{x^2}\right)\right]}{\sqrt{\left(1-\frac{a}{x}\right)\left(1-\frac{b}{x}\right)\left(1-\frac{c}{x}\right)}} \longrightarrow \infty$$

as $x \longrightarrow \infty$.

We have the curve as shown. It consists of a closed loop and an infinite branch.



Case II. $a = b < c$

The equation now becomes

$$y^2 = (x-a)^2(x-c)$$

It is symmetric about the y-axis.
It does not pass through the origin.
It has no asymptotes.

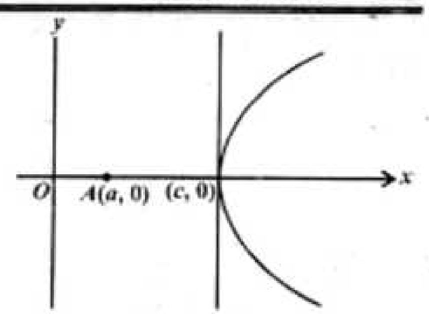
Shifting the origin to $(a, 0)$, equation of the curve becomes

$$y'^2 = x'^2(x'-c)$$

Tangents at the new origin are

$$y'^2 = -cx'^2$$

which are imaginary so that $(a, 0)$ is an isolated point.
The sketch is as shown.



Case III. $a < b = c$.

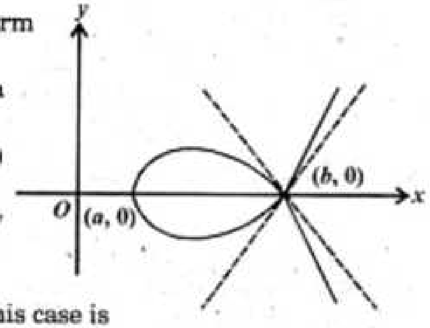
The equation now takes the form

$$y^2 = (x-a)(x-b)^2.$$

It is easily seen that $(b, 0)$ is a node and

$$y = \pm\sqrt{(b-a)}\,(x-b)$$

are the nodal tangents.
The curve may now be easily drawn as shown.



Case IV. $a = b = c$.

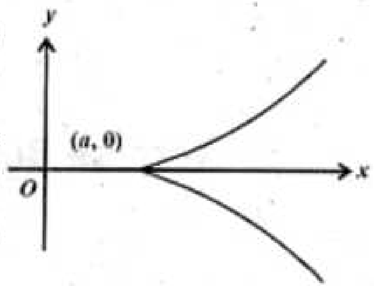
The equation of the curve in this case is

$$y^2 = (x-a)^3$$

Shifting the origin to $(a, 0)$, it takes the from

$$y'^2 = x'^3.$$

Tangents at the new origin are $y'^2 = 0$ which are coincident so that $(a, 0)$ is cusp.
The graph of the curve is as shown.



15. $x = t^2 - t + 2, y = t + 3, -\infty < t < \infty$

Sol. Here parametric equations of the curve can be easily transformed into rectangular equation. Substituting $t = y - 3$ from the second equation into the first, we have

$$x = (y-3)^2 - (y-3) + 2$$

or $$x = y^2 - 7y + 14 = \left(y - \frac{7}{2}\right)^2 + \frac{7}{2}$$

i.e., $$x - \frac{7}{4} = \left(y - \frac{7}{2}\right)^2$$

which is an equation of a parabola with vertex at $\left(\frac{7}{4}, \frac{7}{2}\right)$.

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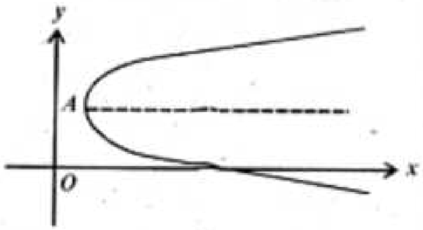
Equation of the axis of the parabola is $y = \frac{7}{2}$ which is parallel to the x-axis. The parabola meets the x-axis at $x = 14$. The shape of the curve is as shown.

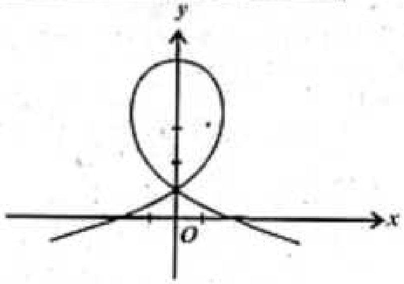


16. $x = t^3 - 3t, y = 4 - t^2, -\infty < t < \infty$

Sol. We construct the following table of values

$t =$	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
$x =$	-2	0	2	0	-2	0	2
$y =$	-0	1	3	4	3	1	0
$\frac{dy}{dx}$	$\frac{4}{9}$	$\frac{1}{\sqrt{3}}$	∞	0	∞	$\frac{-1}{\sqrt{3}}$	$\frac{-4}{9}$

The curve is symmetric about the y-axis. The shape of the curve is as shown.



Exercise Set 7.5 (Page 316)

1. Find the area of the region bounded by the graph of

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (1)$$

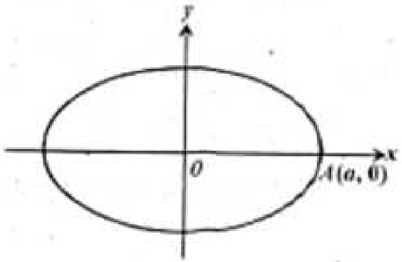
Sol. Here $x = a\cos\theta$ and $y = b\sin\theta$ are parametric equations of the ellipse.

$$dx = -a\sin\theta\, d\theta.$$

Let A be the required area of the region bounded by the graph of (1).
Then

$$A = 4\int_0^a y\,dx$$

$$= 4\int_{\pi/2}^{0} (b\sin\theta)(-a\sin\theta)\,d\theta$$



$$= 4ab\int_0^{\pi/2} \sin^2\theta\,d\theta$$

$$= 4ab\left(\frac{1}{2}\cdot\frac{\pi}{2}\right), \text{ (by Wallis Formula)}$$

$$= \pi ab.$$

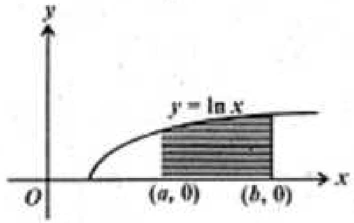
2. Find the area of the region bounded by $y = \ln x$, x-axis, $x = a, x = b$

Sol. Area $= \int_a^b y\,dx = \int_a^b \ln x\,dx$

$$= |x\ln x|_a^b - \int_a^b \frac{x}{x}\,dx$$

$$= b\ln b - a\ln a - \int_a^b 1\,.\,dx$$

$$= b\ln b - a\ln a - b + a$$



3. Find the area of the region bounded by $xy = c^2$, x-axis, $x = a, x = b$.

Sol. Required area $= \int_a^b y\,dx = \int_a^b \frac{c^2}{x}\,dx = c^2\int_a^b \frac{dx}{x}$

$$= c^2\,|\ln x|_a^b = c^2\,[\ln b - \ln a] = c^2 \ln\frac{b}{a}$$

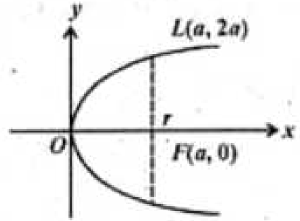
4. Find the area of the region bounded by the graph of the parabola $y^2 = 4ax$ and its latusrectum.

Sol. Required area $= 2\int_0^a y\,dx$

$$= 2\int_0^a \sqrt{4\,ax}\,dx$$

$$= 4\sqrt{a}\int_0^a x^{1/2}\,dx$$

$$= 4\sqrt{a}\left|\frac{x^{3/2}}{3/2}\right|_0^a = 4\sqrt{a}\,.\,\frac{2}{3}\,|x^{3/2}|_0^a = \frac{8}{3}\sqrt{a}\,.\,a^{3/2} = \frac{8a^2}{3}$$



5. Prove that the area of the region bounded by $y = c\cosh\left(\frac{x}{c}\right)$, x-axis, $x = a, x = b$ is $c^2\left[\sinh\left(\frac{b}{c}\right) - \sinh\left(\frac{a}{c}\right)\right]$.

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Sol. Required area $= \int_a^b y\,dx = \int_a^b c\cosh\frac{x}{c}$

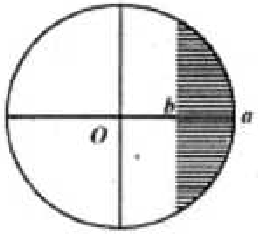
$$= c.c\left|\sinh\frac{x}{c}\right|_a^b = c^2\left[\sinh\left(\frac{b}{c}\right) - \sinh\left(\frac{a}{c}\right)\right]$$

6. Find the area of the smaller segment cut from a circular disc of radius a by a chord at a distance b from the centre, $(a > b)$.

Sol. Let the circular disc be

$$x^2 + y^2 = a^2$$

or $\quad y = \sqrt{a^2 - x^2}$



Required area $= 2\int_b^a y\,dx$

$$= 2\int_b^a \sqrt{a^2 - x^2}\,dx$$

$$= 2\left[\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right]_b^a$$

$$= 2\left[\frac{\pi a^2}{4} - \left\{\frac{b\sqrt{a^2 - b^2}}{2} + \frac{a^2}{2}\arcsin\frac{b}{a}\right\}\right]$$

$$= \frac{\pi a^2}{4} - \frac{b\sqrt{a^2 - b^2}}{2} - a^2\arcsin\frac{b}{a}$$

7. Find the area of the region bounded by the loop of the curve

$$3ay^2 = x(x - a)^2$$

Please see the figure in Problem 1 of Exercise Set 6.4

Sol. Required area $= 2\int_0^a y\,dx = 2\int_0^a \frac{\sqrt{x(x-a)^2}}{\sqrt{3a}}\,dx = \frac{2}{\sqrt{3a}}\int_0^a x^{1/2}(x - a)\,dx$

$$= \frac{2}{\sqrt{3a}}\int_0^a (x^{3/2} - ax^{1/2})\,dx = \frac{2}{\sqrt{3a}}\left|\frac{2}{5}x^{5/2} - a\,\frac{2}{3}x^{3/2}\right|_0^a$$

$$= \frac{2}{\sqrt{3a}}\left[\frac{2}{5}a^{5/2} - \frac{3}{2}a^{5/2}\right] = \frac{2}{\sqrt{3}\sqrt{a}}\left[\frac{2}{5} - \frac{3}{2}\right]a^{5/2}$$

$$= \frac{2}{\sqrt{3}\sqrt{a}}\cdot\left(-\frac{4}{15}\right)a^{5/2}$$

$$= -\frac{8a^2}{\sqrt{3}\,(15)} = \frac{8a^2}{15\sqrt{3}} \text{ (in absolute value).}$$

8. Find the area of the region lying between the curve

$$x^2(x^2 + y^2) = a^2(y^2 - x^2) \qquad (1)$$

and its asymptotes.

Sol. The highest power of y in (1) is y^2. Its coefficient is $x^2 - a^2 = 0$. Hence asymptotes parallel to the y-axis are $x = \pm a$.

Equation of the curve is

$$(x^2 - a^2)y^2 = -a^2x^2 - x^4$$

or $\quad (a^2 - x^2)y^2 = x^2(x^2 + a^2) \qquad$ or $\qquad y^2 = \dfrac{x^2(x^2 + a^2)}{a^2 - x^2}$

or $\quad y = \pm\dfrac{x\sqrt{x^2 + a^2}}{\sqrt{a^2 - x^2}} = \dfrac{x\sqrt{x^2 + a^2}}{\sqrt{a^2 - x^2}}$, (taking +ve sign)

The curve is symmetric about both the axes. We find the area bounded by the x-axis, $x = 0$, $x = a$ and the curve and multiply it by 4.

$$\text{Area} = 4\int_0^a y\,dx = 4\int_0^a \frac{x\sqrt{x^2 + a^2}}{\sqrt{a^2 - x^2}}\,dx$$

$$= 4\int_0^a \frac{x\sqrt{a^2 + x^2}\,.\,\sqrt{a^2 + x^2}}{\sqrt{a^4 - x^4}} = 4\int_0^a \frac{x(a^2 + x^2)}{\sqrt{a^4 - x^4}}\,dx$$

$$= 4\int_0^a \frac{a^2x}{\sqrt{a^4 - x^4}}\,dx + 4\int_0^a \frac{x^3}{\sqrt{a^4 - x^4}}\,dx$$

$$= 4I_1 + 4I_2 \qquad (2)$$

$I_1 = a^2\displaystyle\int_0^a \frac{x\,dx}{\sqrt{a^4 - x^4}}$ $\quad\Big|\quad$ Put $x^2 = a^2\sin\theta$ or $2x\,dx = a^2\cos\theta\,d\theta$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{a^2\cos\theta\,d\theta}{a^2\cos\theta} = \frac{a^2}{2}\int_0^{\pi/2} 1\,.\,d\theta = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4} \qquad (3)$$

$I_2 = \displaystyle\int_0^a x^3(a^4 - x^4)^{-1/2}\,dx$ $\quad\Big|\quad$ Put $a^4 - x^4 = t$ or $-4x^3\,dx = dt$

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$$= \int_{a^4}^{0} -\frac{1}{4} t^{-1/2}\, dt = \frac{1}{4}\int_0^{a^4} t^{-1/2}\, dt$$

$$= \frac{1}{4}\cdot\frac{2}{1}\left|t^{1/2}\right|_0^{a^4} = \frac{1}{2}\sqrt{a^4} = \frac{1}{2}a^2 \qquad (4)$$

Putting the values from (3) and (4) into (2), we have:

Required area $= 4\left(\frac{\pi a^2}{4}\right) + 4\left(\frac{a^2}{2}\right) = \pi a^2 + 2a^2 = a(\pi^2 + 2)$.

9. Find the area of the region bounded by the curve $xy^2 = 4(2 - x)$ and the y-axis.

Sol. Here $y^2 = \frac{4(2-x)}{x}$ or $y = \frac{\pm 2\sqrt{2-x}}{\sqrt{x}}$. Thus the curve is symmetric about the x-axis.

Required area $= 2\int_0^2 y\, dx = 2\int_0^2 \frac{\sqrt{2-x}}{\sqrt{x}}\, dx$

Put $x = 1 + \cos\theta$ or $dx = -\sin\theta\, d\theta$

Now limits of integration are from $-\pi$ to 0

$$\text{Area} = 2\int_{-\pi}^{0} \frac{2.\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \times -\sin\theta\, d\theta = 4\int_0^{-\pi} \frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}} \sin\theta\, d\theta$$

$$= 4\int_0^{-\pi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} . 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta$$

$$= 8\int_0^{-\pi} \sin^2\frac{\theta}{2}\, d\theta \qquad \text{Put } \frac{\theta}{2} = -t \text{ or } d\theta = -2\, dt$$

$$= 8\int_0^{\pi/2} \sin^2 t \times (-2\, dt) = -16\int_0^{\pi/2} \sin^2 t\, dt$$

$$= -16\frac{1}{2}.\frac{\pi}{2} = -4\pi = 4\pi \text{ (in absolute units).}$$

10. Find the area of the region between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.

Sol. We have $y^2(x^2 - a^2) = a^2x^2$ or $y^2 = \frac{a^2x^2}{a^2 - x^2}$

Asymptotes parallel to y-axis are

$x^2 - a^2 = 0$ i.e., $x = \pm a$.

Since the curve is symmetric about both axis, required area.

$$= 4\int_0^a y\, dx = 4\int_0^a \frac{ax}{\sqrt{a^2 - x^2}}\, dx$$

$$= 4\int_0^{\pi/2} \frac{a . a\sin\theta . a\cos\theta\, d\theta}{a\cos\theta}, \text{ putting } x = a\sin\theta, dx = a\cos\theta$$

$$= 4a^2\int_0^{\pi/2} \sin\theta\, d\theta = 4a^2\left|-\cos\theta\right|_0^{\pi/2} = 4a^2.$$

11. Find the area of the region bounded by the loop of the curve

$ay^2 = x^2(a - x)$

Sol. The curve is symmetric about the x-axis.

$$\text{Area of the loop} = 2\int_0^a y\, dx = 2\int_0^a \frac{x\sqrt{a-x}}{\sqrt{a}}\, dx$$

$$= \int_0^{\pi/2} \frac{a\sin^2\theta . \sqrt{a}\cos\theta . 2a\sin\theta\cos\theta\, d\theta}{\sqrt{a}},$$

on putting $x = a\sin^2\theta$ and $dx = 2a\sin\theta\cos\theta\, d\theta$

$$= 4a^2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\, d\theta$$

$$= 4a^2\frac{2.1}{5.3}, \text{ (by Wallis Formula)} = \frac{8a^2}{15}$$

12. Prove that area of the region bounded by the curve $a^2x^2 = y^3(2a - y)$ is equal to that of the circular disc of radius a.

Sol. The curve is symmetric about the y-axis.

$$\text{Required area} = 2\int_0^{2a} x\, dy = 2\int_0^{2a} \frac{y^{3/2}\sqrt{2a-y}}{a}\, dy.$$

Putting $y = 2a\sin^2\theta$, we obtain $dy = 4a\sin\theta\cos\theta\, d\theta$

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and limits of integration are from 0 to $\frac{\pi}{2}$.

Hence area of the region.

$$= 2\int_0^{\pi/2}\frac{(2a)^{3/2}\sin^3\theta\,.\,(2a)^{1/2}\cos\theta\,.\,4a\sin\theta\cos\theta\,d\theta}{a}$$

$$= \frac{2}{a}\int_0^{\pi/2}(2a)^2\,(4a)\sin^4\theta\cos^2\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta$$

$$= 32a^2\,\frac{3\,.\,1\,.\,1}{6\,.\,4\,.\,2}\,\frac{\pi}{2} = \pi a^2 \quad \text{(by Wallis Formula)}$$

= area of the circular disc of radius a.

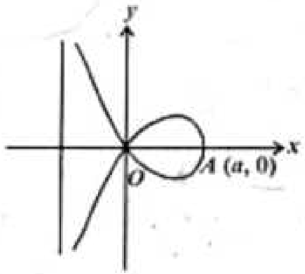
13. Find the area of the region bounded by the loop of the curve $y^2(a + x) = x^2(a - x)$. Also find the area of the region bounded by the curve and its asymptotes.

Sol. The curve $y^2 = x^2\,\frac{a-x}{a+x}$ (1)

is symmetric about the x-axis. $a + x = 0$ is its vertical asymptote. x-intercept of the curve is a

From (1), we have $y = \pm x\sqrt{\frac{a-x}{a+x}}$

Area of the region bounded by the loop



$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}$$

Put $x = a\cos\theta$

or $dx = -a\sin\theta\,d\theta$. Then

$$2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx = 2\int_{\pi/2}^0 a\cos\theta\sqrt{\frac{a-a\cos\theta}{a+a\cos\theta}}(-a\sin\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\cos\theta\,\frac{1-\cos\theta}{\sin\theta}\sin\theta\,d\theta$$

$$= 2a^2\int_0^{\pi/2}\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 2a^2\left[\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 2a^2\left(1 - \frac{\pi}{2}\right) = \frac{a^2(4-\pi)}{2}$$

Area of the region bounded by the curve and its asymptote

$$= 2\int_{-a}^0 x\sqrt{\frac{a-x}{a+x}}\,dx\text{, [since } x \text{ is -ive and } y \text{ is +ive]}$$

Substituting $x = a\cos\theta$ as before, the area

$$= 2a^2\int_\pi^{\pi/2}\left(\cos\theta - \frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= -2a^2\left[\sin\theta - \frac{1}{2}\theta - \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi}$$

$$= -2a^2\left[-\frac{\pi}{2} - 1 + \frac{\pi}{4}\right] = \frac{a^2(4+\pi)}{2}.$$

14. Find the area of region bounded by one arch of the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ and its base.

Sol. Here $x = a(\theta - \sin\theta), dx = a(1 - \cos\theta)\,d\theta$

Limits of integration are from 0 to 2π.

Required area $\int y\,dx$

$$= \int_0^{2\pi} a(1-\cos\theta)\,.\,a(1-\cos\theta)\,d\theta$$

$$= a^2\int_0^{2\pi}(1-\cos\theta)^2\,d\theta = \int_0^{2\pi}\left(2\sin^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{2\pi}\sin^4\frac{\theta}{2}\,d\theta$$

$$= 8a^2\int_0^{2\pi}\sin^4 t\,dt \qquad \left(\text{putting } \frac{\theta}{2} = t, d\theta = 2dt\right)$$

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$$= 16a^2 \int_0^{\pi/2} \sin^4 t\, dt = 16a^2 \frac{3.1}{4.2}\frac{\pi}{2} = 3\pi a^2$$

15. Find the are of the region bounded by the loop of the curve

$$x^4 + y^4 = 2a^2xy.$$

Sol. When changed to polar coordinates, equation of the curve is

$$r^2 = \frac{2a^2 \sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta}$$

Limits of integration for θ in the fist quadrant are 0 to $\frac{\pi}{2}$.

$$\text{Area} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{2a^2\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta} d\theta$$

$$= a^2 \int_0^{\pi/2} \frac{\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta} d\theta = a^2 \int_0^{\pi/2} \frac{\tan\theta\sec^2\theta\, d\theta}{1 + \tan^4\theta}$$

$$= \frac{a^2}{2}\int_0^{\infty} \frac{dt}{1+t^2} \left(\begin{array}{l}\text{putting } \tan^2\theta = t \text{ so that } 2\tan\theta\sec^2\theta\, d\theta \\ \quad = dt \text{ and limits of } t \text{ are } 0 \text{ to } \infty.\end{array}\right)$$

$$= \frac{a^2}{2}\left|\arctan t\right|_0^{\infty} = \frac{a^2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi a^2}{4}$$

16. Find the area of the region bounded by the loop of the curve

$$(x + y)^2 (x^2 + y^2) = 2a^2xy.$$

Sol. Changing into polar coordinates, equation of the curve is

$$r^2 (\cos\theta + \sin\theta)^2 = 2a^2 \cos\theta \sin\theta$$

$$\text{or} \quad r = \frac{2a^2\sin\theta\cos\theta}{(\cos\theta+\sin\theta)^2} = \frac{a^2\sin 2\theta}{\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta}$$

$$= \frac{a^2 \sin 2\theta}{1 + \sin 2\theta}$$

$$\text{Area of the loop} = \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{a^2\sin 2\theta}{1+\sin 2\theta}\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{(1+\sin 2\theta) - 1}{1 + \sin 2\theta} d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} 1 . d\theta - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{1+\sin 2\theta}$$

$$= \frac{a^2}{2}\left|\theta\right|_0^{\pi/2} - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{1+\cos\left(\frac{\pi}{2} - 2\theta\right)}$$

$$= \frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{a^2}{2}\int_0^{\pi/2} \frac{d\theta}{2\cos^2\left(\frac{\pi}{4}-\theta\right)}$$

$$= \frac{\pi a^2}{4} - \frac{a^2}{4}\int_0^{\pi/2} \sec^2\left(\frac{\pi}{4}-\theta\right) d\theta$$

$$= \frac{\pi a^2}{4} + \frac{a^2}{4}\left|\tan\left(\frac{\pi}{4}-\theta\right)\right|_0^{\pi/2}$$

$$= \frac{\pi a^2}{4} + \frac{a^2}{4}\left[\tan\left(-\frac{\pi}{4}\right) - \tan\left(\frac{\pi}{4}\right)\right]$$

$$= \frac{\pi a^2}{4} + \frac{a^2}{4}[-1-1] = \frac{\pi a^2}{4} - \frac{a^2}{2} = a^2\left(\frac{\pi}{2} - \frac{1}{2}\right)$$

17. Show that the area of the region bounded by the circle $r = a$ and the curve $r = a \cos 5\theta$ is equal to three-fourths of the area of the circle.

Sol. Area of the circle $= \pi a^2$

The curve $r = a \cos 5\theta$ has five loops. It is symmetric about the initial line

It has 10 portions and the limits of integration of one portion are from 0 to $\frac{\pi}{10}$. Area of one portion

$$= \frac{1}{2}\int_0^{\pi/10} r^2\, d\theta = \frac{1}{2}\int_0^{\pi/10} a^2\cos^2 5\theta\, d\theta = \frac{a^2}{2}\int_0^{\pi/10} a^2\cos^2 5\theta\, d\theta$$

$$= \frac{a^2}{10}\int_0^{\pi/10} \cos^2 t\, dt \qquad \left(\text{putting } 5\theta = t \ \text{ or } \ d\theta = \frac{1}{5}dt\right)$$

$$= \frac{a^2}{10}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{40}$$

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Hence area of the 10 portions *i.e.*, of all the five loops

$$= 10\left(\frac{\pi a^2}{10}\right) = \frac{\pi a^2}{4}$$

$$\text{Required area} = \pi a^2 - \frac{\pi a^2}{4} = \frac{3}{4}\pi a^2$$

18. Find the area of the region bounded by the cardioid $r = a\,(1 + \cos\theta)$.

Sol. $$\text{Required area} = 2\cdot\frac{1}{2}\int_0^{\pi} r^2\,d\theta = \int_0^{\pi} r^2\,d\theta$$

$$= \int_0^{\pi} a^2(1+\cos\theta)^2\,d\theta = \int_0^{\pi} a^2\left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$$

$$= 4a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\,d\theta$$

Put $\frac{\theta}{2} = t$ or $d\theta = 2dt$ and the limits of integration become 0 and $\frac{\pi}{2}$

$$\text{Required area} = 4a^2\int_0^{\pi/2}\cos^4 t\,.\,2dt$$

$$= \int_0^{\pi/2}\cos^4 t\,dt = 8a^2\,\frac{3\,.\,1}{4\,.\,2}\,\frac{\pi}{2} = \frac{3}{2}\pi a^2$$

19. Find the area of the region bounded by $r^2 = a^2\cos 2\theta$.

Sol. $r^2 = a^2\cos 2\theta$ (1)

(1) is symmetric about the initial line.

It is symmetric about the pole.

It is symmetric about the line. $\theta = \frac{\pi}{2}$.

Putting $r = 0$, we have $\cos 2\theta = 0$ and so

$$2\theta = \frac{\pi}{2} \quad \text{or} \quad \theta = \frac{\pi}{4}$$

This curve has four symmetric portions and the limits of integration of one such portion are 0 to $\frac{\pi}{4}$. Hence area of the region

$$= 4\,.\,\frac{1}{2}\int_0^{\pi/4} r^2\,d\theta = 2\int_0^{\pi/4} a^2\cos 2\theta\,d\theta$$

$$= 2\int_0^{\pi/2} a^2\cos t\,.\,\frac{dt}{2} \qquad \begin{array}{|ll} \text{Put} & 2\theta = t \\ \text{or} & d\theta = \frac{dt}{2} \end{array}$$

$$= a^2\int_0^{\pi/2}\cos t\,dt = a^2\,\left|\sin t\right|_0^{\pi/2} = a^2$$

20. Find the area of the region bounded by the loop of the folium $x^3 + y^3 = 3axy$.

Sol. The curve is symmetric about the line $y = x$

It meets $y = x$ at $(0, 0)$ and $\left(\frac{3a}{2}, \frac{3a}{2}\right)$.

Converting the given equation into polar coordinates, we get

$$r^3(\cos^3\theta + \sin^3\theta) = 3ar^2\sin\theta\cos\theta$$

or $$r = \frac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$$

Putting $r = 0$, we get $\theta = 0$ and $\theta = \frac{\pi}{2}$ as limits of integration.

$$\text{Area of the loop} = \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2}\frac{9a^2\sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\,d\theta$$

$$= \frac{9}{2}a^2\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta$$

Put $1 + \tan^3\theta = z$ so that $3\tan^2\theta\sec^2\theta\,d\theta = dz$ and the limits of integration become $z = 1$ and $z = \infty$

$$\text{Required area} = \frac{3a^2}{2}\int_1^{\infty}\frac{dz}{z^2} = -\frac{3a^2}{2}\left|\frac{1}{z}\right|_1^{\infty} = \frac{3}{2}a^2$$

21. Find the area enclosed by $r = \dfrac{6}{2 - \cos\theta}$

Sol. Here we have

$$2r - r\cos\theta = 6 \qquad (1)$$

Transforming into rectangular coordinates (1) becomes

$$2\sqrt{x^2+y^2} - x = 6$$

or $$2^2(x^2 + y^2) = (x+6)^2$$

or $$4x^2 + 4y^2 = x^2 + 12x + 36$$

i.e., $$3(x^2 - 4x + 4) + 4y^2 = 48$$

or $$3(x-2)^2 + 4y^2 = 48$$

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or $\dfrac{(x-2)^2}{16} + \dfrac{y^2}{12} = 1$

which is an ellipse with semi-major axis 4 and semi-minor axis $\sqrt{12}$. Hence the area enclosed by the ellipse is

$\pi \cdot 4\sqrt{12} = 8\sqrt{3}\,\pi$. **[See Problem 1]**

22. Find the area of the region that lies outside the cardioid $r = 1 + \cos\theta$ and inside the circle $r = 3\cos\theta$.

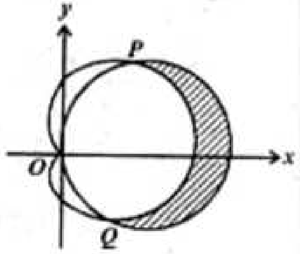
Sol. The required area is the difference of the area enclosed by the circle $r = 3\cos\theta = f(\theta)$ and portion of the area enclosed by $r = 1 + \cos\theta = g(\theta)$. Solving the two equations, we have

$1 + \cos\theta = 3\cos\theta$

or $\cos\theta = \dfrac{1}{2}$ so that $\theta = \pm\dfrac{\pi}{3}$.

Coordinates of the points of intersection are

$P\left(\dfrac{3}{2}, \dfrac{\pi}{3}\right)$ and $Q\left(\dfrac{3}{2}, \dfrac{-\pi}{3}\right)$



$$\text{Required area} = 2\int_0^{\pi/3} \frac{1}{2}[(f(\theta))^2 - g(\theta)^2]\,d\theta$$

$$= \int_0^{\pi/3} [(3\cos\theta)^2 - (1+\cos\theta)^2]\,d\theta$$

$$= \int_0^{\pi/3} (8\cos^2\theta - 2\cos\theta - 1)\,d\theta$$

$$= \int_0^{\pi/3} [4(1+\cos 2\theta) - 2\cos\theta - 1]\,d\theta$$

$$= [4\theta + 2\sin 2\theta - 2\sin\theta - \theta]\,d\theta = \pi$$

23. Find the area of the region inside the graph of $r^2 = 2a^2\cos 2\theta$ and outside the circle $r = a$.

Sol. $r^2 = 2a^2\cos 2\theta$ and $r = a$

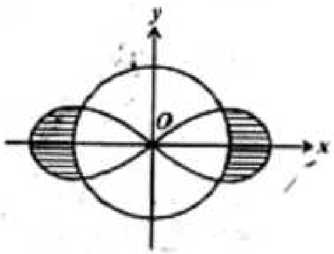
Solving these equations, we have

$1 = 2\cos 2\theta,$

or $\cos 2\theta = \dfrac{1}{2}$

or $2\theta = \dfrac{\pi}{3}, \dfrac{-\pi}{3}$

or $\theta = \dfrac{\pi}{6}, \dfrac{-\pi}{6}$



$$\text{Required area} = \frac{4}{2}\int_0^{\pi/6} (2a^2\cos 2\theta - a^2)\,d\theta$$

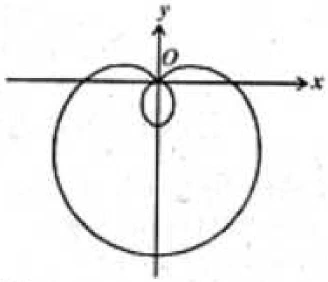
$$= 2\,[a^2\sin 2\theta - a^2\theta]_0^{\pi/6} = 2a^2\left[\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right]$$

$$= a^2\left(\sqrt{3} - \frac{\pi}{3}\right) = \frac{a^2}{3}(3\sqrt{3} - \pi)$$

24. Find the area of the region bounded the smaller loop of the limacon $r = 2 - 3\sin\theta$.

Sol. The left half of the smaller loop of the limacon $r = 2 - 3\sin\theta$ is determined by $\theta = \arcsin\left(\dfrac{2}{3}\right)$ and $\theta = \dfrac{\pi}{2}$

Area enclosed by the smaller loop



$$= 2\int_{\arcsin 2/3}^{\pi/2} \frac{1}{2}(2 - 3\sin\theta)^2\,d\theta$$

$$= \int_{\arcsin 2/3}^{\pi/2} (4 - 12\sin\theta + 9\sin^2\theta)\,d\theta$$

$$= \left[4\theta + 12\cos\theta + \frac{9}{2}\theta - \frac{9}{4}\sin 2\theta\right]_{\arcsin(2/3)}^{\pi/2}$$

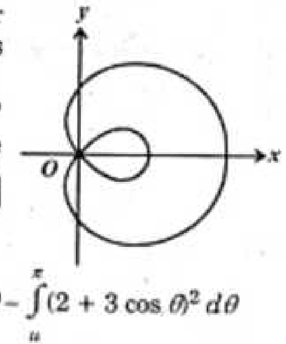
$$= \left[\frac{17}{2}\cdot\frac{\pi}{2} - \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 12\cos\arcsin\frac{2}{3} - \frac{9}{4}\sin\left(2\arcsin\frac{2}{3}\right)\right]$$

$$= \frac{17\pi}{4} - \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 12\cdot\frac{\sqrt{5}}{3} - \frac{9}{4}\cdot 2\cdot\frac{2}{3}\cdot\frac{\sqrt{5}}{3}$$

$$= \frac{17\pi}{4} - \frac{17}{2}\arcsin\left(\frac{2}{3}\right) + 3\sqrt{5}.$$

25. Find the area of the region between the two loops of the limacon $r = 2 + 3\cos\theta$.

Sol. The curve is symmetric about the polar axis. The upper half of the outer loop is determined by $\theta = 0$ to $\theta = \arccos\left(-\dfrac{2}{3}\right)$, where r is positive and lower half of the smaller loop is traced by $\theta = \arccos\left(-\dfrac{2}{3}\right)$ to $\theta = \pi$, where r is negative.



$$\text{Required area} = \int_0^{\arccos(-2/3) = u} (2 + 3\cos\theta)^2\,d\theta - \int_u^{\pi} (2 + 3\cos\theta)^2\,d\theta$$

$$= \left[\frac{17}{2}\theta + \frac{9}{4}\sin 2\theta + 12\sin\theta\right]_0^u - \left[\frac{17}{2}\theta + \frac{9}{4}\sin 2\theta + 12\sin\theta\right]_u^\pi$$

$$= \frac{17}{2}u + \frac{9}{4}\sin 2u + 12\sin u - \left[\frac{17}{2}\pi - \frac{17}{2}u - \frac{9}{4}\sin 2u - 12\sin u\right]$$

$$= \frac{-17}{2}\pi + 17u + \frac{9}{2}\sin 2u + 24\sin u$$

$$= \frac{-17}{2}\pi + 17u + 9\sin u\cos u + 24\sin u$$

$$\left(\text{where } \cos u = -\frac{2}{3} \text{ and } \sin u = \frac{\sqrt{5}}{3}\right)$$

$$= -\frac{17}{2}\pi + 17\arccos\left(\frac{-2}{3}\right) + 9\left(\frac{-2}{3}\right)\left(\frac{\sqrt{5}}{3}\right) + 24\frac{\sqrt{5}}{3}$$

$$= \frac{-17}{2}\pi + 17\arccos\left(\frac{-2}{3}\right) + 6\sqrt{5}.$$

Exercise Set 7.6 (Page 328)

1. Find the length of the arc of the parabola $y^2 = 4ax$ cut off by the straight line $3y = 8x$.

Sol. We have $y^2 = 4ax$ (1)

and $3y = 8x$ (2)

Putting $y = \frac{8x}{3}$ from (2) into (1), we have

$$\left(\frac{8x}{3}\right)^2 = 4ax \quad \text{or} \quad 64x^2 = 36ax \quad \text{or} \quad x(16x - 9a) = 0$$

i.e., $x = 0, \quad x = \frac{9a}{16}$

So corresponding values of y are

$$y = 0, \quad y = \frac{3a}{2}$$

The points of intersection of (1) and (2) area (0, 0) and $\left(\frac{9a}{16}, \frac{3a}{2}\right)$

From (1), $\frac{dx}{dy} = \frac{y}{2a}$

Therefore, $\left(\frac{ds}{dy}\right)^2 = 1 + \left(\frac{dx}{dy}\right)^2 = 1 + \frac{y^2}{4a^2} = \frac{4a^2 + y^2}{4a^2}$

or $\frac{ds}{dy} = \frac{\sqrt{4a^2 + y^2}}{2a}$

Required length of the arc $= \frac{1}{2a}\int_0^{3a/2}\sqrt{4a^2 + y^2}\,dy$

$$= \frac{1}{2a}\left[\frac{y\sqrt{4a^2 + y^2}}{2} + \frac{4a^2}{2}\ln\frac{y + \sqrt{4a^2 + y^2}}{2a}\right]_0^{3a/2}$$

$$= \frac{1}{4a}\left[\frac{3a}{2}\sqrt{4a^2 + \frac{9a^2}{4}} + 4a^2\ln\frac{\left(\frac{3a}{2} + \sqrt{4a^2 + \frac{9a^2}{4}}\right)}{2a} - 4a^2\ln 1\right]$$

$$= \frac{1}{4a}\left[\frac{3a}{2}\cdot\frac{5a}{2} + 4a^2\ln\frac{\frac{3a}{2} + \frac{5a}{2}}{2a}\right] = \frac{1}{4a}\left[\frac{15a^2}{16} + 4a^2\ln 2\right]$$

$$= \frac{a}{4}\left[\frac{15}{4} + 4\ln 2\right] = a\left[\frac{15}{16} + \ln 2\right] = a\left[\ln 2 + \frac{15}{16}\right]$$

2. Show that in the catenary $y = \cosh\left(\frac{x}{c}\right)$, the length s of the arc from the vertex (where $x = 0$) to any point is given by $s = c\sinh\left(\frac{x}{c}\right)$.

Sol. Here $y = c\cosh\frac{x}{c}$, so $\frac{dy}{dx} = \sinh\frac{x}{c}$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \sinh^2\frac{x}{c} = \cosh^2\frac{x}{c}$$

or $\frac{ds}{dx} = \cosh\frac{x}{c}$ or $ds = \cosh\frac{x}{c}\,dx$

Required length $= \int_0^x \cosh\frac{x}{c}\,dx = c\left|\sinh\frac{x}{c}\right|_0^x$

$$= c\left(\sinh\frac{x}{c} - 0\right) = c\sinh\frac{x}{c}$$

3. Find the length of the loop of the curve $3ay^2 = x(a - x)^2$.

Sol. The curve is symmetric about the x-axis and meets the x-axis at $x = 0, x = a$. For the loop above the x-axis, we have

$$y = \frac{1}{\sqrt{3a}}\sqrt{x}(a - x)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{3a}}\left[\frac{a}{2\sqrt{x}} - \frac{3}{2}\sqrt{x}\right] = \frac{1}{2\sqrt{3a}}\left(\frac{a - 3x}{\sqrt{x}}\right)$$

$$1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{1}{12ax}(a^2 - 6ax + 9x^2)$$

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$$= \frac{12ax + a^2 - 6ax + 9x^2}{12ax} = \frac{(a+3x)^2}{12ax}$$

Length of the loop above the x-axis

$$= \int_0^a \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^a \frac{a + 3x}{2\sqrt{3a}\,.\sqrt{x}}\, dx$$

$$= \frac{1}{2\sqrt{3a}}\left[\int_0^a \left(\frac{a}{\sqrt{x}} + 3\sqrt{x}\right)\right] dx = \frac{1}{2\sqrt{3a}}\,[2ax\sqrt{x} + 2x^{3/2}]_0^a$$

$$= \frac{1}{2\sqrt{3a}}\,(2a^{3/2} + 2a^{3/2}) = \frac{2a}{\sqrt{3}}$$

Length of the complete loop $= 2, \frac{2a}{\sqrt{3}} = \frac{4a}{\sqrt{3}}$

4. Find the length of the arc of the curve $x = e^{\theta}\sin\theta$, $y = e^{\theta}\cos\theta$ from $\theta = 0$ to $\theta = \frac{\pi}{2}$.

Sol. $x = e^{\theta}\sin\theta$

$$\frac{dx}{d\theta} = e^{\theta}\sin\theta + e^{\theta}\cos\theta = e^{\theta}(\sin\theta + \cos\theta)$$

$y = e^{\theta}\cos\theta$

$$\frac{dy}{d\theta} = e^{\theta}\cos\theta - e^{\theta}\sin\theta = e^{\theta}(\cos\theta - \sin\theta)$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= e^{2\theta}\,[(\cos\theta + \sin\theta)^2 + (\cos\theta - \sin\theta)^2] = 2\,e^{2\theta}$$

$$\frac{ds}{d\theta} = \sqrt{2}\,e^{\theta}$$

$$s = \sqrt{2}\int_0^{\pi/2} e^{\theta}\, d\theta = \sqrt{2}\,|e^{\theta}|_0^{\pi/2}$$

$$= \sqrt{2}\,|e^{\pi/2} - e^{0}| = \sqrt{2}\,[e^{\pi/2} - 1]$$

5. Show that the length of the arc if the circle $x^2 + y^2 = a^2$ interpreted between the points where $x = a\cos\alpha$ and $x = a\cos\beta$ is $a(\beta - \alpha)$.

Sol. Parametric equations of the given circle arc

$x = a\cos\theta$, $\qquad y = a\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$, $\qquad \frac{dy}{d\theta} = -a\cos\theta$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = a^2\sin^2\theta + a^2\cos^2\theta = a^2$$

or $\frac{ds}{d\theta} = a$ $\quad$ *i.e.*, $\quad ds = a\,d\theta$

$$s = \int_{\alpha}^{\beta} a\, d\theta = a\,|\theta|_{\beta}^{\alpha} = a(\beta - \alpha).$$

6. Show that the length of the arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ between the points for which $\theta = 0$ and $\theta = 2\alpha$ is $s = 4a\sin\alpha$.

Sol. $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, $\frac{dy}{d\theta} = a\sin\theta$

$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= a^2(1 + \cos\theta)^2 + a^2\sin^2\theta$$

$$= a^2[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

$$= 2a^2(1 + \cos\theta) = 4a^2\cos^2\frac{\theta}{2}$$

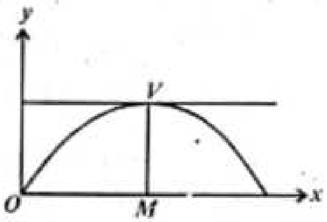
$$\left(\frac{ds}{d\theta}\right)^2 = 4a^2\cos^2\frac{\theta}{2} \text{ or } \frac{ds}{d\theta} = 2a\cos\frac{\theta}{2} \text{ or } ds = 2a\cos\frac{\theta}{2}\,d\theta$$

Hence $$s = 2a\int_0^{2a}\cos\frac{\theta}{2}\,d\theta = 2a\left[2\sin\frac{\theta}{2}\right]_0^{2a}$$

$$= 4a\left[\sin\frac{\theta}{2}\right]_0^{2a} = 4a\sin a.$$

7. Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

Sol. $x = a(\theta - \sin\theta)$



$$\frac{dx}{d\theta} = a(1 - \cos\theta)$$

$y = a(1 - \cos\theta)$

$$\frac{dy}{d\theta} = a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = a^2[(1 - \cos\theta)^2 + \sin^2\theta]$$

$$= a^2[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$$

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$$= a^2[2 - 2\cos\theta] = 4a^2 \sin^2\frac{\theta}{2}$$

or $\dfrac{ds}{d\theta} = 2a\sin\dfrac{\theta}{2}$ or $ds = 2a\sin\dfrac{\theta}{2}\,d\theta$

Length of half of one arch

$$= 2a\int_0^{\pi} \sin\frac{\theta}{2}\,d\theta = 2a\left[-2\cos\frac{\theta}{2}\right]_0^{\pi} = 2a\,[-2\,(0) + 2] = 4a$$

Length of the complete arch $= 2\,(4a) = 8a$.

8. Show that the length in one quadrant of the curve $x^{2/3}+y^{2/3} = a^{2/3}$, is equal to $\dfrac{3a}{2}$ and find the length in one quadrant of the curve

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1. \qquad (1)$$

Sol. Parametric equations of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ are

$x = a\cos^3\theta$, $\quad y = a\sin^3\theta$

$\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\quad \dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$
$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$
$$= 9a^2\sin^2\theta\cos^2\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= 9a^2\sin^2\theta\cos^2\theta$$

or $\dfrac{ds}{d\theta} = 3a\sin\theta\cos\theta$ or $ds = 3a\sin\theta\cos\theta\,d\theta$

$$\text{Required length} = s = 3a\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta = 3a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{3a}{2}$$

Parametric equations of the curve (1) are

$x = a\cos^3\theta$, $\quad y = b\sin^3\theta$

$\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$, $\dfrac{dy}{d\theta} = 3b\sin^2\theta\cos\theta$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$
$$= 9a^2\cos^4\theta\sin^2\theta + 9b\sin^4\theta\cos^2\theta$$
$$= 9\sin^2\theta\cos^2\theta\,(a^2\cos^2\theta + b^2\sin^2\theta)$$

$$\frac{ds}{d\theta} = 3\sin\theta\cos\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}$$

Limits of integration in the first quadrant are from $\theta = 0$ to $\theta = \dfrac{\pi}{2}$

Hence length of the curve in one quadrant is

$$3\int_0^{\pi/2}\sin\theta\cos\theta\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta$$

Put $a^2\cos^2\theta + b^2\sin^2\theta = t^2$

or $(-2a^2\cos\theta\sin\theta + 2b^2\sin\theta\cos\theta)\,d\theta = 2t\,dt$

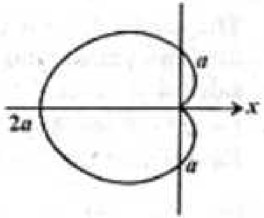
or $(b^2 - a^2)\sin\theta\cos\theta\,d\theta = t\,dt$

or $\sin\theta\cos\theta\,d\theta = \dfrac{t\,dt}{b^2 - a^2}$

$$\text{Required length} = 3\int_a^b \frac{t\,dt}{b^2 - a^2}\,t = \frac{3}{b^2 - a^2}\int_a^b t^2\,dt$$
$$= \frac{3}{b^2 - a^2}\left[\frac{t^3}{3}\right]_a^b = \frac{b^3 - a^3}{b^2 - a^2}$$
$$= \frac{(b-a)(b^2 + ba + a^2)}{(b-a)(b+a)} = \frac{a^2 + ab + b^2}{a + b}$$

9. Find the entire length of the cardioid $r = a\,(1 - \cos\theta)$ and show that the arc of the upper half of the curve is bisected by $\theta = \dfrac{2\pi}{3}$.

Sol. $r = a\,(1 - \cos\theta)$



$$\frac{dr}{d\theta} = a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$
$$= a^2(1 - \cos\theta)^2 + a^2\sin^2\theta$$
$$= a^2[(1 - \cos\theta)^2 + \sin^2\theta]$$
$$= a^2[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]$$
$$= a^2[2 - 2\cos\theta] = 2a^2(1 - \cos\theta)$$
$$= 4a^2\sin^2\frac{\theta}{2}$$

$$\frac{ds}{d\theta} = 2a\sin\frac{\theta}{2}$$

Limits of θ for the upper half of the curve are 0 to π

$$\text{Perimeter of the cardioid} = 2\int_0^{\pi} 2a\sin\frac{\theta}{2}\,d\theta = 4a\int_0^{\pi}\sin\frac{\theta}{2}\,d\theta$$
$$= 8a\left[-\cos\frac{\theta}{2}\right]_0^{\pi} = 8a$$

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Length of the arc from $\theta = 0$ to $\theta = \frac{2\pi}{3}$

$$= \int_0^{2\pi/3} \frac{ds}{d\theta} d\theta = \int_0^{2\pi/3} 2a \sin\frac{\theta}{2} d\theta$$

$$= -4a\left[\cos\frac{\pi}{3} - 1\right] = -4a\left[\frac{1}{2} - 1\right] = 2a$$

= half of the length of the upper portion which is $4a$.

10. Find the length of the curve $r = \sin^2\left(\frac{\theta}{2}\right)$ from (0, 0) to $(1, \pi)$.

Sol. $r = \sin^2\left(\frac{\theta}{2}\right) = \frac{1 - \cos\theta}{2}$

$$\frac{dr}{d\theta} = \frac{\sin\theta}{2}$$

$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{\left(\frac{1-\cos\theta}{2}\right)^2 + \frac{\sin^2\theta}{4}} = \sin\left(\frac{\theta}{2}\right)$$

$$\text{Required length} = \int_0^{\pi} ds = \int_0^{\pi} \sin\left(\frac{\theta}{2}\right) d\theta = \left[2\cos\frac{\theta}{2}\right]_2^{\pi} = 2$$

11. The cardioid $r = a(1 + \cos\theta)$ is divided by the line $4r\cos\theta = 3a$ into two parts. Find the ratio of the lengths of the arcs on the two sides of this line.

Sol. $r = a(1 + \cos\theta)$ (1)

Equation of the line is $4r\cos\theta = 3a$ (2)

i.e., $4x = 3a$ *i.e.*, $x = \frac{3a}{4}$

Solving (1) and (2), we have

$$a(1 + \cos\theta) = \frac{3a}{4\cos\theta}$$

or $4\cos\theta(1 + \cos\theta) = 3$

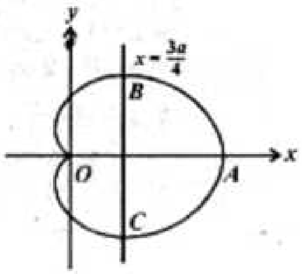
$\Rightarrow$ $4\cos^2\theta + 4\cos\theta - 3 = 0$

or $(2\cos\theta + 3)(2\cos\theta - 1) = 0$

i.e., $\cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{3}{2}$

$\cos\theta = -\frac{3}{2}$ is inadmissible

$\cos\theta = \frac{1}{2}$ gives $\theta = \frac{\pi}{3}, -\frac{\pi}{3}$



Thus the line cuts the curve at the points B and C, where $\theta = \frac{\pi}{3}$ and $\theta = \frac{-\pi}{3}$.

From $r = a(1 + \cos\theta)$, we have

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = a(1 + \cos\theta)^2 + a^2\sin^2\theta$$

$$= 2a^2(1 + \cos\theta) = 4a^2\cos^2\frac{\theta}{2}$$

or $\frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}$,

Length of the curve above the polar axis

$$= \int_0^{\pi} 2a\cos\frac{\theta}{2} d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi} = 4a.$$

Length of the arc AB

$$= \int_0^{\pi/3} 2a\cos\frac{\theta}{2} d\theta = 4a\left[\sin\frac{\theta}{2}\right]_0^{\pi/3} = 2a.$$

Thus the line $r\cos\theta = 3a$ bisects the upper half of the arc of the curve. Since the curve is symmetric about the initial line, the gives line divides the arc of the cardioid into two equal parts.

12. For the epicycloid $x = (a + b)\cos\theta - b\cos\left(\frac{(a+b)}{b}\right)\theta$,

$$y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta,$$

show that $s = \frac{4b(a+b)}{a}\cos\left(\frac{a\theta}{2b}\right)$, where s is measured from the point at which $\theta = \frac{\pi b}{a}$.

Sol. $x = (a + b)\cos\theta - b\cos\left(\frac{a+b}{b}\right)\theta$

$$\frac{dx}{d\theta} = -(a + b)\sin\theta + \frac{b(a+b)}{b}\sin\left(\frac{a+b}{b}\right)\theta$$

$$= -(a + b)\left[\sin\theta - \sin\left(\frac{a+b}{b}\right)\theta\right]$$

$y = (a + b)\sin\theta - b\sin\left(\frac{a+b}{b}\right)\theta$

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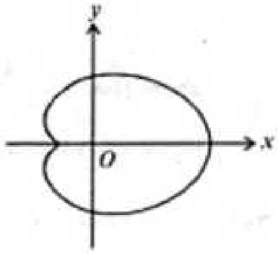
$$\frac{dy}{d\theta} = (a+b)\cos\theta - (a+b)\cos\left(\frac{a+b}{b}\right)\theta$$
$$= (a+b)\left[\cos\theta - \cos\left(\frac{a+b}{b}\right)\theta\right]$$
$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$
$$= (a+b)^2\left[2 - 2\left(\sin\theta\sin\left(\frac{a+b}{b}\right)\theta + \cos\theta\cos\left(\frac{a+b}{b}\right)\theta\right)\right]$$
$$= (a+b)^2\left[2 - 2\cos\left(\frac{a+b}{b} - 1\right)\theta\right]$$
$$= (a+b)^2\left[2 - 2\cos\frac{a}{b}\theta\right] = 2(a+b)^2\left[1 - \cos\frac{a\theta}{b}\right]$$
$$= 4(a+b)^2\sin^2\frac{a\theta}{2b} \quad \text{or} \quad \frac{ds}{d\theta} = 2(a+b)\sin\frac{a\theta}{2b}$$

Length of the arc measured from $\theta = \frac{b\pi}{a}$ to any value of θ

$$= \int_{b\pi/a}^{\theta} 2(a+b)\sin\frac{a\theta}{2b}\,d\theta = 2(a+b)\int_{b\pi/a}^{\theta}\sin\frac{a\theta}{2b}\,d\theta$$
$$= 2(a+b)\frac{2b}{a}\left[-\cos\frac{a\theta}{2b}\right]_{b\pi/a}^{\theta} = -\frac{4b(a+b)}{a}\cos\frac{a\theta}{2b}$$
$$= \frac{4b(a+b)}{a}\cos\frac{a\theta}{2b} \text{ (in absolute value)}$$

13. Show that the perimeter of the limacon $r = a + b\cos\theta$, if $\frac{b}{a}$ is small, is approximately $2\pi a\left(1 + \frac{b^2}{4a^2}\right)$.

Sol. Equation of the curve is



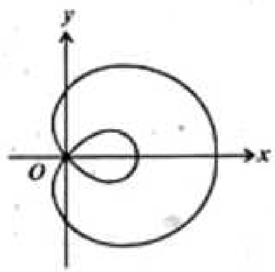
$$r = a + b\cos\theta,\ a > b$$
$$\frac{dr}{d\theta} = -b\sin\theta$$
$$\left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2$$
$$= (a + b\cos\theta)^2 + b^2\sin^2\theta$$
$$= a^2 + b^2 + 2ab\cos\theta$$
$$= a^2\left[1 + \frac{b^2}{a^2} + \frac{2b}{a}\cos\theta\right]$$
$$= a^2(1 + 2k\cos\theta + k^2), \text{ where } k = \frac{b}{a}$$

$$\frac{ds}{d\theta} = a\{1 + (2k\cos\theta + k^2)\}^{1/2}$$
$$= a\left\{1 + \frac{1}{2}(2k\cos\theta + k^2) + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2}(2k\cos\theta + k^2)^2 + \ldots\right\}$$
$$= a\left\{1 + k\cos\theta + \frac{1}{2}k^2 - \frac{1}{2}k^2\cos^2\theta\right\}, \text{ neglecting higher powers of } k$$
$$s = a\int_0^{\pi}\left(1 + k\cos\theta + \frac{1}{2}k^2\sin^2\theta\right)d\theta$$
$$= a\int_0^{\pi} 1\,d\theta + ak\int_0^{\pi}\cos\theta\,d\theta + \frac{ak^2}{2}\int_0^{\pi}\sin^2\theta\,d\theta$$
$$= a\pi + 0 + ak^2\int_0^{\pi/2}\sin^2\theta\,d\theta$$
$$= a\pi + ak^2\frac{1}{2}\frac{\pi}{2} \qquad \text{(by Wallis formula)}$$
$$= a\pi\left(1 + \frac{k^2}{4}\right) = a\pi\left(1 + \frac{b^2}{4a^2}\right).$$

14. Prove that the difference between the lengths of the two loops of the limacon $r = a + b\cos\theta$ is $8a$, where $\frac{a}{b}$ is small.

Sol. Since $a < b$, when $\theta = \arccos\left(-\frac{a}{b}\right) = \alpha$ say, $r = 0$ so that the curve passes through the pole for this value of θ. For upper half of the outer loop, θ varies from 0 to $\arccos\left(-\frac{a}{b}\right) = \alpha$, where r is positive and for lower half of the inner loop, θ varies from α to π, where r is negative.

Difference of the length of these two loops is



$$s = \int_0^{\alpha}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta - \int_{\alpha}^{\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$$
$$= \int_0^{\alpha}[(a + b\cos\theta)^2 + b^2\sin^2\theta]^{1/2}\,d\theta$$
$$- \int_{\alpha}^{\pi}[(a + b\cos\theta)^2 + b^2\sin^2\theta]^{1/2}\,d\theta$$

$$= b\int_0^{\alpha}\left(1+\frac{2a}{b}\cos\theta\right)^{1/2}d\theta - b\int_{\alpha}^{\pi}\left(1+\frac{2a}{b}\cos\theta\right)^{1/2}d\theta$$

$$= b\int_0^{\alpha}\left(1+\frac{a}{b}\cos\theta\right)d\theta - b\int_{\alpha}^{\pi}\left(1+\frac{a}{b}\cos\theta\right)d\theta$$

$$\left(\text{neglecting } \frac{a^2}{b^2} \text{ and higher powers of } \frac{a}{b}\right)$$

$$= b\left[\theta+\frac{a}{b}\sin\theta\right]_0^{\alpha} - b\left[\theta+\frac{a}{b}\sin\theta\right]_{\alpha}^{\pi}$$

$$= b\left[\alpha+\frac{a}{b}\sin\alpha\right] - b\left[\pi-\alpha-\frac{a}{b}\sin\alpha\right]$$

$$= b\left[\alpha-(\pi-\alpha)+\frac{2a}{b}\sin\alpha\right]$$

$$= b\left[2\arccos\left(\frac{-a}{b}\right)-\pi+\frac{2a}{b}\sin\alpha\right]$$

$$= b\left[2\left(\frac{\pi}{2}+\frac{a}{b}\right)-\pi+\frac{2a}{b}\sin\alpha\right], \text{ using Maclaurin's Theorem.}$$

$$= 2a + 2a\left(\sqrt{\frac{b^2-a^2}{b^2}}\right) = 2a + 2a\left(1-\frac{a^2}{b^2}\right)^{1/2} = 2a + 2a = 4a.$$

Total difference of the lengths of the two loops $= 8a$.

15. Prove that the length of the arc of the hyperbolic spiral $r\theta = a$ taken from the point $r = a$ to $r = 2a$ is

$$a\left[\sqrt{5}-\sqrt{2}+\ln\frac{2+\sqrt{8}}{1+\sqrt{5}}\right].$$

Sol. $r = \dfrac{a}{\theta}, \dfrac{dr}{d\theta} = -\dfrac{a}{\theta^2}$

$$\left(\frac{ds}{d\theta}\right)^2 = r^2+\left(\frac{dr}{d\theta}\right)^2 = \frac{a^2}{\theta^2}+\frac{a^2}{\theta^4} = \frac{a^2(\theta^2+1)}{\theta^4}$$

$$\frac{ds}{d\theta} = a\frac{\sqrt{1+\theta^2}}{\theta^2}, ds = \frac{a\sqrt{1+\theta^2}}{\theta^2}d\theta.$$

When $r = a$, $\theta = 1$ and when $r = 2a$, $\theta = \dfrac{1}{2}$

$$s = a\int_1^{1/2}\frac{\sqrt{1+\theta^2}}{\theta^2}d\theta = a\int_1^{1/2}\frac{1+\theta^2}{\theta^2\sqrt{1+\theta^2}}d\theta$$

$$= a\int_1^{1/2}\frac{d\theta}{\theta^2\sqrt{1+\theta^2}} + a\int_1^{1/2}\frac{d\theta}{\sqrt{1+\theta^2}} \qquad (1)$$

Put $\theta = \dfrac{1}{t}$ or $d\theta = -\dfrac{1}{t^2}dt$ in the first integral. Then

$$a\int_1^{1/2}\frac{d\theta}{\theta^2\sqrt{1+\theta^2}} = a\int_1^{2}\frac{-\frac{1}{t^2}dt}{\frac{1}{t^2}\sqrt{1+\frac{1}{t^2}}} = -a\int_1^{2}\frac{-t\,dt}{\sqrt{t^2+1}}$$

$$= -\frac{a}{2}\int_1^2 2t\,(t^2+1)^{-1/2}\,dt = -\frac{2a}{2}\left[\sqrt{t^2+1}\right]_1^2$$

$$= -a\left[\sqrt{5}-\sqrt{2}\right] \qquad (2)$$

Now $$\int_1^{1/2}\frac{d\theta}{\sqrt{1+\theta^2}} = \left[\ln\left(\theta+\sqrt{1+\theta^2}\right)\right]_1^{1/2}$$

$$= \ln\left[\frac{1}{2}+\frac{\sqrt{5}}{2}\right] - \ln\left(1+\sqrt{2}\right)$$

$$= \ln\frac{1+\sqrt{5}}{2(1+\sqrt{2})} = \ln\frac{1+\sqrt{5}}{2+\sqrt{8}} \qquad (3)$$

Substituting from (2) and (3) into (1), we have

$$s = -a\left[\sqrt{5}-\sqrt{2}\right] + a\ln\frac{1+\sqrt{5}}{2+\sqrt{8}}$$

$$= -a\left[\sqrt{5}-\sqrt{2}-\ln\frac{1+\sqrt{5}}{2+\sqrt{8}}\right]$$

$$= -a\left[\sqrt{5}-\sqrt{2}+\ln\frac{2+\sqrt{8}}{1+\sqrt{5}}\right]$$

$$= a\left[\sqrt{5}-\sqrt{2}+\ln\frac{2+\sqrt{8}}{1+\sqrt{5}}\right] \text{ in absolute units.}$$

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16. Show that the intrinsic equation of the catenary

$$y = c\cosh\left(\frac{x}{c}\right) \text{ is } s = c\tan\alpha.$$

Sol. $y = c\cosh\dfrac{x}{c}$ (1)

$$\frac{dy}{dx} = \sinh\frac{x}{c} \quad (2)$$

$$\left(\frac{ds}{d\theta}\right)^2 = 1 + \left(\frac{dr}{d\theta}\right)^2 = 1 + \sinh^2\frac{x}{c} = \cosh^2\frac{x}{c}$$

$$\frac{ds}{dx} = \cosh\frac{x}{c}$$

Thus $s = \displaystyle\int_0^x \cosh\frac{x}{c}\,dx = c\left[\sinh\frac{x}{c}\right]_0^x = c\sinh\frac{x}{c} = c\left(\frac{dy}{dx}\right)$ from (2)

$= c\tan\alpha$

which is the required intrinsic equation.

17. Show that the intrinsic equation of the parabola $x^2 = 4ay$ is

$$s = a\tan\alpha\sec\alpha + a\ln(\tan\alpha + \sec\alpha).$$

Sol. Equation of the curve is $y = \dfrac{x^2}{4a}$ (1)

$$\frac{dy}{dx} = \frac{x}{2a} \quad (2)$$

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{x^2}{4a^2} = \frac{4a^2+x^2}{4a^2}$$

or $\dfrac{ds}{dx} = \dfrac{\sqrt{4a^2+x^2}}{2a}$

$$s = \frac{1}{2a}\int_0^x \sqrt{4a^2+x^2}\,dx$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{4a^2+x^2}}{2} + 2a^2\sinh^{-1}\frac{x}{2a}\right]_0^x$$

$$= \frac{1}{2a}\left[\frac{x\sqrt{4a^2+x^2}}{2} + 2a^2\ln\left(\frac{x+\sqrt{x^2+4a^2}}{2a}\right)\right] \quad (3)$$

$$\frac{dy}{dx} = \frac{x}{2a} = \tan\alpha \quad \text{or} \quad x = 2a\tan\alpha$$

Putting this values of x into (3), we get

$$s = \frac{1}{2a}\left[\frac{2a\tan\alpha\, 2a\sec\alpha}{2} + 2a^2\ln\frac{2a\tan\alpha + 2a\sec\alpha}{2a}\right]$$

$$= a[\tan\alpha\sec\alpha + \ln(\tan\alpha + \sec\alpha)]$$

which is the required intrinsic equation.

18. Show that the intrinsic equation of the astroid

$$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } s = \frac{3a}{2}\sin^2\alpha.$$

Sol. Parametric equations of the curve are

$$x = a\cos^3\theta \quad \text{and} \quad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad (1)$$

and $\dfrac{dy}{d\theta} = 3a\sin^2\theta\sin\theta$ (2)

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$$

$$= 9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta$$

$$= 9a^2\sin^2\theta\cos^2\theta(\cos^2\theta + \sin^2\theta) = 9a^2\sin^2\theta\cos^2\theta$$

or $\dfrac{ds}{d\theta} = 3a\sin\theta\cos\theta$

$$ds = 3a\sin\theta\cos\theta\,d\theta$$

$$s = 3a\int_0^\theta \sin\theta\cos\theta\,d\theta = \frac{3a}{2}\sin 2\theta \quad (3)$$

Dividing (2) by (1), we have

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{3a\cos^2\theta\sin\theta} = \tan\theta$$

i.e., $\tan\alpha = \tan\theta$ or $\theta = \alpha$

Putting $\theta = \alpha$ in (3), we get

$s = \dfrac{3a}{2}\sin^2\alpha$ as the required intrinsic equation.

19. Prove that the intrinsic equation of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta) \text{ is } s = 4a\sin\alpha$$

Sol. Here $x = a(\theta + \sin\theta)$

$$\frac{dx}{d\theta} = a(1 + \cos\theta) \quad (1)$$

$$y = a(1 - \cos\theta)$$

$$\frac{dy}{d\theta} = a\sin\theta$$

Dividing (2) by (1), we get

$$\frac{dy}{dx} = \frac{a \sin \theta}{a(1+\cos\theta)}$$

$$= \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan\frac{\theta}{2}$$

Hence $\tan\alpha = \tan\frac{\theta}{2}$

or $\alpha = \frac{\theta}{2}$ (3)

Now $\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2$

$$= a^2(1+\cos\theta)^2 + a^2\sin^2\theta$$
$$= a^2[(1+\cos\theta)^2 + \sin^2\theta]$$
$$= a^2[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]$$
$$= a^2[1 + 2\cos\theta + 1] = a^2[2 + 2\cos\theta]$$
$$= 2a^2(1+\cos\theta) = 4a^2\cos^2\frac{\theta}{2}$$

$$\frac{ds}{d\theta} = 2a\cos\frac{\theta}{2}$$

$$s = 2a\int_0^\theta \cos\frac{\theta}{2}\,d\theta = 4a\sin\frac{\theta}{2} = 4a\sin\alpha, \text{ from (3)}$$

is the required intrinsic equation.

20. Show that the intrinsic equation of the cardioid $r = a(1+\cos\theta)$ is

$s = 4a\sin\frac{\alpha}{3}$. [Take $\theta = 0$ as the fixed point.].

Sol. Here $\theta = 0$ is the fixed point A on the initial line. The tangent to the curve at A is perpendicular to the initial line. The tangent at P (r, θ) and the tangent at A meet at the point R so that

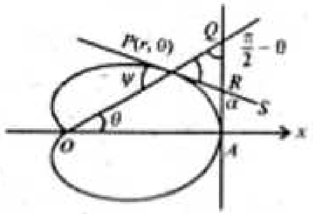
$$\angle ARS = \alpha = \psi + \theta - \frac{\pi}{2} \qquad (1)$$

From the equation of the curve

$r = a(1+\cos\theta)$, we have

$$\frac{dr}{d\theta} = -a\sin\theta$$

Therefore, $\tan\psi = \frac{r}{dr/d\theta}$



$$= \frac{a(1+\cos\theta)}{-a\sin\theta}$$

$$= \frac{2\cos^2\frac{\theta}{2}}{-2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

or $\psi = \frac{\pi}{2} + \frac{\theta}{2}$ (2)

Now $s = \text{Arc } AP = \int_0^\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta$

$$= \int_0^\theta [a^2(1 + 2\cos\theta + \cos^2\theta) + a^2\sin^2\theta]^{1/2}\,d\theta$$

$$= \int_0^\theta \sqrt{2a^2(1+\cos\theta)}\,d\theta = 2a\int_0^\theta \cos\frac{\theta}{2}\,d\theta$$

$$= 4a\sin\frac{\theta}{2} \qquad (3)$$

Eliminate θ and ψ from (1), (2) and (3) to get the required intrinsic equation.

Putting the value of ψ into (2) into (1), we have

$$\alpha = \frac{\pi}{2} + \frac{\theta}{2} + \theta - \frac{\pi}{2} = \frac{3\theta}{2} \quad \text{or} \quad \frac{\theta}{2} = \frac{\alpha}{2}$$

Writing this value of $\frac{\theta}{2}$ into (3), the required equation is

$$s = 4a\sin\frac{\alpha}{3}.$$

Exercise Set 7.7 (Page 334)

Find the radius of curvature at any point on each of the given curves (Problems 1 – 8):

1. $y = c\cosh\left(\frac{x}{c}\right)$ (1)

Sol. $\frac{dy}{dx} = \sinh\frac{x}{c}$

$$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\frac{x}{c}$$

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$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{\left(1+\sinh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cosh\frac{x}{c}} = \frac{\cosh^3\frac{x}{c}}{\frac{1}{c}\cosh\frac{x}{c}}$$

$$= c\cosh^2\left(\frac{x}{c}\right) = c\frac{y^2}{c^2} = \frac{y^2}{c}, \qquad \text{from (1)}$$

2. $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t),\ a > 0$

Sol. $\frac{dx}{dt} = a(-\sin t + \sin t + t\cos t) = a\,t\cos t$

$$\frac{d^2x}{dt^2} = a\cos t - a\,t\sin t$$

$$y = a(\sin t - t\cos t)$$

$$\frac{dy}{dt} = a(\cos t - \cos t + t\sin t) = a\,t\sin t$$

$$\frac{d^2y}{dt^2} = a\sin t + a\,t\cos t$$

$$\rho = \frac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}\right|}$$

$$= \frac{(a^2t^2\cos^2 t + a^2t^2\sin^2 t)^{3/2}}{a\,t\cos t\,(a\sin t + a\,t\cos t) - a\,t\sin t\,(a\cos t - a\,t\sin t)}$$

$$= \frac{a^3t^3}{a^2t^2(\cos^2 t + \sin^2 t)} = \frac{a^3t^3}{a^2t^2} = a\,t.$$

3. $x = a(t - \sin t),\ y = a(1 - \cos t),\ a > 0$

Sol. Differentiating twice the given equations w.r.t. t, we have

$$\frac{dx}{dt} = a(1-\cos t), \quad \frac{d^2x}{dt^2} = a\sin t$$

$$\frac{dy}{dt} = a\sin t, \quad \frac{d^2y}{dt^2} = a\cos t$$

Now $\rho = \dfrac{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]^{3/2}}{\left|\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}\frac{dy}{dt}\right|}$

$$= \frac{[a^2 + a^2\cos^2 t - 2a^2\cos t + a^2\sin^2 t]^{3/2}}{|a^2\cos t\,(1-\cos t) - a^2\sin^2 t|}$$

$$= \frac{[2a^2(1-\cos t)]^{3/2}}{|a^2\cos t - a^2|}$$

$$= 2a\sqrt{1-\cos t} = 4a\sin\left(\frac{t}{2}\right)$$

4. $x = a\cos^3\theta,\ y = a\sin^3\theta$

Sol. $x = a\cos^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$$

$$\frac{d^2x}{d\theta^2} = -3a\cos^3\theta + 6a\cos\theta\sin^2\theta$$

$$y = a\sin^3\theta$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{d^2y}{d\theta^2} = -3a\sin^3\theta + 6a\sin\theta\cos^2\theta$$

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2}-\frac{d^2x}{d\theta^2}\frac{dy}{d\theta}\right|}$$

$$= \frac{(9a^2\cos^4\theta\sin^2\theta + 9a^2\sin^4\theta\cos^2\theta)^{3/2}}{|-18a^2\sin^2\theta\cos^4\theta + 9a^2\sin^4\theta\cos^2\theta + 9a^2\sin^2\theta\cos^4\theta - 18a^2\sin^4\theta\cos^2\theta|}$$

$$= \frac{(9a^2\sin^2\theta\cos^2\theta)^{3/2}}{|9a^2\sin^2\theta\cos^2\theta - 18a^2\sin^2\theta\cos^2\theta|}$$

$$= \frac{27a^3\sin^3\theta\cos^3\theta}{9a^2\sin^2\theta\cos^2\theta} = 3a\sin\theta\cos\theta$$

$$= 3a\left(\frac{y}{a}\right)^{1/3}\left(\frac{x}{a}\right)^{1/3} = 3a\frac{(xy)^{1/3}}{a^{2/3}} = 3(axy)^{1/3}$$

5. $r = 2\cos 2\theta$ at $\theta = \frac{\pi}{12}$

Sol. $\frac{dr}{d\theta} = -4\sin 2\theta,\ \frac{d^2r}{d\theta^2} = -8\cos 2\theta$

At $\theta = \frac{\pi}{12}, r = 2\cos\frac{\pi}{6} = \sqrt{3}$

$$\frac{dr}{d\theta} = -4\sin\frac{\pi}{6} = -2 \quad\text{and}\quad \frac{d^2r}{d\theta^2} = -8\cos\frac{\pi}{6} = -4\sqrt{3}$$

Radius of curvature at the point $\theta = \frac{\pi}{12}$ is

$$\rho = \frac{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2+2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|} = \frac{[3+4]^{3/2}}{3+2(-2)^2-\sqrt{3}(-4\sqrt{3})} = \frac{7\sqrt{7}}{23}$$

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6. $r\theta = a$

Sol. $r = \frac{a}{\theta}$

$$\frac{dr}{d\theta} = -\frac{a}{\theta^2}, \frac{d^2r}{d\theta^2} = \frac{2a}{\theta^3}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|} = \frac{\left(\frac{a^2}{\theta^2} + \frac{a^2}{\theta^4}\right)^{3/2}}{\frac{a^2}{\theta^2} + \frac{2a^2}{\theta^4} - \frac{2a^2}{\theta^4}} = \frac{\left(\frac{a^2\theta^2 + a^2}{\theta^4}\right)^{3/2}}{\frac{a^2}{\theta^2}}$$

$$= \frac{a^3(1+\theta^3)^{3/2} \cdot \theta^2}{\theta^6 \cdot a^2} = \frac{a(1+\theta^2)}{\theta^4}$$

7. $r^n = a^n \sin n\theta$

Sol. Taking ln of both the sides, we get

$$n \ln r = n \ln a + \ln \sin n\theta$$

Differentiating w.r.t. θ, we have

$$\frac{n}{r}\frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} \quad \text{or} \quad \frac{dr}{d\theta} = r \cot n\theta$$

$$\frac{d^2r}{d\theta^2} = -rn \csc^2 n\theta + \cot n\theta \frac{dr}{d\theta} = -rn \csc^2 n\theta + r \cot^2 n\theta$$

$$K = \frac{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}$$

$$= \frac{|r^2 + 2r^2\cot^2 n\theta + r^2 n \csc^2 n\theta - r^2 \cot^2 n\theta}{(r^2 + r^2\cot^2 n\theta)^{3/2}}$$

$$= \frac{r^2(1+n)\csc^2 n\theta}{r^3 \csc^3 n\theta} = \frac{(n+1)}{r}\sin n\theta$$

$$= \frac{(n+1)\frac{r^n}{a^n}}{r} = \frac{(n+1)r^{n-1}}{a^n}.$$

$$p = \frac{a^n}{(n+1)r^{n-1}}$$

8. $r(1 + \cos\theta) = a.$ (1)

Sol. Differentiating (1) w.r.t. θ, we get

$$\frac{dr}{d\theta}(1 + \cos\theta) + r(-\sin\theta) = 0$$

or $$\frac{dr}{d\theta} = \left(\frac{r\sin\theta}{1+\cos\theta}\right) = r\tan\frac{\theta}{2}$$

$$\frac{d^2r}{d\theta^2} = \frac{dr}{d\theta}\tan\frac{\theta}{2} + r\left(\frac{1}{2}\sec^2\frac{\theta}{2}\right) = r\tan^2\frac{\theta}{2} + \frac{r}{2}\sec^2\frac{\theta}{2}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$= \frac{\left[r^2 + r^2\tan^2\frac{\theta}{2}\right]^{3/2}}{r^2 + 2r^2\tan^2\frac{\theta}{2} - r^2\tan^2\frac{\theta}{2} - \frac{r^2}{2}\sec^2\frac{\theta}{2}}$$

$$= \frac{r^3\sec^3\frac{\theta}{2}}{r^2\left(1+\tan^2\frac{\theta}{2}\right) - \frac{r^2}{2}\sec^2\frac{\theta}{2}}$$

$$= 2r\sec\frac{\theta}{2} = 2r\cdot\sqrt{\frac{2r}{a}}, \text{ from (1)}$$

9. Prove that the radius of curvature at the point $(2a, 2a)$ on the curve $x^2y = a(x^2 + y^2)$ is $2a$.

Sol. Let $f(x, y) = a(x^2 + y^2) - x^2y$

$f_x = 2ax - 2xy$, $f_y = 2ay - x^2$

$f_{xx} = 2a$, $f_{yy} = 2a$, $f_{yx} = -2x = f_{xy}$

At $(2a, 2a)$;

$f_x = 4a^2 - 4a(2a) = 4a^2 - 8a^2 = -4a^2$

$f_y = 2a(2a) - (2a)^2 = 0$

$f_{xx} = 2a$

$f_{yy} = 2a$

$f_{yx} = -2(2a) = -4a$

$$\rho = \frac{[(f_x)^2 + (f_y)^2]^{3/2}}{|(f_y)^2(f_{xx}) - 2f_xf_yf_{xy} + (f_x)^2f_{yy}|} = \frac{(16a^4 + 0)^{3/2}}{(-4a^2)^2(2a)} = \frac{64a^6}{32a^5} = 2a$$

10. Prove that for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $\rho = \frac{a^2b^2}{p^3}$, p being length of the perpendicular from the centre of C of the ellipse to the tangent at any point $P(x, y)$ on the ellipse. Prove also that $\rho = \frac{|CQ|^3}{ab}$, where CQ is semi-diameter conjugate to CP.

Sol. Parametric equations of the ellipse are

$$x = a\cos\theta, \quad y = b\sin\theta$$

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$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{d^2x}{d\theta^2} = -a\cos\theta, \qquad \frac{d^2y}{d\theta^2} = -b\sin\theta$$

$$\rho = \frac{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{3/2}}{\left|\frac{dx}{d\theta}\frac{d^2y}{d\theta^2} - \frac{d^2x}{d\theta^2}\frac{dy}{d\theta}\right|} = \frac{(a^2\sin^2\theta + b^2\cos\theta)^{3/2}}{ab\sin^2\theta + ab\cos^2\theta}$$

$$= \frac{(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2}}{ab} \tag{1}$$

Now $\frac{dy}{dx} = -\frac{b\cos\theta}{a\sin\theta}$

Equation of the tangent at $P(a\cos\theta, b\sin\theta)$ is

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

or $\quad ya\sin\theta - ab\sin^2\theta = -b\cos\theta x + ab\cos^2\theta$

or $\quad (b\cos\theta)x + (a\sin\theta)y - ab = 0$

$$p = \frac{ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$$

or $\quad \sqrt{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{ab}{p}$

Putting this value of $\sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$ into (1), we get

$$\rho = \frac{a^3b^3}{p^3}\frac{1}{ab} = \frac{a^2b^2}{p^3}$$

If CQ is the semi-diameter conjugate to CP, then coordinates of Q are $(-a\sin\theta, b\cos\theta)$.

$$|CQ|^2 = a^2\sin^2\theta + b^2\cos^2\theta \tag{2}$$

Putting the value of $a^2\sin^2\theta + b^2\cos^2\theta$ from (2) into (1), we get

$$\rho = \frac{|CQ|^3}{ab}$$

11. Prove that if ρ is the radius of curvature at any point P on the parabola $y^2 = 4ax$ and F is its focus, then ρ^2 varies as $(FP)^3$.

Sol. Here, $\quad f(x, y) = y^2 - 4ax$

$$f_x = -4a,\ f_y = 2y,\ f_{xy} = 0,\ f_{yx} = 0,\ f_{yy} = 2$$

$$\rho = \frac{[(f_x)^2 + (f_y)^2]^{3/2}}{|-(f_y)^2 f_{xx} + 2f_x f_{xy} f_{xy} - (f_x)^2 f_{yy}|}$$

$$= \frac{(16a^2 + 4y^2)^{3/2}}{16a^2\,(2)} = \frac{(16a^2 + 16ax)^{3/2}}{32a^2}$$

$$= \frac{64a^{3/2}(x+a)^{3/2}}{32a^2} = \frac{2(x+a)^{3/2}}{\sqrt{a}}$$

$$\rho^2 = \frac{4(x+a)^3}{a} \tag{1}$$

Now F is $(a, 0)$ and P is (x, y)

$$FP = \sqrt{(x-a)^2 + y^2} = \sqrt{(x-a) + 4ax}$$

$$= \sqrt{(x+a)^2} = x + a \tag{2}$$

From (1) and (2), we get

$$\rho^2 = \frac{4(FP)^3}{a} = \frac{4}{a}(FP)^3 \quad i.e., \quad \rho^2 \text{ varies as } (FP)^3$$

12. Prove that for the cardioid $r = a(1 + \cos\theta)$, $\frac{\rho^2}{r}$ is constant.

Sol. Here $\quad r = a(1 + \cos\theta)$

Taking logarithmic of both sides, we have

$$\ln r = \ln a + \ln(1 + \cos\theta)$$

Differentiating both sides w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta} = -\tan\frac{\theta}{2}$$

$$\frac{dr}{d\theta} = -r\tan\frac{\theta}{2}$$

$$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta}\tan\frac{\theta}{2} - \frac{1}{2}r\sec^2\frac{\theta}{2}$$

$$\rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}\right|}$$

$$= \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{\left|r^2 + 2r^2\tan^2\frac{\theta}{2} - r^2\tan^2\frac{\theta}{2} + \frac{r^2}{2}\sec^2\frac{\theta}{2}\right|}$$

$$= \frac{r^3\sec^3\frac{\theta}{2}}{\frac{3r^2}{2}\left(1 + \tan^2\frac{\theta}{2}\right)} = \frac{2}{3}r\sec\frac{\theta}{2} = \frac{2}{3}r\cdot\sqrt{\frac{2a}{r}}$$

or $\quad \frac{\rho^2}{r} = \frac{8a}{9}$, which is constant.

13. Show that for a pedal curve, $\rho = r\frac{dr}{dp}$.

Sol. We know that $p = r\sin\psi \qquad$ **(6.32)**

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Differentiating w.r.t. r, we have

$$\frac{dp}{dr} = r\cos\psi\,\frac{d\psi}{dr} + \sin\psi$$

$$= r\frac{dr}{ds}\cdot\frac{d\psi}{dr} + r\frac{d\theta}{ds},\left[\text{since } \tan\psi = r\frac{d\theta}{dr},\right.$$

$$\left.\cos\psi = \frac{dr}{\sqrt{dr^2 + r^3\,d\theta^2}} = \frac{dr}{ds} \text{ and } \sin\psi = r\frac{d\theta}{ds}\right]$$

$$\frac{dp}{dr} = r\frac{d\psi}{ds} + r\frac{d\theta}{ds} = r\frac{d}{ds}(\psi + \theta)$$

$$= r\frac{d\alpha}{ds}, \qquad \text{as } \theta + \psi = \alpha \qquad (6.29)$$

$$= \frac{r}{\rho} \quad \text{or} \quad \rho = r\frac{dr}{dp}.$$

Find the radius of curvature at the point (p, r) of each of the given curves (Problems 14 – 16):

14. $p^2 = ar$

Sol. $p = \sqrt{a}\; r^{1/2}$

$$\frac{dp}{dr} = \sqrt{a}\cdot\frac{1}{2}r^{-1/2} = \frac{\sqrt{a}}{2\sqrt{r}}\ ;\quad \frac{dr}{dp} = \frac{2\sqrt{r}}{\sqrt{a}}$$

$$\rho = r\frac{dr}{dp} = \frac{2r^{3/2}}{\sqrt{a}} = \frac{2}{\sqrt{a}}\left(\frac{p}{\sqrt{a}}\right)^3 = \frac{2p^3}{a^2}.$$

15. $\frac{1}{p^2} = \frac{A}{r^2} + B$

Sol. Differentiating w.r.t. p, we get

$$-\frac{2}{p^3} = -\frac{2A}{r^3}\frac{dr}{dp} \quad \text{or} \quad \frac{dr}{dp} = \frac{r^3}{Ap^3}$$

$$\rho = r\frac{dr}{dp} = \frac{r^4}{Ap^3}$$

16. $p^2(a^2 + b^2 - r^2) = a^2b^2$

Sol. Differentiating w.r.t. p, we get

$$2p(a^2 + b^2 - r^2) + p^2\left(-2r\frac{dr}{dp}\right) = 0$$

or $\quad a^2 + b^2 - r^2 - pr\frac{dr}{dp} = 0$

or $\quad r\frac{dr}{dp} = \frac{a^2 + b^2 - r^2}{p} = \frac{a^2b^2}{p^3}$

Hence $\quad \rho = \frac{a^2b^2}{p^3}$

17. If ρ_1, ρ_2 are the radii of curvature at the extremities of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, then prove that

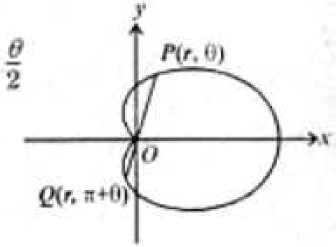
$$\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}.$$

Sol. Here $\quad r = a(1 + \cos\theta)$

From Problem 12, we have

$$\rho = \frac{2}{3}r\sec\frac{\theta}{2} = \frac{2}{3}a(1 + \cos\theta)\sec\frac{\theta}{2}$$

$$= \frac{4a}{3}\cos\frac{\theta}{2}$$



Radius of curvature ρ_1 at $P(r, \theta)$ is

$$\rho_1 = \frac{4a}{3}\cos\frac{\theta}{2}$$

Changing θ into $\theta + \pi$, the radius of curvature ρ_2 at the other extremity of the chord passing through the pole is

$$\rho_2 = \frac{4a}{3}\cos\left(\frac{\theta + \pi}{2}\right) = -\frac{4a}{3}\sin\frac{\theta}{2}$$

Hence $\quad \rho_1^2 + \rho_2^2 = \frac{16a^2}{9}\left(\cos^2\frac{\theta}{2} + \sin\frac{\theta}{2}\right) = \frac{16a^2}{9}$

18. Find the radius of curvature of the curve $r = a(1 + \cos\theta)$ at the point where the tangent is parallel to the initial line.

Sol. Here $\quad r = a(1 + \cos\theta)$

Taking ln of both sides, we have

$$\ln r = \ln a + \ln(1 + \cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

$$r\frac{d\theta}{dr} = -\frac{1 + \cos\theta}{\sin\theta} = \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$$

$$\tan\psi = r\frac{d\theta}{dr} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

or $\quad \psi = \frac{\pi}{2} + \frac{\theta}{2}$

Now, $\alpha = \psi + \theta = \frac{\pi}{2} + \frac{\theta}{2} + \theta = \frac{\pi}{2} + \frac{3\theta}{2}$

For the tangent to be parallel to the initial line, either

$$\alpha = 0 \quad \text{or} \quad \alpha = \pi$$

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$$\alpha = 0 \quad \text{gives} \quad \frac{\pi}{2} + \frac{3\theta}{2} = 0 \quad \Rightarrow \quad \theta = -\frac{\pi}{3}$$

$$\alpha = \pi \quad \text{gives} \quad \frac{\pi}{2} + \frac{3\theta}{2} = \pi \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

From Problem 12, we have

$$\rho = \frac{2r}{3} \sec\frac{\theta}{2} = \frac{2}{3} . 2a \sec^3\frac{\theta}{2} \sec\frac{\theta}{2} = \frac{4a}{3}\cos\frac{\theta}{2}$$

(1) At $\theta = \frac{\pi}{3}$, we hve

$$\rho = \frac{4a}{3}\cos\frac{\pi}{6} = \frac{4a}{3}\left(\frac{\sqrt{3}}{2}\right) = \frac{2\sqrt{3}\,a}{3}$$

(2) At $\theta = -\frac{\pi}{3}$, we have

$$\rho = \frac{4a}{3}\cos\left(-\frac{\pi}{6}\right) = \frac{4a}{3}\cos\frac{\pi}{6} = \frac{2\sqrt{3}\,a}{3}$$

Hence in each case, $\rho = \frac{2\sqrt{3}\,a}{3}$

19. Show that for the parabola $y = ax^2 + bx + c$, the radius of curvature ρ is minimum at its vertex.

Sol. $y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$, $\frac{d^2y}{dx^2} = 2a$.

Radius of curvature ρ at any point is

$$\rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\left|\frac{d^2y}{dx^2}\right|} = \frac{[1 + (2ax + b)^2]^{3/2}}{2a}$$

$$\rho = \frac{(1 + b^2 + 4abx + 4a^2x^2)^{3/2}}{2a}$$

$$\frac{d\rho}{dx} = \frac{3}{4a}[1 + b^2 + 4abx + 4a^2x^2]^{1/2}[4ab + 8a^2x]$$

For extreme values of ρ, we have $\frac{d\rho}{dx} = 0$

This gives $4ab + 8a^2x = 0$ *i.e.*, $x = -\frac{b}{2a}$

It is easy to check that ρ is minimum for this value of x.

If $x = -\frac{b}{2a}$, then

$$y = a\left(-\frac{b}{2a}\right) + b\left(-\frac{b}{2a}\right) + c = \frac{-b^2 + 4ac}{4a}$$

Thus ρ is minimum at $A\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$

Equation of the given parabola may be written as

$$y = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

or $$y + \frac{b^2 - 4ac}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

Therefore, the vertex of the parabola is $\left(-\frac{b}{2a}, -\frac{b^2 - 4ac}{4a}\right)$ and ρ is minimum at this point.

20. Find the point on the curve $y = \ln x$ where the curvature K is maximum.

Sol. $y = \ln x$

$$\frac{dy}{dx} = \frac{1}{x}, \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

Curvature at any point (x, y) of the curve is

$$K = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}} = \frac{\left|-\frac{1}{x^2}\right|}{\left(1 + \frac{1}{x^2}\right)^{3/2}} = \frac{x}{(1 + x^2)^{3/2}}$$

$$\frac{dK}{dx} = \frac{(1 = x^2)^{3/2} - 3x^2(1 + x^2)^{1/2}}{(1 + x^2)^3} = \frac{1 - 2x^2}{(1 + x^2)^{5/2}}$$

For extreme values of K, $\frac{dK}{dx} = 0$ gives $x = \pm\frac{1}{\sqrt{2}}$. The negative sign is not admissible and so $x = \frac{1}{\sqrt{2}}$. We see that $\frac{dK}{dx}$ changes sign from positive to negative around $x = \frac{1}{\sqrt{2}}$. Thus, K is maximum at $x = \frac{1}{\sqrt{2}}$.

For $x = \frac{1}{\sqrt{2}}$, $y = \ln\frac{1}{\sqrt{2}} = -\ln\sqrt{2}$. The required point is $\left(\frac{1}{\sqrt{2}}, -\ln\sqrt{2}\right)$

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Exercise Set 7.8 (Page 338)

1. Find an equation of the osculating circle to the curve $y = \ln x$ at the point $(1, 0)$.

Sol. $y = \ln x$

$$y' = \frac{1}{x}, \qquad y'' = \frac{-1}{x^2}$$

$$y'|_{(1,0)} = 1, \qquad y''|_{(1,0)} = -1$$

Radius of curvature ρ at $(1, 0)$ is

$$\rho = \frac{(1+y'^2)^{3/2}}{|y''|} = \frac{\left(1+\frac{1}{x^2}\right)^{3/2}}{\frac{1}{x^2}} = \frac{(1+x^2)^{3/2}}{x} = 2^{3/2}$$

Centre of curvature (h, k) is

$$h = x - \frac{y'(1+y'^2)}{y''} = 1 - 1\,\frac{1+1}{-1} = 1 + 2 = 3$$

$$k = y + \frac{1+y'^2}{y''} = 0 + \frac{1+1}{-1} = -2.$$

Equation of the osculating circle is

$$(x-3)^2 + (y+2)^2 = (2^{3/2})^2 = 8$$

2. Find an equation of the osculating circle to the hyperbola

$$\frac{x^2}{4} - \frac{y^2}{9} = 1 \text{ at the point } (-2, 0).$$

Sol. Differentiating $\frac{x^2}{4} - \frac{y^2}{9} = 1$, w.r.t. x, we have

$$\frac{x}{2} - \frac{2yy'}{9} = 0 \quad\text{or}\quad 9x - 4yy' = 0 \quad\text{or}\quad y' = \frac{9x}{4y}$$

$$y'' = \frac{9}{4}\left(\frac{y - xy'}{y^2}\right) = \frac{9}{4}\,\frac{\left(y - x\cdot\frac{9x}{4y}\right)}{y^2} = \frac{9}{16}\,\frac{4y^2-9x^2}{y^3}$$

$$\rho = \frac{\left(1+\frac{81x^2}{16y^2}\right)^{3/2}}{\frac{9}{16}\left(\frac{4y^2-9x^2}{y^3}\right)} = \frac{(16y^2+81x^2)^{3/2}}{36(4y^2-9x^2)}$$

$$\rho \text{ at } (-2,0) \text{ is } = \left|\frac{(4\times 81)^{3/2}}{36(-36)}\right| = \frac{9}{2}$$

Center of curvature (h, k)

$$h = x - y'\,\frac{(1+y'^2)}{y''} = x - \frac{9x}{4y}\,\frac{(16y^2+81x^2)y}{9(4y^2-9x^2)}$$

$$= -2 + \frac{9}{2}\,\frac{81\times 4}{9(-9\times 4)} \qquad \text{at } (-2, 0)$$

$$= -2 - \frac{9}{2} = \frac{-13}{2}$$

$$k = y + \frac{1+y'^2}{y''} = 0 \qquad \text{at } (-2, 0)$$

Equation of the osculating circle at $(-2, 0)$ is

$$\left(x + \frac{13}{2}\right)^2 + y^2 = \left(\frac{9}{2}\right)^2.$$

3. Show that the evolute of the ellipse $x = a\cos\theta, y = b\sin\theta$ is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{2/3}.$$

Sol. Here $x = a\cos\theta, \qquad y = b\sin\theta$

$$\frac{dx}{d\theta} = -a\sin\theta, \qquad \frac{dy}{d\theta} = b\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{-b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta$$

$$y'' = \frac{d^2y}{dx^2} = \frac{b}{a}\csc^2\theta\cdot\frac{d\theta}{dx} = \frac{b}{a}\cdot\frac{\csc^2\theta}{-a\sin\theta} = \frac{-b}{a^2}\csc^3\theta$$

If (X, Y) are the coordinates of the center of curvature at $(a\cos\theta, b\sin\theta)$, we have

$$X = x - \frac{y'(1+y'^2)}{y''}$$

$$= a\cos\theta - \frac{-\frac{b}{a}\cot\theta}{-\frac{b}{a^2}\csc^3\theta}\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta\left(1 + \frac{b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$= a\cos\theta - a\cos\theta\sin^2\theta - \frac{b^2}{a}\cos^3\theta$$

$$= a\cos\theta - a\cos\theta(1-\cos^2\theta) - \frac{b^2}{a}\cos^3\theta$$

$$= \left(\frac{a^2-b^2}{a}\right)\cos^3\theta \qquad (1)$$

$$Y = y + \frac{1+y'^2}{y''} = b\sin\theta + \frac{1 + \frac{b^2}{a^2}\cot^2\theta}{-\frac{b}{a^2}\csc^3\theta}$$

$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta\cos^2\theta$$

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$$= b\sin\theta - \frac{a^2}{b}\sin^3\theta - b\sin\theta(1-\sin^2\theta)$$

$$= -\frac{a^2-b^2}{b}\sin^3\theta \qquad (2)$$

For the evolute we have to eliminate θ between (1) and (2). From (1), we get

$$aX = (a^2-b^2)\cos^3\theta$$

or $\quad (aX)^{2/3} = (a^2-b^3)^{2/3}\cos^2\theta \qquad (3)$

From (2), we have

$$bY = -(a^2-b^2)\sin^3\theta$$

or $\quad (bY)^{2/3} = (a^2-b^2)^{2/3}\sin^2\theta \qquad (4)$

Adding (3) and (4), we obtain

$$(aX)^{2/3} + (bY)^{2/3} = (a-b)^{2/3}$$

Changing (X, Y) into current coordinates (x, y) equation of the evolute is

$$(ax)^{2/3} + (by)^{2/3} = (a^2-b^2)^{3/2}$$

4. Find the centre of curvature for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Show that its evolute is $(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}$.

Sol. Parametric equations of the hyperbola are

$$x = a\sec\theta, \qquad y = b\tan\theta$$

$$\frac{dx}{d\theta} = a\sec\theta\tan\theta, \qquad \frac{dy}{d\theta} = b\sec^2\theta$$

$$y' = \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{b\csc\theta}{a}$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{b}{a}\csc\theta\cot\theta\,\frac{d\theta}{dx} = -\frac{b}{a^2}\cot^3\theta$$

If (X, Y) is the centre of curvature at $(a\sec\theta, b\tan\theta)$, then

$$X = x - \frac{y'(1+y'^2)}{y''} = \frac{a^2+b^2}{a}\sec^3\theta \qquad (1)$$

(after substitution and simplification)

$$Y = y + \frac{1+y'^2}{y''} = -\frac{a^2+b^2}{b}\tan^3\theta \qquad (2)$$

From (1), we get

$$aX = (a^2+b^2)\sec^3\theta$$

$$(aX)^{2/3} = (a^2+b^2)^{2/3}\sec^2\theta \qquad (3)$$

From (2), we have

$$bY = -(a^2+b^2)\tan^3\theta$$

or $\quad (bY)^{2/3} = (a^2+b^2)^{2/3}\tan^2\theta \qquad (4)$

Subtracting (4) from (3), we get

$$(aX)^{2/3} - (bY)^{2/3} = (a^2+b^2)^{2/3}$$

Changing (X, Y) into current coordinates equation of the evolute is

$$(ax)^{2/3} - (by)^{2/3} = (a^2+b^2)^{2/3}.$$

5. Prove that the evolute of the hyperbola $2xy = a^2$ is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}.$$

Sol. Here $\quad 2xy = a^2 \quad$ or $\quad y = \frac{a^2}{2x}$

$$y' = \frac{dy}{dx} = -\frac{a^2}{2x^2}, \qquad y'' = \frac{d^2y}{dx^2} = \frac{2a^2}{2x^3} = \frac{a^2}{x^3}$$

If (X, Y) is the centre of curvature at any point (x, y) of the curve, then

$$X = x - \frac{y'(1+y'^2)}{y''} = x + \frac{x}{2}\left(1 + \frac{a^4}{4x^4}\right) = \frac{3x}{2} + \frac{a^4}{8x^3}$$

$$Y = y + \frac{(1+y'^2)}{y''} = \frac{a^2}{2x} + \frac{x^3}{a^2}\left(1 + \frac{a^4}{4x^4}\right) = \frac{3a^2}{4x} + \frac{x^3}{a^2}$$

$$X + Y = \frac{(a^2+2x^2)^3}{8a^2x^3} \qquad \text{(after simplification)}$$

$$\Rightarrow \quad (X+Y)^{2/3} = \frac{(a^2+2x^2)^2}{4a^{4/3}x^2} \qquad (1)$$

Similarly, $\quad (X-Y)^{2/3} = \frac{(a^2-2x^2)^2}{4a^{4/3}x^2} \qquad (2)$

Subtracting (2) from (1), we have

$$(X+Y)^{2/3} - (X-Y)^{2/3} = \frac{8a^2x^2}{4a^{4/3}x^2} = 2a^{2/3}$$

Changing (X, Y) to current coordinates, the required evolute is

$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}.$$

6. Prove that the centres of curvatures at points of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1-\cos\theta)$ lie on an equal cycloid.

Sol. $\quad x = a(\theta - \sin\theta), \qquad y = a(1-\cos\theta)$

$$\frac{dx}{d\theta} = a(1-\cos\theta), \qquad \frac{dy}{d\theta} = a\sin\theta$$

$$y' = \frac{dy}{dx} = \frac{a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

$$y'' = \frac{d^2y}{dx^2} = -\frac{1}{2}\csc^2\frac{\theta}{2}\,\frac{d\theta}{dx} = -\frac{1}{4a\sin^4\frac{\theta}{2}}$$

If (X, Y) is centre of curvature at any point, then

$$X = x - \frac{y'}{y''}(1+y'^2) = a(\theta + \sin\theta) \qquad (1)$$

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$$Y = y + \frac{1+y'^2}{y''} = -a(1-\cos\theta) \qquad (2)$$

Thus parametric equations of evolute of the cycloid are (1) and (2), which represent an equal cycloid.

7. Find the evolute of the four-cusped hypocycloid $x^{2/3} + y^{2/3} = a^{2/3}$ (or $x = a\cos^3\theta, y = a\sin^3\theta$).

Sol. $x = a\cos^3\theta, \quad y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \ \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$y' = \frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta$$

$$y'' = \frac{d^2y}{dx^2} = -\sec^2\theta\frac{d\theta}{dx}$$

$$= -\frac{1}{\cos^2\theta}\left(\frac{1}{-3a\cos^2\theta\sin\theta}\right) = \frac{1}{3a\sin\theta\cos^4\theta}$$

If (X, Y) is the centre of curvature at any point, then

$$X = x - \frac{y'}{y''}(1+y'^2)$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta \qquad (1)$$

(after substitution and simplification)

$$Y = y + \frac{(1+y'^2)}{y''}$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta \qquad (2)$$

(after substitution and simplification)

Adding (1) and (2), we have

$$X + Y = a(\cos^3\theta + \sin^3\theta) + 3a\sin\theta\cos\theta(\sin\theta + \cos\theta)$$
$$= a(\cos\theta + \sin\theta)(\cos^2\theta + \sin^2\theta - \sin\theta\cos\theta)$$
$$+ 3a\sin\theta\cos\theta(\sin\theta + \cos\theta)$$
$$= a(\cos\theta + \sin\theta)[\cos^2\theta + \sin^2\theta + 2\sin\theta\cos\theta]$$
$$= a(\cos\theta + \sin\theta)^3$$

$$(X+Y)^{2/3} = a^{2/3}(\cos\theta + \sin\theta)^2 \qquad (3)$$

Similarly, subtracting (2) from (1), we get

$$X - Y = a(\cos\theta - \sin\theta)^3$$

$$(X-Y)^{2/3} = a^{2/3}(\cos\theta - \sin\theta)^2 \qquad (4)$$

Adding (3) and (4), we have

$$(X+Y)^{2/3} + (X-Y)^{2/3} = a^{3/2}[(\cos\theta + \sin\theta)^2 + (\cos\theta\sin\theta)^2]$$
$$= a^{2/3}[2(\cos^2\theta + \sin^2\theta)]$$
$$= 2a^{2/3}$$

Changing (X, Y) into current coordinates (x, y), we have equation of the evolute as

$$(x+y)^{2/3} + (x-y)^{2/3} = 2a^{2/3}$$

8. Show that the evolute of the tractrix

$$x = a\left[\cos t + \ln\tan\left(\frac{t}{2}\right)\right], y = a\sin t$$

is the catenary $y = a\cosh\left(\frac{x}{a}\right)$.

Sol. $x = a\left[\cos t + \ln\tan\left(\frac{t}{2}\right)\right]$

$$\frac{dx}{dt} = a\left(-\sin t + \frac{\frac{1}{2}\sec^2\frac{t}{2}}{\tan\frac{t}{2}}\right) = a\left(-\sin t + \frac{1}{2\sin\frac{t}{2}\cos\frac{t}{2}}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\left(\frac{1-\sin^2 t}{\sin t}\right) = a\frac{\cos^2 t}{\sin t}$$

$$y = a\sin t$$

$$\frac{dy}{dt} = a\cos t$$

$$y' = \frac{dy}{dx} = \frac{a\cos t \,.\, \sin t}{a\cos^2 t} = \tan t$$

$$y'' = \frac{d^2y}{dx^2} = \sec^2 t\frac{dt}{dx} = \frac{1}{\cos^2 t}\cdot\frac{\sin t}{a\cos^2 t} = \frac{\sin t}{a\cos^4 t}$$

If (X, Y) is the centre of curvature at any point, then

$$X = x - \frac{y'}{y''}(1+y'^2)$$

$$= a\left(\cos t + \ln\tan\frac{t}{2}\right) - \frac{\tan t}{\frac{\sin t}{a\cos^4 t}}(1+\tan^2 t)$$

$$= a\left(\cos t + \ln\tan\frac{t}{2}\right) - a\cos^3 t(1+\tan^2 t)$$

$$= a\left(\cos t + \ln\tan\frac{t}{2}\right) - a\cos^3 t\left(\frac{1}{\cos^2 t}\right)$$

$$= a\cos t + a\ln\tan\frac{t}{2} - a\cos t$$

$$= a\ln\tan\frac{t}{2} \quad \text{or} \quad \tan\frac{t}{2} = e^{X/a} \qquad (1)$$

$$Y = y + \frac{1+y'^2}{y''} = a\sin t + \frac{\sec^2 t}{\frac{\sin t}{a\cos^4 t}}$$

$$= a\sin t + \frac{1}{\cos^2 t}\cdot\frac{a\cos^4 t}{\sin t}$$

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$$= a\sin t + \frac{a\cos^2 t}{\sin t} = \frac{a(\sin^2 t + \cos^2 t)}{\sin t} = \frac{a}{\sin t} \qquad (2)$$

The evolute of the curve is obtained by eliminating t from (1) and (2).

From (2), we get

$$Y = a\frac{1+\tan^2\frac{t}{2}}{2\tan\frac{t}{2}} = a\frac{1+e^{\frac{2X}{a}}}{2e^{\frac{X}{a}}}, \text{ [using (1)]}$$

$$= \frac{a}{2}\left(e^{\frac{X}{a}} + e^{\frac{-X}{a}}\right) = a\left(\frac{e^{\frac{X}{a}} + e^{\frac{-X}{a}}}{2}\right) = a\cosh\frac{X}{a}$$

Hence equation of the evolute is

$$y = a\cosh\frac{x}{a} \quad \text{as required.}$$

9. Show that the centre of curvature at the point $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ of the folium $x^3 + y^3 = 3axy$ is $\left(\frac{21a}{16}, \frac{21a}{16}\right)$.

Sol. Here $f(x, y) = x^3 - y^3 - 3axy$,

$$f_x = 3x^2 - 3ax, \quad f_y = 3y^2 - 3ax$$

$$y' = \frac{-f_x}{f_y} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}$$

$$y' \text{ at } \left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{\frac{3a^2}{2} - \frac{9a^2}{4}}{\frac{9a^2}{4} - \frac{3a^2}{2}} = -1$$

Also f_x at $\left(\frac{3a}{2}, \frac{3a}{2}\right) = \frac{9a^2}{4}$

Again, $f_{xx} = 6a, f_{xy} = -3a, f_{yy} = 6y$

At $\left(\frac{3a}{2}, \frac{3a}{2}\right)$,

$$f_{xy} = -3a, f_{xx} = 6\left(\frac{3a}{2}\right) = 9a \quad \text{and} \quad f_{yy} = 6\left(\frac{3a}{2}\right) = 9a$$

$$y''_{\left(\frac{3a}{2}, \frac{3a}{2}\right)} = \frac{d^2y}{dx^2} = -\frac{(f_y)^2 f_{xx} - 2f_x f_y f_{xy} + (f_x)^2 f_{yy}}{(f_y)^3}$$

$$= -\frac{\frac{81a^4}{16}(9a) - 2\left(\frac{9a^2}{4}\right)\left(\frac{9a^2}{4}\right)(-3a) + \left(\frac{81a^4}{16}\right)(9a)}{\left(\frac{9a^2}{4}\right)^3}$$

$$= -\frac{\frac{729}{16}a^5 + \frac{486a^5}{16} + \frac{729}{16}a^5}{\frac{729}{64}a^6}$$

$$= -\frac{1944a^5}{16} \times \frac{64}{729a^6} = \frac{-32}{3a}$$

If (X, Y) is the centre of curvature, then

$$X = x - \frac{y'}{y''}(1 + y'^2) = \frac{3a}{2} - \frac{-1}{\frac{-32}{3a}}(1+1) = \frac{3a}{2} - \frac{3a}{16} = \frac{21a}{16}$$

$$Y = y + \frac{1+y'^2}{y''} = \frac{3a}{2} + \frac{2}{\frac{-32}{3a}} = \frac{3a}{2} - \frac{3a}{16} = \frac{21a}{16}$$

Hence the center of curvature at $\left(\frac{3a}{2}, \frac{3a}{2}\right)$ is $\left(\frac{21a}{16}, \frac{21a}{16}\right)$.

10. Prove that normals to a curve are tangents to its evolute.

Sol. Let $P(x, y)$ be any point on a curve whose equation is $y = f(x)$. By **Theorem 7.33**, coordinates of the centre of curvature $C(h, k)$ corresponding to the point P are

$$h = x - \rho\sin\alpha \qquad (1)$$

$$k = y + \rho\cos\alpha \qquad (2)$$

Differentiating (1) w.r.t. x, we have

$$\frac{dh}{dx} = 1 - \rho\cos\alpha\frac{d\alpha}{dx} - \sin\alpha\frac{d\rho}{dx}$$

$$= 1 - \frac{ds}{d\alpha}\cdot\frac{dx}{ds}\cdot\frac{d\alpha}{dx} - \sin\alpha\frac{d\rho}{dx}\left(\text{since } \rho = \frac{ds}{d\alpha}, \cos\alpha = \frac{dx}{ds}\right)$$

$$= -\sin\alpha\frac{d\rho}{dx} \qquad (3)$$

From (2), we obtain,

$$\frac{dk}{dx} = \frac{dy}{dx} - \rho\sin\alpha\frac{d\alpha}{dx} + \frac{d\rho}{dx}\cos\alpha$$

$$= \frac{dy}{dx} - \rho\sin\alpha\frac{d\alpha}{ds}\cdot\frac{ds}{dx} + \cos\alpha\frac{d\rho}{dx}$$

$$= \frac{dy}{dx} - \rho\frac{\sin\alpha}{\cos\alpha}\cdot\frac{1}{\rho} + \cos\alpha\frac{d\rho}{dx}$$

$$= \cos\alpha\frac{d\rho}{dx} \qquad (4)$$

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From (3) and (4), we get $\dfrac{dk}{dh} = \dfrac{\frac{dk}{dx}}{\frac{dh}{dx}} = -\cot\alpha$

Now $\dfrac{dk}{dh}$ is slope of the tangent to the evolute at $C\,(h, k)$ and it equals the slope of the normal PC to the curve $y = f(x)$ at $P(x, y)$. Hence the result.

Exercise Set 7.9 (Page 341)

1. Find the envelope of family of lines $y = mx + \dfrac{a}{m}$, m being the parameter.

Sol. $f(x, y, m) = y - mx - \dfrac{a}{m} = 0$ (1)

$f_m(x, y, m) = -x + \dfrac{a}{m^2} = 0$ (2)

From (2), we have

$$m^2 = \frac{a}{x}, \qquad \text{or} \qquad m = \sqrt{\frac{a}{x}}$$

Substituting this value of m into (1), we get

$$y = x\sqrt{\frac{a}{x}} + a\sqrt{\frac{x}{a}} = 2\sqrt{ax}$$

or $y^2 = 4ax$ is the required envelope.

2. Show that the envelope of the family

(a) $f(x, y, t) = At^2 + Bt + C = 0$ is the discriminant $B^2 - 4AC = 0$.

(b) $f(x, y, t) = At^3 + 3Bt^2 + 3Ct + D = 0$ is $(BC - AD)^2 = 4\,(BD - C^2)\,(AC - B^2)$.

Sol.

(a) $f(x, y, t) = At^2 + Bt + C = 0$ (1)

$f_t(x, y, t) = 2At + B = 0$ gives $t = -\dfrac{B}{2A}$

Substitution of this value of t into (1) yields

$$A\left(\frac{B^2}{4A^2}\right) - \frac{B^2}{2A} + C = 0 \qquad \text{or} \qquad B^2 - 4AC = 0.$$

(b) $f(x, y, t) = At^3 + 3Bt^2 + 3Ct + D = 0$

If we write $\beta = At + B$, $H = AC - \beta^2$, $G = A^2D - 3ABC + 2B^3$ the given equation becomes

$g(x, y, \beta) = \beta^3 + 3H\beta + G = 0$ (1)

The envelopes of $g(x, y, \beta) = 0$ and $f(x, y, t) = 0$ are identical.

Now, $g_\beta(x, y, \beta) = 3\beta^2 + 3H = 0$ (2)

We eliminate β between (1) and (2) to get the envelope.

From (2), $\beta = \sqrt{-H}$. Substituting this value of β into (1), we have

$G^2 + 4H^3 = 0$

or $(A^2D - 3ABC + 2B^3)^2 + 4(AC - B^2)^3 = 0$

or $A^4D^2 + 9A^2B^2C^2 + 4B^6 - 6A^3BCD + 4A^2B^3D - 12AB^4C + 4A^3C^3 - 12A^2B^2C^2 + 12ACB^4 - 4B^6 = 0$

or $A^4D^2 - 3A^2B^2C^2 - 6A^3BCD + 4A^2B^3D + 4A^3C^3 = 0$

or $A^2D^2 + B^2C^2 - 2ABCD = 4B^2C^2 + 4ABCD - 4B^3D - 4AC^3$

or $(BC - AD) = 4[-B^2(BD - C) + AC(BD - C^2)]$

$= 4(AC - B^2)\,(BD - C^2)$

3. Find the envelope of the family $y = x\tan\theta - \dfrac{gx^2}{2u^2\cos^2\theta}$, θ being the parameter.

Sol. $f(x, y, \theta) = 2yu^2 - 2u^2x\tan\theta + gx^2\sec^2\theta = 0$ (1)

$f_\theta(x, y, \theta) = -2u^2x\sec^2\theta + 2gx^2\sec^2\theta\tan\theta = 0$ (2)

From (2), $\tan\theta = \dfrac{u^2}{gx}$. Substituting into (1), we obtain

$$2yu^2 - 2u^2x \cdot \frac{u^2}{gx} + gx^2\left(1 + \frac{u^4}{g^2x^2}\right) = 0$$

or $2yu^2g - 2u^4 + g^2x^2 + u^4 = 0$

i.e., $g^2x^2 + 2u^2gy - u^4 = 0$ is the required envelope.

4. Find the envelope of the family $y = mx + \sqrt{a^2m^2 + b^2}$, m being the parameter.

Sol. $f(x, y, m) = (y - mx)^2 - a^2m^2 - b^2 = 0$

or $f(x, y, m) = y^2 - 2mxy + m^2(x^2 - a^2) - b^2 = 0$ (1)

$f_m(x, y, m) = -2xy + 2m(x^2 - a^2) = 0$

This equation gives $m = \dfrac{xy}{x^2 - a^2}$

Putting this value of m into (1), we get

$$y^2 - \frac{2x^2y^2}{x^2 - a^2} + \frac{x^2y^2}{(x^3 - a^2)^2}(x^2 - a^2) - b^2 = 0$$

or $y^2(x^2 - a^2) - 2x^2y^2 + x^2y^2 - b^2(x^2 - a^2) = 0$

i.e., $a^2y^2 + b^2x^2 = a^2b^2$

or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is an equation of the envelope.

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5. Find the envelope of the family of straight lines joining the extremities of a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol. Let C be the centre of the ellipse and CP and CD be its conjugate semi-diameters. If the point P has coordinates $(a\cos\theta, b\sin\theta)$, then D has coordinates $(-a\sin\theta, b\cos\theta)$. Equation of PD is

$$y-b\sin\theta=\frac{b(\sin\theta-\cos\theta)}{a(\cos\theta+\sin\theta)}(x-a\cos\theta)$$

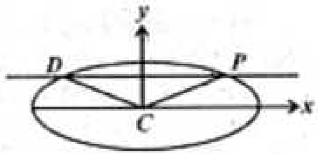
or $$\frac{x}{a}(\sin\theta-\cos\theta)-\frac{y}{b}(\sin\theta+\cos\theta)+1=0 \qquad (1)$$

Differentiating (1) w.r.t. θ, we have

$$\frac{x}{a}(\cos\theta+\sin\theta)-\frac{y}{b}(\cos\theta-\sin\theta)=0 \qquad (2)$$

We eliminate θ between (1) and (2) to obtain the envelope.

From (1), we get



$$\frac{x}{a}(\sin\theta-\cos\theta)-\frac{y}{b}(\sin\theta+\cos\theta)=-1 \qquad (3)$$

Squaring (2) and (3) and adding the results, we have

$$\frac{x^2}{a^2}[(\cos\theta+\sin\theta)^2+(\sin\theta-\cos\theta)^2]$$
$$+\frac{y^2}{b^2}[(\cos\theta-\sin\theta)^2+(\sin\theta+\cos\theta)^2]=1$$

or $\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=1$ is an equation of the envelope.

6. Prove that the envelope of an ellipse having its axes the coordinate axes and the sum of these axes constant and equal to $2a$, is the astroid $x^{2/3}+y^{2/3}=a^{2/3}$.

Sol. Equation of an ellipse with its axes as the coordinate axes is

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1, \text{ where } \quad \alpha+\beta=a$$

Here α, β are parameters. We may eliminate one of the parameters and then proceed as before to find the envelope. Alternatively, we use differentials.

$$f(x, y, \alpha, \beta)=\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}-1=0 \qquad (1)$$

$$g(\alpha, \beta)=\alpha+\beta-a=0 \qquad (2)$$

Using differentials, we have from (1) and (2)

$$\frac{2x^2}{\alpha^3}d\alpha+\frac{2y^2}{\beta^3}d\beta=0$$

and $$d\alpha+d\beta=0$$

Therefore, $$\frac{2x^2}{\alpha^3}=\frac{2y^2}{\beta^3}=\eta \quad \text{(say)} \qquad (3)$$

Elimination of α, β, η from (1), (2) and (3) will give the envelope.

From (3), we have

$$\frac{2x^2}{\alpha^2}=\eta\alpha, \qquad \frac{2y^2}{\beta^2}=\eta\beta$$

or $$2\left(\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}\right)=\eta(\alpha+\beta) \text{ or } 2=\eta a \text{ or } \eta=\frac{2}{a} \qquad (4)$$

From (3) and (4), we have $\frac{2x^2}{\alpha^3}=\eta=\frac{2}{a}$ or $\alpha^3=ax^2$

or $\alpha=a^{1/3}x^{2/3}$

Similarly, $\beta=a^{1/3}y^{2/3}$.

$$\alpha+\beta=a=a^{1/3}(x^{2/3}+y^{2/3})$$

i.e., $x^{2/3}+y^{2/3}=a^{2/3}$ is the required envelope.

7. A straight line of given length slides with its extremities on two fixed straight lines at right angle. Find the envelope of the circle drawn on the sliding line as diameter.

Sol. Let the fixed straight lines at right angles be the coordinates axes. Let $P(a, 0)$ and $Q(0, b)$ be the coordinates of extremies of the sliding line. Centre of PQ is $\left(\frac{a}{2}, \frac{b}{2}\right)$ and $|PQ|=c$. Equation of the circle with PQ as a diameter is

$$\left(x-\frac{a}{2}\right)^2+\left(y-\frac{b}{2}\right)^2=\frac{c^2}{4}$$

i.e., $$x^2+y^2-ax-by=0 \qquad (1)$$

where $$a^2+b^2=c^2 \qquad (2)$$

From (1) and (2), we have

$$x\,da+y\,db=0 \quad \text{and} \quad a\,da+b\,db=0$$

Therefore, $$\frac{x}{a}=\frac{y}{b}=\eta \quad \text{(say)} \qquad (3)$$

To get the envelope, we eliminate a, b, η from (1), (2) and (3).

From (3), we have

$$a=\frac{x}{\eta}, \qquad b=\frac{y}{\eta} \qquad (4)$$

Thus, $$\frac{x^2}{\eta^2}+\frac{y^2}{\eta^2}=a^2+b^2=c^2$$

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or $\quad \eta = \dfrac{\sqrt{x^2+y^2}}{c}$ (5)

From (1) and (4), we get

$$x^2 + y^2 - \frac{x^2+y^2}{\eta} = 0 \quad \text{or} \quad 1 - \frac{1}{\eta} = 0 \tag{6}$$

Writing the value of η from (5) into (6), we have

$$1 - \frac{c}{\sqrt{x^2+y^2}} = 0 \quad \text{or} \quad x^2 + y^2 = c^2$$

as an equation of the envelope.

8. Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters a and b are connected by the relation

(i) $a + b = c$ *(ii)* $a^n + b^n = c^n$

c being a constant.

Sol.

(i) $a + b = c$ gives $b = c - a$

$$f(x, y, a) = \frac{x}{a} + \frac{y}{c-a} - 1 = 0 \tag{1}$$

$$f_a(x, y, a) = \frac{-x}{a^2} + \frac{y}{(c-a)^2} = 0 \tag{2}$$

We eliminate a between (1) and (2) to obtain the envelope.

From (2), we have

$$\sqrt{\frac{x}{y}} = \frac{a}{c-a} \quad \text{which yields} \quad \frac{\sqrt{\frac{x}{y}}}{1+\sqrt{\frac{x}{y}}} = \frac{a}{c}$$

or $\quad a = \dfrac{c\sqrt{\frac{x}{y}}}{1+\sqrt{\frac{x}{y}}} = \dfrac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}$

Substituting this value of a into (1), we get

$$\frac{x(\sqrt{x}+\sqrt{y})}{c\sqrt{x}} + \frac{y(\sqrt{x}+\sqrt{y})}{c\sqrt{y}} - 1 = 0$$

or $\quad \sqrt{x}(\sqrt{x}+\sqrt{y}) + \sqrt{y}(\sqrt{x}+\sqrt{y}) = c$

i.e., $\quad (\sqrt{x}+\sqrt{y})^2 = c$

or $\quad \sqrt{x} + \sqrt{y} = \sqrt{c}$ is the required envelope.

(ii) $\quad \dfrac{x}{a} + \dfrac{y}{b} = 1$ (1)

where $\quad a^n + b^n = c^n$ (2)

Using differentials, from (1) and (2), we have

$$\frac{x}{a^2}da + \frac{y}{b^2}db = 0 \quad \text{and} \quad na^{n-1}da + nb^{n-1}db = 0$$

Therefore, $\quad \dfrac{x/a^2}{na^{n-1}} = \dfrac{y/b^2}{nb^{n-1}} = \eta$ (say) (3)

Elimination of a, b, η from (1), (2) and (3) gives the envelope.

From (3), we obtain

$$\frac{x}{a} = n\eta a^n, \quad \frac{y}{b} = n\eta b^n$$

Adding, $\quad \dfrac{x}{a} + \dfrac{y}{b} = n\eta(a^n + b^n)$

i.e., $\quad 1 = n\eta c^n \quad$ or $\quad \eta = \dfrac{1}{nc^n}$ (4)

Again, from (3), we have

$$a^{n+1} = \frac{x}{n\eta} = xc^n, \text{ using (4)}$$

or $\quad a = x^{\frac{1}{n+1}} c^{\frac{n}{n+1}} \quad$ *i.e.,* $\quad a^n = x^{\frac{n}{n+1}} c^{\frac{n^2}{n+1}}$

Similarly, $\quad b^n = y^{\frac{n}{n+1}} c^{\frac{n^2}{n+1}}$

Adding the last two equations, we get

$$a^n + b^n = c^{\frac{n^2}{n+1}}\left[x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}\right]$$

or $\quad \dfrac{c^n}{c^{\frac{n^2}{n+1}}} = x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}}$

i.e., $\quad x^{\frac{n}{n+1}} + y^{\frac{n}{n+1}} = c^{\frac{n}{n+1}}$ is an equation of the envelope.

9. Find an equation of the normal at any point of the curve with parametric equations $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$. Hence deduce that an equation of the evolute of the curve is $x^2 + y^2 = a^2$.

Sol. $\quad y = a(\sin t - t\cos t)$

$x = a(\cos t - t\sin t)$

$$\frac{dy}{dt} = a(\cos t - \cos t) + t\sin t = a\,t\sin t$$

$$\frac{dx}{dt} = a(-\sin t + \sin t + \cos t) = a\,t\cos t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{\sin t}$$

Slope of the normal at any point $= -\dfrac{\cos t}{\sin t}$

Equation of the normal is

$$y - a(\sin t - t\cos t) = -\frac{\cos t}{\sin t}[x - a(\cos t + t\sin t)]$$

or $\quad y\sin t - a(\sin^2 t - t\sin t\cos t)$

$$= -x\cos t + a(\cos^2 t + t\sin t\cos t)$$

or $\quad x\cos t + y\sin t = a \qquad (1)$

Envelope of (1) is evolute of the given curve

$$f(x, y, t) = x\cos t + y\sin t - a = 0$$

$$f_t(x, y, t) = -x\sin t + y\cos t = 0 \qquad (2)$$

Squaring (1) and (2) and adding the results, we have

$$x^2 + y^2 = a^2$$

as the required evolute of the given curve.

10. Prove that an equation of the normal to the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ may be written in the form $y\cos\theta - x\sin\theta = a\cos 2\theta$. Hence show that the evolute of the curve is $(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}$.

Sol. Parametric equations of the astroid are

$$x = a\sin^3\theta, \qquad y = a\cos^3\theta$$

$$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta, \qquad \frac{dy}{d\theta} = -3a\cos^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = -\cot\theta.$$

Slope of the normal at any point is $\tan\theta$. Equation of the normal is

$$y - a\cos^3\theta = \frac{\sin\theta}{\cos\theta}(x - a\sin^3\theta)$$

or $\quad y\cos\theta - a\cos^4\theta = x\sin\theta - a\sin^4\theta$

or $\quad x\sin\theta - y\cos\theta + a(\cos^4\theta - \sin^4\theta) = 0$

i.e., $\quad x\sin\theta - y\cos\theta + a\cos 2\theta = 0$

is an equation of the normal in the required form.

To find evolute of the curve, we shall find envelope of the normal.

$$f(x, y, \theta) = x\sin\theta - y\cos\theta + a\cos 2\theta = 0 \qquad (1)$$

$$f_\theta(x, y, \theta) = x\cos\theta + y\sin\theta - 2a\sin 2\theta = 0 \qquad (2)$$

We have to eliminate θ between (1) and (2) to obtain the envelope of the normal (which is evolute of the astroid).

From (1) and (2), we have

$$\frac{x}{2a\sin 2\theta\cos\theta - a\sin\theta\cos 2\theta}$$

$$= \frac{y}{a\cos\theta\cos 2\theta + 2a\sin 2\theta\sin\theta} = \frac{1}{\sin^2\theta + \cos^2\theta} = 1$$

Therefore,

$$x = 2a\sin 2\theta\cos\theta - a\sin\theta(1 - 2\sin^2\theta)$$

$$= 4a\sin\theta(1 - \sin^2\theta) - a\sin\theta(1 - 2\sin^2\theta)$$

$$= 3a\sin\theta - 2a\sin^3\theta$$

$$y = a\cos\theta\cos 2\theta + 2a\sin 2\theta\sin\theta$$

$$= a\cos\theta(2\cos^2\theta - 1) + 4a\cos\theta(1 - \cos^2\theta)$$

$$= 3a\cos\theta - 2a\cos^3\theta$$

$$x + y = 3a(\sin\theta + \cos\theta) - 2a(\sin^3\theta + \cos^3\theta)$$

$$= a(\sin\theta + \cos\theta)(1 + \sin 2\theta) \qquad (3)$$

$$x - y = 3a(\sin\theta - \cos\theta) - 2a(\sin^3\theta - \cos^3\theta)$$

$$= a(\sin\theta - \cos\theta)(1 - \sin 2\theta) \qquad (4)$$

From (3) and (4), on squaring, we get

$$(x + y)^2 = a^2(1 + \sin 2\theta)(1 + \sin 2\theta)^2$$

$$= a^2(1 + \sin 2\theta)^3 \qquad (5)$$

$$(x - y)^2 = a^2(1 - \sin 2\theta)(1 - \sin 2\theta)^2$$

$$= a^2(1 - \sin 2\theta)^3 \qquad (6)$$

From (5) and (6), on the taking cube roots, we have

$$(x + y)^{2/3} = a^{2/3}(1 + \sin 2\theta)$$

$$(x - y)^{2/3} = a^{2/3}(1 - \sin 2\theta)$$

Adding the last two equations, we obtain

$$(x + y)^{2/3} + (x - y)^{2/3} = a^{2/3}(1 + \sin 2\theta + 1 - \sin 2\theta) = 2a^{2/3}$$

as evolute of the astroid.

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