Exercise 2.6

Q. 1: Identify the following statements as true or false.

(i) $\sqrt{-3}\sqrt{-3} = 3$ False

(ii) $i^{73} = -i$ False

(iii) $i^{10} = -1$ True

(iv) Complex conjugate of $(-6i + i^2)$ is $(-1 + 6i)$ True

(v) Difference of a complex number $z = a + bi$ and its conjugate is a real number. False

(vi) If $(a-1) - (b+3)i = 5 + 8i$, then $a = 6$ and $b = -11$ True

(vii) Product of a complex number and its conjugate is always a none-negative real number. True

Q. 2: Express each complex number in the standard form $a + bi$, where a and b are real numbers.

(i) $(2+3i) + (7-2i)$
$= 2 + 3i + 7 - 2i$
$= 2 + 7 + 3i - 2i$
$= 9 + i$

(ii) $2(5+4i) - 3(7+4i)$
$= 10 + 8i - 21 - 12i$
$= 10 - 21 + 8i - 12i$
$= -11 - 4i$

(iii) $-(-3+5i) - (4+9i)$
$= 3 - 5i - 4 - 9i$
$= 3 - 4 - 5i - 9i$
$= -1 - 14i$

(iv) $2i^2 + 6i^3 + 3i^{16} - 6i^{19} + 4i^{25} = 2(i^2) + 6(i^2).i + 3(i^2)^8 - 6(i^2)^9.i + 4(i^2)^{12}.i$
$= 2(-1) + 6(-1).i + 3(-1)^8 - 6(-1)^9.i + 4(-1)^{12}.i$
$= -2 - 6i + 3 + 6i + 4i$
$= -2 + 3 + 4i$
$= 1 + 4i$

Q. 3: Simplify and write your answer in the form $a + bi$.

(i) $(-7+3i)(-3+2i)$
$= 21 - 14i - 9i + 6i^2$
$= 21 - 23i + 6(-1)$
$= 21 - 6 - 23i$
$= 15 - 23i$

(ii) $(2-\sqrt{-4})(3-\sqrt{-4})$
$= (2-\sqrt{4}i)(3-\sqrt{4}i)$
$= 6 - 2\sqrt{4}i - 3\sqrt{4}i + (\sqrt{4})^2 i^2$
$= 6 - 5\sqrt{4}i + 4i^2$
$= 6 - 5\sqrt{4}i + 4(-1)$
$= 6 - 5\sqrt{4}i - 4$
$= 2 - 10i$

(iii) $(\sqrt{5} - 3i)^2$
$= (\sqrt{5})^2 - 2(\sqrt{5})(3i) + (3i)^2$
$= 5 - 6\sqrt{5}i + 9i^2$
$= 5 - 6\sqrt{5}i + 9(-1)$
$= 5 - 6\sqrt{5}i - 9$

$= -4 - 6\sqrt{5}i$

(iv) $(2-3i)(\overline{3-2i})$

$= (2-3i)(3+2i)$

$= 6 + 4i - 9i - 6i^2$

$= 6 - 5i - 6(-1)$

$= 6 + 6 - 5i$

$= 12 - 5i$

Q. 4: Simplify and write your answer in the form $a + bi$.

(i) $\frac{-2}{1+i}$

multiply and divide by $1 - i$

$= \frac{-2}{1+i} \times \frac{1-i}{1-i}$

$= \frac{-2(1-i)}{(1+i)(1-i)}$

$= \frac{-2+2i}{1-i^2}$

$= \frac{-2+2i}{1-(-1)}$

$= \frac{-2+2i}{2}$

$= \frac{-2}{2} + \frac{2}{2}i$

$= -1 + i$

(ii) $\frac{2+3i}{4-i}$

multiply and divide by $4 + i$

$= \frac{2+3i}{4-i} \times \frac{4+i}{4+i}$

$= \frac{(2+3i)(4+i)}{(4-i)(4+i)}$

$= \frac{8+2i+12i+3i^2}{16-i^2}$

$= \frac{8+14i+3(-1)}{16-(-1)}$

$= \frac{5+14i}{17}$

$= \frac{5}{17} + \frac{14}{17}i$

$= \frac{5}{17} + \frac{14}{17}i$

(iii) $\frac{9-7i}{3+i}$

multiply and divide by $3 - i$

$= \frac{9-7i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{(9-7i)(3-i)}{(3+i)(3-i)}$

$= \frac{27-9i-21i+7i^2}{9-i^2}$

$= \frac{27-30i+7(-1)}{9-(-1)}$

$= \frac{20-30i}{10}$

$= \frac{20}{10} - \frac{30}{10}i$

$= 2 - 3i$

(iv) $\frac{2-6i}{3+i} - \frac{4+i}{3+i}$

$= \frac{(2-6i)-(4+i)}{3+i}$

$= \frac{2-6i-4-i}{3+i}$

$= \frac{-2-7i}{3+i}$

multiply and divide by $3 - i$

$= \frac{-2-7i}{3+i} \times \frac{3-i}{3-i}$

$= \frac{(-2-7i)(3-i)}{(3+i)(3-i)}$

$= \frac{-6+2i-21i+7i^2}{9-i^2}$

$= \frac{-6-19i+7(-1)}{9-(-1)}$

$= \frac{-13-19i}{10}$

$= \frac{-13}{10} + \frac{-19}{10}i$

(v) $\left(\frac{1+i}{1-i}\right)^2$

$= \left(\frac{1+2i+i^2}{1-2i+i^2}\right)$

$= \frac{1+2i+(-1)}{1-2i+(-1)}$

$= \frac{2i}{-2i}$

= -1

$= -1 + 0i$

(vi) $\frac{1}{(2+3i)(1-i)}$

$= \frac{1}{2-2i+3i-3i^2}$

$= \frac{1}{2+i-3i^2}$

$= \frac{1}{2+i-3(-1)}$

$= \frac{1}{5+i}$

multiply and divide by $5 + i$

$= \frac{1}{5+i} \times \frac{5-i}{5-i}$

$= \frac{5-i}{(5+i)(5-i)}$

$= \frac{5-i}{25-i^2}$

$= \frac{5-i}{25-(-1)}$

$= \frac{5-i}{26}$

$= \frac{5}{26} - \frac{1}{26}i$

Q. 5: Calculate (a) $\bar{z}$ (b) $z + \bar{z}$ (c) $z - \bar{z}$ (d) $z\bar{z}$.

(i) $z = -i$

(a) $\bar{z} = \overline{-i}$

$\bar{z} = i$

(b) $z + \bar{z} = -i + \overline{-i}$

$z + \bar{z} = -i + i$

$z + \bar{z} = 0$

(c) $z - \bar{z} = -i - \overline{-i}$

$z - \bar{z} = -i - i$

$z - \bar{z} = -2i$

(d) $z\bar{z} = (-i)(\overline{-i})$

$z\bar{z} = (-i)(i)$

$z\bar{z} = -i^2$

$z\bar{z} = -(-1)$

$z\bar{z} = 1$

(ii) $z = 2 + i$

(a) $\bar{z} = \overline{2 + i}$

$\bar{z} = \bar{2} + \bar{i}$

$\bar{z} = 2 - i$

(b) $z + \bar{z} = 2 + i + \overline{2 + i}$

$z + \bar{z} = 2 + i + \bar{2} + \bar{i}$

$z + \bar{z} = 2 + i + 2 - i$

$z + \bar{z} = 4$

(c) $z - \bar{z} = 2 + i - (\overline{2 + i})$

$z - \bar{z} = 2 + i - (\bar{2} + \bar{i})$

$z - \bar{z} = 2 + i - (2 - i)$

$z - \bar{z} = 2 + i - 2 + i$

$z - \bar{z} = 2i$

(d) $z\bar{z} = (2 + i)(\overline{2 + i})$

$z\bar{z} = (2 + i)(\bar{2} + \bar{i})$

$z\bar{z} = (2 + i)(2 - i)$

$z\bar{z} = 4 - i^2$

$z\bar{z} = 4 - (-1)$

$z\bar{z} = 4 + 1$

$z\bar{z} = 5$

(iii) $z = \frac{1+i}{1-i}$

$z = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$z = \frac{(1+i)(1+i)}{(1-i)(1+i)}$

$z = \frac{1+2i+i^2}{1-i^2}$

$z = \frac{1+2i+(-1)}{1-(-1)}$

$z = \frac{1+2i-1}{1+1}$

$z = \frac{2i}{2}$

$z = i$

(a) $\bar{z} = \bar{i}$

$\bar{z} = \bar{i}$

$\bar{z} = -i$

(b) $z + \bar{z} = i + \bar{i}$

$z + \bar{z} = i - i$

$z + \bar{z} = 0$

(c) $z - \bar{z} = i - (\bar{i})$

$z - \bar{z} = i - (-i)$

$z - \bar{z} = i + i$

$z - \bar{z} = 2i$

(d) $z\bar{z} = (i)(\bar{i})$

$z\bar{z} = (i)(-i)$

$z\bar{z} = -i^2$

$z\bar{z} = -(-1)$

$z\bar{z} = 1$

(iv) $z = \frac{4-3i}{2+4i}$

$z = \frac{4-3i}{2+4i} \times \frac{2-4i}{2-4i}$

$z = \frac{(4-3i)(2-4i)}{(2+4i)(2-4i)}$

$z = \frac{8-16i-6i+12i^2}{4-16i^2}$

$z = \frac{8-22i+12i^2}{4-16i^2}$

$z = \frac{8-22i+12(-1)}{4-16(-1)}$

$z = \frac{8-22i-12}{4+16}$

$z = \frac{-4-22i}{20}$

$z = -\frac{4}{20} - \frac{22}{20}i$

$z = -\frac{1}{5} - \frac{11}{10}i$

(a) $\bar{z} = \overline{-\frac{1}{5} - \frac{11}{10}i}$

$\bar{z} = \overline{-\frac{1}{5}} + \overline{-\frac{11}{10}i}$

$\bar{z} = -\frac{1}{5} + \frac{11}{10}i$

(b) $z + \bar{z} = -\frac{1}{5} - \frac{11}{10}i + \overline{-\frac{1}{5} - \frac{11}{10}i}$

$$z+\bar{z}=-\frac{1}{5}-\frac{11}{10}i+\overline{-\frac{1}{5}}+\overline{-\frac{11}{10}\imath}$$
$$z+\bar{z}=-\frac{1}{5}-\frac{11}{10}i-\frac{1}{5}+\frac{11}{10}i$$
$$z+\bar{z}=-\frac{2}{5}$$

(c)
$$z-\bar{z}=-\frac{1}{5}-\frac{11}{10}i-\left(\overline{-\frac{1}{5}-\frac{11}{10}\imath}\right)$$
$$z-\bar{z}=-\frac{1}{5}-\frac{11}{10}i-\left(\overline{-\frac{1}{5}}+\overline{-\frac{11}{10}\imath}\right)$$
$$z-\bar{z}=-\frac{1}{5}-\frac{11}{10}i-\left(-\frac{1}{5}+\frac{11}{10}i\right)$$
$$z-\bar{z}=-\frac{1}{5}-\frac{11}{10}i+\frac{1}{5}-\frac{11}{10}i$$
$$z-\bar{z}=-\frac{22}{10}i$$
$$z-\bar{z}=-\frac{11}{5}i$$

(d)
$$z\bar{z}=\left(-\frac{1}{5}-\frac{11}{10}i\right)\left(\overline{-\frac{1}{5}-\frac{11}{10}\imath}\right)$$
$$z\bar{z}=\left(-\frac{1}{5}-\frac{11}{10}i\right)\left(\overline{-\frac{1}{5}}+\overline{-\frac{11}{10}\imath}\right)$$
$$z\bar{z}=\left(-\frac{1}{5}-\frac{11}{10}i\right)\left(-\frac{1}{5}+\frac{11}{10}i\right)$$
$$z\bar{z}=\left(-\frac{1}{5}\right)^2-\left(\frac{11}{10}i\right)^2$$
$$z\bar{z}=\frac{1}{25}-\frac{121}{100}i^2$$
$$z\bar{z}=\frac{1}{25}-\frac{121}{100}(-1)$$
$$z\bar{z}=\frac{1}{25}+\frac{121}{100}$$
$$z\bar{z}=\frac{4+121}{100}$$
$$z\bar{z}=\frac{125}{100}$$
$$z\bar{z}=\frac{5}{4}$$

Q. 6: If $z=2+3i$ and $w=5-4i$

(i) $\overline{z+w}=\bar{z}+\bar{w}$

$$L.H.S=\overline{z+w}$$
$$=\overline{2+3\imath+5-4\imath}$$
$$=\overline{7-\imath}$$
$$=\bar{7}+\overline{-\imath}$$
$$=7+i$$
$$R.H.S=\bar{z}+\bar{w}$$
$$=\overline{2+3\imath}+\overline{5-4\imath}$$
$$=\bar{2}+\overline{3\imath}+\bar{5}+\overline{-4\imath}$$
$$=2-3i+5+4i$$
$$=7+i$$
$$L.H.S=R.H.S$$

(ii) $\overline{z-w}=\bar{z}-\bar{w}$

$$L.H.S=\overline{z-w}$$

$= \overline{(2+3\imath)-(5-4\imath)}$

$= \overline{2+3\imath-5+4\imath}$

$= \overline{-3+7\imath}$

$= \overline{-3}+\overline{7}\imath$

$= -3-7i$

$R.H.S = \bar{z}-\bar{w}$

$= \overline{(2+3\imath)}-\overline{(5-4\imath)}$

$= (\bar{2}+\bar{3}\imath)-(\bar{5}+\overline{-4\imath})$

$= (2-3\imath)-(5+4\imath)$

$= 2-3i-5-4i$

$= -3-7i$

$L.H.S = R.H.S$

(iii) $\overline{zw} = \bar{z}\bar{w}$

$L.H.S = \overline{zw}$

$= \overline{(2+3\imath)(5-4\imath)}$

$= \overline{10-8\imath+15\imath-12\imath^2}$

$= \overline{10-8\imath+15\imath-12(-1)}$

$= \overline{10-8\imath+15\imath+12}$

$= \overline{22+7\imath}$

$= \overline{22}+\overline{7}\imath$

$= 22-7i$

$R.H.S = \bar{z}\bar{w}$

$= \overline{(2+3\imath)}\overline{(5-4\imath)}$

$= (\bar{2}+\bar{3}\imath)(\bar{5}+\overline{-4\imath})$

$= (2-3\imath)(5+4\imath)$

$= 10+8i-15i-12i^2$

$= 10+8i-15i-12(-1)$

$= 10+8i-15i+12$

$= 22-7i$

$L.H.S = R.H.S$

(iv) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

$L.H.S = \overline{\left(\frac{z}{w}\right)}$

$= \overline{\left(\frac{2+3\imath}{5-4\imath}\right)}$

$= \overline{\left(\frac{2+3\imath}{5-4\imath}\times\frac{5+4\imath}{5+4\imath}\right)}$

$= \overline{\left(\frac{(2+3\imath)(5+4\imath)}{(5-4\imath)(5+4\imath)}\right)}$

$= \overline{\left(\frac{10+8\imath+15\imath+12\imath^2}{25-16(-1)}\right)}$

$= \overline{\left(\frac{10+23\imath+12(-1)}{25+16}\right)}$

$$= \overline{\left(\frac{10+23i-12}{41}\right)}$$
$$= \overline{\left(\frac{-2+23i}{41}\right)}$$
$$= \overline{\left(\frac{-2}{41}+\frac{23i}{41}\right)}$$
$$= \frac{\overline{-2}}{41}+\frac{\overline{23i}}{41}$$
$$= \frac{-2}{41}-\frac{23i}{41}$$

$$R.H.S = \frac{\bar{z}}{\bar{w}}$$
$$= \frac{\overline{2+3i}}{\overline{5-4i}}$$
$$= \frac{\bar{2}+\overline{3i}}{\bar{5}+\overline{-4i}}$$
$$= \frac{2-3i}{5+4i}$$
$$= \frac{2-3i}{5+4i}\times\frac{5-4i}{5-4i}$$
$$= \frac{(2-3i)(5-4i)}{(5+4i)(5-4i)}$$
$$= \frac{10-8i-15i+12i^2}{25-16i^2}$$
$$= \frac{10-23i+12(-1)}{25-16(-1)}$$
$$= \frac{10-23i-12}{25+16}$$
$$= \frac{-2-23i}{41}$$
$$= \frac{-2}{41}-\frac{23i}{41}$$

$$L.H.S = R.H.S$$

(v) $\frac{1}{2}(z+\bar{z})$ is the real part of z.

$$\frac{1}{2}(z+\bar{z}) = \frac{1}{2}(2+3i+\overline{2+3i})$$
$$= \frac{1}{2}(2+3i+\bar{2}+\overline{3i})$$
$$= \frac{1}{2}(2+3i+2-3i)$$
$$= \frac{1}{2}(4)$$

$= 2$ which is real part of z.

(vi) $\frac{1}{2i}(z-\bar{z})$ is the imaginary part of z.

$$\frac{1}{2i}(z-\bar{z}) = \frac{1}{2i}(2+3i-\overline{2+3i})$$
$$= \frac{1}{2i}\left(2+3i-(\bar{2}+\overline{3i})\right)$$
$$= \frac{1}{2i}\left(2+3i-(2-3i)\right)$$
$$= \frac{1}{2i}(2+3i-2+3i)$$
$$= \frac{1}{2i}(6i)$$

$= 3$ which is imaginary part of z.

Q. 7: Solve the following equations for real x and y.

(i) $(2-3i)(x+yi) = 4+i$

$$(2-3i)(x+yi) = 4+i$$
$$2x+2yi-3xi-3yi^2 = 4+i$$
$$2x+2yi-3xi-3y(-1) = 4+i$$
$$2x+2yi-3xi-3y(-1) = 4+i$$
$$2x+2yi-3xi+3y = 4+i$$
$$2x+3y+2yi-3xi = 4+i$$
$$(2x+3y)+(-3x+2y)i = 4+i$$

Comparing real and imaginary parts of the equation

$2x+3y = 4$ --------------------------- (i)

$-3x+2y = 1$ --------------------------- (ii)

multiplying equation (i) by 3 and equation (ii) by 2

$6x+9y = 12$ --------------------------- (iii)

$-6x+4y = 2$ --------------------------- (iv)

Adding equation (iii) and (iv)

$+6x+9y = 12$

$-6x+4y = 2$

$13y = 14$ So, $y = \frac{14}{13}$

Putting the value of y in equ (i)

$$2x+3\left(\frac{14}{13}\right) = 4$$
$$2x+\frac{42}{13} = 4$$
$$2x = 4-\frac{42}{13}$$
$$2x = \frac{52-42}{13}$$
$$2x = \frac{10}{13}$$
$$x = \frac{5}{13}$$

(ii) $(3-2i)(x+yi) = 2(x-2yi)+2i-1$

$$(3-2i)(x+yi) = 2(x-2yi)+2i-1$$
$$3x+3yi-2xi-2yi^2 = 2x-4yi+2i-1$$
$$3x+3yi-2xi-2y(-1) = 2x-4yi+2i-1$$
$$3x+3yi-2xi+2y = 2x-4yi+2i-1$$
$$3x+2y+3yi-2xi = 2x-1-4yi+2i$$
$$(3x+2y)+(-2x+3y)i = (2x-1)+(2-4y)i$$

Comparing real and imaginary parts of the equation

$3x+2y = 2x-1$

$3x-2x+2y = -1$

$x+2y = -1$ --------------------------- (i)

$-2x + 3y = 2 - 4y$

$-2x + 3y + 4y = 2$

$-2x + 7y = 2$ ---------------------------- (ii)

Multiplying equation (i) by 2

$2x + 4y = -2$ ---------------------------- (iii)

Adding equation (ii) and (iii)

$-2x + 7y = 2$

$2x + 4y = -2$

$11y = 0$ So, $y = 0$

Putting the value of y in equ (i)

$x + 2(0) = -1$

$x + 0 = -1$

$x = -1$

(iii) $(3 + 4i)^2 - 2(x - yi) = x + yi$

$(3 + 4i)^2 - 2(x - yi) = x + yi$

$9 + 24i + 16i^2 - 2(x - yi) = x + yi$

$9 + 24i + 16(-1) - 2(x - yi) = x + yi$

$9 + 24i - 16 - 2x + 2yi = x + yi$

$-2x - 7 + 2yi + 24i = x + yi$

$(-2x - 7) + (2y + 24)i = x + yi$

Comparing real and imaginary parts of the equation

$-2x - 7 = x$

$-2x - x = 7$

$-3x = 7$

$x = \frac{-7}{3}$

$2y + 24 = y$

$2y - y = -24$

$y = -24$