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Calculus With Analytic Geometry

Our Effort To Surve You Better

Calculus
With
Analytic Geometry

By
S.M Yusuf

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$$p = \frac{\sin 35°17'}{\tan 109°18.8'} = -0.196905109$$

$$q = \frac{\cos 35°17' \tan 21°25.2'}{\sin 109°18.8'} = 0.3393320$$

$$\tan i = p - q = -0.5362371$$

$$i = -28°\ 12.1'$$

The direction of Qibla is 28°12.1′ north of west.

6. Prove that for a place on the equator the direction of Qibla is inclined $\arctan(\tan\phi_o \csc l)$ north of west or north of east according as its classical longitude l is east or west.

Sol. Here $p = \dfrac{\sin 0}{\tan l} = 0, q = \dfrac{\cos 0 \tan \phi_o}{\sin l} = \dfrac{\tan \phi_0}{\sin l}$

Hence $\tan i = \tan \phi_o \csc l$

i.e., $i = \arctan(\tan \phi_o \csc l)$.

7. Prove that for a place on the same parallel of latitude as the Khana-e-Ka'aba the direction of Qibla is inclined at $\arctan\left(\sin \phi_o \tan \dfrac{l}{2}\right)$ north of west or north of east according as its classical longitude l is east or west.

Sol. Here $p = \dfrac{\sin \phi_o}{\tan l}$, since $\phi = \phi_o$

$$q = \frac{\cos \phi_o \tan \phi_o}{\sin l}$$

$$\tan i = p - q = \frac{\sin \phi_o}{\tan l} - \frac{\cos \phi_o \tan \phi_o}{\sin l}$$

$$= \frac{\sin \phi_o \cos l}{\sin l} - \frac{\cos \phi_o \sin \phi_o}{\sin l \cos \phi_o}$$

$$= \frac{\sin \phi_o \cos l}{\sin l} - \frac{\sin \phi_o}{\sin l}$$

$$= \frac{\sin \phi_o}{\sin l}(\cos l - 1) = \frac{\sin \phi_o}{\sin l} 2 \sin^2 \frac{l}{2}$$

$$= \frac{\sin \phi_o}{2 \sin \frac{l}{2} \cos \frac{l}{2}} 2 \sin^2 \frac{l}{2} = \sin \phi_o \tan \frac{l}{2}$$

or $\quad i = \arctan\left(\sin \phi_o \tan \dfrac{l}{2}\right)$.

Chapter 9 FUNCTIONS OF SEVERAL VARIABLES

Exercise Set 9.1 (Page 411)

1. Verify Euler's Theorem for

(*a*) $u = \arcsin\left(\dfrac{x}{y}\right) + \arctan\left(\dfrac{y}{x}\right)$

(*b*) $u = x^n \ln\left(\dfrac{y}{x}\right)$ (*c*) $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}}$

Sol.

(*a*) Here u is a homogeneous function of zero degree. Therefore, by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad (1)$$

Now $u = \arcsin\left(\dfrac{x}{y}\right) + \text{arctna}\left(\dfrac{y}{x}\right)$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1 + \frac{y^2}{x^2}} \cdot \frac{-y}{x^2} = \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{x^2 + y^2}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1 - \frac{x^2}{y^2}}} \cdot \frac{-x}{y^2} + \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{x} = \frac{-x}{y\sqrt{y^2 - x^2}} + \frac{x}{x^2 + y^2}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} - \frac{x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} = 0$$

(*b*) Here u is a homogeneous function of degree n. Therefore, by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad (1)$$

To verify this, we have

$$\frac{\partial u}{\partial x} = nx^{n-1}\ln\frac{y}{x} + x^n \cdot \frac{1}{y/x} \cdot \frac{-y}{x^2} = nx^{n-1}\ln\frac{y}{x} - x^{n-1}$$

and $$\frac{\partial u}{\partial y} = x^n \cdot \frac{1}{y/x} \cdot \frac{1}{x} = \frac{x^n}{y}.$$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nx^n \ln\dfrac{y}{x} - x^n + y\dfrac{x^n}{y}$

$$= nx^n \ln\frac{y}{x} - x^n + x^n = nx^n \ln\frac{y}{x} = nu$$

(c) Here $u = \dfrac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} = \dfrac{x^{1/4}\left[1 + \left(\frac{y}{x}\right)^{1/4}\right]}{x^{1/5}\left[1 + \left(\frac{y}{x}\right)^{1/5}\right]}$

$$= x^{\frac{1}{4}-\frac{1}{5}}\left[\frac{1 + \left(\frac{y}{x}\right)^{1/4}}{1 + \left(\frac{y}{x}\right)^{1/5}}\right] = x^{\frac{1}{20}}\,\frac{1 + \left(\frac{y}{x}\right)^{1/4}}{1 + \left(\frac{y}{x}\right)^{1/5}}$$

Thus u is a homogeneous function of degree $\dfrac{1}{20}$ and hence by Euler's theorem we must have

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}u = \frac{1}{20}u$$

To verify this, we have

$$\frac{\partial u}{\partial x} = \frac{(x^{1/5} + y^{1/5})\cdot\left(\frac{1}{4}x^{-3/4}\right) - (x^{1/4} + y^{1/4})\left(\frac{1}{5}x^{-4/5}\right)}{(x^{1/5} + y^{1/5})^2}$$

and $$\frac{\partial u}{\partial y} = \frac{(x^{1/5} + y^{1/5})\cdot\left(\frac{1}{4}y^{-3/4}\right) - (x^{1/4} + y^{1/4})\left(\frac{1}{5}y^{-4/5}\right)}{(x^{1/5} + y^{1/5})^2}$$

Therefore,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{(x^{1/5}+y^{1/5})\left(\frac{1}{4}x^{1/4}+\frac{1}{4}y^{1/4}\right) - (x^{1/4} + y^{1/4})\left(\frac{1}{5}y^{1/5} + \frac{1}{5}x^{1/5}\right)}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{(x^{1/4} + y^{1/4})(x^{1/5} + y^{1/5})\left(\frac{1}{20}\right)}{(x^{1/5} + y^{1/5})^2}$$

$$= \frac{1}{20}\frac{x^{1/4} + y^{1/4}}{x^{1/5} + y^{1/5}} = \frac{1}{20}u.$$

2. If $u = f\left(\frac{y}{x}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$

Sol. We have $\dfrac{\partial u}{\partial y} = \dfrac{-y}{x^2}f'\left(\frac{y}{x}\right)$

and $\dfrac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\dfrac{1}{x} = \dfrac{1}{x}f'\left(\frac{y}{x}\right)$

Thus $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{-y}{x}f'\left(\frac{y}{x}\right) + \dfrac{y}{x}f'\left(\frac{y}{x}\right) = 0.$

1. If $u = xyf\left(\frac{x}{y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u.$

Sol. We have $\dfrac{\partial u}{\partial x} = y\left[f\left(\frac{x}{y}\right) + xf'\left(\frac{x}{y}\right)\cdot\frac{1}{y}\right], \dfrac{\partial u}{\partial y} = x\left[f\left(\frac{x}{y}\right) - yf'\left(\frac{x}{y}\right)\cdot\frac{x}{y^2}\right]$

Hence $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = xy\left[f\left(\frac{x}{y}\right) + \frac{x}{y}f'\left(\frac{x}{y}\right) + f\left(\frac{x}{y}\right) - \frac{x}{y}f'\left(\frac{x}{y}\right)\right]$

$$= 2xyf\left(\frac{y}{x}\right) = 2u.$$

4. If $z = \arctan\left(\frac{y}{x}\right)$, verify that $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 2u$

Sol. Differentiating partially w.r.t. x, we have

$$\frac{\partial z}{\partial x} = \frac{1}{1 + \frac{y^2}{x^2}}\cdot\frac{-y}{x^2} = \frac{-y}{x^2 + y^2} = (-y)(x^2 + y^2)^{-1}$$

$$\frac{\partial^2 z}{\partial x^2} = (-1)\,(y)\,(x^2 + y^2)^{-2}\,(-2x) = \frac{2xy}{(x^2 + y^2)^2} \qquad (1)$$

Again, $$\frac{\partial z}{\partial y} = \frac{1}{1 + \frac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2 + y^2} = x(x^2 + y^2)^{-1}$$

$$\frac{\partial^2 z}{\partial y^2} = (-1)\,(x)\,(x^2 + y^2)^{-2}\,(2y) = \frac{-2xy}{(x^2 + y^2)^2} \qquad (2)$$

Adding (1) and (2), we get $\dfrac{\partial^2 z}{\partial x^2} + \dfrac{\partial^2 z}{\partial y^2} = 0.$

5. If $u = \arcsin\left(\dfrac{x^2 + y^2}{x + y}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \tan u$

Sol. Writing tx, ty for x, y in the R.H.S. of the given equation, we have

$$\arcsin\frac{t(x^2 + y^2)}{x + y} \neq t\arcsin\frac{x^2 + y^2}{x + y}$$

Hence u is not a homogeneous function.

Let $z = \sin u = \dfrac{x^2 + y^2}{x + y}$.

Then z is a homogeneous function of degree 1.

Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.z \text{ , or } x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = z$$

i.e., $x\cdot\cos u\cdot\dfrac{\partial u}{\partial x} + y\cdot\cos u\cdot\dfrac{\partial u}{\partial y} = z = \sin u$

Dividing by $\cos u$, we get $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \dfrac{\sin u}{\cos u} = \tan u.$

6. If $u = \arcsin\left(\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{y}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$.

Sol. Proceeding as in **Q. 5**, it is easy of see that u is not a homogeneous function.

Let $\quad z = \sin u = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$.

This shows that z is a homogeneous function of zero degree. Therefore, by Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0 \,.\, z = 0 \quad \text{or} \quad x\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + y\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = 0$$

or $\quad x\cos u\dfrac{\partial u}{\partial x} + y\cos u\dfrac{\partial u}{\partial y} = 0.$

i.e., $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0.$

7. If $u = \ln\left(\dfrac{x^2+y^2}{x+y}\right)$, prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1$.

Sol. Here u is a homógeneous function (verify !)

Let $\quad z = e^u = \dfrac{x^2+y^2}{x+y}$. Then z is a homogeneous function of degree 1. By Euler's Theorem, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 1.z$$

or $\quad x\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x} + y\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial y} = z = e^u$

or $\quad x\,.\,e^u\,.\,\dfrac{\partial u}{\partial x} + y\,.\,e^u.\,\dfrac{\partial u}{\partial y} = e^u$

or $\quad x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 1.$

8. If $u = f(x, y)$ is a homogeneous function of degree n, prove that

$$x^2 f_{xx} + 2xy\, f_{xy} + y^2 f_{yy} = n\,(n-1)\,f.$$

Sol. Since f is a homogeneous function of degree n, we have

$$xf_x + yf_y = nf \tag{1}$$

Differentiating (1) w.r.t. x and y respectively, we get

$$xf_{xx} + f_x + yf_{yx} = nf_x \tag{2}$$

and $\quad xf_{xy} + yf_{yx} + f_y = nf_y \quad$ (3)

Assuming $f_{xy} = f_{yx}$ and multiplying (2) by x and (3) by y and adding the results, we have

$$x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} + xf_x + yf_y = n(xf_x + yf_y)$$

or $\quad x^2f_{xx} + 2xyf_{xx} + y^2f_{yy} = (n-1)\,(xf_x + yf_y) = n(n-1)f$, using (1).

9. If $u = f(r)$, where $r = \sqrt{x^2+y^2}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r}f'(r).$$

Sol. We have $\dfrac{\partial u}{\partial x} = f'(r)\,.\,\dfrac{\partial r}{\partial x} = f'(r)\,.\,\dfrac{1}{2}(x^2+y^2)^{-1/2}\,2x = f'(r)\,.\,\dfrac{x}{\sqrt{x^2+y^2}}$

$$\frac{\partial^2 u}{\partial x^2} = f''(r)\,.\,\frac{\partial r}{\partial x}\,.\,\frac{x}{\sqrt{x^2+y^2}} + f'(r)\,\frac{\sqrt{x^2+y^2}\,.\,1 - x\,.\,\frac{1}{2}(x^2+y^2)^{-1/2}\,2x}{(\sqrt{x^2+y^2})^2}$$

$$= f''(r)\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + f'(r)\,\frac{y^2}{(x^2+y^2)^{3/2}}$$

$$= f''(r)\,.\,\frac{x^2}{x^2+y^2} + f'(r)\,\frac{y^2}{(x^2+y^2)^{3/2}} \tag{1}$$

By symmetry, we have

$$\frac{\partial^2 u}{\partial y^2} = f''(r)\,\frac{y^2}{x^2+y^2} + f'(r)\,.\,\frac{x^2}{(x^2+y^2)^{3/2}} \tag{2}$$

Adding (1) and (2), we get

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = f''(r)\left[\frac{x^2+y^2}{x^2+y^2}\right] + f'(r)\left[\frac{x^2+y^2}{(x^2+y^2)^{3/2}}\right]$$

$$= f''(r) + f'(r)\,.\,\frac{r^2}{r^3} = f''(r) + \frac{1}{r}f'(r).$$

10. If $V = \rho^m$, where $\rho^2 = x^2 + y^2 + z^2$, show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m(m+1)\,\rho^{m-2}$$

Sol. We have $\dfrac{\partial V}{\partial x} = m\rho^{m-1}\,.\,\dfrac{\partial \rho}{\partial x} \quad$ (1)

Now, $\quad \rho^2 = x^2 + y^2 + z^2$

Differentiating both the sides w.r.t. x, we have

$$2.\,\rho\,.\,\frac{\partial \rho}{\partial x} = 2x \quad \text{or} \quad \frac{\partial \rho}{dx} = \frac{x}{\rho}$$

Putting this value of $\dfrac{\partial \rho}{\partial x}$ into (1),we get

$$\frac{\partial V}{\partial x} = m\rho^{m-1}\,.\,\frac{x}{\rho} = m\rho^{m-2}\,.\,x$$

Differentiating w.r.t. x, we obtain

$$\frac{\partial^2 V}{\partial x^2} = m\left[\rho^{m-2}\,.\,1 + (m-2)\,\rho^{m-3}\,.\,\frac{\partial \rho}{\partial x}\,.\,x\right]$$

$$= m\left[\rho^{m-2} + (m-2)\,\rho^{m-3}\left(\frac{x}{\rho}\right)x\right]$$

$$= m\,[\rho^{m-2} + (m-2)\rho^{m-4}\,.\,x^2] \qquad (2)$$

By symmetry, we get

$$\frac{\partial^2 V}{\partial y^2} = m\,[\rho^{m-2} + (m-2)\,\rho^{m-4}\,.\,y^2] \qquad (3)$$

and $$\frac{\partial^2 V}{\partial z^2} = m\,[\rho^{m-2} + (m-2)\,\rho^{m-4}\,.\,z^2] \qquad (4)$$

Adding (2), (3) and (4), we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = m\,[3\rho^{m-2} + (m-2)\,(\rho^{m-4})\,(x^2+y^2+z^2)]$$

$$= m\,[3\rho^{m-2} + (m-2)\,\rho^{m-4}\,.\,\rho^2]$$

$$= m\,[3\rho^{m-2} + (m-2)\,\rho^{m-2}]$$

$$= m\,(m+1)\,\rho^{m-2}, \text{ as required.}$$

Exercise Set 9.2 (Page 414)

1. Approximate $\sqrt{299^2 + 399^2}$ by means of differentials.

Sol. Let $f(x,y) = \sqrt{x^2+y^2}$

$$f(300,400) = \sqrt{300^2 + 400^2} = 500$$

$$\Delta f \approx df = f_x\,dx + f_y\,dy$$

$$= \frac{x}{\sqrt{x^2+y^2}}\,dx + \frac{y}{\sqrt{x^2+y^2}}\,dy \qquad (1)$$

With $x = 300, y = 400, dx = \Delta x = -1$

and $dy = \Delta y = -1$, we have from (1)

$$df = \frac{300}{500}(-1) + \frac{400}{500}(-1) = -\frac{7}{5}$$

$$f(x+\Delta x, y+\Delta y) = f(x,y) + \Delta f$$

$$\approx f(x,y) + df$$

or $$f(299,399) = f(300,400) + df = 500 - \frac{7}{5} = 498\frac{3}{5} = 498.6$$

Hence $\sqrt{299^2 + 399^2} = 498.6$

2. If $\theta = \arctan\left(\frac{y}{x}\right)$, use differentials to find an approximate value of θ when $x = 0.95$ and $y = 1.05$.

Sol. $\theta(x,y) = \arctan\left(\frac{y}{x}\right) \Rightarrow \theta(1,1) = \frac{\pi}{4}$

With $x = 1, y = 1, dx = -0.05$ and $dy = 0.05$, we need to find

$$\theta(x+\Delta x, y+\Delta y) = \theta(0.95, 1.05) = \arctan\left(\frac{1.05}{0.95}\right)$$

Now, $\theta(x+\Delta x, y+\Delta y) = \theta(x,y) + \Delta\theta \approx \theta(x,y) + d\theta$

But $$d\theta = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

$$= \left(-\frac{1}{2}\right)(-0.05) + \frac{1}{2}(0.05) = 0.05$$

Therefore, $$\arctan\left(\frac{1.05}{0.95}\right) \approx \theta(1,1) + d\theta = \arctan\frac{1}{1} + d\theta$$

$$= \frac{\pi}{4} + 0.05 = 0.8354$$

3. If $u = \sqrt{x+2y}$ and x changes from 3 to 2.98 while y changes from 0.5 to 0.51, find an approximate value for the change in u.

Sol. We know that $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \qquad (1)$

Here $u = \sqrt{x+2y}$, $x = 3, y = 0.5$

$$\frac{\partial u}{\partial x} = \frac{1}{2}\frac{1}{\sqrt{x+2y}} = \frac{1}{2\sqrt{3+2(0.5)}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

and $$\frac{\partial u}{\partial y} = \frac{1}{2\sqrt{x+2y}}\,.\,2 = \frac{1}{\sqrt{x+2y}} = \frac{1}{\sqrt{3+2(0.5)}} = \frac{1}{2}$$

Also $dx = 2.98 - 3 = -0.02$ and $dy = 0.51 - 0.5 = 0.01$

Substituting these values into (1), we get

$$du = \frac{1}{4}(-0.02) + \frac{1}{2}(0.01) = 0.$$

Hence there is no change in u.

4. If $u = x^2 + y^2 + z^2 + xy^2z^3$ and x changes from 2 to 2.01, y changes from 1 to 1.02 and z changes from -1 to -0.99, find an approximate value for the change in u.

Sol. Here $dx = 2.01 - 2 = 0.01$, $dy = 1.02 - 1 = 0.02$

and $dz = -0.99 - (-1) = 0.01$

Also $$\frac{\partial u}{\partial x} = 2x + y^2z^3 = 3$$

$$\frac{\partial u}{\partial y} = 2y + 2xyz^3 = 2 + 2(2)(1)(-1)^3 = -2$$

and $$\frac{\partial u}{\partial z} = 2z + 3xy^2z^2 = -2 + 6 = 4 \qquad \text{at } x = 2, y = 1, z = -1.$$

Change in $u = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$

$$= 3(0.01) - 2(0.02) + 4(0.01) \doteq 0.03.$$

5. A rectangular plate expands in such a way that its length changes from 10 to 10.03 and its breadth changes from 8 to 8.02. Find an approximate value for the change in its area

Sol. Suppose that the length and breadth are x and y respectively.
Then $dx = 10.03 - 10 = 0.03$, $dy = 8.02 - 8 = 0.02$

Area $A = x \cdot y$

$$dA = \frac{\partial A}{\partial x}dx + \frac{\partial A}{\partial y}dy$$
$$= y\,dx + x\,dy = 8(0.03) + 10\,(0.02)$$
$$= 0.24 + 0.2 = 0.44 \text{ is the change in area}$$

6. The lateral surface of a cone is computed from the formula $S = \pi r\sqrt{r^2 + h^2}$, where r is the radius of the base and h is the height. If r is calculated as 6 with an accuracy of 1 % and h as 8 with an accuracy of 0.25%, with what accuracy will be the area S?

Sol. Here $r = 6, h = 8$

$$dr = 6 \times \frac{1}{100} = 0.06,\; dh = 8 \times \frac{25}{100 \times 100} = 0.02$$

Now as $S = \pi r\sqrt{r^2 + h^2}$,

$$dS = \frac{\partial S}{\partial r}dr + \frac{\partial S}{\partial h}dh$$
$$= \pi\left[\sqrt{r^2 + h^2} + \frac{r^2}{\sqrt{r^2 + h^2}}\right]dr + \pi r\frac{h}{\sqrt{r^2 + h^2}}dh$$
$$= \pi\left[10 + \frac{36}{10}\right][0.06] + \pi.\frac{6 \times 8}{10}0.02$$
$$= \pi[0.6 + 0.216] + \pi(0.096) = \pi[0.912] \qquad (1)$$

But $S = \pi r\sqrt{r^2 + h^2} = \pi\,.\,6\sqrt{36 + 64} = 60\pi$

The change (1) is for 60π. Therefore % change

$$= \frac{\pi(0.912)}{60\pi} \times 100 = 1.5\%.$$

7. The volume V of a rectangular parallelepiped having sides x, y and z is given by the formula $V = xyz$. If this solid is compressed from above so that z is decreased by 2% while x and y each is increased by 0.75% approximately, what percentage change will be in V?

Sol. We have $V = xyz$

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + zx\,dy + xy\,dz$$

Now $dx = \dfrac{75}{100 \times 100}x$, $dy = \dfrac{75}{100 \times 100}y$, $dz = \dfrac{-2}{100}z$

Change in volume

$$= dV = yz\left(\frac{3}{400}x\right) + zx\left(\frac{3}{400}y\right) + xy\left(\frac{-2}{100}z\right) = \frac{-xyz}{200}$$

Percentage change in volume

$$= \frac{dV}{V} \times 100 = \frac{-xyz}{200xyz} \times 100$$
$$= -\frac{1}{2} = -0.5\%, \text{ which is decrease in volume.}$$

8. A formula for the area Δ of a triangle is $\Delta = \frac{1}{2}ab\sin C$, where a, b are two adjacent sides and C is the angle included. Approximately what error is made in computing Δ if a is taken to be 9.1 instead of 9, b is taken to be 4.08 instead of 4 and C is taken to be 30° 3′ instead of 30°?

Sol. $\Delta = \frac{1}{2}ab\sin C$

$$d\Delta = \frac{\partial \Delta}{\partial a}da + \frac{\partial \Delta}{\partial b}db + \frac{\partial \Delta}{\partial C}dC$$

Here $da = 9.1 - 9 = 0.1$

$db = 4.08 - 4 = 0.08$

$$dC = 30^\circ 3' - 30^\circ = 3' = \left(\frac{3}{60}\right)^\circ = \frac{3}{60} \times \frac{\pi}{180} \text{ radians}$$

Hence $d\Delta = \frac{1}{2}(b\sin C)\,da + \frac{1}{2}a\sin C.\,db + \frac{1}{2}ab\cos C.\,dC$

$$= \frac{1}{2}(4)\left(\frac{1}{2}\right)(0.1) + \frac{1}{2}(9)\left(\frac{1}{2}\right)(0.08) + \frac{1}{2}(4)(9)\left(\frac{\sqrt{3}}{2}\right)\left(\frac{3\pi}{60 \times 180}\right)$$
$$= 0.1 + 0.18 + 0.015 = 0.293$$

% change in area $= \dfrac{0.293}{9} \times 100 = 3.25\%$.

9. The dimensions of a box are measured to be 10 in., 12 in. and 15 in. and the measurements are correct to 0.02 in. Find the maximum error if the volume is calculated from the given measurements. Also find the percentage error.

Sol. The volume V of the box with dimension x, y, z (in inches) is

$$V = xyz$$

We shall approximate the maximum error by means of differentials.

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = yz\,dx + zx\,dy + xy\,dz.$$

The maximum error in volume is obtained by taking

$dx = \Delta x = 0.02$, $dy = \Delta y = 0.02$ and $dz = \Delta z = 0.02$ with

$x = 10, y = 12$ and $z = 15$.

Therefore,

$$dV = (12)(15)(0.02) + (10)(15)(0.02) + (10)(12)(0.02) = 9$$

is the maximum error.

$$V = 10 \times 12 \times 15 = 1800 \text{ cu. in.}$$

Relative maximum error $= \dfrac{dV}{V} = \dfrac{9}{1800} = \dfrac{1}{200} = 0.005$

Percentage error $= 0.005 \times 100 = 0.5\%$.

10. Evaluate sine $29^o \cos 28^o \tan 44^o$ by using differentials.

Sol. Let $f(x, y, z) = \sin x \cos y \tan z$

Take $x = \dfrac{\pi}{6}$, $y = \dfrac{\pi}{6}$, $z = \dfrac{\pi}{4}$

$$dx = \frac{-\pi}{180},\ dy = \frac{-\pi}{90},\ dz = \frac{-\pi}{180}$$

With these values, we shall have

$$\begin{aligned}\sin 29^0 \cos 28^0 \tan 44^0 &= f(x + \Delta x, y + \Delta y, z + \Delta z)\\ &= f(x, y, z) + df \qquad (1)\end{aligned}$$

Now

$$df = \cos x \cos y \tan z\, dx - \sin x \sin y \tan z\, dy + \sin x \cos y \sec^2 z\, dz$$

$$= \left(\cos\frac{\pi}{6}\cos\frac{\pi}{6}\tan\frac{\pi}{4}\right)\left(-\frac{\pi}{180}\right) - \left(\sin\frac{\pi}{6}\sin\frac{\pi}{6}\tan\frac{\pi}{4}\right)\left(-\frac{\pi}{90}\right) + \left(\sin\frac{\pi}{6}\cos\frac{\pi}{6}\sec^2\frac{\pi}{4}\right)\left(-\frac{\pi}{180}\right)$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1\left(\frac{-\pi}{180}\right) - \frac{1}{2}\cdot\frac{1}{2}\cdot 1\left(\frac{-\pi}{90}\right) + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 2\left(-\frac{\pi}{180}\right)$$

$$= \frac{\pi}{180}\left[-\frac{3}{4} - \frac{1}{2} - \frac{\sqrt{3}}{2}\right] = 0.0175\,(-1.1160) = -0.01953.$$

$$f\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{4}\right) = \sin\frac{\pi}{6}\cos\frac{\pi}{6}\tan\frac{\pi}{4} = \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot 1 = \frac{\sqrt{3}}{4} = 0.4330.$$

Therefore, $f(29^o, 28^o, 44^o) = \sin 29^o \cos 28^o \tan 44^o$

$$= f\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{4}\right) + df$$

$$= 0.4330 - 0.0195 = 0.4135$$

By actual calculation, $\sin 29^o \cos 28^o \tan 44^o = 0.4132$.

Exercise Set 9.3 (Page 418)

1. If $u = x - y^2, x = 2r - 3s + 4, y = -r + 8s - 5$, find $\dfrac{\partial u}{\partial r}$ and $\dfrac{\partial u}{\partial s}$

Sol. We know that

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r} = (1)(2) + (-2y)(-1) = 2(1 + y)$$

Again, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial s} + \dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial s} = (1)(-3) + (-2y)(8) = -(3 + 16y)$.

2. If $z = \dfrac{\cos y}{x}, x = u^2 - v, y = e^v$, find $\dfrac{\partial z}{\partial u}$, $\dfrac{\partial z}{\partial v}$.

Sol. We have $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial u}$

$$= \frac{-\cos y}{x^2}(2u) + \frac{-\sin y}{x}\cdot 0 = \frac{-2u.\cos y}{x^2}$$

and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v}$

$$= \frac{-\cos y}{x^2}\cdot(-1) + \frac{-\sin y}{x}\cdot e^v = \frac{1}{x^2}[\cos y - x e^v \sin y]$$

$$= \frac{1}{x^2}[\cos y - xy \sin y], \quad \text{since } y = e^v.$$

Find $\dfrac{dy}{dx}$ (Problems 3 – 6):

3. $\sin xy - e^{xy} - x^2y = 0$

Sol. Here $f(x, y) = \sin xy - e^{xy} - x^2y = 0$

$$f_x = y\cos xy - ye^{xy} - 2xy$$

$$f_y = x\cos xy - xe^{xy} - x^2$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{(y\cos xy - ye^{xy} - 2xy)}{x\cos xy - xe^{xy} - x^2} = \frac{y(\cos xy - e^{xy} - 2x)}{x(x + e^{xy} - \cos xy)}$$

4. $3(x^2 + y^2)^2 = 25(x^2 - y^2)$

Sol. $f(x, y) = 3(x^2 + y^2)^2 - 25(x^2 - y^2) = 0$

$$f_x = 6(x^2 + y^2).2x - 50x$$

$$f_z = 6(x^2 + y^2).2y + 50y$$

$$\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{12x(x^2 + y^2) - 50x}{12y(x^2 + y^2) + 50y} = \frac{25x - 6x(x^2 + y^2)}{25y + 6y(x^2 + y^2)}$$

5. $f(x, y) = x^y - y^x = 0$

Sol. $f(x, y) = x^y - y^x$

$$f_x = yx^{y-1} - y^x \ln y$$
$$= yx^{y-1} - x^y \ln y = x^{y-1}(y - x \ln y)$$

and $f_y = x^y \ln x - xy^{x-1} = x^y \ln x - \frac{x}{y}.y^x = x^y \ln x - \frac{x}{y}x^y$

$$= x^y \frac{(y \ln x - x)}{y}$$

Hence $\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{x^{y-1}(y - x \ln y)}{x^y (y \ln x - x)} \cdot y$

$$= \frac{y(y - x \ln y)}{x(x - y \ln x)}.$$

6. $(\tan x)^y + y^{\cot x} = a$

Sol. We have $f(x, y) = (\tan x)^y + y^{\cot x} - a = 0$

$$f_x = y(\tan x)^{y-1}.\sec^2 x - y^{\cot x}.\ln y(\csc^2 x)$$

and $f_y = (\tan x)^y . \ln \tan x + (\cot x) y^{\cot x - 1}$

Hence $\frac{dy}{dx} = \frac{-f_x}{f_y} = -\frac{y \sec^2 x (\tan x)^{y-1} - \csc^2 x . y^{\cot x} \ln y}{(\tan x)^y \ln \tan x + (\cot x) y^{\cot x - 1}}$

7. If $F(x,y,z) = 0$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$

Sol. We know that if $f(x, y) = 0$, then

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \qquad (1)$$

Now in $F(x, y, z) = 0$ we may regard z as a function of x and y. In order to find $\frac{\partial z}{\partial x}$, we treat y as constant and use (1)

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} \qquad (2)$$

Similarly, we get $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$

Here $\frac{dz}{dx}$ and $\frac{dz}{dy}$ are partial derivatives because z is a function of two variables x and y.

8. If $f(x, y, z) = 0$ and $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y} \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

Sol. From $f(x, y) = 0$, we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

From $\phi(y, z) = 0$, we get

$$\frac{dz}{dy} = \frac{-\phi_y}{\phi_z} \qquad (2)$$

$$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx} = \frac{\frac{\partial \phi}{\partial y} \cdot \frac{\partial f}{\partial x}}{\frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y}}$$

Cross multiplying, we obtain

$$\frac{\partial \phi}{\partial z} \cdot \frac{\partial f}{\partial y} \cdot \frac{dz}{dx} = \frac{\partial \phi}{\partial y} \frac{\partial f}{\partial x} \quad \text{or} \quad \frac{\partial f}{\partial y} \cdot \frac{\partial \phi}{\partial z} \cdot \frac{dz}{dx} = \frac{\partial f}{\partial x} \cdot \frac{\partial \phi}{\partial y}$$

9. If $x\sqrt{1-y^2} + y\sqrt{1-x^2} = a$, show that $\frac{d^2y}{dx^2} = \frac{-a}{(1-x^2)^{3/2}}.$

Sol. We have $f(x, y) = x\sqrt{1-y^2} + y\sqrt{1-x^2} - a$

$$f_x = \sqrt{1-y^2} - \frac{1.y.2x}{2\sqrt{1-x^2}} = \frac{\sqrt{1-y^2}\sqrt{1-x^2} - xy}{\sqrt{1-x^2}}$$

$$f_y = \frac{-x}{2\sqrt{1-y^2}}.2y + \sqrt{1-x^2} = \frac{-xy + \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-y^2}}$$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = -\frac{\sqrt{1-x^2}\left(\frac{-1}{2\sqrt{1-y^2}} 2y\right)\frac{dy}{dx} - \sqrt{1-y^2} \times \left(-\frac{1}{2}\frac{2x}{\sqrt{1-x^2}}\right)}{1-x^2}$$

$$= \frac{y\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\frac{dy}{dx} - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.x}{(1-x^2)} = \frac{-y\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\left(\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}\right) - \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}.x}{(1-x^2)}$$

$$= -\frac{(y\sqrt{1-x^2} + x\sqrt{1-y^2})}{(1-x^2)^{3/2}} = \frac{-a}{(1-x^2)^{3/2}}.$$

10. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, prove that

$$\frac{d^2y}{dx^2} = \frac{abc + 2fgh - af^2 - bg^2 - ch^2}{(hx + by + f)^2}$$

Sol. Here $f_x = 2ax + 2hy + 2g$

and $f_y = 2hx + 2by + 2f$

$$\frac{dy}{dx} = -\frac{ax + hy + g}{hx + by + f} \qquad (1)$$

Differentiating (1) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -\frac{(hx + by + f)\left(a + h\frac{dy}{dx}\right) - (ax + hy + g)\left(h + b\frac{dy}{dx}\right)}{(hx + by + f)^2}$$

$$= -\frac{dy}{dx}(h^2x + hby + hf - abx - hby - gb)/(hx + by + f)^2$$

$$\frac{-(ahx + aby + af - ahx - h^2y - hg)}{(hx + by + f)^2}$$

$$= \frac{\frac{ax + hy + g}{hx + by + f}(h^2x + hf - abx - gb) - (aby + af - h^2y - hg)}{(hx + by + f)^2}$$

$$= \frac{(ax+hy+g)(h^2x+hf-abx-gb) - (hx+by+f)(aby+af-h^2y-hg)}{(hx+by+f)^3}$$

$$= \frac{h^2(ax^2+2hxy+by^2+2gx+2fy)-ab(ax^2+2hxy+by^2+2gx+2fy-af^2-bg^2+2fgh)}{(hx+by+f)^3}$$

$$= \frac{h^2(-c)-ab(-c)-af^2-bg^2+2fgh}{(hx+by+f)^3} = \frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^2}$$

11. Find $\frac{d^2y}{dx^2}$ if $x^3 + y^3 = 3axy$

Sol. Here $f(x, y) = x^3 + y^3 - 3axy = 0$

$f_x = 3x^2 - 3ay$ and $f_y = 3y^2 - 3ax$

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = \frac{ay - x^2}{y^2 - ax} \qquad (1)$$

Differentiating (1) w.r.t. x, we have

$$\frac{d^2y}{dx^2} = \frac{(y^2 - ax)\left(a\frac{dy}{dx} - 2x\right) - (ay - x^2)\left(2y\frac{dy}{dx} - a\right)}{(y^2 - ax)^2}$$

$$= \frac{\frac{dy}{dx}(ay^2 - a^2x - 2ay^2 + 2x^2y) - (2xy^2 - 2ax^2 - a^2y + ax^2)}{(y^2 - ax)^2}$$

$$= \frac{\frac{ay - x^2}{y^2 - ax}(2x^2y - ay^2 - a^2x) - (2xy^2 - ax^2 - a^2y)}{(y^2 - ax)^2}$$

$$= 2ax^2y^2 - a^2y^3 - a^3xy - 2x^4y + ax^2y^2 + a^2x^3/(y^2 - ax)^3$$

$$+ \frac{-(2xy^4 - ax^2y^2 - a^2y^3 - 2ax^2y^2 + a^2x^3 + a^3xy)}{(y^2 - ax)^3}$$

$$= \frac{-(2xy^4 - ax^2y^2 - a^2y^3 - 2ax^2y^2 + a^2x^3 + a^3xy)}{(y^2 - ax)^3}$$

$$+ \frac{3a^2x^2 - a^2y^3 - a^3xy - 2x^4y + a^2x^3}{(y^2 - ax)^3}$$

$$= \frac{6ax^2y^2 - 2a^3xy - 2xy(x^3 + y^3)}{(y^2 - ax)^3}$$

$$= \frac{6ax^2y^2 - 2a^3xy - 2xy(3axy)}{(y^2 - ax)^3}$$

$$= \frac{-2a^3xy}{(y^2 - ax)^3} = \frac{2a^3xy}{(ax - y^2)^3}$$

Exercise Set 9.4 (Page 420)

In Problems 1 – 3, find the rate of change of u at the given point and in the given direction.

1. $u = 2xy - \frac{y}{x}$; (1, 2): [2, – 3, 0]

Sol. We have

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2y + \frac{y}{x^2} = 6\\ \frac{\partial u}{\partial y} &= 2x - \frac{1}{x} = 1\\ \frac{\partial u}{\partial z} &= 0\end{aligned}\right\} \text{ at } (1, 2)$$

grad $u = [6, 1, 0]$

Also $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} = [2, -3, 0]$

$$\frac{du}{ds} = \text{Rate of change of } u$$

$$= \frac{\mathbf{v}.\,\mathbf{grad}\,u}{|\mathbf{v}|} = \frac{(2)(6) + (-3)(1) + 0}{\sqrt{4 + 9}} = \frac{9}{\sqrt{13}}.$$

2. $u = ye^{-x}(x^2 + y^2 + z^2 + 1)$; (0; 0; 0); [2, 1, 2]

Sol. Here

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= ye^{-x}(2x - x^2 + y^2 + z^2 + 1) = 0\\ \frac{\partial u}{\partial y} &= e^{-x}(x^2 + z^2 + 1 + 3y^2) = 1\\ \frac{\partial u}{\partial z} &= ye^{-x}(2z) = 0\end{aligned}\right\} \text{ at } (0, 0, 0)$$

grad $u = [0, 1, 0]$

Also $\mathbf{v} = [2, 1, 2]$

$$\frac{du}{ds} = \frac{\mathbf{v}.\,\mathbf{grad}\, u}{|\mathbf{v}|} = \frac{2.0 + 1.1 + 2.0}{\sqrt{4+1+4}} = \frac{1}{\sqrt{9}} = \frac{1}{3}.$$

3. $u = \sinh(x+y) + \cosh z;\ (1;\ 0;\ 1),\qquad [-2, 2, -1]$

Sol.

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= \cosh(x+y) = \cosh 1\\ \frac{\partial u}{\partial y} &= \cosh(x+y) = \cosh 1\\ \frac{\partial u}{\partial z} &= \sinh z = \sinh 1\end{aligned}\right\}\quad \text{at}\quad (1, 0, 1)$$

$\mathbf{grad}\, u = [\cosh 1, \cosh 1, \sinh 1]$

$\mathbf{v} = [-2, 2, -1]$

$$\frac{du}{ds} = \frac{\mathbf{v}.\mathbf{grad}\, u}{|\mathbf{v}|} = \frac{-2\cosh 1 + 2\cosh 1 - \sinh 1}{3}$$

$$= \frac{-\left(\dfrac{e - e^{-1}}{2}\right)}{3} = \frac{-(e^2-1)}{6e} = \frac{1-e^2}{6e}.$$

4. Let $u = x^2 + y^2$. Find the direction of the greatest rate of change of u at (a, b) and the magnitude of this greatest rate of change. Find the direction of no change at (a, b).

Sol. We know that the directional derivative is greatest in the direction of the gradient itself and the magnitude of this greatest directional derivative is the magnitude of the gradient vector. Therefore,

$$\left.\begin{aligned}\frac{\partial u}{\partial x} &= 2x = 2a\\ \frac{\partial u}{\partial y} &= 2y = 2b\end{aligned}\right\}\quad \text{at } (a, b)$$

$$\frac{\partial u}{\partial z} = 0$$

Hence $\mathbf{grad}\, u = [2a, 2b, 0] = 2a\mathbf{i} + 2b\mathbf{j}$ is the direction of greatest rate of change.

Also its magnitude $= \sqrt{4a^2 + 4b^2} = 2\sqrt{a^2+b^2}$.

Again, the directions of no change are the directions perpendicular to $\mathbf{grad}\, u$.

If $[s, t]$ is perpendicular to $\mathbf{grad}\, u$, the $2as + 2bt = 0$

$$\Rightarrow \quad \frac{s}{t} = -\frac{b}{a} \quad \text{or} \quad s = -kb,\ t = ka,$$

$[s, t] = [-kb, ka]$ or $[-b, a]$ is the direction of no change at (a, b), which can also be written as $-b\mathbf{i} + a\mathbf{j}$.

5. If $u = \arctan\left(\dfrac{y}{x}\right)$, find the direction of the greatest rate of change of u at (a, b) and the magnitude of the greatest rate of change. Find also the direction of no change at (a, b).

Sol. We have

$$\frac{\partial u}{\partial x} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\left(\frac{-y}{x^2}\right) = -\frac{y}{x^2+y^2} = \frac{-b}{a^2+b^2}$$

$$\frac{\partial u}{\partial y} = \frac{1}{1+\dfrac{y^2}{x^2}}\cdot\frac{1}{x} = \frac{x}{x^2+y^2} = \frac{a}{a^2+b^2}$$

$\mathbf{grad}\, u =$ Direction of greatest rate of change

$$= \left[\frac{-b}{a^2+b^2}, \frac{a}{a^2+b^2}, 0\right] = \frac{-b}{a^2+b^2}\mathbf{i} + \frac{a}{a^2+b^2}\mathbf{j}$$

Magnitude of greatest rate of change

$$= |\mathbf{grad}\, u| = \sqrt{\frac{a^2+b^2}{(a^2+b^2)^2}} = \frac{1}{\sqrt{a^2+b^2}}.$$

Direction of no change is just a vector perpendicular to $\mathbf{grad}\, u$ which is

$\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, since $\mathbf{v}$ is perpendicular to $\mathbf{grad}\, u$

6. The temperature distribution for the semi-circular plate $x^2 + y^2 \le 1$, $y \ge 0$, is given by the formula $T = 3x^2y - y^3 + 27$ under certain conditions. Find $\dfrac{dT}{ds}$ at $A\left(0, \dfrac{1}{2}\right)$ in the direction of y-axis. Also find (i) $\dfrac{dT}{ds}$ at A in the direction of $[1, -2]$, (ii) the direction of greatest rate of change at A, (iii) the magnitude of the greatest rate of change (iv) the direction of the isothermal through A (the direction of zero rate of change at A).

Sol.

$$\left.\begin{aligned}\frac{dT}{dx} &= 6xy = 0\\ \frac{\partial T}{\partial y} &= 3x^2 - 3y^2 = \frac{-3}{4}\end{aligned}\right\}\quad \text{at}\left(0, \frac{1}{2}\right)$$

$$\mathbf{grad}\, T = \left[0, \frac{-3}{4}\right]$$

Also unit vector in the direction of y-axis is $[0, 1]$.

$$\frac{dT}{ds} = \left[0, \frac{-3}{4}\right].[0, 1] = \frac{-3}{4}$$

Again $\mathbf{v} = [1, -2]$, $|\mathbf{v}| = \sqrt{5}$

Hence (i) $\frac{dT}{ds}$ in the direction of **v**

$$= \frac{\textbf{grad } T.\mathbf{v}}{|\mathbf{v}|} = \frac{\left[0, \frac{-3}{4}\right][1,-2]}{\sqrt{5}} = \frac{3}{2\sqrt{5}}.$$

(ii) The direction of greatest rate of change

$$= \textbf{grad } T = \left[0, \frac{-3}{4}\right].$$

(iii) Magnitude of greatest rate of change

$$= |\textbf{grad } T| = \sqrt{(0)^2 + \left(\frac{-3}{4}\right)^2} = \frac{3}{4}.$$

(iv) Direction of the isothermal through A

= Direction perpendicular to **grad** T

$= [1,0] = i$, since $= [1,0].\left[0, \frac{-3}{4}\right] = 0.$

7. Repeat **Problem 6** with $T = 4x^3y - 4xy^3 + 27z$ and $A\left(\frac{1}{2}, \frac{1}{2}\right)$

Sol. Here $\frac{\partial T}{\partial x} = 12x^2y - 4y^3 = 12\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) - 4\left(\frac{1}{2}\right)^3$

$$= \frac{3}{2} - \frac{1}{2} = 1 \quad \text{at } A\left(\frac{1}{2}, \frac{1}{2}\right)$$

$$\frac{\partial T}{\partial y} = 4x^3 - 12xy^2 = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2 = \frac{1}{2} - \frac{3}{2} = -1.$$

grad $T = [1, -1]$.

As the unit vector in the direction of y – axis is $[0, 1]$, we have

$$\frac{dT}{ds} = [1,-1].[0,1] = -1$$

Now $\mathbf{v} = [1,-2]$, $|\mathbf{v}| = \sqrt{5}$

(i) $\frac{dT}{ds}$ in the direction of **v**

$$= \frac{\textbf{grad } T.\mathbf{v}}{\mathbf{v}} = \frac{[1,-1].[1,-2]}{\sqrt{5}} = \frac{1+2}{\sqrt{5}} = \frac{3}{\sqrt{5}}.$$

(ii) The direction of greatest rate of change

$= \textbf{grad } T = [1,-1] = \mathbf{i} - \mathbf{j}.$

(iii) Magnitude of greatest rate of change

$= |\textbf{grad } T| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2}.$

(iv) Direction of no rate of change at A

= Direction perpendicular to **grad** T

$= [1, 1] = \mathbf{i} + \mathbf{j}$, since $(\mathbf{i} - \mathbf{j}).(\mathbf{i} + \mathbf{j}) = 0.$

Exercise Set 9.5 (Page 424)

In Problems 1 – 10 find equations for the tangent plane and the normal line to the given surface at the indicated point P:

1. $4x^2 - y^2 + 3z^2 = 10, P(2, -3, 1)$

Sol. $f(x, y, z) = 4x^2 - y^2 + 3z^2 - 10$

grad $f = 8x\mathbf{i} - 2y\mathbf{j} + 6z\mathbf{k}$

grad $f|_{P(2,-3,1)} = 16\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$ is a normal vector at P

which is a normal vector of the tangent plane to the surface at $P(2, -3, 1)$. Equation of the tangent plane is

$16(x-2) + 6(y+3) + 6(z-1) = 0$

i.e., $8x + 3y + 3z - 10 = 0$

Equations of the normal line to the surface through $P(2,-3,1)$ are

$x = 2 + 16t, y = -3 + 6t, z = 1 + 6t$

or $\frac{x-2}{8} = \frac{y+3}{3} = \frac{z-1}{3}.$

2. $x^2 + y^2 + z^2 = 14, P(1, -2, 3)$

Sol. $f(x, y, z) = x^2 + y^2 + z^2 - 14$

grad $f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

grad $f|_P = 2\mathbf{i} - 4\mathbf{j} + 6\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at $P(1, -2, 3)$ is

$2(x-1) - 4(y+2) + 6(z-3) = 0$

i.e., $x - 2y + 3z - 14 = 0.$

Equations of the normal line to the surface through P are

$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-3}{3}.$

3. $9x^2 + 4y^2 - z^2 = 36, P(2, 3, 6)$

Sol. $f(x, y, z) = 9x^2 + 4y^2 - z^2 - 36$

grad $f = 18x\mathbf{i} + 8y\mathbf{j} - 2z\mathbf{k}$

grad $f|_P = 36\mathbf{i} + 24\mathbf{j} - 12\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$36(x-2) + 24(y-3) - 12(z-6) = 0$

or $3x + 2y - z - 6 = 0$

Equations of the normal line to the surface through P are

$\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-6}{-1}.$

4. $x^2 - 2y^2 - z^2 = 4, P(-6, 2, \sqrt{24})$

Sol $f(x, y, z) = x^2 - 2y^2 - z^2 - 4$

$\mathbf{grad}\, f = 2x\mathbf{i} - 4y\mathbf{j} - 2z\mathbf{k}$

$\mathbf{grad}\, f\,|_P = -12\mathbf{i} - 8\mathbf{j} - 2\sqrt{24}\,\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$-12(x+6) - 8(y-2) - 2\sqrt{24}\,(z - \sqrt{24})$$

or $\quad 3x + 2y + \sqrt{6}\,z + 2 = 0$

Equations of the normal line to the surface through $P(-6, 2, \sqrt{24})$ are

$$\frac{x+6}{3} = \frac{y-2}{2} = \frac{z-\sqrt{24}}{\sqrt{6}}.$$

5. $z = x^2 + y^2,\ P(-2, 1, 5)$

Sol. $f(x, y, z) = x^2 + y^2 - z$

$\mathbf{grad}\, f = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$

$\mathbf{grad}\, f\,|_P = -4\mathbf{j} + 2\mathbf{i} - \mathbf{k}$ is normal vector at P.

Equation of the tangent plane to the surface at P is

$$-4(x+2) + 2(y-1) - (z-5) = 0$$

or $\quad 4x - 2y + z - 5 = 0$

Equations of the normal line to the surface through P are

$$\frac{x+2}{4} = \frac{y-1}{-2} = \frac{z-5}{1}.$$

6. $xz = 4,\ P(-2, 2, -2)$

Sol. $f(x, y, z) = xz - 4$

$\mathbf{grad}\, f = z\mathbf{i} + 0\mathbf{j} + x\mathbf{k}$

$\mathbf{grad}\, f\,|_P = -2\mathbf{i} + 0\mathbf{j} - 2\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$-2(x+2) + 0(y-2) - 2(z+2) = 0$$

or $\quad x + z + 4 = 0$

Equations of the normal line to the surface through P are

$$\frac{x+2}{1} = \frac{y-2}{0} = \frac{z+2}{1}.$$

7. $x^2 + z^2 = \dfrac{a^2}{h^2}y^2,\ P\left(\dfrac{a}{\sqrt{2}}, h, \dfrac{a}{\sqrt{2}}\right)$

Sol. $f(x, y, z) = x^2 + z^2 - \dfrac{a^2}{h^2}y^2$

$\mathbf{grad}\, f = 2x\mathbf{i} - 2\dfrac{a^2}{h^2}y\mathbf{j} + 2z\mathbf{k}$

$\mathbf{grad}\, f\,|_P = \sqrt{2}\,a\mathbf{i} - 2\dfrac{a^2}{h}\mathbf{j} + \sqrt{2}\,a\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$\sqrt{2}\,a\left(x - \frac{a}{\sqrt{2}}\right) - \frac{2a^2}{h}(y-h) + \sqrt{2}\,a\left(z - \frac{a}{\sqrt{2}}\right) = 0$$

or $\quad \sqrt{2}\,ax - \dfrac{2a^2}{h}y + \sqrt{2}\,az = 0$

or $\quad \sqrt{2}\,h(x+z) - 2ay = 0$

Equations of the normal line to the surface through P are

$$\frac{x - \frac{a}{\sqrt{2}}}{\sqrt{2}\,h} = \frac{y-h}{2a} = \frac{z - \frac{a}{\sqrt{2}}}{\sqrt{2}\,h}.$$

8. $z = e^x \cos y,\ P\left(0, \dfrac{\pi}{2}, 0\right)$

Sol. $f(x, y, z) = e^x \cos y - z$

$\mathbf{grad}\, f = e^x \cos y\,\mathbf{i} - e^x \sin y\,\mathbf{j} - \mathbf{k}$

$\mathbf{grad}\, f\,|_P = 0\mathbf{i} - \mathbf{j} - \mathbf{k}$ is a normal vector at P

Equation of the tangent plane to the surface at P is

$$-1\left(y - \frac{\pi}{2}\right) - z = 0 \quad \text{or} \quad y + z - \frac{\pi}{2} = 0$$

Equations of the normal line to the surface through P are

$$\frac{x}{0} = \frac{y - \frac{\pi}{2}}{1} = \frac{z}{1}.$$

9. $x = \ln\left(\dfrac{y}{2z}\right),\ P(0, 2, 1)$

Sol. $f(x, y, z) = x - \ln\left(\dfrac{y}{2z}\right) = x - \ln y + \ln z + \ln 2$

$\mathbf{grad}\, f = \mathbf{i} - \dfrac{1}{y}\mathbf{j} + \dfrac{1}{z}\mathbf{k}$

$\mathbf{grad}\, f\,|_P = \mathbf{i} - \dfrac{1}{2}\mathbf{j} + \mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$x - \frac{1}{2}(y-2) + z - 1 = 0$$

or $\quad 2x - y + 2z = 0$

Equations of the normal line to the surface through P are

$$\frac{x}{2} = \frac{y-2}{-1} = \frac{z-1}{2}.$$

10. $x^{2/3} + y^{2/3} + z^{2/3} = 9,\quad P(1, 8, -8)$

Sol. $f(x, y, z) = x^{2/3} + y^{2/3} + z^{2/3} - 9$

$$\textbf{grad } f = \frac{2}{3}x^{-1/3}\mathbf{i} + \frac{2}{3}y^{-1/3}\mathbf{j} + \frac{2}{3}z^{-1/3}\mathbf{k}$$

$\textbf{grad } f\,|_P = \frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{1}{3}\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane to the surface at P is

$$\frac{2}{3}(x-1) + \frac{1}{3}(y-8) - \frac{1}{3}(z+8) = 0$$

or $\quad 2x + y - z - 18 = 0$

Equations of the normal line to the surface through P are

$$\frac{x-1}{2} = \frac{y-8}{1} = \frac{z+8}{-1}.$$

11. Find the point on $x^2 - 2y^2 - 4z^2 = 16$ at which the tangent plan parallel to the plane $4x - 2y + 4z = 5$

Sol. Let $P(x_1, y_1, z_1)$ be the required point on the given surface

$f(x, y, z) = x^2 - 2y^2 - 4z^2 - 16 = 0$

$\textbf{grad } f\,|_P = 2x_1\,\mathbf{i} - 4y_1\,\mathbf{j} - 8z_1\,\mathbf{k}$ is a normal vector at P.

Equation of the tangent plane at P is

$$2x_1(x - x_1) - 4y_1(y - y_1) - 8z_1(z - z_1) = 0$$

or $\quad xx_1 - 2yy_1 - 4zz_1 - 16 = 0$

This plane is to be parallel to

$$4x - 2y + 4z - 5 = 0$$

Hence $\frac{x_1}{4} = \frac{-2y_1}{-2} = \frac{-4z_1}{4} = k$ (say)

Therefore, $x_1 = 4k, y_1 = k, z_1 = -k$

Since (x_1, y_1, z_1) lies on the surface, we have

$$16k^2 - 2k^2 - 4k^2 = 16$$

or $\quad k = \pm\frac{4}{\sqrt{10}} = \pm\frac{2\sqrt{2}}{\sqrt{5}}$

Thus $P\left(\frac{8\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}, \frac{-2\sqrt{2}}{\sqrt{5}}\right)$ and $Q\left(\frac{-8\sqrt{2}}{\sqrt{5}}, \frac{-2\sqrt{2}}{\sqrt{5}}, \frac{2\sqrt{2}}{\sqrt{5}}\right)$

are the required points.

12. Two surfaces are said to be **tangent** at a common point P if each has the same tangent plane at P Show that the surfaces $x^2 + z^2 + 4y = 0$ and $x^2 + y^2 + z^2 - 6z + 7 = 0$ are tangent at $P(0, -1, 2)$.

Sol. $f(x, y, z) = x^2 + z^2 + 4y$

$\textbf{grad } f = 2x\mathbf{i} + 2z\mathbf{k} + 4\mathbf{j}$

$\textbf{grad } f\,|_P = 0\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ is a normal vector at P.

Tangent plane to $f(x, y, z) = 0$ at $P(0, -1, 2)$ is

$$4(y + 1) + 4(z - 2) = 0$$

or $\quad y + z - 1 = 0 \qquad (1)$

$g(x, y, z) = x^2 + y^2 + z^2 - 6z + 7$

$\textbf{grad } g = 2x\mathbf{i} + 2y\mathbf{j} + (2z - 6)\mathbf{k}$

$\textbf{grad } g\,|_P = -2\mathbf{j} - 2\mathbf{k}$ is a normal vector at P.

Tangent plane to $g(x, y, z) = 0$ at P is $-2(y + 1) - 2(z - 2) = 0$ or $y + z - 1 = 0$ which is the same as (1). Hence the two surfaces are tangent at P

13. Show that the sphere $x^2 + y^2 + z^2 = 18$ and the cone $x^2 + z^2 = (y - 6)^2$ are tangent along their intersection.

Sol. We find the points of intersection of the two surfaces

$$x^2 + y^2 + z^2 = 18 \quad \text{and} \quad x^2 + z^2 = (y - 6)^2$$

Subtracting, we get

$$y^2 = 18 - (y - 6)^2$$

This gives $y = 3$

If $\quad f(x, y, z) = x^2 + y^2 + z^2 - 18$

then $\textbf{grad } f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$ which shows that $2x_1, 2y_1, 2z_1$ are the direction ratios of the normal to $f(x, y, z) = 0$ at (x_1, y_1, z_1).

But $y_1 = 3$. Therefore direction ratios are $2x_1, 6, 2z_1$,

Again, if $g(x, y, z) = x^2 - (y - 6)^2 + z^2 = 0$

then $\textbf{grad } g = 2x\mathbf{i} - 2(y - 6)\mathbf{j} + 2z\mathbf{k}$

$\textbf{grad } g\,(x_1, y_1, z_1) = 2x_1\mathbf{i} - 2(y_1 - 6)\mathbf{j} + 2z_1\mathbf{k}$ is a normal vector at P.

$2x_1, -2(y_1 - 6), 2z_1$ are the direction ratios of the normal to $g(x, y, z)$ at (x_1, y_1, z_1) But $y_1 = 3$. Therefore, $2x_1, 6, 2z_1$ are the direction ratios of the normal in both the cases which shows that the given sphere and the cone are tangent along their intersection.

14. Show that the surfaces $z = 16 - x^2 - y^2$ and $63z = x^2 + y^2$ intersect orthogonally.

Sol. $f(x, y, z) = x^2 + y^2 + z - 16 = 0 \qquad (1)$

$f_x = 2x, f_y = 2y, f_z = 1$

$g(x, y, z) = x^2 + y^2 - 63z = 0 \qquad (2)$

$g_x = 2x, g_y = 2y, g_z = -63$

Subtracting (2) from (1), we get $z = \frac{1}{4}$.

Eliminating z from (1) and (2), we have $x^2 + y^2 = \frac{63}{4}$

The two surfaces intersect along the curve

$$z = \frac{1}{4},\ x^2 + y^2 = \frac{63}{4}$$

Now, $f_x\, g_x + f_y\, g_y + f_z g_z = 4\,(x^2 + y^2) - 63 = 4\left(\frac{63}{4}\right) - 63 = 0$

Hence the surfaces intersect orthogonally.

15. Prove that all normal lines of the sphere $x^2 + y^2 + z^2 = a^2$ pass through the centre of the sphere.

Sol. Let $P\,(x_1, y_1, z_1)$ be any point on the sphere

$$f\,(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$$

grad $f = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$

grad $f\,|_P = 2x_1\mathbf{i} + 2y_1\mathbf{j} + 2z_1\mathbf{k}$ is a normal vector at P.

Equations of the normal line through P are

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{z_1} \quad \text{or} \quad xy_1 - yx_1 = 0 = yz_1 - zy_1$$

which pass through (0, 0, 0) – the centre of the sphere.

Since (x_1, y_1, z_1) is any point on the sphere, all normal lines pass through the centre of the sphere.

16. Show that the ellipsoid $\frac{x^2}{12} + \frac{y^2}{16} + \frac{z^2}{12} = 1$ and the hyperbolid $\frac{y^2}{3} - x^2 - z^2 = 1$ intersect orthogonally.

Sol. Let $f\,(x, y, z) = \frac{y^2}{16} + \frac{x^2}{12} + \frac{z^2}{12} - 1 = 0$ (1)

and $g\,(x, y, z) = \frac{y^2}{3} - (x^2 + z^2) - 1 = 0$ (2)

$$f_x = \frac{x}{6},\ f_y = \frac{y}{8},\ f_z = \frac{z}{6}$$

$$g_x = -2x,\ g_y = \frac{2y}{3},\ g_z = -2z$$

$$f_x g_x + f_y g_y + f_z g_z = -\frac{x^2}{3} + \frac{y^2}{12} - \frac{z^2}{3} = \frac{y^2}{12} - \frac{1}{3}(x^2 + z^2).$$

The two surfaces intersect at the points where (from (1) and (2))

$$\frac{y^2}{16} + \frac{1}{12}\left(\frac{y^2}{3} - 1\right) - 1 = 0$$

or $13y^2 = 156$ or $y = \pm 2\sqrt{3}$

At $y = \pm 2\sqrt{3}$, $x^2 + z^2 = 3$, from (2)

Hence at all common points of the two surfaces, we have

$$f_x\, g_x + f_y\, g_y + f_z\, g_z = \frac{y^2}{12} - \frac{1}{3}(x^2 + z^2) = 1 - 1 = 0$$

Hence the two surfaces intersect orthogonally.

17. Find the point on $z = 4x^2 + 9y^2$ at which the normal line is parallel to the line through A (–2, 4, 3) and B(5, –1, 2)

Sol. Let $P(x_1, y_1, z_1)$ be the required point.

grad $f\,|\,P = 8x_1\mathbf{i} + 18y_1\mathbf{j} - \mathbf{k}$ is a normal vector at P

Direction ratios of the normal are $8x_1$, $18y_1$, – 1.

Direction ratios of the line AB are 7, – 5, – 1.

Since the normal line is parallel to AB, we have

$$\frac{8x_1}{7} = \frac{18y_1}{-5} = \frac{-1}{-1}$$

Therefore, $x_1 = \frac{7}{8}, y_1 = \frac{-5}{18}$ and $z_1 = \frac{37}{18}$

The required point is $\left(\frac{7}{8}, \frac{-5}{18}, \frac{37}{18}\right)$.

18. Where and at what angle do the cone $x^2 + y^2 = \frac{1}{2}z^2$ and the cylinder $x^2 + y^2 = 4$ intersect?

Sol. For the point of intersection we solve $x^2 + y^2 = \frac{1}{2}z^2$ and $x^2 + y^2 = 4$ simultaneously. We get $z^2 = 8$ or $z = \pm 2\sqrt{2}$. Therefore, intersection of the cone and the cylinder is the circle $x^2 + y^2 = 4$ in the planes $z = 2\sqrt{2}$ and $z = -2\sqrt{2}$.

Now for the cone, we have $f\,(x, y, z) = x^2 + y^2 - \frac{1}{2}z^2 = 0$

grad $f = 2x\mathbf{i} + 2y\mathbf{j} - z\mathbf{k}$ is a normal vector at P

This shows that $2x$, $2y$, $-z$ are the direction ratios of the normal to the tangent plane of the cone at (x, y, z).

For the cylinder, we have

$$g(x, y, z) = x^2 + y^2 - 4 = 0$$

grad $g = 2x\mathbf{i} + 2y\mathbf{j} + 0\mathbf{k}$ is a normal vector at P.

Angle between the normals at the common point is the required angle. Therefore,

$$\cos\theta = \frac{(2x)\,(2x) + (2y)\,(2y) + 0}{\sqrt{4x^2 + 4y^2 + z^2}\,\sqrt{4x^2 + 4y^2 + 0}}$$

$$= \frac{4\,(x^2 + y^2)}{\sqrt{4(x^2 + y^2) + z^2}\,\sqrt{4\,(x^2 + y^2)}}$$

$$= \frac{4\times 4}{\sqrt{4\times 4+8}}\,\frac{1}{\sqrt{4\times 4}},\ \text{as } x^2+y^2=4 \text{ and } z^2=8$$

$$= \frac{16}{4\sqrt{24}} = \frac{4}{2\sqrt{6}} = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

or $\theta = \arccos\sqrt{\frac{2}{3}}$ is the required angle at which the cone and the cylinder intersect.

19. For the surface defined by the parametric equations $x = 2\cosh u \cos v$, $y = 3\cosh u \sin v$, $z = 6\sinh u$, find a vector normal to the surface at the point for which $u = 1$, $v = \frac{\pi}{3}$

Sol. We have

$$\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

$$= (2\cosh u\cos v)\mathbf{i} + (3\cosh u\sin v)\mathbf{j} + (6\sinh u)\mathbf{k}$$

$$\frac{\partial \boldsymbol{r}}{\partial u} = (2\sinh u\cos v)\,\mathbf{i} + (3\sinh u\sin v)\,\mathbf{j} + (6\cosh u)\,\mathbf{k}$$

$$= \left(\frac{1}{2}\times 2\sinh u\right)\mathbf{i} + \left(\frac{\sqrt{3}}{2}\,3\sinh u\right)\mathbf{j} + (6\cosh u)\,\mathbf{k} \ \text{ at } v = \frac{\pi}{3}$$

$$= \sinh u\,\boldsymbol{i} + \frac{3\sqrt{3}}{2}\sinh u\,\mathbf{j} + 6\cosh u\,\mathbf{k}$$

$$\frac{\partial \boldsymbol{r}}{\partial v} = (-2\cosh u.\sin v)\,\mathbf{i} + (3\cosh u\cos v)\,\mathbf{j} + 0\,\mathbf{k}$$

$$= \left(\frac{\sqrt{3}}{2}\times -2\cosh u\right) + \left(\frac{1}{2}\cdot 3\cosh u\right)\mathbf{j} + 0\,\mathbf{k} \ \text{ at } v = \frac{\pi}{3}$$

$$= -\sqrt{3}\cosh u\,\boldsymbol{i} + \frac{3}{2}\cosh u\,\mathbf{j} + 0\mathbf{k}$$

The required vector (normal to the surface) is

$$\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\left|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sinh u & \frac{3\sqrt{3}}{2}\sinh u & 6\cosh u \\ -\sqrt{3}\cosh u & \frac{3}{2}\cosh u & 0 \end{matrix}\right|$$

$$= -9\cosh^2 u\,\mathbf{i} - 6\sqrt{3}\cosh^2 u\mathbf{j} + \left(\frac{3}{2}+\frac{9}{2}\right)\sinh u\cosh u\,\mathbf{k}$$

$$= -9\cosh^2 u\,\mathbf{i} - 6\sqrt{3}\cosh^2 u\,\mathbf{j} + 3\sinh 2u\,\mathbf{k},\ u = 1.$$

$$= [-9\cosh^2 u\mathbf{i},\ -6\sqrt{3}\cosh^2 u\mathbf{j},\ 3\sinh 2u\,\mathbf{k}],\ u = 1.$$

20. For the surface defined by $x = (3+\cos\phi)\cos\theta$, $y = (3+\cos\phi)\sin\theta$, $z = \sin\phi$, $0 \le \theta < 2\pi$, $-\pi < \phi \le \pi$, show that the parametric curves for which ϕ is constant and θ varies are circles in planes parallel to the xy-plane. Also show that the parametric curves for which θ is constant and ϕ varies lie in planes through the z-axis. Find a vector normal to this surface at the point for which $\theta = \frac{\pi}{4}$, $\phi = \frac{2\pi}{3}$.

Sol. First we consider the parametric curves for which ϕ is consta[nt].

Since $x = (3+\cos\phi)\cos\theta$, $y = (3+\cos\phi)\sin\theta$, squaring [an]d adding we get

$$x^2 + y^2 = (3+\cos\phi)^2$$

which represents a circle in the plane z = constant

i.e., a plane parallel to the xy-plane. Again if θ is constant then we have $\frac{x}{\cos\theta} + \frac{-y}{\sin\theta} = 0$ which represents a plane containing the z-axis.

Now if $\boldsymbol{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

$$= (3+\cos\phi)\cos\theta\,\mathbf{i} + (3+\cos\phi)\sin\theta\,\mathbf{j} + \sin\phi\,\mathbf{k},$$

then $\frac{\partial \boldsymbol{r}}{\partial \theta} = -(3+\cos\phi)\sin\theta\,\mathbf{i} + (3+\cos\phi)\cos\theta\,\mathbf{j} + 0\,\mathbf{k}$

$$= -\left(3-\frac{1}{2}\right)\frac{1}{\sqrt{2}}\mathbf{i} + \left(3-\frac{1}{2}\right)\frac{1}{\sqrt{2}}\mathbf{j} + 0\mathbf{k}$$

$$= -\frac{5}{2\sqrt{2}}\mathbf{i} + \frac{5}{2\sqrt{2}}\mathbf{j} + 0\mathbf{k},\ \text{at } \theta = \frac{\pi}{4},\ \phi = \frac{2\pi}{3}$$

$$\frac{\partial \boldsymbol{r}}{\partial \phi} = -\sin\phi\cos\theta\,\mathbf{i} - \sin\phi\sin\theta\,\mathbf{j} + \cos\phi\,\mathbf{k}$$

$$= -\frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\mathbf{i} - \frac{\sqrt{3}}{2\sqrt{2}}\mathbf{j} - \frac{1}{2}\mathbf{k},\ \text{at } \theta = \frac{\pi}{4},\ \phi = \frac{2\pi}{3}$$

Therefore, any vector normal to the surface is

$$\frac{\partial \boldsymbol{r}}{\partial \phi}\times\frac{\partial \boldsymbol{r}}{\partial \theta} = \left|\begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{-\sqrt{3}}{2\sqrt{2}} & \frac{-\sqrt{3}}{2\sqrt{2}} & \frac{-1}{2} \\ \frac{-5}{2\sqrt{2}} & \frac{5}{2\sqrt{2}} & 0 \end{matrix}\right| = \frac{5}{4\sqrt{2}}\mathbf{i} + \frac{5}{4\sqrt{2}}\mathbf{j} - \frac{5\sqrt{3}}{4}\mathbf{k}$$

$$= \left[\frac{5}{4\sqrt{2}},\ \frac{5}{4\sqrt{2}},\ -\frac{5\sqrt{3}}{4}\right]$$

Exercise Set 9.6 (Page 433)

Find the extrema of each of the following (Problems 1–14):

1. $f(x,y) = x^2 - xy + y^2 + 6x$

Sol.
$$f(x, y) = x^2 - xy + y^2 + 6x$$
$$f_x(x, y) = 2x - y + 6$$
$$f_{xx}(x, y) = 2$$
$$f_y(x, y) = -x + 2y$$
$$f_{yy}(x, y) = 2$$
$$f_{xy}(x, y) = f_{yx}(x, y) = -1$$

For critical points, we have
$$2x - y + 6 = 0 \quad (1)$$
$$-x + 2y = 0 \quad (2)$$

Adding two times of equation (2) to (1), we have
$$3y + 6 = 0 \quad \text{or} \quad y = -2$$

Setting $y = -2$ into (2), we get $x = -4$.

The only critical point is $(-4, -2)$

Now $f_{xx}(x, y).f_{yy}(x, y) - [f_{xy}(x, y)]^2 = 4 - 1 > 0$

and $f_{xx}(x, y), f_{yy}(x, y)$ are both positive. $f(x, y)$ has a local minimum at $(-4, -2)$. The minimum value is
$$16 - 8 + 4 - 24 = -12.$$

2. $f(x,y) = \frac{1}{x} + xy - \frac{8}{y}$

Sol. $f(x,y) = \frac{1}{x} + xy - \frac{8}{y}$

$$f_x(x, y) = -\frac{1}{x^2} + y \qquad f_y(x, y) = x + \frac{8}{y^2}$$
$$f_{xx}(x, y) = \frac{2}{x^3} \qquad f_{yy}(x, y) = -\frac{16}{y^3}$$
$$f_{xy}(x, y) = 1 = f_{yx}(x, y)$$

For critical points, we have
$$-\frac{1}{x^2} + y = 0 \quad (1)$$
$$x + \frac{8}{y^2} = 0 \quad (2)$$

Substituting $y = \frac{1}{x^2}$ into (2), we get
$$x + 8x^4 = 0$$
or $\quad x(1 + 8x^3) = 0$

Thus $x = 0 \quad$ or $\quad x = -\frac{1}{2}$

$f(x, y)$ is not defined at $x = 0$ so this value is not admissible.

Putting $x = -\frac{1}{2}$ into (1), we find $y = 4$

A critical point of $f(x, y)$ is $\left(-\frac{1}{2}, 4\right)$

$$f_{xx}\left(-\frac{1}{2}, 4\right) = -16, \; f_{yy}\left(-\frac{1}{2}, 4\right) = -\frac{1}{4}$$
$$f_{xx}\left(-\frac{1}{2}, 4\right) f_{yy}\left(-\frac{1}{2}, 4\right) - \left[f_{xy}\left(-\frac{1}{2}, 4\right)\right]^2 = (-16)\left(-\frac{1}{4}\right) - 1 = 3 > 0$$

But $f_{xx}\left(-\frac{1}{2}, 4\right)$ and $f_{yy}\left(-\frac{1}{2}, 4\right)$ are both negative.

Therefore $\left(-\frac{1}{2}, 4\right)$ is a point of relative maxima.

3. $f(x,y) = 2x^2 + xy^2 - 4x - 1$

Sol.
$$f_x = 4x + y^2 - 4, \quad f_{xx} = 4$$
$$f_y = 2xy, \quad f_{yy} = 2x$$
$$f_{yx} = 2y = f_{xy}$$

For critical points,
$$f_x = y^2 + 4x - 4 = 0 \quad (1)$$
and $\quad f_y = 2xy = 0 \quad (2)$

From (2), we have $x = 0$ or $y = 0$

If $x = 0$, then (1) gives $y = \pm 2$

If $y = 0$, then from (1), we get $x = 1$

Thus critical points, are
$$(1, 0), (0, -2), (0, 2)$$
$$f_{yy}(1, 0) = 2, \quad \text{and} \quad f_{yx}(1, 0) = 0$$

Now, $f_{xx}(1, 0) f_{yy}(1, 0) - [f_{yx}(1, 0)]^2 = 8 > 0$ and $f_{xx}(1, 0) > 0$

Therefore local minima at $(1, 0)$.
$$f_{xx}(0, \pm 2) = 4, f_{yy}(0, \pm 2) = 0 \quad \text{and} \quad f_{yx}(0, \pm 2) = \pm 4$$

Therefore,
$$f_{xx}(0, \pm 2).f_{yy}(0, \pm 2) - [f_{yx}(0, \pm 2)]^2 = -16 < 0$$

Thus $(0, 2)$ and $(0, -2)$ are saddle points.

4. $f(x,y) = x^2 + 6xy + 2y^2 - 6x + 10y - 5$

Sol. $f(x, y) = x^2 + 6xy + 2y^2 - 6x + 10y - 5$

$f_x(x, y) = 2x + 6y - 6 \quad\Big|\quad f_y(x, y) = 6x + 4y + 10$

$f_{xx}(x, y) = 2 \quad\Big|\quad f_{yy}(x, y) = 4$

$$f_{yx}(x, y) = 6 = f_{xy}(x, y)$$

For critical points, we have

$$2x + 6y - 6 = 0$$
$$6x + 4y + 10 = 0$$

i.e., $\quad x + 3y - 3 = 0 \qquad (1)$

$\qquad 3x + 2y + 5 = 0 \qquad (2)$

Subtracting three times of (1) from (2), we get

$$-7y + 14 = 0 \quad \text{or} \quad y = 2$$

Putting this value of y into (1), we find $x = -3$

Thus $(-3, 2)$ is the only critical point.

$$f_{xx}(-3, 2).\, f_{yy}(-3, 2) - [f_{yx}(-3, 2)]^2 = 2 \times 4 - 6^2 < 0$$

Thus $(-3, 2)$ is a saddle point.

5. $f(x, y) = 6x^3y^2 - x^4y^2 - x^3y^3$

Sol. $f_x(x, y) = 18x^2y^2 - 4x^3y^2 - 3x^2y^3$

$f_y(x, y) = 12x^3y - 2x^4y - 3x^3y^2$

For extreme value, we have

$$f_x = x^2y^2(18 - 4x - 3y) = 0$$

and $f_y = x^3y(12 - 2x - 3y) = 0$

i.e., $\quad 18 - 4x - 3y = 0 \qquad (1)$

and $\quad 12 - 2x - 3y = 0 \qquad (2)$

Subtracting (1) from (2), we get

$$-6 + 2x = 0 \quad \text{or} \quad x = 3$$

Substituting $x = 3$ into (2), we have

$$12 - 6 - 3y = 0 \quad \text{or} \quad y = 2$$

Thus critical point is $(3, 2)$.

$$f_{xx} = 36xy^2 - 12x^2y^2 - 6xy^3$$
$$f_{yy} = 12x^3 - 2x^4 - 6x^3y$$
$$f_{yx} = 36x^2y - 8x^3y - 9x^2y^2$$
$$f_{xx}(3, 2) = -144 < 0, f_{yy}(3, 2) = -162 < 0$$
$$f_{yx}(3, 2) = -108$$
$$f_{xx}(3, 2).\, f_{yy}(3, 2) - [f_{xy}(3, 2)]^2 = (-144)(-162) - (-108)^2 > 0$$

Therefore, f has a local maxima at $(3, 2)$.

6. $f(x, y) = 2x^4 + y^2 - x^2 - 2y$

Sol. $f_x = 8x^3 - 2x, \quad f_{xx} = 24x^2 - 2$

$f_y = 2y - 2, \quad f_{yy} = 2$

$f_{xy} = 0 = f_{yx}$

For critical points, we have

$$f_x = 2x(4x^2 - 1) = 0 \quad \text{and} \quad f_y = 2(y - 1) = 0$$

Thus $y = 1$ and $x = 0, \frac{1}{2}, -\frac{1}{2}$.

Critical points are

$$(0, 1)\left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right)$$

$$f_{xx}(0, 1) = -2, f_{xx}\left(-\frac{1}{2}, 1\right) = 4 = f_{xx}\left(\frac{1}{2}, 1\right)$$

$$f_{yy}(0, 1) = f_{yy}\left(-\frac{1}{2}, 1\right) = f_{yy}\left(\frac{1}{2}, 1\right) = 2$$

Thus $f_{xx}(0, 1) f_{yy}(0, 1) - [f_{xy}(0, 1)]^2 = -4 - 0 < 0$

Thus $(0, 1)$ is a saddle point

$$f_{xx}\left(\pm\frac{1}{2}, 1\right) f_{yy}\left(\pm\frac{1}{2}, 1\right) - \left[f_{xy}\left(\pm\frac{1}{2}, 1\right)\right]^2 = 8 - 0 > 0$$

Thus local minima at $\left(-\frac{1}{2}, 1\right), \left(\frac{1}{2}, 1\right)$.

7. $f(x, y) = 18x^2 - 32y^2 - 36x - 128y$

Sol. $f_x = 36x - 36, \quad f_{xx} = 36,$

$f_y = -64y - 128, f_{yy} = -64, f_{xy} = 0 = f_{yx}$

For critical points,

$$f_x = 36(x - 1) = 0$$

and $f_y = -64(y + 2) = 0$

Thus $(1, -2)$ is the only critical point

$$f_{xx}(1, -2).\, f_{yy}(1, -2) - [f_{xy}(1, 2)]^2$$
$$= 36(-64) - 0 < 0$$

Thus $(1, -2)$ is a saddle point.

8. $f(x, y) = e^{-(x^2+y^2+2x)}$

Sol. $f_x = (-2x - 2)\, e^{-(x^2+y^2+2x)}$

$$f_{xx} = (-2x - 2)^2 e^{-(x^2+y^2+2x)} - 2e^{-(x^2+y^2+2x)}$$
$$f_y = -2y\, e^{-(x^2+y^2+2x)}$$
$$f_{yy} = 4y^2 e^{-(x^2+y^2+2x)} - 2e^{-(x^2+y^2+2x)}$$
$$f_{xy} = -2y(-2x + 2)\, e^{-(x^2+y^2+2x)} = f_{yx}.$$

For critical points, we have

$$f_x = (-2x-2)\,e^{-(x^2+y^2+2x)} = 0$$

and $f_y = -2y\,e^{-(x^2+y^2+2x)} = 0$

Therefore, $x = -1, y = 0$ and a critical point is $(-1, 0)$

Now $f_{xx}(-1,0) f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = (-2e)(-2e) > 0$

and $f_{xx}(-1,0) < 0$

Thus $(-1, 0)$ is a point of local maxima.

9. $f(x,y) = 2x^3 + y^2 - 9x^2 - 4y + 12x - 2$

Sol. $f_x = 6x^2 - 18x + 12$, $\quad f_{xx} = 12x - 18$

$f_y = 2y - 4$, $\quad f_{yy} = 2, f_{xy} = 0 = f_{yx}$

For critical points, we have

$$f_x = 6(x^2 - 3x + 2) = 0$$

and $f_y = 2(y-2) = 0$

i.e., $x = 1, 2$ and $y = 2$

Critical points are $(1, 2)$ and $(2, 2)$

Now, $f_{xx}(1,2) f_{yy}(1,2) - [f_{xy}(1,2)]^2 = -12 < 0$

Therefore $(1, 2)$ is a saddle point.

$f_{xx}(2,2) f_{yy}(2,2) - [f_{xy}(2,2)]^2 = 12 > 0$ and $f_{xx}(2,2) > 0$

Hence $(2, 2)$ is a point of local minima.

10. $f(x,y) = x^2 - e^{y^2}$

Sol. $f(x,y) = x^2 - e^{y^2}$

$f_x = 2x$, $\quad f_{xx} = 2$

$f_y = -2y\,e^{y^2}$, $\quad f_{yy} = -4y^2 e^{y^2} - 2e^{y^2}$, $f_{yx} = 0 = f_{xy}$

For critical points,

$f_x = 2x = 0$ and $f_y = -2y\,e^{y^2} = 0$

Therefore $(0, 0)$ is the only critical point.

$f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2 = -4 < 0$

Thus $(0, 0)$ is a saddle point.

11. $f(x,y) = \sin x + \sin y$

Sol. $f_x = \cos x$, $\quad f_{xx} = -\sin x$

$f_y = \cos y$, $\quad f_{yy} = -\sin y, f_{yx} = 0 = f_{xy}$

For critical points

$f_x = \cos x = 0$ and $f_y = \cos y = 0$

Therefore $\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$ is a critical point. (m, n are integers).

$$f_{xx}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) = -\sin\left(m\pi + \frac{\pi}{2}\right) = \begin{cases} -1 & \text{if } m \text{ even} \\ 1 & \text{if } m \text{ odd} \end{cases}$$

Similarly, $f_{yy}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) = -\sin\left(n\pi + \frac{\pi}{2}\right) = \begin{cases} -1 & \text{if } n \text{ even} \\ 1 & \text{if } n \text{ odd} \end{cases}$

Now, $f_{xx}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) \cdot f_{yy}\left(m\pi + \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right) - [f_{yx}]^2$

$$\left.\begin{matrix} > 0 \\ > 0 \\ < 0 \end{matrix}\right\} \begin{matrix} \text{if both } m, n \text{ even} \\ \text{if both } m, n \text{ odd} \\ \text{if one of } m, n \text{ odd and other even.} \end{matrix}$$

Thus (i) maximum for both m, n even

(ii) minimum for both m, n odd

(iii) saddle point otherwise.

12. $f(x,y) = y^2 - 6y\cos x + 6$

Sol. $f_x = 6y\sin x$, $\quad f_{xx} = 6y\cos x$

$f_y = 2y - 6\cos x$, $f_{yy} = 2$

$f_{xy} = 6\sin x = f_{yx}$

For critical points, we have

$$f_x = 6y\sin x = 0 \qquad (1)$$

and $f_y = 2y - 6\cos x = 0 \qquad (2)$

From (1), $y = 0$ or $x = 2n\pi$

Setting $x = 2n\pi$ into (2) we get $y = 3$

$(2n\pi, 3)$ is a critical point.

$f_{xx}(2n\pi, 3) f_{yy}(2n\pi, 3) - [f_{xy}(2n\pi, 3)]^2 = 36 - 0 > 0$

$f_{xx}(2n\pi, 3) = 18 > 0$

Thus f is minimum at $(2n\pi, 3)$.

Setting $y = 0$ into (2), we have

$$x = (2n+1)\frac{\pi}{2}$$

$$f_{xx}f_{yy} - \left[f_{xy}\left(\frac{2n+1}{2}\pi, 0\right)\right]^2 = 0 - 36 < 0$$

Thus $\left(\frac{2n+1}{2}\pi, 0\right)$ is a saddle point.

13. $f(x,y) = \cos x + \cos y + \cos(x+y)$

Sol. $f_x = -\sin x - \sin(x+y)$, $f_{xx} = -\cos x - \cos(x+y)$

$f_y = -\sin y - \sin(x+y), f_{yy} = -\cos y - \cos(x+y)$

$f_{yx} = f_{xy} = -\cos(x+y)$

For critical points,

$-\sin x - \sin(x+y) = 0$ (1)

and $-\sin y - \sin(x+y) = 0$

Therefore, $\sin x = \sin y$

Setting, $\sin x = \sin y$ into (1), we get

$\sin x + \sin x \cos x + \cos x \sin x = 0$

or $\sin x(1 + 2\cos x)$ i.e., $x = 0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}$

$x = 0, \quad y = 0$

$x = \pi, \quad y = \pi$

$x = \frac{2\pi}{3}, \quad y = \frac{2\pi}{3}$

Thus $(0, 0)$ (π, π), $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$ and $\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)$ are the critical points.

$f_{xx}(0,0) f_{yy}(0,0) - [f_{xy}(0,0)]^2 = 4 - 1 > 0$

$f_{xx}(0,0) = -2 < 0$

Therefore maximum at $(0, 0)$.

$$f_{xx}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) f_{yy}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right) - \left[f_{xy}\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)\right]^2 = 1 - \frac{1}{4} > 0$$

Thus minimum at $\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)$. Similarly, minimum at $\left(\frac{4\pi}{3}, \frac{4\pi}{3}\right)$

$f_{xx}(\pi, \pi) f_{yy}(\pi, \pi) - [f_{xy}(\pi, \pi)]^2 = 0 - 1 < 0.$

Hence (π, π) is a saddle point.

14. $f(x, y) = (x+1)(y+1)(x+y+1)$

Sol. $f_x = (y+1)[(x+y+1) + (x+1)]$

$= (y+1)(2x+y+2)$

$f_{xx} = 2(y+1)$

$f_y = (x+1)[(x+y+1) + (y+1)]$

$= (x+1)(x+2y+2)$

$f_{yy} = 2(x+1)$

$f_{xy} = (y+1) + (2x+y+2)$

$= 2x + 2y + 3 = f_{yx}$

For critical points, we have

$f_x = (y+1)(2x+y+2) = 0$ (1)

and $f_y = (x+1)(x+2y+2) = 0$ (2)

From (1), $y + 1 = 0$ or $2x + y + 2 = 0$

i.e., $y = -1$ or $y = -2x - 2$

Setting $y = -1$ into (2), we have

$(x+1)x = 0.$ i.e., $x = 0, -1$

Critical points are $(0, -1)$, $(-1, -1)$

Setting $y = -2x - 2$ into (2), we get $(x+1)(-3x-2) = 0$

i.e., $x = -1$ or $x = -\frac{2}{3}$.

If $x = -1, y = 0$, if $x = -\frac{2}{3}, y = -\frac{2}{3}$

Critical points are $(-1, 0)$, $\left(-\frac{2}{3}, -\frac{2}{3}\right)$

$f_{xx}(0,-1) f_{yy}(0,-1) - [f_{xy}(0,-1)]^2 = 0 - 1 < 0$

$(0, -1)$ is saddle point.

$f_{xx}(-1,-1) f_{yy}(-1,-1) - [f_{xy}(-1,-1)]^2 = 0 - 1 < 0$

$(-1, -1)$ is a saddle point.

$f_{xx}(-1,0) f_{yy}(-1,0) - [f_{xy}(-1,0)]^2 = 0 - 1 < 0$

$(-1, 0)$ is a saddle point.

$$f_{xx}\left(-\frac{2}{3}, -\frac{2}{3}\right) f_{yy}\left(-\frac{2}{3}, -\frac{2}{3}\right) - \left[f_{xy}\left(-\frac{2}{3}, -\frac{2}{3}\right)\right]^2 = \frac{2}{3} \cdot \frac{2}{3} - \frac{1}{9} > 0$$

$\left(-\frac{2}{3}, -\frac{2}{3}\right)$ is a point of relative minimum.

15. Find the point of the sphere $x^2 + y^2 + x^2 + = 49$ that is nearest to the point $(2, 1, 3)$.

Sol. Let $Q(x, y, z)$ be the point on the sphere that is nearest to $(2, 1, 3)$

$d^2 = (x-2)^2 + (y-1)^2 + (z-3)^2.$

d is to be minimized subject to $x^2 + y^2 + z^2 = 49$

or $z^2 = 49 - (x^2 + y^2)$

$d^2 = (x-2)^2 + (y-1)^2 + (\sqrt{49 - x^2 - y^2} - 3)^2 = f(x, y)$

$= x^2 - 4x + 4 + y^2 - 2y + 1 + 49 - x^2 - y^2 + 9 - 6\sqrt{49 - x^2 - y^2}$

$= -4x - 2y + 63 - 6\sqrt{49 - x^2 - y^2}$

$f_x = -4 + \frac{6x}{\sqrt{49 - x^2 - y^2}}$; $f_y = -2 + \frac{6y}{\sqrt{49 - x^2 - y^2}}$

For critical points

$f_x = 0 = -4\sqrt{49 - x^2 - y^2} + 6x$

and $f_y = 0 = -2\sqrt{49 - x^2 - y^2} + 6y$

i.e., $4(49 - x^2 - y^2) = 9x^2$

and $49 - x^2 - y^2 = 9y^2$ (1)

Therefore, $\frac{9x^2}{4} = 9y^2$ or $y = \pm\frac{x}{2}$

Putting this value of y into (1), we get

$$49 - x^2 - \frac{x^2}{4} = \frac{9x^2}{4} \quad \text{or} \quad 7x^2 = 98$$

$$x = \pm\sqrt{14} = \pm\frac{14}{\sqrt{14}}\ ;\ y = \pm\frac{\sqrt{14}}{2} = \pm\frac{7}{\sqrt{14}}$$

$$z^2 = 49 - (x^2 + y^2) = 49 - 14 - \frac{14}{4}$$

or $\quad z = \pm\frac{21}{\sqrt{14}}$

Required point is $\left(\frac{14}{\sqrt{14}}, \frac{7}{\sqrt{14}}, \frac{21}{\sqrt{14}}\right)$

16. The sum of the length and girth (perimeter of a cross-section) of the packages accepted by post office is 270 centimeters. Find the dimensions of the rectangular package of greatest volume that can be sent by post.

Sol. Let x, y, z in centimeters be the length, breadth and height respectively of the package. By the given condition, we have

$$x + 2y + 2z = 270 \qquad (1)$$

Volume V of the package is

$$V = xyz = 2yz(135 - y - z), \text{ using (1)}$$

$$\frac{\partial V}{\partial y} = 2z(135 - y - z) - 2yz = 2z(135 - 2y - z)$$

$$\frac{\partial V}{\partial z} = 2y(135 - y - z) - 2yz = 2z(135 - y - 2z0$$

For extrema, we have

$$\frac{\partial V}{\partial y} = 0 = 2z(135 - 2y - z)$$

$$\frac{\partial V}{\partial z} = 0 = 2y(135 - y - 2z)$$

Solving these equations, we find

$$y = 45, z = 45$$

$$\frac{\partial^2 V}{\partial y^2} = -4z, \frac{\partial^2 V}{\partial z^2} = -4y, \frac{\partial^2 V}{\partial y\, \partial z} = 270 - 4y - 4z^2$$

$$\frac{\partial^2 V}{\partial y^2}\frac{\partial^2 V}{\partial z^2} - \left(\frac{\partial^2 V}{\partial y\, \partial z}\right)^2 = 16yz - (270 - 4y - 4z^2) < 0 \text{ for } y = z = 45$$

Thus V is maximum at $y = z = 45$.

From (1), we have

$$x = 270 - 2(y + z) = 270 - 180 = 90$$

Dimensions of the package of greatest volume are 90 cm × 45 cm × 45 cm.

17. Show that the volume of the largest parallelepiped (with faces parallel to the coordinate planes) that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{8abc}{3\sqrt{3}}$.

Sol. Let the coordinates of one vertex of the parallelepiped be (x, y, z)

The volume V of the parallelepiped is

$$V = 8xyz$$

or $\quad \frac{V^2}{64a^2b^2c^2} = \frac{x^2}{a^2}\cdot\frac{y^2}{b^2}\cdot\frac{z^2}{c^2}$

Let $\quad t = \frac{x^2}{a^2},\ u = \frac{y^2}{b^2},\ w = \frac{z^2}{c^2}$

Then we have to maximize

$$f(t, u, w) = tuw \text{ where } t + u + w = 1$$

or $\quad f(t, u) = tu(1 - t - u)$

$$f_t = u(1 - t - u) - tu = u - u^2 - 2ut, f_{tt} = -2u$$

$$f_u = t(1 - t - u) - tu = t - t^2 - 2ut, f_{uu} = -2t,$$

$$f_{ut} = 1 - 2t - 2u = f_{tu}.$$

For critical points, we have

$$f_t = u\,(1 - u - 2t) = 0$$

and $\quad f_u = t\,(1 - t - 2u) = 0$

Since $u \neq 0$, $t \neq 0$, therefore

$$1 - u - 2t = 0 \qquad (1)$$

and $\quad 1 - t - 2u = 0 \qquad (2)$

Multiply (1) by 2 and subtract the result from (2). We have

$$-1 + 3t = 0 \quad \text{or} \quad t = \frac{1}{3}$$

Substituting $t = \frac{1}{3}$ into (1), we get $u = \frac{1}{3}$.

Finally, we have $w = \frac{1}{3}$

Hence, $\quad \frac{V^2}{64a^2b^2c^2} = tuw = \frac{1}{27}$

or $\quad V = \frac{8abc}{3\sqrt{3}}$

18. Find the minimum distance between the lines $x = t, y = 3 - 2t, z = 1 + 2t \qquad (1)$

and $\quad x = -1 - s, y = s, z = 4 - 3s. \qquad (2)$

Sol. $P(t, 3-2t, 1+2t)$ and $Q(-1-s, s, 4-3s)$ are points on the lines (1) and (2) respectively.

$$|PQ|^2 = (t+s+1)^2 + (3-2t-s)^2 + (2t+3s-3)^2 \quad (3)$$
$$= t^2 + s^2 + 1 + 2ts + 2s + 2t + 9 + 4t^2 + s^2 - 12t - 6s$$
$$+ 4ts + 4t^2 + 9s^2 + 9 - 12t - 18s + 12ts$$
$$= 9t^2 + 11s^2 + 18ts - 22t - 22s + 19 = f(t, s)$$
$$f_t = 18t + 18s - 22, \qquad f_{tt} = 18$$
$$f_s = 22s + 18t - 22, \qquad f_{ss} = 22, \qquad f_{ts} = 12$$

For critical points, we have

$$f_t = 18t + 18s - 22 = 0 \text{ or } 9t + 6s - 11 = 0$$

and $f_s = 18t + 22s - 22 = 0$ or $6t + 11s - 11 = 0$

Solving these equations, we have $s = 0$, $t = \frac{11}{9}$.

$\left(\frac{11}{9}, 0\right)$ is critical point of (3).

Now, $f_{tt} f_{ss} - [f_{ts}]^2 = 18 \times 22 - 122 > 0$

Thus (3) has relative minimum at $\left(\frac{11}{9}, 0\right)$

The points P and Q have coordinates

$$P\left(\frac{11}{9}, \frac{5}{9}, \frac{31}{9}\right) \text{ and } Q(-1, 0, 4)$$

$$|PQ| = \sqrt{\left(\frac{20}{9}\right)^2 + \left(\frac{5}{9}\right)^2 + \left(\frac{-5}{9}\right)^2} = \sqrt{\frac{400}{81} + \frac{25}{81} + \frac{25}{81}} = \frac{\sqrt{50}}{3}$$

is the required minimum distance.

19. Find the dimensions of the largest rectangular box with three of its faces in the coordinate planes and the vertex opposite the origin in the first octant and on the plane $2x + y + 3z = 6$. (1)

Sol. Let x, y, z be the dimensions of the box. Then volume V of the box is

$$V = xyz$$

or $V = f(x, z) = xz(6 - 2x - 3z)$, using (1)

$$f_x = 6z - 4xz - 3z^2, \qquad f_{xx} = -4z$$
$$f_z = 6x - 2x^2 - 6xz, \qquad f_{zz} = -6x$$
$$f_{xz} = 6 - 4x - 6z$$

For critical points, we have

$$f_x = 6z - 4xz - 3z^2 = 0$$

and $f_z = 6x - 2x^2 - 6xz = 0$

or $z(6 - 4x - 3z) = 0$ (1)

and $x(6 - 4x - 3z) = 0$ (2)

$(x \neq 0, \quad z \neq 0)$

Multiply $6 - 2x - 6z = 0$ by 2 and subtract from $6 - 4x - 3z = 0$ to get

$$-6 + 9z = 0 \quad \text{or} \quad z = \frac{2}{3}$$

Setting $z = \frac{2}{3}$ into $6 - 2x - 6z = 0$, we find $x = 1$.

$$y = 6 - 2x - 3z = 6 - 2 - 2 = 2$$

The dimensions of the box are

$$x = 1, \quad y = 2, \quad z = \frac{2}{3}.$$

20. A closed rectangular box with volume 16 ft^3 is to be made of three different materials. The cost of the material for the top and the bottom is Rs. 9 per sq. ft., the cost of material for the front and the back is Rs. 8 per sq ft. and the cost of the material for the other two sides is Rs. 6 per sq. ft Find the dimensions of the box so that the cost of the material is a minimum

Sol. Let the dimensions of the top of the box be x ft by y ft and let its height be z ft. Then $xyz = 16$ (1)

Let C denote the total rupee cost of the material used for constructing the box.

$$C = 18xy + 16xz + 12yz$$
$$= 18xy + 16x \cdot \frac{16}{yx} + 12y \cdot \frac{16}{yx}, \text{ using (1)}$$
$$= 18xy + \frac{256}{y} + \frac{192}{x}$$
$$C_x = 18y - \frac{192}{x^2}, \quad C_{xx} = \frac{384}{x^2}$$
$$C_y = 18x - \frac{256}{y^2}, \quad C_{yy} = \frac{512}{y^3}, C_{xy} = 18 = C_{yx}$$

For critical points, we have

$$C_x = 18y - \frac{192}{x^2} = 0 \text{ and } C_y = 18x - \frac{256}{y^2} = 0$$

or $18x^2y - 192 = 0$ (2)

and $18xy^2 - 256 = 0$ (3)

From (2), $y = \frac{32}{3x^2}$ Substituting into (3), we get

$$18x\left(\frac{32}{3x^2}\right)^2 - 256 = 0$$

or $64(32 - 4x^3) = 0$ or $x = 2$

Therefore, $y = \frac{32}{12} = \frac{8}{3}$.

Critical point is $\left(2, \frac{8}{3}\right)$

$$C_{xx}\left(2, \frac{8}{3}\right) C_{yy}\left(2, \frac{8}{3}\right) - \left[C_{xy}\left(2, \frac{8}{3}\right)\right]^2 = 48 \times 27 - 18^2 > 0$$

Thus $\left(2, \frac{8}{3}\right)$ is a point of local minimum

When $x = 2, y = \frac{8}{3}$, then from (1), $z = 3$

The desired dimensions of the box are 2ft by $\frac{8}{3}$ ft by 3 ft.

Exercise Set 9.7 (Page 438)

1. Find the point of the plane $x + 2y - z = 3$ nearest to the origin.

Sol. The distance of a point (x, y, z) on the plane to the origin is

$$d = \sqrt{x^2 + y^2 + z^2}$$

We shall minimize $d^2 = x^2 + y^2 + z^2$ subject to $x + 2y - z = 3$.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = x^2 + y^2 + z^2 + \lambda(x + 2y - z - 3)$$

$F_x = 2x + \lambda = 0$; $\quad \lambda = -2x$

$F_y = 2y + 2\lambda = 0$; $\quad \lambda = -y$

$F_z = 2z - \lambda = 0$; $\quad \lambda = -2z$

$F_\lambda = x + 2y - z - 3 = 0$

Therefore,

$-2x = -y = 2z$

or $\quad y = 2x \quad$ and $\quad z = -x$

Substituting into $x + 2y - z = 3$, we get $6x = 3$ or $x = \frac{1}{2}$.

The desired point is $\left(\frac{1}{2}, 1, \frac{-1}{2}\right)$.

2. Find the extrema of $f(x, y, z) = xy + z$ subject to $x^2 + y^2 + z^2 = 1$.

Sol. Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = xy + z + \lambda(x^2 + y^2 + z^2 - 1)$$

$F_x = y + 2\lambda x = 0$; $\quad \lambda = \frac{-y}{2x}$

$F_y = x + 2\lambda y = 0$; $\quad \lambda = \frac{-x}{2y}$

$F_z = 1 + 2\lambda z = 0$; $\quad \lambda = -\frac{1}{2z}$

$F_\lambda = x^2 + y^2 + z^2 - 1 = 0 \qquad$ (A)

Therefore,

$$-\frac{1}{2z} = \frac{-x}{2y} = \frac{-y}{2x}$$

and from $\frac{1}{z} = \frac{x}{y}$, we get $y = xz \qquad$ (1)

From $-\frac{1}{2z} = \frac{-y}{2x}$, we have $x = yz \qquad$ (2)

From (1) and (2), we find $z = \pm 1$.

When $z = 1, y = x$ and when $z = -1, y = -x$.

Substituting into (A), we obtain

$x^2 + x^2 + 1 - 1 = 0 \quad$ i.e., $\quad x = 0$.

Thus the critical points are

$(0, 0, 1)$ and $(0, 0, -1)$

Clearly, the maximum and minimum of $f(x, y, z)$ are attained at $(0, 0, 1)$ and $(0, 0, -1)$ respectively.

3. Find the maximum value of $f(x, y, z) = x^4 + y^4 + z^4$ subject to $x + y + z = 1$.

Sol. Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = x^4 + y^4 + z^4 + \lambda(x + y + z - 1)$$

$F_x = 4x^3 + \lambda = 0$; $\quad \lambda = -4x^3$

$F_y = 4y^3 + \lambda = 0$; $\quad \lambda = -4y^3$

$F_z = 4z^3 + \lambda = 0$; $\quad \lambda = -4z^3$

$F_\lambda = x + y + z - 1 = 0$

Therefore, we get

$x^3 = y^3 = z^3 \quad$ or $\quad x = y = z$

Substituting into $x + y + z = 1$, we obtain

$3x = 1 \quad$ or $\quad x = \frac{1}{3}$

Thus the critical point is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Desired value $= \frac{1}{81} + \frac{1}{81} + \frac{1}{81} = \frac{1}{27}$.

4. Find the extrema of $f(x, y, z) = 4x - 3y + 2z$ subject to $x^2 + y^2 - 6z = 0$.

Sol. Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = 4x - 3y + 2z + \lambda(x^2 + y^2 - 6z)$$

$$F_x = 4 + 2\lambda x = 0; \qquad \lambda = \frac{-2}{x}$$

$$F_y = -3 + 2\lambda y = 0; \qquad \lambda = \frac{3}{2y}$$

$$F_z = 2 - 6\lambda = 0; \qquad \lambda = \frac{1}{3}$$

$$F_\lambda = x^2 + y^2 - 6z = 0$$

Thus $\frac{-2}{x} = \frac{1}{3} = \frac{3}{2y}$ and so $x = -6, \quad y = \frac{9}{2}$

Substituting into $x^2 + y^2 = 6z$, we have

$$36 + \frac{81}{4} = 6z \qquad \text{or} \qquad z = \frac{225}{24}$$

The critical point is $\left(-6, \frac{9}{2}, \frac{225}{24}\right)$ and the extreme value is

$$-24 - \frac{27}{2} + \frac{225}{12} = \frac{-225}{12}$$

which is the minimum value.

5. Find the points on $x^2 + y^2 + z^2 = 1$ closest to and farthest from the point (1, 2, 3)

Sol. We have to find the extrema of

$$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$$

subject to $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = (x-1)^2 + (y-2)^2 + (z-3)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$F_x = 2(x-1) + 2\lambda x = 0\,;\ -\lambda = \frac{x-1}{x} \qquad (1)$$

$$F_y = 2(y-2) + 2\lambda y = 0\,;\ -\lambda = \frac{y-2}{y} \qquad (2)$$

$$F_z = 2(z-3) + 2\lambda z = 0\,;\ -\lambda = \frac{z-3}{z} \qquad (3)$$

$$F_\lambda = x^2 + y^2 + z^2 - 1 = 0 \qquad (A)$$

From (1) and (2), we have $\frac{x-1}{x} = \frac{y-2}{y}$ *i.e.,* $y = 2x$

From (1) and (3), we get $\frac{x-1}{x} = \frac{z-3}{z}$ *i.e,* $z = 3x$

The point $(x, y, z) = (x, 2x, 3x)$ lies on (A). Substituting into (A), we find

$$x = \pm\frac{1}{\sqrt{14}}, y = \pm\frac{2}{\sqrt{14}}, z = \pm\frac{3}{\sqrt{14}}$$

When $x = \frac{1}{\sqrt{14}}$ then $y = \frac{2}{\sqrt{14}}, z = \frac{3}{\sqrt{14}}$

When $x = \frac{-1}{\sqrt{14}}$ then $y = \frac{-2}{\sqrt{14}}, z = \frac{-3}{\sqrt{14}}$

The critical points are

$P\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$ and $Q\left(\frac{-1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$

It is easy to see that P is closest to and Q is farthest from (1,2,3).

6. Find the points on the paraboloid $z = 2 - x^2 - y^2$ that is closest to the point (1, 1, 2)

Sol. We have to find the point (x, y, z) on the paraboloid such that $(x-1)^2 + (y-1)^2 + (z-2)^2$ is minimum.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = (x-1)^2 + (y-1)^2 + (z-2)^2 + \lambda(x^2 + y^2 + z - 2)$$

$$F_x = 2(x-1) + 2\lambda x = 0\ ; \quad -\lambda = \frac{x-1}{x}$$

$$F_y = 2(y-1) + 2\lambda y = 0\ ; \quad -\lambda = \frac{y-1}{y}$$

$$F_z = 2(z-2) + \lambda = 0\ ; \quad -\lambda = 2(z-2)$$

$$F_\lambda = x^2 + y^2 + z - 2 = 0$$

Thus $\frac{x-1}{x} = \frac{y-1}{y} = 2(z-2)$ and so we find $x = y$ and $z = \frac{x-1}{2x} + 2$

Substituting into $z = 2 - x^2 - y^2$, we get

$$\frac{x-1}{2x} + 2 = 2 - 2x^2$$

or $\quad 4x^3 + x - 1 = 0$

By inspection, $x = \frac{1}{2}$ and so $y = \frac{1}{2}$ and $z = \frac{1}{2} - 1 + 2 = \frac{3}{2}$

The required point is $\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\right)$.

7. Find the dimensions of a topless box so that the volume is a maximum when the surface area is 24 square metres.

Sol. Let the dimension of the box be x, y and z metres.

Surface area $= xy + 2xz + 2yz$ sq. m. The volume xyz of the box is to be maximized subject to

$$xy + 2xz + 2yz = 24.$$

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = xyz + \lambda(xy + 2xz + 2yz - 24)$$

$$F_x = yz + \lambda y + 2\lambda z \quad = 0 \qquad (1)$$

$$F_y = xz + \lambda x + 2\lambda z = 0 \quad (2)$$
$$F_z = xy + 2\lambda x + 2\lambda y = 0 \quad (3)$$
$$F_\lambda = xy + 2xz + 2yz - 24 = 0 \quad (A)$$

Multiplying (1) by x, (2) by y and (3) by z respectively, we get

$$xyz + \lambda xy + 2\lambda zx = 0 \quad (4)$$
$$xyz + \lambda xy + 2\lambda yz = 0 \quad (5)$$
$$xyz + 2\lambda xz + 2\lambda yz = 0 \quad (6)$$

From (4) and (5), we find $xz = yz$ or $x = y$. From (4) and (6), we get

$$xy = 2yz \text{ so that } x = 2z.$$

Substituting into (A), we have

$$4z^2 + 4z^2 + 4z^2 = 24 \quad \text{or} \quad z = \sqrt{2}$$

Thus dimensions of the box are $2\sqrt{2}, 2\sqrt{2}, \sqrt{2}$ metres.

8. Find the dimensions of the box with maximum volume that can be enclosed by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if the sides of the box are parallel to the coordinate planes.

Sol. Let the corner of the box in the first octant be (x, y, z). Then the dimensions of the box are $2x$, $2y$ and $2z$. We have to find the maximum of $f(x, y, z) = 8xyz$ subject to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda) = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$F_x = 8yz + \frac{2\lambda x}{a^2} = 0 \quad \text{or} \quad 4xyz + \frac{\lambda x^2}{a^2} = 0$$

$$F_y = 8zx + \frac{2\lambda y}{b^2} = 0 \quad \text{or} \quad 4xyz + \frac{\lambda y^2}{b^2} = 0$$

$$F_z = 8xy + \frac{2\lambda z}{c^2} = 0 \quad \text{or} \quad 4xyz + \frac{\lambda z^2}{c^2} = 0$$

$$F_\lambda = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Therefore, we get $\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}$

Since $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, we find that

$$3x^2 = a^2, \quad 3y^2 = b^2, \quad 3z^2 = c^2$$

The corner of the box in first octant has positive coordinates, so this point is $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$. The box with maximum volume has sides $\frac{2a}{\sqrt{3}}, \frac{2b}{\sqrt{3}}, \frac{2c}{\sqrt{3}}$ and the maximum volume is $\frac{8abc}{3\sqrt{3}}$ cu units.

9. Find the minimum volume of a tetrahedron bounded by the coordinate planes and a plane tangent to the sphere $x^2+y^2+z^2 = 1$.

Sol. Let $P(u, v, w)$ be a point on the sphere. An equation of the tangent plane to the sphere at P is $xu + yv + zw - 1 = 0$.

The tetrahedron is bounded by the planes $x = 0, y = 0, z = 0$ and $ux + vy + wz - 1 = 0$.

The vertices of the tetrahedron are

$$(0, 0, 0), \left(\frac{1}{u}, 0, 0\right), \left(0, \frac{1}{v}, 0\right), \left(0, 0, \frac{1}{w}\right)$$

Volume of the tetrahedron

$$= \frac{1}{6}\begin{vmatrix} \frac{1}{u} & 0 & 0 & 1 \\ 0 & \frac{1}{v} & 0 & 1 \\ 0 & 0 & \frac{1}{w} & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix} = \frac{1}{6uvw}$$

We have to minimize $f(u, v, w) = \frac{1}{6uvw}$ subject to $u^2 + v^2 + w^2 = 1$.

Using Lagrange multipliers, we have

$$F(u, v, w, \lambda) = \frac{1}{6uvw} + \lambda(u^2 + v^2 + w^2 - 1)$$

$$F_u = \frac{-vw}{6(uvw)^2} + 2\lambda u = 0$$

$$F_v = \frac{-uw}{6(uvw)^2} + 2\lambda v = 0$$

$$F_w = \frac{-uv}{6(uvw)^2} + 2\lambda w = 0$$

$$F_\lambda = u^2 + v^2 + w^2 - 1 = 0 \quad (1)$$

Thus $u^2 = \frac{1}{12\lambda uvw} = v^2 = w^2$

Substituting into (1), we have $3u^2 = 1$ or $u = \frac{1}{\sqrt{3}} = v = w$

Minimum volume of the tetrahedron is

$$\frac{1}{6\left(\frac{1}{\sqrt{3}}\right)^3} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} \text{ cu. units.}$$

10. Find the point on the line $x + y + 2z - 12 = 0 = x - 3y - 2z + 16$ nearest to the origin.

Sol. Let $P(x, y, z)$ be a point on the line. Then we have to minimize $x^2 + y^2 + z^2$ subject to the given constraints.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda_1, \lambda_2) = x^2+y^2+z^2+\lambda_1(x+y+2z-12)+\lambda_2(x-3y-2z+16)$$

$$F_x = 2x + \lambda_1 + \lambda_2 = 0 \qquad (1)$$

$$F_y = 2y + \lambda_1 - 3\lambda_2 = 0 \qquad (2)$$

$$F_z = 2z + 2\lambda_1 - 2\lambda_2 = 0 \qquad (3)$$

$$F_{\lambda_1} = x + y + 2z - 12 = 0$$

$$F_{\lambda_2} = x - 3y - 2z + 16 = 0$$

From (1) and (2), we get $x - y + 2\lambda_2 = 0$

From (2) and (3), we find $2y - z - 2\lambda_2 = 0$

Adding the last two equations, we have $x + y - z = 0$

Setting $z = x + y$ into the equations of the line, we obtain

$$x + y - 4 = 0 \quad \text{and} \quad -x - 5y + 16 = 0$$

Therefore $y = 3, x = 1$ and $z = 4$

The required point is $(1, 3, 4)$.

11. Find the points that are on both the ellipsoid $x^2 + y^2 + 9z^2 = 25$ and the plane $x + 3y - 2z = 0$ which are closest to and farthest from the origin.

Sol. Let $P(x, y, z)$ be a point on the intersection of the ellipsoid and the plane. Then we have to find the extrema of $d = \sqrt{x^2 + y^2 + z^2}$ or of $d^2 = x^2 + y^2 + z^2$.

Using Lagrange multipliers, we have

$$F(x, y, z, \lambda_1, \lambda_2) = x^2+y^2+z^2+\lambda_1(x^2+y^2+9z^2-25) + \lambda_2(x+3y-2z)$$

$$F_x = 2x + 2\lambda_1 x + \lambda_2 = 0$$

$$F_y = 2y + 2\lambda_1 y + 3\lambda_2 = 0$$

$$F_z = 2z + 18\lambda_1 z - 2\lambda_2 = 0$$

$$F_{\lambda_1} = x^2 + y^2 + 9z^2 - 25 = 0 \qquad (A)$$

$$F_{\lambda_2} = x + 3y - 2z = 0 \qquad (B)$$

or

$$2xyz + 2\lambda_1 xyz + \lambda_2 yz = 0 \qquad (1)$$

$$2xyz + 2\lambda_1 xyz + 3\lambda_2 xz = 0 \qquad (2)$$

$$2xyz + 18\lambda_1 xyz - 2\lambda_2 xy = 0$$

From (1) and (2), we have

$$3\lambda_2 xz - \lambda_2 yz = 0$$

or $3xz - yz = 0$

i.e., $z = 0$ or $y = 3x$

Using $z = 0$ and the two constraints, we get

$x = -3y$ and so $9y^2 + y^2 = 25$

i.e., $y = \pm\frac{5}{\sqrt{10}}$. Therefore, $x = \frac{-15}{\sqrt{10}}$ if $y = \frac{5}{\sqrt{10}}$

and $x = \frac{15}{\sqrt{10}}$ if $y = \frac{-5}{\sqrt{10}}$.

The two points are $A\left(\frac{-15}{\sqrt{10}}, \frac{5}{\sqrt{10}}, 0\right)$, $B\left(\frac{15}{\sqrt{10}}, \frac{-5}{\sqrt{10}}, 0\right)$

If $y = 3x$, then substituting into (B), we find $z = \frac{x + 3y}{2}$

Putting into (A), we get

$$x^2 + 9x^2 + 9\left(\frac{x + 9x}{2}\right) = 25$$

or $x = \pm\frac{5}{\sqrt{235}}$

If $x = \frac{5}{\sqrt{235}}$, then $y = \frac{15}{\sqrt{235}}$ and so $z = \frac{25}{\sqrt{235}}$

If $x = \frac{-5}{\sqrt{235}}$ then $y = \frac{-15}{\sqrt{235}}$ and $z = \frac{-25}{\sqrt{235}}$

The two points are

$$C\left(\frac{5}{\sqrt{235}}, \frac{15}{\sqrt{235}}, \frac{25}{\sqrt{235}}\right), D\left(\frac{-5}{\sqrt{235}}, \frac{-15}{\sqrt{235}}, \frac{-25}{\sqrt{235}}\right)$$

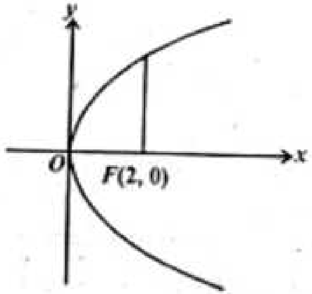
The distance of each of the four points A, B, C and D from $(0, 0, 0)$ is $5, 5, \sqrt{\frac{175}{47}}, \sqrt{\frac{175}{47}}$.

The points A and B are farthest from the origin while the points C and D are closest to the origin.

Exercise Set 9.8 (Page 446)

1. Find the volume generated by revolving the area in the first quadrant bounded by the parabola $y^2 = 8x$ and its latus-rectum about the x-axis.

Sol. Here the limits for x are from $x = 0$ to $x = 2$. Therefore, the required volume is



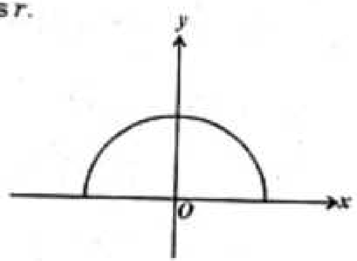
$$= \int_0^2 \pi y^2\, dx$$

$$= \int_0^2 \pi(8x)\, dx = 8\pi \int_0^2 x\, dx$$

$$= 8\pi \left|\frac{x^2}{2}\right|_0^2 = 16\pi.$$

2. Find the volume of a sphere of radius r.

Sol. Volume of the sphere is the volume generated by the circle $x^2 + y^2 = r^2$ when it revolves about the x-axis. Therefore, volume of the sphere



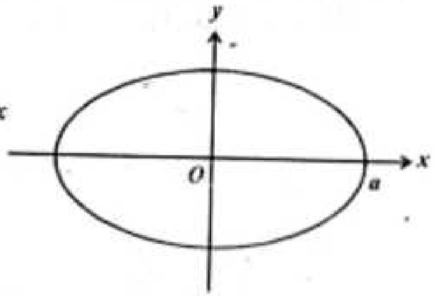
$$= \int_{-r}^{r} \pi y^2\, dx = 2\int_0^r \pi(r^2 - x^2)\, dx$$

$$= 2\pi \left|r^2 x - \frac{x^3}{3}\right|_0^r = 2\pi\left(r^3 - \frac{r^3}{3}\right) = \frac{4\pi r^3}{3}.$$

3. Find the volume of the spheroid formed by the revolution of the area bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the (i) major axis (ii) minor axis.

Sol.

(i) The required volume is



$$= \int_{-a}^{a} \pi y^2\, dx = 2\int_0^a \pi y^2\, dx$$

$$= 2\int_0^a \pi\left(1 - \frac{x^2}{a^2}\right) b^2\, dx$$

$$= 2\frac{b^2}{a^2}\pi \left|a^2 x - \frac{x^3}{3}\right|_0^a = 2\frac{\pi b^2}{a^2}\left(a^3 - \frac{a^3}{3}\right) = \frac{4}{3}\pi a b^2$$

(ii) Here the required volume is

$$= \int_{-b}^{b} \pi x^2\, dy = 2\int_0^b \pi\left(1 - \frac{y^2}{b^2}\right) a^2\, dy$$

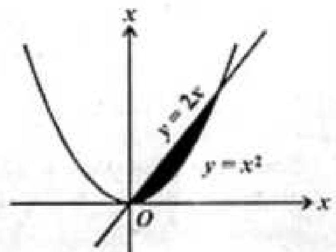
$$= \frac{2\pi a^2}{b^2}\int_0^b (b^2 - y^2)\, dy = \frac{2\pi a^2}{b^2}\left|b^2 y - \frac{y^3}{3}\right|_0^b = \frac{4}{3}\pi b a^2$$

4. Find the volume of the solid generated by revolving the area enclosed by $y = 2x$ and $y = x^2$ about the y-axis.

Sol. The points of intersection of the two equations are given by

$2x = x^2$, or $x = 0,2$

Thus, (0, 0,), (2, 4) are the two points of intersection. The desired volume



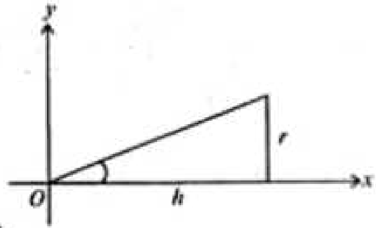
$$= \int_0^4 \left[\pi(\sqrt{y})^2 - \pi\left(\frac{y}{2}\right)^2\right] dy$$

$$= \pi\int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8\pi}{3}$$

5. Find the volume of a right circular cone having base radius r and height h.

Sol. The cone is generated by revolving about the x-axis the area enclosed by

$$y = \frac{r}{h}x,\ x = h \text{ and the } x\text{-axis}$$



$$\text{Desired volume} = \int_0^h \pi y^2\, dx$$

$$= \pi\int_0^h \frac{r^2}{h^2}x^2\, dx$$

$$= \pi\frac{r^2}{h^2}\left[\frac{x^3}{3}\right]_0^h = \frac{\pi r^2 h}{3}.$$

[Solve also by Cross Sectional Method.]

6. The area in the first quadrant bounded by $x = 2y^3 - y^4$ and the y-axis is revolved abut the x-axis. Find the volume of the resulting solid.

Sol. By the shell method, desired volume

$$= \int_0^2 2\pi y(2y^3 - y^4)dy = 2\pi\left[\frac{2y^5}{5} - \frac{y^6}{6}\right]_0^2 = \frac{64\pi}{15}.$$

7. A basin is formed by the revolution of the area bounded by the curve $x^3 = 64y$, $(y > 0)$ about the axis of y. If the depth of the basin is 8 cm, how many cubic cm. of water would it hold?

Sol. The required volume is

$$= \int_0^8 \pi x^2\, dy = \int_0^8 \pi (64y)^{2/3}\, dy = 16\pi \int_0^8 y^{2/3}\, dy = 16\,\pi\left[3.\frac{y^{5/3}}{5}\right]_0^8$$

$$= \frac{48\pi}{5}(8)^{5/3} = \frac{48\pi}{5} \times 32 = \frac{1536}{5}\pi \text{ cubic cm.}$$

8. Show that the volume generated by revolving the area bounded by an arch of the cycloid $x = a\,(\theta - \sin\theta)$, $y = a\,(1 - \cos\theta)$ about its base is $5\,\pi^2 a^3$.

Sol. Required volume

$$= \pi \int_0^{2\pi} y^2\, dx = \pi \int_0^{2\pi} a^2\,(1 - \cos\theta)^2\,(a)\,(1 - \cos\theta)\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 - \cos\theta)^3\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 - 3\cos\theta + 3\cos^2\theta - \cos^3\theta)\, d\theta$$

$$= \pi a^3 \int_0^{2\pi} (1 + 3\cos^2\theta)\, d\theta \text{ as } \int_0^{2\pi} \cos\theta\, d\theta = 0 \text{ and } \int_0^{2\pi} \cos^3\theta\, d\theta = 0$$

$$= \pi a^3 \int_0^{2\pi} \left[1 + \frac{3}{2}(1 + \cos 2\theta)\right] d\theta$$

$$= \pi a^3 \left|\frac{5\theta}{2} + \frac{3}{4}\sin 2\theta\right|_0^{2\pi} = \pi a^3\left[\frac{5}{2}2\pi + 0\right] = 5\pi^2 a^3.$$

9. Find the volume of a right pyramid whose height is h and has a square base with each side of length a.

Sol. Let the axis of the pyramid be along the y-axis. A typical cross section is a square with side s, where s is a function of y.

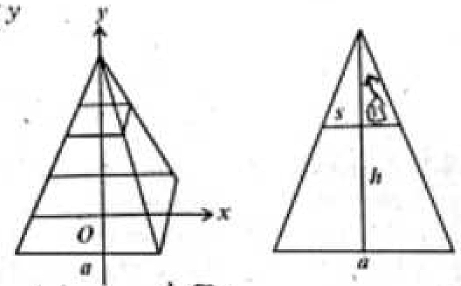
Now $\frac{s}{a} = \frac{y}{h}$ or $s = \frac{a}{h}y$

Area of the cross section

$$= \left(\frac{a}{h}\right)^2 y^2$$

$$\text{Volume} = \int_0^h \left(\frac{a}{h}\right)^2 y^2 dy$$

$$= \frac{a^2}{h^2}\int_0^h y^2\, dy = \frac{1}{3}a^3 h$$



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10. Find the volume of the solid that remains after boring a hole of radius a through the centre of a solid sphere of radius $r > a$.

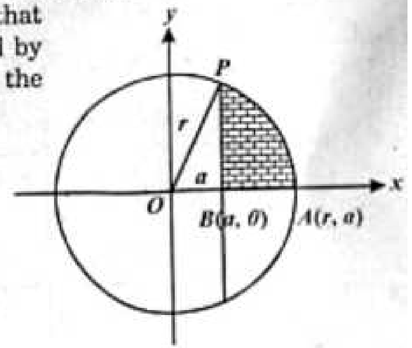
Sol. Suppose the sphere is generated by the right half of the circular disc $x^2 + y^2 \le r^2$ revolved about the y-axis. Let the hole be vertical with its centre line coincident with the y-axis.

The upper half of the solid that remains after boring is generated by revolving the shaded area about the y-axis. The required volume

$$= 2\int_a^r 2\pi x\sqrt{r^2 - x^2}\, dx$$

$$= 4\pi\left[-\frac{1}{3}(r^2 - x^2)^{3/2}\right]_a^r$$

$$= \frac{4\pi}{3}(r^2 - a^2)^{3/2}.$$



11. Find the volume formed by the revolution of the area enclosed by the loop of the curve $y^2 = x^2\frac{a-x}{a+x}$ about the x-axis.

Sol. The required volume is

$$V = \int_0^a \pi y^2 dx = \pi\int_0^a x^2\frac{a-x}{a+x}dx$$

$$= \pi \int_0^a \left(-x^2 + 2ax - 2a^2 + \frac{2a^3}{x+a} \right) dx$$

$$= \pi \left| -\frac{x^3}{3} + ax^2 - 2a^2x + 2a^3 \ln (x+a) \right|_0^a$$

$$= \pi \left(-\frac{4}{3}a^3 + 2a^3 \ln 2a - 2a^3 \ln a \right)$$

$$= 2\pi a^3 \left(\ln 2a - \ln a - \frac{2}{3} \right) = 2\pi a^3 \left(\ln \frac{2a}{a} - \frac{2}{3} \right).$$

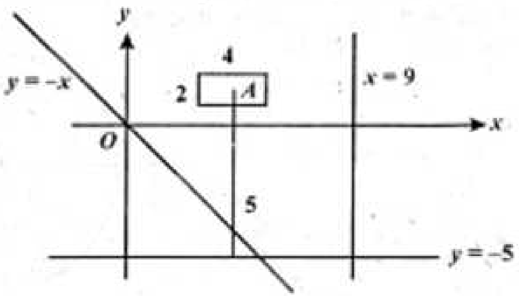
12. A doughnut-shaped solid, called **torus** (or **anchor ring**), is generated by revolving an area enclosed by a circle about a line that does not intersect the circle. Find the volume of the torus if the circle is $(x-b)^2 + y^2 = a^2$ and the line is the y-axis, $(0 < a < b)$.

Sol. Centre of gravity of the circle $(x-b)^2 + y^2 = a^2$ is its centre $(b, 0)$

Area of the circle is revolved about the y-axis. **By Theorem 9.26**, volume of the torus $= \pi a^2 (2\pi b) = 2\pi^2 a^2 b$.

13. A rectangular area whose length and breadth are 4 and 2 units respectively and whose centroid is at (4, 3) is revolved about (i) the straight line $x = 9$, (ii) the straight line $y = -5$ and (iii) the straight line $y = -x$. Find the volume generated in each case.

Sol.



(i) If $A = (4, 3)$ be the centroid of the rectangular area then its distance from the straight line $x = 9$ is $9 - 4 = 5$ units. Therefore, the distance convered by the point A about the line

$x = 9$ in one revolution is $2 . \pi . 5 = 10\pi$

Also area of the rectangle $= 4 \times 2 = 8$ square units.

Required volume $= 8 \times 10\pi$ cubic units.

(ii) Distance of the centroid A from the line $y = -5$ is

$3 - (-5) = 8$ units.

Distance covered by A is one revolution about $y = -5$ is

$2\pi . 8 = 16\pi$ units.

Volume of revolution in this case

$= 8 \times 16\pi = 128\pi$ cubic units

(iii) Perpendicular distance of the centroid $A = (4, 3)$ from the line

$y = -x$ or $x + y = 0$ is $\dfrac{4+3}{\sqrt{2}} = \dfrac{7}{\sqrt{2}}$.

Distance covered by the centroid in one revolution about the line

$$2\pi \frac{7}{\sqrt{2}} = 7\sqrt{2}\,\pi.$$

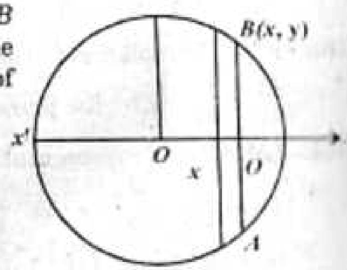
Required volume $= 8 \times 7\sqrt{2}\,\pi = 56\sqrt{2}\,\pi$ cubic units.

14. A solid has a circular base of radius 4 units. Find the volume of the solid if every plane section perpendicular to a fixed diameter is an equilateral triangle

Sol. Since the base is circular, we take the fixed diameter $x'Ox$, *i.e.*, the x-axis with centre at the origin O. If AB represents one section with breadth δx then AB also forms one side of the equilateral triangle. Equation of the circle is

$x^2 + y^2 = 16$.

$|AB| = 2y$.

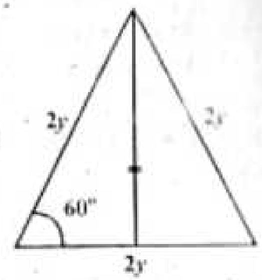


Now area of the equilateral triangle with each side $2y$

$$= \frac{1}{2} 2y . 2y \sin 60^\circ = \sqrt{3}\, y^2$$

Volume of one triangular strip $= \sqrt{3}\, y^2 \,\delta x$

Hence the required volume



$$= \int_{-4}^{4} (\sqrt{3}\, y^2)\, dx = 2\int_0^4 \sqrt{3}\,(16 - x^2) dx$$

$$= 2\sqrt{3} \left| 16x - \frac{x^3}{3} \right|_0^4 = 2\sqrt{3} \left(64 - \frac{64}{3} \right)$$

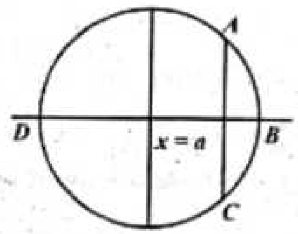
$$= \frac{256\sqrt{3}}{3} \text{ cubic units.}$$

15. Find the volumes of the two portions in which a sphere of radius r is divided by the plane $x = a$, $(a < r)$.

Sol. Sphere, being the solid generated by the circle $x^2 + y^2 = r^2$, when cut by the plane $x = a$, is divided into two portion ABC and ADC.

Volume of the portion ABC

$$= \int_a^r \pi y^2\,dx = \pi \int_a^r (r^2 - x^2)\,dx$$



$$= \pi \left| r^2 x - \frac{x^3}{3} \right|_a^r$$

$$= \pi \left(r^3 - \frac{r^3}{3} - r^2 a + \frac{a^3}{3} \right)$$

$$= \pi \left(\frac{2}{3} r^3 - r^2 a + \frac{a^3}{3} \right) = \frac{2\,\pi}{3} r^3 - \frac{a\pi}{3}(3r^2 - a^2).$$

But the volume of the sphere $= \frac{4}{3}\pi r^3$.

Hence volume of the portion ACD

$$= \frac{4}{3}\pi r^3 - \text{volume of the portion } ABC$$

$$= \frac{4}{3}\pi r^3 - \left[\frac{2\pi}{3} r^3 - \frac{a\pi}{3}(3r^2 - a^2) \right] = \frac{2\pi}{3} r^3 + \frac{a\pi}{3}(3r^2 - a^2).$$

16. Find the volume of the solid cut off from the elliptic paraboloid $\frac{x^2}{16} + \frac{y^2}{25} = z$ by the plane $z = 10$.

Sol. Let AB be a representative ellipse on the paraboloid.

Then $\frac{x^2}{16} + \frac{y^2}{25} = z \Rightarrow \frac{x^2}{16z} + \frac{y^2}{25z} = 1$ is an equation of this ellipse

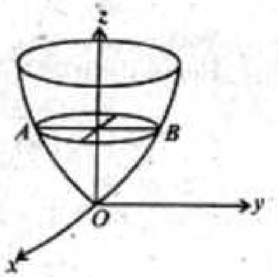
Its area is $\pi (4\sqrt{z})(5\sqrt{z})$,

$\left(\text{using } \pi ab \text{ as the area of the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right)$

\ Volume of the elliptic disc with breadth

δz is $(\pi (4\sqrt{z})(5\sqrt{z})\,\delta z)$.

The required volume is



$$V = \int_0^{10} \pi (4\sqrt{z})(5\sqrt{z})\,dz = 20\,\pi \int_0^{10} z\,dz$$

$$= 20\pi \left| \frac{z^2}{2} \right|_0^{10} = 1000\pi.$$

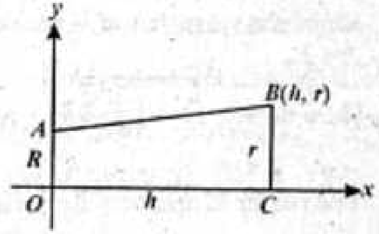
17. Find the volume of a frustrum of a right circular cone of altitude h, lower base radius R and upper base radius r.

Sol. The frustrum of the cone is generated by revolving about x-axis the area enclosed by the line AB, $x = 0$, $x = h$ and the x-axis.

Equation of the line AB is

$$y - R = \frac{r - R}{h} x$$

i.e., $y = R + \frac{r - R}{h} x$



Desired volume $= \int_0^h \pi y^2 dx$

$$= \pi \int_0^h \left(R + \frac{r - R}{h} x \right)^2 dx$$

$$= \frac{\pi h}{3(r - R)} \left[\left(R + \frac{r - R}{h} x \right)^3 \right]_0^h$$

$$= \frac{\pi h}{3(r - R)} (r^3 - R^3) = \frac{1}{3}\pi h (r^2 + rR + R^2).$$

If $R = 0$, the resulting solid is the complete right circular cone and the volume reduces to the familiar formula $\frac{1}{3}\pi r^2 h$.

Exercise Set 9.9 (Page 452)

1. Find the area of the surface of revolution generated by revolving about the x-axis the area bounded by an arc of the parabola $y^2 = 12x$ from $x = 0$ to $x = 3$.

Sol. Required area $A = 2\pi \int y\,ds$

Here $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \left(\frac{6}{y}\right)^2}\,dx$

$$= \sqrt{1 + \frac{36}{12x}}\,dx = \sqrt{\frac{x + 3}{x}}\,dx$$

Hence required area

$$= 2\pi \int_0^3 \sqrt{12x} \sqrt{\frac{x + 3}{x}}\,dx = 2\pi (2\sqrt{3}) \int_0^3 \sqrt{x + 3}\,dx$$

$$= 2\pi (2\sqrt{3}) \left| \frac{(x + 3)^{3/2}}{3/2} \right|_0^3 = 2\pi \frac{4}{\sqrt{3}} [(6)^{3/2} - (3)^{3/2}]$$

$$= 2\pi 4 \frac{3\sqrt{3}}{\sqrt{3}} [2\sqrt{2} - 1] = 12\,(2\sqrt{2} - 1).\,2\pi$$

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$= 24\,(2\sqrt{2}-1)\,\pi$ square units.

2. Find the area of the surface of revolution generated by revolving about the y-axis the area enclosed by the arc of $x = y^3$ from $y = 0$ to $y = 1$.

Sol. Here $ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy = \sqrt{1+9y^4}\,dy$

The required area $A = 2\pi\int_0^1 x\,ds = 2\pi\int_0^1 y^3\sqrt{1+9y^4}\,dy$

Put $1 + 9y^4 = t^2$ or $36y^3dy = 2t\,dt$

or $dy = \frac{1}{18y^3}t\,dt$

Also when $y = 0, t = 1$ and when $y = 1, t = \sqrt{10}$.

Therefore, $A = 2\pi\int_1^{\sqrt{10}}\frac{t}{18}t.dt = \frac{\pi}{9}\left|\frac{t^3}{3}\right|_1^{\sqrt{10}} = \frac{\pi}{27}(10\sqrt{10}-1)$ sq. units

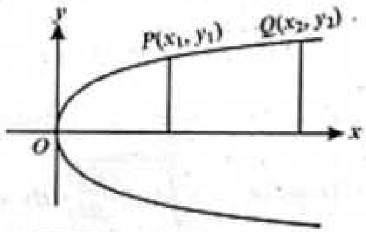
3. Find the surface area of a belt of the paraboloid formed by revolving the area bounded by the curve $y^2 = 4ax$ about the x-axis.

Sol. Let the belt be formed by revolving about x-axis the area enclosed by the curve, the x-axis, $x = x_1$ and $x = x_2$.

Here $ds = \sqrt{1+\left(\frac{2a}{y}\right)^2}\,dx$

$= \sqrt{1+\frac{4a^2}{4ax}}\,dx$

$= \sqrt{\frac{x+a}{x}}\,dx$



Required area

$$A = 2\pi\int y\,ds = 2\pi\int_{x_1}^{x_2}\sqrt{4ax}\sqrt{\frac{x+a}{x}}\,dx$$

$$= 4\pi\sqrt{a}\int_{x_1}^{x_2}(x+a)^{1/2}\,dx$$

$$= 4\pi\sqrt{a}\left|\frac{(x+a)^{3/2}}{3/2}\right|_{x_1}^{x_2} = \frac{8\pi\sqrt{a}}{3}\left((x_2+a)^{3/2}-(x_1+a)^{3/2}\right).$$

4. Find the surface area of a sphere of radius r.

Sol. The required surface area is generated by revolving the circle $x^2 + y^2 = r^2$ about the x-axis.

Here $ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1+\frac{x^2}{y^2}}\,dx$

$= \sqrt{\frac{y^2+x^2}{y^2}}\,dx = \sqrt{\frac{r^2}{r^2-x^2}}\,dx$

Required area

$$A = 2\pi\int y\,ds = 2\pi\int_{-r}^{r}\sqrt{r^2-x^2}\sqrt{\frac{r^2}{r^2-x^2}}\,dx$$

$$= 4\pi\int_0^r r\,dx = 4\pi\,[rx]_0^r = 4\pi r^2.$$

5. Find the area on a sphere of radius r included between two parallel planes at distances r_1 and r_2 from the centre, $(r_1 < r_2 < r)$.

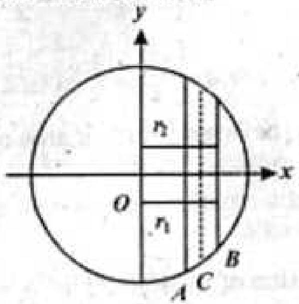
Sol. Let C be any plane intersecting the sphere. If y be the radius of this circle, then we have $y^2 = r^2 - x^2$ where x is the distance of the plane from the origin.

Area of small circular strip at C whose width is δx is

$2\pi y\,\delta s = 2\pi\sqrt{r^2-x^2}\,\delta s$

Required surface area

$$S = \int_{r_1}^{r_2} 2\pi\sqrt{r^2-x^2}\,\frac{ds}{dx}\,dx$$



But $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

$= \sqrt{1+\frac{x^2}{y^2}} = \sqrt{\frac{x^2+y^2}{y^2}}$

$= \sqrt{\frac{r^2}{y^2}} = \frac{r}{y} = \frac{r}{\sqrt{r^2-x^2}}$

Hence $S = \int_{r_1}^{r_2} 2\pi\sqrt{r^2-x^2}\,.\,\frac{r}{\sqrt{r^2-x^2}}\,dx$

$= 2\pi\,[x]_{r_1}^{r_2} = 2\pi\,(r_2 - r_1)$

6. Show that the surface of the solid obtained by revolving the area bounded by the arc of the curve $y = \sin x$ from $x = 0$ to $x = \pi$ about the x-axis is $2\pi\left[\sqrt{2} + \ln(\sqrt{2}+1)\right]$.

Sol. Since $\quad y = \sin x$, so $\dfrac{dy}{dx} = \cos x$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \cos^2 x}\,dx$$

Required area is

$$A = 2\pi\int_0^{\pi} y\,ds = 2\pi\int_0^{\pi} \sin x\,(1 + \cos^2 x)^{1/2}\,dx$$

$$= 4\pi\int_0^{\pi/2} \sin x\,(1 + \cos^2 x)^{1/2}\,dx$$

Put $\cos x = t$ or $-\sin x\,dx = dt$ in the above integral.

Then $A = 4\pi\displaystyle\int_1^0 -(1 + t^2)^{1/2}\,dt = 4\pi\int_0^1 (1 + t^2)^{1/2}\,dt$

$$= 4\pi\left[\frac{t\sqrt{1+t^2}}{2} + \frac{1}{2}\ln\frac{t + \sqrt{t^2+1}}{1}\right]_0^1$$

$$= 4\pi\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln(\sqrt{2}+1)\right] = 2\pi\left[\sqrt{2} + \ln(\sqrt{2}+1)\right].$$

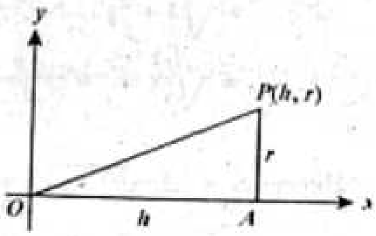
7. Find the lateral surface area of a right circular cone of height h and base radius r.

Sol. A right circular cone is generated by revolving the line OP about the x-axis.

Equation of the line OP is $y = \dfrac{r}{h}x$

Required area

$$= \int_0^h 2\pi y\,ds$$



$$= 2\pi\int_0^h \left(\frac{r}{h}x\right)\sqrt{1 + \frac{r^2}{h^2}}\,dx$$

$$= 2\pi\,\frac{r}{h^2}\sqrt{r^2 + h^2}\left[\frac{x^2}{2}\right]_0^h = \pi r\sqrt{r^2 + h^2}$$

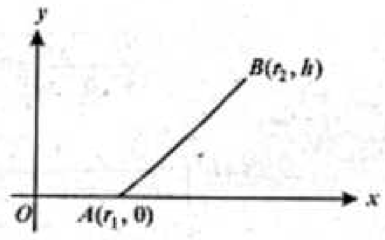
$= \pi r l$, where $l = |OP| =$ slant height of the cone.

8. Find the surface area generated by revolving the line segment between $(r_1, 0)$ and (r_2, h) about the y-axis.

Sol. Equation of the line AB is

$$y = \frac{h}{r_2 - r_1}(x - r_1)$$

or $\quad x = r_1 + \dfrac{r_2 - r_1}{h}y$



Area of the surface generated by revolving AB about the y-axis

$$= 2\pi\int_0^h x\,ds = 2\pi\int_0^h x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy$$

$$= 2\pi\int_0^h x\sqrt{1 + \left(\frac{r_2 - r_1}{h}\right)^2}\,dy$$

$$= 2\pi\int_0^h \left(r_1 + \frac{r_2 - r_1}{h}y\right)\sqrt{1 + \left(\frac{r_2 - r_1}{h}\right)^2}\,dy$$

$$= \frac{2\pi}{h}\sqrt{h^2 + (r_2 - r_1)^2}\left[r_1 y + \frac{r_2 - r_1}{h}\cdot\frac{y^2}{2}\right]_0^h$$

$$= 2\pi\frac{|AB|}{h}\left[r_1 h + \frac{r_2 - r_1}{2}h\right] = 2\pi\,|AB|\,\frac{r_1 + r_2}{2}.$$

9. Prove that the surface area of the prolate spheroid formed by the revolution of an area enclosed by an ellipse of eccentricity e about its major axis is

$$2\,(\text{Area of the ellipse}) \times \left(\sqrt{1 - e^2} + \frac{\arcsin e}{e}\right).$$

Sol. Let the ellipse be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{b^4x^2}{a^4y^2}}\,dx = \frac{\sqrt{a^4y^2 + b^4x^2}}{a^2y}\,dx$$

Required area

$$A = 2\pi \int y\, ds = 2\pi \int_{-a}^{a} \frac{y\sqrt{a^4y^2 + b^4x^2}}{a^2y}\, dx$$

$$= \frac{4\pi}{a^2} \int_0^a \sqrt{a^4\left(\frac{a^2 - x^2}{a^2}\right) b^2 + b^4x^2}\; dx$$

$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - x^2\,(a^2 - b^2)}\, dx$$

But $b^2 = a^2\,(1 - e^2)$ or $a^2 - b^2 = a^2e^2$

$$\text{Hence } A = \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - a^2e^2x^2}\, dx$$

$$= \frac{4\pi b}{a} \int_0^a \sqrt{a^2 - e^2x^2}\, dx = \frac{4\pi b}{a} e \int_0^a \sqrt{\frac{a^2}{e^2} - x^2}\, dx$$

$$= \frac{4\pi b}{a} e \left[\frac{x\sqrt{\frac{a^2}{e^2} - x^2}}{2} + \frac{a^2}{2e^2} \arcsin \frac{xe}{a}\right]_0^a$$

$$= \frac{4\pi b}{a} e \left[\frac{a\sqrt{a^2 - e^2a^2}}{2e} + \frac{a^2}{2e^2} \arcsin e\right]$$

$$= 2\pi ab\left[\sqrt{1 - e^2} + \frac{1}{e} \arcsin e\right]$$

$$= 2(\text{Area of the ellipse})\left[\sqrt{1 - e^2} + \frac{1}{e} \arcsin e\right].$$

10. Prove that the surface area of the ellipsoid formed by the revolution of the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about its major axis is $2\pi\left(a^2 + \frac{b^2}{e} \ln\sqrt{\frac{1+e}{1-e}}\right)$, e being the eccentricity of the ellipse.

Sol. Here, as in Problem 9,

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \frac{a^4\,y^2}{b^4\,x^2}}\, dy = \frac{\sqrt{b^4\,x^2 + a^4\,y^2}}{b^2x}\, dy$$

Required area

$$A = 2\pi \int_{-b}^{b} \frac{\sqrt{b^4x^2 + a^4\,y^2}}{b^2}\, dy$$

$$= \frac{4\pi}{b^2} \int_0^b \sqrt{b^2\,(a^2\,b^2 - a^2\,y^2) + a^4y^2}\, dy$$

$$= \frac{4\pi}{b^2} a \int_0^b \sqrt{b^4 + (a^2 - b^2)\,y^2}\, dy$$

$$= \frac{4\pi a}{b^2} \int_0^b \sqrt{b^4 + a^2\,e^2y^2}\, dy = \frac{4\pi a^2e}{b^4} \int_0^b \sqrt{\frac{b^4}{a^2e^2} + y^2}\, dy$$

$$= \frac{4\pi a^2e}{b^2} \cdot \left[\frac{y\sqrt{\frac{b^4}{a^2e^2} + y^2}}{2} + \frac{b^4}{2a^2e^2} \ln \frac{y + \sqrt{y^2 + \frac{b^4}{a^2e^2}}}{b^2/ae}\right]_0^b$$

$$= \frac{2\pi a^2e}{b^2}\left[\frac{b\sqrt{b^4 + a^2e^2b^2}}{ae} + \frac{b^4}{a^2e^2} \ln \frac{bae + \sqrt{b^2a^2e^2 + b^4}}{b^2}\right]$$

$$= \frac{2\pi a^2e}{b^2}\left[\frac{b^2\sqrt{b^2 + a^2e^2}}{ae} + \frac{b^4}{a^2e^2} \ln \frac{ae + \sqrt{a^2e^2 + b^2}}{b}\right]$$

$$= 2\pi a^2\, e\left[\frac{a}{ae} + \frac{b^2}{a^2e^2} \ln \frac{ae + a}{b}\right] = 2\pi\left[a^2 + \frac{b^2}{e} \ln \frac{a(1 + e)}{a\sqrt{1 - e^2}}\right]$$

$$= 2\pi\left[a^2 + \frac{b^2}{e} \ln \sqrt{\frac{1 + e}{1 - e}}\right], \text{ using } b^2 = a^2\,(1 - e^2).$$

11. Find the area of the surface generated by revolving the curve $x = e^\theta \sin\theta, y = e^\theta \cos\theta$, from $\theta = 0$ to $\theta = \pi$ about the y-axis.

Sol. $\frac{dx}{d\theta} = e^\theta\,(\sin\theta + \cos\theta), \frac{dy}{d\theta} = e^\theta\,(\cos\theta - \sin\theta)$

$$\frac{ds}{d\theta} = \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = (2e^{2\theta})^{1/2} = \sqrt{2}\, e^\theta$$

$$\text{Desired area} = 2\pi \int_0^\pi x\,ds = 2\pi \int_0^\pi e^\theta \sin\theta . \sqrt{2}\, e^\theta\, d\theta$$

$$= 2\sqrt{2}\,\pi \int_0^{\pi} e^{2\theta}\sin\theta\, d\theta$$

$$= 2\sqrt{2}\,\pi\,\frac{1}{5}\left[(2e^{2\theta}\sin\theta - \cos\theta)\right]_0^{\pi} = \frac{2\sqrt{2}\,\pi\,[2e^{2\pi}+1]}{5}$$

12. Find the area of the surface generated by revolving the area enclosed by the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ about $y = 0$.

Sol. Here $\frac{dx}{d\theta} = a(1-\cos\theta)$, $\quad \frac{dy}{d\theta} = a\sin\theta$

$$\frac{ds}{d\theta} = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2} = \sqrt{a^2(1-\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{(1-\cos\theta)^2 + \sin^2\theta} = a\sqrt{2 - 2\cos\theta} = 2a\sin\frac{\theta}{2}$$

Required surface area is

$$A = 2\int_0^{\pi} 2\pi y\frac{ds}{d\theta}d\theta = 4\pi\int_0^{\pi} a(1-\cos\theta)\cdot 2a\sin\frac{\theta}{2}d\theta$$

$$= 8\pi a^2\int_0^{\pi} 2\sin^2\frac{\theta}{2}\cdot\sin\frac{\theta}{2}d\theta = 16\pi a^2\int_0^{\pi}\sin^3\frac{\theta}{2}d\theta$$

Put $\frac{\theta}{2} = z$ or $d\theta = 2dz$

Then $A = 16\pi a^2 \int_0^{\pi/2} \sin^3 z\cdot(2dz)$

$$= 32\pi a^2\cdot\frac{2}{3} \quad \text{(by Wallis formula)}$$

$$= \frac{64\pi a^2}{3}.$$

13. Show that the surface area formed by revolving the area enclosed by the loop of the curve $3ay^2 = x(x-a)^2$ about the x-axis is $\pi a^2/3$.

Sol. Here we have $6ay\frac{dy}{dx} = (x-a)^2 + 2x(x-a)$

$$= (x-a)(3x-a)$$

or $\quad \frac{dy}{dx} = \frac{(x-a)(3x-a)}{6ay}$

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}dx = \sqrt{1+\frac{(x-a)^2(3x-a)^2}{36a^2y^2}}dx$$

$$= \sqrt{1+\frac{(3x-a)^2}{12a\cdot x}}dx = \sqrt{\frac{12ax+(3x-a)^2}{12ax}}dx$$

$$= \sqrt{\frac{(3x+a)^2}{12ax}}dx = \frac{3x+a}{\sqrt{12ax}}dx$$

Hence the required area $= 2\pi\int_0^a \sqrt{\frac{x}{3a}}(x-a)\left(\frac{3x+a}{\sqrt{12ax}}\right)dx$

$$= \frac{2\pi}{6a}\int_0^a (3x^2 - 2ax - a^2)\,dx$$

$$= \frac{\pi}{3a}\left[x^3 - ax^2 - a^2x\right]_0^a = \frac{\pi}{3a}\cdot a^3 = \frac{\pi}{3}a^2.$$

14. Find the surface area of the **torus** formed by revolving a disc of radius a about a straight line in its plane at a distance b from the centre $(b > a)$.

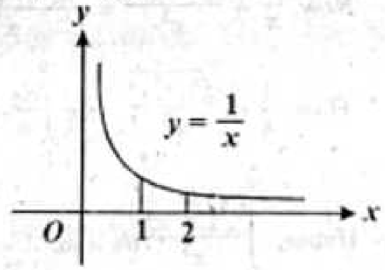
Sol. The circumference of the disc $= 2\pi a$. Also the c.g. describes a circle of radius b. Hence the required area

$$= (2\pi a)(2\pi b) = 4\pi^2 ab.$$

15. The curve $y = \frac{1}{x}$, $1 \le x \le 2$, is rotated about the x-axis. Find the area of the resulting surface. If $1 \le x < \infty$, show that the volume of solid generated is finite but its surface area is infinite.

Sol. Required area $= 2\pi\int_1^2 y\,ds = 2\pi\int_1^2 \frac{1}{x}\sqrt{1+\left(\frac{dy}{dx}\right)^2}dx$

$$= 2\pi\int_1^2 \frac{\sqrt{1+x^4}}{x^3}dx$$



Put $x^2 = \tan\theta$, $2x\,dx = \sec^2\theta\,d\theta$.

Then $I = \int\frac{\sqrt{1+x^4}}{x^3}dx$

$$= \int\frac{\sec\theta\sec^2\theta\,d\theta}{2\tan^2\theta} = \int\frac{d\theta}{2\sin^2\theta\cos\theta}$$

Again put $\sin\theta = u$, $\cos\theta\, d\theta = du$, so that

$$I = \int \frac{du}{2u^2(1-u^2)} = \int\left(\frac{1}{u^2+\frac{1}{1-u^2}}\right)du = -\frac{1}{u}+\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|$$

$$= \frac{-1}{\sin\theta}+\frac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|$$

$$= -\left(\frac{\sqrt{1+x^4}}{x^2}\right)+\frac{1}{2}\ln\left|\frac{1+\frac{x^2}{\sqrt{1+x^4}}}{1-\frac{x^2}{\sqrt{1+x^4}}}\right|$$

$$\text{Area} = 2\pi\left[-\frac{\sqrt{1+x^4}}{x^2}+\frac{1}{2}\ln\frac{\sqrt{1+x^4}+x^2}{\sqrt{1+x^4}-x^2}\right]_1^2$$

$$= 2\pi\left[-\frac{\sqrt{17}}{4}+\frac{1}{2}\ln\frac{\sqrt{17}+4}{\sqrt{17}-4}+\sqrt{2}-\frac{1}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

If $1 \le x < \infty$, then

$$\text{Volume} = \int_1^\infty \pi y^2\,dx = \pi\int_1^\infty \frac{dx}{x^2} = \pi\left[-\frac{1}{x}\right]_\infty^1 = \pi.$$

$$\text{And area} = 2\pi\int_1^\infty \frac{\sqrt{1-x^4}}{x^3}\,dx$$

We know that if $f(x) \le g(x)$ and if

$$\int_1^\infty f(x)\,dx = \infty, \text{ then } \int_1^\infty g(x)\,dx = \infty.$$

Now, $\dfrac{-1}{x}+\dfrac{\sqrt{1+x^4}}{x^3} = \dfrac{-x^2+\sqrt{1+x^4}}{x^3} \ge 0$ for all $x \ge 1$.

Thus, $\dfrac{1}{x} < \dfrac{\sqrt{1+x^4}}{x^3}$. But $\displaystyle\int_1^\infty \frac{dx}{x} = \infty$.

$$\text{Hence, } \int_1^\infty \frac{\sqrt{1+x^4}}{x^3}\,dx = \infty.$$

Exercise Set 9.10 (Page 456)

1. Find the area of solid generated by revolving the circle $r = a$, $0 \le \theta \le \frac{\pi}{4}$, about the polar axis.

Sol. $x = a\cos\theta, y = a\sin\theta$ are parametric equations of the circle $r = a$

$$\left(\frac{ds}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2 = a^2$$

$$\text{Desired area} = 2\pi\int_0^{\pi/4} y\,ds = 2\pi\int_0^{\pi/4} a\sin\theta\cdot a\,d\theta$$

$$= 2\pi a^2[-\cos\theta]_0^{\pi/4} = 2\pi a^2\left(1-\frac{1}{\sqrt{2}}\right) = 2\pi a^2\left(1-\frac{1}{\sqrt{2}}\right)$$

2. Find the area of the surface generated by revolving $r = 2a\sin\theta$ about the polar axis.

Sol. $r = 2a\sin\theta = f(\theta)$

$$f'(\theta) = \frac{dr}{d\theta} = 2a\cos\theta$$

$$\left(\frac{ds}{d\theta}\right)^2 = [f(\theta)]^2+[f'(\theta)]^2 = 4a^2\sin^2\theta+4a^2\cos^2\theta = 4a^2$$

$$ds = 2a\,d\theta.$$

$$\text{Required area} = 2\,.\,2\pi\int_0^{\pi/2} f(\theta)\sin\theta\,ds$$

$$= 4\pi\int_0^{\pi/2} 2a\sin^2\theta\,.\,2a\,d\theta = 16\pi a^2\frac{1}{2}\frac{\pi}{2} = 4\pi^2a^2.$$

3. The arc of the spiral $r = e^\theta$ from (1, 0) to (e, 1) is revolved about the line $\theta = \frac{\pi}{2}$. Find the area of the resulting surface.

Sol. $r = f(\theta) = e^\theta$

$$\left(\frac{ds}{d\theta}\right)^2 = [f(\theta)]^2+[f'(\theta)]^2 = e^{2\theta}+e^{2\theta} = 2e^{2\theta}$$

$$\text{Required area} = 2\pi\int_0^1 x\,ds = 2\pi\int_0^1 e^\theta\cos\theta\sqrt{2}\,e^\theta\,d\theta$$

$$= 2\sqrt{2}\,\pi \int_0^1 e^{2\theta} \cos\theta\, d\theta$$

$$= 2\sqrt{2}\,\pi \left[\frac{e^{2\theta}}{5}(2\cos\theta + \sin\theta)\right]_0^1$$

$$= \frac{2\sqrt{2}\,\pi}{5}\,[e^2(2\cos 1 + \sin 1) - 2].$$

4. Find the surface area of the solid generated by revolving the curve $r = e^{\theta/2}$ about the polar axis, $0 \le \theta \le \pi$.

Sol. $r = f(\theta) = e^{\theta/2}$

$$f'(\theta) = \frac{1}{2} e^{\theta/2}$$

$$\left(\frac{ds}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2 = e^{\theta} + \frac{1}{4} e^{\theta} = \frac{5}{4} e^{\theta}.$$

$$\text{Required area} = 2\pi \int_0^{\pi} e^{\theta/2} \sin\theta \frac{\sqrt{5}}{2} e^{\theta}\, d\theta$$

$$= \sqrt{5}\,\pi \int_0^{\pi} e^{\theta} \sin\theta\, d\theta = \frac{\sqrt{5}\pi}{2}\,[e^{\theta}(\sin\theta - \cos\theta)]_0^{\pi}$$

$$= \frac{\sqrt{5}\pi}{2}(e^{\pi} + 1).$$

5. Prove that the volume of the solid generated by the revolution of the area enclosed by the limacon $r = a + b\cos\theta$, $(a > b)$ is $\frac{4}{3}\pi a\,(a^2 + b^2)$.

Sol. Required volume

$$= \frac{2}{3}\pi \int_0^{\pi} r^3 \sin\theta\, d\theta = \frac{2}{3}\pi \int_0^{\pi} (a + b\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi\left[-\frac{(a + b\cos\theta)^4}{4b}\right]_0^{\pi} = \frac{-2}{3}\pi\left[\frac{(a-b)^4 - (a+b)^4}{4b}\right]$$

$$= -\frac{2}{3}\frac{\pi}{4b}\,[a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 - a^4 - 4a^3b - 6a^2b^2 - 4ab^3 - b^4]$$

$$= \frac{\pi}{6b}\,[8a^3b + 8ab^3] = \frac{4}{3}\pi a\,[a^2 + b^2]$$

6. Find the volume and area of the surface of the solid generated by the revolution of the area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line.

Sol. Required volume V is given by

$$V = \int_0^{\pi/4} \frac{2\pi}{3} r^3 \sin\theta\, d\theta = \frac{4\pi}{3}\int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \sin\theta\, d\theta$$

$$= \frac{4\pi a^3}{3}\int_0^{\pi/4} (2\cos^2\theta - 1)^{3/2} \sin\theta\, d\theta$$

Put $\sqrt{2}\cos\theta = z$ or $-\sqrt{2}\sin\theta\, d\theta = dz$

$$V = \frac{4\pi a^3}{3}\int_{\sqrt{2}}^{1} (z^2 - 1)^{3/2}\left(-\frac{dz}{\sqrt{2}}\right) = \frac{4\pi a^3}{3\sqrt{2}}\int_1^{\sqrt{2}} (z^2 - 1)^{3/2}\, dz$$

Now let $I = \int (z^2 - 1)^{3/2}\, dz = z(z^2 - 1)^{3/2} - \int z \cdot \frac{3}{2}(z^2 - 1)^{1/2}\, 2z\, dz$

$$= z(z^2 - 1)^{3/2} - 3\int z^2 (z^2 - a)^{1/2}\, dz$$

$$= z(z^2 - 1)^{3/2} - 3\int (z^2 - 1 + 1)(z^2 - 1)^{1/2}\, dz$$

$$= z(z^2 - 1)^{3/2} - 3\int (z^2 - 1)^{3/2}\, dz - 3\int (z^2 - 1)^{1/2}\, dz$$

Thus $4I = z(z^2 - 1)^{3/2} - 3\int (z^2 - 1)^{1/2}\, dz$

$$= z(z^2 - 1)^{3/2} - 3\left[\frac{z(z^2 - 1)^{1/2}}{2} - \frac{1}{2}\ln\left(z + (z^2 - 1)^{1/2}\right)\right]$$

or $$I = \frac{z(z^2 - 1)^{3/2}}{4} - \frac{3}{8}\left[z(z^2 - 1)^{1/2} - \ln\left(z + \sqrt{z^2 - 1}\right)\right]$$

Hence, $$V = \frac{4\pi a^3}{3\sqrt{2}}\left[\left|\frac{z(z^2 - 1)^{3/2}}{4} - \frac{3}{8}\left[z(z^2-1)^{1/2} - \ln\left(z + \sqrt{z^2 - 1}\right)\right]\right|\right]_1^{\sqrt{2}}$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\left[\frac{\sqrt{2}}{4} - \frac{3}{8}(\sqrt{2}) - \ln(\sqrt{2} + 1)\right]$$

$$= \frac{\pi a^3}{3\sqrt{2}}\left[\sqrt{2} - \frac{3\sqrt{2}}{2} + \frac{3}{2}\ln(\sqrt{2} + 1)\right]$$

$$= \frac{\pi a^3}{3\sqrt{2}}\left[\frac{-\sqrt{2}}{2} + \frac{3}{2}\ln(\sqrt{2}+1)\right]$$

$$= \frac{\pi a^3}{2}\left[\frac{1}{\sqrt{2}}\ln(\sqrt{2}+1) - \frac{1}{3}\right].$$

Surface S is given by

$$S = 2\int_0^{\pi/4} 2\pi(r\sin\theta)\,ds = 4\pi\int_0^{\pi/4} r\sin\theta\,\frac{ds}{d\theta}\,d\theta,$$

where $\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$

$$= a\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} = \frac{a}{\sqrt{\cos 2\theta}}$$

$$S = 4\pi\int_0^{\pi/4} a.\sqrt{\cos 2\theta}.\sin\theta.\frac{a}{\sqrt{\cos 2\theta}}\,d\theta$$

$$= 4\pi a^2\int_0^{\pi/4}\sin\theta\,d\theta = -4\pi a^2\,|\cos\theta|_0^{\pi/4}$$

$$= -4\pi a^2\left[\frac{1}{\sqrt{2}} - 1\right] = (4 - 2\sqrt{2})\,\pi a^2.$$

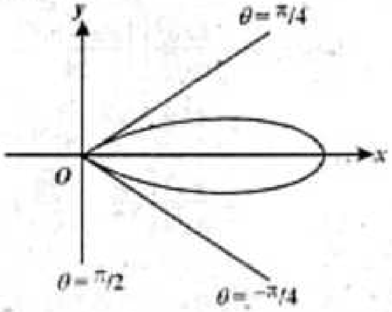
7. Show that the volume of the solid formed by revolving the area bounded by one loop of the curve $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$ is $\frac{\pi^2 a^3}{4\sqrt{2}}$.

Sol. Here the limits of integration are from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$

Therefore, the required volume is

$$V = \int_{-\pi/4}^{\pi/4}\frac{2}{3}\pi r^3\cos\theta\,d\theta$$

$$= \frac{4}{3}\pi.\int_0^{\pi/4} r^3\cos\theta\,d\theta$$



$$= \frac{4}{3}\pi a^3\int_0^{\pi/4}(\cos 2\theta)^{3/2}\cos\theta\,d\theta$$

Put $\sqrt{2}\sin\theta = \sin t$ or $\sqrt{2}\cos\theta\,d\theta = \cos t\,dt$ and we have

$$V = \frac{4\pi a^3}{3}\int_0^{\pi/2}(1-\sin^2 t)^{3/2}\frac{\cos t\,dt}{\sqrt{2}}$$

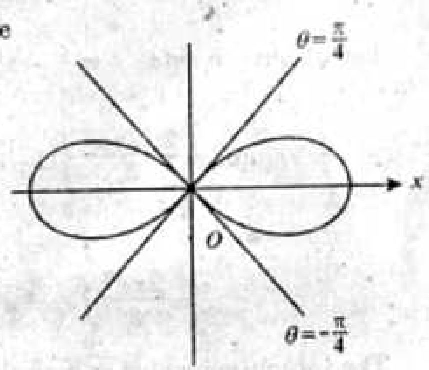
or $$V = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}\cos^3 t\cos t\,dt = \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}\cos^4 t\,dt$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}, \text{(by Wallis formula)} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$

8. The area enclosed by the lemniscate $r^2 = a^2\cos 2\theta$ revolves about a tangent at the pole. Show that the volume and area of the surface of the solid generated are respectively $\frac{1}{4}\pi^2 a^3$ and $4\pi a^2$.

Sol. Tangent to the curve at the pole are $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{\pi}{4}$.

For one loop (on the right hand side) θ varies from $\frac{-\pi}{4}$ to $\frac{\pi}{4}$. Therefore, the volume generated by one loop is



$$= \int_{-\pi/4}^{\pi/4}\frac{2}{3}\pi r^3\sin\left(\theta + \frac{\pi}{4}\right)d\theta$$

$$= \frac{2}{3}\pi\int_{-\pi/4}^{\pi/4} a^3(\cos 2\theta)^{3/2}\left[\frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)\right]d\theta$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^{3/2} \cos\theta\, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\cos 2\theta)^{3/2} \sin\theta\, d\theta \right]$$

$$= \frac{\sqrt{2}}{3}\pi a^3 \left[\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (2\cos^2\theta - 1)^{3/2} \sin\theta\, d\theta \right]$$

The first integral is even (for if we replace θ by $-\theta$ we get the same integral), therefore we can replace it by

$$2\int_0^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta$$

The second integral is odd, it vanishes and so the volume is

$$= \frac{\sqrt{2}\pi a^3}{3} . 2\int_0^{\frac{\pi}{4}} (1 - 2\sin^2\theta)^{3/2} \cos\theta\, d\theta$$

Let $\sqrt{2}\sin\theta = \sin\phi$ or $\sqrt{2}\cos\theta\, d\theta = \cos\phi\, d\phi$.

$$\text{Volume} = \frac{2\sqrt{2}\pi a^3}{3} \int_0^{\frac{\pi}{4}} (1 - \sin^2\phi)^{3/2} \frac{1}{\sqrt{2}} \cos\phi\, d\phi$$

$$= \frac{2\pi a^3}{3} \int_0^{\frac{\pi}{4}} \cos^4\phi\, d\phi = \frac{2\pi a^3}{3} \cdot \frac{3\cdot 1}{4\cdot 2}\frac{\pi}{2} = \frac{\pi^2 a^3}{8}.$$

The volume generated by both the loops is

$$2 . \frac{\pi^2 a^3}{8} = \frac{\pi^2 a^3}{4}.$$

Now we find the surface area generated by the given curve. Surface area generated by one loop is

$$S = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r \sin\left(\theta + \frac{\pi}{4}\right) \frac{ds}{d\theta} d\theta$$

Here $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2\cos 2\theta + \frac{a^2\sin^2 2\theta}{\cos 2\theta}}$$

$$= a\sqrt{\frac{\cos^2 2\theta + \sin^2 2\theta}{\cos 2\theta}} = \frac{a^2}{r}.$$

$$\text{Hence } S = 2\pi \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r \left[\frac{1}{\sqrt{2}}(\sin\theta + \cos\theta)\right] \frac{a^2}{r} d\theta$$

$$= \frac{2\pi a^2}{\sqrt{2}} \left| \sin\theta - \cos\theta \right|_{-\pi/4}^{\pi/4} = \frac{2\pi a^2}{\sqrt{2}}\left[\frac{2}{\sqrt{2}}\right] = 2\pi a^2$$

The surface generated by both the loops is $2.2\pi a^2 = 4\pi a^2$.

9. Find the area of the surface generated by revolving the area enclosed by the upper half of the cardioid $r = a(1 - \cos\theta)$ about the initial line.

Sol. The required surface S is

$$= \int_0^{\pi} 2\pi r \sin\theta \frac{ds}{d\theta} d\theta$$

Here $$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2 - 2\cos\theta}$$

$$= a \cdot \sqrt{2} . \sqrt{2}\sin\frac{\theta}{2} . = 2a\sin\frac{\theta}{2}$$

$$\text{Hence } S = \int_0^{\pi} 2\pi\left(a(1 - \cos\theta) . \sin\theta . 2a\sin\frac{\theta}{2}\right) d\theta$$

$$= 4\pi a^2 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right) 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} . \sin\frac{\theta}{2} d\theta$$

$$= 16\pi a^2 \int_0^{\pi} \sin^4\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$$

$$= 16\pi a^2\, 2 . \left[\frac{\sin^5 \theta/2}{5}\right]_0^{\pi} = \frac{32}{5}\pi a^2.$$