Exercise 2.4

Q. 1: If α, β are the roots of the equation $x^2 + px + q = 0$, then evaluate.

(i) $\alpha^2 + \beta^2$

sum of roots $= \alpha + \beta \quad = -\frac{b}{a} \quad = -\frac{p}{1} \quad = -p$

Product of roots $= \alpha\beta \quad = \frac{c}{a} \quad = \frac{q}{1} \quad = q$

So,

$\alpha^2 + \beta^2$

adding and subtracting $2\alpha\beta$

$= \alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta$

$= (-p)^2 - 2(q)$

$= p^2 - 2q$

(ii) $\alpha^3\beta + \alpha\beta^3$

sum of roots $= \alpha + \beta \quad = -\frac{b}{a} \quad = -\frac{p}{1} \quad = -p$

Product of roots $= \alpha\beta \quad = \frac{c}{a} \quad = \frac{q}{1} \quad = q$

So,

$\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$

adding and subtracting $2\alpha\beta$

$= \alpha\beta(\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta)$

$= \alpha\beta[(\alpha + \beta)^2 - 2\alpha\beta]$

$= q[(-p)^2 - 2(q)]$

$= q[p^2 - 2q]$

$= qp^2 - 2q^2$

(iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

sum of roots $= \alpha + \beta \quad = -\frac{b}{a} \quad = -\frac{p}{1} \quad = -p$

Product of roots $= \alpha\beta \quad = \frac{c}{a} \quad = \frac{q}{1} \quad = q$

So,

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta}$

adding and subtracting $2\alpha\beta$

$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{\alpha\beta}$

$= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$= \frac{(-p)^2 - 2(q)}{q}$

$= \frac{p^2}{q} - \frac{2q}{q}$

$= \frac{p^2}{q} - 2$

Q. 2: If α, β are the roots of the equation $4x^2 - 5x + 6 = 0$, then find the values of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$

sum of roots $= \alpha + \beta = -\frac{b}{a} = -\frac{-5}{4} = \frac{5}{4}$

Product of roots $= \alpha\beta = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$

So,

$$\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{5/4}{3/2} = \frac{5}{4} \times \frac{2}{3} = \frac{5}{6}$$

(ii) $\alpha^2\beta^2$

sum of roots $= \alpha + \beta = -\frac{b}{a} = -\frac{-5}{4} = \frac{5}{4}$

Product of roots $= \alpha\beta = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$

So,

$$\alpha^2\beta^2 = (\alpha\beta)^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$$

(iii) $\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2}$

sum of roots $= \alpha + \beta = -\frac{b}{a} = -\frac{-5}{4} = \frac{5}{4}$

Product of roots $= \alpha\beta = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$

So,

$$\frac{1}{\alpha^2\beta} + \frac{1}{\alpha\beta^2} = \frac{\alpha + \beta}{\alpha^2\beta^2} = \frac{\alpha + \beta}{(\alpha\beta)^2} = \frac{5/4}{(3/2)^2} = \frac{5/4}{9/4} = \frac{5}{4} \times \frac{4}{9} = \frac{5}{9}$$

(iv) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$

sum of roots $= \alpha + \beta = -\frac{b}{a} = -\frac{-5}{4} = \frac{5}{4}$

Product of roots $= \alpha\beta = \frac{c}{a} = \frac{6}{4} = \frac{3}{2}$

So,

$$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha} = \frac{\alpha^3+\beta^3}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)(\alpha^2-\alpha\beta+\beta^2)}{(\alpha\beta)^2}$$

adding and subtracting $2\alpha\beta$

$$= \frac{(\alpha+\beta)(\alpha^2-\alpha\beta-2\alpha\beta+2\alpha\beta+\beta^2)}{(\alpha\beta)^2}$$

$$= \frac{(\alpha+\beta)(\alpha^2+2\alpha\beta+\beta^2-3\alpha\beta)}{(\alpha\beta)^2}$$

$$= \frac{(\alpha+\beta)((\alpha+\beta)^2-3\alpha\beta)}{(\alpha\beta)^2}$$

$$= \frac{(5/4)\left((5/4)^2-3(3/2)\right)}{(3/2)^2}$$

$$= \frac{(5/4)(25/16-9/2)}{9/4}$$

$$= \frac{(5/4)(25-72/16)}{9/4}$$

$$= \frac{(5/4)(-47/16)}{9/4}$$

$$= \frac{(-235/64)}{9/4}$$

$$= \frac{-235}{64}\times\frac{4}{9}$$

$$= \frac{-235}{144}$$

Q. 3: If α, β are the roots of the equation $lx^2+mx+n=0$, then find the values of

(i) $\alpha^3\beta^2+\alpha^2\beta^3$

sum of roots $=\alpha+\beta$ $=-\frac{b}{a}$ $=-\frac{m}{l}$ $=-\frac{m}{l}$

Product of roots $=\alpha\beta$ $=\frac{c}{a}$ $=\frac{n}{l}$ $=\frac{n}{l}$

So,

$$\alpha^3\beta^2+\alpha^2\beta^3 = \alpha^2\beta^2(\alpha+\beta)$$

$$= (\alpha\beta)^2(\alpha+\beta)$$

$$= \left(\frac{n}{l}\right)^2\left(-\frac{m}{l}\right)$$

$$= \frac{n^2}{l^2}\times\left(-\frac{m}{l}\right)$$

$$= -\frac{n^2m}{l^3}$$

(ii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$

sum of roots $=\alpha+\beta$ $=-\frac{b}{a}$ $=-\frac{m}{l}$ $=-\frac{m}{l}$

Product of roots $=\alpha\beta$ $=\frac{c}{a}$ $=\frac{n}{l}$ $=\frac{n}{l}$

So,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2} = \frac{\alpha^2+\beta^2}{\alpha^2\beta^2}$$

Adding and subtracting $2\alpha\beta$

$$= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$= \frac{\left(-m/_l\right)^2 - 2\left(n/_l\right)}{\left(n/_l\right)^2}$$

$$= \frac{m^2/_{l^2} - 2n/_l}{n^2/_{l^2}}$$

$$= \frac{m^2 - 2nl/_{l^2}}{n^2/_{l^2}}$$

$$= \frac{m^2 - 2nl}{l^2} \times \frac{l^2}{n^2}$$

$$= \frac{m^2 - 2nl}{n^2}$$