Exercise 2.1

Q. 1: Find the discriminant of the following given quadratic equations:

(i) $2x^2 + 3x - 1 = 0$

a = 2, b = 3, c = -1

$$\begin{aligned} Disc &= b^2 - 4ac \\ &= (3)^2 - 4(2)(-1) \\ &= 9 + 8 \\ &= 17 \end{aligned}$$

(ii) $6x^2 - 8x + 3 = 0$

a = 6, b = -8, c = 3

$$\begin{aligned} Disc &= b^2 - 4ac \\ &= (-8)^2 - 4(6)(3) \\ &= 64 - 72 \\ &= -8 \end{aligned}$$

(iii) $9x^2 - 30x + 25 = 0$

a = 9, b = -30, c = 25

$$\begin{aligned} Disc &= b^2 - 4ac \\ &= (-30)^2 - 4(9)(25) \\ &= 900 - 900 \\ &= 0 \end{aligned}$$

(iv) $4x^2 - 7x - 2 = 0$

a = 4, b = -7, c = -2

$$\begin{aligned} Disc &= b^2 - 4ac \\ &= (-7)^2 - 4(4)(-2) \\ &= 49 + 32 \\ &= 81 \end{aligned}$$

Q. 2: Find the nature of the roots of the following given quadratic equations and verify the result by solving the equations:

(i) $x^2 - 23x + 120 = 0$

a = 1, b = -23, c = 120

$$\begin{aligned} Disc &= b^2 - 4ac \\ &= (-23)^2 - 4(1)(120) \\ &= 529 - 480 \\ &= 49 \end{aligned}$$

Disc > 0 and a perfect square, So the roots are rational (real) and unequal.

Now, Verify by using quadratic formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-(-23)\pm\sqrt{(-23)^2-4(1)(120)}}{2(1)}$$
$$= \frac{23\pm\sqrt{529-480}}{2}$$
$$= \frac{23\pm\sqrt{49}}{2}$$
$$= \frac{23\pm7}{2}$$

$x = \frac{23+7}{2}$ and $x = \frac{23-7}{2}$

$x = 15$ and $x = 8$

(ii) $2x^2 + 3x + 7 = 0$

a = 2, b = 3, c = 7

$$Disc = b^2 - 4ac$$
$$= (3)^2 - 4(2)(7)$$
$$= 9 - 56$$
$$= -47$$

Disc < 0, So the roots are imaginary (complex conjugates).

Now, Verify by using quadratic formula

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$= \frac{-(3)\pm\sqrt{-47}}{2(2)}$$
$$= \frac{-3\pm\sqrt{-47}}{4}$$

(iii) $16x^2 - 24x + 9 = 0$

a = 16, b = -24, c = 9

$$Disc = b^2 - 4ac$$
$$= (-24)^2 - 4(16)(9)$$
$$= 576 - 576$$
$$= 0$$

Disc = 0, So the roots are rational (real) and equal.

Now, Verify by competing square method

$$16x^2 - 24x + 9 = 0$$
$$(4x)^2 - 2(4x)(3) + (3)^2 = 0$$
$$(4x-3)^2 = 0$$
$$(4x-3)(4x-3) = 0$$

$x = \frac{3}{4}$ and $x = \frac{3}{4}$

(iv) $3x^2 + 7x - 13 = 0$

a = 3, b = 7, c = -13

$$
\begin{aligned}
Disc &= b^2 - 4ac \\
&= (7)^2 - 4(3)(-13) \\
&= 49 + 156 \\
&= 205
\end{aligned}
$$

Disc > 0, So the roots are Irrational (real) and unequal.

Now, Verify by using quadratic formula

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(7) \pm \sqrt{205}}{2(3)} \\
&= \frac{-7 \pm \sqrt{205}}{6}
\end{aligned}
$$

$x = \frac{-7+\sqrt{205}}{6}$ and $x = \frac{-7-\sqrt{205}}{6}$

Q. 3: For what value of k, the expression

$k^2x^2 + 2(k+1)x + 4$ is perfect square.

The given equation is perfect square if Disc = 0.

So,

$$
\begin{aligned}
\text{Disc} &= 0 \\
b^2 - 4ac &= 0 \\
[2(k+1)]^2 - 4(k^2)(4) &= 0 \\
4(k^2 + 2k + 1) - 16k^2 &= 0 \\
4k^2 + 8k + 4 - 16k^2 &= 0 \\
-12k^2 + 8k + 4 &= 0
\end{aligned}
$$

a = -12, b = 8, c = 4

$$
\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(8) \pm \sqrt{(8)^2 - 4(-12)(4)}}{2(-12)} \\
&= \frac{-(8) \pm \sqrt{64 + 192}}{2(-12)} \\
&= \frac{-8 \pm \sqrt{256}}{-24}
\end{aligned}
$$

$k = \frac{-8+16}{-24}$ and $k = \frac{-8-16}{-24}$

$k = \frac{8}{-24}$ and $k = \frac{-24}{-24}$

$k = \frac{-1}{3}$ and $k = 1$

Q. 4: Find the value of k, if the roots of the following equations are equal.

(i) $(2k-1)x^2 + 3kx + 3 = 0$

The given equation have equal roots if Disc = 0.

So,

$$\begin{aligned}
\text{Disc} &= 0\\
b^2 - 4ac &= 0\\
[3k]^2 - 4(2k-1)(3) &= 0\\
9k^2 - 24k + 12 &= 0
\end{aligned}$$

a = 9, b = -24, c = 12

$$\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
&= \frac{-(-24) \pm \sqrt{(-24)^2 - 4(9)(12)}}{2(9)}\\
&= \frac{24 \pm \sqrt{576-432}}{2(9)}\\
&= \frac{24 \pm \sqrt{144}}{18}\\
&= \frac{24 \pm 12}{18}
\end{aligned}$$

$k = \frac{24+12}{18}$ and $k = \frac{24-12}{18}$

$k = \frac{36}{18}$ and $k = \frac{12}{18}$

$k = 2$ and $k = \frac{2}{3}$

(ii) $x^2 + 2(k+2)x + (3k+4) = 0$

The given equation have equal roots if Disc = 0.

So,

$$\begin{aligned}
\text{Disc} &= 0\\
b^2 - 4ac &= 0\\
[2(k+2)]^2 - 4(1)(3k+4) &= 0\\
4(k^2+4k+4) - 12k - 16 &= 0\\
4k^2 + 16k + 16 - 12k - 16 &= 0\\
4k^2 + 4k &= 0
\end{aligned}$$

a = 4, b = 4, c = 0

$$\begin{aligned}
k &= \frac{-b \pm \sqrt{b^2-4ac}}{2a}\\
&= \frac{-(4) \pm \sqrt{(4)^2 - 4(4)(0)}}{2(4)}\\
&= \frac{-4 \pm \sqrt{16-0}}{8}\\
&= \frac{-4 \pm \sqrt{16}}{8}
\end{aligned}$$

$k = \frac{-4+4}{8}$ and $k = \frac{-4-4}{8}$

$k = 0$ and $k = -1$

(iii) $(3k+2)x^2 - 5(k+1)x + (2k+3) = 0$

The given equation have equal roots if Disc = 0.

So,

$$\text{Disc} = 0$$
$$b^2 - 4ac = 0$$
$$[-5(k+1)]^2 - 4(3k+2)(2k+3) = 0$$
$$25(k^2+2k+1) - 4(6k^2+9k+4k+6) = 0$$
$$25k^2+50k+25-4(6k^2+13k+6) = 0$$
$$25k^2+50k+25-24k^2-52k-24 = 0$$
$$k^2-2k+1 = 0$$
$$(k-1)^2 = 0$$
$$(k-1)(k-1) = 0$$

$k = 1$ and $k = 1$

Q. 5: Show that the equation $x^2 + (mx+c)^2 = a^2$ has equal roots,

if $c^2 = a^2(1+m^2)$

$$x^2 + (mx+c)^2 = a^2$$
$$x^2 + m^2x^2 + 2mcx + c^2 = a^2$$
$$(m^2+1)x^2 + (2mc)x + (c^2-a^2) = 0$$

The given equation have equal roots if Disc = 0.

So,

$$\text{Disc} = 0$$
$$b^2 - 4ac = 0$$
$$(2mc)^2 - 4(m^2+1)(c^2-a^2) = 0$$
$$4m^2c^2 - 4(m^2c^2 - m^2a^2 + c^2 - a^2) = 0$$
$$4m^2c^2 - 4m^2c^2 - 4(-m^2a^2 + c^2 - a^2) = 0$$
$$-4(-m^2a^2 + c^2 - a^2) = 0$$
$$-m^2a^2 + c^2 - a^2 = 0$$
$$c^2 = m^2a^2 + a^2$$
$$c^2 = a^2(1+m^2)$$

Hence proved.

Q. 6: Find the condition that the roots of the equation $(mx+c)^2 - 4ax = 0$ are equal.

$$(mx+c)^2 - 4ax = 0$$
$$m^2x^2 + 2mcx + c^2 - 4ax = 0$$
$$(m^2)x^2 + (2mc-4a)x + (c^2) = 0$$

The given equation have equal roots if Disc = 0.

So,

$$\text{Disc} = 0$$
$$b^2 - 4ac = 0$$
$$(2mc-4a)^2 - 4(m^2)(c^2) = 0$$
$$4m^2c^2 - 16mac + 16a^2 - 4m^2c^2 = 0$$

$16a^2 = 16mac$

$a = mc$

Q. 7: If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal, then $a = 0$ and $a^3 + b^3 + c^3 = 3abc$

$(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$

The given equation have equal roots if Disc = 0.

So,

$$\text{Disc} = 0$$
$$b^2 - 4ac = 0$$
$$\left(2(a^2 - bc)\right)^2 - 4(c^2 - ab)(b^2 - ac) = 0$$
$$4(a^4 - 2a^2bc + b^2c^2) - 4(b^2c^2 - ac^3 - ab^3 + a^2bc) = 0$$
$$4[(a^4 - 2a^2bc + b^2c^2) - (b^2c^2 - ac^3 - ab^3 + a^2bc)] = 0$$
$$a^4 - 2a^2bc + b^2c^2 - b^2c^2 + ac^3 + ab^3 - a^2bc = 0$$
$$a^4 - 2a^2bc + ac^3 + ab^3 - a^2bc = 0$$
$$a(a^3 - 2abc + c^3 + b^3 - abc) = 0$$
$$a(a^3 - 3abc + c^3 + b^3) = 0$$

$a = 0$ and $a^3 - 3abc + c^3 + b^3 = 0$

$a = 0$ and $a^3 + b^3 + c^3 = 3abc$

Q. 8: Show that the roots of the following equations are rational.

(i) $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$

$$\begin{aligned}
\text{Disc} &= b^2 - 4ac \\
&= [b(c - a)]^2 - 4\left(a(b - c)\right)\left(c(a - b)\right) \\
&= (bc - ab)^2 - 4ac(b - c)(a - b) \\
&= b^2c^2 - 2ab^2c + a^2b^2 - 4ac(ab - b^2 - ac + bc) \\
&= b^2c^2 - 2ab^2c + a^2b^2 - 4a^2bc + 4ab^2c + 4a^2c^2 - 4abc^2 \\
&= b^2c^2 + a^2b^2 - 4a^2bc + 2ab^2c + 4a^2c^2 - 4abc^2 \\
&= b^2c^2 + a^2b^2 + 2ab^2c - 4abc^2 - 4a^2bc + 4a^2c^2 \\
&= (bc + ab)^2 - 4ac(bc + ab) + 4a^2c^2 \\
&= (bc + ab)^2 - 2(bc + ab)(2ac) + (2ac)^2 \\
&= (bc + ab - 2ac)^2
\end{aligned}$$

as Disc is a perfect square so the roots are rational.

(ii) $(a + 2b)x^2 + 2(a + b + c)x + (a + 2c) = 0$

$$\begin{aligned}
\text{Disc} &= b^2 - 4ac \\
&= [2(a + b + c)]^2 - 4(a + 2b)(a + 2c) \\
&= 4(a + b + c)^2 - 4(a + 2b)(a + 2c)
\end{aligned}$$

$= 4(a+b+c)^2 - 4(a^2 + 2ac + 2ab + 4bc)$
$= 4(a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - a^2 - 2ca - 2ab - 4bc)$
$= 4(b^2 + c^2 - 2bc)$
$= 4(b-c)^2$
$= [2(b-c)]^2$

as Disc is a perfect square so the roots are rational.

Q. 9: For all values of k, prove that the roots of the equation
$x^2 - 2\left(k + \frac{1}{k}\right)x + 3 = 0$, $k \neq 0$ are real.

Disc $= b^2 - 4ac$
$= \left[-2\left(k+\frac{1}{k}\right)\right]^2 - 4(1)(3)$
$= 4\left(k+\frac{1}{k}\right)^2 - 4(3)$
$= 4\left(\left(k+\frac{1}{k}\right)^2 - 3\right)$

For all values of k the Disc > 0, so the roots of the equation are real.

Q. 10: Show that the roots of the equation
$(b-c)x^2 + (c-a)x + (a-b) = 0$ are real.

Disc $= b^2 - 4ac$
$= (c-a)^2 - 4(b-c)(a-b)$
$= (c-a)^2 - 4(ab - b^2 - ac + bc)$
$= (c^2 - 2ac + a^2 - 4ab + 4b^2 + 4ac - 4bc)$
$= (c^2 + 2ac + a^2 - 4ab - 4bc + 4b^2)$
$= ((c+a)^2 - 4b(c+a) + 4b^2)$
$= (c+a)^2 - 2(c+a)(2b) + (2b)^2$
$= (c+a-2b)^2$

as Disc is greater than zero and perfect square so the roots are rational and real.