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Calculus With Analytic Geometry

Calculus
With
Analytic Geometry

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Chapter 6

PLANE CURVES - I

Exercise Set 6.1 (Page 234)

Examine whether each of the given equations represents two straight lines. If so, find the equation of each straight line (Problems 1 – 5):

1. $10xy + 8x - 15y - 12 = 0$

Sol. Here $a = 0, b = 0, h = 5, g = 4, f = -\frac{15}{2}, c = -12,$

Now, $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 0 & 5 & 4 \\ 5 & 0 & -\frac{15}{2} \\ 4 & -\frac{15}{2} & -12 \end{vmatrix} = -5(-60 + 30) + 4\left(-\frac{75}{2}\right)$

$= 150 - 150 = 0$

Hence the given equation represents two straight lines.

$10xy + 8x - 15y - 12 = 0$, my be written as

$2x(5y + 4) - 3(5y + 4) = 0$

or $(2x - 3)(5y + 4) = 0$

Thus $2x - 3 = 0, 5y + 4 = 0$ are the lines represented by the given equation.

2. $2x^2 - xy + 5x - 2y + 2 = 0.$

Sol. Here, $a = 2, b = 0, h = -\frac{1}{2}, g = \frac{5}{2}, f = -1, c = 2,$

$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 2 & -\frac{1}{2} & \frac{5}{2} \\ -\frac{1}{2} & 0 & -1 \\ \frac{5}{2} & -1 & 2 \end{vmatrix} = 2(-1) + \frac{1}{2}\left(-1 + \frac{5}{2}\right) + \frac{5}{2}\left(\frac{1}{2}\right)$

$= -2 + \frac{3}{4} + \frac{5}{4} = -2 + 2 = 0$

Thus the given equation represents two straight lines.

Now to obtain an equation of each line, we solve the given equation as a quadratic in x. The equation may be written as

$2x^2 + (5 - y)x + (2 - 2y) = 0$

$$x = \frac{-(5-y) \pm \sqrt{(5-y)^2 - 8(2-2y)}}{4} = \frac{(y-5) \pm \sqrt{y^2+6y+9}}{4}$$

$$= \frac{(y-5) \pm \sqrt{(y+3)^2}}{4} = \frac{(y-5) \pm (y+3)}{4}$$

$$= \frac{y-5+y+3}{4}, \frac{y-5-y-3}{4} = \frac{y-1}{2}, -2$$

Equations of the lines are

$$2x - y + 1 = 0 \quad \text{and} \quad x + 2 = 0$$

3. $6x^2 - 17xy - 3y^2 + 22x + 10y - 8 = 0$

Sol. Here $a = 6, b = -3, h = \frac{-17}{2}, g = 11, f = 5, c = -8,$

and $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 6(-3)(-8) + 2 \times 5 \times 11 \times \left(\frac{-17}{2}\right) - 6(5)^2 - (-3)(11)^2 - (-8)\left(\frac{-17}{2}\right)^2$$

$$= 144 - 55 \times 17 - 6 \times 25 + 3 \times 121 + 8 \cdot \frac{289}{4}$$

$$= 144 - 935 - 150 + 363 + 578 = 1085 - 1085 = 0$$

Thus the given equation represents a pair of lines. The equation may be written as

$$6x^2 + (22 - 17y)x + (-3y^2 + 10y - 8) = 0$$

$$x = \frac{-(22-17y) \pm \sqrt{(22-17y)^2 - 24(-3y^2+10y-8)}}{12}$$

$$= \frac{-(22-17y) \pm \sqrt{484 + 289y^2 - 748y + 72y^2 - 240y + 192}}{12}$$

$$= \frac{-(22-17y) + \sqrt{361y^2 - 988y + 676}}{12}$$

$$= \frac{(-22+17y) \pm (19y-26)}{12} = -4 + 3y, \frac{2-y}{6}$$

Equations of the lines are

$$x - 3y + 4 = 0 \quad \text{and} \quad 6x + y - 2 = 0$$

4. $10x^2 - 23xy - 5y^2 - 29x + 32y + 21 = 0$

Sol. Here $a = 10, b = -5, h = \frac{-23}{2}, g = \frac{-29}{2}, f = 16, c = 21,$

Now, $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= -10 \times (-5) \times 21 + 2\left(16 \times \frac{-29}{2} \times \frac{-23}{2}\right) - 10 \times (16)^2 - (-5)\left(\frac{-29}{2}\right)^2 - 21\left(\frac{-23}{2}\right)^2$$

$$= -1050 + 5336 - 2560 + \frac{4205}{4} - \frac{11109}{4}$$

$$= 1726 + \frac{4205 - 11109}{4} = 1726 + \frac{-6904}{4} = 1726 - 1726 = 0$$

Hence the equation represents a pair of lines. The given equation can be written as

$$10x^2 - (23y + 29)x + (-5y^2 + 32y + 21) = 0$$

$$x = \frac{(23y+29) \pm \sqrt{(23y+29)^2 - 40(-5y^2+32y+21)}}{20}$$

$$= \frac{(23y+29) \pm \sqrt{529y^2 + 1334y + 841 + 200y^2 - 1280y - 840}}{20}$$

$$= \frac{(23y+29) \pm \sqrt{729y^2 + 54y + 1}}{20} = \frac{(23y+29) \pm (27y+1)}{20}$$

$$= \frac{5y+3}{2}, \frac{-y+7}{5}$$

Equations of the lines are

$$2x - 5y - 3 = 0 \quad \text{and} \quad 5x + y - 7 = 0$$

5. $6x^2 - 15y^2 - xy + 16x + 24y = 0$

Sol. Here $a = 6, b = -15, h = \frac{-1}{2}, g = 8, f = 12, c = 0,$

and $abc + 2fgh - af^2 - bg^2 - ch^2$

$$= 0 - 96 - 6 \times 144 + 15 \times 64 - 0$$

$$= -96 - 864 + 960 = -960 + 960 = 0$$

The given equation represents a pair of lines. It may be written as

$$6x^2 + (-y + 16)x + (-15y^2 + 24y) = 0$$

$$x = \frac{-(16-y) \pm \sqrt{(16-y)^2 - 24(-15y^2 + 24y)}}{12}$$

$$= \frac{(y-16) \pm \sqrt{256 + y^2 - 32y + 360y^2 - 576y}}{12}$$

$$= \frac{(y-16) \pm \sqrt{361y^2 - 608y + 256}}{12} = \frac{(y-16) \pm (19y-16)}{12}$$

$$= \frac{20y-32}{12}, \frac{-18y}{12} \quad i.e., \quad x = \frac{5y-8}{3}, \frac{-3y}{2}$$

Equations of the lines are

$$3x - 5y + 8 = 0 \quad \text{and} \quad 2x + 3y = 0$$

For what value of λ will each of the following equations represent a pair of straight lines? (Problems 6 – 8):

6. $\lambda x^2 - 10xy + 12y^2 + 5x - 16y - 3 = 0$

Sol. Here $a = \lambda, h = -5, b = 12, g = \frac{5}{2}, f = -8, c = -3.$

The given equation represents two straight lines if

$$\begin{vmatrix} \lambda & -5 & 5/2 \\ -5 & 12 & -8 \\ 5/2 & -8 & -3 \end{vmatrix} = 0$$

or $\quad \lambda(-36-64) + 5(15+20) + \frac{5}{2}(40-30) = 0$

or $\quad -100\lambda + 175 + 25 = 0$

or $\quad -100\lambda = -200 \quad$ *i.e.*, $\quad \lambda = 2$

7. $\lambda xy + 5x + 3y + 2 = 0 \qquad (1)$

Sol. Here $a = 0, b = 0, h = \frac{\lambda}{2}, g = \frac{5}{2}, f = \frac{3}{2}, c = 2,$

(1) represents two straight lines if

$$\begin{vmatrix} 0 & \frac{\lambda}{2} & \frac{5}{2} \\ \frac{\lambda}{2} & 0 & \frac{3}{2} \\ \frac{5}{2} & \frac{3}{2} & 2 \end{vmatrix} = 0 \text{ or } -\frac{\lambda}{2}\left(\lambda - \frac{15}{4}\right) + \frac{5}{2} \times \frac{3\lambda}{4} = 0$$

or $\quad -\frac{\lambda^2}{2} + \frac{15\lambda}{8} + \frac{15\lambda}{8} = 0 \quad$ or $\quad -\frac{\lambda^2}{2} + \frac{30\lambda}{8} = 0$

or $\quad -2\lambda^2 + 15\lambda = 0 \quad$ or $\quad \lambda(-2\lambda + 15) = 0$

i.e., $\quad \lambda = 0, \quad \frac{15}{2}$

If $\lambda = 0$, the given equation is linear and represents a straight line.

Hence (1) represents a pair of lines if $\lambda = \frac{15}{2}$.

8. $4x^2 - 9y^2 - 2(8+\lambda)x - 18y = 29 + 2\lambda.$

Sol. The given equation can be written as

$$4x^2 - 9y^2 - 2(8+\lambda)x - 18y - (29 + 2\lambda) = 0 \qquad (1)$$

Here $a = 4, b = -9, h = 0, g = -(8+\lambda), f = -9, c = -(29 + 2\lambda),$

(1) will represent a pair of lines if

$$\begin{vmatrix} 4 & 0 & -(8+\lambda) \\ 0 & -9 & -9 \\ -(8+\lambda) & -9 & -(29+2\lambda) \end{vmatrix} = 0$$

or $\quad 4[9(29 + 2\lambda) - 9 \times 9] - (8+\lambda)[0 - (-9) \times (-(8+\lambda))] = 0$

or $\quad 36(20 + 2\lambda) + 9(8+\lambda)^2 = 0 \Rightarrow 4(20 + 2\lambda) + (8+\lambda)^2 = 0$

or $\quad 80 + 8\lambda + 64 + \lambda^2 + 16\lambda = 0$

or $\quad \lambda^2 + 24\lambda + 144 = 0 \quad$ or $\quad (\lambda + 12)^2 = 0$

Thus $\lambda = -12$

Find the angle between each of the following pairs of lines (Problem 9 – 13):

9. $x^2 - 2xy \tan\theta - y^2 = 0$

Sol. Here $a = 1, b = -1$ so that $a + b = 0$

Thus the two lines are perpendicular.

10. $3x^2 + 7xy + 2y^2 = 0$

Sol. Here $a = 3, b = 2, h = \frac{7}{2}$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{\frac{49}{4} - 3 \times 2}}{3+2} = \frac{2}{5} \times \sqrt{\frac{25}{4}} = \frac{2}{5} \times \frac{5}{2} = 1$$

Thus $\quad \theta = 45^\circ$

11. $11x^2 + 16xy - y^2 = 0$

Sol. Here $a = 11, h = 8, b = -1$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{64 + 11}}{11 - 1} = \frac{2 \times 5\sqrt{3}}{10} = \sqrt{3}$$

Hence $\quad \theta = 60^\circ$

12. $x^2 + 4xy + y^2 - 6x - 3 = 0$

Sol. Here $a = 1, b = 1, h = 2$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b} = \frac{2\sqrt{4-1}}{1+1} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

Thus $\quad \theta = 60^\circ$

13. $6x^2 + xy - y^2 - 21x - 8y + 9 = 0$

Sol. Here $a = 6, b = -1, \ h = \frac{1}{2}$

$$\tan\theta = \frac{2\sqrt{\frac{1}{4} + 6}}{6 + (-1)} = \frac{2 \times \frac{5}{2}}{5} = 1$$

Thus $\theta = 45^\circ$

14. Show that $16xy - 6x + 8y - 3 = 0$ represents a pair of straight lines. Also prove that this together with the coordinate axes form a rectangle and find the area enclosed by the rectangle.

Sol. $16xy - 6x + 8y - 3 = 0 \qquad (1)$

Here $a = 0, b = 0, h = 8, g = -3, f = 4, c = -3$, so

$$\begin{vmatrix} 0 & 8 & -3 \\ 8 & 0 & 4 \\ -3 & 4 & -3 \end{vmatrix} = -8(-24 + 12) + (-3)(32 - 0) = 96 - 96 = 0$$

Thus (1) represents a pair of straight lines.

(1) can be written as

$$2x(8y-3)+1\,.\,(8y-3)=0 \;\Rightarrow\; (2x+1)(8y-3)=0$$

i.e., equations of lines are $2x+1=0$ (2)

$8y-3=0$ (3)

As $a+b=0$, so these lines are perpendicular.

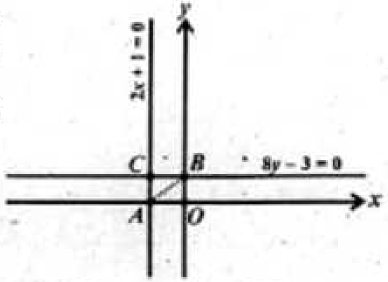
(2) is perpendicular to the x-axis and cuts it at $A\left(-\frac{1}{2}, 0\right)$

(3) is perpendiculars to the y-axis and cuts it at $B\left(0, \frac{3}{8}\right)$

Thus these lines together with the coordinates axes form a rectangle.

Let C be the point of intersection of (2) and (3).

Then C is $\left(-\frac{1}{2}, \frac{3}{8}\right)$.



Area of rectangle $AOBC = 2 \times$ area of the triangle AOB.

$$= 2\,.\left(\frac{1}{2}\begin{vmatrix} -\frac{1}{2} & 0 & 1 \\ 0 & 0 & 1 \\ 0 & \frac{3}{8} & 1 \end{vmatrix}\right)$$

$$= -\frac{1}{2}\left(0-\frac{3}{8}\right)+1(0)=\frac{3}{16}$$

Thus the required area is $\frac{3}{16}$ square units.

15. Show that an equation of the rectangular hyperbola $x^2-y^2=1$ referred to its asymptotes as axes is $x'y'=-\frac{1}{2}$.

Sol. Equation of the asymptotes of the hyperbola

$x^2-y^2=1$ (1)

is $x^2-y^2=0$

i.e., $x-y=0$ and $x+y=0$ are asymptotes of (1)

These are the new axes. The line $x-y$ or $y=x$ makes an angle of 45° with the x-axis. Therefore the new axes are obtained by rotating the original axes through an angle of 45°. Equations of transformation are

$$x = x'\cos 45^\circ - y'\cos 45^\circ$$
$$y = x'\sin 45^\circ + y'\cos 45^\circ$$

i.e., $x=\dfrac{x'-y'}{\sqrt{2}}$, $y=\dfrac{x'+y'}{\sqrt{2}}$

Substituting into the equation (1) of the hyperbola, we get

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2-\left(\frac{x'+y'}{\sqrt{2}}\right)^2=1$$

or $[(x'-y')+(x'+y')]\,[(x'-y')-(x'+y')]=2$

or $(2x')(-2y')=2 \;\Rightarrow\; -4x'y'=2$

or $x'y'=-\frac{1}{2}$ as required.

Analyze and graph the conic represented by each of the following equations (Problems 16 – 25):

16. $\sqrt{x}+\sqrt{y}=1$

Sol. Squaring both sides, we have

$$x+y+2\sqrt{xy}=1$$

or $x+y-1=-2\sqrt{xy}$

Again squaring, we obtain

$$x^2+2xy+y^2-2x-2y+1=4xy$$

or $x^2-2xy+y^2-2x-2y+1=0$ (1)

Here $a=1=b$ and $a-b=0$, so $2\theta=90^\circ \Rightarrow \theta=45^\circ$

To remove the product term xy, we rotate the axes through an angle of 45°. Equations of transformation are

$$x=\frac{x'-y'}{\sqrt{2}},\quad y=\frac{x'+y'}{\sqrt{2}}$$

Substituting into (1), we get

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2-2\frac{(x'-y')(x'+y')}{2}+\left(\frac{x'+y'}{\sqrt{2}}\right)^2-2\left(\frac{x'-y'}{\sqrt{2}}\right)-2\left(\frac{x'+y'}{\sqrt{2}}\right)+1=0$$

or $2y'^2=2\sqrt{2}\,x'-1$

or $y'^2=\sqrt{2}\left(x'-\dfrac{1}{2\sqrt{2}}\right)$

or $Y^2=\sqrt{2}\,X$, where $y'=Y$ and $x'-\dfrac{1}{2\sqrt{2}}=X$

This represents a parabola.

Axis: $Y=0 \Rightarrow y'=0 \Rightarrow \dfrac{-x+y}{\sqrt{2}}=0$ or $x-y=0$

Focus: $X=\dfrac{\sqrt{2}}{4}=\dfrac{1}{2\sqrt{2}}$, $Y=0$

i.e., $x'-\dfrac{1}{2\sqrt{2}}=\dfrac{1}{2\sqrt{2}} \Rightarrow x'=\dfrac{1}{\sqrt{2}}$ and $y'=0$

or $\quad x' = \dfrac{x+y}{\sqrt{2}} = \dfrac{1}{\sqrt{2}} \quad$ and $\quad \dfrac{-x+y}{\sqrt{2}} = 0$

$$\left.\begin{array}{ll}\text{i.e.,} & x+y=1 \\ \text{and} & x-y=0\end{array}\right\} \Rightarrow x = \frac{1}{2},\ y = \frac{1}{2}$$

Thus $\left(\frac{1}{2}, \frac{1}{2}\right)$ is focus of the given parabola.

Vertex: $X = 0,\ Y = 0$

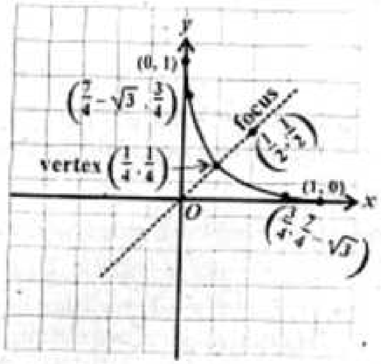
i.e., $\quad x' - \dfrac{1}{2\sqrt{2}} = 0$

$\Rightarrow x' = \dfrac{1}{2\sqrt{2}}$ and $y' = 0$

or $\quad \dfrac{x+y}{\sqrt{2}} = \dfrac{1}{2\sqrt{2}}$

$\Rightarrow \quad x + y = \dfrac{1}{2}$ and $\dfrac{-x+y}{\sqrt{2}} = 0$

These equations yield $x = \dfrac{1}{4}, y = \dfrac{1}{4}$



i.e., $\left(\frac{1}{4}, \frac{1}{4}\right)$ is the vertex. The graph of the parabola is as shown.

17. $xy = 1$ (1)

Sol. In order to eliminate the xy term, let the axes be rotated through an angle θ where

$$\tan 2\theta = \frac{2h}{a-b}$$

Here $a - b = 0$, therefore $2\theta = 90^\circ \quad \Rightarrow \quad \theta = 45^\circ$

Equations of transformation are

$$x = x'\cos 45^\circ - y'\sin 45^\circ = \frac{x'-y'}{\sqrt{2}}$$

$$y = y'\sin 45^\circ + y'\cos 45^\circ = \frac{x'+y'}{\sqrt{2}}$$

Substituting into (1), we have

$$\frac{x'^2 - y'^2}{2} = 1 \quad \text{i.e.,} \quad \frac{x'^2}{2} - \frac{y'^2}{2} = 1$$

which is standard form of a rectangular hyperbola. Its centre is (0, 0), i.e.,

$$x' = 0 \Rightarrow x\cos 45^\circ + y\sin 45^\circ = \frac{x+y}{\sqrt{2}} \Rightarrow 0 \Rightarrow x + y = 0$$

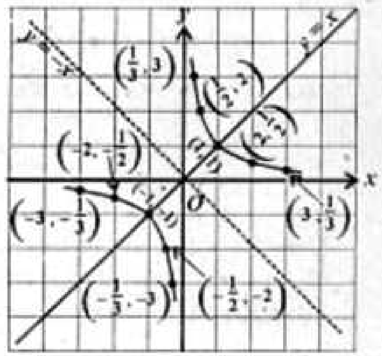
and $\quad y' = 0 \Rightarrow -x\cos 45^\circ + y\sin 45^\circ = \dfrac{-x+y}{\sqrt{2}} = 0 \Rightarrow -x + y = 0$

Solving the equations, we get $x = 0, y = 0$. Thus the centre of (1) is (0, 0) referred to xy-system.

Vertices: $x' = \pm\sqrt{2}\ ,\ y' = 0 \quad$ i.e. $\pm\sqrt{2} = \dfrac{x+y}{\sqrt{2}}$

$\Rightarrow x + y = \pm 2, 0 = \dfrac{-x+y}{\sqrt{2}} \Rightarrow -x + y = 0$

Solving $x + y = 2$
and $-x + y = 0$,
we get $x = 1, y = 1$
Solving $x + y = -2$
and $-x + y = 0$, we get
$x = -1, y = -1$



The vertices are (1, 1) and (−1, −1) in the original system of axis.

Equation of the transverse axis is

$y' = 0 \quad$ i.e., $\quad x = y$

and conjugate axis is $x' = 0$

i.e., $\quad x = -y$

Joint equation of the asymptotes is $\dfrac{x'^2}{2} - \dfrac{y'^2}{2} = 0$

or $\quad x' + y' = 0$ and $x' - y' = 0$ are asymptotes referred to the $x'y'$-system.

i.e., $\quad \dfrac{x+y}{\sqrt{2}} + \dfrac{-x+y}{\sqrt{2}} = 0 \Rightarrow y = 0$

and $\quad \dfrac{x+y}{\sqrt{2}} - \dfrac{-x+y}{\sqrt{2}} = 0 \Rightarrow x = 0$

Thus $y = 0$ and $x = 0$ are asymptotes in the xy-system.

The graph of the conic is as shown.

18. $xy + x - 2y + 3 = 0$ (1)

Sol. Here $a = 0 = b$ so the axes should be rotated through an angle of 45° to eliminate the products term xy.

$x = \dfrac{x'-y'}{\sqrt{2}}\ ,\ y = \dfrac{x'+y'}{\sqrt{2}}$ are the equations of transformation.

Substituting into (1), we have

$$\frac{x'-y'}{\sqrt{2}} \cdot \frac{x'+y'}{\sqrt{2}} + \frac{x'-y'}{\sqrt{2}} - 2 \cdot \frac{x'+y'}{\sqrt{2}} + 3 = 0$$

or $\quad x'^2 - y'^2 + \sqrt{2}\,(x' - y') - 2\sqrt{2}\,(x' + y') + 6 = 0$

i.e., $\quad x'^2 - y'^2 - \sqrt{2}\,x' - 3\sqrt{2}\,y' + 6 = 0$

or $\quad \left(x'^2 - \sqrt{2}\,x' + \frac{1}{2}\right) - \left(y'^2 + 3\sqrt{2}\,y' + \frac{9}{2}\right) = -6 - 4$

or $\left(x'-\frac{1}{\sqrt{2}}\right)^2-\left(y'+\frac{3}{\sqrt{2}}\right)^2=-10$ (2)

Let $X=x'-\frac{1}{\sqrt{2}}$, $Y=y'+\frac{3}{\sqrt{2}}$, so that (2) becomes

$\frac{Y^2}{10}-\frac{X^2}{10}=1$. This is a rectangular hyperbola with transverse axis $X=0$ and conjugate axis $Y=0$.

i.e., $x'-\frac{1}{\sqrt{2}}=0$ and $y'+\frac{3}{\sqrt{2}}=0$

or $\frac{x+y}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$ and $\frac{-x+y}{\sqrt{2}}+\frac{3}{\sqrt{2}}=0$

or $x+y-1=0$ and $-x+y+3=0$

Centre: The point of intersection of $x+y-1=0$ and $-x+y+3=0$ is $(2,-1)$, *i.e.*, the centre of the conic is $(2,-1)$.

Vertices: $Y=\pm\sqrt{10}, X=0$

i.e., $y'+\frac{3}{\sqrt{2}}=\pm\sqrt{10}$, $x'-\frac{1}{\sqrt{2}}=0$

or $\frac{-x+y}{\sqrt{2}}+\frac{3}{\sqrt{2}}=\pm\sqrt{10}$, $\frac{x+y}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0$

$$x+y-1=0$$

$$-x+y+3=\pm\sqrt{10}\times\sqrt{2}=\pm 2\sqrt{5}$$

Adding the above equations, we get $2y+2=\pm 2\sqrt{5}$

or $y=-1\pm\sqrt{5}$

If $y=-1+\sqrt{5}$, then $x=1-y$

$$=1+1-\sqrt{5}=2-\sqrt{5}$$

If $y=-1-\sqrt{5}$, then $x=1-y$

$$=1+1+\sqrt{5}=2+\sqrt{5}$$

Thus the vertices are

$$(2-\sqrt{5},-1+\sqrt{5}),\ (2+\sqrt{5},-1-\sqrt{5})$$

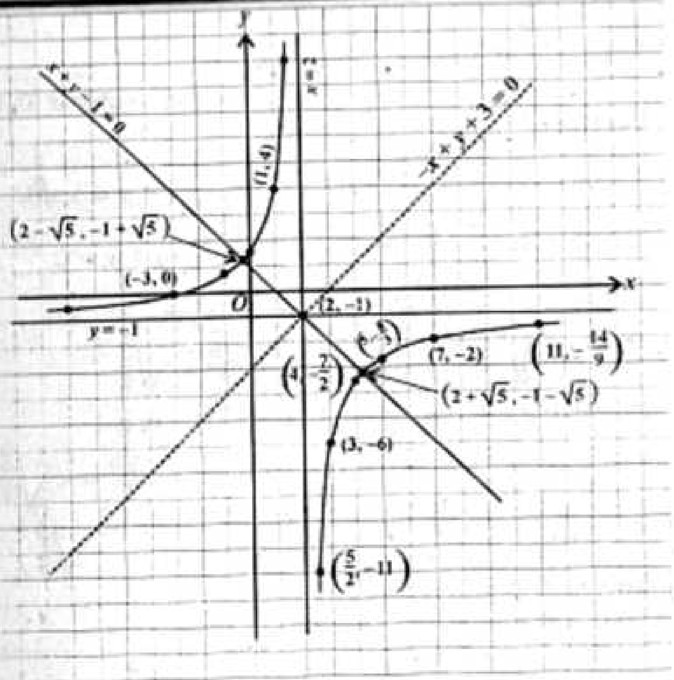
Asymptotes: $Y^2-X^2=0$

or $(Y-X)(Y+X)=0$

or $\left(y'+\frac{3}{\sqrt{2}}-x'+\frac{1}{\sqrt{2}}\right)\left(y'+\frac{3}{\sqrt{2}}+x'-\frac{1}{\sqrt{2}}\right)=0$

i.e., $\frac{-x+y}{\sqrt{2}}+\frac{3}{\sqrt{2}}-\frac{x+y}{\sqrt{2}}+\frac{1}{\sqrt{2}}=0 \Rightarrow -2x+4=0$

$$\Rightarrow x-2=0$$



or $\frac{-x+y}{\sqrt{2}}+\frac{3}{\sqrt{2}}+\frac{x+y}{\sqrt{2}}-\frac{1}{\sqrt{2}}=0 \Rightarrow 2y+2=0 \Rightarrow y+1=0$

Thus $x-2=0$ and $y+1=0$ are asymptotes.

The graph of the conic is as shown above.

10. $5x^2-2xy+5y^2-12=0$ (1)

Sol. Since $a-b=5-5=0$, the axes are to be rotated through an angle of 45° to remove the xy term.

$$5\left(\frac{x'-y'}{\sqrt{2}}\right)^2-2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right)-5\left(\frac{x'+y'}{\sqrt{2}}\right)^2-12=0$$

$$\frac{5}{2}(x'^2+y'^2-2x'y')-(x'^2-y'^2)+\frac{5}{2}(x'^2+y'^2+2x'y')-12=0$$

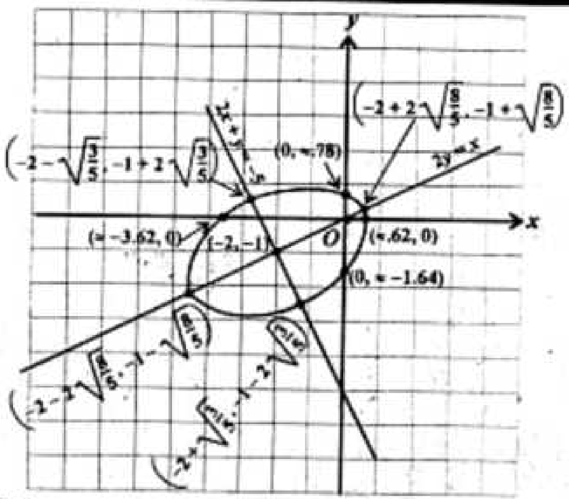
$$5x'^2+5y'^2-x'^2+y'^2=12 \quad\Rightarrow\quad 4x'^2+6y'^2=12$$

or $\frac{x'^2}{3}+\frac{y'^2}{2}=1$ which is an ellipse.

Major axis: $y'=0$ *i.e.,* $\frac{-x+y}{\sqrt{2}}=0$ or $y=x$

Minor axis: $x'=0$ *i.e.,* $\frac{x+y}{\sqrt{2}}=0$ or $y=-x$

Centre: $x'=0,\ y'=0$



Ends of the minor axis:

$$X = 0\ ,\ \ Y = \pm\sqrt{3}$$

$\Rightarrow \quad x' + \sqrt{5} = 0, y' = \pm\sqrt{3}$

or $\quad \dfrac{2x+y}{\sqrt{5}} + \sqrt{5} = 0 \quad , \quad \dfrac{-x+2y}{\sqrt{5}} = \pm\sqrt{3}$

or $\quad 2x + y + 5 = 0 \quad , \quad -x + 2y = \pm\sqrt{15}$

Solving the equations, we get

$$\left(-2 - \sqrt{\frac{3}{5}}, -1 + 2\sqrt{\frac{3}{5}}\right) \text{ and } \left(-2 + \sqrt{\frac{3}{5}}, -1 - 2\sqrt{\frac{3}{5}}\right)$$

as the end points of minor axis.

The graph of the curve is as shown above.

21. $x^2 - 2xy + y^2 - 2\sqrt{2}x - 2\sqrt{2}y + 2 = 0 \qquad (1)$

Sol. Here, $a = 1$, $b = 1$, $h = -1$

$ab - h^2 = 0$. Hence (1) is a parabola.

Since $a - b = 0$, the angle of ration is $\theta = 45°$

Equations of transformation are

$$\left.\begin{aligned} x &= \frac{x'-y'}{\sqrt{2}} \\ y &= \frac{x'+y'}{\sqrt{2}} \end{aligned}\right\}, \qquad \begin{aligned} x' &= \frac{x+y}{\sqrt{2}} \\ y' &= \frac{-x+y}{\sqrt{2}} \end{aligned}$$

Transformed equation (1) is

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 - 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2$$

$$-2\sqrt{2}\left(\frac{x'-y'}{\sqrt{2}}\right) - 2\sqrt{2}\left(\frac{x'+y'}{\sqrt{2}}\right) + 2 = 0$$

$$\frac{x'^2+y'^2}{2} - x'y' - (x'^2 - y'^2) + \frac{x'^2+y'^2}{2} + x'y' - 4x' + 2 = 0$$

or $\quad 2y'^2 - 4x' + 2 = 0$ or $y'^2 = 2\left(x' - \dfrac{1}{2}\right)$

or $\quad Y^2 = 2X$, where $Y = y'$, $X = x' - \dfrac{1}{2}$. This is a parabola.

Axis: $Y = 0 \Rightarrow y' = 0 \Rightarrow y' = \dfrac{-x+y}{\sqrt{2}} = 0 \quad$ *i.e.*, $x - y = 0$.

Vertex: $X = 0$, $Y = 0 \qquad$ *i.e.*, $x' = \dfrac{1}{2}, y' = 0$ in $x'y'$-system

or $\quad \dfrac{x+y}{\sqrt{2}} = \dfrac{1}{2} \Rightarrow x + y = \dfrac{1}{\sqrt{2}}\Big],\ \dfrac{-x+y}{\sqrt{2}} \Rightarrow -x + y = 0$

Solving the equations, we get $\left(\dfrac{1}{2\sqrt{2}}, \dfrac{1}{2\sqrt{2}}\right)$ as vertex in the xy-system.

Focus: $X = \dfrac{1}{2}, Y = 0$

i.e., $x' - \dfrac{1}{2} = \dfrac{1}{2}$

and $y' = 0$

$\Rightarrow x' = 1$ and $y' = 0$

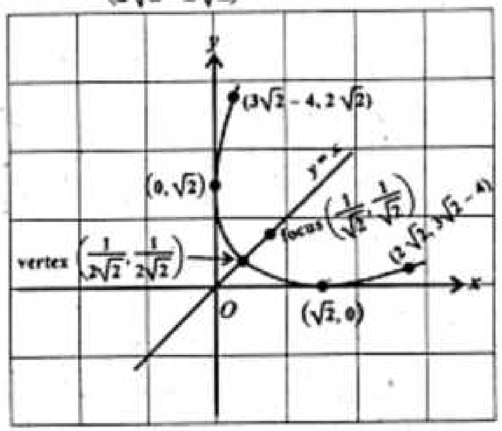
in $x'y'$-system

$x' = 1 \Rightarrow \dfrac{x+y}{\sqrt{2}} = 1$,

$y = 0 \Rightarrow \dfrac{-x+y}{\sqrt{2}} = 0$

or $\quad x + y = \sqrt{2}$

and $x + y = 0$



Solving the equations, we get

$$x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2}}, \text{ i.e., } \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ is focus.}$$

The graph is as shown above.

22. $9x^2 + 24xy + 16y^2 - 125y = 0$

Sol. $9x^2 + 24xy + 16y^2 - 125y = 0 \qquad (1)$

Here $a = 9, b = 16, h = 12$

$ab - h^2 = 0$. Thus (1) represents a parabola

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a-b} = \frac{24}{-7}$$

i.e., $12\tan^2\theta - 7\tan\theta - 12 = 0$

or $\tan\theta = \dfrac{7 \pm \sqrt{49+576}}{24} = \dfrac{7 \pm 25}{24} = \dfrac{4}{3}, -\dfrac{3}{4}$

We choose θ such that $\tan\theta = \frac{4}{3}$ and so $\sin\theta = \frac{4}{5}$, $\cos\theta = \frac{3}{5}$

Equations of transformation are

$$x = x'\cos\theta - y'\sin\theta = \frac{3x'-4y'}{5}$$

$$y = x'\sin\theta + y'\cos\theta = \frac{4x'+3y'}{5}$$

$$x' = \frac{3x+4y}{5}, \; y' = \frac{-4x+3y}{5}$$

Substituting for x, y into (1), we have

$$\frac{9}{25}(3x'-4y')^2 + \frac{24}{25}(3x'-4y')(4x'+3y') + \frac{16}{25}(4x'+3y')^2 - \frac{625}{25}(4x'+3y') = 0$$

$(81+288+256)x'^2 + (144-288+144)y'^2 + (-216-168+384)x'y' - 625(4x'+3y') = 0$

i.e., $x'^2 - 4x' - 3y' = 0$ or $(x'-2)^2 = 3\left(y' + \frac{4}{3}\right)$

or $X^2 = 3Y$, where $X = x'-2$, $Y = y' + \frac{4}{3}$

Axis: $X = 0$ or $x' = 2$ in the $x'y'$-system

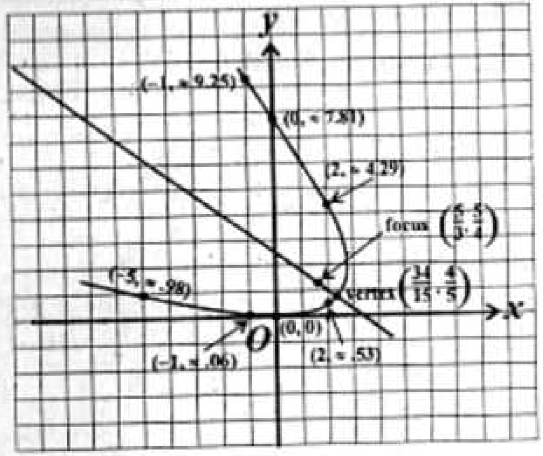
or $3x + 4y = 10$ in the xy-system

Focus: $\left(0, \frac{3}{4}\right)$ in the XY-system. *i.e.,* $x' - 2 = 0$, $y' + \frac{4}{3} = \frac{3}{4}$

or $x' = 2$, $y' = -\frac{7}{12}$ in the $x'y'$ system

$$\left.\begin{aligned} \text{or} \quad \frac{3x+4y}{5} &= 2 \;\Rightarrow\; 3x+4y = 10 \\ \text{and} \quad \frac{-4x+3y}{5} &= \frac{-7}{12} \;\Rightarrow\; -4x+3y = -\frac{35}{\sqrt{12}} \end{aligned}\right\} \text{in the } xy\text{-system}$$

Solving these equations, we get $y = \frac{5}{4}$, $x = \frac{5}{3}$



Thus focus is $\left(\frac{5}{3}, \frac{5}{4}\right)$ in the xy-system

Vertex: (0, 0) in the XY-system

or $\left(2, \frac{-4}{3}\right)$ in the $x'y'$-system

$$\left.\begin{aligned} \text{i.e.,} \quad \frac{3x+4y}{5} &= 2 \;\Rightarrow\; 3x+4y = 10, \\ \frac{-4x+3y}{5} &= \frac{-4}{3} \;\Rightarrow\; -4x+3y = -\frac{20}{3} \end{aligned}\right\} \text{in the } xy\text{-system}$$

Solving these equations, we get $y = \frac{4}{3}$, $x = \frac{34}{15}$

Thus vertex is $\left(\frac{34}{15}, \frac{4}{5}\right)$ in the xy-system

The graph of the conic is as shown above.

38. $2x^2 + 6xy + 10y^2 - 11 = 0$ (1)

Sol. Here $a = 2$, $b = 10$, $h = 3$

$h^2 - ab = 9 - 20 < 0$

Hence the conic is an ellipse.

i.e., $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{6}{2-10}$ *i.e.,* $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{6}{-8} = \dfrac{-3}{4}$

or $3\tan^2\theta - 8\tan\theta - 3 = 0$

or $\tan\theta = \dfrac{8 \pm \sqrt{64+36}}{6} = 3, -\dfrac{1}{3}$

We choose θ such that $\tan\theta = 3$

Therefore, $\sin\theta = \dfrac{3}{\sqrt{10}}$, $\cos\theta = \dfrac{1}{\sqrt{10}}$

Equations of transformation are

$$x = x'\cos\theta - y'\sin\theta = \frac{x' - 3y'}{\sqrt{10}}$$

$$y = x'\sin\theta + y'\cos\theta = \frac{3x' + y'}{\sqrt{10}}$$

and $x' = \dfrac{x + 3y}{\sqrt{10}}$, $y' = \dfrac{-3x + y}{\sqrt{10}}$

Substituting for x, y into (1), we have

$$2\left(\frac{x' - 3y'}{\sqrt{10}}\right)^2 + 6\left(\frac{x' - 3y'}{\sqrt{10}}\right)\left(\frac{3x' + y'}{\sqrt{10}}\right) + 10\left(\frac{3x' + y'}{\sqrt{10}}\right)^2 - 11 = 0$$

or $\quad \left(\frac{1}{5} + \frac{9}{5} + 9\right)x'^2 + \left(\frac{9}{5} - \frac{9}{5} + 1\right)y'^2 - 11 = 0$

or $\quad 11x'^2 + y'^2 - 11 = 0 \quad$ *i.e.,* $\quad \dfrac{x'^2}{1} + \dfrac{y'^2}{(\sqrt{11^2})} = 1$

The major axis of the ellipse lies along the y'-axis

Centre: $x' = 0$, $y' = 0$ *i.e.,* $x + 3y = 0$ and $-3x + y = 0$

which give $x = 0$, $y = 0$

Foci $= (0, \mp c)$ (where $c = \sqrt{11 - 1} = \sqrt{10}$) , in $x'y'$-system

i.e., $\quad \dfrac{x + 3y}{\sqrt{10}} = 0$

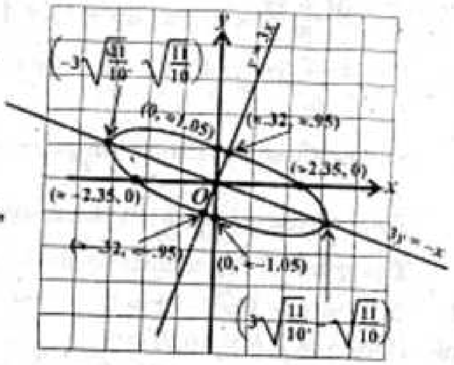
and $\quad \dfrac{-3x + y}{\sqrt{10}} = \mp\sqrt{10}$

or $\quad x + 3y = 0$

and $\quad -3x + y = \mp 10$

Solving the above equations, we get

$(3, -1)$ and $(-3, 1)$ as foci.



Major axis: $x' = 0$,

i.e., $\quad x + 3y = 0$

Minor axis: $y' = 0$,

i.e., $\quad 3x - y - 0$

Vetrices: $A' = (0, -\sqrt{11})$, $A = (0, \sqrt{11})$ in the $x'y'$-system

To find A', we have

or $\quad x + 3y = 0$ and $\dfrac{-3x + y}{\sqrt{10}} = -\sqrt{11}$ in the xy-system

which give $A' = \left(3\sqrt{\frac{11}{10}}, -\sqrt{\frac{11}{10}}\right)$

Similarly, $x^3 + 3y = 0$ and $\dfrac{-3x + y}{\sqrt{10}} = \sqrt{11}$ in the xy-system

which yield $A = \left(-3\sqrt{\frac{11}{10}}, \sqrt{\frac{11}{10}}\right)$

The graph is as shown above.

24. $x^2 - 4xy + 4y^2 + 5\sqrt{5}\,y + 1 = 0 \qquad (1)$

Sol. Here $\tan 2\theta = \dfrac{2h}{a - b} = \dfrac{-4}{1 - 4} = \dfrac{-4}{-3} = \dfrac{4}{3}$

or $\quad \dfrac{2\tan\theta}{1 - \tan^2\theta} = \dfrac{4}{3} \quad \Rightarrow \quad 3\tan\theta = 2 - 2\tan^2\theta$

or $\quad 2\tan^2\theta + 3\tan\theta - 2 = 0$

$\quad \tan\theta = \dfrac{-3 \pm \sqrt{9 + 16}}{4} = \dfrac{-3 \pm 5}{4}$

i.e., $\quad \tan\theta = \dfrac{1}{2}$, -2

We take $\tan\theta = \dfrac{1}{2}$ so that $\sin\theta = \dfrac{1}{\sqrt{5}}$, $\cos\theta = \dfrac{2}{\sqrt{5}}$

Equations of transformation are

$$x = \frac{2x' - y'}{\sqrt{5}} \quad , \quad y = \frac{x' + 2y'}{\sqrt{5}}$$

Transformed equation (1) is

$$\left(\frac{2x' - y'}{\sqrt{5}}\right)^2 - 4\left(\frac{2x' - y'}{\sqrt{5}}\right)\left(\frac{x' + 2y'}{\sqrt{5}}\right)$$

$$+ 4\left(\frac{x' + 2y'}{\sqrt{5}}\right)^2 + 5\sqrt{5}\left(\frac{x' + 2y'}{\sqrt{5}}\right) + 1 = 0$$

$$\left(\frac{4}{5} - \frac{8}{5} + \frac{4}{5}\right)x'^2 + \left(\frac{1}{5} + \frac{8}{5} + \frac{16}{5}\right)y'^2 + \left(-\frac{4}{5} - \frac{12}{5} + \frac{16}{5}\right)x'y'$$

$$+ 5x' + 10y' + 1 = 0$$

or $\quad 5y'^2 + 10y' = -5x - 1 \quad \Rightarrow \quad y'^2 + 2y' = -x' - \dfrac{1}{5}$

or $\quad (y' + 1)^2 = -x' - \dfrac{1}{5} + 1 = -\left(x' - \dfrac{4}{5}\right)$

Now $Y^2 = -X$ where $X = x' - \dfrac{4}{5}$ $\quad Y = y' + 1$

This is a parabola.

Vertex: $X = 0$, $\qquad Y = 0$

$\quad X = 0 \quad \Rightarrow \quad x' - \dfrac{4}{5} = 0 \quad \Rightarrow \quad x' = \dfrac{4}{5}$

And $\quad Y = 0 \quad \Rightarrow \quad y' + 1 = 0 \quad \Rightarrow \quad y' = -1$

$\quad x' = \dfrac{4}{5} \quad \Rightarrow \quad \dfrac{2x + y}{\sqrt{5}} = \dfrac{4}{5}$

and $y' = -1 \Rightarrow \dfrac{-x+2y}{\sqrt{5}} = -1$

i.e., $2x + y = \dfrac{4}{\sqrt{5}}$ and $-x + 2y = -\sqrt{5}$

Solving these equations, we have $x = \dfrac{13}{5\sqrt{5}}$, $y = \dfrac{-6}{5\sqrt{5}}$

Axis: $Y = 0$

$$Y = 0 \Rightarrow y' + 1 = 0 \Rightarrow \frac{-x+2y}{\sqrt{5}} = -1$$

i.e., $-x + 2y = -\sqrt{5}$ in the xy-system.

Focus: $X = -\dfrac{1}{4}$, $Y = 0$, *i.e.*, $x' - \dfrac{4}{\sqrt{5}} = -\dfrac{1}{4}$

and $y' = -1$

or $\dfrac{2x+y}{\sqrt{5}} = \dfrac{4}{\sqrt{5}} - \dfrac{1}{4} \Rightarrow 2x + y = 4 - \dfrac{\sqrt{5}}{4}$

and $-x + 2y = -\sqrt{5}$

Solving the equations, we get

$$x = \frac{8}{5} + \frac{1}{2\sqrt{5}}, \quad y = \frac{4}{5} - \frac{9}{4\sqrt{5}}$$

25. $16x^2 - 24xy + 9y^2 + 100x - 200y + 100 = 0$ (1)

Sol. Here $\tan 2\theta = \dfrac{2h}{a-b} = \dfrac{-24}{16-9} = \dfrac{-24}{7}$.

i.e., $\dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{-24}{7}$ or $\dfrac{\tan\theta}{1-\tan^2\theta} = \dfrac{-12}{7}$

or $12\tan^2\theta - 7\tan\theta - 12 = 0$

$$\tan\theta = \frac{7 \pm \sqrt{49+576}}{24} = \frac{7+25}{24} = \frac{4}{3}, -\frac{3}{4}$$

We take $\tan\theta = \dfrac{4}{3}$ so that $\sin\theta = \dfrac{4}{5}$, $\cos\theta = \dfrac{3}{5}$

Equations of transformation are

$$x = \frac{3x' - 4y'}{5}, \quad y = \frac{4x' + 3y'}{5}$$

Substituting into (1), we get

$$16\left(\frac{3x'-4y'}{5}\right)^2 - 24\left(\frac{3x'-4y'}{5}\right)\left(\frac{4x'+3y'}{5}\right)$$
$$+ 9\left(\frac{4x'-3y'}{5}\right)^2 + 100\left(\frac{3x'-4y'}{5}\right)$$
$$- 200\left(\frac{4x'+3y'}{5}\right) + 100 = 0$$

or $\left(\dfrac{144}{25} - \dfrac{288}{25} + \dfrac{144}{25}\right)x'^2 + \left(\dfrac{256}{25} + \dfrac{288}{25} + \dfrac{81}{25}\right)y'^2$

$+ (60 - 160)x' + (-80 - 120)y' + 100 = 0$

or $25y' - 100x' - 200y' + 100 = 0$

or $25x'^2 - 200y' = 100(x - 1)$

or $y'^2 - 8y' + 16 = 4(x - 1) = 16$

or $(y' - 4)^2 = 4(x' + 3)$

or $Y^2 = 4X$

where $Y = y' - 4$, $X = x' + 3$

This is a parabola with vertex $X = 0$, $Y = 0$

i.e., $x' = -3$, $y' = 4$ in the $x'y'$-system.

But $x = \dfrac{3x'-4y'}{5} = \dfrac{-9-16}{5} = -5$

$y = \dfrac{4x'+3y'}{5} = \dfrac{-12+12}{5} = 0$

Thus $(-5, 0)$ is the vertex in the xy-system.

Axis: $Y = 0$ *i.e.*, $y' = 4$

or $\dfrac{-4x+3y}{5} = 4$ or $-4x + 3y = 20 \Rightarrow 4x - 3y + 20 = 0$

Focus: $Y = 0$, $X = 1$

or $y' = 4$ and $x' + 3 = 1 \Rightarrow x' = -2$

i.e., $(-2, 4)$ in the $x'y'$- system. or $x = \dfrac{-6-16}{5}$, $y = \dfrac{-8+12}{5}$

i.e., $\left(-\dfrac{22}{5}, \dfrac{4}{5}\right)$ in the xy-system.

Exercise Set 6.2 (Page 245)

Find equations of tangent and normal to each of the following curves at the indicated point (Problems 1 – 4):

1. $y^2 = 4ax$ at $(a, -2a)$ (1)

Sol. $y^2 = 4ax$

Differentiating (1) w.r.t, x, we have

$$2y\frac{dy}{dx} = 4a \quad \text{or} \quad \frac{dy}{dx} = \frac{2a}{y}$$

$\left(\dfrac{dy}{dx}\right)_{(a,-2a)} = \dfrac{2a}{-2a} = -1$ which is the slope of the tangent at $(a, -2a)$.

Slope of the normal at $(a, -2a) = -\dfrac{1}{-1} = 1$

Hence equation of the tangent to (1) at $(a, -2a)$ is

$$y - (-2a) = -1\,(x - a) \quad \text{or} \quad y + 2a = -x + a$$

i.e., $x + y + a = 0$

Equation of the normal to (1) at $(a, -2a)$ is

$$y - (-2a) = 1\,(x - a) \quad \text{or} \quad y + 2a = x - a$$

or $x - y - 3a = 0$

2. $xy = c^2$ at $\left(cp, \frac{c}{p}\right)$

Sol. $xy = c^2$ (1)

Differentiating (1) w.r.t. x, we get

$$x\frac{dy}{dx} + y = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y}{x}$$

$$\left(\frac{dy}{dx}\right)_{\left(cp,\frac{c}{p}\right)} = -\frac{c/p}{cp} = -\frac{1}{p^2}$$

which is the slope of the tangent at $\left(cp, \frac{c}{p}\right)$.

Slope of the normal at $\left(cp, \frac{c}{p}\right) = p^2$

Hence equation of the required tangent is

$$\left(y - \frac{c}{p}\right) = -\frac{1}{p^2}(x - cp)$$

or $p^2\left(y - \frac{c}{p}\right) = -(x - cp) \Rightarrow p^2y - cp = -x + cp$

or $x + p^2y = 2cp$

Equation of the normal is $\left(y - \frac{c}{p}\right) = p^2\,(x - cp)$

or $py - c = p^3x - cp^4$ or $p^3x - py = c(p^4 - 1)$

3. $x(x^2 + y^2) - ay^2 = 0$ at $x = \frac{a}{2}$

Sol. $x^3 + xy^2 - ay^2 = 0$

Let $f(x, y) = x^3 + xy^2 - ay^2 = 0$.

When $x = \frac{a}{2}$, $f\left(\frac{a}{2}, y\right) = \left(\frac{a}{2}\right)^3 + \frac{a}{2}(y^2) - ay^2 = 0$

or $\frac{a^3}{8} - \frac{a}{2}y^2 = 0$ or $y^2 = \frac{a^2}{4}$ or $y = \pm\frac{a}{2}$

Hence the points at which the tangents and normal are required are

$$\left(\frac{a}{2}, \frac{a}{2}\right), \left(\frac{a}{2}, \frac{-a}{2}\right)$$

Now $\frac{\partial f}{\partial x} = 3x^2 + y^2$ and $\frac{\partial f}{\partial y} = 2xy - 2ay$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 + y^2}{2xy - 2ay}$$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{2},\frac{a}{2}\right)} = \frac{3\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2}{2\left(\frac{a}{2}\right)^2 - 2a\left(\frac{a}{2}\right)} = 2$$

which is the slope of the tangent to the curve at $\left(\frac{a}{2}, \frac{a}{2}\right)$

Equation of the tangent at $\left(\frac{a}{2}, \frac{a}{2}\right)$ is

$$y - \frac{a}{2} = 2\left(x - \frac{a}{2}\right)$$

or $y - \frac{a}{2} = 2x - a \Rightarrow 2x - y - a + \frac{a}{2} = 0$

or $4x - 2y - a = 0$.

Slope of the normal at $\left(\frac{a}{2}, \frac{a}{2}\right) = -\frac{1}{2}$

Equation of the normal at $\left(\frac{a}{2}, \frac{a}{2}\right)$ is

$$y - \frac{a}{2} = -\frac{1}{2}\left(x - \frac{a}{2}\right) \Rightarrow 2y - a = -x + \frac{a}{2}$$

or $2x + 4y = 3a$

$$\left(\frac{dy}{dx}\right)_{\left(\frac{a}{2},-\frac{a}{2}\right)} = -\frac{3\left(\frac{a}{2}\right)^2 + \left(\frac{-a}{2}\right)^2}{2\left(\frac{a}{2}\right)\left(\frac{-a}{2}\right) - 2a\left(\frac{-a}{2}\right)} = -\frac{a^2}{-\frac{a^2}{2} + a^2} = -2$$

which is the slope of the tangent to the curve at $\left(\frac{a}{2}, \frac{-a}{2}\right)$

Equation of the tangent at $\left(\frac{a}{2}, \frac{-a}{2}\right)$ is

$$y + \frac{a}{2} = -2\left(x - \frac{a}{2}\right) \Rightarrow 2y + a = -2\,(2x - a)$$

or $4x + 2y - a = 0$

Slope of the normal at $\left(\frac{a}{2}, \frac{-a}{2}\right) = \frac{1}{2}$

Equation of the normal at $\left(\frac{a}{2}, \frac{-a}{2}\right)$ is

$$y + \frac{a}{2} = \frac{1}{2}\left(x - \frac{a}{2}\right) \Rightarrow y + \frac{a}{2} = \frac{x}{2} - \frac{a}{4}$$

or $\quad 2x - 4y = 3a$

4. $c^2(x^2 + y^2) = x^2y^2$ at $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$

Sol. $c^2(x^2 + y^2) = x^2y^2$ or $\frac{c^2}{x^2} + \frac{c^2}{y^2} = 1$

Its parametric equations are $x = \frac{c}{\cos\theta}$, $y = \frac{c}{\sin\theta}$

$$\frac{dx}{d\theta} = \frac{c\sin\theta}{\cos^2\theta}, \quad \frac{dy}{d\theta} = -\frac{c\cos\theta}{\sin^2\theta}$$

$$\frac{dy}{dx} = \left(-\frac{c\cos\theta}{\sin^2\theta}\right)\left(\frac{\cos^2\theta}{c\sin\theta}\right) = -\frac{\cos^3\theta}{\sin^3\theta} = -\cot^3\theta \quad \text{which is}$$

the slope of the tangent to the curve at the point $\left(\frac{c}{\cos\theta}, \frac{c}{\sin\theta}\right)$

Equation of the tangent is

$$y - \frac{c}{\sin\theta} = -\cot^3\theta\left(x - \frac{c}{\cos\theta}\right)$$

or $$y - \frac{c}{\sin\theta} = -\frac{\cos^3\theta}{\sin^3\theta}\left(x - \frac{c}{\cos\theta}\right)$$

$$= -x\frac{\cos^3\theta}{\sin^3\theta} + \frac{c\cos^2\theta}{\sin^3\theta}$$

or $\quad y\sin^3\theta - c\sin^2\theta = -x\cos^3\theta + c\cos^2\theta$

or $\quad x\cos^3\theta + y\sin^3\theta = c$

Slope of the normal at the given point is $\tan^3\theta$.

Equation of the normal is

$$y - \frac{c}{\sin\theta} = \tan^3\theta\left(x - \frac{c}{\cos\theta}\right)$$

$$= \frac{\sin^3\theta}{\cos^3\theta}\left(x - \frac{c}{\cos\theta}\right)$$

or $$y\cos^3\theta - \frac{c\cos^3\theta}{\sin\theta} = x\sin^3\theta - \frac{c\sin^3\theta}{\cos\theta}$$

or $$x\sin^3\theta - y\cos^3\theta = \frac{c\sin^3\theta}{\cos\theta} - \frac{c\cos^3\theta}{\sin\theta} = \frac{c\sin^4\theta - c\cos^4\theta}{\sin\theta\cos\theta}$$

$$= \frac{c(\sin^2\theta - \cos^2\theta)}{\sin\theta\cos\theta} = -\frac{2c\cos 2\theta}{\sin 2\theta} = -2c\cot 2\theta$$

or $\quad x\sin^3\theta - y\cos^3\theta + 2c\cot 2\theta = 0$

Find the points where the tangent is parallel to the x-axis and where it is parallel to the y-axis for each of the given curves (Problems 5 – 7):

5. $x^3 + y^3 = a^3$ (1)

Sol. Differentiating (1) w.r.t. x, we get

$$3x^2 + 3y^2\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x^2}{y^2}$$

For tangent to be parallel to the x-axis, $\frac{dy}{dx} = 0$

i.e., $x^2 = 0$ or $x = 0$

Putting $x = 0$ in (1), we get $y^3 = a^3 \Rightarrow y = a$

The tangent is parallel to the x-axis at $(0, a)$.

For tangent to be parallel to the y-axis or perpendicular to the x-axis,

$\frac{dy}{dx} = \infty$ which requires $y^2 = 0$ *i.e.,* $y = 0$

Putting $y = 0$ in (1), we have $x^3 = a^3 \Rightarrow x = a$

The tangent is perpendicular to the x-axis at $(a, 0)$.

6. $x^3 + y^3 = 3axy$ (1)

Sol. Differentiating (1) w.r.t. x, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3ay + 3ax\frac{dy}{dx}$$

or $$(y^2 - ax)\frac{dy}{dx} = ay - x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$$

For tangent to be parallel to the x-axis, $\frac{dy}{dx} = 0$

Therefore, $ay - x^2 = 0 \Rightarrow y = \frac{x^2}{a}$ (2)

Solving (1) and (2), we have

$$x^3 + \left(\frac{x^2}{a}\right)^3 = 3ax\left(\frac{x^2}{a}\right) \Rightarrow \frac{x^6}{a^3} = 2x^3$$

or $\quad x^3 = 2a^3$ which gives $x = 2^{1/3}a$

Putting $x = 2^{1/3}a$ in (2), we get

$$y = \frac{(2^{1/3}a)^2}{a} = \frac{2^{2/3}a^2}{a} = 2^{2/3}a$$

Hence the tangent is parallel to the x-axis at

$$(2^{1/3}a, 2^{2/3}a) = (\sqrt[3]{2}a, \sqrt[3]{4}a)$$

Now for tangent to be parallel to the y-axis, $\frac{dy}{dx} = \infty$

which requires $y^2 - ax = 0$ or $x = \dfrac{y^2}{a}$

(3)

Solving (1) and (3), we have

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right)y \quad \text{or} \quad \frac{y^6}{a^3} + y^3 = 3y^3$$

i.e., $y^3 = 2a^3$ which gives $y = 2^{1/3}\,a$

Putting $y = 2^{1/3}\,a$ in (3), we get $x = 2^{2/3}\,a$

Hence the tangent is parallel to the y-axis at

$$(2^{2/3}\,a,\ 2^{1/3}\,a) = (\sqrt[3]{4}\,a,\ \sqrt[3]{2}\,a)$$

7. $25x^2 + 12xy + 4y^2 = 1$ (1)

Sol. $f(x, y) = 25x^2 + 12xy + 4y^2 - 1 = 0$

$$\frac{\partial f}{\partial x} = 50x + 12y,\ \frac{\partial f}{\partial y} = 12x + 8y$$

$$\frac{dy}{dx} = -\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}} = -\frac{50x + 12y}{12x + 8y} = -\frac{25x + 6y}{6x + 4y}$$

For tangent to be the parallel to the x-axis, $\dfrac{dy}{dx} = 0$

which implies $25x + 6y = 0$ or $x = -\dfrac{6}{25}\,y$ (2)

Putting $x = \dfrac{-6}{25}\,y$ in (1), we get

$$25\left(-\frac{6}{25}y\right)^2 + 12\left(-\frac{6}{25}y\right)y + 4y^2 = 1$$

or $\quad \dfrac{36}{25}y^2 - \dfrac{72}{25}y^2 + 4y^2 = 1 \Rightarrow 36y^2 - 72y^2 + 100y^2 = 25$

or $\quad 64y^2 = 25 \quad \Rightarrow \quad y = \pm\dfrac{5}{8}$

Putting $y = \dfrac{5}{8}$ in (2), we obtain $x = -\dfrac{6}{25} \times \dfrac{5}{8} = -\dfrac{3}{20}$

Putting $y = -\dfrac{5}{8}$ in (2), we have

$$x = -\frac{6}{25} \times \left(-\frac{5}{8}\right) = \frac{3}{20}$$

Hence tangent is parallel to the x-axis at

$$\left(-\frac{3}{20}, \frac{5}{8}\right) \text{ and } \left(\frac{3}{20}, -\frac{5}{8}\right)$$

Now for tangent to be parallel to the y-axis, $\dfrac{dy}{dx} = \infty$

which requires that $6x + 4y = 0$ or $y = -\dfrac{3}{2}x$ (3)

Putting $y = -\dfrac{3}{2}x$ in (1), we obtain

$$25x^2 + 12x\left(-\frac{3}{2}x\right) + 4\left(-\frac{3}{2}x\right)^2 = 1$$

or $\quad 25x^2 - 18x^2 + 9x^2 = 1$ or $16x^2 = 1 \Rightarrow x = \pm\dfrac{1}{4}$

Putting $x = \dfrac{1}{4}$ in (3), we have $y = -\dfrac{3}{2}\left(\dfrac{1}{4}\right) = -\dfrac{3}{8}$

and putting $x = -\dfrac{1}{4}$ in (3), we have

$$y = -\frac{3}{2}\left(-\frac{1}{4}\right) = \frac{3}{8}$$

Hence tangents are parallel to the y-axis at

$$\left(\frac{1}{4}, -\frac{3}{8}\right) \text{ and } \left(-\frac{1}{4}, \frac{3}{8}\right).$$

8. If $p = x\cos\theta + y\sin\theta$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1,$$

prove that $p^n = (a\cos\theta)^n + (b\sin\theta)^n$.

Sol. $f(x, y) = \left(\dfrac{x}{a}\right)^{\frac{n}{n-1}} + \left(\dfrac{y}{b}\right)^{\frac{n}{n-1}} - 1 = 0$

(1)

$$\frac{\partial f}{\partial x} = \frac{n}{n-1}\left(\frac{x}{a}\right)^{\frac{n}{n-1}-1}\left(\frac{1}{a}\right) = \frac{n}{(n-1)\,a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}$$

$$\frac{\partial f}{\partial y} = \frac{n}{n-1}\left(\frac{x}{b}\right)^{\frac{n}{n-1}-1}\left(\frac{1}{b}\right) = \frac{n}{(n-1)\,b}\left(\frac{x}{b}\right)^{\frac{1}{n-1}}$$

Equation of the tangent is $Y - y = -\dfrac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}(X - x)$

or $\quad (X - x)\dfrac{\partial f}{\partial f} + (Y - y)\dfrac{\partial f}{\partial y} = 0$

Substituting the values, we have

$$\frac{(X-x)}{a}\cdot\frac{n}{n-1}\cdot\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+\frac{(Y-y)}{b}\cdot\frac{n}{(n-1)}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}=0$$

or $$\frac{X-x}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+\frac{Y-y}{b}\left(\frac{x}{b}\right)^{\frac{1}{n-1}}=0$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}-\left(\frac{x}{b}\right)^{\frac{n}{n-1}}+\frac{Y}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}-\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=0$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}+\frac{Y}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}=\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1 \quad \text{(From (1))}$$

This is identical to $X\cos\theta+Y\sin\theta=p$

and so, on comparing coefficients of like terms, we get

$$\frac{\frac{1}{a}\left(\frac{x}{a}\right)^{\frac{1}{n-1}}}{\cos\theta}=\frac{\frac{1}{b}\left(\frac{y}{b}\right)^{\frac{1}{n-1}}}{\sin\theta}=\frac{1}{p}$$

or $$\left(\frac{x}{a}\right)^{\frac{1}{n-1}}=\frac{a\cos\theta}{p} \qquad (2)$$

and $$\left(\frac{y}{b}\right)^{\frac{1}{n-1}}=\frac{b\sin\theta}{p} \qquad (3)$$

Raising both the sides in (2) and (3) to the power n, we have

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}}=\frac{(a\cos\theta)^n}{p^n} \qquad (4)$$

$$\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=\frac{(b\sin\theta)^n}{p^n} \qquad (5)$$

Adding (4) and (5), we get

$$\frac{(a\cos\theta)^n}{p^n}+\frac{(b\sin\theta)^n}{p^n}=\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1$$

or $(a\cos\theta)^n+(b\sin\theta)^n=p^n$.

9. The tangent at any point on the curve $x^3+y^3=2a^3$ makes intercepts p and q on the coordinate axes. Show that

$$p^{-3/2}+q^{-3/2}=2^{-1/2}a^{-3/2}$$

Sol. $f(x,y)=x^3+y^3-2a^3=0 \qquad (1)$

$$\frac{\partial f}{\partial x}=3x^2,\ \frac{\partial f}{\partial y}=3y^2$$

Equation of the tangent at any point on the curve is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0$$

Substituting the values, we have

$$(X-x)\,3x^2+(Y-y)\,3y^2=0$$

or $Xx^2+Yy^2=x^3+y^3$

or $Xx^2+Yy^2=2a^3$, (using (1)) $\qquad (2)$

The tangent cuts off length p on the X-axis.

Putting $X=p$ and $Y=0$ into (2), we get

$$px^2+0=2a^3 \Rightarrow x^2=\frac{2a^3}{p} \text{ or } x=\left(\frac{2a^3}{p}\right)^{1/2} \qquad (3)$$

Also the tangent cuts off length q on the Y-axis.

Putting $X=0$ and $Y=q$ in (2), we have

$$0+qy^2=2a^3 \Rightarrow y^2=\frac{2a^3}{q} \Rightarrow y=\left(\frac{2a^3}{q}\right)^{1/2} \qquad (4)$$

Raising both the sides of (3) and (4) to the power 3 and adding the results, we get

$$x^3+y^3=\left(\frac{2a^3}{p}\right)^{3/2}+\left(\frac{2a^3}{q}\right)^{3/2}=\frac{2^{3/2}a^{9/2}}{p^{3/2}}+\frac{2^{3/2}a^{9/2}}{q^{3/2}}$$

or $2a^3=2^{3/2}a^{9/2}(p^{-3/2}+q^{-3/2})$

or $2^{-1/2}a^{-3/2}=p^{-3/2}+q^{-3/2}$ which is the required condition.

Find the angle of intersection of the given curves (Problems 10 – 12):

Note: The angle of intersection of two curves at a point of intersection of the curves is the angle between the tangents to the two curves at that point.

10. The parabolas $y^2=4ax$ and $x^2=4ay$ at the point other than (0, 0).

Sol. $y^2=4ax \qquad (1)$

$x^2=4ay \qquad (2)$

$x=\dfrac{y^2}{4a} \qquad (3)$

From (1), putting $x=\dfrac{y^2}{4a}$ in (2), we get

$$\left(\frac{y^2}{4a}\right)^2=4ay \quad \text{or} \quad y^4=64a^3y$$

i.e., $y(y^3-64a^3)=0$ or $y(y-4a)(y^2+4ay+16a^2)=0$

or $y=0,\ 4a$ (real values)

Putting $y=0$ in (3), we have $x=0$

Putting $y = 4a$ in (3), we get $x = \frac{16a^2}{4a} = 4a$

Therefore, the points of intersection of (1) and (2) are (0, 0) and (4a, 4a).

Now $y^2 = 4ax$ gives

$$2y\frac{dy}{dx} = 4a$$

or $\frac{dy}{dx} = \frac{2a}{y}$

or $\left(\frac{dy}{dx}\right)_{(4a,\,4a)} = \frac{2a}{4a}$

$= \frac{1}{2} = m_1$ (say)

And $x^2 = 4ay$ gives

$$2x = 4a\frac{dy}{dx}$$

or $\frac{dy}{dx} = \frac{x}{2a}$

or $\left(\frac{dy}{dx}\right)_{(4a,\,4a)} = \frac{4a}{2a}$

$= 2 = m_2$ (say)

The angle of intersection θ is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} \qquad (1)$$

$$= \frac{2 - \frac{1}{2}}{1 + 2\frac{1}{2}} = \frac{\frac{3}{2}}{2} = \frac{3}{4} \quad i.e., \quad \theta = \text{arc}\tan\left(\frac{3}{4}\right)$$

11. $x^2 - y^2 = a^2$, $x^2 + y^2 = a^2\sqrt{2}$

Sol. $x^2 - y^2 = a^2$ (1)

$x^2 + y^2 = \sqrt{2}\,a^2$ (2)

Adding (1) and (2), we get

$$2x^2 = (\sqrt{2} + 1)a^2 \text{ or } x^2 = \left(\frac{\sqrt{2}+1}{2}\right)a^2 \Rightarrow x = \pm\sqrt{\frac{\sqrt{2}+1}{2}}\,a$$

From (1), $y^2 = x^2 - a^2 = \frac{\sqrt{2}+1}{2}a^2 - a^2 = \frac{\sqrt{2}-1}{2}a^2$

or $y = \pm\sqrt{\frac{\sqrt{2}-1}{2}}\,a$

Thus the four points of intersection are

$$\left(\pm\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,\pm\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$$

Let θ be the angle from l_1 to l_2 where l_1 is the tangent to (1) and l_2 is the tangent to (2) at (x_1, y_1) which is any point of intersection.

From (1), $2x - 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$

and $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = \frac{x_1}{y_1} = m_1$ (say)

From (2), $\frac{dy}{dx} = -\frac{x}{y}$ and $\left(\frac{dy}{dx}\right)_{(x_1,y_1)} = -\frac{x_1}{y_1} = m_2$ (say)

Now $\tan\theta = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{-\frac{x_1}{y_1} - \frac{x_1}{y_1}}{1 + \left(-\frac{x_1}{y_1}\right)\left(\frac{x_1}{y_1}\right)} = \frac{-\frac{2x_1}{y_1}}{1 - \frac{x_1^2}{y_1^2}}$

$$= \frac{2x_1 y_1}{x_1^2 - y_1^2} = \frac{2x_1 y_1}{a^2}, \text{ since } x_1^2 - y_1^2 = a^2$$

As $2\left(\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right)\left(\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right) = \frac{2\sqrt{(\sqrt{2}+1)(\sqrt{2}-1)}}{2}a^2 = a^2$

and $2\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right)\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\right) = a^2$, so

$\tan\theta = \frac{a^2}{a^2} = 1$ *i.e.*, $\theta = 45°$ at the points $\left(\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$

and $\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,-\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$,

But $2x_1y_1 = -a^2$ for the points $\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$

and $\left(-\sqrt{\frac{\sqrt{2}+1}{2}}\,a\,,-\sqrt{\frac{\sqrt{2}-1}{2}}\,a\right)$, and so $\tan\theta = \frac{-a^2}{a^2} = -1$

i.e., $\theta = 135°$ at these points.

12. $y^2 = ax$ and $x^3 + y^3 = 3axy$

Sol. $y^2 = ax$ (1)

and $x^3 + y^3 = 3axy$ (2)

From (1), $x = \frac{y^2}{a}$ (3)

Putting this value of x into (2), we get

$$\left(\frac{y^2}{a}\right)^3 + y^3 = 3a\left(\frac{y^2}{a}\right).y$$

or $y^6 - 2a^3y^3 = 0 \Rightarrow y^3(y^3 - 2a^3) = 0 \Rightarrow y = 0\,,2^{1/3}a$

or $\frac{y^6}{a^3} + y^3 = 3y^3$ or $y^3 + a^3 = 3a^3$ or $y = 2^{1/3}a$

Now for $y = 0$ in (3), we get $x = 0$ and putting $y = 2^{1/3}a$ in (3), we have

$$x = \frac{(2^{1/3}a)^2}{a} = 2^{2/3}a.$$

The point of intersection of (1) and (2), say P, is

$P(2^{2/3}a,\ 2^{1/3}a)$

Differentiating (1) w.r.t. x, we get

$$2y\frac{dy}{dx} = a$$

or $\quad \dfrac{dy}{dx} = \dfrac{a}{2y}$

i.e., $\left(\dfrac{dy}{dx}\right)_P = \dfrac{a}{2.2^{1/3}a} = \dfrac{1}{2^{4/3}}$

i.e., $\tan\theta_1 = \dfrac{1}{2^{4/3}} = m_1$,

where θ_1 is the angle between the tangent to (1) and the x-axis at the point of intersection.

Differentiating (1) w.r.t. x, we have

$$3x^2 + 3y^2\frac{dy}{dx} = 3a\left[x\frac{dy}{dx} + y\right]$$

or $\quad (3y^2 - 3ax)\dfrac{dy}{dx} = 3ay - 3x^2$

or $\quad \dfrac{dy}{dx} = \dfrac{ay - x^2}{y^2 - ax}$

i.e., $\left(\dfrac{dy}{dx}\right)_P = \dfrac{a\,.\,2^{1/3}a - (2^{2/3}a)^2}{(2^{1/3}a)^2 - a.\,2^{2/3}a}$

$$= \frac{2^{1/3}a^2 - 2^{4/3}a^2}{2^{2/3}a^2 - 2^{2/3}a^2}$$

$$= \frac{2^{1/3}a^2 - 2^{4/3}a^2}{0}$$

i.e., $\theta_2 = 90°$,

where θ_2, is the angle between the tangent to (2) and the x-axis at the point of intersection.

If θ is the required angle, then $\theta = 90° - \theta_1$

i.e., $\tan\theta = \tan(90° - \theta_1)$

$$= \cot\theta_1 = \frac{1}{\tan\theta_1} = 2^{4/3}$$

Hence $\quad \theta = \arctan(2^{4/3})$

13. Find the condition that the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ should intersect orthogonally.

Sol. Let (x_1, y_1) be the point of intersection of the given curves

$$ax^2 + by^2 = 1 \qquad (1)$$

and $\quad a_1x^2 + b_1y^2 = 1 \qquad (2)$

Differentiating (1) w.r.t, x, we have

$$2ax + 2by\frac{dy}{dx} = 0$$

or $\quad \dfrac{dy}{dx} = -\dfrac{ax}{by}$ *i.e.*, $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = -\dfrac{ax_1}{by_1} = m_1$ (say)

Similarly, from (2), $\dfrac{dy}{dx} = -\dfrac{a_1x}{b_1x}$

i.e., $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)} = \dfrac{-a_1x_1}{b_1y_1} = m_2$ (say)

Since the curves cut orthogonally, we have

$$m_1 m_2 = -1$$

i.e., $\left(-\dfrac{ax_1}{by_1}\right)\left(-\dfrac{a_1x_1}{b_1y_1}\right) = -1$

or $\quad \dfrac{aa_1x_1^2}{bb_1y_1^2} = -1 \qquad (3)$

Now, from (1) and (2), by subtraction, we get

$$(a - a_1)x^2 + (b - b_1)y^2 = 0$$

or $\quad \dfrac{x_1^2}{y_1^2} = -\dfrac{b - b_1}{a - a_1}$ at (x_1, y_1)

Substituting this value into (3), we have

$$\frac{aa_1(b - b_1)}{bb_1(a - a_1)} = 1 \quad \text{or} \quad aa_1(b - b_1) = bb_1(a - a_1)$$

Dividing both sides by aa_1bb_1, we get

$$\frac{b - b_1}{bb_1} = \frac{a - a_1}{aa_1} \quad \text{or} \quad \frac{1}{b_1} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{a}$$

or $\quad \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{a_1} - \dfrac{1}{b_1}$ is the required condition.

14. Show that the pedal equation of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{is} \quad \frac{1}{p} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$$

Sol. The given equation is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \qquad (1)$

Let (x_1, y_1) lie on (1). Then $\dfrac{x_1^2}{a^2} + \dfrac{y_1^2}{b^2} = 1 \qquad (2)$

Equation of the tangent to (1) at (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad i.e., \quad \frac{x_1}{a^2}x + \frac{y_1}{b^2}y - 1 = 0$$

$$p = \frac{|-1|}{\sqrt{\dfrac{x_1^2}{a^4} + \dfrac{y_1^2}{b^4}}} \quad \text{or} \quad \frac{1}{p^2} = \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} \tag{3}$$

Also $x_1^2 + y_1^2 = r^2$ (4)

Now we eliminate x_1^2, y_1^2 from (2), (3) and (4).

From (4) and (2), we have

$$x_1^2 + y_1^2 - r^2 = 0 \quad \text{and} \quad b^2x_1^2 + ay_1^2 - a^2b^2 = 0$$

Therefore, $\dfrac{x_1^2}{-a^2b^2 + a^2r^2} = \dfrac{y_1^2}{-b^2r^2 + a^2b^2} = \dfrac{1}{a^2 - b^2}$

or $\quad x_1^2 = \dfrac{a^2(r^2 - b^2)}{a^2 - b^2} \quad ; \quad y_1^2 = \dfrac{-b^2(r^2 - a^2)}{a^2 - b^2}$

Substituting these values into (3), we get

$$\frac{1}{p^2} = \frac{r^2 - b^2}{a^2(a^2 - b^2)} - \frac{r^2 - a^2}{b^2(a^2 - b^2)}$$

$$= \frac{b^2r^2 - b^4 - a^2r^2 + a^4}{a^2b^2(a^2 - b^2)} = \frac{a^4 - b^4}{a^2b^2(a^2 - b^2)} - \frac{r^2(a^2 - b^2)}{a^2b^2(a^2 - b^2)}$$

$$= \frac{a^2 + b^2}{a^2b^2} - \frac{r^2}{a^2b^2} = \frac{1}{b^2} + \frac{1}{a^2} - \frac{r^2}{a^2b^2}$$

is the required pedal equation.

15. Show that the pedal equation of the curve

$$c^2(x^2 + y^2) = x^2y^2 \text{ is } \frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

Sol. The given equation is $c^2(x^2 + y^2) = x^2y^2$ (1)

Divide through by $c^2x^2y^2$ to have from (1)

$$\frac{1}{x^2y^2}(x^2 + y^2) = \frac{1}{c^2} \quad \text{or} \quad \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{c^2} \tag{2}$$

Now differentiating (2) w.r.t. x, we get

$$-\frac{2}{x^3} - \frac{2}{y^3}\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{y^3}{x^3}$$

If (x_1, y_1) lies on (1), then

$$\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = -\frac{y_1^3}{x_1^3} \text{ and also } c^2(x_1^2 + y_1^2) = x_1^2y_1^2$$

Equation of the tangent to (1) at (x_1, y_1) is

$$y - y_1 = -\frac{y_1^3}{x_1^3}(x - x_1) \quad \text{or} \quad -\frac{y}{y_1^3} + \frac{y_1}{y_1^3} = \frac{x}{x_1^3} - \frac{x_1}{x_1^3}$$

$$\text{or} \quad -\frac{y}{y_1^3} + \frac{1}{y_1^2} = \frac{x}{x_1^3} - \frac{1}{x_1^2} \qquad \text{or} \quad \frac{x}{x_1^3} + \frac{y}{y_1^3} - \frac{1}{x_1^2} - \frac{1}{y_1^2} = 0$$

$$p = \frac{\left|-\left(\dfrac{1}{x_1^2} + \dfrac{1}{y_1^2}\right)\right|}{\sqrt{\dfrac{1}{x_1^6} + \dfrac{1}{y_1^6}}} = \frac{\dfrac{x_1^2 + y_1^2}{x_1^2y_1^2}}{\sqrt{\dfrac{x_1^6 + y_1^6}{x_1^6y_1^6}}}$$

$$p^2 = \frac{(x_1^2 + y_1^2)^2}{x_1^4y_1^4} \times \frac{x_1^6y_1^6}{x_1^6 + y_1^6} = \frac{x_1^2y_1^2(x_1^2 + y_1^2)^2}{(x_1^2 + y_1^2)^3 - 3x_1^2y_1^2(x_1^2 + y_1^2)}$$

$$= \frac{c^2r^2 \cdot r^4}{r^6 - 3c^2r^2 \cdot r^2} = \frac{c^2r^2}{r^2 - 3c^2}, \text{ since } x_1^2 + y_1^2 = r^2$$

$$\text{or} \quad \frac{1}{p^2} = \frac{r^2 - 3c^2}{c^2r^2} = \frac{1}{c^2} - \frac{3}{r^2} \Rightarrow \frac{1}{p^2} + \frac{3}{r^2} = \frac{1}{c^2}$$

which is the required pedal equation.

16. Show that from any point three normals can be drawn to a parabola $y^2 = 4ax$ and the sum of the slopes of the three normals is zero.

Sol. Equation of any normal to $y^2 = 4ax$ is

$$y = mx - 2am - am^3. \quad \text{(Example 7)}$$

If it passes through a given point (h, k), then

$$k = mh - 2am - am^3$$

$$\text{or} \quad am^3 + 0m^2 - m(2a - h) + k = 0 \tag{1}$$

This is a cubic in m which gives the slopes of the three normals which pass through the point (h, k).

Hence three normals, can be drawn from any point (h, k) to the parabola $y^2 = 4ax$.

If m_1, m_2, m_3 are roots of the cubic (1), then

$$m_1 + m_2 + m_3 = \frac{-0}{a} = 0$$

Hence the sum of the slopes of the three normals to the parabola $y^2 = 4ax$ drawn from any point is zero.

17. Show that the tangents at the ends of a focal chord of a parabola intersect at right angles on the directrix.

Sol. Let $y^2 = 4ax$ be equation of a parabola. If t_1, t_2 are the extremities of a focal chord then $t_1t_2 = -1$. Also tangents at t_1, t_2 are

$$t_1y = x + at_1^2 \tag{1}$$

and $t_2 y = x + at_2^2$ (2)

The product of the slopes of (1) and (2)

$$= \frac{1}{t_1} \cdot \frac{1}{t_2}$$

$$= -1, \text{ (since } t_1 t_2 = -1)$$

Therefore, the tangents to (1) and (2) are perpendicular to each other. The point of intersection of (1) and (2) is

$[at_1 t_2, a(t_1 + t_2)]$

i.e., $[-a, a(t_1 + t_2)]$

which clearly lies on the directrix $x = -a$.

Hence the result.

18.(a) Show that the tangent at the vertex of a diameter of a parabola is parallel to the chords bisected by the diameter.

Sol.

(a) Let an equation of the parabola be $y^2 = 4ax$ and m be the slope of the parallel chords.

Equation of the diameter bisecting the chords is

$$y = \frac{2a}{m} \quad (1)$$

Let (x_1, y_1) be the vertex of the diameter.

Since (x_1, y_1) lies on (1), therefore

$$y_1 = \frac{2a}{m} \quad \text{or} \quad m = \frac{2a}{y_1} \quad (2)$$

The equation of the tangent at the vertex (x_1, y_1) is

$yy_1 = 2a(x + x_1)$.

The slope of the tangent is $= \frac{2a}{y_1} = m =$ the slope of the parallel chords.

Hence the tangent is parallel to the chords.

18.(b) Prove that the tangents at the ends of any chord of a parabola meet on the diameter which bisects the chord.

Sol. Let an equation of the parabola be $y^2 = 4ax$ and let equation of a chord PQ be $y = mx + c$. Suppose $P = (x_1, y_1)$ and $Q = (x_2, y_2)$.

Equations of the tangents at P and Q are respectively.

$yy_1 = 2a(x + x_1)$

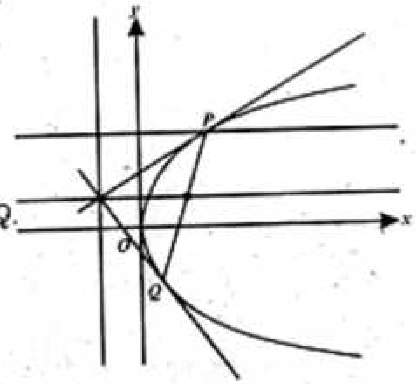
$yy_2 = 2a(x + x_2)$

Ordinate of the point of intersection of these tangents is

$$y = \frac{2a(x_1 - x_2)}{y_1 - y_2} \quad (1)$$

But $m = \frac{y_1 - y_2}{x_1 - x_2} =$ slope of PQ.

Therefore, (1) is $y = \frac{2a}{m}$



By Example 10, $y = \frac{2a}{m}$ is equation of the diameter bisecting the chords parallel to PQ. Thus the tangents at P and Q meet on this diameter.

19. Find the condition that the straight line $lx + my + n = 0$ may touch the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Also find the coordinates of the point of contact.

Sol. Let us assume that $lx + my + n = 0$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1).

The tangent at (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

or $b^2 x_1 x + a^2 y_1 y - a^2 b^2 = 0$. This must be identical to

$lx + my + n = 0$

Therefore, $\frac{b^2 x_1}{l} = \frac{a^2 y_1}{m} = \frac{-a^2 b^2}{n}$

or $x_1 = \frac{-a^2 l}{n}$ and $y_1 = \frac{-b^2 m}{n}$

The point of contact is $\left(\frac{-a^2 l}{n}, \frac{-b^2 m}{n}\right)$

It lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so

$\frac{a^4 l^2}{n^2 a^2} + \frac{b^4 m^2}{n^2 b^2} = 1$ *i.e.*, $a^2 l^2 + b^2 m^2 = n^2$ is the required condition.

20. Show that the condition that normal at the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) on the ellipse $\frac{x_2}{a_2} + \frac{y^2}{b^2} = 1$ may be concurrent is

$$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

Sol. Equation of the normal at (h, k) on the ellipse can be written as

$$\frac{a^2 x}{h} - \frac{b^2 y}{k} = a^2 - b^2, \text{ (Example 12)}$$

or $\quad a^2kx - b^2hy = (a^2 - b^2)\, hk.$

Therefore, equations of the three normals at (x_1, y_1), (x_2, y_2), (x_3, y_3) are

$$a^2y_1 - b^2x_1y - (a^2 - b^2)\, x_1 y_1 = 0 \qquad (1)$$

$$a^2y_2 - b^2x_2y - (a^2 - b^2)\, x_2 y_2 = 0 \qquad (2)$$

and $\quad a^2y_3 - b^2x_3y - (a^2 - b^2)\, x_3 y_3 = 0 \qquad (3)$

Solving (2) and (3), we have

$$x = \frac{(a^2 - b^2)\, x_2 x_3(y_3 - y_2)}{a^2(x_2y_3 - x_3y_2)}, \; y = \frac{(a^2 - b^2)\, y_2 y_3(x_3 - x_2)}{b^2(x_2y_3 - x_3y_2)}$$

Putting these values of x and y into (1), we get the condition of concurrency of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1), (x_2, y_2) and (x_3, y_3). Hence

$$a^2y_1\left(\frac{(a^2 - b^2)\, x_2x_3(y_3 - y_2)}{a^2(x_2y_3 - x_3y_2)}\right) - b^2x_1\left(\frac{(a^2 - b^2)\, y_2y_3(x_3 - x_2)}{a^2(x_2y_3 - x_3y_2)}\right) - (a^2 - b^2)x_1y_1 = 0$$

$$x_1(y_2x_3y_3 - y_3x_2y_2) - y_1(x_2x_3y_3 - x_3x_2y_2) + x_1y_1(x_2y_3 - x_3y_2) = 0$$

or $$\begin{vmatrix} x_1 & y_1 & x_1y_1 \\ x_2 & y_2 & x_2y_2 \\ x_3 & y_3 & x_3y_3 \end{vmatrix} = 0$$

21. If a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, with centre C, meets the major and minor axes in T and t, prove that $\frac{a^2}{CT^2} + \frac{b^2}{Ct^2} = 1$

Sol. Tangent to the ellipse at the point 'θ' is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

This meets the major axis where $y = 0$

i.e., $\frac{x}{a}\cos\theta = 1 \quad$ or $\quad x = \frac{a}{\cos\theta}$

or $\quad CT = \frac{a}{\cos\theta} \quad$ or $\quad \cos\theta = \frac{a}{CT} \qquad (1)$

Similarly, (1) meets the minor axis where $x = 0$

i.e., $\frac{y}{b}\sin\theta = 1 \quad$ or $\quad y = \frac{b}{\sin\theta}$

i.e., $\quad Ct = \frac{b}{\sin\theta} \quad$ or $\quad \sin\theta = \frac{b}{Ct} \qquad (2)$

Squaring (1) and (2) and adding the results, we get

$$\cos^2\theta + \sin^2\theta = 1 = \frac{a^2}{CT^2} + \frac{b^2}{Ct^2} \text{ as required.}$$

22. Show that the locus of the point of intersection of tangents at two points on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x_2}{a_2} + \frac{y_2}{b_2} = \sec^2\lambda$, where 2λ is the difference of the eccentric angles of the two points.

Sol. Let the two points on the ellipse be

$$(a\cos\alpha, b\sin\alpha), \; (a\cos\beta, \text{be}\sin\beta)$$

Equations of the tangents at these point are

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1$$

and $\quad \frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1$

The point of intersection of the tangents is

$$x = \frac{a(\sin\alpha - \sin\beta)}{\sin(\alpha - \beta)} = \frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}$$

$$y = \frac{-b(\cos\alpha - \cos\beta)}{\sin(\alpha - \beta)} = \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \qquad (1)$$

Now $\alpha - \beta = 2\lambda$ (Given)

Squaring the equations in (1) and adding the results, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\cos^2\left(\frac{\alpha+\beta}{2}\right) + \sin^2\left(\frac{\alpha+\beta}{2}\right)}{\cos^2\left(\frac{\alpha-\beta}{2}\right)} = \frac{1}{\cos^2\lambda} = \sec^2\lambda$$

23. Show that locus of the feet of the perpendiculars from the foci on any tangent to an ellipse is the auxiliary circle and product of the lengths of perpendiculars is equal to the square of the semi-minor axis.

Sol. Let an equation of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Equation of a tangent to the ellipse, in the m – form, is

$$y = mx + \sqrt{a^2m^2+b^2} \qquad (1)$$

Slope of any line perpendicular to it is $-\frac{1}{m}$

Equation of the perpendicular F_1P from F_1 $(-ae, 0)$ on the tangent (1) is

$$y-0=-\frac{1}{m}(x+ae) \qquad (2)$$

To find the locus of P, the foot of the perpendicular [*i.e.*, point of intersection of (1) and (2)], we have to eliminate m between (1) and (2). (1) and (2) may be written as

$$y-mx=\sqrt{a^2m^2+b^2} \qquad (3)$$

and $\quad x+my=-ae \qquad (4)$

Squaring (3) and (4), we have

$$y^2-2mxy+m^2x^2=a^2m^2+b^2 \qquad (5)$$

and $\quad x^2+2mxy+m^2y^2=a^2e^2 \qquad (6)$

Adding (5) and (6), we get

$$x^2(1+m^2)+y^2(1+m^2)=a^2m^2+b^2+a^2e^2$$
$$=a^2m^2+a^2(1-e^2)+a^2e^2$$
$$=a^2(1+m^2)$$

or $\quad x^2+y^2=a^2$, which is the required locus.

This is auxiliary circle of the given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Similarly, the locus of Q, the foot of the perpendicular from the other focus F_2 on the tangent (1), is the auxiliary circle.

Now we prove that $|F_1P|\cdot|F_2Q|=CB^2$.

Equation of the tangent is $y=mx+\sqrt{a^2m^2+b^2}$

or $\quad mx-y+\sqrt{a^2m^2+b^2}=0 \qquad (1)$

$|F_1P|$ = perpendicular from $(-ae, 0)$ on (1)

$$=\frac{\left|-aem+\sqrt{a^2m^2+b^2}\right|}{\sqrt{1+m^2}}$$

and $\quad |F_2Q|$ = perpendicular from $(ae, 0)$ on (1)

$$=\frac{\left|aem+\sqrt{a^2m^2+b^2}\right|}{\sqrt{1+m^2}}$$

$$|F_1P|\cdot|F_2Q|=\frac{a^2m^2+b^2-a^2e^2m^2}{1+m^2}=\frac{a^2m^2(1-e^2)+b^2}{1+m^2}$$

$$=\frac{m^2b^2+b^2}{1+m^2}=\frac{b^2(1+m^2)}{1+m^2}=b^2=CB^2.$$

24. Prove that the area enclosed by the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant.

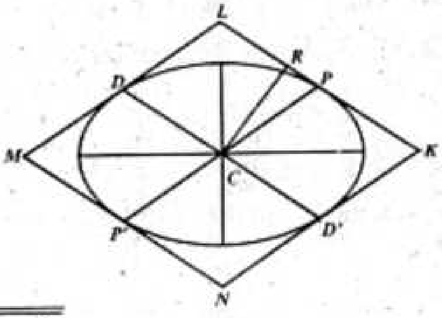
Sol. Let $P'CP$ and $D'CD$ be conjugate diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then

$P=(a\cos\theta, b\sin\theta)$ and $D=(-a\sin\theta, b\cos\theta)$.

Let $KLMN$ be the parallelogram formed by the tangents at P, D, P' and D'.

Area $KLMN$ = 4 area $CPLD$
$= 4\,|CR|\cdot|PL|$
$= 4|CR|\cdot|CD|$

where $|CR|$ = length of the perpendicular from C on the tangent at P.



Equation of the tangent at P is

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta-1=0$$

Therefore, $$|CR|=\frac{|-1|}{\sqrt{\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}}}$$

$$=\frac{ab}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}=\frac{ab}{|CD|},$$

since $|CD|=\sqrt{(-a\sin\theta-0)^2+(b\cos\theta-0)^2}$

Hence, area $KLMN = 4\,|CR|\cdot|CD| = 4\cdot\frac{ab}{|CD|}\cdot|CD|$

$= 4ab$, which is a constant

25. The hyperbolas $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ are said to be conjugate to each other. If e and e' are eccentricities of a hyperbola and its **conjugate**, prove that $\frac{1}{e^2}+\frac{1}{e'^2}=1$

Sol. The eccentricity e of the hyperbola

Thus slope of the normal is $-\dfrac{a\sin\theta}{b}$

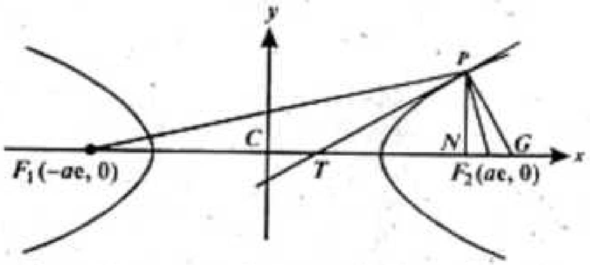
Equation of the normal at P is

$$y - b\tan\theta = -\frac{a\sin\theta}{b}(x - a\sec\theta)$$

or $\quad by - b^2\tan\theta = -a\sin\theta(x - a\sec\theta)$

$$by\cot\theta - b^2 = -a^x\cos\theta + a^2$$

$$ax\cos\theta + by\cot\theta = a^2 + b^2 \quad\text{or}\quad \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$



Suppose that the normal at P $(a\sec\theta, b\tan\theta)$ meets the transverse axis in G. Then coordinates of G are

$$\left(\frac{(a^2+b^2)\sec\theta}{a}, 0\right)$$

As $\dfrac{a^2+b^2}{a}\sec\theta = \dfrac{a^2e^2}{a}\sec\theta = ae^2\sec\theta$, so it is $(ae^2\sec\theta, 0)$

Now $(F_2P)^2 = (a\sec\theta - ae)^2 + b^2\tan^2\theta$

$$= (a\sec\theta - ae)^2 + b^2(\sec^2\theta - 1)$$
$$= a^2\sec^2\theta - 2a^2e\sec\theta + a^2e^2 + b^2\sec^2\theta - b^2$$
$$= (a^2+b^2)\sec^2\theta - 2a^2e\sec\theta + a^2e^2 - b^2$$
$$= a^2e^2\sec^2\theta - 2a^2e\sec\theta + a^2 = [a(e\sec\theta - 1)]^2$$

$$(F_2G)^2 = (ae^2\sec\theta - ae)^2 = [ae(e\sec\theta - 1)]^2$$
$$= e^2[a(e\sec\theta - 1)]^2$$

Therefore, $(F_2G)^2 = e^2(F_2P)^2 \Rightarrow |F_2G| = e|F_2P|$

Similarly, $|F_1G| = e|F_1P|$

Therefore, $\dfrac{|F_2G|}{|F_1G|} = \dfrac{|F_2P|}{|F_1P|}$

Hence the normal PG is external bisector of the angle F_1PF_2.

Since the tangent PT at P is perpendicular to the normal PG, therefore, PT bisects the interior angle F_1PF_2.

Exercise Set 6.3 (Page 250)

In Problems 1 – 8, express the given equations in rectangular coordinates:

1. $r^2 = a^2\sin 2\theta$

Sol. $r^2 = a^2\sin 2\theta$ or $r^2 = 2a^2\sin\theta\cos\theta$

Multiplying both sides by r^2, we have

$$r^4 = 2a^2(r\sin\theta)(r\cos\theta) \quad\text{or}\quad (x^2+y^2)^2 = 2a^2xy$$

2. $r^4\sin 4\theta = a^4$

Sol. $r^4\sin 4\theta = a^4$ or $r^4(2\sin 2\theta\cos 2\theta) = a^4$

i.e., $\quad 4r^4\sin\theta\cos\theta(\cos^2\theta - \sin^2\theta) = a^4$

or $\quad 4(r\cos\theta)(r\sin\theta)(r^2\cos^2\theta - r^2\sin^2\theta) = a^4$

or $\quad 4xy(x^2 - y^2) = a^4$

3. $r^2 = a^2\cos 2\theta$

Sol. $r^2 = a^2\cos 2\theta$ or $r^2 = a^2(\cos^2\theta - \sin^2\theta)$

Multiplying both sides by r^2, we have

$$r^4 = a^2(r^2\cos^2\theta - r^2\sin^2\theta)$$
$$= a^2[(r\cos\theta)^2 - (r\sin\theta)^2]$$

or $\quad (x^2+y^2)^2 = a^2(x^2 - y^2)$

4. $r = 2a\sin\theta\tan\theta$

Sol. $r = 2a\sin\theta\tan\theta$ or $r^2 = 2a(r\sin\theta)\dfrac{(r\sin\theta)}{r\cos\theta}$

i.e., $\quad (x^2+y^2) = \dfrac{2ay^2}{x} \Rightarrow x(x^2+y^2) = 2ay^2$

5. $r = 1 - \cos\theta$

Sol. $r = 1 - \cos\theta$ (1)

Multiplying both sides of (1) by r, we get

$$r^2 = r - r\cos\theta \quad\text{or}\quad r^2 + r\cos\theta = r$$

i.e., $\quad x^2 + y^2 + x = \sqrt{x^2+y^2}$

or $\quad (x^2+y^2+x)^2 = x^2 + y^2$

6. $r^2(4\sin^2\theta - 9\cos^2\theta) = 36$

Sol. $r^2(4\sin^2\theta - 9\cos^2\theta) = 36$ or $4r^2\sin^2\theta - 9r^2\cos^2\theta = 36$

i.e., $\quad 4y^2 - 9x^2 = 36$ or $\dfrac{y^2}{9} - \dfrac{x^2}{4} = 1$

7. $r = \dfrac{8}{2 - \cos\theta}$

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is given by $b^2=a^2(e^2-1)$ i.e., $\frac{b^2}{a^2}=e^2-1$ (1)

The eccentricity e' of the conjugate hyperbola

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1$ is given by $a^2=b^2(e'^2-1)$ i.e., $\frac{a^2}{b^2}=e'^2-1$ (2)

Multiplying (1) and (2) together, we have

$$(e^2-1)(e'^2-1)=1 \quad \text{or} \quad e^2e'^2=e^2+e'^2$$

or $\frac{1}{e^2}+\frac{1}{e'^2}=1.$

26. Show that the asymptotes of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and the lines drawn from any point on the hyperbola parallel to the asymptotes form a parallelogram of constant area $\frac{ab}{2}$.

Sol. Let $P(x_1, y_1)$ be any point on $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

Equations of the asymptotes of the hyperbola are

$$y=\frac{b}{a}x \quad (1)$$

and $$y=-\frac{b}{a}x \quad (2)$$

Equation of the line PR parallel to (2) is

$$y-y_1=-\frac{b}{a}(x-x_1) \quad (3)$$

Solving (1) and (3), we have

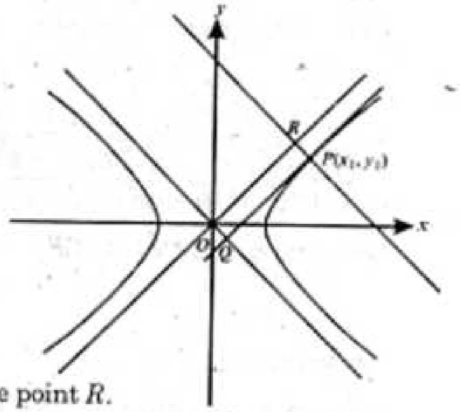
$$\frac{b}{a}x-y_1=-\frac{b}{a}x+\frac{b}{a}x_1$$

$$\Rightarrow \frac{2b}{a}x=y_1+\frac{b}{a}x_1$$

or $$x=\frac{a}{2b}\left(y_1+\frac{b}{a}x_1\right)$$

and so $$y=\frac{b}{a}\,\frac{a}{2b}\left(y_1+\frac{b}{a}x_1\right)$$

$$=\frac{1}{2}\left(y_1+\frac{b}{a}x_1\right)$$



These are the coordinates of the point R.

The line PQ parallel to (1) meets the asymptotes (2) at Q.

We need the area of the parallelogram $OQPR$.

Required area = 2 area $\Delta\, OPR$.

$$=\begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ \frac{1}{2}\left(y_1+\frac{bx_1}{a}\right) & \frac{a}{2b}\left(y_1+\frac{b}{a}x_1\right) & 1 \end{vmatrix}$$

$$=\frac{x_1}{2a}(ay_1+bx_1)-\frac{ay_1(ay_1+bx_1)}{2ab}$$

$$=\frac{1}{2ab}(ab\,x_1y_1+b^2x_1^2-a^2y_1^2-ab\,x_1y_1)=\frac{1}{2ab}[b^2x_1^2-a^2y_1^2]$$

$$=\frac{1}{2ab}(a^2b^2), \text{ since } \frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}=1$$

$$=\frac{ab}{2}$$

27. Show that the normal to the rectangular hyperbola $xy=c^2$ at the point 't' meets the curve again at the point 't'' such that $t^3t'=-1$.

Note: The point $\left(ct, \frac{c}{t}\right)$ on $xy=c^2$ is referred to as the point 't'.

Sol. Equation of the normal at the point 't' is

$$t^3x-ty+c-ct^4=0$$

Let this normal cut the hyperbola again at the point 't''. Then

$\left(ct', \frac{c}{t'}\right)$ lies on this normal

Hence $t^3ct'-t\frac{c}{t'}+c-ct^4=0$

or $t'^2t^3-t+t'-t't^4=0$

or $t'(t^3t'+1)-t(1+t't^3)=0$

or $(t'-t)(t^3t'+1)=0$

But $t\neq t'$, so $t^3t'+1=0 \Rightarrow t^3t'=-1$

28. Prove that if P is any point on a hyperbola with foci F_1 and F_2, then the tangent at P bisects the angle F_1PF_2.

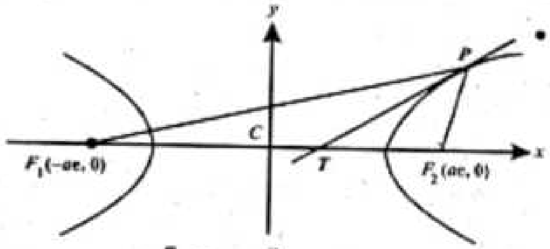
Sol. Let the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad (1)$$

Let $P(a\sec\theta, b\tan\theta)$ be any point on (1). Foci of (1) are $F_2(ae, 0)$ and $F_1(-ae, 0)$.

Equation of the tangent at P is

$$\frac{x\sec\theta}{a}-\frac{y\tan\theta}{b}=1$$



It cuts the x-axis at $T\left[\frac{a}{\sec\theta}, 0\right]$ i.e., $(a\cos\theta, 0)$

$$(PF_2)^2 = (a\sec\theta - ae)^2 + (b\tan\theta)^2$$
$$= a^2\sec^2\theta + a^2e^2 + b^2\tan^2\theta - 2a^2e\sec\theta$$
$$= a^2\sec^2\theta + (a^2 + b^2) + b^2\tan^2\theta - 2a^2e\sec\theta$$
$$= a^2\sec^2\theta + a^2 + b^2\sec^2\theta - 2a^2e\sec\theta$$
$$= (a^2 + b^2)\sec^2\theta + a^2 - 2a^2e\sec\theta$$
$$= a^2e^2\sec^2\theta + a^2 - 2a^2e\sec\theta$$
$$= (ae\sec\theta - a)^2 = a^2(e\sec\theta - 1)^2$$

Similarly $(PF_1)^2 = a^2(-e\sec\theta - 1)^2$ (changing e into $-e$)
$$= a^2(e\sec\theta + 1)^2$$
$$\frac{(PF_2)^2}{(PF_1)^2} = \frac{a^2(e\sec\theta - 1)^2}{a^2(e\sec\theta + 1)^2} = \frac{(e - \cos\theta)^2}{(e + \cos\theta)^2}$$
$$\frac{|PF_2|}{|PF_1|} = \frac{|e - \cos\theta|}{|e + \cos\theta|} \qquad (2)$$

Further, $(F_2T)^2 = (a\cos\theta - ae)^2 = a^2(\cos\theta - e)^2$
$$(F_1T)^2 = (a\cos\theta + ae)^2 = a^2(\cos\theta + e)^2$$
$$\frac{|F_2'T|}{|F_1T|} = \frac{|e - \cos\theta|}{|e + \cos\theta|} \qquad (3)$$

From (2) and (3), it is clear that the tangent divides F_2F_1 internally in the ratio of PF_2 and PF_1.

Hence PT bisects the angle F_2PF_1.

29. Find an equation of a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the form $\frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1$. Show that the product of lengths of perpendiculars on it from the foci is constant.

Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (1)

The point $x = a\cosh\theta$, $y = b\sinh\theta$ lies on (1). Thus any point on (1) is $P(a\cosh\theta, b\sinh\theta)$. Equation of the tangent to (1) at the point P is
$$\frac{x \cdot a\cosh\theta}{a^2} - \frac{y \cdot b\sinh\theta}{b^2} = 1 \text{ or } \frac{x}{a}\cosh\theta - \frac{y}{b}\sinh\theta = 1 \qquad (2)$$
as desired.

The foci of the hyperbola (1) are $F_2(a, e, 0)$, $F'_1(-ae, 0)$.

Let F_2Q, F'_1R be perpendiculars from F_2 and F'_1 to (2). Then
$$|F_2Q| = \frac{|e\cosh\theta - 1|}{\sqrt{\frac{\cosh^2\theta}{a^2} + \frac{\sinh^2\theta}{b^2}}}$$
$$|F_1R| = \frac{|-e\cosh\theta - 1|}{\sqrt{\frac{\cosh^2\theta}{a^2} + \frac{\sinh^2\theta}{b^2}}} = \frac{|e\cosh\theta + 1|}{\sqrt{\frac{\cosh^2\theta}{a^2} + \frac{\sinh^2\theta}{b^2}}}$$
$$F_2Q.F_1R = \frac{(e^2\cosh^2\theta - 1)a^2b^2}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$
$$= \frac{a^2b^2\left(\frac{a^2 + b^2}{a^2}\right)\cosh^2\theta - a^2b^2}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$
$$= \frac{a^2b^2\cosh^2\theta + b^4\cosh^2\theta - a^2b^2}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$
$$= \frac{a^2b^2(\cosh^2\theta - 1) + b^4\cosh^2\theta}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$
$$= \frac{b^2(a^2\sinh^2\theta + b^2\cosh^2\theta)}{b^2\cosh^2\theta + a^2\sinh^2\theta}$$
$= b^2$, which is a constant.

30. Find an equation of a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ in the form $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$. Prove that the normal is external bisector of the angle between the focal distances of its foot.

Sol. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (1)

Differentiating (1) w.r.t. x we get
$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = \frac{2x}{a^2} \times \frac{b^2}{2y} = \frac{b^2x}{a^2y}$$

At $P(a\sec\theta, b\tan\theta)$,
$$\frac{dy}{dx} = \frac{b^2a\sec\theta}{a^2b\tan\theta} = \frac{b}{a\sin\theta}$$

Sol. $r = \dfrac{8}{2 - \cos\theta}$ can be written as $2r - r\cos\theta = 8$

$$\text{i.e.,}\quad 2\sqrt{x^2 + y^2} - x = 8 \Rightarrow 2\sqrt{x^2 + y^2} = x + 8$$
$$\text{or}\quad 4(x^2 + y^2) = x^2 + 16x + 64$$
$$\text{or}\quad 3x^2 + 4y^2 - 16x - 64 = 0$$

8. $r = 2\sin\theta + 3\cos\theta$

Sol. $r = 2\sin\theta + 3\cos\theta$

Multiplying both sides by r, we get
$$r^2 = 2r\sin\theta + 3r\cos\theta$$
$$\text{or}\quad x^2 + y^2 = 2y + 3x$$

Transform the given equation in polar coordinates (Problems 9 – 15).

9. $xy = a$

Sol. $xy = a$

Putting $x = r\cos\theta, y = r\sin\theta$, we have
$$(r\cos\theta)(r\sin\theta) = a$$
$$\text{or}\quad r^2\sin\theta\cos\theta = a \Rightarrow r^2 = a\sec\theta\csc\theta$$

10. $y^2 = 4x$

Sol. $y^2 = 4x$

Putting $x = r\cos\theta, y = r\sin\theta$, we have
$$(r\sin\theta)^2 = 4r\cos\theta$$
$$\text{or}\quad r^2\sin^2\theta - 4r\cos\theta = 0 \Rightarrow r\sin^2\theta - 4\cos\theta = 0$$
$$\text{or}\quad r = 4\cot\theta\csc\theta$$

11. $y = \dfrac{x}{x + 1}$

Sol. $y = \dfrac{x}{x + 1}$ can be written as $xy + y = x$ and its polar form is
$$(r\cos\theta)(r\sin\theta) + r\sin\theta = r\cos\theta$$
Dividing both sides by r, we have
$$r\cos\theta\sin\theta + \sin\theta = \cos\theta$$
$$\Rightarrow\quad r\sin\theta\cos\theta = \cos\theta - \sin\theta$$
$$\text{or}\quad r = \frac{\cos\theta - \sin\theta}{\sin\theta\cos\theta} = \csc\theta - \sec\theta$$

12. $x^2 + y^2 - 8x + 6y + 7 = 0$

Sol. $x^2 + y^2 - 8x + 6y + 7 = 0$
$$\text{or}\quad (r\cos\theta)^2 + (r\sin\theta)^2 - 8r\cos\theta + 6r\sin\theta + 7 = 0$$
$$\text{or}\quad r^2 - 8r\cos\theta + 6r\sin\theta + 7 = 0$$
$$\Rightarrow\quad r^2 - 2r(4\cos\theta - 3\sin\theta) + 7 = 0$$

13. $(x^2 + y^2)y^2 = a^2x^2$

Sol. $(x^2 + y^2)y^2 = a^2x^2$ (1)

Putting $x = r\cos\theta, y = r\sin\theta$ in (1) we have
$$r^2 \,.\, r^2\sin^2\theta = a^2r^2\cos^2\theta$$
Dividing both sides by r^2, we have
$$r^2\sin^2\theta = a^2\cos^2\theta$$
$$\Rightarrow\quad r^2 = a^2\cot^2\theta$$

14. $x^3 + 4x^2 + xy^2 - 4y^2 = 0$

Sol. $x^3 + 4x^2 + xy^2 - 4y^2 = 0$ can be written as
$$x(x^2 + y^2) + 4(x^2 - y^2) = 0$$
Putting $x = r\cos\theta, y = r\sin\theta$, we have
$$r\cos\theta(r^2) + 4(r^2\cos^2\theta - r^2\sin^2\theta) = 0$$
$$\text{or}\quad r^2[r\cos\theta + 4(\cos^2\theta - \sin^2\theta)] = 0$$
$$\Rightarrow\quad r\cos\theta + 4\cos 2\theta = 0$$
$$\text{or}\quad r + 4\cos 2\theta\sec\theta = 0$$

15. $x^4 + 2x^2y^2 + y^4 - 6x^2y + 2y^3 = 0$

Sol. $x^4 + 2x^2y^2 + y^4 - 6x^2y + 2y^3 = 0$ can be written as
$$(x^2 + y^2)^2 - 2y(3x^2 - y^2) = 0$$
$$\text{or}\quad r^4 - 2r\sin\theta(3r^2\cos^2\theta - r^2\sin^2\theta) = 0$$
Dividing both sides by r^3, we get
$$r - 2\sin\theta(3\cos^2\theta - \sin^2\theta) = 0.$$

Exercise Set 6.4 (Page 256)

In Problems 1 – 6, identify and graph the given polar equations:

1. $r = \dfrac{4}{1 + \cos\theta}$

Sol. $r = \dfrac{4}{1 + \cos\theta}$ can be written as $\dfrac{4}{r} = 1 + \cos\theta$

Comparing it with $\dfrac{l}{r} = 1 + e\cos\theta$, we have
$$l = 4 = \text{length of the semi latusrectum},\ e = 1$$
Hence, the conic is a parabola.

Some points on the graph in polar form are
$$(2, 0), \left(\frac{8}{3}, \frac{\pi}{3}\right), \left(4, \frac{\pi}{2}\right), \left(8, \frac{2\pi}{3}\right), \left(8, \frac{4\pi}{3}\right)\left(4, \frac{3\pi}{2}\right) \text{ and } \left(\frac{8}{3}, \frac{5\pi}{3}\right).$$

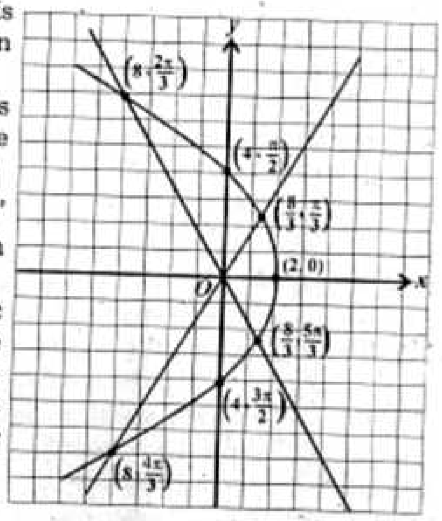
The graph is as shown,

$y^2 = -8(x - 2)$ is equation of the conic in cartesian system.

(2, 0) is vertex and (0, 0) is focus. Points shown in the graph are (2, 0), $\left(\frac{4}{3}, \pm\frac{4\sqrt{3}}{3}\right)$, (0, ±4) and (−4, ±4 $\sqrt{3}$) in cartesian system.

Note: The conic is symmetric about the initial line. The upper half of the conic can be described by taking some special values of θ from 0 to $\frac{2\pi}{3}$

Using symmetry, the lower half can be traced.



2. $r = \dfrac{10}{1 - \sin\theta}$

Sol. $r = \dfrac{10}{1 - \sin\theta}$ can be written as $\dfrac{10}{r} = 1 - \sin\theta$

or $\dfrac{10}{r} = r - \cos\left(\dfrac{\pi}{2} - \theta\right)$

Comparing it with $\dfrac{l}{r} = 1 - e\cos\theta$, we have $l = 10$, $e = 1$

Hence the conic is a parabola.

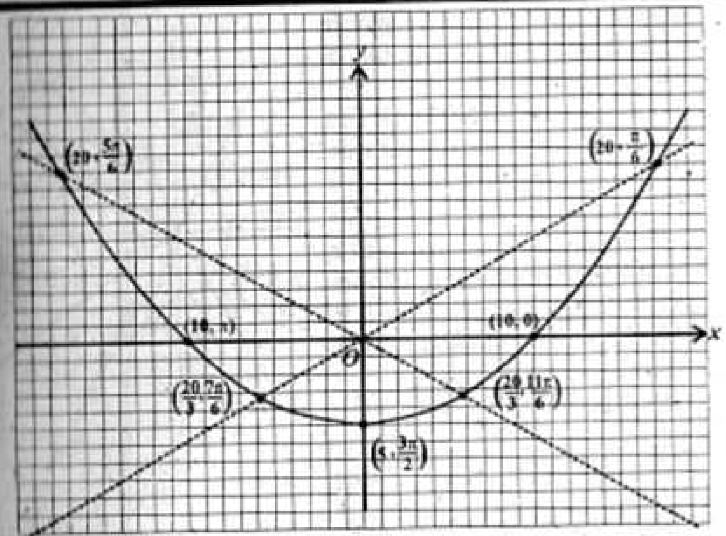
We plot the following points (in polar form)

$(10, 0), \left(20, \frac{\pi}{6}\right), \left(20, \frac{5\pi}{6}\right), (10, \pi), \left(\frac{20}{3}, \frac{7\pi}{6}\right), \left(5, \frac{3\pi}{2}\right)\left(\frac{20}{3}, \frac{11\pi}{6}\right)$.

Now joining the plotted points smoothly, we get the graph of the given conic.

Equation of the conic in cartesian coordinate system is $x^2 = 20(y + 5)$. The vertex of the parabola is (0, −5) and its focus is (0, 0). The points shown in the graph in cartesian system are

$(\pm 10, 0), (\pm 10\sqrt{3}, 10), \left(\pm\frac{10\sqrt{3}}{3}, -\frac{10}{3}\right), (0, -5).$



The orientation of the curve is in the positive direction of the y-axis.

Note: The conic is symmetric about the y-axis.

3. $r = \dfrac{8}{3 - \cos\theta}$

Sol. $r = \dfrac{8}{3 - \cos\theta}$ can be written as

$\dfrac{8}{r} = 3 - \cos\theta = 3\left(1 - \dfrac{1}{3}\cos\theta\right)$

or $\dfrac{\frac{8}{3}}{r} = 1 - \dfrac{1}{3}\cos\theta$

Comparing it with $\dfrac{l}{r} = 1 - e\cos\theta$, we have

$l = \dfrac{8}{3}$; $e = \dfrac{1}{3}$ which is less than 1

Hence the conic is an ellipse.

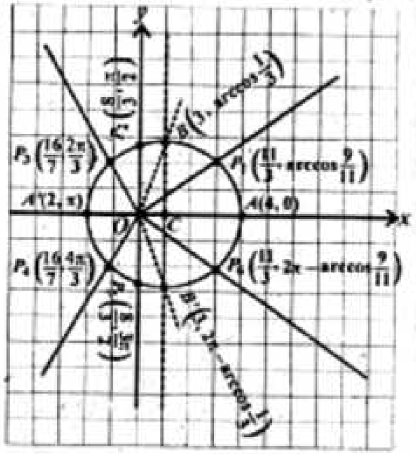
We plot the following points (in polar form)

$A(4, 0), P_1\left(\frac{11}{3}, \text{arcsos}\,\frac{9}{11}\right), B\left(3, \arccos\frac{1}{3}\right), P_2\left(\frac{8}{3}, \frac{\pi}{2}\right),$

$P_3\left(\frac{16}{7}, \frac{2\pi}{3}\right), A'(2, \pi), P_4\left(\frac{16}{7}, \frac{4\pi}{3}\right), P_5\left(\frac{8}{3}, \frac{3\pi}{2}\right),$

$B'\left(3, 2\pi - \arccos\frac{1}{3}\right), P_6\left(\frac{11}{3}, 2\pi - \arccos\frac{9}{11}\right)$

Joining the plotted points smoothly, we get the graph of the given conic.



Equation of the conic in cartesian coordinate system is

$$\frac{(x-1)^2}{9} + \frac{y^2}{8} = 1$$

The centre of the conic is $C(1, 0)$ and vertices are $A(4, 0)$ and $A'(-2, 0)$. The points shown in the graph in cartesian system are $A(4, 0)$, $P_1\left(3, \frac{2\sqrt{10}}{3}\right)$, $B(1, 2\sqrt{2})$, $P_2\left(0, \frac{8}{3}\right)$, $P_3\left(-\frac{8}{7}, \frac{8\sqrt{3}}{7}\right)$, $A'(-2, 0)$, $P_4\left(-\frac{8}{7}, \frac{-8\sqrt{3}}{7}\right)$, $P_5\left(0, -\frac{8}{3}\right)$, $B'(1, -2\sqrt{2})$, $P_6\left(\frac{11}{3}, \frac{-2\sqrt{10}}{3}\right)$

4. $r = \dfrac{9}{2 + \sin\theta}$

Sol. $r = \dfrac{9}{2 + \sin\theta}$ can be written as

$$\frac{9}{r} = 2 + \sin\theta = 2 - \cos\left(\frac{\pi}{2} + \theta\right) = 2\left\{1 - \frac{1}{2}\cos\left(\frac{\pi}{2} + \theta\right)\right\}$$

$$\frac{\frac{9}{2}}{r} = 1 - \frac{1}{2}\cos\left(\frac{\pi}{2} + \theta\right)$$

Comparing it with $\frac{l}{r} = 1 - e\cos\theta$, we have

$l = \frac{9}{2}$ = length of the semi-latusrectum, $e = \frac{1}{2}$ which is less than 1

Hence the conic is an ellipse.

We plot the following points (in polar form)

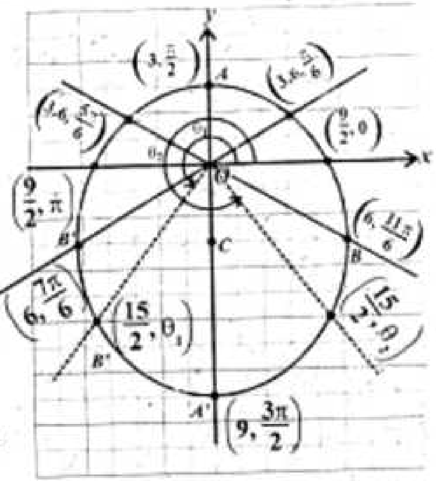
$\left(\frac{9}{2}, 0\right), \left(3.6, \frac{\pi}{6}\right), \left(3, \frac{\pi}{2}\right), \left(3.6, \frac{5\pi}{6}\right)$

$\left(\frac{9}{2}, \pi\right), \left(6, \frac{7\pi}{6}\right), \left(\frac{15}{2}, \theta_1\right), \left(9, \frac{3\pi}{2}\right)$,

$\left(\frac{15}{2}, \theta_2\right), \left(6, \frac{11\pi}{6}\right)$ where $\tan\theta_1 = \frac{4}{3}$ (θ_1 is in the 3rd quadrant) and

$\tan\theta_2 = -\frac{4}{3}$ (θ_2 is in the 4rth quadrant).

Joining the plotted points smoothly we get the graph of the given conic.



Equation of the conic in cartesian coordinate system is $\frac{x^2}{27} + \frac{(y+3)^2}{36} = 1$.

The cente of the conic is $(0, -3)$.

A, A', B and B' are $(0, 3)$, $(0, -9)$, $(3\sqrt{3}, -3)$ and $(-3\sqrt{3}, -3)$ in the cartesian system respectively.

Note: The conic is symmetric about the y-axis. The left half of the conic can be traced by taking some special values of θ from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$. Using symmetry, the other half can be traced.

5. $r = \dfrac{6}{1 - 2\sin\theta}$

Sol. $r = \dfrac{6}{1 - 2\sin\theta}$ can be written as $\frac{6}{r} = 1 - 2\sin\theta = 1 - 2\cos\left(\frac{\pi}{2} - \theta\right)$

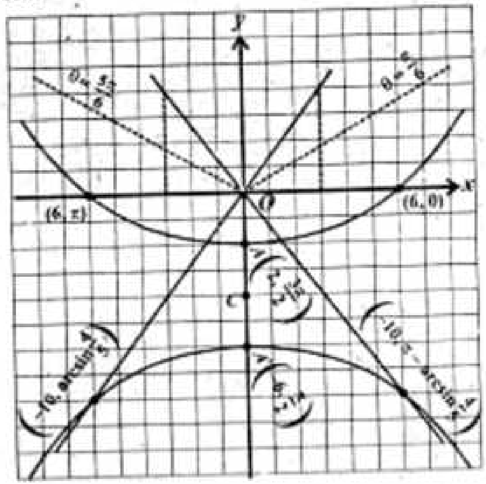
Comparing it with $\frac{l}{r} = 1 - e\cos\theta$, we have

$l = 6$, $e = 2$ which is greater than 1

Hence the conic is a hyperbola.

We plot the following points (in polar form)

$(6, 0)$, $\left(-10, \arcsin\frac{4}{5}\right)$, $\left(-6, \frac{\pi}{2}\right)$, $\left(-10, \pi - \arcsin\frac{4}{5}\right)$, $(6, \pi)$, $\left(2, \frac{3\pi}{2}\right)$



Joining the plotted points smoothly, we get the graph of the given conic. Equation of the conic in cartesian coordinate system is

$$\frac{(y+4)^2}{4} - \frac{x^2}{12} = 1$$

The centre of the conic is (0, −4). A and A' are (0, −2) and (0, −6) respectively.

6. $r = \dfrac{10}{2 + 3\cos\theta}$

Sol. $r = \dfrac{10}{2 + 3\cos\theta}$ can be written as

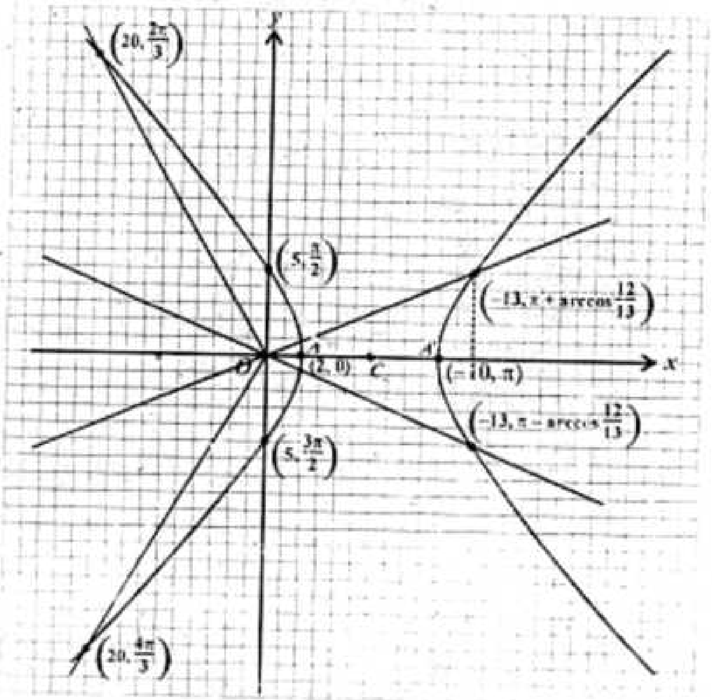
$$\frac{10}{r} = 2 + 3\cos\theta = 2\left(1 + \frac{3}{2}\cos\theta\right)$$

Thus $\dfrac{5}{r} = 1 + \dfrac{3}{2}\cos\theta$

Comparing it with $\dfrac{l}{r} = 1 + e\cos\theta$, we have

$$l = 5,\ e = \frac{3}{2} \text{ which is greater than 1}$$

Hence the conic is a hyperbola.



We plot the following points (in polar form),

$(2, 0), \left(5, \frac{\pi}{2}\right), \left(20, \frac{2\pi}{3}\right), \left(-13, \pi - \arccos\frac{12}{13}\right), (-10, \pi),$

$\left(-13, \pi + \arccos\frac{12}{13}\right), \left(20, \frac{4\pi}{3}\right), \left(5, \frac{3\pi}{2}\right)$

Joining the plotted points smoothly, we get the graph of the given conic.

Equation of the conic in the cartesian coordinate system is

$$\frac{(x-6)^2}{16} - \frac{y^2}{20} = 1$$

The centre of the conic is (6, 0).

A and A' are (2, 0) and (10, 0) in cartesian system respectively.

7. Show that in any conic the sum of the reciprocals of the segments of any focal chord is constant.

Sol. Let PQ be any focal chord of the conic $\dfrac{l}{r} = 1 - e\cos\theta$, making angle α with the polar line along the positive direction of the x-axis.

Then P is (FP, α) and Q is $(FQ, \pi + \alpha)$.

Now, $\dfrac{l}{FP} = 1 - e\cos\alpha \Rightarrow \dfrac{1}{FP} = \dfrac{1 - e\cos\alpha}{l}$

and $\dfrac{l}{FQ} = 1 - e\cos(\pi + \alpha) = 1 + e\cos\alpha \Rightarrow \dfrac{1}{FQ} = \dfrac{1 + e\cos\alpha}{l}$

Hence $\dfrac{1}{FP} + \dfrac{1}{FQ} = \dfrac{1 - e\cos\alpha}{l} + \dfrac{1 + e\cos\alpha}{l}$

$= \dfrac{2}{l}$, which is constant.

8. If PFP' QFQ' are two perpendicular focal chords of a conic, prove that $\dfrac{1}{|PF|\,.\,|FP'|} + \dfrac{1}{|QF|\,.\,|FQ'|}$, is constant.

Sol. Let P be (FP, α)

Then P' is $(FP', \pi + \alpha)$, Q is $\left(FQ, \frac{\pi}{2} + \alpha\right)$ and Q' is $\left(FQ', \frac{3\pi}{2} + \alpha\right)$

Now $\dfrac{1}{|PF|} = \dfrac{1 - e\cos\alpha}{l}$, $\dfrac{1}{|FP'|} = \dfrac{1 - e\cos(\pi + \alpha)}{l} = \dfrac{1 + e\cos\alpha}{l}$

Thus $\dfrac{1}{|PF|\,.\,|FP'|} = \dfrac{1 - e^2\cos^2\alpha}{l^2}$

$$\frac{1}{|QF|} = \frac{1 - e\cos\left(\frac{\pi}{2} + \alpha\right)}{l} = \frac{1 + e\sin\alpha}{l}$$

$$\frac{1}{|FQ'|} = \frac{1 - e\cos\left(\frac{3\pi}{2} + \alpha\right)}{l} = \frac{1 - e\sin\alpha}{l}$$

Thus $\dfrac{1}{|QF|\,.\,|FQ'|} = \dfrac{1 - e^2\sin^2\alpha}{l^2}$

$$\frac{1}{|PF|\,.\,|FP'|} + \frac{1}{|QF|\,.\,|FQ'|} = \frac{1 - e^2\cos^2\alpha}{l^2} + \frac{1 - e^2\sin^2\alpha}{l^2}$$

$$= \frac{2 - e^2}{l^2}\text{, which is constant}$$

9. If PF_2Q, PF_1R be two chords of an ellipse through the foci F_2, F_1, show that

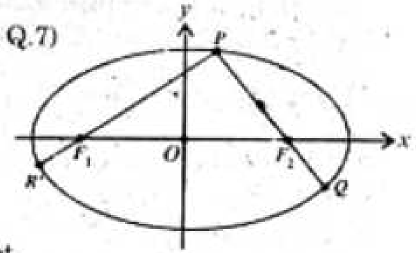
$$\frac{|PF_2|}{|F_2Q|} + \frac{|PF_1|}{|F_1R|}$$

is independent of the position of P.

Sol. As PF_2Q is a focal chord, so

$$\frac{1}{|PF_2|} + \frac{1}{|F_2Q|} = \frac{2}{l} \text{ (See Q.7)}$$

or $\quad 1 + \dfrac{|PF_2|}{|F_2Q|} = \dfrac{2|PF_2|}{l}$



Similarly $1 + \dfrac{|PF_1|}{|F_1R|} = \dfrac{2|PF_1|}{l}$

Adding the above results, we get

$$2 + \frac{|PF_2|}{|F_2Q|} + \frac{|PF_1|}{|F_1R|} = \frac{2|PF_2|}{l} + \frac{2|PF_1|}{l} = \frac{2(|PF_2| + |PF_1|)}{l}$$

$$= \frac{2(2a)}{l} = \frac{4a}{l}$$

or $\quad \dfrac{|PF_2|}{|F_2Q|} + \dfrac{|PF_1|}{|F_1R|} = \dfrac{4a}{l} - 2$ which is independent of the position of P.

Express each of the given equations in polar form and find the eccentricity and equation of the directrix. (Problems 10 – 12):

10. $y^2 = 4 - 4x$

Sol. The equation in polar coordinates is

$$r^2\sin^2\theta = 4 - 4r\cos\theta$$

or $\quad r^2\sin^2\theta + 4r\cos\theta - 4 = 0$

$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta + 16\sin^2\theta}}{2\sin^2\theta} = \frac{-4\cos\theta \pm 4}{2(1-\cos^2\theta)}$$

$$= \frac{4 - 4\cos\theta}{2(1-\cos^2\theta)}\text{, neglecting the -ve sign.}$$

$$= \frac{4(1-\cos\theta)}{2(1-\cos\theta)(1+\cos\theta)} = \frac{2}{1+\cos\theta}$$

This is a parabola with eccentricity $e = 1$

Equation of the directrix is $x = 2$ or $r\cos\theta = 2 \Rightarrow r = 2\sec\theta$

11. $3y^2 - 16y - x^2 + 16 = 0$

Sol. The given equation in polar form is

$$3r^2\sin^2\theta - 16r\sin\theta - r^2\cos^2\theta + 16 = 0$$

or $\quad r^2(3\sin^2\theta - \cos^2\theta) - 16r\sin\theta + 16 = 0$

$$r = \frac{16\sin\theta \pm \sqrt{256\sin^2\theta - 192\sin^2\theta + 64\cos^2\theta}}{2(3\sin^2\theta - \cos^2\theta)}$$

$$= \frac{16\sin\theta \pm 8}{2(4\sin^2\theta - 1)}$$

or $\quad r = \dfrac{8(2\sin\theta - 1)}{2(2\sin\theta - 1)(2\sin\theta + 1)}$ (neglecting the +ve sign)

$$= \frac{4}{2\sin\theta + 1} = \frac{4}{1 + 2\sin\theta}$$

This is a hyperbola with $e = 2$

Equation of directrix is $y = \dfrac{4}{2} = 2$

i.e., $\quad r\sin\theta = 2 \Rightarrow r = 2\csc\theta$

12. $8x^2 + 9y^2 + 4x - 4 = 0$

Sol. The equation in polar form is

$$8r^2\cos^2\theta + 9r^2\sin^2\theta + 4r\cos\theta - 4 = 0$$

or $\quad r^2(8\cos^2\theta + 9\sin^2\theta) + 4r\cos\theta - 4 = 0$

$$r = \frac{-4\cos\theta \pm \sqrt{16\cos^2\theta + 128\cos^2\theta + 144\sin^2\theta}}{2(8\cos^2\theta + 9\sin^2\theta)}$$

$$= \frac{-4\cos\theta \pm 12}{2(8\cos^2\theta + 9 - 9\cos^2\theta)} = \frac{-2\cos\theta \pm 6}{9 - \cos^2\theta}$$

or $\quad r = \dfrac{6 - 2\cos\theta}{(3-\cos\theta)(3+\cos\theta)}$ (neglecting –ve sign)

$$= \frac{2}{3+\cos\theta} = \frac{\frac{2}{3}}{1 + \frac{1}{3}\cos\theta}$$

This is an ellipse with $e = \frac{1}{3}$

Equation of directrix is $x = \frac{2/3}{1/3} = 2$

i.e., $r \cos \theta = 2 \Rightarrow r = 2 \sec \theta$.

Exercise Set 6.5 (Page 261)

Sketch the graph of each of the given curves:

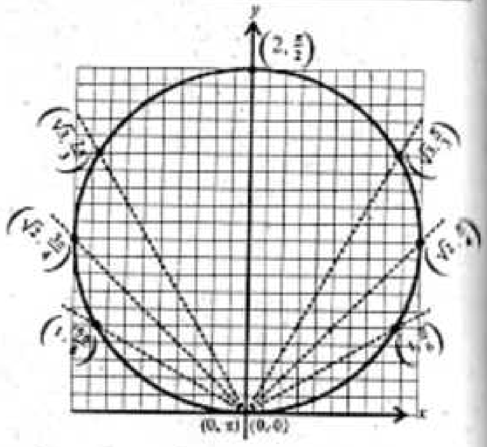
1. $r = 2 \sin \theta$

Sol. If (r, θ) is replaced by $(-r, -\theta)$ in $r = 2 \sin \theta$, the equation remains unchanged. Hence the curve is symmetric about the y-axis.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	1	$\sqrt{2}$	$\sqrt{3}$	2	$\sqrt{3}$	$\sqrt{2}$	1	0

The graph is a circle of radius 1 and passes through the pole having its centre (0, 1) on the y-axis. The graph of the curve is as shown.



2. $r = 3 \cos \theta$

Sol. Replacing (r, θ) by $(r, -\theta)$ in $r = 3 \cos \theta$, we have

$$r = 3 \cos(-\theta) = 3 \cos \theta$$

i.e., the equation remains unchanged, so the curve is symmetric about the initial line (the x-axis).

Table of values

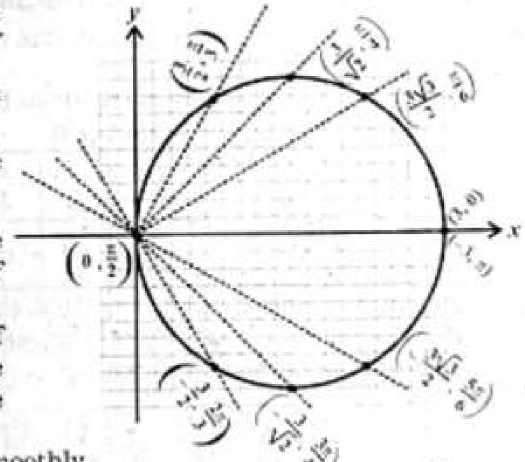
θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	3	$\frac{3\sqrt{3}}{2}$	$\frac{3}{\sqrt{2}}$	$\frac{3}{2}$	0	$-\frac{3}{2}$	$-\frac{3}{\sqrt{2}}$	$-\frac{3\sqrt{3}}{2}$	-3

The graph of the given equation is a circle of radius $\frac{3}{2}$ having its centre $\left(\frac{3}{2}, 0\right)$ on the initial line.

It passes through the pole O. The graph of the curve is as shown.

Note: The upper half of the curve can be traced by joining the plotted points shown above the initial line smoothly.

Using symmetry other half of the curve can be traced.



3. $r = a(1 - \sin \theta)$ $a > 0$ **(Cardioid)**

Sol. On changing (r, θ) to $(r, \pi - \theta)$, the equation $r = a(1 - \sin \theta)$, remains unchanged.

Hence the curve is symmetric about the y-axis.

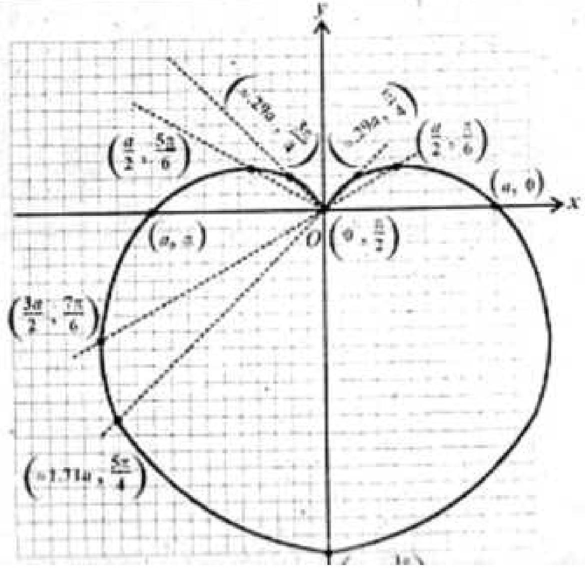
Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$
r	a	$\frac{a}{2}$	$0.29a$	0	$0.29a$	$\frac{a}{2}$	a	$\frac{3a}{2}$	$1.71a$	$2a$

Joining the plotted points from $\theta = \frac{\pi}{2}$ to $\theta = \frac{3\pi}{2}$ smoothly, we get the left half of the curve.

Using symmetry the remaining part is traced.

The graph of the curve is as shown.



4. $r = a(\pm 1 + \cos\theta)$ $a > 0$ **(cardioids)**

Sol. On changing (r, θ) by to $(r, -\theta)$, the equations $r = a(\pm 1 + \cos\theta)$, remain unchanged.

Hence the curve are symmetric about the initial line $\theta = 0$

Table of values for $r = a(1 + \cos\theta)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
r	$2a$	$1.87a$	$1.71a$	$\frac{a}{2}$	a	$\frac{a}{2}$	$0.29a$	0

Joining the plotted points smoothly, the upper half of the curve is the traced. Using symmetry, the remaining part is traced. The graph of the curve is as shown.

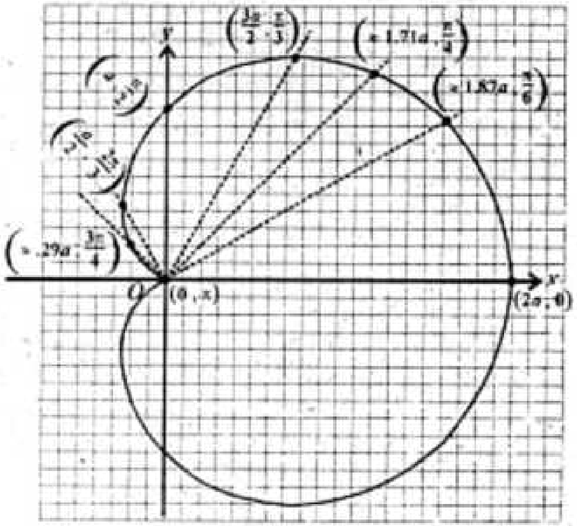
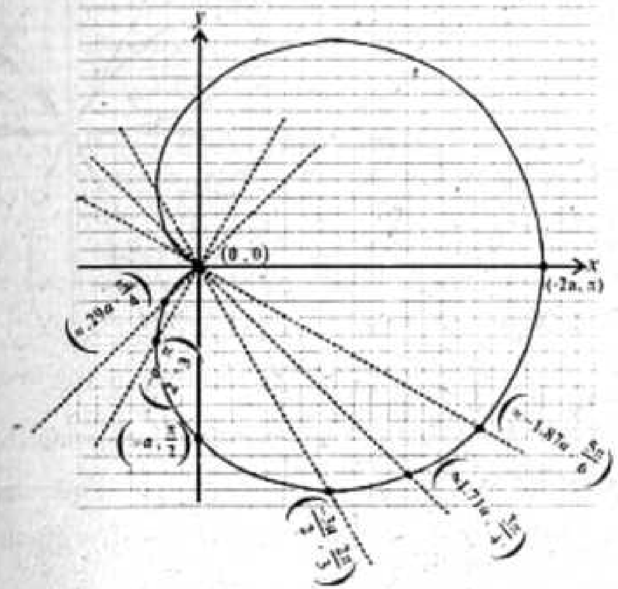


Table of values for $r = a(-1 + \cos\theta)$

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
r	0	$-0.29a$	$\frac{-a}{2}$	$-a$	$\frac{-3a}{2}$	$-1.71a$	$-1.87a$	$-2a$

The lower half of the curve is traced with help of the plotted points. Using symmetry, the remaining part is traced. The graph of the curve is as shown.



5. $r = a\sin 3\theta$, $a > 0$ **(three-leafed rose)**

Sol.

Table of values

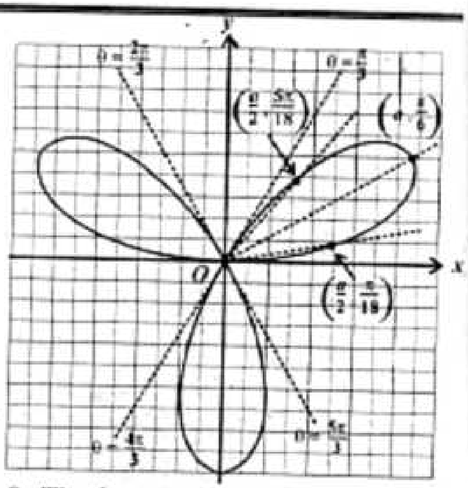
θ	0	$\frac{\pi}{18}$	$\frac{\pi}{6}$	$\frac{5\pi}{18}$	$\frac{\pi}{3}$	$\frac{7\pi}{18}$	$\frac{\pi}{2}$	$\frac{11\pi}{18}$	$\frac{2\pi}{3}$
r	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0

θ	$\frac{2\pi}{3}$	$\frac{13\pi}{18}$	$\frac{5\pi}{6}$	$\frac{17\pi}{18}$	π
r	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0

The curve is symmetric about the y-axis. As θ increases from 0 to $\frac{\pi}{6}$, r increases from 0 to a. As θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r decreases from a to 0.

The loop in the first quadrant is described.

As θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, r decreases from 0 to $-a$ to 0. When θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r increase from $-a$ to 0. The loop below the pole is described. As θ increases $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r increases from 0 to a. When θ increases from $\frac{5\pi}{6}$ to π, r decreases from a to 0. The loop in the 2nd quadrant is described.



The loop in the 1st quadrant is traced by joining the plotted the points $(0, 0)$, $\left(\frac{a}{2}, \frac{\pi}{18}\right)$, $\left(a, \frac{\pi}{6}\right)$, $\left(\frac{a}{2}, \frac{5\pi}{18}\right)$ etc. Following the above procedure, the remaining loops are traced. The graph of the curve is as shown. above.

6. $r = a\cos 3\theta$, $a > 0$ **(three-leafed rose)**

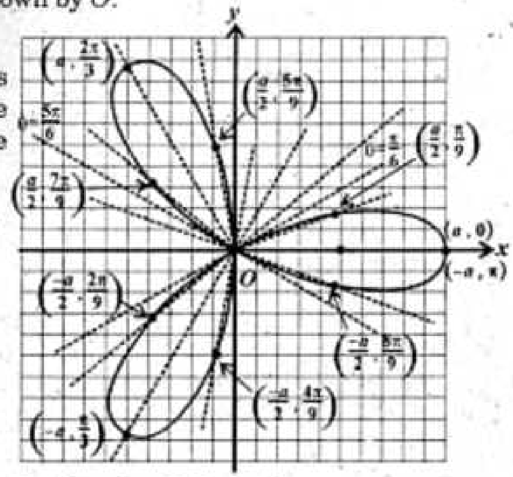
Sol. It is symmetric about the initial line. As θ increases from 0 to $\frac{\pi}{6}$, r decreases a to 0. The half loop in the 1st quadrant is traced. As θ increases from $\frac{\pi}{6}$ to $\frac{\pi}{3}$, r is negative and varies from 0 to $-a$. When θ increases from $\frac{\pi}{3}$ to $\frac{\pi}{2}$, r varies from $-a$ to 0. Thus the loop in the 3rd quadrant is described. As θ increases from $\frac{\pi}{2}$ to $\frac{2\pi}{3}$, r varies from 0 to a. When θ increases from $\frac{2\pi}{3}$ to $\frac{5\pi}{6}$, r varies from a to 0. Thus the loop in the 2nd quadrant is described. As θ increases from $\frac{5\pi}{6}$ to π, r varies from 0 to $-a$. Thus half loop in the 4th quadrant is described.

Table of values

θ	0	$\frac{\pi}{9}$	$\frac{\pi}{6}$	$\frac{2\pi}{9}$	$\frac{\pi}{3}$	$\frac{4\pi}{9}$	$\frac{\pi}{2}$	$\frac{5\pi}{9}$	$\frac{2\pi}{3}$	$\frac{7\pi}{9}$	$\frac{5\pi}{6}$	$\frac{8\pi}{9}$	π
r	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$

$\left(0, \frac{\pi}{6}\right)$, $\left(0, \frac{\pi}{2}\right)$, $\left(0, \frac{5\pi}{6}\right)$ are shown by O.

Joining the plotted points smoothly, three loops are traced. The graph of the curve is as shown.



7. $r = a\cos 2\theta$, $a > 0$ **(four-leafed rose)**

Sol. The curve is symmetric about the initial line.

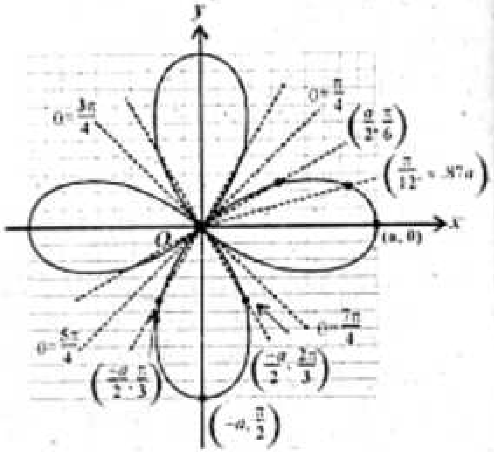
Table of values

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$
r	a	$\approx 87a$	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0

θ	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
r	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a

As θ increases from 0 to $\frac{\pi}{4}$, r decreases from a to 0. The half loop in the 1st quadrant is described.

As θ increases from $\frac{\pi}{4}$ to $\frac{\pi}{2}$, r decreases from 0 to $-a$. When θ increases from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, r increases from $-a$ to 0. Thus the loop below, the pole is described. Other details are left for the reader. $\left(0, \frac{\pi}{4}\right)$, $\left(0, \frac{3\pi}{4}\right)$, $\left(0, \frac{5\pi}{4}\right)$, $\left(0, \frac{7\pi}{4}\right)$ are shown by O.



8. $r = a\cos 5\theta$, $a > 0$ **(five-leafed rose)**

Sol.

Table of values

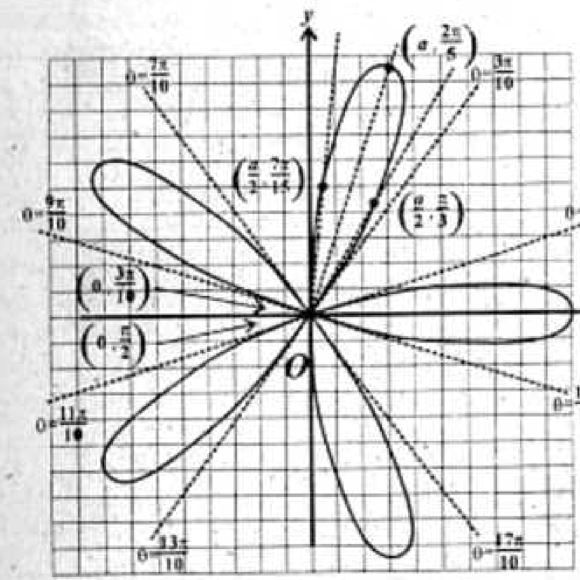
θ	0	$\frac{\pi}{15}$	$\frac{\pi}{10}$	$\frac{2\pi}{15}$	$\frac{\pi}{5}$	$\frac{4\pi}{15}$	$\frac{3\pi}{10}$	$\frac{\pi}{3}$	$\frac{2\pi}{5}$	$\frac{7\pi}{15}$	$\frac{\pi}{2}$
r	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0

half loop in the 1st quadrant; loop in the 3rd quadrant; loop in the 1st quadrant

θ	$\frac{\pi}{2}$	$\frac{8\pi}{15}$	$\frac{3}{5}$	$\frac{2\pi}{3}$	$\frac{7\pi}{10}$	$\frac{11\pi}{15}$	$\frac{4\pi}{5}$	$\frac{13\pi}{15}$	$\frac{9\pi}{10}$	$\frac{14\pi}{15}$	π
r	0	$\frac{-a}{2}$	$-a$	$\frac{-a}{2}$	0	$\frac{a}{2}$	a	$\frac{a}{2}$	0	$\frac{-a}{2}$	$-a$

loop in the 4th quadrant; loop in the 2nd quadrant; half loop in the 4th quadrant

The loop between the angle $\theta = \frac{3\pi}{10}$ and $\theta = \frac{\pi}{2}$ is plotted. Other loops are shown in the graph. Readers are advised to write the details.



9. $r^2 = a\sin 2\theta$ $a > 0$ **(lemniscate)**

Sol. Since $r^2 = (-r^2)$, the graph is symmetric about the pole. As $r^2 \geq 0$, the function is defined for those values of θ for which $\sin 2\theta \geq 0$. If $\theta \in [0, 2\pi]$ then $\sin 2\theta \geq 0$ if and only if θ is in $\left[0, \frac{\pi}{2}\right]$ or in $\left[\pi, \frac{3\pi}{2}\right]$.

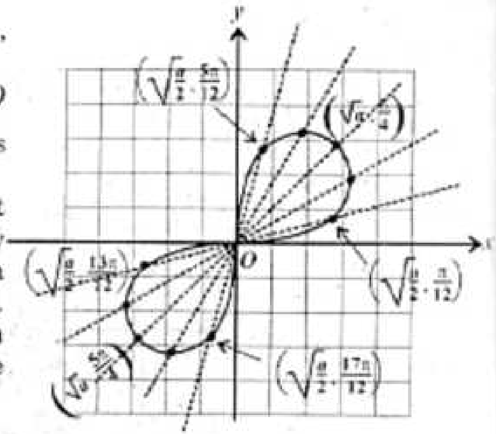
Table of values

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$	$\frac{\pi}{2}$
$r = \sqrt{a\sin 2\theta}$	0	$\sqrt{\frac{a}{2}}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{a}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{\frac{a}{2}}$	0

θ	π	$\frac{13\pi}{12}$	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{17\pi}{12}$	$\frac{3\pi}{2}$
$r = \sqrt{a\sin 2\theta}$	0	$\sqrt{\frac{a}{2}}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{a}$	$\sqrt{a \cdot \frac{\sqrt{3}}{2}}$	$\sqrt{\frac{a}{2}}$	0

The points $(0, 0)$, $\left(0, \frac{\pi}{2}\right)$, $(0, \pi)$, $\left(0, \frac{3\pi}{2}\right)$ are shown by O (the pole). The graph is as shown.

Note: The loop in the Ist quadrant can be drawn by joining the plotted points in the first quadrant smoothly. Using symmetry the loop in the 3rd quadrant can be traced.



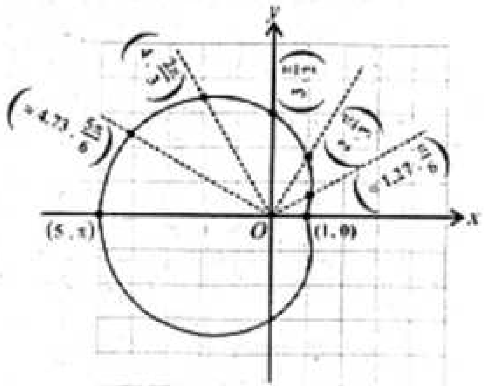
10. $r = 3 - 2\cos\theta$ **(limacon)**

Sol. On changing (r, θ) into $(r - \theta)$, the equation of the curve remains unchanged. Thus curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	1	≈ 1.27	2	3	4	≈ 4.73	5

The upper half of the curve is drawn by joining the plotted points smoothly. Using symmetry remaining part is traced. The graph of the curve is as shown.



11. $r = 3 + 4\cos\theta$ **(limacon)**

Sol. On changing (r, θ) into $(r, -\theta)$ the equation of the curve remains unchanged. Thus the curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{36}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
r	7	$3+2\sqrt{3}$	5	3	1	$3-2\sqrt{3}$	-1	$3-2\sqrt{3}$	1	3	5	7

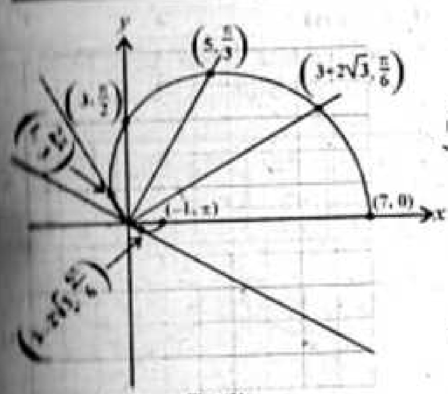


Fig. (1)

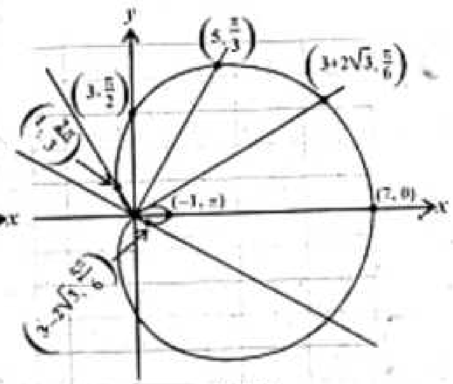


Fig. (2)

As θ varies from 0 to π, the half of the curve is described as shown in the figure (1). Using symmetry, the other half of the curve is traced. The graph of the curve is as shown in the figure (2).

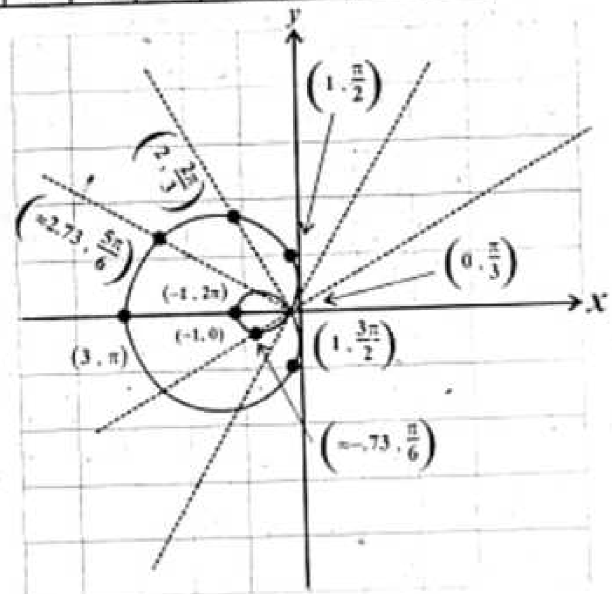
12. $r = 1 - 2\cos\theta$ **(limacon)**

Sol. On changing (r, θ) into $(r, -\theta)$, the equation of the curve remains unchanged. Therefore the curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
r	-1	-0.73	0	1	2	2.73	3	1	-1

The half part of the curve is drawn by joining the plotted points starting from $(-1, 0)$ to $(3, \pi)$ smoothly. Other half is traced by using symmetry. The graph of the curve is as shown.



13. $r = 3 + 2\sin\theta$ **(limacon)**

Sol. If (r, θ) are replaced by $(r, \pi, -\theta)$, the equation of the curve remains unchanged. Hence the curve is symmetric about the y-axis.

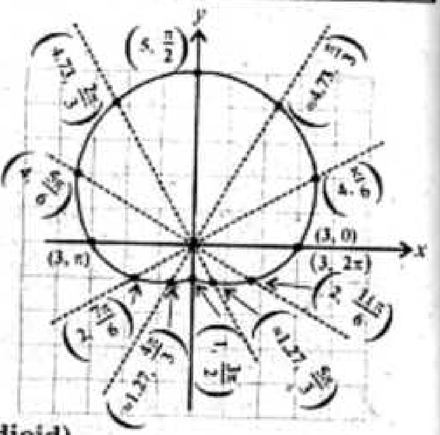
Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
r	3	4	4.73	5	4.73	4	3	2	1.27	1	1.27	2	3

The plotted points are joined smoothly drawn the required graph.

The graph of the curve is as shown.

Note: The half curve can be drawn by joining the plotted points starting from $\left(5, \frac{\pi}{2}\right)$ to $\left(1, \frac{3\pi}{2}\right)$ smoothly. Using symmetry, other half can be traced.



14. $r = -a(1 + \cos\theta)$ $a > 0$ **(cardioid)**

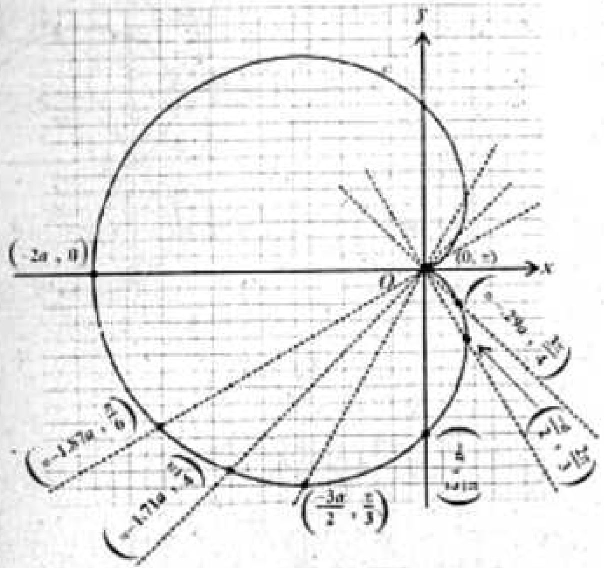
Sol. Since $\cos(-\theta) = \cos\theta$, the curve is symmetric about the initial line.

Table of values

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
r	$-2a$	$-1.87a$	$-1.71a$	$\frac{-3a}{2}$	$-a$	$\frac{-a}{2}$	$-0.29a$	0

The lower half of the graph is drawn by joining the plotted points of the table smoothly, Remaining part is traced by using symmetry.

The graph of the curve is as shown.



15. $r = a\sin\frac{\theta}{2}, a > 0$

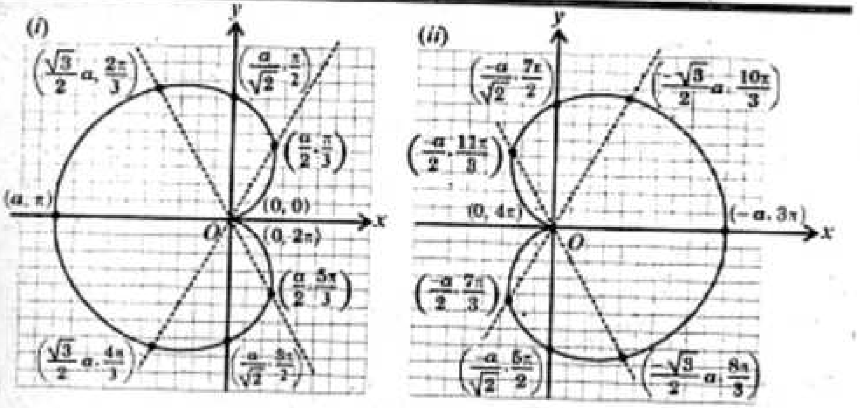
Sol. Since $a\sin\left(\pi - \frac{\theta}{2}\right) = a\sin\left(\frac{\theta}{2}\right) = r$, the curve is symmetric about the y-axis.

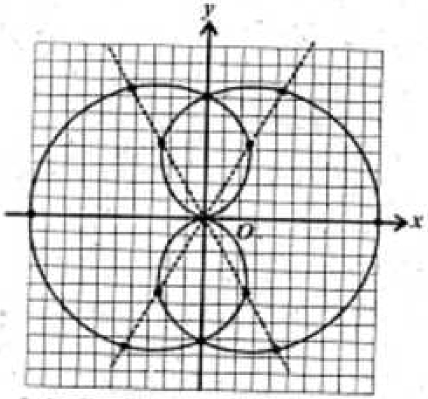
(i) Table of values

θ	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	2π
r	0	$\frac{a}{2}$	$\frac{a}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}a$	a	$\frac{\sqrt{3}}{2}a$	$\frac{a}{\sqrt{2}}$	$\frac{a}{2}$	0

(ii)

θ	$\frac{7\pi}{3}$	$\frac{5\pi}{2}$	$\frac{8\pi}{3}$	3π	$\frac{10\pi}{3}$	$\frac{7\pi}{2}$	$\frac{11\pi}{3}$	4π
r	$\frac{-a}{2}$	$\frac{-a}{\sqrt{2}}$	$\frac{-\sqrt{3}}{2}a$	$-a$	$\frac{-\sqrt{3}}{2}a$	$\frac{-a}{\sqrt{2}}$	$\frac{-a}{2}$	-0





The first graph is drawn by joining the plotted points of the table (*i*) smoothly. The second graph is got by joining the plotted points of the table (*ii*), smoothly. The combined graph is shown in the last figure.

Note: After drawing the graph according to the table (*i*), other part of the graph can be traced by using symmetry.

Exercise Set 6.6 (Page 266)

Find ψ for each of the given curves (Problems 1 – 4):

1. $r = a(1 - \cos\theta)$ (1)

Sol. Taking ln of both sides of (1), we get

$$\ln r = \ln a + \ln(1 - \cos\theta)$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{1-\cos\theta}(\sin\theta) = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

Therefore, $r\dfrac{d\theta}{dr} = \tan\dfrac{\theta}{2}$ and so $\tan\psi = \tan\dfrac{\theta}{2}$

Hence $\psi = \dfrac{\theta}{2}$

2. $r = -5\csc\theta$

Sol. $\dfrac{dr}{d\theta} = -5(-\csc\theta\cot\theta) = 5\csc\theta\cot\theta$

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{-5\csc\theta}{5\csc\theta\cot\theta} = -\tan\theta = \tan(\pi - \theta)$$

Hence $\psi = \pi - \theta$

3. $\dfrac{2a}{r} = 1 + \sin\theta$

Sol. $\dfrac{2a}{r} = 1 + \sin\theta \Rightarrow r = \dfrac{2a}{1+\sin\theta}$, so

$$\frac{dr}{d\theta} = \frac{-2a\cos\theta}{(1+\sin\theta)^2}$$

$$\tan\psi = \frac{r}{dr/d\theta} = \frac{2a}{1+\sin\theta}\left(-\frac{(1+\sin\theta)^2}{2a\cos\theta}\right) = -\frac{1+\sin\theta}{\cos\theta}$$

$$= -\frac{\cos^2\left(\frac{\theta}{2}\right) + \sin^2\left(\frac{\theta}{2}\right) + 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right)}$$

$$= -\frac{\cos\left(\frac{\theta}{2}\right) + \sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right) - \sin\left(\frac{\theta}{2}\right)} = -\frac{1 + \tan\left(\frac{\theta}{2}\right)}{1 - \tan\left(\frac{\theta}{2}\right)}$$

$$= -\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) = \tan\left(\pi - \left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$$

Hence $\psi = \dfrac{3\pi}{4} - \dfrac{\theta}{2}$

4. $r = \dfrac{3}{2(1 - \cos\theta)}$

Sol. $r = \dfrac{3}{2(1-\cos\theta)} = \dfrac{3}{2}(1 - \cos\theta)^{-1}$.

$$\frac{dr}{d\theta} = \frac{3}{2}(-1)\,.\,(1-\cos\theta)^{-2} \times -(-\sin\theta) = \frac{3}{2}\left[\frac{-\sin\theta}{(1-\cos\theta)^2}\right]$$

$$\tan\psi = r\,.\frac{d\theta}{dr} = \frac{3}{2}\cdot\frac{1}{1-\cos\theta}\times\frac{2}{3}\left[-\frac{(1-\cos\theta)^2}{\sin\theta}\right]$$

$$= -\frac{1-\cos\theta}{\sin\theta} = -\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\tan\frac{\theta}{2}$$

i.e., $\tan\psi = \tan\left(\pi - \frac{\theta}{2}\right) \Rightarrow \psi = \pi - \frac{\theta}{2}$

Find the measure of the angle of intersection of the given curves (Problems 5 – 10):

5. $r = 1$ and $r = 2\sin\theta$

Sol. $r = 1$ (1)

and $r = 2\sin\theta$ (2)

Solving (1) and 2), we get

$$1 = 2\sin\theta \text{ or } \sin\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Thus the points of intersection of (1) and (2) are $\left(1, \frac{\pi}{6}\right)$ and $\left(1, \frac{5\pi}{6}\right)$

From (1), $\frac{dr}{d\theta} = 0$ and $\tan\psi_1 = \frac{r}{dr/d\theta} = \frac{1}{0} \Rightarrow \psi_1 = \frac{\pi}{2}$

From (2), $\frac{dr}{d\theta} = 2\cos\theta$ and $\tan\psi_2 = \frac{r}{dr/d\theta} = \frac{2\sin\theta}{2\cos\theta} = \tan\theta$

$\Rightarrow \quad \psi_2 = \theta$

Now at $\left(1, \frac{\pi}{6}\right)$, $\psi_1 = \frac{\pi}{2}$ and $\psi_2 = \frac{\pi}{6}$

Therefore $\psi_1 - \psi_2 = \frac{\pi}{2} - \frac{\pi}{6} = \frac{3\pi - \pi}{6} = \frac{\pi}{3}$

At $\left(1, \frac{5\pi}{6}\right)$, $\psi_1 = \frac{\pi}{2}$ and $\psi_2 = \frac{5\pi}{6}$, therefore

$$\psi_2 - \psi_1 = \frac{5\pi}{6} - \frac{\pi}{2} = \frac{5\pi - 3\pi}{6} = \frac{\pi}{3}$$

6. $r = a\theta$ and $r\theta = a$

Sol. $r = a\theta$ (1)

and $r\theta = a$ (2)

Solving (1) and (2), we get

$$a\theta\,.\,\theta = a \quad \text{or} \quad \theta^2 = 1 \quad \text{or} \quad \theta = \pm 1$$

From (1), we have

$$\ln r = \ln a + \ln\theta$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} \Rightarrow r\,.\frac{d\theta}{dr} = \theta$$

i.e., $\tan\psi_1 = \theta$

From (2), we get

$$\ln r + \ln\theta = \ln a$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} + \frac{1}{\theta} = 0 \text{ or } \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta}$$

i.e, $r\,.\frac{d\theta}{dr} = -\theta \Rightarrow \tan\psi_2 = -\theta$

At $\theta = 1$, $\tan\psi_1 = 1$ and $\tan\psi_2 = -1$

Thus $\psi_1 = \frac{\pi}{4}$ and $\psi_2 = \frac{3\pi}{4}$

The required angle is $\psi_2 - \psi_1 = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$

At $\theta = -1$, $\tan\psi_1 = -1$ and $\tan\psi_2 = 1$

Thus $\psi_1 = \frac{3\pi}{4}$, $\psi_2 = \frac{\pi}{4}$

The required angle is $\psi_1 - \psi_2 = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{\pi}{2}$

7. $r = \frac{a\theta}{1+\theta}$ and $r = \frac{a}{1+\theta^2}$

Sol. $r = \frac{a\theta}{1+\theta}$ (1)

and $r = \frac{a}{1+\theta^2}$ (2)

Solving (1) and (2), we have

$$\frac{a\theta}{1+\theta} = \frac{a}{1+\theta^2} \text{ or } \frac{\theta}{1+\theta} = \frac{1}{1+\theta^2} \Rightarrow \theta + \theta^3 = 1 + \theta$$

or $\theta^3 = 1 \quad \Rightarrow \quad \theta = 1$ (3)

Taking ln of (1), we get

$$\ln r = \ln a + \ln\theta - \ln(1+\theta)$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\theta} - \frac{1}{1+\theta} = \frac{1+\theta-\theta}{\theta(1+\theta)} = \frac{1}{\theta(1+\theta)}$$

or $r\frac{d\theta}{dr} = \theta(1+\theta) \Rightarrow \tan\psi_1 = \theta(1+\theta)$

At $\theta = 1$, $\tan\psi_1 = 1(1+1) = 2$ (4)

Taking logarithm of both sides of (2), we have

$$\ln r = \ln a - \ln(1+\theta^2)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\theta}{1+\theta^2} \text{ or } r\frac{d\theta}{dr} = -\frac{1+\theta^2}{2\theta} \Rightarrow \tan\psi_2 = -\frac{1+\theta^2}{2\theta}$$

At $\quad \theta = 1, \tan\psi_2 - \frac{1+(1)^2}{2(1)} = -\frac{2}{2} = -\frac{2}{2} = -1$

If β is the angle between two curves, then

$$\tan\beta = \frac{\tan\psi_1 - \tan\psi_2}{1+\tan\psi_1\tan\psi_2} = \frac{2+1}{1-2} = \frac{3}{-1} = -3$$

i.e, $\quad \beta = \arctan(-3)$

8. $\quad r = ae^{\theta}$ and $re^{\theta} = b$

Sol.

$$r = ae^{\theta} \qquad (1)$$

Taking logarithms

$$\ln r = \ln a + \ln e^{\theta}$$
$$= \ln a + \theta$$

Differentiating w.r.t., 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} = 1 \Rightarrow \frac{r\,d\theta}{dr} = 1$$

i.e., $\quad \tan\psi_1 = 1$

or $\quad \psi_1 = \frac{\pi}{4}$

$$re^{\theta} = b \qquad (2)$$

Taking logarithms

$$\ln r + \ln e^{\theta} = \ln b$$
$$\ln r + \theta = \ln b$$

Differentiating w.r.t. 'θ'

$$\frac{1}{r}\frac{dr}{d\theta} + 1 = 0 \Rightarrow r\frac{d\theta}{dr} = -1$$

i.e., $\quad \tan\psi_2 = -1$

or $\quad \psi_2 = \frac{3\pi}{4}$

At the point of intersection $\left(\sqrt{ab}, \ln\sqrt{\frac{b}{a}}\right)$ of (1) and (2), the required angle $= \psi_2 - \psi_1 = \frac{3\pi}{4} - \left(\frac{\pi}{4}\right) = \frac{\pi}{2}$

9. $\quad r = a(1-\cos\theta)$ and $r = \frac{a}{1-\cos\theta}$

Sol. $\quad r = a(1-\cos\theta) \qquad (1)$

and $\quad r = \frac{a}{1-\cos\theta} \qquad (2)$

Solving (1) and (2), we have

$$a(1-\cos\theta) = \frac{a}{1-\cos\theta} \Rightarrow (1-\cos\theta)^2 = 1$$

or $\quad 1-\cos\theta = \pm 1$

or $\quad 1-\cos\theta = 1 \Rightarrow \cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}, \frac{3\pi}{2}$

or $\quad 1-\cos\theta = -1 \Rightarrow \cos\theta = 2$ which is not possible.

The points of intersection of (1) and (2) are $\left(a, \frac{\pi}{2}\right)$ and $\left(a, \frac{3\pi}{2}\right)$

From (1) $\ln r = \ln a + \ln(1-\cos\theta)$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}(-(-\sin\theta) = \frac{\sin\theta}{1-\cos\theta}$$

or $\quad r\,.\frac{d\theta}{dr} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} \Rightarrow \tan\psi_1 = \tan\frac{\theta}{2} \Rightarrow \psi_1 = \frac{\theta}{2}$

From (2), $\ln r = \ln a - \ln(1-\cos\theta)$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = 0 - \frac{1}{1-\cos\theta}.(-(-\sin\theta) = -\frac{\sin\theta}{1-\cos\theta}$$

or $\quad r\frac{d\theta}{dr} = -\frac{1-\cos\theta}{\sin\theta} = -\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$$\tan\psi_2 = -\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \tan\frac{\theta}{2} = \tan\left(\pi - \frac{\theta}{2}\right)$$

Thus $\tan\psi_2 = \tan\left(\pi - \frac{\theta}{2}\right) \Rightarrow \psi_2 = \left(\pi - \frac{\theta}{2}\right)$

At $\quad \left(a, \frac{\pi}{2}\right), \psi_1 = \frac{\pi/2}{2} = \frac{\pi}{4}$ and $\psi_2 = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$

Thus $\psi_2 - x_1 = \frac{3\pi}{4} - \frac{\pi}{4} = \frac{2\pi}{4} = \frac{\pi}{2}$

At $\quad \left(a, \frac{3\pi}{2}\right), \psi_1 = \frac{\frac{3\pi}{2}}{2} = \frac{3\pi}{4}$ and $\psi_2 = \pi - \frac{3\pi}{4} = \frac{\pi}{4}$

Thus $\psi_1 - \psi_2 = \frac{3\pi}{4} - \left(\frac{\pi}{4}\right) = \frac{2\pi}{4} = \frac{\pi}{2}$

10. $\quad r = \cos 2\theta$ and $r = \sin\theta$ at $\left(\frac{1}{2}, \frac{\pi}{6}\right)$

Sol. $\quad r = \cos 2\theta \qquad (1)$

and $\quad r = \sin\theta \qquad (2)$

Differentiating (1) w.r.t. θ, we have

$$\frac{dr}{d\theta} = -(\sin 2\theta).2 = -2\sin 2\theta$$

$$\tan\psi_1 = r\,.\frac{d\theta}{dr} = \cos^2\theta \times \left(-\frac{1}{2\sin 2\theta}\right) = -\frac{1}{2}\cot 2\theta$$

At $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\tan \psi_1 = -\frac{1}{2} \cdot \cot \frac{\pi}{3} = -\frac{1}{2} \cdot \frac{1}{\sqrt{3}} = -\frac{1}{2\sqrt{3}}$

Differentiating (2) w.r.t. θ, we get

$$\frac{dr}{d\theta} = \cos \theta$$

$$\tan \psi_2 = r\frac{d\theta}{dr} = \frac{\sin \theta}{\cos \theta} = \tan \theta \Rightarrow \psi_2 = \theta$$

At $\left(\frac{1}{2}, \frac{\pi}{6}\right)$, $\psi_2 = \frac{\pi}{6}$

If β is the angle between the two curves then

$$\tan \beta = \frac{\tan \psi_2 - \tan \psi_1}{1 + \tan \psi_2 \tan \psi_1} = \frac{\frac{1}{\sqrt{3}} + \frac{1}{2\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}\left(-\frac{1}{2\sqrt{3}}\right)} = \frac{\frac{\sqrt{3}}{2}}{1 - \frac{1}{6}} = \frac{\frac{\sqrt{3}}{2}}{\frac{5}{6}}$$

$$= \frac{\sqrt{3}}{2} \times \frac{6}{5} = \frac{3\sqrt{3}}{5} \Rightarrow \beta = \arctan\left(\frac{3\sqrt{3}}{5}\right)$$

Find the pedal equation of each of the given curves (Problems 111 – 15):

11. $\frac{l}{r} = 1 + e \cos \theta$

Sol. Taking logarithm, we have

$$\ln l - \ln r = \ln (1 + e \cos \theta)$$

Differentiating w.r.t. θ, we have

$$-\frac{1}{r}\frac{dr}{d\theta} = \frac{-e \sin \theta}{1 + e \cos \theta} \Rightarrow \frac{dr}{d\theta} = \frac{r (e \sin \theta)}{1 + e \cos \theta} = \frac{r^2 e \sin \theta}{l}$$

Now, $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left[\frac{dr}{d\theta}\right]^2 = \frac{1}{r^2} + \frac{1}{r^4}\frac{r^4 e^2 \sin^2\theta}{l^2}$

$$= \frac{1}{r^2} + \frac{e^2 \sin^2\theta}{l^2} = \left[\frac{1 + e \cos \theta}{l}\right]^2 + \frac{e^2 \sin^2\theta}{l^2}$$

$$= \frac{1 + 2e \cos \theta + e^2 \cos^2\theta + e^2 \sin^2\theta}{l^2} = \frac{1 + 2e \cos \theta + e^2}{l^2}$$

$$= \frac{1}{l^2}\left[1 + e^2 + 2\left(\frac{l}{r} - 1\right)\right], \text{ since } e \cos \theta = \frac{l}{r} - 1.$$

$$= \frac{1}{l^2}\left[e^2 - 1 + \frac{2l}{r}\right] = \frac{e^2 - 1}{l^2} + \frac{2}{l\,r}$$

which is the required pedal equation.

12. $r = a\theta$

Sol. $\frac{dr}{d\theta} = a$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}a^2$$

$= \frac{1}{r^2} + \frac{a^2}{r^4}$ which is the required pedal equation.

13. $r = a \sin m\theta$

Sol. $\ln r = \ln a + \ln \sin m\theta$ (1)

Differentiating (1) w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{m \cos m\theta}{\sin m\theta} \text{ or } \frac{dr}{d\theta} = \frac{m\, r \cos m\theta}{\sin m\theta}$$

Now $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$

$$= \frac{1}{r^2} + \frac{1}{r^4}\frac{m^2 r^2 \cos^2 m\,\theta}{\sin^2 m\,\theta} = \frac{1}{r^2} + \frac{m^2}{r^2}\left[\frac{1 - \frac{r^2}{a^2}}{\frac{r^2}{a^2}}\right]$$

$$= \frac{1}{r^2} + \frac{m^2}{r^2}\left\{\frac{a^2 - r^2}{r^2}\right\} = \frac{1}{r^2} + \frac{m^2}{r^4}(a^2 - r^2)$$

or $\frac{1}{p^2} = \frac{1}{r^4}[r^2 + m^2 (a^2 - r^2)]$

i.e., $r^4 = p^2[a^2 m^2 + r^2(1 - m^2)]$

which is the required pedal equation.

14. $r = a + b \cos \theta$

Sol. Taking logarithm on both sides, we get

$$\ln r = \ln (a + b \cos \theta)$$

Differentiating w.r.t. θ, we have

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-b \sin \theta}{a + b \cos \theta}$$

$$\frac{dr}{d\theta} = -\frac{b\, r \sin \theta}{a + b \cos \theta} = -\frac{b\, r \sin \theta}{r} = -b \sin \theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{1}{r^4}b^2 \sin^2\theta$$

$$= \frac{1}{r^4}[r^2 + b^2 (1 - \cos^2 \theta)] = \frac{1}{r^4}[r^2 + b^2 - b^2 \cos^2\theta]$$

$$= \frac{1}{r^4}[r^2 + b^2 - (r - a)^2] = \frac{1}{r^4}[r^2 + b^2 - r^2 - a^2 + 2ar]$$

$$= \frac{1}{r^4}[b^2 - a^2 + 2ar].$$

Thus $r^4 = (b^2 - a^2 + 2ar)\, p^2$ is the required pedal equation.

15. $r = a(1 - \sin\theta)$

Sol. $r = a(1 - \sin\theta) \Rightarrow r - a = -a\sin\theta$ (1)

$$\frac{dr}{d\theta} = -a\cos\theta$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$$= \frac{1}{r^2} + \frac{a^2\cos^2\theta}{r^4} = \frac{1}{r^4}[r^2 + a^2(1 - \sin^2\theta)]$$

i.e, $\frac{1}{p^2} = \frac{1}{r^4}[r^2 + a^2 - a^2\sin^2\theta]$ (2)

For the pedal equation, we have to eliminate θ between (1) and (2). From (1),

$$a^2\sin^2\theta = (r - a)^2$$

Thus $\frac{1}{p^2} = \frac{1}{r^4}[r^2 + a^2 - (r - a)^2] = \frac{1}{r^4}(2ar) = \frac{2a}{r^3}$

i.e., $r^3 = 2p^2 a$ is the required pedal equation.

16. Show that the curves $r^m = a^m\cos m\theta$ and $r^m = a^m\sin m\theta$ cut each other orthogonally.

Sol. $r^m = a^m\cos m\theta$ and $r^m = a^m\sin m\theta$

Taking ln, we have

$m\ln r = m\ln a + \ln\cos m\theta$

Differentiating w.r.t. θ, we get

$$m\cdot\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\cos m\theta}(-m\sin m\theta),$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{\sin m\theta}{\cos m\theta} = -\tan m\theta$$

or $r\cdot\frac{d\theta}{dr} = -\cot m\theta$

i.e., $\tan\psi_1 = \tan\left(\frac{\pi}{2} + m\theta\right)$

$\Rightarrow$ $\psi_1 = \frac{\pi}{2} + m\theta$

Taking ln, we get

$m\ln r = m\ln a + \ln\sin m\theta$

Differentiating w.r.t. θ, we have

$$m\cdot\frac{1}{r}\frac{dr}{d\theta} = \frac{1}{\sin m\theta}(m\cos m\theta)$$

or $\frac{1}{r}\frac{dr}{d\theta} = \cot m\theta$

or $r\cdot\frac{d\theta}{dr} = \tan m\theta$

i.e, $\tan\psi_2 = \tan m\theta$

$\Rightarrow$ $\psi_2 = m\theta$

Thus the angle between the two curves is $\psi_1 - \psi_2 = \frac{\pi}{2}$

Hence the two curves cut each other orthogonally.

17. Show that the tangents to the cardioid $r = a(1 + \cos\theta)$ at the points $\theta = \frac{\pi}{3}$ and $\theta = \frac{2\pi}{3}$ are respectively parallel and perpendicular to the initial line.

Sol. $r = a(1 + \cos\theta)$ (1)

Taking logarithm, we have

$$\ln r = \ln a + \ln(1 + \cos\theta)$$

Differentiating w.r.t. θ, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1 + \cos\theta}$$

or $r\frac{d\theta}{dr} = -\frac{1 + \cos\theta}{\sin\theta} = -\frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = -\cot\frac{\theta}{2}$

i.e., $\tan\psi = -\cot\frac{\theta}{2} = \tan\left[\frac{\pi}{2} + \frac{\theta}{2}\right] \Rightarrow \psi = \frac{\pi}{2} + \frac{\theta}{2}$

(i) At $\theta = \frac{\pi}{3}$, $\psi = \frac{\pi}{2} + \frac{\pi}{6} = \frac{3\pi + \pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

Now $\alpha = \psi + \theta = \frac{2\pi}{3} + \frac{\pi}{3} = \pi$

Thus the tangent to (1) at $\theta = \frac{\pi}{3}$ is parallel to the initial line.

(ii) At $\theta = \frac{2\pi}{3}$, $\psi = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$

$$\alpha = \psi + \theta = \frac{5\pi}{6} + \frac{2\pi}{3} = \frac{9\pi}{6} = \frac{3\pi}{2}$$

Therefore, the tangent to (1) at $\theta = \frac{2\pi}{3}$ is perpendicular to the initial line.

18. Show that $\tan\psi = \dfrac{x\frac{dy}{dx} - y}{y\frac{dy}{dx} + x}$

Sol. Let the initial line Ox be taken as the x-axis. Let the polar coordinates of a point P on the curve $r = f(\theta)$ be (r, θ) and suppose that the cartesian coordinates of P are (x, y). Then

$$x = r\cos\theta, y = r\sin\theta$$

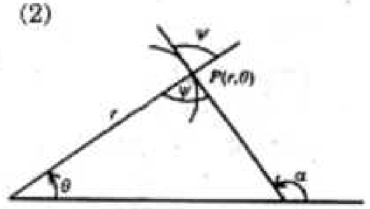
and $\tan\theta = \frac{y}{x}$ (1)

If α is the angle which the tangent at P makes with Ox, then

$$\frac{dy}{dx} = \tan\alpha \qquad (2)$$

From the figure, we have $\psi = \alpha - \theta$

Hence $\tan\psi = \tan(\alpha - \theta)$



$$= \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$$

$$= \frac{\frac{dy}{dx} - \frac{y}{x}}{1 + \frac{dy}{dx}\cdot\frac{y}{x}} = \frac{x\frac{dy}{dx} - y}{x + y\frac{dy}{dx}}, \text{ from (1) and (2)}$$

19. Show that at any point of the lemeniscate $r^2 = a^2\cos 2\theta$, $0 \le \theta \le \frac{\pi}{4}$, the measure of the angle between the radius vector and the outward-pointed normal is 2θ.

Sol. $r^2 = a^2\cos 2\theta$

Taking logarithms, we have

$$2\ln r = 2\ln a + \ln\cos 2\theta$$

Differentiating w.r.t. θ, we get

$$\frac{2}{r}\frac{dr}{d\theta} = \frac{-2\sin 2\theta}{\cos 2\theta} \Rightarrow r\frac{d\theta}{dr} = -\cot 2\theta$$

or $\quad \tan\psi = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$

Thus $\psi = \frac{\pi}{2} + 2\theta$

The tangent makes an angle $\frac{\pi}{2} + 2\theta$ with the radius vector. As the normal is perpendicular to the tangent, it will make an angle 2θ with the radius vector.

20. Find an equation (in rectangular coordinates) of the line tangent to:

(*i*) $r = \sin 2\theta$ at $\left(1, \frac{\pi}{4}\right)$ (*ii*) $r = 1 + \cos\theta$ at $\left(1, \frac{\pi}{2}\right)$

Sol.

(*i*) $r = \sin 2\theta$

$$\frac{dr}{d\theta} = (\cos 2\theta) \,.\, 2 = 2\cos 2\theta \quad \text{and}$$

at $\left(1, \frac{\pi}{4}\right), \frac{dr}{d\theta} = 2\cos\frac{\pi}{2} = 2(0) = 0$

Putting $\frac{dr}{d\theta} = 0, r = 1, \theta = \frac{\pi}{4}$ in $\frac{dy}{dx} = \dfrac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$, we have

$$\frac{dy}{dx} = \frac{0\,.\sin\frac{\pi}{4} + 1\,.\cos\frac{\pi}{4}}{0\,.\cos\frac{\pi}{4} - 1\,.\sin\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}} = -1$$

Now at $\left(1, \frac{\pi}{4}\right)$, $x = 1.\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $y = 1\,.\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

Thus the equation of the line tangent to (*i*) at $\left(1, \frac{\pi}{4}\right)$ is

$$y - \frac{1}{\sqrt{2}} = -1\left(x - \frac{1}{\sqrt{2}}\right) \text{ in rectangular coordinates}$$

$$\sqrt{2}y - 1 = -(\sqrt{2}x - 1) \Rightarrow \sqrt{2}\,x + \sqrt{2}y - 2 = 0$$

or $\quad x + y - \sqrt{2} = 0$

(*ii*) $r = 1 + \cos\theta$

$\frac{dr}{d\theta} = -\sin\theta$ and at $\left(1, \frac{\pi}{2}\right), \frac{dr}{d\theta} = -\sin\frac{\pi}{2} = -1$

Putting $\frac{dr}{d\theta} = -1, r = 1, \theta = \frac{\pi}{2}$ in

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}, \text{ we have}$$

$$\frac{dy}{dx} = \frac{(-1)\sin\frac{\pi}{2} + 1\,.\cos\frac{\pi}{2}}{(-1)\cos\frac{\pi}{2} - 1\,.\sin\frac{\pi}{2}} = \frac{-1}{-1} = 1$$

Now at $\left(1, \frac{\pi}{2}\right)$, $x = 1\,.\cos\frac{\pi}{2} = 1\,.\,0 = 0$ and $y = 1\,.\sin\,.\frac{\pi}{2} = 1$

Thus an equation of the line tangent to (*ii*) at $\left(1, \frac{\pi}{2}\right)$ in rectangular coordinates is

$$y - 1 = 1\,(x - 0) \Rightarrow x - y + 1 = 0.$$

Exercise Set 6.7 (Page 275)

Find parametric equations of the given curves (Problems 1 – 3):

1. $r = a \sin 2\theta, 0 \le \theta \le 2\pi$

Sol. $r = a \sin 2\theta$ (1)

We know that $x = r\cos\theta, y = r\sin\theta$

Writing the value of r from (1) in the above equations, we have

$$x = a\sin 2\theta \cos\theta, y = a\sin 2\theta \sin\theta$$

as the parametric equations of (1), θ being the parameter.

2. $r = \theta$

Sol. Putting $r = \theta$ in $x = r\cos$ and $y = r\sin\theta$, we have

$$x = \theta\cos\theta, y = \theta\sin\theta$$

which are the desired parametric equations.

3. $r = 2 + 3\sin\theta$.

Sol. Parametric equations of the given limacon are

$$x = r\cos\theta = (2 + 3\sin\theta)\cos\theta,$$
$$y = r\sin\theta = (2 + 3\sin\theta)\sin\theta$$

4. Show that the equations $x = a + r\cos\theta, y = b + r\sin\theta$ are parametric equations for a circle with centre (a, b) and radius $|r|$.

Sol. From the given equations, we have

$$x - a = r\cos\theta$$
$$y - b = r\sin\theta$$

Squaring the above equations and adding the results, we get

$$(x-a)^2 + (y-b)^2 = r^2(\cos^2\theta + \sin^2\theta)$$
$$= r^2$$

which is a circle with centre (a, b) and radius $|r|$.

5. Show that the curve whose parametric equations are

$$\left.\begin{array}{l} x = a\cos\theta + h \\ y = b\sin\theta + k \end{array}\right\}, \ 0 \le \theta \le 2\pi$$

is an ellipse with centre (h, k).

Sol. $x - h = a\cos\theta$ or $\dfrac{x-h}{a} = \cos\theta$

$y - k = b\sin\theta$ or $\dfrac{y-k}{b} = \sin\theta$

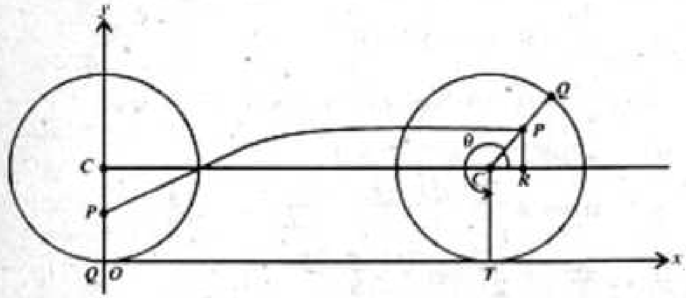
Squaring these equations and adding the results, we have

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

which is an ellipse with centre at (h, k).

6. A wheel of radius a rolls on a straight line without slipping or sliding. Let P be a fixed point on the wheel, at a distance b from the centre of the wheel. Find parametric equations of the curve described by the point P. The curve is called a **trochoid**. Hence deduce parametric equations for a cycloid.

Sol. Let x-axis be taken the line on which the wheel rolls. In the initial position, let the centre C of the wheel be on the y-axis and the point P is on the y-axis below C. Let the point Q on the rim of the wheel that lies on the radial line CP coincide with the origin O.



Let the wheel turn through an angle θ (the radial line CQ turns through an angle θ) in moving from its initial position to some general position as shown in the figure. Let $\angle RCP = \phi$.

Then $\theta + \phi = \dfrac{3\pi}{2}$ (1)

In the general position, let P have coordinates (x, y).

Now $x = OT + CR = a\theta + b\cos\phi$ (2)

$y = TC + RP = a + b\sin\phi$ (3)

Substituting the value of ϕ from (1) into (2) and (3), we get

$$x = a\theta + b\cos\left(\frac{3\pi}{2} - \theta\right) = a\theta - b\sin\theta \quad (4)$$

$$y = a + b\sin\left(\frac{3\pi}{2} - \theta\right) = a - b\cos\theta \quad (5)$$

as parametric equations of the trochoid.

For a cycloid, the point P is on the rim of the wheel so that $b = a$ and from (4)(and (5), we have

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$$

as equations of the cycloid.

7. Find the points at which $r = 1 + \cos\theta$ has horizontal and vertical tangents.

Sol. Parametric equations of the curve are

$x = (1 + \cos\theta)\cos\theta = \cos\theta + \cos^2\theta$
$y = (1 + \cos\theta)\sin\theta = \sin\theta + \sin\theta\cos\theta$

$$\frac{dx}{d\theta} = -\sin\theta - 2\sin\theta\cos\theta$$

$$\frac{dy}{d\theta} = \cos\theta + \cos^2\theta - \sin^2\theta$$

Therefore, $\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta}$

The curve has horizontal tangent at the points where

$$\frac{dy}{d\theta} = 0 \text{ but } \frac{dx}{d\theta} \neq 0$$

Now, $\dfrac{dy}{d\theta} = 0$ implies $\cos\theta + \cos^2\theta - \sin^2\theta = 0$

or $2\cos^2\theta + \cos\theta - 1 = 0$

or $\cos\theta = \dfrac{-1 \pm \sqrt{1+8}}{4} = \dfrac{-1 \pm 3}{4} = -1, \dfrac{1}{2}$

If $\cos\theta = \dfrac{1}{2}$, then $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

If $\cos\theta = -1, \theta = \pi$

But $\dfrac{dx}{d\theta} = 0$ for $\theta = \pi$ and it is non-zero for $\theta = \dfrac{\pi}{3}, \dfrac{5\pi}{3}$

Thus, tangents are horizontal at

$$\left(\frac{3}{2}, \frac{\pi}{3}\right), \left(\frac{3}{2}, \frac{5\pi}{3}\right)$$

Tangents are vertical at points where $\dfrac{dx}{d\theta} = 0$ but $\dfrac{dy}{d\theta} \neq 0$

Now $\dfrac{dx}{d\theta} = 0$ implies $-\sin\theta - 2\sin\theta\cos\theta = 0$

or $\sin\theta(1 + 2\cos\theta) = 0 \Rightarrow \sin\theta = 0$ implies $\theta = 0, \pi$,

and $1 + 2\cos\theta = 0$ gives $\cos\theta = -\dfrac{1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

$\dfrac{dy}{d\theta} \neq 0$ at $\theta = 0, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$

Hence, tangents are vertical at

$$(2, 0), \left(\frac{1}{2}, \frac{2\pi}{3}\right), \left(\frac{1}{2}, \frac{4\pi}{3}\right)$$

8. Find the points on the curve

$x(t) = t^2 + 4, \ y(t) = 3t^2 - 6t + 2$

where tangents are horizontal and vertical.

Sol. $\dfrac{dx}{dt} = 2t, \dfrac{dy}{dt} = 6t - 6$

Hence $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{6(t-1)}{2t} = \dfrac{3(t-1)}{t}$

For vertical tangents, $\dfrac{dx}{dt} = 0$ but $\dfrac{dy}{dt} \neq 0$

$\dfrac{dx}{dt} = 0$ implies $t = 0$ and $\dfrac{dy}{dt} \neq 0$ at $t = 0$

The curve has vertical tangent at $x(0) = 4, y(0) = 2$.
i.e., at $(4, 2)$.

For horizontal tangents $\dfrac{dy}{dt} = 0$ but $\dfrac{dx}{dt} \neq 0$

Now $\dfrac{dy}{dt} = 0$ gives $3(t-1) = 0$ or $t = 1$

$\dfrac{dx}{dt} \neq 0$ at $t = 1$

The tangent is horizontal at

$x(1) = 5, y(1) = -1$ *i.e.*, at $(5, -1)$.

Find equations of the tangent and normal to each of the given curve at the indicated point (Problem 9 – 11):

9. $x = 2a\cos\theta - a\cos 2\theta, \ y = 2a\sin\theta - a\sin 2\theta$ at $\theta = \dfrac{\pi}{2}$

Sol. $\dfrac{dx}{d\theta} = -2a\sin\theta + 2a\sin 2\theta$

and $\dfrac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta$

Therefore, $\dfrac{dy}{dx} = \dfrac{2a\cos\theta - 2a\cos 2\theta}{2a\sin 2\theta - 2a\sin\theta} = \dfrac{\cos\theta - \cos 2\theta}{\sin 2\theta - \sin\theta}$

$$= \frac{2\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{3\theta}{2}} = \tan\frac{3\theta}{2}$$

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = \tan\frac{3}{2}\left[\frac{\pi}{2}\right] = \tan\frac{3\pi}{4} = -1$$

At $\theta = \dfrac{\pi}{2}$

$$x = 2a\cos\frac{\pi}{2} - a\cos 2\left[\frac{\pi}{2}\right] = a$$

$$y = 2a\sin\frac{2\pi}{2} - a\sin 2\left(\frac{\pi}{2}\right) = 2a$$

Hence equation of the tangent at $\theta = \frac{\pi}{2}$ is

$$y - 2a = -(x - a)$$

i.e., $x + y = 3a$.

Slope of the normal at $\theta = \frac{\pi}{2}$ is 1

Equation of the normal at $(a, 2a)$ is $y - 2a = x - a$

or $x - y + a = 0$.

10. $x = \dfrac{2at^2}{1+t^2}, y = \dfrac{2at^3}{1+t^2}$ at $t = \frac{1}{2}$

Sol. At $t = \frac{1}{2}$, we have $x = \dfrac{2a \cdot \frac{1}{4}}{1+\frac{1}{4}} = \dfrac{\frac{a}{2}}{\frac{5}{4}} = \dfrac{2a}{5}$ and $y = \dfrac{2a \cdot \left(\frac{1}{8}\right)}{1+\frac{1}{4}} = \dfrac{\frac{a}{4}}{\frac{5}{4}} = \dfrac{a}{5}$

Now, $x = \dfrac{2at^2}{1+t^2} = 2a\left[1 - \dfrac{1}{1+t^2}\right]$

$$\frac{dx}{dt} = 2a\left[\frac{2t}{(1+t^2)^2}\right] = \frac{4at}{(1+t^2)^2} \quad (1)$$

$$y = \frac{2at^3}{1+t^2} = 2a\left[t - \frac{t}{1+t^2}\right]$$

$$\frac{dy}{dt} = 2a\left[1 - \frac{(1+t^2) - t(2t)}{(1+t^2)^2}\right]$$

$$= 2a\left[1 - \frac{1-t^2}{(1+t^2)^2}\right] = 2a\left[\frac{(1+t^2)^2 - 1 + t^2}{(1+t^2)^2}\right]$$

$$= 2a\frac{3t^2 + t^4}{(1+t^2)^2} = \frac{2at^2(3+t^2)}{(1+t^2)^2} \quad (2)$$

Dividing (2) by (1), we have

$$\frac{dy}{dx} = \frac{2at^2(3+t)^2}{(1+t^2)^2} \times \frac{(1+t^2)}{4at} = \frac{t(3+t^2)}{2}$$

$$\left(\frac{dy}{dx}\right)_{t=\frac{1}{2}} = \frac{\frac{1}{2}\left(3+\frac{1}{4}\right)}{2} = \frac{13}{16}$$

Equation of the required tangent is

$$y - \frac{a}{5} = \frac{13}{16}\left[x - \frac{2a}{5}\right] = \frac{13x}{16} - \frac{13a}{40}$$

$$\frac{13x}{16} - y = \frac{13a}{40} - \frac{a}{5} = \frac{a}{8}$$

or $13x - 16y = 2a$.

Slope of the normal at $t = \frac{1}{2}$ *i.e.*, at the point $\left(\frac{2a}{5}, \frac{a}{5}\right)$ is $\frac{-16}{13}$

Equation of the normal at this point is

$$y - \frac{a}{5} = \frac{-16}{13}\left(x - \frac{2a}{5}\right)$$

$$13y - \frac{13a}{5} = -16x + \frac{29a}{5}$$

or $16x + 13y = \dfrac{32a}{5} + \dfrac{13a}{5}$ *i.e.*, $16x + 13y = 9a$

11. $x = (t-1)^{3/2}$, $y = 3t$ at $t = 5$

Sol. $\dfrac{dx}{dt} = \dfrac{3}{2}(t-1)^{1/2}$, $\dfrac{dy}{dt} = 3$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3}{\frac{3}{2}(t-1)^{1/2}} = \frac{2}{\sqrt{t-1}}$$

$$\left.\frac{dy}{dx}\right|_{t=5} = \frac{2}{\sqrt{5-1}} = 1$$

The point $P(x, y)$ at $t = 5$ is $P(8, 15)$

Equation of the tangent at P is

$y - 15 = 1(x - 8)$ or $x - y + 7 = 0$

Equation of the normal at P is

$y - 15 = -1(x - 8)$

or $x + y - 23 = 0$

12. Show that the normal at any point of the curve

$$x = a\cos\theta + a\theta\sin\theta, y = a\sin\theta - a\theta\cos\theta$$

is at a constant distance from the origin.

Sol. $\dfrac{dx}{d\theta} = -a\sin\theta + a\sin\theta + a\theta\cos\theta = a\theta\cos\theta$

$\dfrac{dy}{d\theta} = a\cos\theta - a\cos\theta + a\theta\sin\theta = a\theta\sin\theta$

Therefore, $\dfrac{dy}{dx} = \dfrac{a\theta\sin\theta}{a\theta\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$

Slope of the normal $= -\dfrac{\cos\theta}{\sin\theta}$

Equation of the normal is

$$y - a\sin\theta + a\theta\cos\theta = -\frac{\cos\theta}{\sin\theta}(x - a\cos\theta - a\theta\sin\theta)$$

or $y\sin\theta - a\sin^2\theta + a\theta\sin\theta\cos\theta$

$$= -x\cos\theta + a\cos^2\theta + a\theta\sin\theta\cos\theta$$

or $\quad x\cos\theta + y\sin\theta - a = 0 \qquad (1)$

Length of the perpendicular from (0, 0) to (1)

$$= \frac{|-a|}{\sqrt{\cos^2\theta+\sin^2\theta}} = a \quad \text{which is constant}$$

13. Prove that an equation of the normal to the astroid $x^{2/3}+y^{2/3} = a^{2/3}$ can be written in the form

$x\sin t - y\cos t + a\cos t = 0$, t being parameter.

Sol. $x^{2/3}+y^{2/3} = a^{2/3} \qquad (1)$

Differentiating (1) w.r.t. x, we have

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x^{-1/3}}{y^{-1/3}} = -\frac{y^{1/3}}{x^{1/3}}$$

Slope of the normal $= \dfrac{x^{1/3}}{y^{1/3}}$

Let this slope be denoted by $\tan t$, then

$$\frac{x^{1/3}}{y^{1/3}} = \tan t$$

or $\quad x^{1/3} = y^{1/3}\tan t \qquad (2)$

Putting this value of $x^{1/3}$ in (1), we get

$$y^{2/3}\tan^2 t + y^{2/3} = a^{2/3}$$

or $\quad y^{2/3}(1+\tan^2 t) = a^{2/3} \Rightarrow y^{2/3}\sec^2 t = a^{2/3}$

or $\quad \dfrac{y^{2/3}}{a^{2/3}} = \cos^2 t \Rightarrow \dfrac{y}{a} = \cos^3 t$ or $y = a\cos^3 t$

From (2), $x^{1/3} = (a\cos^3 t)^{1/3}\tan t = a^{1/3}\cos t\dfrac{\sin t}{\cos t} = a^{1/3}\sin t$

or $\quad x = a\sin^3 t$

Thus we have to find equation of the normal at $(a\sin^3 t, a\cos^3 t)$

Slope of the normal $= \dfrac{x^{1/3}}{y^{1/3}} = \dfrac{a^{1/3}\sin t}{a^{1/3}\cos t} = \dfrac{\sin t}{\cos t}$

Hence the equation of the normal is

$$y - a\cos^3 t = \frac{\sin t}{\cos t}(x - a\sin^3 t)$$

or $\quad y\cos t - a\cos^4 t = x\sin t - a\sin^4 t$

or $\quad x\sin t - y\cos t = -a(\cos^2 t - \sin^2 t) = -\cos 2t$

$x\sin t - y\cos t\, t + a\cos 2t = 0$ as required.

14. Show that the pedal equation of the curve

$x = a\cos^3\theta,\ y = a\sin^3\theta$

is $\quad r^2 = a^2 - 3p^2$

Sol. $x = a\cos^3\theta$ and $y = a\sin^3\theta$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta \quad\text{and}\quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

$$\frac{dy}{dx} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta}$$

Equation of the tangent at $(a\cos^3\theta, a\sin^3\theta)$ is

$$y - a\sin^3\theta = -\frac{\sin\theta}{\cos\theta}(x - a\cos^3\theta)$$

or $\quad y\cos\theta - a\sin^3\theta\cos\theta = -x\sin\theta + a\sin\theta\cos^3\theta$

or $\quad x\sin\theta + y\cos\theta - a\sin\theta\cos\theta(\sin^2\theta+\cos^2\theta) = 0$

or $\quad x\sin\theta + y\cos\theta - a\sin\theta\cos\theta = 0$

$$p = \frac{|-a\sin\theta\cos\theta|}{\sqrt{\sin^2\theta+\cos^2\theta}} \quad\text{or}\quad p^2 = a^2\sin^2\theta\cos^2\theta \qquad (1)$$

Now $r^2 = x^2 + y^2 = a^2\cos^6\theta + a^2\sin^6\theta$

$= a^2(\cos^2\theta+\sin^2\theta)(\cos^4\theta - \sin^2\theta\cos^2\theta + \sin^4\theta)$

$= a^2[(\cos^2\theta+\sin^2\theta)^2 - 3\sin^2\theta\cos^2\theta]$

$= a^2(1 - 3\sin^2\theta\cos^2\theta)$

or $\quad 3a^2\sin^2\theta\cos^2\theta = a^2 - r^2 \qquad (2)$

Using (2), we get from (1)

$3p^2 = a^2 - r^2 \quad$ or $\quad r^2 = a^2 - 3p^2$

i.e., $\quad r^2 = a^2 - 3p^2$

which is the required pedal equation.

15. Prove that the pedal equation of the curve

$x = 2a\cos\theta - a\cos 2\theta,\ y = 2a\sin\theta - a\sin 2\theta$

is $\quad 9(r^2 - a^2) = 8p^2$

Sol. From $x = 2a\cos\theta - a\cos 2\theta$, we have

$$\frac{dx}{d\theta} = -2a\sin\theta + 2a\sin 2\theta = 2a(\sin 2\theta - \sin\theta)$$

$$= 4a\sin\frac{\theta}{2}\cos\frac{3\theta}{2} \qquad (1)$$

From $y = 2a\sin\theta - a\sin 2\theta$, we get

$$\frac{dy}{d\theta} = 2a\cos\theta - 2a\cos 2\theta = 2a(\cos\theta - \cos 2\theta)$$

$$= 4a\sin\frac{\theta}{2}\sin\frac{3\theta}{2} \qquad (2)$$

Dividing (2) by (1), we have

$$\frac{dy}{dx} = \frac{4a\sin\frac{\theta}{2}\sin\frac{3\theta}{2}}{4a\sin\frac{\theta}{2}\cos\frac{3\theta}{2}} = \frac{\sin\frac{3\theta}{2}}{\cos\frac{3\theta}{2}}$$

Equation of the tangent at $(2a\cos\theta - a\cos 2\theta, 2a\sin\theta - a\sin 2\theta)$ is

$$y - 2a\sin\theta + a\sin 2\theta = \frac{\sin\frac{3\theta}{2}}{\cos\frac{3\theta}{2}}(x - 2a\cos\theta + a\cos 2\theta)$$

or $$y\cos\frac{3\theta}{2} - 2a\sin\theta\cos\frac{3\theta}{2} + a\sin 2\theta\cos\frac{3\theta}{2}$$

$$= x\sin\frac{3\theta}{2} - 2a\cos\theta\sin\frac{3\theta}{2} + a\cos 2\theta\sin\frac{3\theta}{2}$$

or $$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\left\{\sin\frac{3\theta}{2}\cos\theta - \sin\theta\cos\frac{3\theta}{2}\right\}$$

$$-a\left\{\sin 2\theta\cos\frac{3\theta}{2} - \cos 2\theta\sin\frac{3\theta}{2}\right\} = 0$$

or $$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 2a\sin\frac{\theta}{2} - a\sin\frac{\theta}{2} = 0$$

or $$x\sin\frac{3\theta}{2} - y\cos\frac{3\theta}{2} - 3a\sin\frac{\theta}{2} = 0$$

$$p = \frac{\left|-3a\sin\frac{\theta}{2}\right|}{\sqrt{\sin^2\frac{3\theta}{2} + \cos^2\frac{3\theta}{2}}} \text{ or } p^2 = 9a^2\sin^2\frac{\theta}{2} \tag{3}$$

Now, $r^2 = x^2 + y^2$

$$= (2a\cos\theta - a\cos 2\theta)^2 + (2a\sin\theta - a\sin 2\theta)^2$$
$$= 4a^2\cos^2\theta + a^2\cos^2 2\theta - 4a^2\cos 2\theta\cos\theta$$
$$+ 4a^2\sin^2\theta + a^2\sin^2 2\theta - 4a^2\sin 2\theta\sin\theta$$
$$= 4a^2 + a^2 - 4a^2[\cos 2\theta\cos\theta + \sin\theta\sin 2\theta]$$
$$= 5a^2 - 4a^2\cos\theta$$
$$= 5a^2 - 4a^2\left[1 - 2\sin^2\frac{\theta}{2}\right] = a^2 + 8a^2\sin^2\frac{\theta}{2}$$

or $$r^2 - a^2 = 8a^2\sin^2\frac{\theta}{2} \tag{4}$$

From (3), $$\sin^2\frac{\theta}{2} = \frac{p^2}{9a^2} \tag{5}$$

From (4) and (5), we have

$$r^2 - a^2 = 8a^2\left(\frac{p^2}{9a^2}\right) = \frac{8p^2}{9}$$

or $9(r^2 - a^2) = 8p^2$ which is the required equation.

16. Show that the pedal equation of the curve

$$x = ae^\theta(\sin\theta - \cos\theta),\ y = ae^\theta(\sin\theta + \cos\theta)$$

is $r = \sqrt{2}p$.

Sol. $x = ae^\theta(\sin\theta - \cos\theta)$

$$\frac{dx}{d\theta} = ae^\theta(\sin\theta - \cos\theta) + ae^\theta(\cos\theta + \sin\theta) = 2ae^\theta\sin\theta \tag{1}$$

$$y = ae^\theta(\sin\theta + \cos\theta)$$

$$\frac{dy}{d\theta} = ae^\theta(\sin\theta + \cos\theta) + ae^\theta(\cos\theta - \sin\theta) = 2ae^\theta\cos\theta \tag{2}$$

From (1) and (2), we have

$$\frac{dy}{dx} = \frac{2a\,e^\theta\cos\theta}{2a\,e\sin\theta} = \frac{\cos\theta}{\sin\theta}$$

Equation of the tangent at $(ae^\theta(\sin\theta - \cos\theta), ae^\theta(\sin\theta + \cos\theta))$ is

$$y - ae^\theta(\sin\theta + \cos\theta) = \frac{\cos\theta}{\sin\theta}[x - ae^\theta(\sin\theta - \cos\theta)]$$

or $$y\sin\theta - ae^\theta(\sin^2\theta + \sin\theta\cos\theta)$$
$$= x\cos\theta - ae^\theta(\cos\theta\sin\theta - \cos^2\theta)$$

or $$y\sin\theta - ae^\theta\sin^2\theta = x\cos\theta + ae^\theta\cos^2\theta$$

or $$x\cos\theta - y\sin\theta + ae^\theta = 0$$

$$p = \frac{ae^\theta}{\sqrt{\cos^2\theta + \sin^2\theta}} = ae^\theta \tag{3}$$

Now $$r^2 = x^2 + y^2 = a^2e^{2\theta}[(\sin\theta - \cos\theta)^2 + (\sin\theta + \cos\theta)^2]$$
$$= a^2e^{2\theta}[2(\sin^2\theta + \cos^2\theta)] = 2a^2e^{2\theta} \tag{4}$$

Squaring (3), we have

$$p^2 = a^2e^{2\theta} \tag{5}$$

From (4) and (5), we get

$r^2 = 2p^2 \Rightarrow r = \sqrt{2}p$ which is the required pedal equation

17. Prove that the pedal equation of the curve

$$x = a(3\cos\theta - \cos^3\theta),\ y = a(3\sin\theta - \sin^3\theta)$$

is $3p^2(7a^2 - r^2) = (10a^2 - r^2)^2$.

Sol. $x = a(3\cos\theta - \cos^3\theta)$

$$\frac{dx}{d\theta} = a[-3\sin\theta + 3\cos^2\theta\sin\theta]$$
$$= -3a\sin\theta(1 - \cos^2\theta) = -3a\sin^3\theta$$

$$y = a(3\sin\theta - \sin^3\theta)$$

$$\frac{dy}{d\theta} = a[3\cos\theta - 3\sin^2\theta\cos\theta]$$
$$= 3a\ \text{cso}\ \theta(1 - \sin^2\theta) = 3a\cos^3\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\cos^3\theta}{-3a\sin^3\theta} = -\frac{\cos^3\theta}{\sin^3\theta}$$

Equation of the tangent at $(a(3\cos\theta - \cos^3\theta), a(3\sin\theta - \sin^3\theta))$ is

$$y - a(3\sin\theta - \sin^3\theta) = -\frac{\cos^3\theta}{\sin^3\theta}[x - a(3\cos\theta - \cos^3\theta)]$$

or $\quad y\sin^3\theta - a\sin^3\theta(3\sin\theta - \sin^3\theta)$

$\quad = -x\cos^3\theta + a\cos^3\theta(3\cos\theta - \cos^3\theta)$

or $\quad y\sin^3\theta - 3a\sin^4\theta + a\sin^6\theta = -x\cos^3\theta + 3a\cos^4\theta - a\cos^6\theta$

or $\quad x\cos^3\theta + y\sin^3\theta - 3a(\sin^4\theta + \cos^4\theta + a(\sin^6\theta + \cos^6\theta) = 0$

Now $\sin^4\theta + \cos^4\theta = (\sin^2\theta + \cos^2\theta)^2 - 2\sin^2\theta\cos^2\theta$

$\quad = 1 - 2\sin^2\theta\cos^2\theta$

and $\sin^6\theta + \cos^6\theta = (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta\cos^2\theta(\sin^2\theta + \cos^2\theta)$

$\quad = 1 - 3\sin^2\theta\cos^2\theta$

Hence equation of the tangent can be written as

$x\cos^3\theta + y\sin^3\theta - 3a(1 - 2\sin^2\theta\cos^2\theta) + a(1 - 3\sin^2\theta\cos^2\theta) = 0$

or $\quad x\cos^3\theta + y\sin^3\theta - (2a - 3a\sin^2\theta\cos^2\theta) = 0 \quad (1)$

Length of the perpendicular from the origin to (1) is

$$p = \frac{|-(2a - 3a\sin^2\theta\cos^2\theta)|}{\sqrt{\cos^6\theta + \sin^6\theta}} = \frac{|-(2a - 3a\sin^2\theta\cos^2\theta)|}{\sqrt{1 - 3\sin^2\theta\cos^2\theta}}$$

or $$p^2 = \frac{(2a - 3a\sin^2\theta\cos^2\theta)^2}{1 - 3\sin^2\theta\cos^2\theta} \quad (2)$$

Now $r^2 = x^2 + y^2 = a^2(3\cos\theta - \cos^3\theta)^2 + a^2(3\sin\theta - \sin^3\theta)^2$

$\quad = a^2(9\cos^2\theta + \cos^6\theta - 6\cos^4\theta) + a^2[9\sin^2\theta + \sin^6\theta - 6\sin^4\theta]$

$\quad = a^2[9(\cos^2\theta + \sin^2\theta) + (\cos^6\theta + \sin^6\theta) - 6(\cos^4\theta + \sin^4\theta)]$

$\quad = a^2[9 + 1 - 3\sin^2\theta\cos^2\theta - 6(1 - 2\sin^2\theta\cos^2\theta)]$

$\quad = a^2[10 - 3\sin^2\theta\cos^2\theta - 6 + 12\sin^2\cos^2\theta]$

$\quad = a^2[4 + 9\sin^2\theta\cos^2\theta] = 4a^2 + 9a^2\sin^2\theta\cos^2\theta$

or $\quad 9a^2\sin^2\theta\cos^2\theta = r^2 - 4a^2 \quad$ or $\quad \sin^2\theta\cos^2\theta = \dfrac{r^2 - 4a^2}{9a^2}$

Putting this value in (2), we get

$$p^2 = \frac{\left(2a - \frac{r^2 - 4a^2}{3a}\right)^2}{1 - \frac{r^2 - 4a^2}{3a^2}} = \frac{(10a^2 - r^2)^2}{9a^2} \times \frac{3a^2}{7a^2 - r^2}$$

$$= \frac{(10a^2 - r^2)^2}{7a^2 - r^2} \times \frac{1}{3} \quad \text{or} \quad 3p^2(7a^2 - r^2) = (10a^2 - r^2)^2$$

18. If $x = a\cos g(t)$, $y = b\sin g(t)$, prove that

$$xy^2\frac{d^2y}{dx^2} = b^2\frac{dy}{dx}$$

Sol. $x = a\cos g(t)$ gives

$$\frac{dx}{dt} = (-a\sin g(t))\,g'(t)$$

$y = b\sin g(t)$ gives

$$\frac{dy}{dt} = (b\cos g(t))\,.\,g'(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = -\frac{b}{a}\frac{\cos g(t)}{\sin g(t)} = \frac{b\,.\,\frac{x}{a}}{a\,.\,\frac{y}{b}} = -\frac{b^2}{a^2}\cdot\frac{x}{y}$$

$$\frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y\,.\,1 - x\frac{dy}{dx}}{y^2}\right] = -\frac{b^2}{a^2}\left[\frac{y - x\left(-\frac{b^2}{a^2}\frac{x}{y}\right)}{y^2}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{a^2y^2 + b^2x^2}{a^2y^3}\right]$$

$$= -\frac{b^2}{a^2}\left[\frac{a^2b^2\sin^2 g(t) + b^2a^2\cos^2 g(t)}{a^2y^3}\right]$$

$$= -\frac{b^2}{a^2}\cdot\frac{a^2b^2}{a^2y^3} = -\frac{b^4}{a^2y^3}$$

Now, $xy^2\dfrac{d^2y}{dx^2} = xy^2\left(\dfrac{-b^4}{a^2y^3}\right) = -\dfrac{b^4}{a^2}\dfrac{x}{y}$

$$= b^2\left(-\frac{b^2}{a^2}\frac{x}{y}\right) = b^2\frac{dy}{dx} \text{ as required.}$$