Our Effort To Surve You Better

Calculus
With
Analytic Geometry

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Calculus With Analytic Geometry

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When you know better you do better!!

Put $\tan\frac{x}{2} = z$ so that $dx = \frac{2\,dz}{1+z^2}$ and $\sin x = \frac{2z}{1+z^2}$

$$\int \csc x\,dx = \int \frac{1}{\sin x}\,dx = \int \frac{1+z^2}{2z}\cdot\frac{2\,dz}{1+z^2} = \int \frac{1}{z}\,dz = \ln|z|$$

$$= \ln\left|\tan\frac{x}{2}\right| = \ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right| = \frac{1}{2}\cdot 2\ln\left|\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right|$$

$$= \frac{1}{2}\ln\left|\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\right| = \frac{1}{2}\ln\left|\frac{1-\cos x}{1+\cos x}\right|.$$

Chapter 5

THE DEFINITE INTEGRAL

Exercise Set 5.1 (Page 172)

Evaluate the following by definition (Problems 1 – 8):

1. $\int_{-1}^{1} x\,dx$

Sol. Consider a partition P of $[-1, 1]$ into n subintervals of equal length

i.e., $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$. The subintervals are

$$\left[-1, -1+\frac{2}{n}\right], \left[-1+\frac{2}{n}, -1+\frac{4}{n}\right], \dots, \left[-1+\frac{2(n-1)}{n}, -1+\frac{2n}{n}\right]$$

Take c_r as the right endpoint of each subinterval. Then

$$S(P,f) = S(P,x) = \sum_{r=1}^{n} \Delta x\, f(c_r)$$

$$= \frac{2}{n}\left[f\left(-1+\frac{2}{n}\right) + f\left(-1+\frac{4}{n}\right) + \dots + f\left(-1+\frac{2n}{n}\right)\right]$$

$$= \frac{2}{n}\left[\left(-1+\frac{2}{n}\right) + \left(-1+\frac{4}{n}\right) + \dots + \left(-1+\frac{2n}{n}\right)\right]$$

$$= \frac{2}{n}\left[-n + \frac{2}{n}(1+2+\dots+n)\right]$$

$$= \frac{2}{n}\left[-n + \frac{2}{n}\frac{(n+1)\cdot n}{2}\right] = \frac{2}{n}(-n+n+1) = \frac{2}{n}$$

$$\int_{-1}^{1} x\,dx = \lim_{n\to\infty}\sum_{r=1}^{n}\Delta x\cdot f(c_r) = \lim_{n\to\infty}\frac{2}{n} = 0.$$

2. $\int_{a}^{b} \frac{1}{x}\,dx$

Sol. This problem could be solved by the method of **Examples**. However, we give another method which is more elegant.

Let $P = \{a, a(\Delta x), a(\Delta x)^2, \dots, a(\Delta x)^{n-1}, a(\Delta x)^n = b\}$, be a partition of $[a, b]$, where $\Delta x = \left(\frac{b}{a}\right)^{1/n}$. The subintervals, into which $[a, b]$ is subdivided are

$[a, a\,\Delta x], [a\Delta x, a(\Delta x)^2], \dots, [a(\Delta x)^{n-1}, a(\Delta x)^n]$

Taking c_r as the left endpoint of each subinterval, we have

$$S(P, f) = S\left(P, \frac{1}{x}\right)$$

$$= a(\Delta x - 1)\cdot\frac{1}{a} + a\,\Delta x\,(\Delta x - 1)\cdot\frac{1}{a\,\Delta x}$$

$$+ a\,(\Delta x)^2(\Delta x - 1)\cdot\frac{1}{a(\Delta x)^2} + \dots + a(\Delta x)^{n-1}(\Delta x - 1)\cdot\frac{1}{a(\Delta x)^{n-1}}$$

$$= (\Delta x - 1)\,[1 + 1 + \dots + 1,\ n \text{ terms}] = n\,(\Delta x - 1) \qquad (1)$$

As $n \to \infty$, $\Delta x \to 1$, $\left[\text{since } \Delta x = \left(\frac{b}{a}\right)^{1/n}\right]$ so that the length of each subinterval approaches zero.

Now, $\Delta x = \left(\frac{b}{a}\right)^{1/n}$ or $\ln \Delta x = \frac{1}{n}\ln\left(\frac{b}{a}\right)$ *i.e.*, $n = \frac{1}{\ln \Delta x}\cdot\ln\left(\frac{b}{a}\right)$

Hence (1) becomes

$$S(P, f) = \left(\ln\frac{b}{a}\right)\cdot\left(\frac{\Delta x - 1}{\ln \Delta x}\right)$$

Taking limits as $\Delta x \to 1$, we get

$$\int_a^b \frac{1}{x}\,dx = \left(\ln\frac{b}{a}\right)\lim_{\Delta x \to 1}\frac{\Delta x - 1}{\ln \Delta x} \qquad \left(\frac{0}{0}\right)$$

$$= \left(\ln\frac{b}{a}\right)\cdot 1 = \ln b - \ln a$$

3. $\int_a^b x^2\,dx$

Sol. Let $P = \{a, a + \Delta x, a + 2\Delta x, \dots, a + (n-1)\Delta x, a + n\,\Delta x = b\}$ be a partition of $[a, b]$, where $\Delta x = \frac{b-a}{n}$.

Taking left endpoint of each subinterval as c_r, we have

$$S(P, f) = S(P, x^2) = \Delta x a^2 + \Delta x(a + \Delta x)^2 + \Delta x(a + 2\Delta x)^2 + \dots + \Delta x(a + (n-1)\,\Delta x)^2$$

$$= \Delta x[na^2 + 2a\Delta x(1 + 2 + \dots + (n-1)) + \Delta x^2(1^2 + 2^2 + \dots + (n-))^2]$$

$$= \Delta x\left[na^2 + 2a\Delta x\,\frac{(n-1)\,n}{2} + \Delta x^2\,\frac{(n-1)\cdot n\cdot(2n-1)}{6}\right]$$

$$= \frac{b-a}{n}\cdot na^2 + a\,\frac{(b-a)^2}{n^2}\,(n-1)\cdot n + \frac{(b-a)^3}{n^3}\cdot\frac{(n-1)\,n\,(2n-1)}{6}$$

since $\Delta x = \frac{b-a}{n}$

$$= a^2(b-a) + a(b-a)^2\cdot\left(1 - \frac{1}{n}\right) + \frac{(b-a)^3}{6}\cdot\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

Taking limits as $\Delta x \to 0$ and $n \to \infty$, we have

$$\lim_{\substack{n \to \infty \\ \Delta x \to 0}} S(P, f) = \int_a^b x^2\,dx$$

$$= a^2(b-a) + a(b-a)^2\lim_{n \to \infty}\left(1 - \frac{1}{n}\right) + \frac{(b-a)^3}{6}\cdot\lim_{n \to \infty}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$$= (b-a)\,a^2 + a(b-a)^2\cdot 1 + \frac{(b-a)^3}{6}\cdot 2$$

$$= (b-a)\left[a^2 + a\,(b-a) + \frac{(b-a)^2}{3}\right] = (b-a)\left[ab + \frac{b^2 - 2ab + a^2}{3}\right]$$

$$= \frac{b-a}{3}\,[3ab + b^2 - 2ab + a^2]$$

$$= \frac{b-a}{3}\,(b^2 + ab + a^2) = \frac{b^3 - a^3}{3}$$

4. $\int_a^b \frac{dx}{\sqrt{x}}$

Sol. Let $\Delta x = \frac{b-a}{n}$. Then taking left end of each subinterval as c_r, we have

$$S(P, f) = S\left(P, \frac{1}{\sqrt{x}}\right) = \Delta x\left[\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a + \Delta x}} + \dots + \frac{1}{\sqrt{a + (n-1)\,\Delta x}}\right]$$

Now $\lim_{\Delta x \to 0}\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} = \frac{1}{2}\cdot x^{\frac{1}{2}-1} = \frac{1}{2}\cdot\frac{1}{\sqrt{x}}$

$$\text{Therefore, } \frac{1}{2} = \lim_{\Delta x \to 0}\frac{\sqrt{x + \Delta x} - \sqrt{x}}{\frac{\Delta x}{\sqrt{x}}} \qquad (1)$$

Putting $x = a, a + \Delta x, a + 2\Delta x, \dots, a + (n-1)\Delta x$ in (1), we obtain

$$\frac{1}{2} = \lim_{\Delta x \to 0}\frac{\sqrt{a + \Delta x} - \sqrt{a}}{\frac{\Delta x}{\sqrt{a}}}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{a+2\Delta x} - \sqrt{a+\Delta x}}{\dfrac{\Delta x}{\sqrt{a+\Delta x}}}$$

$= \ldots \quad \ldots \quad \ldots$

$= \ldots \quad \ldots \quad \ldots$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{a+n\Delta x} - \sqrt{a+(n-1)\Delta x}}{\dfrac{\Delta x}{\sqrt{a+(n-1)\Delta x}}}$$

$$= \lim_{\Delta x \to 0} \frac{\text{Sum of } n \text{ numerators}}{\text{Sum of } n \text{ denominators}}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{a+n\,\Delta x} - \sqrt{a}}{\Delta x\left(\dfrac{1}{\sqrt{a}} + \dfrac{1}{\sqrt{a+\Delta x}} + \ldots + \dfrac{1}{\sqrt{a+(n-1)\Delta x}}\right)}$$

or $\displaystyle \lim_{\Delta x \to 0} \Delta x\left(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{a+\Delta x}} + \ldots + \frac{1}{\sqrt{a+(n-1)\Delta x}}\right)$

$$= \lim_{\Delta x \to 0} S\left(P, \frac{1}{\sqrt{x}}\right) = \lim_{\Delta x \to 0} 2\left[\sqrt{a+n\,\Delta x} - \sqrt{a}\right]$$

Hence $\displaystyle \int_a^b \frac{dx}{\sqrt{x}} = \lim_{\substack{\Delta x \to 0 \\ n \to \infty}} 2\left[\sqrt{a + \frac{n(b-a)}{n}} - \sqrt{a}\right]$

$$= 2(\sqrt{a+b-a} - \sqrt{a}) = 2(\sqrt{b} - \sqrt{a})$$

5. $\displaystyle \int_a^b \sin x\,dx$

Sol. Let $\Delta x = \dfrac{b-a}{n}$. Then taking left endpoint of each subinterval as c_r, we have

$S(P, f) = S(P, \sin x) = \Delta x\,[\sin a + \sin(a+\Delta x) + \sin(a+2\Delta x) + \ldots + \sin(a+(n-1)\Delta x)]$

$$= \Delta x \frac{\sin\left(a + \dfrac{n-1}{2}\Delta x\right)\sin\dfrac{n\,\Delta x}{2}}{\sin\dfrac{\Delta x}{2}} \qquad \text{(From Trignometry)}$$

$$= \frac{\dfrac{\Delta x}{2}\left[\cos\left(a - \dfrac{\Delta x}{2}\right) - \cos\left(a + \dfrac{2n-1}{2}\Delta x\right)\right]}{\sin\dfrac{\Delta x}{2}}$$

Taking limits of both sides as $n \to \infty$ and $\Delta x \to 0$, we get

$$\int_a^b \sin x\,dx = \lim_{\Delta x \to 0} \frac{\dfrac{\Delta x}{2}}{\sin\dfrac{\Delta x}{2}} \times \lim_{\substack{n \to \infty \\ \Delta x \to 0}} \left[\cos\left(a - \frac{\Delta x}{2}\right) - \cos\left(a + \frac{2n-1}{2}\cdot\frac{b-a}{n}\right)\right]$$

$$= 1 \times [\cos a - \cos[(a + (b-a))] = \cos a - \cos b$$

6. $\displaystyle \int_a^b \sin^2 x\,dx$

Sol. Let $\Delta x = \dfrac{b-a}{n}$

Then taking left endpoint of each subinterval as c_r, we have

$S(P, f) = S(P, \sin^2 x)$

$= \Delta x\,[\sin^2 a + \sin^2(a+\Delta x) + \sin^2(a+2\Delta x) + \ldots + \sin^2(a+(n-1)\Delta x)]$

$$= \Delta x\left[\frac{1-\cos 2a}{2} + \frac{1-\cos(2a+2\Delta x)}{2} + \frac{1-\cos(2a+4\Delta x)}{2} + \ldots + \frac{1-\cos(2a+(2n-2)\Delta x)}{2}\right]$$

$$= \Delta x\left[\frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2}, n \text{ terms}\right] - \frac{\Delta x}{2}[\cos 2a + \cos(2a+2\Delta x) + \cos(2a+4\Delta x) + \ldots + (2a+(2n-2)\Delta x)]$$

$$= \frac{n\,\Delta x}{2} - \frac{\dfrac{\Delta x}{2}[\cos(2a+(n-1)\Delta x)\sin n\,\Delta x}{\sin \Delta x}$$

$$= \frac{n\,\Delta x}{2} - \frac{\Delta x}{\sin\Delta x}\cdot\frac{1}{4}[\sin(2a+(2n-1)\Delta x) - \sin(2a-\Delta x)]$$

$$= \frac{n}{2}\cdot\frac{b-a}{n} - \frac{\Delta x}{\sin\Delta x}\cdot\frac{1}{4}\left[\sin\left(2a+(2n-1)\frac{b-a}{n}\right) - \sin(2a-\Delta x)\right]$$

Taking limit as $n \to \infty$ and $\Delta x \to 0$, we have

$$\int_a^b \sin^2 x\,dx = \frac{b-a}{2} - 1\cdot\frac{1}{4}[\sin(2a+2(b-a)) - \sin 2a]$$

$$= \frac{b-a}{2} - \frac{1}{4}[\sin 2b - \sin 2a]$$

7. $\displaystyle \int_a^b \cosh x\,dx$

Sol. We have, $S(P, f) = S(P, \cosh x)$

$= \Delta x[\cosh a + \cosh(a+\Delta x) + \cosh(a+2\Delta x) + \ldots + \cosh(a+(n-1)\Delta x)]$

$$= \Delta x\left[\frac{e^a+e^{-a}}{2}+\frac{e^{a+\Delta x}+e^{-(a+\Delta x)}}{2}+\ldots+\frac{e^{a+(n-1)\Delta x}+e^{-(a+(n-1)\Delta x)}}{2}\right]$$

$$= \frac{\Delta x}{2}\left[e^a+e^{a+\Delta x}+\ldots+e^{a+(n-1)\Delta x}\right]+\frac{\Delta x}{2}\left[e^{-a}+e^{-(a+\Delta x)}+\ldots+e^{-(a+(n-1)\Delta x)}\right]$$

$$= \frac{\Delta x}{2}\,e^{a}\left[1+e^{\Delta x}+e^{2\Delta x}+\ldots+e^{(n-1)\Delta x}\right]+\frac{\Delta x}{2}\,e^{-a}\left[1+e^{-\Delta x}+e^{-2\Delta x}+\ldots+e^{-(n-1)\Delta x}\right]$$

$$= \frac{\Delta x}{2}\cdot e^a\left[\frac{1-e^{n\Delta x}}{1-e^{\Delta x}}\right]+\frac{\Delta x}{2}\cdot e^{-a}\left[\frac{1-e^{-n\Delta x}}{1-e^{-\Delta x}}\right]$$

$$= \frac{e^a}{2}\left(1-e^{n\Delta x}\right)\cdot\frac{\Delta x}{1-e^{\Delta x}}+\frac{e^{-a}}{2}\left(1-e^{-n\Delta x}\right)\cdot\frac{\Delta x}{1-e^{-\Delta x}}$$

$$= \frac{e^a}{2}\left(1-e^{b-a}\right)\cdot(-1)\cdot\frac{\Delta x}{e^{\Delta x}-1}+\frac{e^{-a}}{2}\left(1-e^{-b+a}\right)\cdot\frac{-\Delta x}{e^{-\Delta x}-1}$$

Taking limits as $\Delta x \to 0$, we get

$$\int_a^b \cosh x\,dx = \frac{e^a-e^b}{2}\times(-1)\cdot 1+\frac{e^{-a}-e^{-b}}{2}\times 1$$

$$= \frac{1}{2}\left[-e^a+e^b+e^{-a}-e^{-b}\right]=\frac{e^b-e^{-b}}{2}-\frac{e^a-e^{-a}}{2}$$

$$= \sinh b-\sinh a$$

8. $\displaystyle\int_0^{\pi/2}\cos x\,dx$

Sol. We subdivide $\left[0,\frac{\pi}{2}\right]$ into n subintervals each of length $\Delta x = \frac{\pi}{2n}$

Then $S(P,f) = S(P,\cos x)$

$$= \Delta x\left[\cos 0+\cos\Delta x+\cos 2\Delta x+\ldots+\cos(n-1)\Delta x\right]$$

where c_r has been taken as left endpoint of each subinterval.

$$= \Delta x\,\frac{\cos\left(0+\frac{n-1}{2}\Delta x\right)\sin\frac{n\Delta x}{2}}{\sin\frac{\Delta x}{2}}=\frac{\frac{\Delta x}{2}\left[\sin\frac{2n-1}{2}\cdot\Delta x+\sin\frac{\Delta x}{2}\right]}{\sin\frac{\Delta x}{2}}$$

Taking limits of both sides as $n\to\infty$ and $\Delta x\to 0$, we have

$$\int_0^{\pi/2}\cos x\,dx=\lim_{\Delta x\to 0}\frac{\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}}\times\lim_{\substack{\Delta x\to 0\\ n\to\infty}}\left(\sin\frac{2n-1}{2}\cdot\Delta x+\sin\frac{\Delta x}{2}\right)$$

$$= 1\times\lim_{n\to\infty}\sin\left(\frac{2n-1}{2}\cdot\frac{\pi}{2n}\right)$$

$$= \lim_{n\to\infty}\sin\left(1-\frac{1}{2n}\right)\frac{\pi}{2}=\sin\frac{\pi}{2}=1$$

Determine the limit of each of the following as $n\to\infty$ (Problems 9 – 15):

9. $\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}+\ldots+\dfrac{1}{n+n}$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\frac{1}{1+\frac{3}{n}}+\ldots+\frac{1}{1+\frac{n}{n}}\right]$$

$$= \int_0^1\frac{dx}{1+x}=\left[\ln(1+x)\right]_0^1=\ln 2-\ln 1=\ln 2$$

10. $\dfrac{n}{n^2}+\dfrac{n}{n^2+1^2}+\dfrac{n}{n^2+2^2}+\ldots+\dfrac{n}{n^2+(n-1)^2}$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[1+\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\ldots+\frac{1}{1+\left(\frac{n-1}{n}\right)^2}\right]$$

$$= \int_0^1\frac{dx}{1+x^2}=\left[\arctan x\right]_0^1=\arctan 1-\arctan 0=\frac{\pi}{4}$$

11. $\dfrac{n}{(n+1)^2}+\dfrac{n}{(n+2)^2}+\ldots+\dfrac{n}{(n+n)^2}$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{\left(1+\frac{1}{n}\right)^2}+\frac{1}{\left(1+\frac{2}{n}\right)^2}+\ldots+\frac{1}{\left(1+\frac{n}{n}\right)^2}\right]$$

$$\int_0^1\frac{dx}{(1+x)^2}=\left[-\frac{1}{1+x}\right]_0^1=-\frac{1}{2}+1=\frac{1}{2}$$

12. $\dfrac{1}{n\sqrt{n}}\left[\sqrt{n+1}+\sqrt{n+2}+\ldots+\sqrt{n+n}\right]$

Sol. The given limit is

$$\lim_{n\to\infty}\frac{1}{n}\left[\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}+\ldots+\sqrt{1+\frac{n}{n}}\right]$$

$$= \int_0^1 \sqrt{1+x}\,dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1 = \frac{2}{3}(2^{3/2}-1) = \frac{2}{3}(2\sqrt{2}-1)$$

13. $\dfrac{1}{n} + \dfrac{1}{\sqrt{n^2-1}} + \dfrac{1}{\sqrt{n^2-2^2}} + \ldots + \dfrac{1}{\sqrt{n^2-(n-1)^2}}$

Sol. The given limit is

$$\lim_{n\to\infty} \frac{1}{n}\left[\frac{1}{\sqrt{1-\left(\frac{0}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{1}{n}\right)^2}} + \frac{1}{\sqrt{1-\left(\frac{2}{n}\right)^2}} + \ldots + \frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}}\right]$$

$$= \int_0^1 \frac{dx}{\sqrt{1-x^2}} = [\arcsin x]_0^1 = \arcsin 1 - \arcsin 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

14. $\left\{\left(1+\frac{1^2}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\ldots\left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

Sol. Let $y = \left\{\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^3}\right)\left(1+\frac{3^2}{n^2}\right) + \ldots + \left(1+\frac{n^2}{n^2}\right)\right\}^{1/n}$

$$\ln y = \frac{1}{n}\left[\ln\left(1+\frac{1}{n^2}\right) + \ln\left(1+\frac{2^2}{n^2}\right) + \ln\left(1+\frac{3^2}{n^2}\right) + \ldots + \ln\left(1+\frac{n^2}{n^2}\right)\right]$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \frac{1}{n}\left[\ln\left(1+\left(\frac{1}{n}\right)^2\right) + \ln\left(1+\left(\frac{2}{n}\right)^2\right)\right.$$

$$\left. + \ln\left(1+\left(\frac{3}{n}\right)^2\right) + \ldots + \left(1+\left(\frac{n}{n}\right)^2\right)\right]$$

$$= \int_0^1 \ln(1+x^2)\,dx = [x\ln(1+x^2)]_0^1 - \int_0^1 x \times \frac{2x}{1+x^2}\,dx$$

$$= \ln 2 - 2\int_0^1 \frac{x^2}{1+x^2} = \ln 2 - 2\int_0^1 \left(1 - \frac{1}{1+x^2}\right)dx$$

$$= \ln 2 - 2[x - \arctan x]_0^1 = \ln 2 - 2[1 - \arctan 1]$$

$$= \ln 2 - 2\left(1 - \frac{\pi}{4}\right) = \ln 2 - 2 + \frac{\pi}{2} = \ln 2 + \left(\frac{\pi}{2} - 2\right)\ln e$$

$$\lim_{n\to\infty} y = \ln 2 + \ln e^{\frac{\pi}{2}-2} = \ln(2e^{\frac{\pi}{2}-2})$$

Therefore, $y = 2\,e^{(\pi-4)/2}$

15. $\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\ldots\left(1+\frac{n}{n}\right)^{1/n}$

Sol. $\ln y = \ln\left(1+\frac{1}{n}\right) + \frac{1}{2}\ln\left(1+\frac{2}{n}\right) + \ldots + \frac{1}{n}\ln\left(1+\frac{n}{n}\right)$

$$= \sum_{r=1}^{n} \frac{1}{r}\ln\left(1+\frac{r}{n}\right) = \sum_{r=1}^{n} \frac{1}{n}\cdot\frac{n}{r}\ln\left(1+\frac{r}{n}\right)$$

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n}\left(\frac{n}{r}\ln\left(1+\frac{r}{n}\right)\right)$$

$$= \int_0^1 \frac{1}{x}\ln(1+x)\,dx = \int_0^1 \frac{1}{x}\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right)dx$$

$$= \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\right)dx = \left[x - \frac{x^2}{2^2} + \frac{x^3}{3^3} - \frac{x^4}{4^2} + \ldots\right]_0^1$$

$$= 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) - 2\left(\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{6^2} + \ldots\right)$$

$$= \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\right) - 2 \times \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right)$$

$$= \frac{\pi^2}{6} - \frac{1}{2}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{6} - \frac{\pi^2}{12} = \frac{\pi^2}{12}$$

Thus, $\lim\limits_{n\to\infty} y = e^{\frac{\pi^2}{12}}$.

Exercise Set 5.2 (Page 181)

Evaluate the following integrals (Problems 1 – 11):

1. $\int_0^6 f(x)\,dx$, where $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ 3x-2 & \text{if } x \geq 2 \end{cases}$

Sol. $\int_0^6 f(x)\,dx = \int_0^2 f(x)\,dx + \int_2^6 f(x)\,dx = \int_0^2 x^2\,dx + \int_2^6 (3x-2)\,dx$

$$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^6 = \frac{8}{3} + (54-12) - (6-4)$$

$$= \frac{8}{3} + 42 - 2 = \frac{8}{3} + 40 = \frac{128}{3}$$

2. $\int_{-1}^{5} |x-2|\, dx$

Sol. $\int_{-1}^{5} |x-2|\, dx = \int_{-1}^{2} |x-2|\, dx + \int_{2}^{5} |x-2|\, dx$

$$= \int_{-1}^{2} -(x-2)\, dx + \int_{2}^{5} (x-2)\, dx$$

$$= \left[-\frac{x^2}{2} + 2x\right]_{-1}^{2} + \left[\frac{x^2}{2} - 2x\right]_{2}^{5}$$

$$= \left[-2 + 4 - \left(\frac{-1}{2} - 2\right)\right] + \left[\frac{25}{2} - 10 - (2-4)\right]$$

$$= \left(2 + \frac{5}{2}\right) + \left(\frac{5}{2} + 2\right) = \frac{9}{2} + \frac{9}{2} = 9$$

3. $\int_{0}^{3\pi/4} |\cos x|\, dx$

Sol. $\int_{0}^{3\pi/4} |\cos x|\, dx = \int_{0}^{\pi/2} |\cos x|\, dx + \int_{\pi/2}^{3\pi/4} |\cos x|\, dx$

$$= \int_{0}^{\pi/2} \cos x\, dx + \int_{\pi/2}^{3\pi/4} (-\cos x)\, dx$$

$$= [\sin x]_{0}^{\pi/2} + [-\sin x]_{\pi/2}^{3\pi/4}$$

$$= \left(\sin\frac{\pi}{2} - \sin 0\right) + \left[\left(-\sin\frac{3\pi}{4}\right) - \left(-\sin\frac{\pi}{2}\right)\right]$$

$$= [1-0] + \left(-\frac{1}{\sqrt{2}}\right) - (-1) = 1 - \frac{1}{\sqrt{2}} + 1 = 2 - \frac{1}{\sqrt{2}}$$

4. $\int_{0}^{\pi} \cos^{2n+1} x\, dx$

Sol. $\int_{0}^{\pi} \cos^{2n+1} x\, dx = \int_{0}^{\pi/2} \cos^{2n+1} x\, dx + \int_{0}^{\pi/2} \cos^{2n+1} (\pi - x)\, dx$, by Theorem 5.10

$$= \int_{0}^{\pi/2} \cos^{2n+1} x\, dx + \int_{0}^{\pi/2} (-\cos x)^{2n+1}\, dx$$

$$= \int_{0}^{\pi/2} \cos^{2n+1} x\, dx - \int_{0}^{\pi/2} \cos^{2n+1} x\, dx \text{ (since } (-\cos x)^{2n+1} = -\cos^{2n+1} x)$$

$$= 0$$

Alternative Method:

$I = \int_{0}^{\pi} \cos^{2n+1} x\, dx = \int_{0}^{\pi} \cos^{2n+1} (\pi - x)\, dx$, by Theorem 5.9

$$= \int_{0}^{\pi} (-\cos x)^{2n+1}\, dx = -\int_{0}^{\pi} \cos^{2n+1} x\, dx$$

or $\quad 2I = 0 \Rightarrow I = 0$

5. $\int_{0}^{\pi/4} \frac{\sec^2\theta\, dx}{\tan x - \tan\theta},\ \theta > \frac{\pi}{4}$

Sol. Put $\tan x = z$ so that $\sec^2 x\, dx = dz$ or $dx = \frac{1}{1+z^2} dz$

When $x = 0$, $z = 0$ and when $x = \frac{\pi}{4}, z = 1$. Then

$$I = \int_{0}^{\pi/4} \frac{\sec^2\theta\, dx}{\tan x - \tan\theta} = \sec^2\theta \int_{0}^{1} \frac{dz}{(z - \tan\theta)(1+z^2)}$$

$$= \sec^2\theta \int_{0}^{1} \left[\frac{\frac{1}{\sec^2\theta}}{z - \tan\theta} + \frac{-\frac{1}{\sec^2\theta}\cdot z - \frac{\tan\theta}{\sec^2\theta}}{1+z^2}\right] dz$$

$$= \int_{0}^{1} \left(\frac{dz}{z - \tan\theta} - \frac{z\, dz}{1+z^2} - \frac{\tan\theta}{1+z^2}\right) dz$$

$$= \ln\left[|z - \tan\theta|\right]_0^1 - \left[\frac{1}{2}\ln(1+z^2)\right]_0^1 - [\tan\theta \arctan z]_0^1$$

$$I = |1 - \tan\theta| - \ln|0 - \tan\theta| - \frac{1}{2}(\ln 2 - \ln 1) - \tan\theta\cdot\left(\frac{\pi}{4} - 0\right)$$

$$I = \ln\left|\frac{\tan\theta - 1}{\tan\theta}\right| - \ln\sqrt{2} - \frac{\pi}{4}\tan\theta$$

Alternative Answer:

$$\frac{\tan\theta-1}{\tan\theta}=\frac{\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin\frac{\pi}{4}}{\cos\frac{\pi}{4}}}{\dfrac{\sin\theta}{\cos\theta}}=\frac{\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}}{\sin\theta\cos\frac{\pi}{4}}=\frac{\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)}{\sin\theta}$$

Therefore, $I=\ln\left|\dfrac{\sqrt{2}\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\theta}\right|-\ln\sqrt{2}-\dfrac{\pi}{4}\tan\theta$

$$=\ln\sqrt{2}+\ln\left|\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\theta}\right|-\ln\sqrt{2}-\frac{\pi}{4}\tan\theta$$

$$=\ln\left|\frac{\sin\left(\frac{\pi}{4}-\theta\right)}{\sin\theta}\right|-\frac{\pi}{4}\tan\theta.$$

6. $\displaystyle\int_0^{\pi/2}\tan x\ln(\sin x)\,dx$

Sol. $I=\displaystyle\int_0^{\pi/2}\tan x\ln(\sin x)\,dx=\int_0^{\pi/2}\frac{\sin x}{\cos x}\ln\sqrt{1-\cos^2 x}\,dx.$

Put $z=\cos x$ so that $dz=-\sin x\,dx$.

When $x=0, z=1$ and when $x=\frac{\pi}{2}, z=0$. Then

$$I=-\int_1^0\frac{\ln\sqrt{1-z^2}}{z}\,dz=\frac{1}{2}\int_0^1\frac{\ln(1-z^2)}{z}\,dz$$

$$=\frac{1}{2}\int_0^1\frac{1}{z}\left(-z^2-\frac{z^4}{2}-\frac{z^6}{3}-\ldots\right)dz=-\frac{1}{2}\int_0^1\left(z+\frac{z^3}{2}+\frac{z^5}{3}+\ldots\right)dz$$

$$=-\frac{1}{2}\left[\frac{z^2}{2}+\frac{z^4}{2\cdot 4}+\frac{z^6}{3\cdot 6}+\ldots\right]_0^1=-\frac{1}{2}\left[\frac{1}{2}+\frac{1}{2\cdot 4}+\frac{1}{3\cdot 6}+\ldots\right]$$

$$=-\frac{1}{2^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\right]=-\frac{1}{4}\left(\frac{\pi^2}{6}\right)=-\frac{\pi^2}{24}$$

7. $\displaystyle\int_0^{2\pi}\frac{dx}{5+3\cos x}$

Sol. $\displaystyle\int_0^{2\pi}\frac{dx}{5+3\cos x}=2\int_0^{\pi}\frac{dx}{5+3\cos(2\pi-x)}$, by Theorem 5.11 (*i*)

$$=2\int_0^{\pi}\frac{dx}{5+3\cos x}=\frac{2}{\sqrt{5^2-3^2}}\left|\arccos\frac{3+5\cos x}{5+3\cos x}\right|_0^{\pi}$$

$$=\frac{1}{2}\left[\arccos\frac{3-5}{5-3}-\arccos\frac{3+5}{5+3}\right]$$

$$=\frac{1}{2}[\arccos -1)-\arccos(1)]$$

$$=\frac{1}{2}[\pi-0]=\frac{\pi}{2}$$

8. $\displaystyle\int_0^1\arctan\left(\frac{2x-1}{1+x-x^2}\right)dx$

Sol. $I=\displaystyle\int_0^1\arctan\left(\frac{2x-1}{1+x-x^2}\right)dx$

$$=\int_0^1\arctan\frac{2(1-x)-1}{1+(1-x)-(1-x)^2}\,dx,\ \text{by Theorem 5.9}$$

$$=\int_0^1\arctan\frac{1-2x}{1+x-x^2}=-\int_0^1\arctan\frac{2x-1}{1+x-x^2}=-I$$

or $2I=0$ *i.e.,* $I=0$

9. $\displaystyle\int_{-\pi/4}^{\pi/4}\frac{2x^3-x}{(x^2+1)(x-1)(x+1)}\,dx$

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)} dx$ · Then

$$I = \int_{-\pi/4}^{0} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)} dx + \int_{0}^{\pi/4} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)} dx, \text{ by Theorem 5.8}$$

$= I_1 + I_2$

Now putting $x = -t$ i.e., $dx = -dt$ in I_1, we have

$$I_1 = \int_{\pi/4}^{0} \frac{2(-t)^3 - (-t)}{((-t)^2+1)(-t-1)(-t+1)} \cdot -dt$$

$$\left(\text{since } t = \frac{\pi}{4} \text{ when } x = -\frac{\pi}{4} \text{ and } t = 0 \text{ when } x = 0\right)$$

$$= \int_{\pi/4}^{0} \frac{-2t^3 + t}{(t^2+1)(t+1)(t-1)} \cdot -dt = \int_{\pi/4}^{0} \frac{2t^3 - t}{(t^2+1)(t-1)(t+1)} dt$$

$$= -\int_{0}^{\pi/4} \frac{2t^3 - t}{(t^2+1)(t-1)(t+1)} dt, \qquad \text{by Theorem 5.7}$$

$$= -\int_{0}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)} dx, \qquad \text{by Theorem 5.6}$$

Thus $I = -\int_{0}^{\pi/4} \frac{2x^3 - x}{(x^2+1)(x-1)(x+1)} dx + \int_{0}^{\pi/4} \frac{2x^3-x}{(x^2+1)(x-1)(x+1)} dx = 0$

10. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$

Sol. By (5.9), we have $I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)} dx$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x} dx = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

or $\quad 2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx$

Put $\cos x = z$ so that $-\sin x\, dx = dz$

$z = 1$ when $x = 0$; $z = -1$ when $x = \pi$

$$2I = -\int_1^{-1} \frac{\pi\, dz}{1 + z^2} = \pi \int_{-1}^{1} \frac{dz}{1+z^2} = \pi [\arctan z]_{-1}^{1}$$

$$= \pi[\arctan(1) - \arctan(-1)] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$= \pi \cdot \frac{\pi}{2} \quad \text{or} \quad I = \frac{\pi^2}{4}$$

11. $\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx$

Sol. $I = \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx \qquad (1)$

Put $9 - x = 3 + y$ or $x = 6 - y$ so that $dx = -dy$

When $x = 2, y = 4$ and when $x = 4, y = 2$

$$I = \int_4^2 \frac{\sqrt{\ln(3+y)}\,(-dy)}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}} = -\int_4^2 \frac{\sqrt{\ln(3+y)}\, dy}{\sqrt{\ln(3+y)} + \sqrt{\ln(9-y)}}$$

$$= \int_2^4 \frac{\sqrt{\ln(3+y)}\, dy}{\sqrt{\ln(9-y)} + \sqrt{\ln(3+y)}} = \int_2^4 \frac{\sqrt{\ln(3+x)}\, dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} \qquad (2)$$

Adding (1) and (2), we get

$$2I = \int_2^4 \frac{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)} + \sqrt{\ln(3+x)}} dx$$

$$= \int_2^4 1 \cdot dx = [x]_2^4 = 4 - 2 = 2 \quad \text{or} \quad I = 1$$

In Problems 12 – 26, show that:

12. $\int_0^{\pi/2} \ln(\tan x)\,dx = 0$

Sol. $$\int_0^{\pi/2} \ln(\tan x)\,dx = \int_0^{\pi/2} \ln\left(\frac{\sin x}{\cos x}\right)dx = \int_0^{\pi/2} \ln(\sin x)\,dx - \int_0^{\pi/2} \ln(\cos x)\,dx$$

$$= \int_0^{\pi/2} \ln\left[\sin\left(\frac{\pi}{2} - x\right)\right]dx - \int_0^{\pi/2} \ln(\cos x)\,dx$$

$$= \int_0^{\pi/2} \ln(\cos x)\,dx - \int_0^{\pi/2} \ln(\cos x)\,dx = 0$$

13. $\int_0^{\pi/2} \sin 2x \ln(\tan x)\,dx = 0$

Sol. $$\int_0^{\pi/2} \sin 2x \ln(\tan x)\,dx = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \ln \tan\left(\frac{\pi}{2} - x\right)dx$$

$$= \int_0^{\pi/2} \sin(\pi - 2x) \ln \cot x\,dx$$

$$= \int_0^{\pi/2} \sin 2x \ln \cot x\,dx$$

$$= \int_0^{\pi/2} \sin 2x \ln(\tan x)^{-1}\,dx$$

$$= -\int_0^{\pi/2} \sin 2x \ln \tan x\,dx$$

i.e., $2\int_0^{\pi/2} \sin 2x \ln \tan x\,dx = 0$

or $\int_0^{\pi/2} \sin 2x \ln \tan x\,dx = 0$ as required

14. $\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\,dx = \frac{\pi^2}{4}$

Sol. $$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\,dx = \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1 + \cos^2(\pi - x)}\,dx$$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\,dx = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\,dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\,dx$$

or $$2I = \pi \int_0^{\pi} \frac{\sin x\,dx}{1 + \cos^2 x}\,dx$$

Put $\cos x = t$ in the integral on R.H.S. so that $-\sin x\,dx = dt$.
When $x = 0, t = 1$ and when $x = \pi, t = -1$.

$$2I = -\pi \int_1^{-1} \frac{dt}{1 + t^2} = \pi \int_{-1}^{1} \frac{dt}{1 + t^2} = 2\pi \int_0^1 \frac{dt}{1 + t^2}$$

$$= 2\pi\,[\arctan t]_0^1 = 2\pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{2} \quad \text{or} \quad I = \frac{\pi^2}{4}$$

15. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \frac{\pi}{4}$

Sol. Let $$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\sqrt{\sin\left(\frac{\pi}{2} - x\right)} + \sqrt{\cos\left(\frac{\pi}{2} - x\right)}}$$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\,dx$$

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$$

$$= \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}}\, dx = \int_0^{\pi/2} 1 \cdot dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

Hence $I = \frac{\pi}{4}$

16. $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1)$

Sol. Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x}\, dx = \int_0^{\pi/2} \frac{\sin^2 x\left(\frac{\pi}{2}-x\right) dx}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}$

$$= \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x}$$

$$2I = \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{\sin x + \cos x}\, dx = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \frac{\frac{1}{\sqrt{2}}\, dx}{\frac{1}{\sqrt{2}} \cdot \sin x + \frac{1}{\sqrt{2}} \cos x} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \int_0^{\pi/2} \csc\left(x + \frac{\pi}{4}\right) dx$$

$$= \frac{1}{\sqrt{2}} \left|\ln \tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|_0^{\pi/2} = \frac{1}{\sqrt{2}}\left[\ln \tan \frac{3\pi}{8} - \ln \tan \frac{\pi}{8}\right]$$

$$= \frac{1}{\sqrt{2}} \ln \frac{\tan \frac{3\pi}{8}}{\tan \frac{\pi}{8}} = \frac{1}{\sqrt{2}} \ln \frac{\sin \frac{3\pi}{8} \cos \frac{\pi}{8}}{\sin \frac{\pi}{8} \cos \frac{3\pi}{8}} = \frac{1}{\sqrt{2}} \ln \frac{\sin \frac{\pi}{2} + \sin \frac{\pi}{4}}{\sin \frac{\pi}{2} - \sin \frac{\pi}{4}}$$

$$= \frac{1}{\sqrt{2}} \ln \frac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}} \ln \frac{\sqrt{2}+1}{\sqrt{2}-1}$$

$$= \frac{1}{\sqrt{2}} \ln \frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}$$

$$= \frac{1}{\sqrt{2}} \ln (\sqrt{2}+1)^2 = \sqrt{2} \ln(\sqrt{2}+1)$$

Hence $I = \frac{\sqrt{2}}{2} \ln(\sqrt{2}+1) = \frac{1}{\sqrt{2}} \ln(\sqrt{2}+1)$

17. $\displaystyle\int_0^{\pi} \frac{x\, dx}{1 + \sin x} = \pi$

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x\, dx}{1 + \sin x}$. Then by Theorem 5.9, we have

$$I = \int_0^{\pi} \frac{(\pi - x)}{1 + \sin(\pi - x)} = \int_0^{\pi} \frac{\pi - x}{1 + \sin x}\, dx$$

$$= \int_0^{\pi} \frac{\pi}{1 + \sin x} \cdot dx - \int_0^{\pi} \frac{x}{1 + \sin x}\, dx = \pi \int_0^{\pi} \frac{1}{1 + \sin x}\, dx - I$$

or $\displaystyle 2I = \pi \int_0^{\pi} \frac{1}{1 + \sin x}\, dx \Rightarrow I = \frac{\pi}{2} \int_0^{\pi} \frac{1}{1 + \sin x}\, dx$ (1)

Now $\displaystyle\int_0^{\pi} \frac{1}{1 + \sin x}\, dx = 2 \int_0^{\pi/2} \frac{1}{1 + \sin x}\, dx$, by Theorem 5.11

$\displaystyle = 2 \int_0^{\pi/2} \frac{1}{1 + \sin\left(\frac{\pi}{2} - x\right)}\, dx$ by Theorem 5.9

$$= 2\int_0^{\pi/2} \frac{1}{1+\cos x}\,dx = 2\int_0^{\pi/2} \frac{1}{2\cos^2\frac{x}{2}}\,dx = \int_0^{\pi/2} \sec^2\frac{x}{2}\,dx$$

$$= \int_0^{\pi/4} (\sec^2 t)\cdot 2dt \qquad \text{(by setting } \frac{x}{2} = t \text{, then } dx = 2dt\text{. When } x = \frac{\pi}{2},$$

$$t = \frac{\pi}{4}, \text{ when } x = 0, t = 0$$

$$= 2\left[\tan t\right]_0^{\pi/4} = 2[1-0] = 2.$$

Thus $I = \int_0^{\pi} \frac{x\,dx}{1+\sin x} = \frac{\pi}{2}\cdot 2 = \pi$ (by using (1)

18. $\int_0^{\pi/2}\left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \pi\ln 2$

Sol. $\int_0^{\pi/2}\left(\frac{\theta}{\sin\theta}\right)^2 d\theta = \int_0^{\pi/2}\theta^2(\operatorname{cosec}^2\theta)\,d\theta = \left[\theta^2(-\cot\theta)\right]_0^{\pi/2} - \int_0^{\pi/2}(-\cot\theta)\cdot 2\theta\,d\theta$

$$= 0 + 2\int_0^{\pi/2}\theta\left(\frac{\cos\theta}{\sin\theta}\right)d\theta \quad \left(\because \left(\frac{\pi}{2}\right)^2 \cot\frac{\pi}{2} = 0 \text{ and } \lim_{\theta\to 0}\frac{\theta^2\cos\theta}{\sin\theta} = 0\right)$$

$$= 2\left[\theta\cdot\ln\sin\theta\right]_0^{\pi/2} - 2\int_0^{\pi/2}(\ln\sin\theta)\cdot 1\,d\theta$$

$$= 0 - 2\int_0^{\pi/2}\ln\sin\theta\,d\theta \left(\because \frac{\pi}{2}\sin\frac{\pi}{2} = 0 \text{ and } \lim_{\theta\to 0}\frac{\ln\sin\theta}{1/\theta} = \lim_{\theta\to 0}\left(-\frac{\theta^2\cos\theta}{\sin\theta}\right) = 0\right)$$

$$= -2\left(-\frac{\pi}{2}\ln 2\right) = \pi\ln 2, \text{ (see Example 9)}$$

19. $\int_0^{\pi/2}\ln(\tan\theta + \cot\theta)\,d\theta = \pi\ln 2$

Sol. Let $I = \int_0^{\pi/2}\ln(\tan\theta+\cot\theta)\,d\theta = \int_0^{\pi/2}\ln\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)d\theta$

$$= \int_0^{\pi/2}\ln\frac{1}{\sin\theta\cos\theta}\,d\theta = -\int_0^{\pi/2}\ln\sin\theta\,d\theta - \int_0^{\pi/2}\ln\cos\theta\,d\theta$$

$$= -\int_0^{\pi/2}\ln\sin\theta\,d\theta - \int_0^{\pi/2}\ln\cos\left(\frac{\pi}{2}-\theta\right)d\theta$$

$$= -2\int_0^{\pi/2}\ln\sin\theta\,d\theta = -2\left(\frac{-\pi}{2}\ln 2\right) = \pi\ln 2.$$

20. $\int_0^{\pi} x\ln(\sin x)\,dx = \frac{\pi^2}{2}\ln\left(\frac{1}{2}\right)$

Sol. Let $I = \int_0^{\pi} x\ln\sin x\,dx = \int_0^{\pi}(\pi-x)\ln\sin(\pi-x)\,dx$

$$= \int_0^{\pi}(\pi-x)\ln\sin x\,dx = \int_0^{\pi}\pi\ln\sin x\,dx - \int_0^{\pi}x\ln\sin x\,dx$$

or $2I = \int_0^{\pi}\pi\ln\sin x\,dx = \pi\int_0^{\pi}\ln\sin x\,dx$

$$= 2\pi\int_0^{\pi/2}\ln\sin x\,dx, \text{ by Theorem 5.11 } (i)$$

$$= 2\pi\cdot\left(-\frac{\pi}{2}\ln 2\right) = \pi^2\cdot(-1)\ln(2) = \pi^2\ln\frac{1}{2}$$

or $I = \frac{\pi^2}{2}\ln\frac{1}{2}$

21. $\int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx = \frac{\pi}{8}\ln 2$

Sol. $I = \int_0^1 \frac{\ln(1+x)}{1+x^2}\,dx$

Put $x = \tan\theta$ so that $dx = \sec^2\theta\,d\theta$

When $x = 0$, $\theta = 0$ and when $x = 1$, $\theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{1+\tan^2\theta}\cdot\sec^2\theta\, d\theta = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

$$= \int_0^{\pi/4} \ln\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right], \text{ by Theorem 5.9}$$

$$= \int_0^{\pi/4} \ln\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right] d\theta = \int_0^{\pi/4} \ln\left[\frac{2}{1+\tan\theta}\right] d\theta$$

$$= \int_0^{\pi/4} \ln 2\, d\theta - \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta$$

or $\quad 2\int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \ln 2 \int_0^{\pi/4} 1\cdot d\theta = \ln 2\,[\theta]_0^{\pi/4} = \frac{\pi}{4}\ln 2$

or $\quad I = \int_0^{\pi/4} \ln(1+\tan\theta)\, d\theta = \frac{\pi}{8}\ln 2$

22. $\displaystyle\int_0^{\pi/2} \sin x \ln(\sin x)\, dx = \ln\left(\frac{2}{e}\right)$

Sol. $I = \displaystyle\int_0^{\pi/2} \sin x \ln(\sin x)\, dx$

$$= \int_0^{\pi/2} \ln\sqrt{1-\cos^2 x}\cdot \sin x\, dx$$

Put $z = \cos x$ so that, $dz = -\sin x\, dx$

When $x = 0$, $z = 1$ and when $x = \frac{\pi}{2}$, $z = 0$. Then

$$I = -\int_1^0 \ln\sqrt{1-z^2}\, dz = \frac{1}{2}\int_0^1 \ln(1-z^2)\, dz$$

$$= -\frac{1}{2}\int_0^1 \left(z^2+\frac{z^4}{2}+\frac{z^6}{3}+\dots\right) dz = -\frac{1}{2}\left[\frac{z^3}{3}+\frac{z^5}{2\cdot 5}+\frac{z^7}{3\cdot 7}+\dots\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{3}+\frac{1}{2\cdot 5}+\frac{1}{3\cdot 7}+\dots\right] = -\left[\frac{1}{2\cdot 3}+\frac{1}{4\cdot 5}+\frac{1}{6\cdot 7}+\dots\right]$$

$$= -\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\dots\right]$$

$$= -\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\dots$$

$$= \left[1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\dots\right]-1$$

$$= \ln 2 - 1 = \ln 2 - \ln e = \ln\frac{2}{e}$$

23. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\, dx = \frac{\pi}{4}$

Sol. Let $\displaystyle I = \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\, dx = \int_0^{\pi/2} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\, dx$

$$= \int_0^{\pi/2} \frac{\sin x}{\cos x+\sin x}\, dx$$

or $\displaystyle 2I = \int_0^{\pi/2} \frac{\sin x+\cos x}{\cos x+\sin x}\, dx = \int_0^{\pi/2} 1\cdot dx = \frac{\pi}{2}$

or $\displaystyle I = \int_0^{\pi/2} \frac{\cos x}{\sin x+\cos x}\, dx = \frac{\pi}{4}$

24. $\displaystyle\int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx = \frac{\pi^2}{2}-\pi$

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x\sin x\, dx}{1+\sin x} = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)\, dx}{1+\sin(\pi-x)}$

$$= \int_0^{\pi} \frac{(\pi-x)\sin x\, dx}{1+\sin x} = \int_0^{\pi} \frac{\pi\sin x}{1+\sin x}\, dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx$$

or $$2I = \pi\int_0^{\pi}\frac{\sin x\,dx}{1+\sin x} = \pi\int_0^{\pi}\left(1-\frac{1}{1+\sin x}\right)dx$$

$$= \pi\int_0^{\pi}1\cdot dx - \pi\int_0^{\pi}\frac{dx}{1+\sin x} = \pi[x]_0^{\pi} - \pi\int_0^{\pi}\frac{dx}{1+\sin x}$$

$$= \pi\cdot(\pi-0) - \pi\cdot\int_0^{\pi}\frac{dx}{1+\sin x}$$

$$= \pi^2 - \pi\cdot 2 \qquad \left(\int_0^{\pi}\frac{dx}{1+\sin x} = 2 \quad \text{by Problem 17}\right)$$

Therefore $I = \dfrac{\pi^2}{2} - \pi$.

25. $$\int_0^{\pi/2}\frac{\sin x - \cos x}{1+\sin x\cos x} = 0$$

Sol. Let $$I = \int_0^{\pi/2}\frac{\sin x - \cos x}{1+\sin x\cos x}dx = \int_0^{\pi/2}\frac{\sin\left(\frac{\pi}{2}-x\right)-\cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cos\left(\frac{\pi}{2}-x\right)}dx$$

$$= \int_0^{\pi/2}\frac{\cos x - \sin x}{1+\cos x\sin x}dx = -I$$

or $2I = 0$ *i.e.,* $I = 0$

26. $$\int_0^{\pi/2}\frac{\sin^2 x\,dx}{1+\sin x\cos x} = \frac{\pi}{3\sqrt{3}}$$

Sol. Let $$I = \int_0^{\pi/2}\frac{\sin^2 x\,dx}{1+\sin x\cos x}$$

$$= \int_0^{\pi/2}\frac{\sin^2\left(\frac{\pi}{2}-x\right)dx}{1+\sin\left(\frac{\pi}{2}-x\right)\operatorname{cod}\left(\frac{\pi}{2}-x\right)} = \int_0^{\pi/2}\frac{\cos^2 x}{1+\sin x\cos x}dx$$

Therefore, $$2I = \int_0^{\pi/2}\frac{\sin^2 x+\cos^2 x}{1+\sin x\cos x}dx = \int_0^{\pi/2}\frac{dx}{1+\frac{1}{2}\sin 2x}$$

$$2I = \int_0^{\pi/4}\frac{dx}{1+\frac{1}{2}\sin 2x} + \int_{\pi/4}^{\pi/2}\frac{dx}{1+\frac{1}{2}\sin 2x}$$

Putting $x = \dfrac{\pi}{2} - z$ in $\displaystyle\int_{\pi/4}^{\pi/2}\frac{dx}{1+\frac{1}{2}\sin 2x}$, we can prove that

$$\int_{\pi/4}^{\pi/2}\frac{dx}{1+\frac{1}{2}\sin 2x} = \int_0^{\pi/4}\frac{dx}{1+\frac{1}{2}\sin 2x}$$

Therefore $2I = 2\displaystyle\int_0^{\pi/4}\frac{dx}{1+\frac{1}{2}\sin 2x}$ or $I = \displaystyle\int_0^{\pi/4}\frac{dx}{1+\frac{1}{2}\sin 2x}$

Now we put $\tan x = t$ so that $\sec^2 x\,dx = dt$ or $dx = \dfrac{1}{1+t^2}dt$ and $\sin 2x = \dfrac{2t}{1+t^2}$. When $x = 0$, $t = 0$ and when $x = \dfrac{\pi}{4}$, $t = 1$.

Thus $$I = \int_0^1\frac{1}{1+\frac{1}{2}\cdot\frac{2t}{1+t^2}}\cdot\frac{1}{1+t^2}dt = \int_0^1\frac{dt}{t^2+t+1}$$

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$$I = \int_0^1 \frac{1}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\,dt = \frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2t+1}{\sqrt{3}}\right)\right]_0^1$$

$$I = \frac{2}{\sqrt{3}}\left[\arctan\sqrt{3} - \arctan\frac{1}{\sqrt{3}}\right] = \frac{2}{\sqrt{3}}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{2}{\sqrt{3}}\cdot\frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

27. Let f and g be integrable on $[a, b]$ and suppose that $f(x) \le g(x)$ for all $x \in [a, b]$. Show that $\int_a^b f(x)\,dx \le \int_a^a g(x)\,dx$.

Sol. Let F and G be antiderivatives of f and g respectively. Then $F' = f$ and $G' = g$ and $-F + G$ is antiderivative of $-f + g$.

We first show that if $f(x) \ge 0$, then $\int_a^b f(x) \ge 0$.

As $F'(x) = f(x) \ge 0$ on $[a, b]$, so, F is an increasing function on $[a, b] \Rightarrow F(b) \ge F(a)$. Therefore,

$$\int_a^b f(x)\,dx = F(b) - F(a) \ge 0$$

Now
$$\int_a^b [g(x) - f(x)]\,dx = [G(x) - F(x)]_a^b$$
$$= G(b) - G(a) - [F(b) - F(a)]$$
$$= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$$

But $g(x) - f(x) \ge 0$. Therefore,

$$\int_a^b [g(x) - f(x)]\,dx = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \ge 0$$

$$\Rightarrow \quad \int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$$

<u>Alternative Method:</u>

Let $P = [a = x_0 < x_1 < \dots < x_{n-1} < x_n = b]$ be a partition of $[a, b]$. Since $f(x) \le g(x)$ on $[a, b]$, we have $f(x) \le g(x)$ on each subinterval Δx_i · Hence $\sum_{i=1}^{n} f(c_i)\,\Delta x_i \le \sum_{i=1}^{n} g(c_i)\,\Delta x_i$, $c_i \in \Delta x_i$

Taking limits as $n \to \infty$, $\|P\| \to 0$, we get $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.

Exercise Set 5.3 (Page 189)

Determine whether the following improper integral converge. Evaluate the integrals that converge (Problems 1 – 33):

1. $\int_0^\infty e^{-x}\,dx$

Sol. $\int_0^t e^{-x}\,dx = [-e^{-x}]_0^t = -e^{-t} + 1$

Therefore, $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty} \int_0^t e^{-x}\,dx$

$= \lim_{t\to\infty} (-e^{-t} + 1) = 1$ and the given integral converges.

2. $\int_0^\infty e^{-x}\sin x\,dx$

Sol. $\int e^{-x}\sin x\,dx = e^{-x}(-\cos x) - \int(-\cos x)(-e^{-x})\,dx$

$= -e^{-x}\cos x - \int e^{-x}\cos x\,dx$

$= -e^{-x}\cos x - [e^{-x}(\sin x) - \int \sin x \cdot (-e^{-x})\,dx$

$= -e^{-x}\cos - e^{-x}\sin x - \int e^{-x}\sin x\,dx$

Therefore, $\int e^{-x}\sin x\,dx = \frac{-e^{-x}}{2}(\sin x + \cos x)$

and $\int_0^\infty e^{-x}\sin x\,dx = \lim_{t\to\infty}\int_0^t e^{-x}\sin x\,dx$

$= \lim_{t\to\infty}\left[\frac{-e^{-x}}{2}(\sin x + \cos x)\right]_0^t = \lim_{t\to\infty}\left[-\frac{e^{-t}(\sin t + \cos t)}{2} + \frac{1}{2}\right]$

$= \frac{1}{2}$ Thus the given integral converges

3. $\int_{-\infty}^{0} \frac{dx}{1+x^2}$

Sol. $\displaystyle\int_{-\infty}^{0}\frac{dx}{1+x^2} = \lim_{t\to-\infty}\int_{t}^{0}\frac{dx}{1+x^2} = \lim_{t\to-\infty}[\arctan t]_t^0 = \lim_{t\to-\infty}[-\arctan t]$

$$= -(\lim_{t\to-\infty}\arctan t) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Thus the given integral converges.

4. $\displaystyle\int_0^{\infty} e^{-2x}\cos 2x\,dx$

Sol. $\displaystyle\int_0^{\infty} e^{-2x}\cos 2x\,dx = \frac{e^{-2x}}{8}[-2\cos 2x + 2\sin 2x]$, by Example 13 Page 140.

Hence $\displaystyle\int_0^{\infty} e^{-2x}\cos 2x\,dx = \lim_{t\to\infty}\int_0^{t} e^{-2x}\cos 2x\,dx$

$$= \lim_{t\to\infty}\left[\frac{e^{-2x}}{8}(-2\cos 2x + 2\sin 2x)\right]_0^t$$

$$= \lim_{t\to\infty}\left[\frac{e^{-2t}}{8}(-2\cos 2t + 2\sin 2t) + \frac{1}{4}\right]$$

$= \dfrac{1}{4}$. (Since $\displaystyle\lim_{t\to\infty} e^{-2t} = \lim_{t\to\infty}\frac{1}{e^{2t}} = 0$)

Thus the given integral converges.

5. $\displaystyle\int_{-\infty}^{0}\frac{dx}{(2x-1)^3}$

Sol. $\displaystyle\int_{-\infty}^{0}\frac{dx}{(2x-1)^3} = \lim_{t\to-\infty}\int_{t}^{0}\frac{dx}{(2x-1)^3} = \lim_{t\to-\infty}\left[\frac{-1}{4(2x-1)^2}\right]_t^0$

$$= \lim_{t\to-\infty}\left[-\frac{1}{4} + \frac{1}{4(2t-1)^2}\right] = -\frac{1}{4}$$

Thus the given integral converges.

6. $\displaystyle\int_{-\infty}^{2} e^{2x}\,dx$

Sol. $\displaystyle\int_{-\infty}^{2} e^{2x}\,dx = \lim_{t\to-\infty}\int_{t}^{2} e^{2x}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}e^{2x}\right]_t^2 = \lim_{t\to-\infty}\left[\frac{1}{2}e^4 - \frac{1}{2}e^{2t}\right] = \frac{e^4}{2}$

Thus the given integral converges.

7. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2}$

Sol. $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \int_{-\infty}^{0}\frac{dx}{1+x^2} + \int_{0}^{\infty}\frac{dx}{1+x^2}$

Now, $\displaystyle\int_{0}^{\infty}\frac{dx}{1+x^2} = \lim_{t\to-\infty}\int_{0}^{t}\frac{dx}{1+x^2} = \lim_{t\to-\infty}[\arctan x]_0^t = \frac{\pi}{2}$

And $\displaystyle\int_{-\infty}^{0}\frac{dx}{1+x^2} = \frac{\pi}{2}$, by **Problem 3.**

Therefore, $\displaystyle\int_{-\infty}^{\infty}\frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi$ and the given integral converges.

8. $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{\sqrt{x^2+2}}$

Sol. $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{\sqrt{x^2+2}} = \int_{-\infty}^{0}\frac{x\,dx}{\sqrt{x^2+2}} + \int_{0}^{\infty}\frac{x\,dx}{\sqrt{x^2+2}}$

Now, $\displaystyle\int_{-\infty}^{0}\frac{x\,dx}{\sqrt{x^2+2}} = \lim_{t\to-\infty}\int_{t}^{0}\frac{x\,dx}{\sqrt{x^2+2}} = \lim_{t\to-\infty}[\sqrt{x^2+2}]_t^0$

$$= \lim_{t\to-\infty}[\sqrt{2} - \sqrt{t^2+2}] = -\infty$$

Similarly, $\displaystyle\int_{0}^{\infty}\frac{x\,dx}{\sqrt{x^2+2}} = \infty$. Hence, $\displaystyle\int_{-\infty}^{\infty}\frac{x\,dx}{\sqrt{x^2+2}}$ diverges.

9. $\displaystyle\int_{-\infty}^{\infty} x^3\,dx$

Sol. $\int_{-\infty}^{\infty} x^3\,dx = \int_{-\infty}^{0} x^3\,dx + \int_{0}^{\infty} x^3\,dx$

$$\int_{-\infty}^{0} x^3\,dx = \lim_{t\to-\infty}\int_t^0 x^3\,dx = \lim_{t\to-\infty}\left[\frac{x^4}{4}\right]_t^0 = -\lim_{t\to-\infty}\left(\frac{t^4}{4}\right) = -\infty$$

$$\int_0^{\infty} x^3\,dx = \lim_{t\to\infty}\int_0^t x^3\,dx = \lim_{t\to\infty}\left[\frac{x^4}{4}\right]_t^0 = \lim_{t\to\infty}\left(\frac{t^4}{4}\right) = \infty$$

Thus the given integral diverges.

10. $\int_{-\infty}^{\infty}\frac{x}{(x^4+1)}\,dx$

Sol. In $\int_{-\infty}^{\infty}\frac{x}{x^4+1}\,dx$, put $x^2 = z$ so that $x\,dx = \frac{1}{2}\,dz$, then

$$\int\frac{x}{x^4+1}\,dx = \frac{1}{2}\int\frac{dz}{1+z^2} = \frac{1}{2}\arctan z = \frac{1}{2}\arctan x^2$$

Now $\int_{\infty}^{-\infty}\frac{x}{1+x^4}\,dx = \int_{-\infty}^{0}\frac{x\,dx}{1+x^4} + \int_0^{\infty}\frac{x\,dx}{1+x^4}$.

$$= \lim_{t\to-\infty}\int_t^0\frac{x\,dx}{1+x^4} + \lim_{t\to\infty}\int_0^t\frac{x\,dx}{1+x^4}$$

$$= \lim_{t\to-\infty}\left[\frac{1}{2}\arctan x^2\right]_t^0 + \lim_{t\to\infty}\left[\frac{1}{2}\arctan x^2\right]_0^t$$

$$= \lim_{t\to-\infty}\left[0 - \frac{1}{2}\arctan t^2\right] + \lim_{t\to\infty}\left(\frac{1}{2}\arctan t^2 - 0\right)$$

$$= -\frac{1}{2}\left(\frac{\pi}{2}\right) + \frac{1}{2}\left(\frac{\pi}{2}\right) = -\frac{\pi}{4} + \frac{\pi}{4} = 0$$

Thus the given integral converges.

11. $\int_0^1\frac{dx}{x}$

Sol. $\int_0^1\frac{dx}{x} = \lim_{t\to0^+}\int_t^1\frac{dx}{x} = \lim_{t\to0^+}[\ln x]_t^1 = \lim_{t\to0^+}[-\ln t] = \infty$

Thus the given integral diverges.

12. $\int_0^a\frac{dx}{x\sqrt{a^2-x^2}}$

Sol. $\int_0^a\frac{dx}{x\sqrt{a^2-x^2}} = \int_0^{a/2}\frac{dx}{x\sqrt{a^2-x^2}} + \int_{a/2}^{0}\frac{dx}{x\sqrt{a^2-x^2}}$

Now $\int_0^{a/2}\frac{dx}{x\sqrt{a^2-x^2}} = \lim_{t\to0^+}\int_t^{a/2}\frac{dx}{x\sqrt{a^2-x^2}} = \lim_{t\to0^+}\left[-\frac{1}{a}\ln\frac{a+\sqrt{a^2-x^2}}{x}\right]_t^{a/2}$

$$= \lim_{t\to0^+}\left[-\frac{1}{a}\ln(2+\sqrt{3}) + \frac{1}{a}\ln\left(\frac{a+\sqrt{a^2-t^2}}{t}\right)\right]$$

$$= -\frac{1}{a}\ln(2+\sqrt{3}) + \frac{1}{a}\cdot\infty \quad \left(\begin{array}{l}\text{As } \dfrac{a+\sqrt{a^2-t^2}}{t}\to\infty \text{ when } t\to0^+\\ \text{so } \ln\dfrac{a+\sqrt{a^2+t^2}}{t}\to\infty \text{ when } t\to0^+\end{array}\right)$$

$$= \infty$$

And $\int_{a/2}^{0}\frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\ln(2+\sqrt{3})$ (The details are left for the students)

Thus $\int_0^a\frac{dx}{x\sqrt{a^2-x^2}}$ is not finite and it diverges.

13. $\int_0^1\frac{dx}{(x-1)^2}$

Sol. $\int_0^1\frac{dx}{(x-1)^2} = \lim_{t\to1^-}\int_0^t\frac{dx}{(x-1)^2} = \lim_{t\to1^-}\left[\frac{-1}{x-1}\right]_0^t$

$$= \lim_{t\to1^-}\left[\frac{-1}{t-1}-1\right] = \infty$$

Hence the given integral diverges.

14. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

Sol. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to1^-}[\arcsin x]_0^t$

$$= \lim_{t\to1^-}[\arcsin t - \arcsin 0] = \arcsin 1 - 0 = \frac{\pi}{2}$$

Thus the given integral converges.

15. $\displaystyle\int_0^e x^2 \ln x\, dx$

Sol. $\displaystyle\int_0^e x^2 \ln x\, dx = \lim_{t\to0^+}\int_t^e x^2\ln x\, dx = I$

Now $\displaystyle\int (\ln x)\, x^2\, dx = (\ln x)\frac{x^3}{3} - \int\frac{x^3}{3}\cdot\frac{1}{x}\,dx = \frac{x^3}{3}\ln x - \frac{1}{3}\cdot\frac{x^3}{3}$

$$I = \lim_{t\to0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_t^e = \lim_{t\to0^+}\left[\frac{e^3}{3}-\frac{e^3}{9}-\frac{t^3}{3}\ln t+\frac{t^3}{9}\right]$$

$$= \frac{2e^3}{9}\quad\left(\begin{array}{l}\text{since } \displaystyle\lim_{t\to0^+}\frac{\ln t}{\frac{1}{t^3}} = \lim_{t\to0^+}\left(\frac{\frac{1}{t}}{-3\times\frac{1}{t^4}}\right)\\ \qquad\qquad = \left(\displaystyle\lim_{t\to0^+}\frac{t^3}{-3}\right) = 0\end{array}\right)$$

Thus the given integral converges.

16. $\displaystyle\int_{-1}^8 \frac{dx}{x^{1/3}}$

Sol. $\displaystyle\int_{-1}^8\frac{dx}{x^{1/3}} = \int_{-1}^0\frac{dx}{x^{1/3}} + \int_0^8\frac{dx}{x^{1/3}} = \lim_{t\to0^-}\int_{-1}^t\frac{dx}{x^{1/3}} + \lim_{t\to0^+}\int_t^8\frac{dx}{x^{1/3}}$

$$= \lim_{t\to0^-}\left[\frac{3}{2}x^{2/3}\right]_{-1}^t + \lim_{t\to0^+}\left[\frac{3}{2}x^{2/3}\right]_t^8$$

$$= \lim_{t\to0^-}\left[\frac{3}{2}t^{2/3}-\frac{3}{2}(-1)^{2/3}\right] + \lim_{t\to0^+}\left[\frac{3}{2}(8)^{2/3}-\frac{3}{2}t^{2/3}\right]$$

$$= -\frac{3}{2}+6 = \frac{9}{2}$$

Hence the given integral converges.

17. $\displaystyle\int_{-2}^2\frac{dx}{x}$

Sol. $\displaystyle\int_{-2}^2\frac{dx}{x} = \int_{-2}^0\frac{dx}{x} + \int_0^2\frac{dx}{x} = \lim_{t\to0^-}\int_{-2}^t\frac{dx}{x} + \lim_{t\to0^+}\int_t^2\frac{dx}{x}$

$$= \lim_{t\to0^-}[\ln|x|]_{-2}^t + \lim_{t\to0^+}[(\ln|x|)]_t^2$$

But both the limits are not finite and so the given integral diverges.

18. $\displaystyle\int_0^3\frac{dx}{x^2+2x-3}$

Sol. $\displaystyle\int_0^3\frac{dx}{(x+3)(x-1)} = \int_0^1\frac{dx}{(x+3)(x-1)} + \int_1^3\frac{dx}{(x+3)(x-1)}$

Now $\displaystyle\int_0^1\frac{dx}{(x+3)(x-1)} = \lim_{t\to1^-}\int_0^t\frac{1}{4}\left(\frac{1}{x-1}-\frac{1}{x+3}\right)dx$

$$= \lim_{t\to1^-}\left(\left[\frac{1}{4}\ln|x-1|\right]_0^t - \left[\frac{1}{4}\ln|x+3|\right]_0^t\right)$$

$$= \lim_{t\to1^-}\left(\frac{1}{4}\ln|1-t|\right) - \frac{1}{4}\ln\frac{4}{3} \text{ which is not finite.}$$

Hence $\displaystyle\int_0^3\frac{dx}{x^2+2x-3}$ diverges.

19. $\displaystyle\int_0^{\pi/2}\frac{\cos x}{\sqrt{1-\sin x}}\,dx$

Sol. $\int_0^{\pi/2} \frac{\cos x}{\sqrt{1-\sin x}}\, dx = \lim_{t \to (\frac{\pi}{2})^-} \int_0^t (1-\sin x)^{-1/2} \cos x\, dx$

$= \lim_{t \to \frac{\pi}{2}^-} [-2\sqrt{1-\sin x}]_0^t = \lim_{t \to \frac{\pi}{2}^-} [-2\sqrt{1-\sin t} + 2] = 2$

Thus the given integral converges.

20. $\int_0^2 \frac{x}{x^2-5x+6}\, dx$

Sol. $\int_0^2 \frac{x}{x^2-5x+6}\, dx = \int_0^2 \left(\frac{3}{x-3} - \frac{2}{x-2}\right) dx = 3\int_0^2 \frac{dx}{x-3} - 2\int_0^2 \frac{dx}{x-2}$

Now $\int_0^2 \frac{dx}{x-2} = \lim_{t \to 2^-} \int_0^t \frac{dx}{x-2} = \lim_{t \to 2^-} [\ln (x-2)]_0^t$

$= \lim_{t \to 2^-} [\ln (t-2) - \ln (-2)] = \lim_{t \to 2^-} \ln \left(\frac{2-t}{2}\right)$

$= \lim_{t \to 2^-} \ln \left(1 - \frac{t}{2}\right)$ which is not finite.

Hence the given integral diverges.

21. $\int_0^\infty \frac{\ln (1+x^2)}{1+x^2}\, dx$

Sol. $I = \int_0^\infty \frac{\ln (1+x^2)}{1+x^2}\, dx = \lim_{t \to \infty} \int_0^t \frac{\ln (1+x^2)}{1+x^2}\, dx$

To evaluate $\int_0^t \frac{\ln (1+x^2)}{1+x^2}\, dx$, we put $x = \tan \theta$ so that

$dx = \sec^2\theta\, d\theta$ When $x = 0$, $\theta = 0$ and $\theta = \arctan t$ when $x = t$

Now $\int_0^t \frac{\ln (1+x^2)}{1+x^2}\, dx = \int_0^{\arctan t} \frac{\ln (\sec^2\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$ $\left(\begin{array}{l}\text{since } 1+x^2 \\ = 1+\tan^2\theta = \sec^2\theta\end{array}\right)$

and $I = \lim_{t \to \infty} \int_0^t \frac{\ln (1+x^2)}{1+x^2}\, dx = (-2) \lim_{t \to \infty} \int_0^{\arctan t} \ln (\cos \theta)\, d\theta$

$= -2\int_0^{\pi/2} \ln (\cos \theta)\, d\theta = -2\left(-\frac{\pi}{2} \ln 2\right)$ (by Example 9 Page 179)

$= \pi \ln 2.$

Thus the given integral converges.

22. $\int_0^\infty \frac{x\, dx}{(1+x)(1+x^2)} dx$

Sol. $I = \int_0^\infty \frac{x\, dx}{(1+x)(1+x^2)} = \lim_{t \to \infty} \int_0^t \frac{x\, dx}{(1+x)(1+x^2)}$

To evaluate $\int_0^t \frac{x\, dx}{(1+x)(1+x^2)}$, we put $x = \tan \theta$ so that $dx = \sec^2\theta\, d\theta$

When $x = 0$, $\theta = 0$ and $\theta = \arctan t$ when $x = t$

Now $\int_0^t \frac{x\, dx}{(1+x)(1+x^2)} = \int_0^{\arctan t} \frac{\tan \theta}{(1+\tan \theta)\sec^2\theta} \cdot \sec^2\theta\, d\theta$

(since $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$)

$= \int_0^{\arctan t} \frac{\sin \theta}{\sin \theta + \cos \theta}$

and $I = \lim_{t \to \infty} \int_0^t \frac{x\, dx}{(1+x)(1+x^2)} = \lim_{t \to \infty} \int_0^{\arctan t} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta$

$= \int_0^{\pi/2} \frac{\sin \theta}{\sin \theta + \cos \theta} d\theta = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\sin\left(\frac{\pi}{2} - \theta\right) + \cos\left(\frac{\pi}{2} - \theta\right)} d\theta$

$$= \int_0^{\pi/2} \frac{\cos\theta}{\cos\theta + \sin\theta} d\theta$$

Hence $2I = \int_0^{\pi/2} \frac{\sin\theta + \cos\theta}{\sin\theta + \cos\theta} d\theta = \int_0^{\pi/2} 1 \cdot d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

i.e., $I = \frac{\pi}{4}$ and the given integral converges.

23. $\int_{-\infty}^{0} \frac{e^x}{1+e^x} dx$

Sol. $I = \int_{-\infty}^{0} \frac{e^x}{1+e^x} dx = \lim_{t \to -\infty} \int_t^0 \frac{e^x}{1+e^x} dx$

To evaluate $\int_t^0 \frac{e^x}{1+e^x} dx$, we put $e^x = z$ so that $e^x\, dx = dz$.

When $x = 0$, $z = 1$ and $z = e^t$ when $x = t$

Now $\int_t^0 \frac{e^x}{1+e^x} = \int_{e^t}^1 \frac{dz}{1+z} = [\ln(1+z)]_{e^t}^1 = \ln 2 - \ln(1+e^t)$

and $\lim_{t \to -\infty} \int_t^0 \frac{e^x}{1+e^x} dx = \lim_{t \to -\infty} [\ln 2 - \ln(1+e^t)]$

$= \ln 2 - 0 = \ln 2$, because $\lim_{t \to -\infty} (1+e^t) = 1$

Thus the given integral converges.

24. $\int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$

Sol. $I = \int_0^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\left[\int_0^{\infty} \frac{dx}{x^2+b^2} - \int_0^{\infty} \frac{dx}{x^2+a^2}\right]$

Let $I_1 = \int_0^{\infty} \frac{dx}{x^2+b^2} = \lim_{t \to \infty} \int_0^t \frac{dx}{x^2+b^2}$

We put $x = b\tan\theta$ so that $dx = b\sec^2\theta\, d\theta$.

When $x = 0$, $\theta = 0$ and $\theta = \arctan\frac{t}{b}$ when $x = t$.

Now $\int_0^t \frac{dx}{x^2+b^2} = \int_0^{\arctan\frac{t}{b}} \frac{b\sec^2\theta\, d\theta}{b^2\sec^2\theta} \left(\begin{matrix}\text{because } x^2+b^2 \\ = b^2\tan^2\theta + b^2 = b^2\sec^2\theta\end{matrix}\right)$

$$= \frac{1}{b}\int_0^{\arctan\frac{t}{b}} 1 \cdot d\theta = \frac{1}{b}[\theta]_0^{\arctan\frac{t}{b}} = \frac{1}{b}\left(\arctan\frac{t}{b} - 0\right)$$

and $\lim_{t \to \infty} \int_0^t \frac{dx}{x^2+b^2} = \frac{1}{b}\lim_{t \to \infty}\left(\arctan\frac{t}{b}\right) = \frac{1}{b}\cdot\frac{\pi}{2} = \frac{\pi}{2b}$

Similarly, $\lim_{t \to \infty} \int_0^t \frac{dx}{x^2+b^2} = \frac{\pi}{2a}$

Thus $I = \frac{1}{a^2-b^2}\left[\frac{\pi}{2b} - \frac{\pi}{2a}\right] = \frac{1}{a^2-b^2}\cdot\frac{\pi}{2}\left(\frac{a-b}{ab}\right) = \frac{\pi}{2ab(a+b)}$

and the given integral converges.

25. $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx$

Sol. $\int_{\pi/4}^{\pi/2} \frac{\sin x}{\sqrt{\cos x}} dx = \lim_{t \to \frac{\pi}{2}}\left[-\int_{\pi/4}^{t} (\cos x)^{-1/2}(-\sin x)\, dx\right]$

$= \lim_{t \to \frac{\pi}{2}} \int_t^{\pi/4} (\cos x)^{-1/2}(-\sin x)\, dx = \lim_{t \to \frac{\pi}{2}} [2(\cos x)^{1/2}]_t^{\pi/4}$

$= \lim_{t \to \frac{\pi}{2}}\left[2\left(\frac{1}{\sqrt{2}}\right)^{1/2} - (\cos t)^{1/2}\right]$

$$= 2 \cdot \frac{1}{2^{1/4}} - 0, \text{ because } \sqrt{\cos t} \to 0 \text{ when } t \to \frac{\pi}{2}$$

$= 2^{3/4}$. Thus the given integral converges.

26. $\int\limits_{-\infty}^{\infty} xe^{-x^2}\,dx$

Sol. $I = \int\limits_{-\infty}^{\infty} xe^{-x^2}\,dx = \int\limits_{-\infty}^{0} xe^{-x^2}\,dx + \int\limits_{0}^{\infty} xe^{-x^2}\,dx$

Now $\int xe^{-x^2}\,dx = \frac{1}{2}\int e^{-z}\,dz$, by putting $x^2 = z$

$$= -\frac{1}{2}e^{-z} = -\frac{1}{2}e^{-x^2}$$

Therefore, $I = \lim\limits_{t\to-\infty} \frac{1}{2}[-e^{-x^2}]_t^0 + \lim\limits_{t\to\infty}\frac{1}{2}[-e^{-x^2}]_0^t$

$$= \lim_{t\to-\infty}\frac{1}{2}[-1 + e^{-t^2}] + \lim_{t\to\infty}\frac{1}{2}[-e^{-t^2} + 1]$$

$$= \frac{1}{2}[-1 + 0 + 0 + 1] = 0$$

Thus the given integral converges.

27. $\int\limits_{-\infty}^{\infty} \frac{dx}{x^2 + 6x + 12}$

Sol. $I = \int\limits_{-\infty}^{\infty}\frac{dx}{x^2+6x+12} = \int\limits_{-\infty}^{\infty}\frac{dx}{(x+3)^2+(\sqrt{3})^2} + \int\limits_{0}^{\infty}\frac{dx}{(x+3)^2+(\sqrt{3})^2}$

Now $\int \frac{dx}{(x+3)^2 + (\sqrt{3})^2} = \frac{1}{\sqrt{3}}\arctan\left(\frac{x+3}{\sqrt{3}}\right)$

Therefore, $I = \frac{1}{\sqrt{3}}\lim\limits_{t\to-\infty}\left[\arctan\frac{x+3}{\sqrt{3}}\right]_t^0 + \lim\limits_{t\to\infty}\left[\arctan\frac{x+3}{\sqrt{3}}\right]_0^t$

$$= \frac{1}{\sqrt{3}}\lim_{t\to-\infty}\left[\arctan\sqrt{3} - \arctan\frac{t+3}{\sqrt{3}}\right]$$

$$+ \frac{1}{\sqrt{3}}\lim_{t\to\infty}\left[\arctan\frac{t+3}{\sqrt{3}} - \arctan\sqrt{3}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} - \left(-\frac{\pi}{2}\right)\right] + \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{3}\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{3} + \frac{\pi}{2} + \frac{\pi}{2} - \frac{\pi}{3}\right] = \frac{1}{\sqrt{3}}\cdot\pi = \frac{\pi}{\sqrt{3}}$$

The given integral converges.

28. $\int\limits_{-1}^{1}\frac{dx}{x^2}$

Sol. $I = \int\limits_{-1}^{0}\frac{dx}{x^2} + \int\limits_{0}^{1}\frac{dx}{x^2} = \lim\limits_{t\to0^-}\int\limits_{-1}^{t}\frac{dx}{x^2} + \lim\limits_{t\to0^+}\int\limits_{t}^{1}\frac{dx}{x^2}$

As $\int\frac{dx}{x^2} = \int x^{-2}\,dx = \frac{x^{-1}}{-1} = -\frac{1}{x}$, so

$$I = \lim_{t\to0^-}\left[-\frac{1}{x}\right]_{-1}^{t} + \lim_{t\to0^+}\left[-\frac{1}{x}\right]_t^1$$

$$= \lim_{t\to0^-}\left[-\frac{1}{t} - 1\right] + \lim_{t\to0^+}\left[-1 + \frac{1}{t}\right] = \infty$$

Thus the given integral diverges.

29. $\int\limits_{2}^{\infty}\frac{dx}{x(\ln x)^3}$

Sol. $I = \int\limits_{2}^{\infty}\frac{dx}{x(\ln x)^3} = \lim\limits_{t\to\infty}\int\limits_{2}^{t}\frac{dx}{x(\ln x)^3} = \lim\limits_{t\to\infty}\left[-\frac{1}{2(\ln x)^2}\right]_2^t$

$$= \lim_{t\to\infty}\left[-\frac{2}{2(\ln t)^3} + \frac{1}{2(\ln 2)^2}\right] = \frac{1}{2(\ln 2)^2}$$

Thus the given integral converges.

30. $\int\limits_{0}^{\infty} xe^{-x}\,dx$

Sol. $\int\limits_{0}^{\infty} xe^{-x}\,dx = \lim\limits_{t\to\infty}\int\limits_{0}^{t} xe^{-x}\,dx$

$$= \lim_{t\to\infty}[-xe^{-x} - e^{-x}]_0^t \quad \text{(integrating by parts)}$$

$$= \lim_{t\to\infty}[te^{-t} - e^{-t} + 1] = \lim_{t\to\infty}\left[\frac{-(t+1)}{e^t} + 1\right] = 0 + 1 = 1$$

Thus the given integral converges.

31. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)}$

Sol. $\displaystyle\int \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)} = \int \frac{2z\,dz}{z\,(z+1)}$, (on putting $\sqrt{x} = z$)

$$= 2\int \frac{dz}{z+1} = 2\ln|z+1| = 2\ln(\sqrt{x}+1)$$

$$\int_1^\infty \frac{dx}{\sqrt{x}\,(\sqrt{x}+1)} = \lim_{t\to\infty}\left[2\ln(\sqrt{x}+1)\right]_1^t$$

$$= \lim_{t\to\infty}\left[2\ln\left(\frac{\sqrt{t}+1}{2}\right)\right] = \infty$$

Thus the given integral diverges.

32. $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Sol. $\displaystyle I = \int_0^1 e^{\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 e^{\sqrt{x}}\cdot\frac{1}{\sqrt{x}}\,dx$

As $\displaystyle\int_0^1 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2e^{\sqrt{x}}$, so

$$I = \lim_{t\to 0^+}\left[2e^{\sqrt{x}}\right]_t^1 = \lim_{t\to 0^+}\left(2e - 2e^{\sqrt{t}}\right) = 2e - 2 = 2(e-1)$$

Thus the given integral converges.

33. $\displaystyle\int_0^\infty \frac{x^3}{x^3+1}\,dx$

Sol. $\displaystyle I = \int_0^\infty \frac{x^3}{x^3+1}\,dx = \lim_{t\to\infty}\int_0^t \frac{x^3}{x^3+1}\,dx$

Now $\displaystyle\int \frac{x^3}{x^3+1}\,dx = \int\left(1 - \frac{1}{x^3+1}\right)dx$

$$= \int\left(1 - \frac{1}{(x+1)(x^2-x+1)}\right)dx$$

$$= \int\left(1 - \frac{1}{3(x+1)} + \frac{x-2}{3(x^2-x+1)}\right)dx$$

$$= x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\int\frac{2x-1-3}{x^2-x+1}\,dx$$

As $\displaystyle\int\frac{2x-1-3}{x^2-x+1}\,dx = \int\frac{2x-1}{x^2-x+1} - \int\frac{3\,dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \ln(x^2-x+1) - \frac{3\times 2}{\sqrt{3}}\arctan\frac{(2x-1)}{\sqrt{3}}, \text{ so}$$

$$\int\frac{x^3}{x^3+1}\,dx = x - \frac{1}{3}\ln|x+1| + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{\sqrt{3}}\arctan\frac{(2x-1)}{\sqrt{3}} \text{ and}$$

$$I = \lim_{t\to\infty}\left(\left[x - \frac{1}{3}\ln(x+1) + \frac{1}{6}\ln(x^2-x+1) - \frac{1}{3}\operatorname{arc}\left(\frac{2x-1}{3}\right)\right]_0^t\right)$$

which does not exist. Hence the given integral diverges.

34. Let $\displaystyle I_n = \int_0^\infty x^n e^{-x}\,dx$, where n is a positive integer. Prove that

$I_n = nI_{n-1}$. Hence show that $I_n = n!$.

Sol. $\displaystyle I_n = \int_0^\infty x^n e^{-x}\,dx = \lim_{t\to\infty}\int_0^t x^n e^{-x}\,dx$

Integrating $\int x^n e^{-x}\,dx$ by parts, we have

$$\int x^n e^{-x}\,dx = x^n(-e^{-x}) - \int(-e^{-x})\cdot nx^{n-1}\,dx = -\frac{x^n}{e^x} + n\int x^{n-1}\cdot e^{-x}\,dx$$

Now $\displaystyle I_n = \lim_{t\to\infty}\left[-\frac{x^n}{e^x} + n\int_0^t x^{n-1}e^{-x}\,dx\right]$

$$= \lim_{t\to\infty}\left[-\frac{x^n}{e^x}\right]_0^t + n\lim_{t\to\infty}\int_0^t x^{n-1}\cdot e^{-x}\,dx$$

$$= \lim_{t\to\infty}\left[-\frac{t^n}{e^t}\right] + n\int_0^\infty x^{n-1}\cdot e^{-x}\,dx$$

Applying L'Hospital's rule successively, we have

$$\lim_{t\to\infty}\left[-\frac{t^n}{e^t}\right] = -\lim_{t\to\infty}\left(\frac{n(n-1)(n-2)\ldots.1}{e^t}\right) = 0$$

Thus $I_n = 0 + n\,I_{n-1} = n\,I_{n-1}$

Continuing in this way, $I_{n-1} = (n-1)\,I_{n-2}$, $I_{n-1} = (n-2)\,I_{n-3}\cdots$

Hence, $I_n = n(n-1)(n-2)\ldots I_1$, where $I_1 = \int_0^\infty x\,e^{-x}\,dx = 1$ by Problem 30.

$$= n(n-1)(n-2)\ldots 1 = n!$$

35. Evaluate $\int_1^5 [x]\,dx$, where $[x]$ denotes the greatest integer less than or equal to x.

Sol. The function $[x]$ is defined as under

$$\begin{aligned}[x] &= 1 \text{ if } 1 \le x < 2\\ &= 2 \text{ if } 2 \le x < 3\\ &= 3 \text{ if } 3 \le x < 4\\ &= 4 \text{ if } 4 \le x < 5\\ &= 5 \text{ if } 5 \le x = 5\end{aligned}$$

Therefore, $I = \lim\limits_{\varepsilon\to 0}\int_1^{2-\varepsilon} 1\cdot dx + \lim\limits_{\varepsilon\to 0}\int_2^{3-\varepsilon} 2dx + \lim\limits_{\varepsilon\to 0}\int_3^{4-\varepsilon} 3dx$

$$+ \lim_{\varepsilon\to 0}\int_4^{5-\varepsilon} 4dx + \lim_{\varepsilon\to 0}\int_{5-\varepsilon}^{5} 5dx$$

$$= \lim_{\varepsilon\to 0}[x]_1^{2-\varepsilon} + \lim_{\varepsilon\to 0}[2x]_2^{3-\varepsilon} + \lim_{\varepsilon\to 0}[3x]_3^{4-\varepsilon}$$

$$+ \lim_{\varepsilon\to 0}[4x]_4^{5-\varepsilon} + \lim_{\varepsilon\to 0}[5x]_{5-\varepsilon}^{5}$$

$$= \lim_{\varepsilon\to 0}[(2-\varepsilon-1) + (2(3-\varepsilon)-2.2) + (3(4-\varepsilon)-3.3)$$

$$+ (4(5-\varepsilon)-4.4) + (515) - 5(5-\varepsilon)]$$

$$= 1 + 2 + 3 + 4 + 0 = 10$$

Exercise Set 5.4 (Page 200)

Evaluate (Problems 1 – 21):

1. $\int \frac{\sec^4 x}{\tan^5 x}\,dx$

Sol. Put $\tan x = z$ so that $\sec^2 x\,dx = dz$

$$I = \int \frac{\sec^2 x}{\tan^5 x}\cdot \sec^2 x\,dx = \int \frac{1+z^2}{z^5}\,dz = \int\left(\frac{1}{z^5}+\frac{1}{z^3}\right)dz$$

$$= -\frac{1}{4z^4} - \frac{1}{2z^2} = -\frac{1}{4\tan^4 x} - \frac{1}{2\tan^2 x}$$

2. $\int \sin^2 x\cos^4 x\,dx$

Sol. We connect $\int \sin^p x\cos^q x\,dx$ with $\int \sin^p x\cos^{q-2} x\,dx$.

Here, $P = \sin^{p+1} x\cos^{q-1} x$

$$\frac{dP}{dx} = (p+1)\sin^p x\cos^q x - (q-1)\cos^{q-2} x\sin^{p+2} x$$

$$= (p+1)\sin^p x\cos^q x - (q-1)\cos^{q-2} x\sin^p (1-\cos^2 x)$$

$$= (p+1)\sin^p x\cos^q x + (q-1)\sin^p x\cos^q x$$

$$-(q-1)\sin^p x\cos^{q-2} x$$

$$= (p+q)\sin^p x\cos^q x - (q-1)\sin^p x\cos^{q-2} x$$

$$P = (p+q)\int \sin^p x\cos^q x\,dx - (q-1)\int \sin^p x\cos^{q-2} x\,dx$$

Hence

$$\int \sin^p x\cos^q x\,dx = \frac{\sin^{p+1} x\cos^{q-1} x}{p+q} + \frac{q-1}{p+q}\int \sin^p x\cos^{q-2} x\,dx$$

Put $p = 2, q = 4$ in this formula to get

$$I = \int \sin^2 x\cos^4 x\,dx = \frac{\sin^3 x\cos^3 x}{6} + \frac{1}{2}\int \sin^2 x\cos^2 x\,dx$$

$$\int \sin^2 x\cos^2 x\,dx = \frac{\sin^3 x\cos x}{4} + \frac{1}{4}\int \sin^2 x\cos^0 x\,dx$$

$$= \frac{1}{4}\sin^3 x\cos x + \frac{1}{4}\int \frac{1-\cos 2x}{2}\,dx$$

$$= \frac{1}{4}\sin^3 x\cos x + \frac{x}{8} - \frac{\sin 2x}{16}$$

Therefore,

$$I = \frac{1}{6}\sin^3 x\cos^3 x + \frac{1}{8}\sin^3 x\cos x + \frac{x}{16} - \frac{\sin x\cos x}{16}$$

3. $\int \sin^6 x\cos^2 x\,dx$

Sol. We have the reduction formula

$$\int \sin^p x\cos^q x\,dx = -\frac{\sin^{p-1}x\cos^{q+1}x}{p+q} + \frac{p-1}{p+q}\int \sin^{p-2}\cos^q x\,dx$$

Put $p = 6, q = 2$. Then

$$I = \int \sin^6 x\cos^2 x\,dx = -\frac{1}{8}\sin^5 x\cos^3 x + \frac{5}{8}\int \sin^4 x\cos^2 x\,dx,$$

$$\int \sin^4 x\cos^2 x\,dx = -\frac{1}{6}\sin^3 x\cos^3 x + \frac{1}{2}\int \sin^2 x\cos^2 x\,dx,$$

$$\int \sin^2 x\cos^2 x\,dx = -\frac{1}{4}\sin x\cos^3 x + \frac{1}{4}\int \cos^2 x\,dx$$

$$= -\frac{1}{4}\sin x\cos^3 x + \frac{1}{4}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)$$

$$I = -\frac{1}{8}\sin^5 x\cos^3 x + \frac{5}{8}\left[-\frac{1}{6}\sin^3 x\cos^3 x + \frac{1}{2}\left(-\frac{1}{4}\sin x\cos^3 x\right) + \frac{1}{8}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)\right]$$

$$= -\frac{1}{8}\sin^5 x\cos^3 x - \frac{5}{48}\sin^3 x\cos^3 x - \frac{5}{64}\sin x\cos^3 x + \frac{5}{64}\left(\frac{x}{2} + \frac{\sin 2x}{4}\right)$$

4. $\int \sin^{1/2} x\cos^3 x\,dx$

Sol. Put $\sin x = z^2$ so that $\cos x\,dx = 2z\,dz$. Then

$$I = \int \sin^{1/2} x\cos^3 x\,dx = \int \sin^{1/2} x\cos^2 x\cdot\cos x\,dx$$

$$= \int z(1 - z^4)\cdot 2z\,dz$$

$$= \int 2(z^2 - z^6)\,dz = \frac{2}{3}z^3 - \frac{2}{7}z^7 = \frac{2}{3}\sin^{3/2} x - \frac{2}{7}\sin^{7/2} x$$

5. $\int \sec^2 x\csc^3 x\,dx = \int \csc^3 x\cdot\sec^2 x\,dx$

Sol. Integrate by parts (taking $\csc^3 x$ as first function). Then

$$I = \int \csc^3 x\cdot\sec^2 x\,dx = \csc^3 x\tan x - \int(-3\csc^3 x\cot x)\cdot\tan x\,dx$$

Now $\int \csc^3 x\,dx = -\frac{1}{2}\csc x\cot x + \frac{1}{2}\int \csc x\,dx,$

(by the reduction formula)

$$= -\frac{1}{2}\csc x\cot c + \frac{1}{2}\ln|\csc x - \cot x|$$

$$I = \csc^3 x\tan x + 3\left[-\frac{1}{2}\csc x\cot x + \frac{1}{2}\ln|\csc x - \cot x|\right]$$

$$= \csc^3 x\tan x - \frac{3}{2}\csc x\cot x + \frac{3}{2}\ln[\csc x - \cot x]$$

6. $\int \tan^3 x\sec^5 x\,dx$

Sol. Put $\sec x = z$, so that $\sec x\tan x\,dx = dz$. Then

$$I = \int \tan^3 x\sec^5 x\,dx = \int \tan^2 x\cdot\sec^4 x(\sec x\tan x)\,dx$$

$$= \int (z^2 - 1)z^4\,dz = \frac{z^7}{7} - \frac{z^5}{5} = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x$$

7. $\int \cot^5 x\csc^4 x\,dx$

Sol. Put $\cot x = z$, so that $-\csc^2 x\,dx = dz$. Then

$$I = \int \cot^5 x\csc^4 x\,dx = -\int \cot^5 x\csc^2 x(-\csc^2 x)\,dx$$

$$= -\int z^5(1 + z^2)\,dz = -\frac{z^6}{6} - \frac{z^8}{8} = -\frac{1}{6}\cot^6 x - \frac{1}{8}\cot^8 x$$

8. $\int \frac{\sin^2 x}{\cos^5 x}\,dx$

Sol. $I = \int \frac{\sin^2 x}{\cos^5 x}\,dx = \int \tan^2 x\sec^3 x\,dx = \int(-1 + \sec^2 x)\sec^3 x\,dx$

$$= \int \sec^5 x\,dx - \int \sec^3 x\,dx$$

$$\int \sec^5 x\,dx = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx \quad \text{(by the reduction formula)}$$

$$I = \frac{\sec^3 x\tan x}{4} + \frac{3}{4}\int \sec^3 x\,dx - \int \sec^3 x\,dx$$

$$= \frac{\sec^3 x\tan x}{4} - \frac{1}{4}\int \sec^3 x\,dx$$

Now $\int \sec^3 x\,dx = \frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\,dx$

(by the reduction formula)

$$= \frac{\sec x\tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|$$

$$\text{Thus } I = \frac{\sec^3 x \tan x}{4} - \frac{1}{4}\left[\frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x|\right]$$

$$= \frac{\sec^3 x \tan x}{4} - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x|$$

9. $\int_{\pi/4}^{\pi/2} \cot^4 x\, dx$

Sol. By the reduction formula $\int \cot^4 x\, dx = -\frac{\cot^3 x}{3} - \int \cot^2 x\, dx$

$$\int \cot^4 x\, dx = -\frac{\cot^3 x}{3} - \int (\operatorname{cosec}^2 x - 1)\, dx = -\frac{\cot^3 x}{3} + \cot x + x$$

$$\int_{\pi/4}^{\pi/2} \cot^4 x\, dx = \left[-\frac{\cot^3 x}{3} + \cot x + x\right]_{\pi/4}^{\pi/2} = \frac{\pi}{2} - \left[-\frac{1}{3} + 1 + \frac{\pi}{4}\right] = \frac{\pi}{4} - \frac{2}{3}$$

10. $\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x\, dx$

Sol. Put $\csc x = z$, so that $-\csc x \cot x\, dx = dz$

$$I = \int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x\, dx = -\int_{\pi/4}^{\pi/2} (\csc^2 x - 1)\csc^2 x\,(-\csc x \cot x)\, dx$$

$$= -\int_{\sqrt{2}}^{1} (z^2 - 1) z^2\, dz = \int_{1}^{\sqrt{2}} (z^4 - z^2)\, dz$$

$$= \left[\frac{z^5}{5} - \frac{z^3}{3}\right]_1^{\sqrt{2}} = \left(\frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3}\right) - \left(\frac{1}{5} - \frac{1}{3}\right)$$

$$= \sqrt{2}\left(\frac{4}{5} - \frac{2}{3}\right) - \left(-\frac{2}{15}\right) = \frac{2}{15}(\sqrt{2} + 1)$$

11. $\int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right) dx$

Sol. Put $\frac{x}{2} = z$, so that $dx = 2\,dz$. Then

$$I = \int_0^{\pi/2} \tan^5\left(\frac{x}{2}\right) dx = 2\int_0^{\pi/4} \tan^5 z\, dz$$

Now $\int \tan^5 z\, dz = \frac{\tan^4 z}{4} - \int \tan^3 z dz = \frac{\tan^4 z}{4} - \left[\frac{\tan^2 z}{2} - \int \tan z dz\right]$

$$= \frac{\tan^4 z}{4} - \frac{\tan^2 z}{2} + \ln \sec z$$

$$\text{Thus } I = 2\left[\frac{\tan^4 z}{4} - \left(\frac{\tan^2 z}{2}\right) + \ln \sec z\right]_0^{\pi/4}$$

$$= 2\left[\left(\frac{1}{4} - \frac{1}{2} + \ln\sqrt{2}\right) - 0\right] \text{ because } \ln \sec 0 = \ln 1 = 0$$

$$= 2\left(-\frac{1}{4}\right) + 2\log\sqrt{2} = -\frac{1}{2} + \ln 2$$

12. $\int_0^a (a^2 - x^2)^{5/2}\, dx$

Sol. Put $x = a\sin\theta$, so that $dx = a\cos\theta\, d\theta$. Then

$$I = \int_0^a (a^2 - x^2)^{5/2}\, dx = \int_0^{\pi/2} (a^2 - a^2\sin^2\theta)^{5/2} \cdot a\cos\theta\, d\theta$$

$$= a^6 \int_0^{\pi/2} \cos^6\theta\, d\theta = a^6 \cdot \frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2}, \quad \text{(Wallis' formula)}$$

$$= \frac{5\pi a^6}{32}$$

13. $\int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx$

Sol. $\int_0^{\pi} \frac{\sin^4 x}{(1+\cos x)^2}\, dx = \int_0^{\pi} \frac{(\sin^2 x)^2}{(1+\cos x)^2}\, dx = \int_0^{\pi} \frac{(1-\cos x)^2(1+\cos x)^2}{(1+\cos x)^2}\, dx$

$$= \int_0^{\pi} (1-\cos x)^2\, dx = \int_0^{\pi} (1 - 2\cos x + \cos^2 x)\, dx$$

$$= \int_0^{\pi} \left(1 - 2\cos x + \frac{1+\cos 2x}{2}\right) dx$$

$$= \left[\frac{3}{2}x - 2\sin x + \frac{\sin 2x}{4}\right]_0^{\pi} = \frac{3}{2}\pi.$$

14. $\int_0^{\pi/4} \sin^4 2x\, dx$

Sol. $\int_0^{\pi/4} \sin^4 2x\,dx = \int_0^{\pi/2} \sin^4 t \cdot \frac{dt}{2}$, putting $2x = t$ so that, $2\,dx = dt$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^4 t\,dt = -\frac{1}{2}\cdot\frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{3\pi}{32}$$

15. $\int_0^{\pi/6} \sin^6 3x\,dx$

Sol. Put $3x = \theta$ so that $3\,dx = d\theta$ or $dx = \frac{1}{3}d\theta$

Now, when $x = 0,\ \theta = 0$ and when $x = \frac{\pi}{6},\ \theta = \frac{\pi}{2}$

Therefore, $\int_0^{\pi/6} \sin^6 3x\,dx = \int_0^{\pi/2} (\sin^6\theta).\frac{1}{3}d\theta = \frac{1}{3}\int_0^{\pi/2} \sin^6\theta\,d\theta$

$$= \frac{1}{3}\cdot\frac{5\cdot 3\cdot 1}{6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{5\pi}{96} \text{ (By Wallis' formula)}$$

16. $\int_0^{\pi/8} \sin^5 4x \cos^4 4x\,dx$

Sol. Put $4x = z$ so that $dx = \frac{dz}{4}$

When $x = 0$, $z = 0$ and when $x = \frac{\pi}{8}, z = \frac{\pi}{2}\cdot$ Then

$$I = \int_0^{\pi/8} \sin^5 4x \cos^4 4x\,dx = \frac{1}{4}\int_0^{\pi/2} \sin^5 z \cos^4 z\,dz$$

Here p is odd and q is even, so

Using $\int_0^{\pi/2} \sin^p x \cos^q x\,dx = \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots}$, we have

$$\frac{1}{4}\int_0^{\pi/2} \sin^5 z \cos^4 z\,dz = \frac{1}{4}\left[\frac{(5-1)(5-3)(4-1)(4-3)}{(5+4)(5+4-2)(5+4-4)(5+4-6)}\right]$$

$$= \frac{1}{4}\left(\frac{4}{9}\cdot\frac{2}{7}\cdot\frac{3}{5}\cdot\frac{1}{3}\right) = \frac{2}{315}$$

17. $\int_0^{\pi/4} \cos^2 2x\,dx$

Sol. Put $2x = \theta$ so that $dx = \frac{1}{2}d\theta$

When $x = 0,\ \theta = 0$ and when $x = \frac{\pi}{4}, \theta = \frac{\pi}{2}$

Therefore, $\int_0^{\pi/4} \cos^2 2x\,dx = \int_0^{\pi/2} (\cos^2\theta)\cdot\frac{1}{2}d\theta = \frac{1}{2}\int_0^{\pi/2} \cos^2\theta\,d\theta$

$$= \frac{1}{2}\cdot\left(\frac{2-1}{2}\cdot\frac{\pi}{2}\right) = \frac{\pi}{8} \quad \text{(Here } n = 2\text{)}$$

18. $\int_0^{\pi/6} \cos^3 3x\,dx$

Sol. Put $3x = \theta$ so that $dx = \frac{1}{3}d\theta$

When $x = 0,\ \theta = 0$ and when $x = \frac{\pi}{6}, \theta = \frac{\pi}{2}$

Therefore, $\int_0^{\pi/6} \cos^3 3x\,dx = \int_0^{\pi/2} \cos^3\theta\frac{1}{3}d\theta$

$$= \frac{1}{3}\int_0^{\pi/2} \cos^3\theta\,d\theta = \frac{1}{3}\cdot\left(\frac{3-1}{3}\right), \text{ (by Wallis' formula)}$$

$$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}$$

19. $\int_0^{\pi/3} \sin^2 6x \cos^4 3x\,dx$

Sol. $I = \int_0^{\pi/3} (2\sin 3x\cos 3x)^2 \cos^4 3x\,dx$

$$= 4\int_0^{\pi/3} \sin^2 3x \cos^6 3x\,dx$$

Put $3x = z$, so that $dx = \frac{dz}{3}$. When $x = 0, z = 0$

and when $x = \frac{\pi}{3}, z = \pi$

$I = \frac{4}{3}\int_0^{\pi} \sin^2 z \cos^6 z\,dz = \frac{8}{3}\int_0^{\pi/2} \sin^2 z \cos^6 z\,dz$, by Theorem5.11 (*i*)

Using $\displaystyle\int_0^{\pi/2} \sin^p x \cos^q x\, dx = \frac{(p-1)(p-3)\ldots(q-1)(q-3)\ldots}{(p+q)(p+q-2)\ldots}\cdot\frac{\pi}{2}$,

we get $\displaystyle\frac{8}{3}\int_0^{\pi/2}\sin^2 z\cos^6 z\, dz = \frac{8}{3}\cdot\left(\frac{1\cdot5\cdot3\cdot1}{8\cdot6\cdot4\cdot2}\cdot\frac{\pi}{2}\right) = \frac{5}{96}\pi$

20. $\displaystyle\int_{\pi/3}^{\pi/2}\frac{\cos^2 x}{\sin x}\, dx$

Sol. Put $\cos x = z$, then $-\sin x\, dx = dz$

or $\displaystyle -\sin^2 x\cdot\frac{1}{\sin x}dx = dz \Rightarrow \frac{1}{\sin x}dx = -\frac{1}{1-z^2}dz$

When $x = \frac{\pi}{3}, z = \frac{1}{2}$ and when $x = \frac{\pi}{2}, z = 0$

$$I = \int_{\pi/3}^{\pi/2}\frac{\cos^2 x}{\sin x}dx = \int_{1/2}^{0} z^2\cdot\left(-\frac{1}{1-z^2}\right)dz = \int_{1/2}^{0}\left(1-\frac{1}{1-z^2}\right)dz$$

$$= \int_{1/2}^{0}\left[1-\frac{1}{2}\left(\frac{1}{1+z}+\frac{1}{1-z}\right)\right]dz = [z]_{1/2}^{0} - \frac{1}{2}\left[\ln\left(\frac{1+z}{1-z}\right)\right]_{1/2}^{0}$$

$$= \left(0-\frac{1}{2}\right)-\frac{1}{2}\left[\ln 1 - \ln\left(\frac{3/2}{1/2}\right)\right] = -\frac{1}{2}+\frac{1}{2}\ln 3 = -\frac{1}{2}+\ln\sqrt{3}$$

21. $\displaystyle\int_0^1\frac{x^6\, dx}{\sqrt{1-x^2}}$

Sol. Put $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$

When $x = 0, \theta = 0$ and when $x = 1, \theta = \frac{\pi}{2}$

Therefore, $\displaystyle\int_0^1\frac{x^6\, dx}{\sqrt{1-x^2}} = \int_0^{\pi/2}\frac{\sin^6\theta\cdot\cos\theta\, d\theta}{\cos\theta} = \int_0^{\pi/2}\sin^6\theta\, d\theta$

$\displaystyle = \frac{5\cdot3\cdot1}{6\cdot4\cdot2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$ (By Walls formula)

22. Show that

$$\int\sec^{2n+1}x\, dx = \frac{\sec^{2n-1}x\tan x}{2n} + \left(1-\frac{1}{2n}\right)\int\sec^{2n-1}x\, dx.$$

Sol. $\displaystyle\int\sec^{2n+1}x\, dx = \int\sec^{2n-1}x\cdot\sec^2 x\, dx$

$\displaystyle = \sec^{2n-1}x\cdot\tan x - \int\tan x\cdot(2n-1)\sec^{2n-2}x\cdot\sec x\tan x\, dx$

(Integrating by parts)

$\displaystyle = \sec^{2n-1}x\cdot\tan x - (2n-1)\int\sec^{2n-1}x\tan^2 x\, dx$

$\displaystyle = \sec^{2n-1}x\cdot\tan x - (2n-1)\int\sec^{2n-1}x(\sec^2 x - 1)\, dx$

$\displaystyle = \sec^{2n-1}x\,.\tan x - (2n-1)\int\sec^{2n+1}x\, dx + (2n-1)\int\sec^{2n-1}x\, dx$

Therefore, $\displaystyle 2n\int\sec^{2n+1}x\, dx = \sec^{2n-1}x\cdot\tan x - (2n-1)\int\sec^{2n-1}x\, dx$

or $\displaystyle\int\sec^{2n+1}x\, dx = \frac{\sec^{2n-1}x\tan x}{2n} + \left(1-\frac{1}{2n}\right)\int\sec^{2n-1}x\, dx$

23. Obtain a reduction formula for $\displaystyle\int\frac{dx}{(a^2+x^2)^n}$, where n is an integer.

Show that $\displaystyle\int_0^\infty\frac{dx}{(1+x^2)^5} = \frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{\pi}{2}$

Sol. $\displaystyle\int\frac{dx}{(a^2+x^2)^n} = \int(a^2+x^2)^{-n}\cdot 1dx$

$\displaystyle = (a^2+x^2)^{-n}x - \int x\cdot(-n)(a^2+x^2)^{-n-1}\,2x\, dx$ (Integrating by parts)

$\displaystyle = x(a^2+x^2)^{-n} + 2n\int x^2(a^2+x^2)^{-n-1}\, dx$

$\displaystyle = x(a^2+x^2)^{-n} + 2n\int(a^2+x^2-a^2)(a^2+x^2)^{-n-1}$

(Writing x^2 as $a^2+x^2-a^2$)

$\displaystyle = x(a^2+x^2)^{-n} + 2n\int(a^2+x^2)^{-n}\, dx - 2na^2\int(a^2+x^2)^{-n-1}\, dx$

or $\displaystyle 2na^2\int\frac{dx}{(a^2+x^2)^{n+1}} = x(a^2+x^2)^{-n} + (2n-1)\int\frac{dx}{(a^2+x^2)^n}$

Changing n into $n-1$, we get

$$2(n-1)a^2\int\frac{dx}{(a^2+x^2)^n} = \frac{x}{(a^2+x^2)^{n-1}} + (2n-3)\int\frac{dx}{(a^2+x^2)^{n-1}}$$

Dividing both sides by $2(n-1)a^2$, we get

$$\int \frac{dx}{(a^2+x^2)^n} = \frac{x}{2(n-1)a^2(a^2+x^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int \frac{dx}{(a^2+x^2)^{n-1}}$$

Integrating between the limits 0 to ∞

$$\int_0^\infty \frac{dx}{(a^2+x^2)^n} = \frac{2n-3}{2a^2(n-1)}\int_0^\infty \frac{dx}{(a^2+x^2)^{n-1}}$$

Taking $a = 1$, and $n = 5$, we have

$$\int_0^\infty \frac{dx}{(1+x^2)^5} = \frac{7}{2\cdot 4}\int_0^\infty \frac{dx}{(1+x^2)^4} \qquad (1)$$

$$\int_0^\infty \frac{dx}{(1+x^2)^4} = \frac{5}{2\cdot 3}\int_0^\infty \frac{dx}{(1+x^2)^3} \qquad (2)$$

$$\int_0^\infty \frac{dx}{(1+x^2)^3} = \frac{3}{2\cdot 2}\int_0^\infty \frac{dx}{(1+x^2)^2} \qquad (3)$$

$$\int_0^\infty \frac{dx}{(1+x^2)^2} = \frac{1}{2\cdot 1}\int_0^\infty \frac{dx}{(1+x^2)} \qquad (4)$$

$$\int_0^\infty \frac{dx}{1+x^2} = \lim_{n\to\infty} \left|\arctan x\right|_0^\infty = \frac{\pi}{2} \qquad (5)$$

Multiplying vertically (1) to (5), we get

$$\int_0^\infty \frac{dx}{(1+x^2)^5} = \frac{7\cdot 5\cdot 3\cdot 1}{8\cdot 6\cdot 4\cdot 2}\cdot\frac{\pi}{2} = \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{\pi}{2}.$$

24. If I_n denotes $\int_0^1 x^p(1-x^q)^n\,dx$, where p, q and n are positive, prove that $(qn+p+1)\,I_n = qnI_{n-1}$. Evaluate I_n when n is a positive integer.

Sol. $\int x^p(1-x^q)^n\,dx = (1-x^q)^n\frac{x^{p+1}}{p+1} - \int \frac{x^{p+1}}{p+1}\cdot n(1-x^q)^{n-1}(-q)x^{q-1}\,dx$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} + \frac{qn}{p+1}\int x^{p+q}(1-x^q)^{n-1}\,dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} - \frac{qn}{p+1}\int x^p(1-x^q-1)(1-x^q)^{n-1}\,dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} - \frac{qn}{p+1}\int x^p[(1-x^q)^n-(1-x^q)^{n-1}]\,dx$$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} - \frac{qn}{p+1}\int x^p(1-x^q)^n\,dx + \frac{qn}{p+1}\int x^p(1-x^q)^{n-1}\,dx$$

Therefore, $\left(1+\frac{qn}{p+1}\right)\int x^p(1-x^p)^n\,dx$

$$= \frac{x^{p+1}(1-x^q)^n}{p+1} + \frac{qn}{p+1}\int x^p(1-x^q)^{n-1}\,dx$$

or $\quad (qn+p+1)\int x^p(1-x^q)^n\,dx$

$$= x^{p+1}(1-x^q)^n + qn\int x^p(1-x^q)^{n-1}\,dx$$

Integrating between the limits 0 and 1, we have

$$(qn+p+1)\int_0^1 x^p(1-x^q)^n\,dx$$

$$= [x^{p+1}(1-x^q)^n]_0^1 + qn\int_0^1 x^p(1-x^q)^{n-1}\,dx$$

Thus $\quad (qn+p+1)I_n = qn\,I_{n-1}$ or $I_n = \frac{qn}{qn+p+1}I_{n-1}$

Now $I_{n-1} = \frac{q(n-1)}{q(n-1)+p+1}I_{n-2}$

$$I_{n-2} = \frac{q(n-2)}{q(n-2)+p+1}I_{n-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_3 = \frac{3q}{3q+p+1}I_2 \quad ; \quad I_2 = \frac{2q}{2q+p+1}I_1$$

$$I_1 = \frac{q}{q+p+1}I_0\,; \qquad I_0 = \int_0^1 x^p\,dx = \frac{1}{p+1}$$

Multiplying vertically, we have

$$I_n = \frac{q^n\cdot n!}{(qn+p+1)\{q(n-1)+p+1\}\,\{q(n-2)+p+1\}\ldots(2q+p+1)\,(q+p+1)\,(p+1)}$$

which is the required value.

25. Obtain a reduction formula for $\int \frac{x^n}{\sqrt{1-x^2}}\,dx$ and hence evaluate $\int \frac{x^3}{\sqrt{1-x^2}}\,dx$.

Sol. We connect $\int x^m (1-x^2)^{-1/2}\,dx$ with $\int x^{m-2} (1-x^2)^{-1/2}\,dx$

Here $P = x^{m-1}(1-x^2)^{1/2}$

$$\frac{dP}{dx} = (m-1)x^{m-2}(1-x^2)^{1/2} + x^{m-1}\cdot\frac{1}{2}(1-x^2)^{-1/2}(-2x)$$

$$= (m-1)x^{m-2}(1-x^2)^{1/2} - x^m (1-x^2)^{-1/2}$$

Integrating, we get

$P = (m-1)\int x^{m-2}(1-x^2)^{1/2}\,dx - \int x^m (1-x^2)^{-1/2}\,dx$ or

$$x^{m-1}(1-x^2)^{1/2} = (m-1)\int x^{m-2}(1-x^2)^{-1/2}(1-x^2)\,dx - \int x^m (1-x)^{-1/2}dx$$

$$= (m-1)\int x^{m-2}(1-x^2)^{-1/2}dx - (m-1)\int x^m (1-x^2)^{-1/2}dx - \int x^m (1-x^2)^{-1/2}dx$$

$$= (m-1)\int x^{m-2}(1-x^2)^{-1/2}\,dx - m\int x^m (1-x^2)^{-1/2}\,dx$$

or $\quad m\int x^m (1-x^2)^{-1/2}\,dx$

$$= -x^{m-1}(1-x^2)^{1/2} + (m-1)\int x^{m-2}(1-x^2)^{-1/2}\,dx$$

Therefore,

$$\int x^m (1-x^2)^{-1/2}\,dx = \frac{-x^{m-1}(1-x^2)^{1/2}}{m} + \frac{m-1}{m}\int \frac{x^{m-2}}{\sqrt{1-x^2}}\,dx$$

or $$\int \frac{x^m}{\sqrt{1-x^2}}\,dx = \frac{-x^{m-1}(1-x^2)^{1/2}}{m} + \frac{m-1}{m}\int \frac{x^{m-2}}{\sqrt{1-x^2}}\,dx$$

Putting $m = 3$ in the above equation, we have

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{2}{3}\int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{1}{3}\int (-2x)(1-x^2)^{-1/2}\,dx$$

$$= -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{1}{3}\,\frac{(1-x^2)^{1/2}}{1/2}$$

$$= -\frac{x^2(1-x^2)^{1/2}}{3} + \frac{2}{3}\sqrt{1-x^2}$$

26. Calculate the value of $\int_0^{2a} x^m \sqrt{2ax - x^2}\,dx$, n being a positive integer.

Hence or otherwise calculate the values of

(i) $\int_0^{2a} x\sqrt{2ax - x^2}\,dx$

(ii) $\int_0^{2a} x^4\sqrt{2ax - x^2}\,dx$

Sol. $\int_0^{2a} x^m \sqrt{2ax - x^2}\,dx = \int_0^{2a} x^{m+1/2}(2a-x)^{1/2}\,dx = I_m$ (say)

We connect $\int x^{m+1/2}(2a-x)^{1/2}\,dx$ with $\int x^{m-1/2}(2a-x)^{1/2}\,dx$

Here $P = x^{m+1/2}(2a-x)^{3/2}$

$$\frac{dP}{dx} = \left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{3/2} + x^{m+1/2}\cdot\frac{3}{2}(2a-x)^{1/2}(-1)$$

$$= \left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{1/2}(2a-x) - \frac{3}{2}x^{m+1/2}(2a-x)^{1/2}$$

$$= 2a\left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{1/2} - \left(m+\frac{1}{2}\right)x^{m+1/2}(2a-x)^{1/2} - \frac{3}{2}x^{m+1/2}(2a-x)^{1/2}$$

$$= 2a\left(m+\frac{1}{2}\right)x^{m-1/2}(2a-x)^{1/2} - (m+2)x^{m+1/2}(2a-x)^{1/2}\,dx$$

Integrating, we have

$$P = x^{m+1/2}(2a-x)^{3/2}$$

$$= a(2m+1)\int x^{m-1/2}(2a-x)^{1/2}\,dx - (m+2)\int x^{m+1/2}(2a-x)^{1/2}\,dx$$

or $\quad (m+2)\int x^{m+1/2}(2a-x)^{1/2}\,dx$

$$= -x^{m+1/2}(2a-x)^{3/2} + a(2m+1)\int x^{m-3/2}(2a-x)^{3/2}dx$$

or $(m+2)\int_0^{2a} x^{m+1/2}(2a-x)^{1/2}dx = 0 + a(2m+1)\int_0^{2a} x^{m-1/2}(2a-x)^{1/2}dx$

Thus $(m+2)I_m = a(2m+1)I_{m-1}$ or $I_m = \frac{(2m+1)a}{m+2}I_{m-1}$ (1)

Similarly, $I_{m-1} = \frac{(2m-1)a}{m+1}I_{m-2}$ (2)

$$I_{m-2} = \frac{(2m-3)a}{m} I_{m-3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_3 = \frac{7a}{5} I_2\ ,\ I_2 = \frac{5a}{4} I_1\ ,\ I_1 = \frac{3a}{3} I_0$$

$$I_0 = \int_0^{2a} x^{1/2}(2a-x)^{1/2}$$

$$= \int_0^{2a} \sqrt{2ax - x^2}\,dx.$$ Putting $x = 2a\sin^2\theta$, so that

$dx = 4a\sin\theta\cos\theta\,d\theta$, we have

$$I_0 = \int_0^{\pi/2} \sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}\cdot(4a\sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} \sqrt{4a^2\sin^2\theta(1-\sin^2\theta)}\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2a\sin\theta\cos\theta\cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 8a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta = 8a^2\,\frac{1\cdot 1}{4\cdot 2}\,\frac{\pi}{2} = \frac{a^2\pi}{2}$$

Multiplying vertically, we get

$$I_m = \frac{(2m+1)(2m-1)\ldots 5\cdot 3}{(m+2)(m+1)\ldots 4.\,3}\,a^{m+2}\cdot\frac{\pi}{2} \qquad \text{(A)}$$

(i) Putting $m = 1$ in (A), we have

$$I_1 = \int_0^{2a} x\sqrt{2ax-x^2}\,dx = \frac{3}{3}\cdot a^3\,\frac{\pi}{2} = \frac{a^3\pi}{2}$$

(ii) Putting $m = 4$ in (A), we have

$$I_4 = \int_0^{2a} x^4\sqrt{2ax-x^2}\,dx = \frac{9\cdot 7\cdot 5\cdot 3}{6\cdot 5\cdot 4\cdot 3}\cdot a^6\cdot\frac{\pi}{2} = \frac{21a^6\pi}{16}$$

27. If $I_n = \int x^n(a^2-x^2)^{1/2}\,dx$, prove that

$$I_n = \frac{x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}a^2 I_{n-2}$$

Hence evaluate $\int_0^a x^4\sqrt{a^2-x^2}\,dx$.

Sol. Connect $\int x^n(a^2-x^2)^{1/2}\,dx$ with $\int x^{n-2}(a^2-x^2)^{1/2}\,dx$

Here $P = x^{n-1}(a^2-x^2)^{3/2}$

$$\frac{dP}{dx} = (n-1)x^{n-2}(a^2-x^2)^{3/2} + x^{n-1}\cdot\frac{3}{2}(a^2-x^2)^{1/2}(-2x)$$

$$= (n-1)x^{n-2}(a^2-x^2)^{1/2}(a^2-x^2) - 3x^n(a^2-x^2)^{1/2}$$

$$= (n-1)a^2x^{n-2}(a^2-x^2)^{1/2} - (n-1)x^n(a^2-x^2)^{1/2} - 3x^n(a^2-x^2)^{1/2}$$

$$= (n-1)a^2x^{n-2}(a^2-x^2)^{1/2} - (n-1+3)x^n(a^2-x^2)^{1/2}$$

Integrating, we get

$$P = x^{n-1}(a^2-x^2)^{3/2} = (n-1)a^2\int x^{n-2}(a^2-x^2)^{1/2}dx - (n+2)\int x^n(a^2-x^2)^{1/2}dx$$

or $\quad (n+2)\int x^n(a^2-x^2)^{1/2}\,dx$

$$= -x^{n-1}(a^2-x^2)^{3/2} + (n-1)a^2\int x^{n-2}(a^2-x^2)^{1/2}\,dx$$

Therefore, $$I_n = \frac{-x^{n-1}(a^2-x^2)^{3/2}}{n+2} + \frac{n-1}{n+2}a^2 I_{n-2}$$

$$I_4 = \frac{-x^3(a^2-x^2)^{3/2}}{6} + \frac{3}{6}\cdot a^2 I_2 \text{ and } I_2 = -\frac{x(a^2-x^2)^{1/2}}{4} + \frac{1}{4}a^2 I_0$$

Hence $$\int_0^a x^4(a^2-x^2)^{1/2}\,dx = \frac{1}{2}a^2\int_0^a x^2(a^2-x^2)^{1/2}\,dx$$

$$= \frac{1}{2}a^2\cdot\left(\frac{1}{4}a^2\int_0^a\sqrt{a^2-x^2}\,dx\right)$$

$$= \frac{a^4}{8}\left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\arcsin\frac{x}{a}\right]_0^a$$

$$= \frac{a^4}{8}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^6}{32}$$

28. Prove that $\int x^m(\ln x)^n\,dx = \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx$.

Hence calculate

(i) $\int x^m(\ln x)^3\,dx$ $\qquad$ (ii) $\int_0^1 x^m(\ln x)^n\,dx$

Sol. $\int x^m(\ln x)^n\,dx = (\ln x)^n\cdot\frac{x^{m+1}}{m+1} - \int\frac{x^{m+1}}{m+1}\cdot n(\ln x)^{n-1}\cdot\frac{1}{x}\,dx$

(Integrating by parts)

$$= \frac{x^{m+1}(\ln x)^n}{m+1} - \frac{n}{m+1}\int x^m(\ln x)^{n-1}\,dx \qquad (1)$$

(*i*) Putting $n = 3$ in (1), we get

$$\int x^m (\ln x)^3\,dx = \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3}{m+1}\int x^m (\ln x)^2\,dx$$

$$= \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3}{m+1}\left[\frac{x^{m+1}(\ln x)^2}{m+1} - \frac{2}{m+1}\int x^m \ln x\,dx\right]$$

$$= \frac{x^{m+1}(\ln x)^3}{m+1} - \frac{3x^{m+1}(\ln x)^2}{(m+1)^2} + \frac{6}{(m+1)^2}\left[\ln x \cdot \frac{x^{m+1}}{m+1} - \int \frac{x^m}{m+1}\,dx\right]$$

$$= \frac{x^{m+1}}{m+1}\left[(\ln x)^3 - \frac{3(\ln x)^2}{m+1} + \frac{6\ln x}{(m+1)^2} - \frac{6}{(m+1)^3}\right]$$

(*ii*) Again, from (1), we have

$$\int_0^1 x^m (\ln x)^n\,dx = -\frac{n}{m+1}\int_0^1 x^m (\ln x)^{n-1}\,dx$$

i.e., $I_{m,n} = -\dfrac{n}{m+1} I_{m,n-1}$

$$I_{m,n-1} = -\frac{n-1}{m+1} I_{m,n-2}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$I_{m,2} = -\frac{2}{m+1} I_{m,1}\ ;$$

$$I_{m,1} = -\frac{1}{m+1} I_{m,0}\ ;$$

$$I_{m,0} = \int_0^1 x^m\,dx = \left[\frac{x^{m+1}}{m+1}\right]_0^1 = \frac{1}{m+1}$$

Multiplying vertically, we get $I_{m,n} = \dfrac{(-1)^n\, n!}{(m+1)^{n+1}}$

29. Prove that

$$\int_0^{\pi/2} \cos^m x \sin nx\,dx = \frac{1}{m+n} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\,dx$$

Hence evaluate $\displaystyle\int_0^{\pi/2} \cos^5 x \sin 3x\,dx$.

Sol. $\int \cos^m x \sin nx\,dx$

$$= \cos^m x\left(-\frac{\cos nx}{n}\right) - \int m\cos^{m-1} x(-\sin x)\left(-\frac{\cos nx}{n}\right)dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^{m-1} x \sin x \cos nx\,dx \qquad (1)$$

Since $\sin(n-1)x = \sin nx \cos x - \cos nx \sin x$, we have

$\cos nx \sin x = \sin nx \cos x - \sin(n-1)x$.

Putting this value of $\cos nx \sin x$ into (1), we get

$$\int \cos^m x \sin nx\,dx = -\frac{\cos^m x \cos nx}{n}$$

$$-\frac{m}{n}\int \cos^{m-1} x\,[\sin nx \cos x - \sin(n-1)x]\,dx$$

$$= -\frac{\cos^m x \cos nx}{n} - \frac{m}{n}\int \cos^m x \sin nx\,dx$$

$$+ \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\,dx$$

Transposition yields $\left(1 + \dfrac{m}{n}\right)\displaystyle\int \cos^m x \sin nx\,dx$

$$= -\frac{\cos^m x \cos nx}{n} + \frac{m}{n}\int \cos^{m-1} x \sin(n-1)x\,dx$$

or $\displaystyle\int \cos^m x \sin nx\,dx$

$$= -\frac{\cos^m x \cos nx}{m+n} + \frac{m}{m+n}\int \cos^{m-1} x \sin(n-1)x\,dx$$

Hence $\displaystyle\int_0^{\pi/2} \cos^m x \sin nx\,dx$

$$= \left[-\frac{\cos^m x \cos nx}{m+n}\right]_0^{\pi/2} + \frac{m}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\,dx$$

$$= \frac{1}{m+n} + \frac{n}{m+n}\int_0^{\pi/2} \cos^{m-1} x \sin(n-1)x\,dx.$$

Putting $m = 5$ and $n = 3$ in the above formula we have

$$\int_0^{\pi/2} \cos^5 x \sin 3x\,dx = \frac{1}{5+3} + \frac{5}{5+3}\int_0^{\pi/2} \cos^4 x \sin 2x\,dx$$

$$= \frac{1}{8} + \frac{5}{8}\int_0^{\pi/2} \cos^4 x \sin 2x\,dx \text{ and}$$

$$\int_0^{\pi/2} \cos^4 x \sin 2x\,dx = \frac{1}{4+2} + \frac{4}{4+2}\int_0^{\pi/2} \cos^3 x \sin x\,dx$$

$$= \frac{1}{6} + \frac{2}{3}\left[-\frac{\cos^4 x}{4}\right]_0^{\pi/2} = \frac{1}{6} + \frac{2}{3}\left[-\left(0 - \frac{1}{4}\right)\right] = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$$

Thus $\int_0^{\pi/2} \cos^5 x \sin 3x\, dx = \frac{1}{8} + \frac{5}{8}\cdot\frac{1}{3} = \frac{3+5}{24} = \frac{1}{3}$

30. Find a reduction formula for $\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}}\, dx$.

Sol. We connect the given integral with $\frac{x^{n-2}}{\sqrt{ax^2 + 2bx + c}}\, dx$

Hence $P = x^{n-2+1}(ax^2 + 2bx + c)^{-1/2+1} = x^{n-1}(ax^2 + 2bx + c)^{1/2}$

$$\frac{dP}{dx} = (n-1)x^{n-2}\cdot(ax^2 + 2bx + c)^{1/2} + x^{n-1}\cdot\frac{2ax + 2b}{2\sqrt{ax^2 + 2bx + c}}$$

$$= \frac{(n-1)x^{n-2}(ax^2 + 2bx + c) + x^{n-1}(ax + b)}{\sqrt{ax^2 + 2bx + c}}$$

$$= \frac{(n-1)(ax^n + 2bx^{n-1} + cx^{n-2}) + ax^n + bx^{n-1}}{\sqrt{ax^2 + 2bx + c}}$$

$$= \frac{an\,x^n}{\sqrt{ax^2 + 2bx + c}} + \frac{b(2n-1)x^{n-1}}{\sqrt{ax^2 + 2bx + c}} + \frac{c(n-1)x^{n-2}}{\sqrt{ax^2 + 2bx + c}}$$

Integrating, we obtain

$$P = x^{n-1}\sqrt{ax^2 + 2bx + c} = an\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}}\, dx + \int \frac{b(2n-1)x^{n-1}}{\sqrt{ax^2 + 2bx + c}}\, dx + \int \frac{c(n-1)x^{n-2}}{\sqrt{ax^2 + 2bx + c}}\, dx$$

or $$\int \frac{x^n}{\sqrt{ax^2 + 2bx + c}}\, dx = \frac{x^{n-1}\sqrt{ax^2 + 2bx + c}}{an} - \frac{b(2n-1)}{an}\int \frac{x^{n-1}}{\sqrt{ax^2 + 2bx + c}}\, dx - \frac{c(n-1)}{an}\int \frac{x^{n-2}}{\sqrt{ax^2 + 2bx + c}}\, dx$$

is the required reduction formula.

Exercise Set 5.5 (Page 211)

In each of Problems (1 – 12), use the trapezoidal rule to approximate the given integral:

1. $\int_1^4 \frac{dx}{x} = \ln 4$ with $n = 3$

Sol. Here length of each subinterval is $\frac{4-1}{3} = 1$. The interval $[1, 4]$ is partitioned into subintervals by the points

$x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4$

The corresponding function values are as under:

n	x_n	$f(x_n)$
0	1	1
1	2	$\frac{1}{2}$
2	3	$\frac{1}{3}$
3	4	$\frac{1}{4}$

Substituting into the trapezoidal rule, we have

$$\int_1^4 \frac{dx}{x} \approx \frac{4-1}{3}\left[\frac{1}{2}(1) + \frac{1}{2} + \frac{1}{3} + \frac{1}{2}\left(\frac{1}{4}\right)\right] = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = 1 + \frac{11}{24} = \frac{35}{24}$$

≈ 1.4583 By calculator, $\ln 4 \approx 1.3863$

2. $\int_0^{\pi/3} \cos x\, dx = \frac{\sqrt{3}}{2}$ with $n = 4$

Sol. Length of each subinterval is $\frac{\frac{\pi}{3} - 0}{4} = \frac{\pi}{12}$

The interval $\left[0, \frac{\pi}{2}\right]$ is partitioned by the points

$x_0 = 0,\ x_1 = \frac{\pi}{12},\ x_2 = \frac{2\pi}{12} = \frac{\pi}{6},\ x_3 = \frac{3\pi}{12} = \frac{\pi}{4},\ x_4 = \frac{\pi}{3}$

$f(x_0) = \cos 0 = 1, f(x_1) = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$

$f(x_2) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, f(x_3) = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $f(x_4) = \cos\frac{\pi}{3} = \frac{1}{2}$

$$\int_0^{\pi/3}\cos x\,dx \approx \frac{\pi}{12}\left[\frac{1}{2}(1) + \frac{\sqrt{3}+1}{2\sqrt{2}} + \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} + \frac{1}{2}\left(\frac{1}{2}\right)\right]$$

$$\approx \frac{\pi}{12}\left[\frac{1}{2} + \frac{1}{4} + \frac{\sqrt{3}+1+\sqrt{2}\cdot\sqrt{3}+2}{2\sqrt{2}}\right]$$

$$\approx \frac{\pi}{12}\left[0.75 + \frac{\sqrt{3}+\sqrt{6}+3}{2\sqrt{2}}\right]$$

$$\approx \frac{\pi}{12}\left[0.75 + \frac{1.7321+2.4495+3}{2.8284}\right] = \frac{\pi}{12}\left[0.75 + \frac{7.1816}{2.8284}\right]$$

$$\approx 0.2618\,[0.75 + 2.5391] = 0.2618\,(3.2891)$$

≈ 0.8611 By calculator $\frac{\sqrt{3}}{2} \approx 0.8660$

3. $\int_0^2 e^{-x^2}\,dx$ with $n = 4$

Sol. Length of each subinterval $= \frac{2-0}{4} = \frac{1}{2}$

The interval [0, 2] is partitioned into subintervals by the points

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1, x_3 = \frac{3}{2},\ x_4 = 2$$

$$f(x_0) = 1, f(x_1) = e^{-1/4} \approx 0.7788, f(x_2) = e^{-1} = 0.3679$$

$$f(x_3) = e^{-9/4} = 0.1054,\ f(x_4) = e^{-4} = 0.0183$$

$$\int_0^2 e^{-x^2}\,dx \approx \frac{2-0}{4}\left[\frac{1}{2}(1) + 0.7788 + 0.3679 + 0.1054 + \frac{1}{2}(0.01830)\right]$$

$$\approx \frac{1}{2}(1.7613) \approx .8807 \approx 0.88$$

4. $\int_0^4 x^2\,dx$ with $n = 8$

Sol. Length of subinterval $= \frac{4-0}{8} = \frac{1}{2}$

Points of the partitions are

$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2,\ x_5 = \frac{5}{2}$

$x_6 = 3,\ x_7 = \frac{7}{2},\ x_8 = 4.$

$$\int_0^4 x^2\,dx \approx \frac{1}{2}\left[\frac{1}{2}f(x_0) + f(x_1) + \dots + f(x_7) + \frac{1}{2}f(x_8)\right]$$

$$\approx \frac{1}{2}\left[0 + 0.25 + 1 + 2.25 + 4 + 6.25 + 9 + 12.25 + \frac{16}{2}\right] = \frac{43}{2}$$

$$\approx 21.5$$

5. $\int_0^{\pi}\sin x\,dx$ with $n = 6$

Sol. Length of each subinterval $= \frac{\pi}{6}$

Points of the partition of $[0, \pi]$ are

$$x_0 = 0,\ x_1 = \frac{\pi}{6},\ x_2 = \frac{\pi}{3},\ x_3 = \frac{\pi}{2},\ x_4 = \frac{2\pi}{3},\ x_5 = \frac{5\pi}{6},\ x_6 = \pi$$

$$f(x_0) = 0,\ f(x_1) = \sin\frac{\pi}{6} = \frac{1}{2},\ f(x_2) = \frac{\sqrt{3}}{2},\ f(x_3) = 1,$$

$$f(x_4) = \frac{\sqrt{3}}{2},\ f(x_5) = \frac{1}{2},\ f(x_6) = 0$$

$$\int_0^{\pi}\sin x\,dx \approx \frac{\pi}{6}\left[\frac{1}{2}(0) + \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 + \frac{\sqrt{3}}{2} + \frac{1}{2} + \frac{1}{2}(0)\right]$$

$$\approx \frac{\pi}{6}(2 + \sqrt{3}) \approx \pi\left(\frac{3.732}{6}\right) \approx (3.14159)\,(0.6228) \approx 1.9541$$

6. $\int_0^2 \frac{dx}{1+x^3}$ with $n = 4$

Sol. Length of each interval $= \frac{2-0}{4} = \frac{1}{2}$

Points of a subdivision of [0, 2] are

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$$

$$\int_0^2 \frac{dx}{1+x^3} \approx \frac{1}{2}\left[\frac{1}{2}(1) + \frac{8}{9} + \frac{1}{2} + \frac{8}{35} + \frac{1}{2}\left(\frac{1}{9}\right)\right]$$

$$\approx \frac{1}{2}[0.5 + 0.88889 + 0.5 + 0.22857 + 0.05556] = \frac{1}{2}(2.17302)$$

$$\approx 1.0865$$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$ with $n = 4$

Sol. Length of each subinterval $= \dfrac{1-0}{4} = \dfrac{1}{4}$

Points of a subdivision of $[0, 1]$ are

$x_0 = 0,\ x_1 = \dfrac{1}{4},\ x_2 = \dfrac{1}{2},\ x_3 = \dfrac{3}{4},\ x_4 = 1$

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} \approx \frac{1}{4}\left[\frac{1}{2}f(0) + f\left(\frac{1}{4}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{3}{4}\right) + \frac{1}{2}f(1)\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{2}\left(\frac{1}{2}\right) + \frac{4}{\sqrt{63}} + \frac{2}{\sqrt{15}} + \frac{4}{\sqrt{55}} + \frac{1}{2}\frac{1}{\sqrt{3}}\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{4} + \frac{4}{21}\cdot\sqrt{7} + \frac{2}{15}\cdot\sqrt{15} + \frac{4}{55}\cdot\sqrt{55} + \frac{1}{6}\cdot\sqrt{3}\right]$$

$$\approx \frac{1}{4}(0.2500+0.5040+0.5164+0.5394+0.2887) = \frac{1}{4}(2.0985)$$

$$\approx 0.5246$$

8. $\displaystyle\int_{-2}^{2} (2x^2 + 1)\,dx$ with $n = 4$

Sol. Length of each subinterval $= \dfrac{2-(-2)}{4} = 1$

Points of subdivision of $[-2, 2]$ are

$x_0 = -2,\ x_1 = -1\ \ x_2 = 0,\ x_3 = 1,\ x_4 = 2$

$$\int_{-2}^{2} (2x^2 + 1)\,dx \approx 1\cdot\left[1.\frac{1}{2}f(-2) + f(-1) + f(0) + f(1) + \frac{1}{2}f(2)\right]$$

$$\approx \frac{1}{2}(9) + 3 + 1 + 3 + \frac{9}{2} = 16.$$

9. $\displaystyle\int_1^5 \frac{dx}{x^2}$ with $n = 4$

Sol. Length of each subinterval $= \dfrac{5-1}{4} = 1$

Points of subdivision of $[1, 5]$ are

$x_0 = 1,\ x_1 = 2,\ x_2 = 3,\ x_3 = 4,\ x_4 = 5$

$$\int_5^1 \frac{dx}{x^2} \approx 1\cdot\left[\frac{1}{2}f(1) + f(2) + f(3) + f(4) + \frac{1}{2}f(5)\right]$$

$$\approx \frac{1}{2}(1) + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{2}\left(\frac{1}{25}\right)$$

$$\approx 0.5000 + 0.2500 + 0.1111 + 0.0625 + 0.0200 = 0.9436$$

10. $\displaystyle\int_0^1 e^{-x}\,dx$ with $n = 6$

Sol. Length of each subinterval $= \dfrac{1-0}{6} = \dfrac{1}{6}$

Points of subdivision of $[0, 1]$ are

$x_0 = 0,\ x_1 = \dfrac{1}{6},\ x_2 = \dfrac{1}{3},\ x_3 = \dfrac{1}{2},\ x_4 = \dfrac{2}{3},\ x_5 = \dfrac{5}{6},\ x_6 = 1$

$$\int_0^1 e^{-x}\,dx \approx \frac{1}{6}\left[\frac{1}{2}f(0) + f\left(\frac{1}{6}\right) + f\left(\frac{1}{3}\right) + f\left(\frac{1}{2}\right) + f\left(\frac{2}{3}\right) + f\left(\frac{5}{6}\right) + \frac{1}{2}f(1)\right]$$

$$\approx \frac{1}{6}\left[\frac{1}{2}\left(\frac{1}{1}\right) + \frac{1}{1.18136} + \frac{1}{1.3956} + \frac{1}{1.6487} + \frac{1}{1.9477} + \frac{1}{2}\left(\frac{1}{2.71828}\right)\right]$$

$$\approx \frac{1}{6}[0.5+0.8465+0.7165+0.6065+0.5134+0.4346+0.1839] = \frac{1}{6}[3.8014]$$

$$\approx 0.6336 \qquad \left(\text{Note: } \int_0^1 e^{-x}\,dx = 1 - \frac{1}{e} \approx 1 - 0.3679 = 0.6321\right)$$

11. $\displaystyle\int_1^2 \ln x\,dx$ with $n = 4$

Sol. Length of each subinterval $= \dfrac{2-1}{4} = \dfrac{1}{4}$

Points of subdivision of $[1, 2]$ are

$x_0 = 1,\ x_1 = \dfrac{5}{4},\ x_2 = \dfrac{6}{4} = \dfrac{3}{2},\ x_3 = \dfrac{7}{4},\ x_4 = 2$

$$\int_1^2 \ln x\,dx \approx \frac{1}{4}\left[\frac{1}{2}f(1) + f\left(\frac{5}{4}\right) + f\left(\frac{3}{2}\right) + f\left(\frac{7}{4}\right) + \frac{1}{2}f(2)\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{2}\ln(1) + \ln(1.25) + \ln(1.5) + \ln(1.75) + \frac{1}{2}\ln 2\right]$$

$$\approx \frac{1}{4}\left[\frac{1}{2}(0) + 0.2231 + 0.4055 + 0.5596 + \frac{1}{2}(0.6931)\right]$$

$$\approx \frac{1.5348}{4} = 0.3837$$

$$\left(\text{Note: } \int_1^2 \ln x\,dx = 2\ln 2 - 1 = \ln 4 - 1 \approx 1.3863 - 1 = .3863\right)$$

12. $\displaystyle\int_0^2 \frac{1}{\sqrt{1+x^2}}\,dx$ with $n = 4$

Sol. Length of each subinterval $= \dfrac{2-0}{4} = \dfrac{1}{2}$

Points of subdivision of [0, 2] are

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2$$

$$\int_0^2 \frac{1}{\sqrt{1+x^2}}\,dx \approx \frac{1}{2}\left[\frac{1}{2}f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + \frac{1}{2}f(2)\right]$$

$$\approx \frac{1}{2}\left[\frac{1}{2}(1) + \frac{2}{\sqrt{5}} + \frac{1}{\sqrt{2}} + \frac{2}{\sqrt{13}} + \frac{1}{2}\left(\frac{1}{\sqrt{5}}\right)\right]$$

$$\approx \frac{1}{2}\left[\frac{1}{2}(1) + 0.9444 + 0.7071 + 0.5547 + \frac{1}{2}(0.4472)\right] = \frac{2.9298}{2}$$

$$\approx 1.4649 \quad \left(\text{Note: } \int_0^2 \frac{1}{\sqrt{1+x^2}}\,dx = \ln(\sqrt{5}+2) \approx \ln 4.2361 \approx 1.4436\right)$$

13. Use Simpson's rule to approximate the integrals of Problems 3, 4, 10, 11 and 12.

Sol.

(3) $\displaystyle\int_0^2 e^{-x^2}\,dx$ with $n = 4$

Length of each subinterval is $\frac{1}{2}$ · Using the function values as in Problem 3, and substituting into Simpson's rule, we have

$$\int_2^0 e^{-x^2}\,dx \approx \frac{2-0}{3\times 4}[1 + 4(0.7788) + 2(0.3679) + 4(0.1055) + 0.0183]$$

$$\approx \frac{1}{6}[1 + 3.1152 + 0.7358 + 0.4220 + 0.0183] = \frac{1}{6}(5.2913)$$

$$\approx 0.8819$$

(4) $\displaystyle\int_0^4 x^2\,dx \approx \frac{4-0}{3\times 8}[0 + 4(0.25) + 2(1) + 4(2.25) + 2(4) + 4(6.25)$

$+ 2(9) + 4(12.25) + 16]$

$$\approx \frac{1}{6}[1 + 2 + 9 + 8 + 25 + 18 + 49 + 16] = \frac{128}{6}$$

$$\approx 21.3333$$

(10) $\displaystyle\int_0^1 e^{-x}\,dx$ with $n = 6$

By Simpson's rule, we have

$$\int_0^1 e^{-x}\,dx \approx \frac{1-0}{3\times 6}[1 + 4(0.8465) + 2(0.7165) + 4(0.6065) + 2(0.5134) + 4(0.4346) + 0.3679]$$

$$\approx \frac{1}{18}[1+3.3360+1.4330+2.4260+1.0268+1.7384+0.3679] = \frac{1}{18}(11.3781)$$

$$\approx 0.6321$$

(11) $\displaystyle\int_1^2 \ln x\,dx$ with $n = 4$

By Simpson's rule, we have

$$\int_1^2 \ln x\,dx \approx \frac{1-0}{3\times 4}[0 + 4(0.2231) + 2(0.4055) + 4(0.5596) + 0.6931]$$

$$\approx \frac{1}{12}[0.8924 + 0.8110 + 2.2384 + 0.6931] = \frac{1}{12}[4.6349]$$

$$\approx 0.3862$$

(12) $\displaystyle\int_0^2 \frac{dx}{\sqrt{1+x^3}}$ with $n = 4$

By Simpson's rule, the given integral

$$\int_0^2 \frac{dx}{\sqrt{1+x^3}} \approx \frac{2-0}{3\times 4}[1 + 4(0.9444) + 2(0.7071) + 4(0.5547) + 0.4472]$$

$$\approx \frac{1}{6}[1 + 3.7776 + 1.4142 + 2.2188 + 0.4472] = \frac{1}{6}(8.8578)$$

$$\approx 1.4763$$

Find a bound on the error in approximating the given integral using (i) the trapezoidal rule (ii) Simpson's rule. (Problems 14 – 16):

14. $\displaystyle\int_{-1}^2 x^5\,dx$ with $n = 10$

Sol.

(i) Here $f(x) = x^5$, $f'(x) = 5x^4$, $f''(x) = 20x^3$

$f'''(x) = 60x^2$, $f^{(4)}(x) = 120x$

Now max $|f''(x) = 20x^3|$ on $[-1, 2]$ is attained at $x = 2$ and the maximum value $M = 20 \times 8 = 160$

$$\text{Maximum error} = \frac{(b-a)^3 M}{12n^2} = \frac{[2-(-1)]^3 M}{12 \cdot 10^2} = \frac{27 \times 160}{12 \times 100} = \frac{36}{10} = 3.6$$

(ii) For Simpson's rule,

$$\begin{aligned} M &= \max. |f^{(4)}(x)| \\ &= \max. |120x| \quad \text{on } [-1, 2] \\ &= 240 \end{aligned}$$

$$\text{Maximum error} = \frac{M(b-a)^5}{180n^4} = \frac{240 \times 3^5}{180 \times 10^4}$$

$$= \frac{240 \times 243}{180 \times 10000} = \frac{324}{10000} = 0.0324$$

15. $\displaystyle\int_1^3 \frac{dx}{x}$ with $n = 10$

Sol. $\displaystyle\int_1^3 \frac{dx}{x}$ with $n = 10$

(i) $f(x) = \dfrac{1}{x}$, $f'(x) = \dfrac{-1}{x^2}$, $f''(x) = \dfrac{2}{x^3}$

$f'''(x) = \dfrac{-6}{x^4}$, $f^{(4)}(x) = \dfrac{24}{x^5}$

Now max. $|f''(x)| = \max \left|\dfrac{2}{x^3}\right| = 2$ on $[1, 3]$

$$\text{Maximum error} = \frac{M(b-a)^3}{12n^2}$$

$$= \frac{2 \times 2^3}{12 \times 100} = \frac{1}{3}\left(\frac{4}{100}\right) = \frac{1}{3}(.04) \approx 0.01333$$

(ii) Max. $|f^{(4)}(x)| = \max \left|\dfrac{24}{x^5}\right| = 24$ on $[1, 3]$

$$\text{Maximum error} = \frac{M(b-a)^5}{180n^4} = \frac{24 \times 2^5}{180 \times 10^4}$$

$$= \frac{2 \times 32}{15 \times 10000} = \frac{1}{15}(.0064) \approx .00043$$

16. $\displaystyle\int_0^2 \frac{dx}{\sqrt{1+x}}$ with $n = 8$

Sol.

(i) $f(x) = (1+x)^{-1/2}$, $f'(x) = -\dfrac{1}{2}(1+x)^{-3/2}$

$f''(x) = \left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)(1+x)^{-5/2}$, $f'''(x) = \left(-\dfrac{1}{2}\right)\left(-\dfrac{3}{2}\right)\left(-\dfrac{5}{2}\right)(1+x)^{-7/2}$

$f^{(4)}(x) = \left(-\dfrac{1}{2}\right)\cdot\left(-\dfrac{3}{2}\right)\cdot\left(-\dfrac{5}{2}\right)\cdot\left(-\dfrac{7}{2}\right)(1+x)^{-9/2}$

$$M = \max. |f''(x)| = \max_{x \in [0,2]} \left|\frac{3}{4} \cdot \frac{1}{(1+x)^{5/2}}\right| = \frac{3}{4}$$

$$\text{Maximum error} = \frac{M(b-a)^3}{12n^2} = \frac{\frac{3}{4} \cdot 2^3}{12(8)^2}$$

$$= \frac{3}{4} \times \frac{1}{12 \times 8} = \frac{1}{128} = 0.0078125$$

(ii) $M = \max. |f^{(4)}(x)| = \displaystyle\max_{x \in [0,2]} \left|\frac{105}{16} \cdot \frac{1}{(1+x)^{9/2}}\right| = \frac{105}{16}$

$$\text{Maximum error} = \frac{M(b-a)^5}{180n^4}$$

$$= \frac{105 \times 32}{16 \times 180 \times 64 \times 64} = \frac{7}{192 \times 128} \approx 0.0002848$$

17. With $n = 8$, find the area under the semicircle $y = \sqrt{4 - x^2}$ and above the x-axis by (i) the trapezoidal rule (ii) Simpson's rule.

Sol. Required area

$$A = \int_{-2}^{2} \sqrt{4 - x^2}\, dx \quad \text{with} \quad n = 8$$

Length of each subinterval $= \dfrac{2-(-2)}{8} = \dfrac{1}{2}$

Points of subdivision of $[-2, 2]$ are

$x_0 = -2$, $x_1 = -\dfrac{3}{2}$, $x_2 = -1$, $x_3 = -\dfrac{1}{2}$, $x_4 = 0$,

$x_5 = \dfrac{1}{2}$, $x_6 = 1$, $x_7 = \dfrac{3}{2}$, $x_8 = 2$

$f(x_0) = f(-2) = 0$, $f(x_1) = f\left(-\dfrac{3}{2}\right) \approx 1.32287$

$$f(x_2) = f(-1) \approx 1.73210, \quad f(x_3) = f\left(-\frac{1}{2}\right) \approx 1.9365$$

$$f(x_4) = f(0) = 2, \quad f(x_5) = f\left(\frac{1}{2}\right) = 1.9365$$

$$f(x_6) = f(1) = 1.7321, \quad f(x_7) = f\left(\frac{3}{2}\right) = 1.3229$$

$$f(x_8) = f(2) = 0$$

(i) By the trapezoidal rule

$$A \approx \frac{1}{2}\left[\frac{1}{2}(0)+1.3229+1.7321+1.9365+2+1.9365+1.7321+1.3229+\frac{1}{2}(0)\right]$$

$$\approx \frac{1}{2}(11.9830) \approx 5.9915$$

(ii) By Simpson's rule,

$$A \approx \frac{1}{6}[0 + 4(1.3229) + 2(1.7321) + 4(1.9365) + 2(2)$$
$$+ 4(1.9365) + 2(1.7321) + 4(1.3229)+0]$$

$$\approx \frac{1}{6}[5.2916+3.4642+7.7460+4+7.7460+3.4642+5.2916] = \frac{37.0036}{6}$$

$$\approx 6.1673$$

Actual area $= 2\int_0^2 \sqrt{4-x^2}\,dx$

Put $x = 2\sin\theta$ so that $dx = 2\cos\theta\, d\theta$

When $x = 0$, $\theta = 0$ and when $x = 2$, $\theta = \frac{\pi}{2}$

$$\text{Area} = 2\int_0^{\pi/2} (2\cos\theta)\, 2\cos\theta\, d\theta = 8\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 8\int_0^{\pi/2} \frac{1+\cos 2\theta}{2}\, d\theta = 8\left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2}$$

$$= 8\left[\left(\frac{\pi}{4}+0\right)-0\right] = 8\left(\frac{\pi}{4}\right) = 2\pi \approx 2(3.14159) \approx 6.2832$$

18. A reading of the velocity of a ship was made every quarter hour as shown below:

Time t (hours)	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{7}{4}$	2
Velocity $v(t)$ (mph)	19.5	24.3	34.2	40.5	38.4	26.2	18	16	8

Estimate the distance travelled by the ship during the 2-hour period.

Sol. The total distance travelled by the ship during the 2-hour period is

$$\int_0^2 v(t)\,dt.$$

We approximate this integral by the trapezoidal rule.

Here length of each subinterval is $\frac{2-0}{8} = \frac{1}{4}$. The points of subdivision and the corresponding values are as in the table above. Substituting into the trapezoidal rule, we have

$$\int_0^2 v(t)dt \approx \frac{1}{4}\left[\frac{1}{2}(19.5)+24.3+34.2+40.5+38.4+26.2+18+16+\frac{1}{2}(8)\right]$$

$$\approx \frac{1}{4}[9.75+24.30+34.20+40.50+38.40+26.20+18.00+16.00+4.00] = \frac{1}{4}(211.35)$$

$$\approx 52.8375 \approx 52.84$$

The total distance travelled by the ship $\approx$ 52.84 miles.