Exercise 4.2

Resolve into partial fractions.

1. $\frac{x^2-3x+1}{(x-2)(x-1)^2} = \frac{A}{(x-2)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2}$ ..... (i)

multiplying by $(x-2)(x-1)^2$ we get

$$x^2 - 3x + 1 = A(x-1)^2 + B(x-1)(x-2) + C(x-2) \text{ ..... (ii)}$$
$$x^2 - 3x + 1 = A(x^2 - 2x + 1) + B(x^2 - 2x - x + 2) + C(x-2)$$
$$x^2 - 3x + 1 = A(x^2 - 2x + 1) + B(x^2 - 3x + 2) + C(x-2) \text{ ..... (iii)}$$

put $x = 2$ in (ii)

$$(2)^2 - 3(2) + 1 = A(2-1)^2 + B(2-1)(2-2) + C(2-2)$$
$$4 - 6 + 1 = A(1)^2$$
$$-1 = A$$
$$A = -1$$

put $x = 1$ in (ii)

$$(1)^2 - 3(1) + 1 = A(1-1)^2 + B(1-1)(1-2) + C(1-2)$$
$$1 - 3 + 1 = C(-1)$$
$$-1 = C(-1)$$
$$-1 = -C$$
$$C = 1$$

Now, comparing coefficients of equation (iii)

x^2; $A + B = 1$

as A = -1

$$-1 + B = 1$$
$$B = 2$$

put the values in (i) we get

$$\frac{x^2-3x+1}{(x-2)(x-1)^2} = \frac{-1}{(x-2)} + \frac{2}{(x-1)} + \frac{1}{(x-1)^2}$$

2. $\frac{x^2+7x+11}{(x+3)(x+2)^2} = \frac{A}{(x+3)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2}$ ..... (i)

multiplying by $(x+3)(x+2)^2$ we get

$$x^2 + 7x + 11 = A(x+2)^2 + B(x+2)(x+3) + C(x+3) \text{ ..... (ii)}$$
$$x^2 + 7x + 11 = A(x^2 + 4x + 4) + B(x^2 + 3x + 2x + 6) + C(x+3)$$
$$x^2 + 7x + 11 = A(x^2 + 4x + 4) + B(x^2 + 5x + 6) + C(x+3) \text{ ..... (iii)}$$

put $x = -3$ in (ii)

$$(-3)^2 + 7(-3) + 11 = A(-3+2)^2 + B(-3+2)(-3+3) + C(-3+3)$$
$$9 - 21 + 11 = A(-1)^2$$
$$-1 = A$$
$$A = -1$$

put $x = -2$ in (ii)

$$(-2)^2 + 7(-2) + 11 = A(-2+2)^2 + B(-2+2)(-2+3) + C(-2+3)$$

$$4-14+11 = C(1)$$
$$1 = C(1)$$
$$1 = C$$
$$C = 1$$

Now, comparing coefficients of equation (iii)

x^2; $A+B = 1$

as A = -1

$-1+B = 1$

$B = 2$

put the values in (i) we get

$$\frac{x^2+7x+11}{(x+3)(x+2)^2} = \frac{-1}{(x+3)} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2}$$

3. $$\frac{9}{(x-1)(x+2)^2} = \frac{A}{(x-1)} + \frac{B}{(x+2)} + \frac{C}{(x+2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

multiplying by $(x-1)(x+2)^2$ we get

$$9 = A(x+2)^2 + B(x+2)(x-1) + C(x-1) \ldots\ldots\ldots \text{(ii)}$$
$$9 = A(x^2+4x+4) + B(x^2-x+2x-2) + C(x-1)$$
$$9 = A(x^2+4x+4) + B(x^2+x+6) + C(x-1) \ldots\ldots\ldots \text{(iii)}$$

put $x = 1$ in (ii)

$$9 = A(1+2)^2 + B(1+2)(1-1) + C(1-1)$$
$$9 = A(3)^2$$
$$9 = 9A$$
$$A = 1$$

put $x = -2$ in (ii)

$$9 = A(-2+2)^2 + B(-2+2)(2-1) + C(-2-1)$$
$$9 = C(-2-1)$$
$$9 = C(-3)$$
$$9 = -3C$$
$$C = -3$$

Now, comparing coefficients of equation (iii)

x^2; $A+B = 0$

as A = 1

$1+B = 0$

$B = -1$

put the values in (i) we get

$$\frac{9}{(x-1)(x+2)^2} = \frac{1}{(x-1)} + \frac{-1}{(x+2)} + \frac{-3}{(x+2)^2}$$

4. $$\frac{x^4+1}{x^2(x-1)} = \frac{x^4+1}{x^3-x^2} = x+1+\frac{x^2+1}{x^3-x^2} \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{x^2+1}{x^3-x^2} = \frac{A}{(x-1)} + \frac{B}{x} + \frac{C}{x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

multiplying by $(x-1)x^2$ we get

$$x^2+1 = Ax^2 + Bx(x-1) + C(x-1) \ldots\ldots\ldots \text{(iii)}$$

$$x^2 + 1 = Ax^2 + B(x^2 - x) + C(x - 1) \text{ ..... (iv)}$$

put $x = 1$ in (iii)

$$(1)^2 + 1 = A(1)^2 + B(1)(1-1) + C(1-1)$$
$$1 + 1 = A(1)^2$$
$$2 = A$$
$$A = 2$$

put $x = 0$ in (iii)

$$(0)^2 + 1 = A(0)^2 + B(0)(0-1) + C(0-1)$$
$$1 = C(-1)$$
$$C = -1$$

Now, comparing coefficients of equation (iv)

x^2; $A + B = 1$

as A = 2

$$2 + B = 1$$
$$B = -1$$

put the values in (ii) we get

$$\frac{x^2+1}{x^3-x^2} = \frac{2}{(x-1)} + \frac{-1}{x} + \frac{-1}{x^2}$$

put this in (i) we get

$$\frac{x^4+1}{x^2(x-1)} = x + 1 + \frac{2}{(x-1)} - \frac{1}{x} - \frac{1}{x^2}$$

5. $\frac{7x+4}{(3x+2)(x+1)^2} = \frac{A}{(3x+2)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2}$ ..... (i)

multiplying by $(3x + 2)(x + 1)^2$ we get

$$7x + 4 = A(x + 1)^2 + B(3x + 2)(x + 1) + C(3x + 2) \text{ ..... (ii)}$$
$$7x + 4 = A(x^2 + 2x + 1) + B(3x^2 + 3x + 2x + 2) + C(x - 1)$$
$$7x + 4 = A(x^2 + 2x + 1) + B(3x^2 + 5x + 2) + C(x - 1) \text{ ..... (iii)}$$

put $x = -\frac{2}{3}$ in (ii)

$$7\left(-\frac{2}{3}\right) + 4 = A\left(-\frac{2}{3} + 1\right)^2 + B\left(3\left(-\frac{2}{3}\right) + 2\right)\left(-\frac{2}{3} - 1\right) + C\left(3\left(-\frac{2}{3}\right) + 2\right)$$
$$-\frac{14}{3} + 4 = A\left(-\frac{2}{3} + 1\right)^2$$
$$\frac{-14+12}{3} = A\left(\frac{-2+3}{3}\right)^2$$
$$\frac{-2}{3} = A\left(\frac{1}{3}\right)^2$$
$$\frac{-2}{3} = \frac{1}{9}A$$
$$A = -6$$

put $x = -1$ in (ii)

$$7(-1) + 4 = A(-1 + 1)^2 + B(3(-1) + 2)(-1 - 1) + C(3(-1) + 2)$$
$$-7 + 4 = C(-3 + 2)$$
$$-3 = C(-1)$$
$$-3 = -C$$

$C = 3$

Now, comparing coefficients of equation (iii)

x^2; $A + 3B = 0$

as A = -6

$-6 + 3B = 0$

$3B = 6$

$B = 2$

put the values in (i) we get

$$\frac{7x+4}{(3x+2)(x+1)^2} = \frac{-6}{(3x+2)} + \frac{2}{(x+1)} + \frac{3}{(x+1)^2}$$

6. $$\frac{1}{(x+1)(x-1)^2} = \frac{A}{(x+1)} + \frac{B}{(x-1)} + \frac{C}{(x-1)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(i)}$$

multiplying by $(x+1)(x-1)^2$ we get

$1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$ ..... (ii)

$1 = A(x^2 - 2x + 1) + B(x^2 - 1) + C(x+1)$ ..... (iii)

put $x = -1$ in (ii)

$1 = A(-1-1)^2 + B(-1+1)(-1-1) + C(-1+1)$

$1 = A(-2)^2$

$1 = 4A$

$A = \frac{1}{4}$

put $x = 1$ in (ii)

$1 = A(1-1)^2 + B(1+1)(1-1) + C(1+1)$

$1 = C(1+1)$

$1 = C(2)$

$1 = 2C$

$C = \frac{1}{2}$

Now, comparing coefficients of equation (iii)

x^2; $A + B = 0$

as A = $\frac{1}{4}$

$\frac{1}{4} + B = 0$

$B = -\frac{1}{4}$

put the values in (i) we get

$$\frac{1}{(x+1)(x-1)^2} = \frac{1}{4(x+1)} + \frac{-1}{4(x-1)} + \frac{1}{2(x-1)^2}$$

7. $$\frac{3x^2+15x+16}{(x+2)^2} = \frac{3x^2+15x+16}{x^2+4x+4} = 3 + \frac{3x+4}{x^2+4x+4} \quad \ldots\ldots\ldots\ldots\ldots \text{(i)}$$

$$\frac{3x+4}{x^2+4x+4} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(ii)}$$

multiplying by $(x+2)^2$ we get

$3x + 4 = A(x+2) + B$ ..... (iii)

$3x + 4 = A(x+2) + B$ ..... (iv)

put $x = -2$ in (iii)

$3(-2)+4 = A(-2+2)+B$

$-6+4 = B$

$-2 = B$

$B = -2$

Now, comparing coefficients of equation (iv)

x; $A = 3$

put the values in (ii) we get

$$\frac{3x+4}{x^2+4x+4} = \frac{3}{(x+2)} + \frac{-2}{(x+2)^2}$$

put this in (i) we get

$$\frac{3x^2+15x+16}{(x+2)^2} = 3 + \frac{3}{(x+2)} - \frac{2}{(x+2)^2}$$

8. $$\frac{1}{(x^2-1)(x+1)} = \frac{1}{(x-1)(x+1)^2} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+1)^2} \quad \text{..... (i)}$$

multiplying by $(x-1)(x+1)^2$ we get

$1 = A(x+1)^2 + B(x-1)(x+1) + C(x-1)$ ..... (ii)

$1 = A(x^2+2x+1) + B(x^2-1) + C(x-1)$ ..... (iii)

put $x = 1$ in (ii)

$1 = A(1+1)^2 + B(1-1)(1+1) + C(1-1)$

$1 = A(2)^2$

$1 = 4A$

$A = \frac{1}{4}$

put $x = -1$ in (ii)

$1 = A(-1+1)^2 + B(-1-1)(-1+1) + C(-1-1)$

$1 = C(-1-1)$

$1 = C(-2)$

$1 = -2C$

$C = \frac{-1}{2}$

Now, comparing coefficients of equation (iii)

x^2; $A + B = 0$

as A = $\frac{1}{4}$

$\frac{1}{4} + B = 0$

$B = -\frac{1}{4}$

put the values in (i) we get

$$\frac{1}{(x^2-1)(x+1)} = \frac{1}{4(x-1)} + \frac{-1}{4(x+1)} + \frac{-1}{2(x+1)^2}$$