Exercise 3.4

Q. 1: Prove that a : b = c : d, if

(i) $\frac{4a+5b}{4a-5b} = \frac{4c+5d}{4c-5d}$

By componendo-dividendo

$$\frac{(4a+5b)+(4a-5b)}{(4a+5b)-(4a-5b)} = \frac{(4c+5d)+(4c-5d)}{(4c+5d)-(4c-5d)}$$

$$\frac{4a+5b+4a-5b}{4a+5b-4a+5b} = \frac{4c+5d+4c-5d}{4c+5d-4c+5d}$$

$$\frac{8a}{10b} = \frac{8c}{10d}$$

Multiplying by $\frac{10}{8}$

$$\frac{a}{b} = \frac{c}{d}$$

$$a : b = c : d$$

(ii) $\frac{2a+9b}{2a-9b} = \frac{2c+9d}{2c-9d}$

By componendo-dividendo

$$\frac{(2a+9b)+(2a-9b)}{(2a+9b)-(2a-9b)} = \frac{(2c+9d)+(2c-9d)}{(2c+9d)-(2c-9d)}$$

$$\frac{2a+9b+2a-9b}{2a+9b-2a+9b} = \frac{2c+9d+2c-9d}{2c+9d-2c+9d}$$

$$\frac{4a}{18b} = \frac{4c}{18d}$$

Multiplying by $\frac{18}{4}$

$$\frac{a}{b} = \frac{c}{d}$$

$$a : b = c : d$$

(iii) $\frac{ac^2+bd^2}{ac^2-bd^2} = \frac{c^3+d^3}{c^3-d^3}$

By componendo-dividendo

$$\frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)} = \frac{(c^3+d^3)+(c^3-d^3)}{(c^3+d^3)-(c^3-d^3)}$$

$$\frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2} = \frac{c^3+d^3+c^3-d^3}{c^3+d^3-c^3+d^3}$$

$$\frac{2ac^2}{2bd^2} = \frac{2c^3}{2d^3}$$

$$\frac{ac^2}{bd^2} = \frac{c^3}{d^3}$$

Multiplying by $\frac{d^2}{c^2}$

$$\frac{a}{b} = \frac{c}{d}$$

$$a : b = c : d$$

(iv) $\frac{a^2c+b^2d}{a^2c-b^2d} = \frac{ac^2+bd^2}{ac^2-bd^2}$

By componendo-dividendo

$$\frac{(a^2c+b^2d)+(a^2c-b^2d)}{(a^2c+b^2d)-(a^2c-b^2d)} = \frac{(ac^2+bd^2)+(ac^2-bd^2)}{(ac^2+bd^2)-(ac^2-bd^2)}$$

$$\frac{a^2c+b^2d+a^2c-b^2d}{a^2c+b^2d-a^2c+b^2d} = \frac{ac^2+bd^2+ac^2-bd^2}{ac^2+bd^2-ac^2+bd^2}$$

$$\frac{2a^2c}{2b^2d} = \frac{2ac^2}{2bd^2}$$
$$\frac{a^2c}{b^2d} = \frac{ac^2}{bd^2}$$

Multiplying by $\frac{bd}{ac}$

$$\frac{a}{b} = \frac{c}{d}$$
$$a : b = c : d$$

(v) $\frac{pa+qb}{pa-qb} = \frac{pc+qd}{pc-qd}$

By componendo-dividendo

$$\frac{(pa+qb)+(pa-qb)}{(pa+qb)-(pa-qb)} = \frac{(pc+qd)+(pc-qd)}{(pc+qd)-(pc-qd)}$$
$$\frac{pa+qb+pa-qb}{pa+qb-pa+qb} = \frac{pc+qd+pc-qd}{pc+qd-pc+qd}$$
$$\frac{2pa}{2qb} = \frac{2pc}{2qd}$$
$$\frac{pa}{qb} = \frac{pc}{qd}$$

Multiplying by $\frac{q}{p}$

$$\frac{a}{b} = \frac{c}{d}$$
$$a : b = c : d$$

(vi) $\frac{a+b+c+d}{a+b-c-d} = \frac{a-b+c-d}{a-b-c+d}$

By componendo-dividendo

$$\frac{(a+b+c+d)+(a+b-c-d)}{(a+b+c+d)-(a+b-c-d)} = \frac{(a-b+c-d)+(a-b-c+d)}{(a-b+c-d)-(a-b-c+d)}$$
$$\frac{a+b+c+d+a+b-c-d}{a+b+c+d-a-b+c+d} = \frac{a-b+c-d+a-b-c+d}{a-b+c-d-a+b+c-d}$$
$$\frac{2a+2b}{2c+2d} = \frac{2a-2b}{2c-2d}$$
$$\frac{a+b}{c+d} = \frac{a-b}{c-d}$$
$$\frac{a+b}{a-b} = \frac{c+d}{c-d}$$

By componendo-dividendo

$$\frac{(a+b)+(a-b)}{(a+b)-(a-b)} = \frac{(c+d)+(c-d)}{(c+d)-(c-d)}$$
$$\frac{a+b+a-b}{a+b-a+b} = \frac{c+d+c-d}{c+d-c+d}$$
$$\frac{2a}{2b} = \frac{2c}{2d}$$
$$\frac{a}{b} = \frac{c}{d}$$
$$a : b = c : d$$

(vii) $\frac{2a+3b+2c+3d}{2a+3b-2c-3d} = \frac{2a-3b+2c-3d}{2a-3b-2c+3d}$

By componendo-dividendo

$$\frac{(2a+3b+2c+3d)+(2a+3b-2c-3d)}{(2a+3b+2c+3d)-(2a+3b-2c-3d)} = \frac{(2a-3b+2c-3d)+(2a-3b-2c+3d)}{(2a-3b+2c-3d)-(2a-3b-2c+3d)}$$
$$\frac{2a+3b+2c+3d+2a+3b-2c-3d}{2a+3b+2c+3d-2a-3b+2c+3d} = \frac{2a-3b+2c-3d+2a-3b-2c+3d}{2a-3b+2c-3d-2a+3b+2c-3d}$$

$$\frac{4a+6b}{4c+6d} = \frac{4a-6b}{4c-6d}$$

$$\frac{4a+6b}{4a-6b} = \frac{4c+6d}{4c-6d}$$

By componendo-dividendo

$$\frac{(4a+6b)+(4a-6b)}{(4a+6b)-(4a-6b)} = \frac{(4c+6d)+(4c-6d)}{(4c+6d)-(4c-6d)}$$

$$\frac{4a+6b+4a-6b}{4a+6b-4a+6b} = \frac{4c+6d+4c-6d}{4c+6d-4c+6d}$$

$$\frac{8a}{12b} = \frac{8c}{12d}$$

$$\frac{2a}{3b} = \frac{2c}{3d}$$

Multiplying by $\frac{3}{2}$

$$\frac{a}{b} = \frac{c}{d}$$

$$a : b = c : d$$

(viii) $\frac{a^2+b^2}{a^2-b^2} = \frac{ac+bd}{ac-bd}$

By componendo-dividendo

$$\frac{(a^2+b^2)+(a^2-b^2)}{(a^2+b^2)-(a^2-b^2)} = \frac{(ac+bd)+(ac-bd)}{(ac+bd)-(ac-bd)}$$

$$\frac{a^2+b^2+a^2-b^2}{a^2+b^2-a^2+b^2} = \frac{ac+bd+ac-bd}{ac+bd-ac+bd}$$

$$\frac{2a^2}{2b^2} = \frac{2ac}{2bd}$$

$$\frac{a^2}{b^2} = \frac{ac}{bd}$$

Multiplying by $\frac{b}{a}$

$$\frac{a}{b} = \frac{c}{d}$$

$$a : b = c : d$$

Q. 2: Using theorem of componendo-dividendo

(i) Find the value of $\frac{x+2y}{x-2y} + \frac{x+2z}{x-2z}$, if $x = \frac{4yz}{y+z}$

$$x = \frac{4yz}{y+z} \text{ ------------ (i)}$$

From equation (i)

$$x = \frac{2y \times 2z}{y+z}$$

$$\frac{x}{2y} = \frac{2z}{y+z}$$

By applying componendo-dividendo theorem

$$\frac{x+2y}{x-2y} = \frac{2z+y+z}{2z-y-z}$$

$$\frac{x+2y}{x-2y} = \frac{y+3z}{z-y} \text{ ----------- (ii)}$$

From equation (i)

$$x = \frac{2y \times 2z}{y+z}$$

$$\frac{x}{2z} = \frac{2y}{y+z}$$

By applying componendo-dividendo theorem

$$\frac{x+2z}{x-2z} = \frac{2y+y+z}{2y-y-z}$$

$$\frac{x+2z}{x-2z} = \frac{z+3y}{y-z} \text{ ----------- (iii)}$$

Adding equation (ii) and (iii)

$$\frac{x+2y}{x-2y} + \frac{x+2z}{x-2z} = \frac{y+3z}{z-y} + \frac{z+3y}{y-z}$$

$$= -\frac{y+3z}{y-z} + \frac{z+3y}{y-z}$$

$$= \frac{z+3y}{y-z} - \frac{y+3z}{y-z}$$

$$= \frac{z+3y-y-3z}{y-z}$$

$$= \frac{2y-2z}{y-z}$$

$$= \frac{2(y-z)}{y-z}$$

$$= 2$$

(ii) Find the value of $\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p}$, if $m = \frac{10np}{n+p}$

$$m = \frac{10np}{n+p} \text{ ------------ (i)}$$

From equation (i)

$$m = \frac{5n \times 2p}{n+p}$$

$$\frac{m}{5n} = \frac{2p}{n+p}$$

By applying componendo-dividendo theorem

$$\frac{m+5n}{m-5n} = \frac{2p+n+p}{2p-n-p}$$

$$\frac{m+5n}{m-5n} = \frac{3p+n}{p-n} \text{ ----------- (ii)}$$

From equation (i)

$$m = \frac{2n \times 5p}{n+p}$$

$$\frac{m}{5p} = \frac{2n}{n+p}$$

By applying componendo-dividendo theorem

$$\frac{m+5p}{m-5p} = \frac{2n+n+p}{2n-n-p}$$

$$\frac{m+5p}{m-5p} = \frac{3n+p}{n-p} \text{ ----------- (iii)}$$

Adding equation (ii) and (iii)

$$\frac{m+5n}{m-5n} + \frac{m+5p}{m-5p} = \frac{3p+n}{p-n} + \frac{3n+p}{n-p}$$

$$= -\frac{3p+n}{n-p} + \frac{3n+p}{n-p}$$

$$= \frac{3n+p}{n-p} - \frac{3p+n}{n-p}$$

$$= \frac{3n+p-3p-n}{n-p}$$

$$= \frac{2n-2p}{n-p}$$

$= \frac{2(n-p)}{n-p}$

$= 2$

(iii) Find the value of $\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b}$, if $x = \frac{12ab}{a-b}$

$x = \frac{12ab}{a-b}$ ------------ (i)

From equation (i)

$x = \frac{6a \times 2b}{a-b}$

$\frac{x}{6a} = \frac{2b}{a-b}$

By applying componendo-dividendo theorem

$\frac{x+6a}{x-6a} = \frac{2b+a-b}{2b-a+b}$

$\frac{x+6a}{x-6a} = \frac{a+b}{3b-a}$

$\frac{x-6a}{x+6a} = \frac{3b-a}{a+b}$ ----------- (ii)

From equation (i)

$x = \frac{6b \times 2a}{a-b}$

$\frac{x}{6b} = \frac{2a}{a-b}$

By applying componendo-dividendo theorem

$\frac{x+6b}{x-6b} = \frac{2a+a-b}{2a-a+b}$

$\frac{x+6b}{x-6b} = \frac{3a-b}{a+b}$

$\frac{x+6b}{x-6b} = \frac{3a-b}{a+b}$ ----------- (iii)

Subtracting equation (iii) from (ii)

$\frac{x-6a}{x+6a} - \frac{x+6b}{x-6b} = \frac{3b-a}{a+b} - \frac{3a-b}{a+b}$

$= \frac{3b-a}{a+b} - \frac{3a-b}{a+b}$

$= \frac{-4a+4b}{a+b}$

$= \frac{4(b-a)}{a+b}$

(iv) Find the value of $\frac{x-3y}{x+3y} - \frac{x+3z}{x-3z}$, if $x = \frac{3yz}{y-z}$

$x = \frac{3yz}{x-y}$ ------------ (i)

From equation (i)

$x = \frac{3y \times z}{y-z}$

$\frac{x}{3y} = \frac{z}{y-z}$

By applying componendo-dividendo theorem

$\frac{x+3y}{x-3y} = \frac{z+y-z}{z-y+z}$

$\frac{x+3y}{x-3y} = \frac{y}{2z-y}$

$\frac{x-3y}{x+3y} = \frac{2z-y}{y}$ ----------- (ii)

From equation (i)

$$x = \frac{3z \times y}{y-z}$$

$$\frac{x}{3z} = \frac{y}{y-z}$$

By applying componendo-dividendo theorem

$$\frac{x+3z}{x-3z} = \frac{y+y-z}{y-y+z}$$

$$\frac{x+3z}{x-3z} = \frac{2y-z}{z}$$

$$\frac{x+3z}{x-3z} = \frac{2y-z}{z} \text{ ----------- (iii)}$$

Subtracting equation (iii) from (ii)

$$\frac{x-3y}{x+3y} - \frac{x+3z}{x-3z} = \frac{2z-y}{y} - \frac{2y-z}{z}$$

$$= \frac{z(2z-y)-y(2y-z)}{yz}$$

$$= \frac{2z^2-yz-2y^2+yz}{yz}$$

$$= \frac{2\left(z^2-y^2\right)}{yz}$$

(v) Find the value of $\frac{s-3p}{s+3p} + \frac{s+3q}{s-3q}$, if $s = \frac{6pq}{p-q}$

$$s = \frac{6pq}{p-q} \text{ ------------ (i)}$$

From equation (i)

$$s = \frac{3p \times 2q}{p-q}$$

$$\frac{s}{3p} = \frac{2q}{p-q}$$

By applying componendo-dividendo theorem

$$\frac{s+3p}{s-3p} = \frac{2q+p-q}{2q-p+q}$$

$$\frac{s+3p}{s-3p} = \frac{p+q}{3q-p}$$

$$\frac{s-3p}{s+3p} = \frac{3q-p}{p+q} \text{ ----------- (ii)}$$

From equation (i)

$$s = \frac{2p \times 3q}{p-q}$$

$$\frac{s}{3q} = \frac{2p}{p-q}$$

By applying componendo-dividendo theorem

$$\frac{s+3q}{s-3q} = \frac{2p+p-q}{2p-p+q}$$

$$\frac{s+3q}{s-3q} = \frac{3p-q}{p+q}$$

$$\frac{s+3q}{s-3q} = \frac{3p-q}{p+q} \text{ ----------- (iii)}$$

Adding equation (ii) and (iii)

$\frac{s-3p}{s+3p}+\frac{s+3q}{s-3q} = \frac{3q-p}{p+q}+\frac{3p-q}{p+q}$

$= \frac{3q-p+3p-q}{p+q}$

$= \frac{2p+2q}{p+q}$

$= \frac{2(p+q)}{p+q}$

$= 2$

(vi) Solve $\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2} = \frac{12}{13}$

$\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2} = \frac{12}{13}$

$\frac{(x-2)^2-(x-4)^2}{(x-2)^2+(x-4)^2} = \frac{12}{13}$

By componendo-dividendo theorem

$\frac{(x-2)^2-(x-4)^2+(x-2)^2+(x-4)^2}{(x-2)^2-(x-4)^2-(x-2)^2-(x-4)^2} = \frac{12+13}{12-13}$

$\frac{2(x-2)^2}{-2(x-4)^2} = \frac{25}{-1}$

$\frac{(x-2)^2}{(x-4)^2} = 25$

Taking root on b.s

$\frac{x-2}{x-4} = \pm 5$

$\frac{x-2}{x-4} = 5$; $\frac{x-2}{x-4} = -5$

$x-2 = 5(x-4)$; $x-2 = -5(x-4)$

$x-2 = 5x-20$; $x-2 = -5x+20$

$x-5x = 2-20$; $x+5x = 2+20$

$-4x = -18$; $6x = 22$

$x = \frac{-18}{-4}$; $x = \frac{22}{6}$

$x = \frac{9}{2}$; $x = \frac{11}{3}$

S.S = $\left\{\frac{9}{2}, \frac{11}{3}\right\}$

(vii) Solve $\frac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}} = 2$

$\frac{\sqrt{x^2+2}+\sqrt{x^2-2}}{\sqrt{x^2+2}-\sqrt{x^2-2}} = 2$

By componendo-dividendo theorem

$\frac{\sqrt{x^2+2}+\sqrt{x^2-2}+\sqrt{x^2+2}-\sqrt{x^2-2}}{\sqrt{x^2+2}+\sqrt{x^2-2}-\sqrt{x^2+2}+\sqrt{x^2-2}} = \frac{2+1}{2-1}$

$\frac{2\sqrt{x^2+2}}{2\sqrt{x^2-2}} = \frac{3}{1}$

$\frac{\sqrt{x^2+2}}{\sqrt{x^2-2}} = 3$

Taking square on b.s

$\frac{x^2+2}{x^2-2} = 9$

$x^2+2 = 9(x^2-2)$

$x^2 + 2 \quad = 9x^2 - 18$

$x^2 - 9x^2 \quad = -18 - 2$

$-8x^2 \quad = -20$

$x^2 \quad = \frac{-20}{-8}$

$x^2 \quad = \frac{5}{2}$

$x \quad = \pm\sqrt{\frac{5}{2}}$

If we check the given equation for this value the value doesn't satisfy the equation so the given solution is extraneous. So, s.s = { }

(viii) Solve $\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}} = \frac{1}{3}$

$\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}} \quad = \frac{1}{3}$

By componendo-dividendo theorem

$\frac{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}+\sqrt{x^2+8p^2}+\sqrt{x^2-p^2}}{\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}-\sqrt{x^2+8p^2}-\sqrt{x^2-p^2}} = \frac{1+3}{1-3}$

$\frac{2\sqrt{x^2+8p^2}}{-2\sqrt{x^2-p^2}} \quad = \frac{4}{-2}$

$\frac{\sqrt{x^2+8p^2}}{\sqrt{x^2-p^2}} \quad = 2$

Taking square on b.s

$\frac{x^2+8p^2}{x^2-p^2} \quad = 4$

$x^2 + 8p^2 \quad = 4(x^2 - p^2)$

$x^2 + 8p^2 \quad = 4x^2 - 4p^2$

$x^2 - 4x^2 \quad = -4p^2 - 8p^2$

$-3x^2 \quad = -12p^2$

$x^2 \quad = 4p^2$

$x \quad = \pm 2p$

S.S = $\{2p, -2p\}$

(ix) Solve $\frac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3} = \frac{13}{14}$

$\frac{(x+5)^3-(x-3)^3}{(x+5)^3+(x-3)^3} \quad = \frac{13}{14}$

By componendo-dividendo theorem

$\frac{(x+5)^3-(x-3)^3+(x+5)^3+(x-3)^3}{(x+5)^3-(x-3)^3-(x+5)^3-(x-3)^3} \quad = \frac{13+14}{13-14}$

$\frac{2(x+5)^3}{-2(x-3)^3} \quad = \frac{27}{-1}$

$\frac{(x+5)^3}{(x-3)^3} \quad = 27$

Taking cube root on b.s

$\frac{x+5}{x-3} \quad = 3$

$x + 5 \quad = 3(x - 3)$

$$
\begin{aligned}
x + 5 &= 3x - 9 \\
x - 3x &= -9 - 5 \\
-2x &= -14 \\
x &= 7
\end{aligned}
$$

S.S = {7}