Exercise 2.2

Q. 1: Find the cube roots of -1, 8, -27, 64.

cube root of -1:

let $x^3 = -1$

$x^3 + 1 = 0$

$x^3 + 1^3 = 0$

$(x+1)(x^2 - x + 1^2) = 0$

$x + 1 = 0$ and $x^2 - x + 1 = 0$

$x = -1$ and $x^2 - x + 1 = 0$

applying quadratic formula a = 1, b = -1, c = 1

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1-4}}{2}$$

$$= -\frac{-1 \pm \sqrt{-3}}{2}$$

$x = -\frac{-1+\sqrt{-3}}{2}$ and $x = -\frac{-1-\sqrt{-3}}{2}$

$x = -\omega$ and $x = -\omega^2$

so the cube roots of -1 are -1, $-\omega$, $-\omega^2$

cube root of 8:

let $x^3 = 8$

$x^3 - 8 = 0$

$x^3 - 2^3 = 0$

$(x-2)(x^2 + 2x + 2^2) = 0$

$x - 2 = 0$ and $x^2 + 2x + 4 = 0$

$x = 2$ and $x^2 + 2x + 4 = 0$

applying quadratic formula a = 1, b = 2, c = 4

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{4-16}}{2}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

$$= \frac{-2 \pm \sqrt{4 \times -3}}{2}$$

$$= \frac{-2 \pm 2\sqrt{-3}}{2}$$

$= 2\left(\frac{-1\pm\sqrt{-3}}{2}\right)$

$x = 2\left(\frac{-1+\sqrt{-3}}{2}\right)$ and $x = 2\left(\frac{-1-\sqrt{-3}}{2}\right)$

$x = 2\omega$ and $x = 2\omega^2$

so the cube roots of 8 are 2, 2ω, $2\omega^2$

cube root of -27:

let $x^3 = -27$

$x^3 + 27 = 0$

$x^3 + 3^3 = 0$

$(x+3)(x^2 - 3x + 3^2) = 0$

$x + 3 = 0$ and $x^2 - 3x + 9 = 0$

$x = -3$ and $x^2 - 3x + 9 = 0$

applying quadratic formula a = 1, b = -3, c = 9

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(-3)\pm\sqrt{(-3)^2-4(1)(9)}}{2(1)}$

$= \frac{3\pm\sqrt{9-36}}{2}$

$= \frac{3\pm\sqrt{-27}}{2}$

$= \frac{3\pm\sqrt{9\times-3}}{2}$

$= \frac{3\pm3\sqrt{-3}}{2}$

$= -3\left(\frac{-1\pm\sqrt{-3}}{2}\right)$

$x = -3\left(\frac{-1+\sqrt{-3}}{2}\right)$ and $x = -3\left(\frac{-1-\sqrt{-3}}{2}\right)$

$x = -3\omega$ and $x = -3\omega^2$

so the cube roots of -27 are -3, -3ω, $-3\omega^2$

cube root of 64:

let $x^3 = 64$

$x^3 - 64 = 0$

$x^3 - 4^3 = 0$

$(x-4)(x^2 + 4x + 4^2) = 0$

$x - 4 = 0$ and $x^2 + 4x + 16 = 0$

$x = 4$ and $x^2 + 4x + 16 = 0$

applying quadratic formula a = 1, b = 4, c = 16

$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$

$= \frac{-(4)\pm\sqrt{(4)^2-4(1)(16)}}{2(1)}$

$= \frac{-4 \pm \sqrt{16-64}}{2}$

$= \frac{-4 \pm \sqrt{-48}}{2}$

$= \frac{-4 \pm \sqrt{16 \times -3}}{2}$

$= \frac{-4 \pm 4\sqrt{-3}}{2}$

$= 4\left(\frac{-1 \pm \sqrt{-3}}{2}\right)$

$x = 4\left(\frac{-1+\sqrt{-3}}{2}\right)$ and $x = 4\left(\frac{1-\sqrt{-3}}{2}\right)$

$x = 4\omega$ and $x = 4\omega^2$

so the cube roots of 64 are 4, 4ω, $4\omega^2$

Q. 2: Evaluate:

(i) $(1-\omega-\omega^2)^7 = \left(1-(\omega+\omega^2)\right)^7$

as $1+\omega+\omega^2 = 0$ and $\omega+\omega^2 = -1$

$= \left(1-(-1)\right)^7$

$= (1+1)^7$

$= (2)^7$

$= 128$

(ii) $(1-3\omega-3\omega^2)^5 = \left(1-3(\omega+\omega^2)\right)^5$

as $1+\omega+\omega^2 = 0$ and $\omega+\omega^2 = -1$

$= \left(1-3(-1)\right)^5$

$= (1+3)^5$

$= (4)^5$

$= 1024$

(iii) $(9+4\omega+4\omega^2)^3 = \left(9+4(\omega+\omega^2)\right)^3$

as $1+\omega+\omega^2 = 0$ and $\omega+\omega^2 = -1$

$= \left(9+4(-1)\right)^3$

$= (9-4)^3$

$= (5)^3$

$= 125$

(iv) $(2+2\omega-2\omega^2)(3-3\omega+3\omega^2) = (2(1+\omega)-2\omega^2)\left(-3\omega+3(1+\omega^2)\right)$

as $1+\omega+\omega^2 = 0$ and $1+\omega^2 = -\omega$ and $1+\omega = -\omega^2$

$= (2(-\omega^2)-2\omega^2)\left(-3\omega+3(-\omega)\right)$

$= (-2\omega^2-2\omega^2)(-3\omega-3\omega)$

$= (-4\omega^2)(-6\omega)$

$= 24\omega^3$

as $\omega^3 = 1$

$= 24(1)$

$= 24$

(v) $\left(-1+\sqrt{-3}\right)^6+\left(-1-\sqrt{-3}\right)^6$ $=\left(2\times\frac{-1+\sqrt{-3}}{2}\right)^6+\left(2\times\frac{-1-\sqrt{-3}}{2}\right)^6$

$=64(\omega)^6+64(\omega^2)^6$

$=64(\omega)^6+64(\omega)^{12}$

$=64(\omega^3)^2+64(\omega^3)^4$

as $\omega^3=1$

$=64(1)^2+64(1)^4$

$=64+64$

$=128$

(vi) $\left(\frac{-1+\sqrt{-3}}{2}\right)^9+\left(\frac{-1-\sqrt{-3}}{2}\right)^9$ $=(\omega)^9+(\omega^2)^9$

$=(\omega)^9+(\omega)^{18}$

$=(\omega^3)^3+(\omega^3)^6$

as $\omega^3=1$

$=(1)^3+(1)^6$

$=1+1$

$=2$

(vii) $\omega^{37}+\omega^{38}-5$ $=\omega^{36}.\omega+\omega^{36}.\omega^2-5$

$=(\omega^3)^{12}.\omega+(\omega^3)^{12}.\omega^2-5$

$=(\omega^3)^{12}.\omega+(\omega^3)^{12}.\omega^2-5$

$=1.\omega+1.\omega^2-5$

$=\omega+\omega^2-5$

as $1+\omega+\omega^2=0$ and $\omega+\omega^2=-1$

$=-1-5$

$=-6$

(viii) $\omega^{-13}+\omega^{-17}$ $=\frac{1}{\omega^{13}}+\frac{1}{\omega^{17}}$

$=\frac{1}{\omega^{12}.\omega}+\frac{1}{\omega^{15}.\omega^2}$

$=\frac{1}{(\omega^3)^4.\omega}+\frac{1}{(\omega^3)^5.\omega^2}$

as $\omega^3=1$

$=\frac{1}{(1)^4.\omega}+\frac{1}{(1)^5.\omega^2}$

$=\frac{1}{\omega}+\frac{1}{\omega^2}$

$=\frac{\omega+\omega^2}{\omega^3}$

as $\omega^3=1$ and $1+\omega+\omega^2=0$ and $\omega+\omega^2=-1$

$=\frac{-1}{1}$

$=-1$

Q. 3: Prove that $(x^3+y^3)=(x+y)(x+\omega y)(x+\omega^2 y)$

R.H.S $=(x+y)(x+\omega y)(x+\omega^2 y)$

$=(x+y)(x^2+\omega^2xy+\omega xy+\omega^3y^2)$

$=(x+y)(x^2+(\omega^2+\omega)xy+(\omega^3)y^2)$

as $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$ and $\omega + \omega^2 = -1$

$= (x + y)(x^2 + (-1)xy + (1)y^2)$

$= (x + y)(x^2 - xy + y^2)$

as $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

$= x^3 + y^3 =$ L.H.S

Q. 4: Prove that $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

R.H.S $= (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$

$= (x + y + z)(x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz + \omega^2 xz + \omega^4 yz + \omega^3 z^2)$

$= (x + y + z)(x^2 + \omega^2 xy + \omega xz + \omega xy + \omega^3 y^2 + \omega^2 yz + \omega^2 xz + \omega^3.\omega yz + \omega^3 z^2)$

as $\omega^3 = 1$

$= (x + y + z)(x^2 + \omega^2 xy + \omega xz + \omega xy + y^2 + \omega^2 yz + \omega^2 xz + \omega yz + z^2)$

$= (x + y + z)(x^2 + y^2 + z^2 + (\omega + \omega^2)xy + (\omega + \omega^2)yz + (\omega + \omega^2)xz)$

as $1 + \omega + \omega^2 = 0$ and $\omega + \omega^2 = -1$

$= (x + y + z)(x^2 + y^2 + z^2 + (-1)xy + (-1)yz + (-1)xz)$

$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$

as $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - xz)$, So

$= x^3 + y^3 + z^3 - 3xyz =$ L.H.S

Q. 5: Prove that $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots\ldots\ldots 2n\ factors = 1$

L.H.S $= (1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^8) \ldots\ldots\ldots 2n\ factors$

$= (1 + \omega)(1 + \omega^2)(1 + \omega^3.\omega)(1 + \omega^6.\omega^2) \ldots\ldots\ldots 2n\ factors$

$= (1 + \omega)(1 + \omega^2)(1 + (\omega^3).\omega)(1 + (\omega^3)^2.\omega^2) \ldots\ldots\ldots 2n\ factors$

as $\omega^3 = 1$

$= (1 + \omega)(1 + \omega^2)(1 + (1).\omega)(1 + (1)^2.\omega^2) \ldots\ldots\ldots 2n\ factors$

$= (1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2)(1 + \omega)(1 + \omega^2) \ldots\ldots\ldots 2n\ factors$

$= (1 + \omega^2 + \omega + \omega^3)(1 + \omega^2 + \omega + \omega^3)(1 + \omega^2 + \omega + \omega^3) \ldots\ldots\ldots 2n\ factors$

$= (1 + \omega + \omega^2 + \omega^3)(1 + \omega + \omega^2 + \omega^3)(1 + \omega + \omega^2 + \omega^3) \ldots\ldots\ldots 2n\ factors$

as $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$

$= (0 + 1)(0 + 1)(0 + 1) \ldots\ldots\ldots 2n\ factors$

$= (1)(1)(1) \ldots\ldots\ldots 2n\ factors$

multiplying 1 up to 2n times we get

$= 1 =$ R.H.S