Exercise 4.4

Resolve into partial fractions.

1. $\frac{x^3}{(x^2+4)^2} = \frac{Ax+B}{(x^2+4)} + \frac{Cx+D}{(x^2+4)^2}$ .... (i)

multiplying by $(x^2+4)^2$ we get

$x^3 = (Ax+B)(x^2+4) + (Cx+D)$ .... (ii)

$x^3 = Ax(x^2+4) + B(x^2+4) + Cx + D$

$x^3 = A(x^3+4x) + B(x^2+4) + Cx + D$ .... (iii)

Now, comparing coefficients of equation (iii)

$x^3;\quad A = 1$

$x^2;\quad B = 0$

$x;\quad 4A + C = 0$

as A = 1

$4 + C = 0$

$C = -4$

$const;\ 4B + D = 0$

as B = 0

$0 + D = 0$

$D = 0$

put the values in (i) we get

$\frac{x^3}{(x^2+4)^2} = \frac{x}{(x^2+4)} + \frac{-4x}{(x^2+4)^2}$

2. $\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$ .... (i)

multiplying by $(x+1)(x^2+1)^2$ we get

$x^4+3x^2+x+1 = A(x^2+1)^2 + (Bx+C)(x^2+1)(x+1) + (Dx+E)(x+1)$ .... (ii)

$x^4+3x^2+x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x^2+x+1) + (Dx+E)(x+1)$

$x^4+3x^2+x+1 = A(x^4+2x^2+1) + Bx(x^3+x^2+x+1) + C(x^3+x^2+x+1) + Dx(x+1)$
$+E(x+1)$

$x^4+3x^2+x+1 = A(x^4+2x^2+1) + B(x^4+x^3+x^2+x) + C(x^3+x^2+x+1) + D(x^2+x)$
$+E(x+1)$ .... (iii)

Put $x = -1$ in equation (ii)

$(-1)^4 + 3(-1)^2 + (-1) + 1 = A((-1)^2+1)^2$

$1 + 3 - 1 + 1 = A(1+1)^2$

$4 = A(2)^2$

$4 = 4A$

$A = 1$

Now, comparing coefficients of equation (iii)

$x^4;\quad A + B = 1$

As A = 1

$1 + B \quad = 1$

$B \quad = 0$

$x^3; \quad B + C \quad = 0$

As B = 0

$0 + C \quad = 0$

$C \quad = 0$

$x^2; \quad 2A + B + C + D = 3$

As A = 1, B = 0, C = 0

$2 + 0 + 0 + D = 3$

$2 + D \quad = 3$

$D \quad = 1$

$x; \quad B + C + D + E = 1$

As B = 0, C = 0, D = 1

$0 + 0 + 1 + E = 1$

$1 + E \quad = 1$

$E \quad = 0$

put the values in (i) we get

$$\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{1}{(x+1)} + \frac{0+0}{(x^2+1)} + \frac{x+0}{(x^2+1)^2}$$

$$\frac{x^4+3x^2+x+1}{(x+1)(x^2+1)^2} = \frac{1}{(x+1)} + \frac{x}{(x^2+1)^2}$$

3. $\frac{x^2}{(x+1)(x^2+1)^2} = \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$ .... (i)

multiplying by $(x+1)(x^2+1)^2$ we get

$x^2 \quad = A(x^2+1)^2 + (Bx+C)(x^2+1)(x+1) + (Dx+E)(x+1)$ .... (ii)

$x^2 \quad = A(x^4+2x^2+1) + (Bx+C)(x^3+x^2+x+1) + (Dx+E)(x+1)$

$x^2 \quad = A(x^4+2x^2+1) + Bx(x^3+x^2+x+1) + C(x^3+x^2+x+1) + Dx(x+1) + E(x+1)$

$x^2 \quad = A(x^4+2x^2+1) + B(x^4+x^3+x^2+x) + C(x^3+x^2+x+1) + D(x^2+x) + E(x+1)$ ...... (iii)

Put $x = -1$ in equation (ii)

$(-1)^2 = A((-1)^2+1)^2$

$1 \quad = A(1+1)^2$

$1 \quad = A(2)^2$

$1 \quad = 4A$

$A \quad = \frac{1}{4}$

Now, comparing coefficients of equation (iii)

$x^4; \quad A + B \quad = 0$

As A = $\frac{1}{4}$

$\frac{1}{4} + B \quad = 0$

$B \quad = -\frac{1}{4}$

$x^3; \quad B + C \quad = 0$

As B = $-\frac{1}{4}$

$-\frac{1}{4} + C = 0$

$C \quad = \frac{1}{4}$

$x^2; \quad 2A + B + C + D = 1$

As A = $\frac{1}{4}$, B = $-\frac{1}{4}$, C = $\frac{1}{4}$

$\frac{1}{2} - \frac{1}{4} + \frac{1}{4} + D \quad = 1$

$\frac{1}{2} + D \quad = 1$

$D \quad = 1 - \frac{1}{2}$

$D \quad = \frac{1}{2}$

$x; \quad B + C + D + E = 0$

As B = $-\frac{1}{4}$, C = $\frac{1}{4}$, D = $\frac{1}{2}$

$-\frac{1}{4} + \frac{1}{4} + \frac{1}{2} + E \quad = 0$

$\frac{1}{2} + E \quad = 0$

$E \quad = -\frac{1}{2}$

put the values in (i) we get

$$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{\frac{1}{4}}{(x+1)} + \frac{-\frac{1}{4}x+\frac{1}{4}}{(x^2+1)} + \frac{\frac{1}{2}x-\frac{1}{2}}{(x^2+1)^2}$$

$$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{\frac{1}{4}}{(x+1)} + \frac{\frac{-x+1}{4}}{(x^2+1)} + \frac{\frac{x-1}{2}}{(x^2+1)^2}$$

$$\frac{x^2}{(x+1)(x^2+1)^2} = \frac{1}{4(x+1)} + \frac{-x+1}{4(x^2+1)} + \frac{x-1}{2(x^2+1)^2}$$

4. $\frac{x^2}{(x-1)(x^2+1)^2} = \frac{A}{(x-1)} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$ .... (i)

multiplying by $(x-1)(x^2+1)^2$ we get

$x^2 \quad = A(x^2+1)^2 + (Bx+C)(x^2+1)(x-1) + (Dx+E)(x-1)$ .... (ii)

$x^2 \quad = A(x^4+2x^2+1) + (Bx+C)(x^3-x^2+x-1) + (Dx+E)(x-1)$

$x^2 \quad = A(x^4+2x^2+1) + Bx(x^3-x^2+x-1) + C(x^3-x^2+x-1) + Dx(x-1) + E(x-1)$

$x^2 \quad = A(x^4+2x^2+1) + B(x^4-x^3+x^2-x) + C(x^3-x^2+x-1) + D(x^2-x) + E(x-1)$ ...... (iii)

Put $x = 1$ in equation (i)

$(1)^2 \quad = A((1)^2+1)^2$

$1 \quad = A(1+1)^2$

$1 \quad = A(2)^2$

$1 \quad = 4A$

$A \quad = \frac{1}{4}$

Now, comparing coefficients of equation (iii)

$x^4; \quad A + B \; = 0$

As A = $\frac{1}{4}$

$\frac{1}{4} + B \quad = 0$

$B \quad = -\frac{1}{4}$

$x^3; \quad -B + C = 0$

As B = $-\frac{1}{4}$

$\frac{1}{4} + C \quad = 0$

$C \quad = -\frac{1}{4}$

$x^2; \quad 2A + B - C + D = 1$

As A = $\frac{1}{4}$, B = $-\frac{1}{4}$, C = $-\frac{1}{4}$

$\frac{1}{2} - \frac{1}{4} + \frac{1}{4} + D \quad = 1$

$\frac{1}{2} + D \quad = 1$

$D \quad = 1 - \frac{1}{2}$

$D \quad = \frac{1}{2}$

$x; \quad -B + C - D + E = 0$

As B = $-\frac{1}{4}$, C = $-\frac{1}{4}$, D = $\frac{1}{2}$

$+\frac{1}{4} - \frac{1}{4} - \frac{1}{2} + E \quad = 0$

$-\frac{1}{2} + E \quad = 0$

$E \quad = \frac{1}{2}$

put the values in (i) we get

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{\frac{1}{4}}{(x-1)} + \frac{-\frac{1}{4}x - \frac{1}{4}}{(x^2+1)} + \frac{\frac{1}{2}x + \frac{1}{2}}{(x^2+1)^2}$$

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{\frac{1}{4}}{(x-1)} + \frac{\frac{-x-1}{4}}{(x^2+1)} + \frac{\frac{x+1}{2}}{(x^2+1)^2}$$

$$\frac{x^2}{(x-1)(x^2+1)^2} = \frac{1}{4(x-1)} - \frac{x+1}{4(x^2+1)} + \frac{x+1}{2(x^2+1)^2}$$

5. $\frac{x^4}{(x^2+2)^2} = \frac{x^4}{x^4+4x+4}$

By long division

$$\frac{x^4}{x^4+4x+4} = 1 + \frac{-4x^2-4}{x^4+4x+4} \qquad = 1 - \frac{4x^2+4}{(x^2+2)^2} \text{ .... (i)}$$

$$\frac{4x^2+4}{(x^2+2)^2} = \frac{Ax+B}{(x^2+2)} + \frac{Cx+D}{(x^2+2)^2} \qquad \text{.............................. (ii)}$$

multiplying by $(x^2+2)^2$ we get

$4x^2 + 4 \quad = (Ax + B)(x^2 + 2) + (Cx + D)$

$4x^2 + 4 \quad = (Ax + B)(x^2 + 2) + (Cx + D)$

$4x^2 + 4 \quad = Ax(x^2 + 2) + B(x^2 + 2) + Cx + D$

$4x^2 + 4 \quad = A(x^3 + 2x) + B(x^2 + 2) + Cx + D$ .... (iii)

Now, comparing coefficients of equation (iii)

$x^3; \quad A \quad = 0$

x^2; $B = 4$

x; $2A + C = 0$

As A = 0

$0 + C = 0$

$C = 0$

$const$; $2B + D = 4$

As B = 4

$8 + D = 4$

$D = -4$

put the values in (ii) we get

$$\frac{4x^2+4}{(x^2+2)^2} = \frac{0+4}{(x^2+2)} + \frac{0-4}{(x^2+2)^2}$$

$$\frac{4x^2+4}{(x^2+2)^2} = \frac{4}{(x^2+2)} - \frac{4}{(x^2+2)^2}$$

put the values in (i) we get

$$\frac{x^4}{(x^2+2)^2} = 1 - \left[\frac{4}{(x^2+2)} - \frac{4}{(x^2+2)^2}\right]$$

$$= 1 - \frac{4}{(x^2+2)} + \frac{4}{(x^2+2)^2}$$

6. $\frac{x^5}{(x^2+1)^2} = \frac{x^5}{x^4+2x+1}$

By long division

$$\frac{x^5}{x^4+2x+1} = x + \frac{-2x^3-x}{x^4+2x+1} \qquad = x - \frac{2x^3+x}{(x^2+1)^2} \text{ .... (i)}$$

$$\frac{2x^3+x}{(x^2+1)^2} = \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+1)^2} \text{ .... (ii)}$$

multiplying by $(x^2+1)^2$ we get

$2x^3 + x = (Ax + B)(x^2 + 1) + (Cx + D)$

$2x^3 + x = (Ax + B)(x^2 + 1) + (Cx + D)$

$2x^3 + x = Ax(x^2 + 1) + B(x^2 + 1) + Cx + D$

$2x^3 + x = A(x^3 + x) + B(x^2 + 1) + Cx + D$ .... (iii)

Now, comparing coefficients of equation (iii)

x^3; $A = 2$

x^2; $B = 0$

x; $A + C = 1$

As A = 2

$2 + C = 1$

$C = -1$

$const$; $B + D = 0$

As B = 0

$0 + D = 0$

$D = 0$

put the values in (ii) we get

$$\frac{2x^3+x}{(x^2+1)^2} = \frac{2x+0}{(x^2+1)} + \frac{-x+0}{(x^2+1)^2}$$

$$\frac{2x^3+x}{(x^2+1)^2} = \frac{2x}{(x^2+1)} - \frac{x}{(x^2+1)^2}$$

put the values in (i) we get

$$\frac{2x^3+x}{(x^2+1)^2} = x - \left[\frac{2x}{(x^2+1)} - \frac{x}{(x^2+1)^2}\right]$$

$$= x - \frac{2x}{(x^2+1)} + \frac{x}{(x^2+1)^2}$$