# CHAPTER 8 TECHNIQUES OF INTEGRATION

## 8.1 BASIC INTEGRATION FORMULAS

1. $\int \frac{16x\,dx}{\sqrt{8x^2+1}}$; $\begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix}$ $\to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{8x^2+1} + C$

2. $\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$; $\begin{bmatrix} u = 1+3\sin x \\ du = 3\cos x\,dx \end{bmatrix}$ $\to \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + C = 2\sqrt{1+3\sin x} + C$

3. $\int 3\sqrt{\sin v}\cos v\,dv$; $\begin{bmatrix} u = \sin v \\ du = \cos v\,dv \end{bmatrix}$ $\to \int 3\sqrt{u}\,du = 3 \cdot \frac{2}{3} u^{3/2} + C = 2(\sin v)^{3/2} + C$

4. $\int \cot^3 y \csc^2 y\,dy$; $\begin{bmatrix} u = \cot y \\ du = -\csc^2 y\,dy \end{bmatrix}$ $\to \int u^3(-du) = -\frac{u^4}{4} + C = \frac{-\cot^4 y}{4} + C$

5. $\int_0^1 \frac{16x\,dx}{8x^2+2}$; $\begin{bmatrix} u = 8x^2+2 \\ du = 16x\,dx \\ x=0 \Rightarrow u=2,\quad x=1 \Rightarrow u=10 \end{bmatrix}$ $\to \int_2^{10} \frac{du}{u} = [\ln|u|]_2^{10} = \ln 10 - \ln 2 = \ln 5$

6. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 z\,dz}{\tan z}$; $\begin{bmatrix} u = \tan z \\ du = \sec^2 z\,dz \\ z = \frac{\pi}{4} \Rightarrow u = 1,\ z = \frac{\pi}{3} \Rightarrow u = \sqrt{3} \end{bmatrix}$ $\to \int_1^{\sqrt{3}} \frac{1}{u}\,du = [\ln|u|]_1^{\sqrt{3}} = \ln\sqrt{3} - \ln 1 = \ln\sqrt{3}$

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$; $\begin{bmatrix} u = \sqrt{x}+1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix}$ $\to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln(\sqrt{x}+1) + C$

8. $\int \frac{dx}{x-\sqrt{x}} = \int \frac{dx}{\sqrt{x}(\sqrt{x}-1)}$; $\begin{bmatrix} u = \sqrt{x}-1 \\ du = \frac{1}{2\sqrt{x}}\,dx \\ 2\,du = \frac{dx}{\sqrt{x}} \end{bmatrix}$ $\to \int \frac{2\,du}{u} = 2\ln|u| + C = 2\ln\left|\sqrt{x}-1\right| + C$

9. $\int \cot(3-7x)\,dx$; $\begin{bmatrix} u = 3-7x \\ du = -7\,dx \end{bmatrix}$ $\to -\frac{1}{7}\int \cot u\,du = -\frac{1}{7}\ln|\sin u| + C = -\frac{1}{7}\ln|\sin(3-7x)| + C$

10. $\int \csc(\pi x - 1)\,dx$; $\begin{bmatrix} u = \pi x - 1 \\ du = \pi\,dx \end{bmatrix}$ $\to \int \csc u \cdot \frac{du}{\pi} = \frac{-1}{\pi}\ln|\csc u + \cot u| + C$
$= -\frac{1}{\pi}\ln|\csc(\pi x - 1) + \cot(\pi x - 1)| + C$

11. $\int e^\theta \csc(e^\theta + 1)\,d\theta$; $\begin{bmatrix} u = e^\theta + 1 \\ du = e^\theta\,d\theta \end{bmatrix}$ $\to \int \csc u\,du = -\ln|\csc u + \cot u| + C = -\ln\left|\csc(e^\theta+1) + \cot(e^\theta+1)\right| + C$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx$; $\begin{bmatrix} u = 3+\ln x \\ du = \frac{dx}{x} \end{bmatrix}$ $\to \int \cot u\,du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C$

13. $\int \sec \frac{t}{3}\, dt;\ \begin{bmatrix} u = \frac{t}{3} \\ du = \frac{dt}{3} \end{bmatrix} \to \int 3 \sec u\, du = 3 \ln |\sec u + \tan u| + C = 3 \ln \left|\sec \frac{t}{3} + \tan \frac{t}{3}\right| + C$

14. $\int x \sec (x^2 - 5)\, dx;\ \begin{bmatrix} u = x^2 - 5 \\ du = 2x\, dx \end{bmatrix} \to \int \frac{1}{2} \sec u\, du = \frac{1}{2} \ln |\sec u + \tan u| + C$
$= \frac{1}{2} \ln |\sec (x^2 - 5) + \tan (x^2 - 5)| + C$

15. $\int \csc (s - \pi)\, ds;\ \begin{bmatrix} u = s - \pi \\ du = ds \end{bmatrix} \to \int \csc u\, du = -\ln |\csc u + \cot u| + C = -\ln |\csc (s - \pi) + \cot (s - \pi)| + C$

16. $\int \frac{1}{\theta^2} \csc \frac{1}{\theta}\, d\theta;\ \begin{bmatrix} u = \frac{1}{\theta} \\ du = \frac{-d\theta}{\theta^2} \end{bmatrix} \to \int -\csc u\, du = \ln |\csc u + \cot u| + C = \ln \left|\csc \frac{1}{\theta} + \cot \frac{1}{\theta}\right| + C$

17. $\int_0^{\sqrt{\ln 2}} 2xe^{x^2}\, dx;\ \begin{bmatrix} u = x^2 \\ du = 2x\, dx \\ x = 0 \Rightarrow u = 0,\ x = \sqrt{\ln 2} \Rightarrow u = \ln 2 \end{bmatrix} \to \int_0^{\ln 2} e^u\, du = [e^u]_0^{\ln 2} = e^{\ln 2} - e^0 = 2 - 1 = 1$

18. $\int_{\pi/2}^{\pi} \sin (y)\, e^{\cos y}\, dy;\ \begin{bmatrix} u = \cos y \\ du = -\sin y\, dy \\ y = \frac{\pi}{2} \Rightarrow u = 0,\ y = \pi \Rightarrow u = -1 \end{bmatrix} \to \int_0^{-1} -e^u\, du = \int_{-1}^{0} e^u\, du = [e^u]_{-1}^{0} = 1 - e^{-1} = \frac{e-1}{e}$

19. $\int e^{\tan v} \sec^2 v\, dv;\ \begin{bmatrix} u = \tan v \\ du = \sec^2 v\, dv \end{bmatrix} \to \int e^u\, du = e^u + C = e^{\tan v} + C$

20. $\int \frac{e^{\sqrt{t}}\, dt}{\sqrt{t}};\ \begin{bmatrix} u = \sqrt{t} \\ du = \frac{dt}{2\sqrt{t}} \end{bmatrix} \to \int 2e^u\, du = 2e^u + C = 2e^{\sqrt{t}} + C$

21. $\int 3^{x+1}\, dx;\ \begin{bmatrix} u = x + 1 \\ du = dx \end{bmatrix} \to \int 3^u\, du = \left(\frac{1}{\ln 3}\right) 3^u + C = \frac{3^{(x+1)}}{\ln 3} + C$

22. $\int \frac{2^{\ln x}}{x}\, dx;\ \begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \end{bmatrix} \to \int 2^u\, du = \frac{2^u}{\ln 2} + C = \frac{2^{\ln x}}{\ln 2} + C$

23. $\int \frac{2^{\sqrt{w}}\, dw}{2\sqrt{w}};\ \begin{bmatrix} u = \sqrt{w} \\ du = \frac{dw}{2\sqrt{w}} \end{bmatrix} \to \int 2^u\, du = \frac{2^u}{\ln 2} + C = \frac{2^{\sqrt{w}}}{\ln 2} + C$

24. $\int 10^{2\theta}\, d\theta;\ \begin{bmatrix} u = 2\theta \\ du = 2\, d\theta \end{bmatrix} \to \int \frac{1}{2} 10^u\, du = \frac{10^u}{2 \ln 10} + C = \frac{1}{2}\left(\frac{10^{2\theta}}{\ln 10}\right) + C$

25. $\int \frac{9\, du}{1 + 9u^2};\ \begin{bmatrix} x = 3u \\ dx = 3\, du \end{bmatrix} \to \int \frac{3\, dx}{1 + x^2} = 3 \tan^{-1} x + C = 3 \tan^{-1} 3u + C$

26. $\int \frac{4\, dx}{1 + (2x + 1)^2};\ \begin{bmatrix} u = 2x + 1 \\ du = 2\, dx \end{bmatrix} \to \int \frac{2\, du}{1 + u^2} = 2 \tan^{-1} u + C = 2 \tan^{-1} (2x + 1) + C$

27. $\int_0^{1/6} \frac{dx}{\sqrt{1 - 9x^2}};\ \begin{bmatrix} u = 3x \\ du = 3\, dx \\ x = 0 \Rightarrow u = 0,\ x = \frac{1}{6} \Rightarrow u = \frac{1}{2} \end{bmatrix} \to \int_0^{1/2} \frac{1}{3} \frac{du}{\sqrt{1 - u^2}} = \left[\frac{1}{3} \sin^{-1} u\right]_0^{1/2} = \frac{1}{3}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{18}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}} = \left[\sin^{-1} \frac{t}{2}\right]_0^1 = \sin^{-1}\left(\frac{1}{2}\right) - 0 = \frac{\pi}{6}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}}$; $\begin{bmatrix} u = s^2 \\ du = 2s\,ds \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1} s^2 + C$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$; $\begin{bmatrix} u = 2\ln x \\ du = \frac{2\,dx}{x} \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(2\ln x) + C$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}} = \int \frac{6\,dx}{5x\sqrt{x^2-\frac{1}{25}}} = \frac{6}{5} \cdot 5\sec^{-1}|5x| + C = 6\sec^{-1}|5x| + C$

32. $\int \frac{dr}{r\sqrt{r^2-9}} = \frac{1}{3}\sec^{-1}\left|\frac{r}{3}\right| + C$

33. $\int \frac{dx}{e^x+e^{-x}} = \int \frac{e^x\,dx}{e^{2x}+1}$; $\begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \rightarrow \int \frac{du}{u^2+1} = \tan^{-1} u + C = \tan^{-1} e^x + C$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}} = \int \frac{e^y\,dy}{e^y\sqrt{(e^y)^2-1}}$; $\begin{bmatrix} u = e^y \\ du = e^y\,dy \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1} e^y + C$

35. $\int_1^{e^{\pi/3}} \frac{dx}{x\cos(\ln x)}$; $\begin{bmatrix} u = \ln x \\ du = \frac{dx}{x} \\ x = 1 \Rightarrow u = 0,\ x = e^{\pi/3} \Rightarrow u = \frac{\pi}{3} \end{bmatrix} \rightarrow \int_0^{\pi/3} \frac{du}{\cos u} = \int_0^{\pi/3} \sec u\,du = [\ln|\sec u + \tan u|]_0^{\pi/3}$
$= \ln\left|\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\right| - \ln|\sec 0 + \tan 0| = \ln\left(2+\sqrt{3}\right) - \ln(1) = \ln\left(2+\sqrt{3}\right)$

36. $\int \frac{\ln x\,dx}{x+4x\ln^2 x} = \int \frac{\ln x\,dx}{x(1+4\ln^2 x)}$; $\begin{bmatrix} u = \ln^2 x \\ du = \frac{2}{x}\ln x\,dx \end{bmatrix} \rightarrow \int \frac{1}{2}\frac{du}{1+4u} = \frac{1}{8}\ln|1+4u| + C = \frac{1}{8}\ln(1+4\ln^2 x) + C$

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2} = 8\int_1^2 \frac{dx}{1+(x-1)^2}$; $\begin{bmatrix} u = x-1 \\ du = dx \\ x = 1 \Rightarrow u = 0,\ x = 2 \Rightarrow u = 1 \end{bmatrix} \rightarrow 8\int_0^1 \frac{du}{1+u^2} = 8\left[\tan^{-1} u\right]_0^1$
$= 8\left(\tan^{-1} 1 - \tan^{-1} 0\right) = 8\left(\frac{\pi}{4} - 0\right) = 2\pi$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10} = 2\int_2^4 \frac{dx}{(x-3)^2+1}$; $\begin{bmatrix} u = x-3 \\ du = dx \\ x = 2 \Rightarrow u = -1,\ x = 4 \Rightarrow u = 1 \end{bmatrix} \rightarrow 2\int_{-1}^1 \frac{du}{u^2+1} = 2\left[\tan^{-1} u\right]_{-1}^1$
$= 2\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = 2\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right] = \pi$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}} = \int \frac{dt}{\sqrt{1-(t-2)^2}}$; $\begin{bmatrix} u = t-2 \\ du = dt \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(t-2) + C$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}} = \int \frac{d\theta}{\sqrt{1-(\theta-1)^2}}$; $\begin{bmatrix} u = \theta-1 \\ du = d\theta \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + C = \sin^{-1}(\theta-1) + C$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}} = \int \frac{dx}{(x+1)\sqrt{(x+1)^2-1}}$; $\begin{bmatrix} u = x+1 \\ du = dx \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|x+1| + C,$
$|u| = |x+1| > 1$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$ ; $\begin{bmatrix} u = x-2 \\ du = dx \end{bmatrix} \to \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C$
$= \sec^{-1}|x-2| + C, |u| = |x-2| > 1$

43. $\int (\sec x + \cot x)^2\, dx = \int (\sec^2 x + 2\sec x \cot x + \cot^2 x)\, dx = \int \sec^2 x\, dx + \int 2\csc x\, dx + \int (\csc^2 x - 1)\, dx$
$= \tan x - 2\ln|\csc x + \cot x| - \cot x - x + C$

44. $\int (\csc x - \tan x)^2\, dx = \int (\csc^2 x - 2\csc x \tan x + \tan^2 x)\, dx = \int \csc^2 x\, dx - \int 2\sec x\, dx + \int (\sec^2 x - 1)\, dx$
$= -\cot x - 2\ln|\sec x + \tan x| + \tan x - x + C$

45. $\int \csc x \sin 3x\, dx = \int (\csc x)(\sin 2x \cos x + \sin x \cos 2x)\, dx = \int (\csc x)(2\sin x \cos^2 x + \sin x \cos 2x)\, dx$
$= \int (2\cos^2 x + \cos 2x)\, dx = \int [(1+\cos 2x) + \cos 2x]\, dx = \int (1 + 2\cos 2x)\, dx = x + \sin 2x + C$

46. $\int (\sin 3x \cos 2x - \cos 3x \sin 2x)\, dx = \int \sin(3x - 2x)\, dx = \int \sin x\, dx = -\cos x + C$

47. $\int \frac{x}{x+1}\, dx = \int \left(1 - \frac{1}{x+1}\right) dx = x - \ln|x+1| + C$

48. $\int \frac{x^2}{x^2+1}\, dx = \int \left(1 - \frac{1}{x^2+1}\right) dx = x - \tan^{-1} x + C$

49. $\int_{\sqrt{2}}^{3} \frac{2x^3}{x^2-1}\, dx = \int_{\sqrt{2}}^{3} \left(2x + \frac{2x}{x^2-1}\right) dx = \left[x^2 + \ln|x^2-1|\right]_{\sqrt{2}}^{3} = (9 + \ln 8) - (2 + \ln 1) = 7 + \ln 8$

50. $\int_{-1}^{3} \frac{4x^2-7}{2x+3}\, dx = \int_{-1}^{3} \left[(2x-3) + \frac{2}{2x+3}\right] dx = \left[x^2 - 3x + \ln|2x+3|\right]_{-1}^{3} = (9 - 9 + \ln 9) - (1 + 3 + \ln 1) = \ln 9 - 4$

51. $\int \frac{4t^3 - t^2 + 16t}{t^2+4}\, dt = \int \left[(4t-1) + \frac{4}{t^2+4}\right] dt = 2t^2 - t + 2\tan^{-1}\left(\frac{t}{2}\right) + C$

52. $\int \frac{2\theta^3 - 7\theta^2 + 7\theta}{2\theta - 5}\, d\theta = \int \left[(\theta^2 - \theta + 1) + \frac{5}{2\theta-5}\right] d\theta = \frac{\theta^3}{3} - \frac{\theta^2}{2} + \theta + \frac{5}{2}\ln|2\theta - 5| + C$

53. $\int \frac{1-x}{\sqrt{1-x^2}}\, dx = \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x\, dx}{\sqrt{1-x^2}} = \sin^{-1} x + \sqrt{1-x^2} + C$

54. $\int \frac{x + 2\sqrt{x-1}}{2x\sqrt{x-1}}\, dx = \int \frac{dx}{2\sqrt{x-1}} + \int \frac{dx}{x} = (x-1)^{1/2} + \ln|x| + C$

55. $\int_0^{\pi/4} \frac{1+\sin x}{\cos^2 x}\, dx = \int_0^{\pi/4} (\sec^2 x + \sec x \tan x)\, dx = [\tan x + \sec x]_0^{\pi/4} = \left(1 + \sqrt{2}\right) - (0 + 1) = \sqrt{2}$

56. $\int_0^{1/2} \frac{2-8x}{1+4x^2}\, dx = \int_0^{1/2} \left(\frac{2}{1+4x^2} - \frac{8x}{1+4x^2}\right) dx = \left[\tan^{-1}(2x) - \ln|1+4x^2|\right]_0^{1/2}$
$= (\tan^{-1} 1 - \ln 2) - (\tan^{-1} 0 - \ln 1) = \frac{\pi}{4} - \ln 2$

57. $\int \frac{dx}{1+\sin x} = \int \frac{(1-\sin x)}{(1-\sin^2 x)}\, dx = \int \frac{(1-\sin x)}{\cos^2 x}\, dx = \int (\sec^2 x - \sec x \tan x)\, dx = \tan x - \sec x + C$

58. $1 + \cos x = 1 + \cos\left(2 \cdot \frac{x}{2}\right) = 2\cos^2 \frac{x}{2} \Rightarrow \int \frac{dx}{1+\cos x} = \int \frac{dx}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{x}{2}\right) dx = \tan\frac{x}{2} + C$

59. $\int \frac{1}{\sec\theta + \tan\theta}\, d\theta = \int d\theta$; $\begin{bmatrix} u = 1 + \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{du}{u} = \ln|u| + C = \ln|1 + \sin\theta| + C$

60. $\int \frac{1}{\csc\theta + \cot\theta}\, d\theta = \int \frac{\sin\theta}{1+\cos\theta}\, d\theta;\ \begin{bmatrix} u = 1+\cos\theta \\ du = -\sin\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{-du}{u} = -\ln|u| + C = -\ln|1+\cos\theta| + C$

61. $\int \frac{1}{1-\sec x}\, dx = \int \frac{\cos x}{\cos x - 1}\, dx = \int \left(1 + \frac{1}{\cos x - 1}\right) dx = \int \left(1 - \frac{1+\cos x}{\sin^2 x}\right) dx = \int \left(1 - \csc^2 x - \frac{\cos x}{\sin^2 x}\right) dx$
$= \int (1 - \csc^2 x - \csc x \cot x)\, dx = x + \cot x + \csc x + C$

62. $\int \frac{1}{1-\csc x}\, dx = \int \frac{\sin x}{\sin x - 1}\, dx = \int \left(1 + \frac{1}{\sin x - 1}\right) dx = \int \left(1 + \frac{\sin x + 1}{(\sin x - 1)(\sin x + 1)}\right) dx$
$= \int \left(1 - \frac{1+\sin x}{\cos^2 x}\right) dx = \int \left(1 - \sec^2 x - \frac{\sin x}{\cos^2 x}\right) dx = \int (1 - \sec^2 x - \sec x \tan x)\, dx = x - \tan x - \sec x + C$

63. $\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx = \int_0^{2\pi} \left|\sin \frac{x}{2}\right| dx;\ \begin{bmatrix} \sin\frac{x}{2} \ge 0 \\ \text{for } 0 \le \frac{x}{2} \le 2\pi \end{bmatrix} \rightarrow \int_0^{2\pi} \sin\left(\frac{x}{2}\right) dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = -2(\cos\pi - \cos 0)$
$= (-2)(-2) = 4$

64. $\int_0^{\pi} \sqrt{1-\cos 2x}\, dx = \int_0^{\pi} \sqrt{2}\,|\sin x|\, dx;\ \begin{bmatrix} \sin x \ge 0 \\ \text{for } 0 \le x \le \pi \end{bmatrix} \rightarrow \sqrt{2}\int_0^{\pi} \sin x\, dx = \left[-\sqrt{2}\cos x\right]_0^{\pi}$
$= -\sqrt{2}(\cos\pi - \cos 0) = 2\sqrt{2}$

65. $\int_{\pi/2}^{\pi} \sqrt{1+\cos 2t}\, dt = \int_{\pi/2}^{\pi} \sqrt{2}\,|\cos t|\, dt;\ \begin{bmatrix} \cos t \le 0 \\ \text{for } \frac{\pi}{2} \le t \le \pi \end{bmatrix} \rightarrow \int_{\pi/2}^{\pi} -\sqrt{2}\cos t\, dt = \left[-\sqrt{2}\sin t\right]_{\pi/2}^{\pi}$
$= -\sqrt{2}\left(\sin\pi - \sin\frac{\pi}{2}\right) = \sqrt{2}$

66. $\int_{-\pi}^{0} \sqrt{1+\cos t}\, dt = \int_{-\pi}^{0} \sqrt{2}\left|\cos\frac{t}{2}\right| dt;\ \begin{bmatrix} \cos\frac{t}{2} \ge 0 \\ \text{for } -\pi \le t \le 0 \end{bmatrix} \rightarrow \int_{-\pi}^{0} \sqrt{2}\cos\frac{t}{2}\, dt = \left[2\sqrt{2}\sin\frac{t}{2}\right]_{-\pi}^{0}$
$= 2\sqrt{2}\left[\sin 0 - \sin\left(-\frac{\pi}{2}\right)\right] = 2\sqrt{2}$

67. $\int_{-\pi}^{0} \sqrt{1-\cos^2\theta}\, d\theta = \int_{-\pi}^{0} |\sin\theta|\, d\theta;\ \begin{bmatrix} \sin\theta \le 0 \\ \text{for } -\pi \le \theta \le 0 \end{bmatrix} \rightarrow \int_{-\pi}^{0} -\sin\theta\, d\theta = [\cos\theta]_{-\pi}^{0} = \cos 0 - \cos(-\pi)$
$= 1 - (-1) = 2$

68. $\int_{\pi/2}^{\pi} \sqrt{1-\sin^2\theta}\, d\theta = \int_{\pi/2}^{\pi} |\cos\theta|\, d\theta;\ \begin{bmatrix} \cos\theta \le 0 \\ \text{for } \frac{\pi}{2} \le \theta \le \pi \end{bmatrix} \rightarrow \int_{\pi/2}^{\pi} -\cos\theta\, d\theta = [-\sin\theta]_{\pi/2}^{\pi} = -\sin\pi + \sin\frac{\pi}{2} = 1$

69. $\int_{-\pi/4}^{\pi/4} \sqrt{\tan^2 y + 1}\, dy = \int_{-\pi/4}^{\pi/4} |\sec y|\, dy;\ \begin{bmatrix} \sec y \ge 0 \\ \text{for } -\frac{\pi}{4} \le y \le \frac{\pi}{4} \end{bmatrix} \rightarrow \int_{-\pi/4}^{\pi/4} \sec y\, dy = [\ln|\sec y + \tan y|]_{-\pi/4}^{\pi/4}$
$= \ln\left|\sqrt{2}+1\right| - \ln\left|\sqrt{2}-1\right|$

70. $\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy = \int_{-\pi/4}^{0} |\tan y|\, dy;\ \begin{bmatrix} \tan y \le 0 \\ \text{for } -\frac{\pi}{4} \le y \le 0 \end{bmatrix} \rightarrow \int_{-\pi/4}^{0} -\tan y\, dy = [\ln|\cos y|]_{-\pi/4}^{0} = -\ln\left(\frac{1}{\sqrt{2}}\right)$
$= \ln\sqrt{2}$

71. $\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx = \int_{\pi/4}^{3\pi/4} (\csc^2 x - 2\csc x\cot x + \cot^2 x)\, dx = \int_{\pi/4}^{3\pi/4} (2\csc^2 x - 1 - 2\csc x\cot x)\, dx$
$= [-2\cot x - x + 2\csc x]_{\pi/4}^{3\pi/4} = \left(-2\cot\frac{3\pi}{4} - \frac{3\pi}{4} + 2\csc\frac{3\pi}{4}\right) - \left(-2\cot\frac{\pi}{4} - \frac{\pi}{4} + 2\csc\frac{\pi}{4}\right)$
$= \left[-2(-1) - \frac{3\pi}{4} + 2\left(\sqrt{2}\right)\right] - \left[-2(1) - \frac{\pi}{4} + 2\left(\sqrt{2}\right)\right] = 4 - \frac{\pi}{2}$

72. $\int_0^{\pi/4} (\sec x + 4\cos x)^2\, dx = \int_0^{\pi/4} \left[\sec^2 x + 8 + 16\left(\frac{1+\cos 2x}{2}\right)\right] dx = \left[\tan x + 16x - 4\sin 2x\right]_0^{\pi/4}$
$= \left(\tan\frac{\pi}{4} + 4\pi - 4\sin\frac{\pi}{2}\right) - (\tan 0 + 0 - 4\sin 0) = 5 + 4\pi$

73. $\int \cos\theta \csc(\sin\theta)\, d\theta;\ \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \csc u\, du = -\ln|\csc u + \cot u| + C$
$= -\ln|\csc(\sin\theta) + \cot(\sin\theta)| + C$

74. $\int \left(1 + \frac{1}{x}\right)\cot(x + \ln x)\, dx;\ \begin{bmatrix} u = x + \ln x \\ du = \left(1 + \frac{1}{x}\right) dx \end{bmatrix} \rightarrow \int \cot u\, du = \ln|\sin u| + C = \ln|\sin(x + \ln x)| + C$

75. $\int (\csc x - \sec x)(\sin x + \cos x)\, dx = \int (1 + \cot x - \tan x - 1)\, dx = \int \cot x\, dx - \int \tan x\, dx$
$= \ln|\sin x| + \ln|\cos x| + C$

76. $\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx = \begin{bmatrix} u = \frac{x}{2} + \ln 5 \\ 2\, du = dx \end{bmatrix} = 6\int \sinh u\, du = 6\cosh u + C = 6\cosh\left(\frac{x}{2} + \ln 5\right) + C$

77. $\int \frac{6\, dy}{\sqrt{y}\,(1+y)};\ \begin{bmatrix} u = \sqrt{y} \\ du = \frac{1}{2\sqrt{y}}\, dy \end{bmatrix} \rightarrow \int \frac{12\, du}{1+u^2} = 12\tan^{-1} u + C = 12\tan^{-1}\sqrt{y} + C$

78. $\int \frac{dx}{x\sqrt{4x^2-1}} = \int \frac{2\, dx}{2x\sqrt{(2x)^2-1}};\ \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + C = \sec^{-1}|2x| + C$

79. $\int \frac{7\, dx}{(x-1)\sqrt{x^2-2x-48}} = \int \frac{7\, dx}{(x-1)\sqrt{(x-1)^2-49}};\ \begin{bmatrix} u = x - 1 \\ du = dx \end{bmatrix} \rightarrow \int \frac{7\, du}{u\sqrt{u^2-49}} = 7 \cdot \frac{1}{7}\sec^{-1}\left|\frac{u}{7}\right| + C$
$= \sec^{-1}\left|\frac{x-1}{7}\right| + C$

80. $\int \frac{dx}{(2x+1)\sqrt{4x^2+4x}} = \int \frac{dx}{(2x+1)\sqrt{(2x+1)^2-1}};\ \begin{bmatrix} u = 2x + 1 \\ du = 2\, dx \end{bmatrix} \rightarrow \int \frac{du}{2u\sqrt{u^2-1}} = \frac{1}{2}\sec^{-1}|u| + C$
$= \frac{1}{2}\sec^{-1}|2x+1| + C$

81. $\int \sec^2 t \tan(\tan t)\, dt;\ \begin{bmatrix} u = \tan t \\ du = \sec^2 t\, dt \end{bmatrix} \rightarrow \int \tan u\, du = -\ln|\cos u| + C = \ln|\sec u| + C = \ln|\sec(\tan t)| + C$

82. $\int \frac{dx}{x\sqrt{3+x^2}} = -\frac{1}{3}\operatorname{csch}^{-1}\left|\frac{x}{\sqrt{3}}\right| + C$

83. (a) $\int \cos^3\theta\, d\theta = \int (\cos\theta)(1 - \sin^2\theta)\, d\theta;\ \begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int (1 - u^2)\, du = u - \frac{u^3}{3} + C = \sin\theta - \frac{1}{3}\sin^3\theta + C$

(b) $\int \cos^5\theta\, d\theta = \int (\cos\theta)(1 - \sin^2\theta)^2\, d\theta = \int (1 - u^2)^2\, du = \int (1 - 2u^2 + u^4)\, du = u - \frac{2}{3}u^3 + \frac{u^5}{5} + C$
$= \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta + C$

(c) $\int \cos^9\theta\, d\theta = \int (\cos^8\theta)(\cos\theta)\, d\theta = \int (1 - \sin^2\theta)^4(\cos\theta)\, d\theta$

84. (a) $\int \sin^3\theta\, d\theta = \int (1 - \cos^2\theta)(\sin\theta)\, d\theta;\ \begin{bmatrix} u = \cos\theta \\ du = -\sin\theta\, d\theta \end{bmatrix} \rightarrow \int (1 - u^2)(-du) = \frac{u^3}{3} - u + C$
$= -\cos\theta + \frac{1}{3}\cos^3\theta + C$

(b) $\int \sin^5\theta\, d\theta = \int (1 - \cos^2\theta)^2(\sin\theta)\, d\theta = \int (1 - u^2)^2(-du) = \int (-1 + 2u^2 - u^4)\, du$
$= -\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C$

(c) $\int \sin^7 \theta \, d\theta = \int (1-u^2)^3(-du) = \int (-1+3u^2-3u^4+u^6)\, du = -\cos\theta + \cos^3\theta - \frac{3}{5}\cos^5\theta + \frac{\cos^7\theta}{7} + C$

(d) $\int \sin^{13}\theta \, d\theta = \int (\sin^{12}\theta)(\sin\theta)\, d\theta = \int (1-\cos^2\theta)^6(\sin\theta)\, d\theta$

85. (a) $\int \tan^3\theta \, d\theta = \int (\sec^2\theta - 1)(\tan\theta)\, d\theta = \int \sec^2\theta \tan\theta \, d\theta - \int \tan\theta \, d\theta = \frac{1}{2}\tan^2\theta - \int \tan\theta\, d\theta$
$= \frac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$

(b) $\int \tan^5\theta \, d\theta = \int (\sec^2\theta - 1)(\tan^3\theta)\, d\theta = \int \tan^3\theta \sec^2\theta \, d\theta - \int \tan^3\theta\, d\theta = \frac{1}{4}\tan^4\theta - \int \tan^3\theta\, d\theta$

(c) $\int \tan^7\theta \, d\theta = \int (\sec^2\theta - 1)(\tan^5\theta)\, d\theta = \int \tan^5\theta \sec^2\theta \, d\theta - \int \tan^5\theta\, d\theta = \frac{1}{6}\tan^6\theta - \int \tan^5\theta\, d\theta$

(d) $\int \tan^{2k+1}\theta \, d\theta = \int (\sec^2\theta - 1)(\tan^{2k-1}\theta)\, d\theta = \int \tan^{2k-1}\theta \sec^2\theta \, d\theta - \int \tan^{2k-1}\theta\, d\theta;$

$\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \to \int u^{2k-1}\, du - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}u^{2k} - \int \tan^{2k-1}\theta\, d\theta = \frac{1}{2k}\tan^{2k}\theta - \int \tan^{2k-1}\theta\, d\theta$

86. (a) $\int \cot^3\theta \, d\theta = \int (\csc^2\theta - 1)(\cot\theta)\, d\theta = \int \cot\theta \csc^2\theta \, d\theta - \int \cot\theta \, d\theta = -\frac{1}{2}\cot^2\theta - \int \cot\theta\, d\theta$
$= -\frac{1}{2}\cot^2\theta - \ln|\sin\theta| + C$

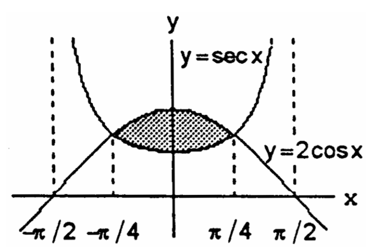
(b) $\int \cot^5\theta \, d\theta = \int (\csc^2\theta - 1)(\cot^3\theta)\, d\theta = \int \cot^3\theta \csc^2\theta \, d\theta - \int \cot^3\theta\, d\theta = -\frac{1}{4}\cot^4\theta - \int \cot^3\theta\, d\theta$

(c) $\int \cot^7\theta \, d\theta = \int (\csc^2\theta - 1)(\cot^5\theta)\, d\theta = \int \cot^5\theta \csc^2\theta \, d\theta - \int \cot^5\theta\, d\theta = -\frac{1}{6}\cot^6\theta - \int \cot^5\theta\, d\theta$

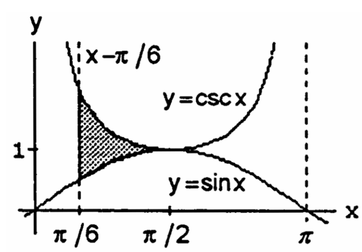
(d) $\int \cot^{2k+1}\theta \, d\theta = \int (\csc^2\theta - 1)(\cot^{2k-1}\theta)\, d\theta = \int \cot^{2k-1}\theta \csc^2\theta \, d\theta - \int \cot^{2k-1}\theta\, d\theta;$

$\begin{bmatrix} u = \cot\theta \\ du = -\csc^2\theta\, d\theta \end{bmatrix} \to -\int u^{2k-1}\, du - \int \cot^{2k-1}\theta\, d\theta = -\frac{1}{2k}u^{2k} - \int \cot^{2k-1}\theta\, d\theta$
$= -\frac{1}{2k}\cot^{2k}\theta - \int \cot^{2k-1}\theta\, d\theta$

87. $A = \int_{-\pi/4}^{\pi/4} (2\cos x - \sec x)\, dx = [2\sin x - \ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4}$
$= \left[\sqrt{2} - \ln\left(\sqrt{2}+1\right)\right] - \left[-\sqrt{2} - \ln\left(\sqrt{2}-1\right)\right]$
$= 2\sqrt{2} - \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\sqrt{2} - \ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right)$
$= 2\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)$




88. $A = \int_{\pi/6}^{\pi/2} (\csc x - \sin x)\, dx = [-\ln|\csc x + \cot x| + \cos x]_{\pi/6}^{\pi/2}$
$= -\ln|1+0| + \ln\left|2+\sqrt{3}\right| - \frac{\sqrt{3}}{2} = \ln\left(2+\sqrt{3}\right) - \frac{\sqrt{3}}{2}$



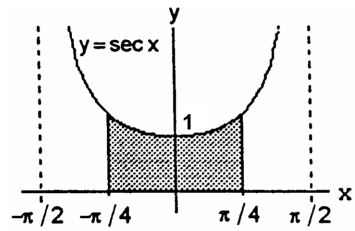

89. $V = \int_{-\pi/4}^{\pi/4} \pi(2\cos x)^2\, dx - \int_{-\pi/4}^{\pi/4} \pi\sec^2 x\, dx = 4\pi \int_{-\pi/4}^{\pi/4} \cos^2 x\, dx - \pi \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$
$= 2\pi \int_{-\pi/4}^{\pi/4} (1+\cos 2x)\, dx - \pi\,[\tan x]_{-\pi/4}^{\pi/4} = 2\pi\left[x + \frac{1}{2}\sin 2x\right]_{-\pi/4}^{\pi/4} - \pi[1-(-1)]$
$= 2\pi\left[\left(\frac{\pi}{4}+\frac{1}{2}\right) - \left(-\frac{\pi}{4}-\frac{1}{2}\right)\right] - 2\pi = 2\pi\left(\frac{\pi}{2}+1\right) - 2\pi = \pi^2$

90. $V = \int_{\pi/6}^{\pi/2} \pi\csc^2 x\, dx - \int_{\pi/6}^{\pi/2} \pi\sin^2 x\, dx = \pi \int_{\pi/6}^{\pi/2} \csc^2 x\, dx - \frac{\pi}{2}\int_{\pi/6}^{\pi/2} (1-\cos 2x)\, dx$
$= \pi\,[-\cot x]_{\pi/6}^{\pi/2} - \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_{\pi/6}^{\pi/2} = \pi\left[0 - \left(-\sqrt{3}\right)\right] - \frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right) - \left(\frac{\pi}{6} - \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\right)\right]$
$= \pi\sqrt{3} - \frac{\pi}{2}\left(\frac{2\pi}{6} + \frac{\sqrt{3}}{4}\right) = \pi\left(\frac{7\sqrt{3}}{8} - \frac{\pi}{6}\right)$

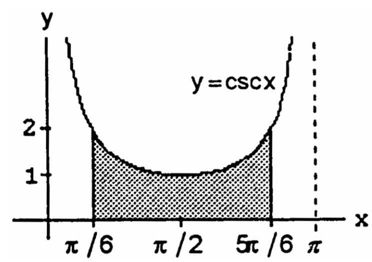
91. $y = \ln(\cos x) \Rightarrow \frac{dy}{dx} = -\frac{\sin x}{\cos x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1$; $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$= \int_0^{\pi/3} \sqrt{1 + (\sec^2 x - 1)}\, dx = \int_0^{\pi/3} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/3} = \ln\left|2 + \sqrt{3}\right| - \ln|1 + 0| = \ln\left(2 + \sqrt{3}\right)$

92. $y = \ln(\sec x) \Rightarrow \frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \tan^2 x = \sec^2 x - 1$; $L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$= \int_0^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4} = \ln\left|\sqrt{2} + 1\right| - \ln|1 + 0| = \ln\left(\sqrt{2} + 1\right)$

93. $M_x = \int_{-\pi/4}^{\pi/4} \left(\frac{1}{2} \sec x\right)(\sec x)\, dx = \frac{1}{2} \int_{-\pi/4}^{\pi/4} \sec^2 x\, dx$
$= \frac{1}{2}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2}[1 - (-1)] = 1$;
$M = \int_{-\pi/4}^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4}$
$= \ln\left|\sqrt{2} + 1\right| - \ln\left|\sqrt{2} - 1\right| = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$
$= \ln\left(\frac{\left(\sqrt{2}+1\right)^2}{2-1}\right) = \ln\left(3 + 2\sqrt{2}\right)$; $\overline{x} = 0$ by
symmetry of the region, and $\overline{y} = \frac{M_x}{M} = \frac{1}{\ln\left(3+2\sqrt{2}\right)}$




94. $M_x = \int_{\pi/6}^{5\pi/6} \left(\frac{1}{2} \csc x\right)(\csc x)\, dx = \frac{1}{2} \int_{\pi/6}^{5\pi/6} \csc^2 x\, dx$
$= \frac{1}{2}[-\cot x]_{\pi/6}^{5\pi/6} = \frac{1}{2}\left[-\left(-\sqrt{3}\right) - \left(-\sqrt{3}\right)\right] = \sqrt{3}$;
$M = \int_{\pi/6}^{5\pi/6} \csc x\, dx = [-\ln|\csc x + \cot x|]_{\pi/6}^{5\pi/6}$
$= -\ln\left|2 - \sqrt{3}\right| - \left(-\ln\left|2 + \sqrt{3}\right|\right) = \ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|$
$= \ln\left(\frac{\left(2+\sqrt{3}\right)^2}{4-3}\right) = 2\ln\left(2 + \sqrt{3}\right)$; $\overline{x} = \frac{\pi}{2}$ by symmetry
of the region, and $\overline{y} = \frac{M_x}{M} = \frac{\sqrt{3}}{2\ln\left(2+\sqrt{3}\right)}$




95. $\int \csc x\, dx = \int (\csc x)(1)\, dx = \int (\csc x)\left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx = \int \frac{\csc^2 x + \csc x \cot x}{\csc x + \cot x}\, dx$;
$\left[\begin{array}{c} u = \csc x + \cot x \\ du = (-\csc x \cot x - \csc^2 x)\, dx \end{array}\right] \rightarrow \int \frac{-du}{u} = -\ln|u| + C = -\ln|\csc x + \cot x| + C$

96. $\left[(x^2 - 1)(x + 1)\right]^{-2/3} = \left[(x - 1)(x + 1)^2\right]^{-2/3} = (x - 1)^{-2/3}(x + 1)^{-4/3} = (x + 1)^{-2}\left[(x - 1)^{-2/3}(x + 1)^{2/3}\right]$
$= (x + 1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} = (x + 1)^{-2}\left(1 - \frac{2}{x+1}\right)^{-2/3}$

(a) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x + 1)^{-2}\left(1 - \frac{2}{x+1}\right)^{-2/3} dx$; $\left[\begin{array}{c} u = \frac{1}{x+1} \\ du = -\frac{1}{(x+1)^2}\, dx \end{array}\right]$
$\rightarrow \int -(1 - 2u)^{-2/3}\, du = \frac{3}{2}(1 - 2u)^{1/3} + C = \frac{3}{2}\left(1 - \frac{2}{x+1}\right)^{1/3} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(b) $\int \left[(x^2 - 1)(x + 1)\right]^{-2/3} dx = \int (x + 1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3} dx$; $u = \left(\frac{x-1}{x+1}\right)^k$
$\Rightarrow du = k\left(\frac{x-1}{x+1}\right)^{k-1} \frac{[(x+1)-(x-1)]}{(x+1)^2}\, dx = 2k \frac{(x-1)^{k-1}}{(x+1)^{k+1}}\, dx$; $dx = \frac{(x+1)^2}{2k}\left(\frac{x+1}{x-1}\right)^{k-1} du$
$= \frac{(x+1)^2}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du$; then, $\int \left(\frac{x-1}{x+1}\right)^{-2/3} \frac{1}{2k}\left(\frac{x-1}{x+1}\right)^{1-k} du = \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{(1/3-k)} du$
$= \frac{1}{2k}\int \left(\frac{x-1}{x+1}\right)^{k(1/3k-1)} du = \frac{1}{2k}\int u^{(1/3k-1)}\, du = \frac{1}{2k}(3k)\, u^{1/3k} + C = \frac{3}{2} u^{1/3k} + C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3} + C$

(c) $\int [(x^2-1)(x+1)]^{-2/3}\,dx = \int (x+1)^{-2}\left(\frac{x-1}{x+1}\right)^{-2/3}dx;$

$$\begin{bmatrix} u = \tan^{-1} x \\ x = \tan u \\ dx = \frac{du}{\cos^2 u}\end{bmatrix} \rightarrow \int \frac{1}{(\tan u+1)^2}\left(\frac{\tan u - 1}{\tan u + 1}\right)^{-2/3}\left(\frac{du}{\cos^2 u}\right) = \int \frac{1}{(\sin u+\cos u)^2}\left(\frac{\sin u - \cos u}{\sin u + \cos u}\right)^{-2/3} du;$$

$$\begin{bmatrix} \sin u + \cos u = \sin u + \sin\left(\frac{\pi}{2}-u\right) = 2\sin\frac{\pi}{4}\cos\left(u-\frac{\pi}{4}\right) \\ \sin u - \cos u = \sin u - \sin\left(\frac{\pi}{2}-u\right) = 2\cos\frac{\pi}{4}\sin\left(u-\frac{\pi}{4}\right)\end{bmatrix} \rightarrow \int \frac{1}{2\cos^2\left(u-\frac{\pi}{4}\right)}\left[\frac{\sin\left(u-\frac{\pi}{4}\right)}{\cos\left(u-\frac{\pi}{4}\right)}\right]^{-2/3} du$$

$$= \frac{1}{2}\int \tan^{-2/3}\left(u-\frac{\pi}{4}\right)\sec^2\left(u-\frac{\pi}{4}\right)du = \frac{3}{2}\tan^{1/3}\left(u-\frac{\pi}{4}\right)+C = \frac{3}{2}\left[\frac{\tan u - \tan\frac{\pi}{4}}{1+\tan u\tan\frac{\pi}{4}}\right]^{1/3}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(d) $u = \tan^{-1}\sqrt{x} \Rightarrow \tan u = \sqrt{x} \Rightarrow \tan^2 u = x \Rightarrow dx = 2\tan u\left(\frac{1}{\cos^2 u}\right)du = \frac{2\sin u}{\cos^3 u}\,du = -\frac{2d(\cos u)}{\cos^3 u};$

$x-1 = \tan^2 u - 1 = \frac{\sin^2 u - \cos^2 u}{\cos^2 u} = \frac{1-2\cos^2 u}{\cos^2 u};\ x+1 = \tan^2 u + 1 = \frac{\cos^2 u + \sin^2 u}{\cos^2 u} = \frac{1}{\cos^2 u};$

$$\int (x-1)^{-2/3}(x+1)^{-4/3}\,dx = \int \frac{(1-2\cos^2 u)^{-2/3}}{(\cos^2 u)^{-2/3}}\cdot\frac{1}{(\cos^2 u)^{-4/3}}\cdot\frac{-2d(\cos u)}{\cos^3 u}$$

$$= \int (1-2\cos^2 u)^{-2/3}\cdot(-2)\cdot\cos u\cdot d(\cos u) = \frac{1}{2}\int (1-2\cos^2 u)^{-2/3}\cdot d\left(1-2\cos^2 u\right)$$

$$= \frac{3}{2}(1-2\cos^2 u)^{1/3}+C = \frac{3}{2}\left[\frac{\left(\frac{1-2\cos^2 u}{\cos^2 u}\right)}{\left(\frac{1}{\cos^2 u}\right)}\right]^{1/3}+C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(e) $u = \tan^{-1}\left(\frac{x-1}{2}\right) \Rightarrow \frac{x-1}{2} = \tan u \Rightarrow x+1 = 2(\tan u + 1) \Rightarrow dx = \frac{2\,du}{\cos^2 u} = 2d(\tan u);$

$$\int (x-1)^{-2/3}(x+1)^{-4/3}\,dx = \int (\tan u)^{-2/3}(\tan u+1)^{-4/3}\cdot 2^{-2}\cdot 2\cdot d(\tan u)$$

$$= \frac{1}{2}\int\left(1-\frac{1}{\tan u+1}\right)^{-2/3} d\left(1-\frac{1}{\tan u+1}\right) = \frac{3}{2}\left(1-\frac{1}{\tan u+1}\right)^{1/3}+C = \frac{3}{2}\left(1-\frac{2}{x+1}\right)^{1/3}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(f) $$\begin{bmatrix} u = \cos^{-1} x \\ x = \cos u \\ dx = -\sin u\,du\end{bmatrix} \rightarrow -\int \frac{\sin u\,du}{\sqrt[3]{(\cos^2 u - 1)^2(\cos u+1)^2}} = -\int \frac{\sin u\,du}{(\sin^{4/3} u)\left(2^{2/3}\cos\frac{u}{2}\right)^{4/3}}$$

$$= -\int \frac{du}{(\sin u)^{1/3}\left(2^{2/3}\cos\frac{u}{2}\right)^{4/3}} = -\int \frac{du}{2\left(\sin\frac{u}{2}\right)^{1/3}\left(\cos\frac{u}{2}\right)^{5/3}} = -\frac{1}{2}\int\left(\frac{\cos\frac{u}{2}}{\sin\frac{u}{2}}\right)^{1/3}\frac{du}{\left(\cos^2\frac{u}{2}\right)}$$

$$= -\int \tan^{-1/3}\left(\frac{u}{2}\right)d\left(\tan\frac{u}{2}\right) = -\frac{3}{2}\tan^{2/3}\frac{u}{2}+C = \frac{3}{2}\left(-\tan^2\frac{u}{2}\right)^{1/3}+C = \frac{3}{2}\left(\frac{\cos u - 1}{\cos u + 1}\right)^{1/3}+C$$

$$= \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

(g) $$\int [(x^2-1)(x+1)]^{-2/3}\,dx;\ \begin{bmatrix} u = \cosh^{-1} x \\ x = \cosh u \\ dx = \sinh u\end{bmatrix} \rightarrow \int \frac{\sinh u\,du}{\sqrt[3]{(\cosh^2 u - 1)^2(\cosh u+1)^2}}$$

$$= \int \frac{\sinh u\,du}{\sqrt[3]{(\sinh^4 u)(\cosh u+1)^2}} = \int \frac{du}{\sqrt[3]{(\sinh u)\left(4\cosh^4\frac{u}{2}\right)}} = \frac{1}{2}\int \frac{du}{\sqrt[3]{\sinh\left(\frac{u}{2}\right)\cosh^5\left(\frac{u}{2}\right)}}$$

$$= \int \left(\tanh\frac{u}{2}\right)^{-1/3}d\left(\tanh\frac{u}{2}\right) = \frac{3}{2}\left(\tanh\frac{u}{2}\right)^{2/3}+C = \frac{3}{2}\left(\frac{\cosh u - 1}{\cosh u + 1}\right)^{1/3}+C = \frac{3}{2}\left(\frac{x-1}{x+1}\right)^{1/3}+C$$

## 8.2 INTEGRATION BY PARTS

1. $u = x,\ du = dx;\ dv = \sin\frac{x}{2}\,dx,\ v = -2\cos\frac{x}{2};$

$$\int x\sin\frac{x}{2}\,dx = -2x\cos\frac{x}{2} - \int\left(-2\cos\frac{x}{2}\right)dx = -2x\cos\left(\frac{x}{2}\right)+4\sin\left(\frac{x}{2}\right)+C$$

2. $u = \theta,\ du = d\theta;\ dv = \cos\pi\theta\,d\theta,\ v = \frac{1}{\pi}\sin\pi\theta;$

$$\int \theta\cos\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta - \int\frac{1}{\pi}\sin\pi\theta\,d\theta = \frac{\theta}{\pi}\sin\pi\theta + \frac{1}{\pi^2}\cos\pi\theta + C$$

3.

$$\begin{array}{lcl} & & \cos t \\ t^2 & \xrightarrow{(+)} & \sin t \\ 2t & \xrightarrow{(-)} & -\cos t \\ 2 & \xrightarrow{(+)} & -\sin t \\ 0 & & \end{array}$$

$$\int t^2 \cos t\, dt = t^2 \sin t + 2t \cos t - 2 \sin t + C$$

4.

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$$\int x^2 \sin x\, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$$

5. $u = \ln x,\ du = \frac{dx}{x};\ dv = x\, dx,\ v = \frac{x^2}{2}$;

$$\int_1^2 x \ln x\, dx = \left[\frac{x^2}{2} \ln x\right]_1^2 - \int_1^2 \frac{x^2}{2}\,\frac{dx}{x} = 2 \ln 2 - \left[\frac{x^2}{4}\right]_1^2 = 2 \ln 2 - \frac{3}{4} = \ln 4 - \frac{3}{4}$$

6. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^3\, dx,\ v = \frac{x^4}{4}$;

$$\int_1^e x^3 \ln x\, dx = \left[\frac{x^4}{4} \ln x\right]_1^e - \int_1^e \frac{x^4}{4}\,\frac{dx}{x} = \frac{e^4}{4} - \left[\frac{x^4}{16}\right]_1^e = \frac{3e^4+1}{16}$$

7. $u = \tan^{-1} y,\ du = \frac{dy}{1+y^2};\ dv = dy,\ v = y$;

$$\int \tan^{-1} y\, dy = y \tan^{-1} y - \int \frac{y\, dy}{(1+y^2)} = y \tan^{-1} y - \frac{1}{2} \ln\left(1+y^2\right) + C = y \tan^{-1} y - \ln \sqrt{1+y^2} + C$$

8. $u = \sin^{-1} y,\ du = \frac{dy}{\sqrt{1-y^2}};\ dv = dy,\ v = y$;

$$\int \sin^{-1} y\, dy = y \sin^{-1} y - \int \frac{y\, dy}{\sqrt{1-y^2}} = y \sin^{-1} y + \sqrt{1-y^2} + C$$

9. $u = x,\ du = dx;\ dv = \sec^2 x\, dx,\ v = \tan x$;

$$\int x \sec^2 x\, dx = x \tan x - \int \tan x\, dx = x \tan x + \ln |\cos x| + C$$

10. $\int 4x \sec^2 2x\, dx;\ [y = 2x] \ \to\ \int y \sec^2 y\, dy = y \tan y - \int \tan y\, dy = y \tan y - \ln |\sec y| + C$
$= 2x \tan 2x - \ln |\sec 2x| + C$

11.

$$\begin{array}{lcl} & & e^x \\ x^3 & \xrightarrow{(+)} & e^x \\ 3x^2 & \xrightarrow{(-)} & e^x \\ 6x & \xrightarrow{(+)} & e^x \\ 6 & \xrightarrow{(-)} & e^x \\ 0 & & \end{array}$$

$$\int x^3 e^x\, dx = x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C = \left(x^3 - 3x^2 + 6x - 6\right) e^x + C$$

12.

$$\begin{array}{lcl} & & e^{-p} \\ p^4 & \xrightarrow{(+)} & -e^{-p} \\ 4p^3 & \xrightarrow{(-)} & e^{-p} \\ 12p^2 & \xrightarrow{(+)} & -e^{-p} \\ 24p & \xrightarrow{(-)} & e^{-p} \\ 24 & \xrightarrow{(+)} & -e^{-p} \\ 0 & & \end{array}$$

$$\int p^4 e^{-p}\, dp = -p^4 e^{-p} - 4p^3 e^{-p} - 12p^2 e^{-p} - 24pe^{-p} - 24e^{-p} + C$$
$$= \left(-p^4 - 4p^3 - 12p^2 - 24p - 24\right) e^{-p} + C$$

13.

$$\begin{array}{lcl} & & e^{x} \\ x^2 - 5x & \xrightarrow{(+)} & e^{x} \\ 2x - 5 & \xrightarrow{(-)} & e^{x} \\ 2 & \xrightarrow{(+)} & e^{x} \\ 0 & & \end{array}$$

$$\int \left(x^2 - 5x\right) e^x\, dx = \left(x^2 - 5x\right) e^x - (2x - 5)e^x + 2e^x + C = x^2 e^x - 7xe^x + 7e^x + C$$
$$= \left(x^2 - 7x + 7\right) e^x + C$$

14.

$$\begin{array}{lcl} & & e^{r} \\ r^2 + r + 1 & \xrightarrow{(+)} & e^{r} \\ 2r + 1 & \xrightarrow{(-)} & e^{r} \\ 2 & \xrightarrow{(+)} & e^{r} \\ 0 & & \end{array}$$

$$\int \left(r^2 + r + 1\right) e^r\, dr = \left(r^2 + r + 1\right) e^r - (2r + 1)\, e^r + 2e^r + C$$
$$= \left[\left(r^2 + r + 1\right) - (2r + 1) + 2\right] e^r + C = \left(r^2 - r + 2\right) e^r + C$$

15.

$$\begin{array}{lcl} & & e^{x} \\ x^5 & \xrightarrow{(+)} & e^{x} \\ 5x^4 & \xrightarrow{(-)} & e^{x} \\ 20x^3 & \xrightarrow{(+)} & e^{x} \\ 60x^2 & \xrightarrow{(-)} & e^{x} \\ 120x & \xrightarrow{(+)} & e^{x} \\ 120 & \xrightarrow{(-)} & e^{x} \\ 0 & & \end{array}$$

$$\int x^5 e^x\, dx = x^5 e^x - 5x^4 e^x + 20x^3 e^x - 60x^2 e^x + 120xe^x - 120e^x + C$$
$$= \left(x^5 - 5x^4 + 20x^3 - 60x^2 + 120x - 120\right) e^x + C$$

16.
$$\begin{array}{lcl} & & e^{4t} \\ t^2 & \xrightarrow{(+)} & \frac{1}{4} e^{4t} \\ 2t & \xrightarrow{(-)} & \frac{1}{16} e^{4t} \\ 2 & \xrightarrow{(+)} & \frac{1}{64} e^{4t} \\ 0 & & \end{array}$$

$$\int t^2 e^{4t}\, dt = \frac{t^2}{4} e^{4t} - \frac{2t}{16} e^{4t} + \frac{2}{64} e^{4t} + C = \frac{t^2}{4} e^{4t} - \frac{t}{8} e^{4t} + \frac{1}{32} e^{4t} + C$$
$$= \left(\frac{t^2}{4} - \frac{t}{8} + \frac{1}{32}\right) e^{4t} + C$$

17.
$$\begin{array}{lcl} & & \sin 2\theta \\ \theta^2 & \xrightarrow{(+)} & -\frac{1}{2}\cos 2\theta \\ 2\theta & \xrightarrow{(-)} & -\frac{1}{4}\sin 2\theta \\ 2 & \xrightarrow{(+)} & \frac{1}{8}\cos 2\theta \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta = \left[-\frac{\theta^2}{2}\cos 2\theta + \frac{\theta}{2}\sin 2\theta + \frac{1}{4}\cos 2\theta\right]_0^{\pi/2}$$
$$= \left[-\frac{\pi^2}{8}\cdot(-1) + \frac{\pi}{4}\cdot 0 + \frac{1}{4}\cdot(-1)\right] - \left[0 + 0 + \frac{1}{4}\cdot 1\right] = \frac{\pi^2}{8} - \frac{1}{2} = \frac{\pi^2 - 4}{8}$$

18.
$$\begin{array}{lcl} & & \cos 2x \\ x^3 & \xrightarrow{(+)} & \frac{1}{2}\sin 2x \\ 3x^2 & \xrightarrow{(-)} & -\frac{1}{4}\cos 2x \\ 6x & \xrightarrow{(+)} & -\frac{1}{8}\sin 2x \\ 6 & \xrightarrow{(-)} & \frac{1}{16}\cos 2x \\ 0 & & \end{array}$$

$$\int_0^{\pi/2} x^3 \cos 2x\, dx = \left[\frac{x^3}{2}\sin 2x + \frac{3x^2}{4}\cos 2x - \frac{3x}{4}\sin 2x - \frac{3}{8}\cos 2x\right]_0^{\pi/2}$$
$$= \left[\frac{\pi^3}{16}\cdot 0 + \frac{3\pi^2}{16}\cdot(-1) - \frac{3\pi}{8}\cdot 0 - \frac{3}{8}\cdot(-1)\right] - \left[0 + 0 - 0 - \frac{3}{8}\cdot 1\right] = -\frac{3\pi^2}{16} + \frac{3}{4} = \frac{3(4-\pi^2)}{16}$$

19. $u = \sec^{-1} t$, $du = \frac{dt}{t\sqrt{t^2-1}}$; $dv = t\, dt$, $v = \frac{t^2}{2}$;
$$\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt = \left[\frac{t^2}{2}\sec^{-1} t\right]_{2/\sqrt{3}}^{2} - \int_{2/\sqrt{3}}^{2}\left(\frac{t^2}{2}\right)\frac{dt}{t\sqrt{t^2-1}} = \left(2\cdot\frac{\pi}{3} - \frac{2}{3}\cdot\frac{\pi}{6}\right) - \int_{2/\sqrt{3}}^{2}\frac{t\, dt}{2\sqrt{t^2-1}}$$
$$= \frac{5\pi}{9} - \left[\frac{1}{2}\sqrt{t^2-1}\right]_{2/\sqrt{3}}^{2} = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \sqrt{\frac{4}{3}-1}\right) = \frac{5\pi}{9} - \frac{1}{2}\left(\sqrt{3} - \frac{\sqrt{3}}{3}\right) = \frac{5\pi}{9} - \frac{\sqrt{3}}{3} = \frac{5\pi - 3\sqrt{3}}{9}$$

20. $u = \sin^{-1}(x^2)$, $du = \frac{2x\, dx}{\sqrt{1-x^4}}$; $dv = 2x\, dx$, $v = x^2$;
$$\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx = \left[x^2 \sin^{-1}(x^2)\right]_0^{1/\sqrt{2}} - \int_0^{1/\sqrt{2}} x^2 \cdot \frac{2x\, dx}{\sqrt{1-x^4}} = \left(\frac{1}{2}\right)\left(\frac{\pi}{6}\right) + \int_0^{1/\sqrt{2}} \frac{d\left(1-x^4\right)}{2\sqrt{1-x^4}}$$
$$= \frac{\pi}{12} + \left[\sqrt{1-x^4}\right]_0^{1/\sqrt{2}} = \frac{\pi}{12} + \sqrt{\frac{3}{4}} - 1 = \frac{\pi + 6\sqrt{3} - 12}{12}$$

21. $I = \int e^{\theta} \sin\theta\, d\theta$; $[u = \sin\theta,\ du = \cos\theta\, d\theta;\ dv = e^{\theta}\, d\theta,\ v = e^{\theta}] \Rightarrow I = e^{\theta}\sin\theta - \int e^{\theta}\cos\theta\, d\theta$;
$[u = \cos\theta,\ du = -\sin\theta\, d\theta;\ dv = e^{\theta}\, d\theta,\ v = e^{\theta}] \Rightarrow I = e^{\theta}\sin\theta - \left(e^{\theta}\cos\theta + \int e^{\theta}\sin\theta\, d\theta\right)$
$= e^{\theta}\sin\theta - e^{\theta}\cos\theta - I + C' \Rightarrow 2I = (e^{\theta}\sin\theta - e^{\theta}\cos\theta) + C' \Rightarrow I = \frac{1}{2}(e^{\theta}\sin\theta - e^{\theta}\cos\theta) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

22. $I = \int e^{-y} \cos y \, dy$; $[u = \cos y, du = -\sin y \, dy; dv = e^{-y} \, dy, v = -e^{-y}]$
$\Rightarrow I = -e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy = -e^{-y} \cos y - \int e^{-y} \sin y \, dy$; $[u = \sin y, du = \cos y \, dy;$
$dv = e^{-y} \, dy, v = -e^{-y}] \Rightarrow I = -e^{-y} \cos y - \left(-e^{-y} \sin y - \int (-e^{y}) \cos y \, dy\right) = -e^{-y} \cos y + e^{-y} \sin y - I + C'$
$\Rightarrow 2I = e^{-y}(\sin y - \cos y) + C' \Rightarrow I = \frac{1}{2}(e^{-y} \sin y - e^{-y} \cos y) + C$, where $C = \frac{C'}{2}$ is another arbitrary constant

23. $I = \int e^{2x} \cos 3x \, dx$; $\left[u = \cos 3x; du = -3 \sin 3x \, dx, dv = e^{2x} \, dx; v = \frac{1}{2} e^{2x}\right]$
$\Rightarrow I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2} \int e^{2x} \sin 3x \, dx$; $\left[u = \sin 3x, du = 3 \cos 3x, dv = e^{2x} \, dx; v = \frac{1}{2} e^{2x}\right]$
$\Rightarrow I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{2}\left(\frac{1}{2} e^{2x} \sin 3x - \frac{3}{2} \int e^{2x} \cos 3x \, dx\right) = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x - \frac{9}{4} I + C'$
$\Rightarrow \frac{13}{4} I = \frac{1}{2} e^{2x} \cos 3x + \frac{3}{4} e^{2x} \sin 3x + C' \Rightarrow \frac{e^{2x}}{13}(3 \sin 3x + 2 \cos 3x) + C$, where $C = \frac{4}{13} C'$

24. $\int e^{-2x} \sin 2x \, dx$; $[y = 2x] \rightarrow \frac{1}{2} \int e^{-y} \sin y \, dy = I$; $[u = \sin y, du = \cos y \, dy; dv = e^{-y} \, dy, v = -e^{-y}]$
$\Rightarrow I = \frac{1}{2}\left(-e^{-y} \sin y + \int e^{-y} \cos y \, dy\right)$ $[u = \cos y, du = -\sin y; dv = e^{-y} \, dy, v = -e^{-y}]$
$\Rightarrow I = -\frac{1}{2} e^{-y} \sin y + \frac{1}{2}\left(-e^{-y} \cos y - \int (-e^{-y})(-\sin y) \, dy\right) = -\frac{1}{2} e^{-y}(\sin y + \cos y) - I + C'$
$\Rightarrow 2I = -\frac{1}{2} e^{-y}(\sin y + \cos y) + C' \Rightarrow I = -\frac{1}{4} e^{-y}(\sin y + \cos y) + C = -\frac{e^{-2x}}{4}(\sin 2x + \cos 2x) + C$, where
$C = \frac{C'}{2}$

25. $\int e^{\sqrt{3s+9}} \, ds$; $\begin{bmatrix} 3s + 9 = x^2 \\ ds = \frac{2}{3} x \, dx \end{bmatrix} \rightarrow \int e^{x} \cdot \frac{2}{3} x \, dx = \frac{2}{3} \int x e^{x} \, dx$; $[u = x, du = dx; dv = e^{x} \, dx, v = e^{x}]$;
$\frac{2}{3} \int x e^{x} \, dx = \frac{2}{3}\left(x e^{x} - \int e^{x} \, dx\right) = \frac{2}{3}(x e^{x} - e^{x}) + C = \frac{2}{3}\left(\sqrt{3s+9}\, e^{\sqrt{3s+9}} - e^{\sqrt{3s+9}}\right) + C$

26. $u = x, du = dx; dv = \sqrt{1-x} \, dx, v = -\frac{2}{3}\sqrt{(1-x)^3}$;
$\int_0^1 x\sqrt{1-x} \, dx = \left[-\frac{2}{3}\sqrt{(1-x)^3}\, x\right]_0^1 + \frac{2}{3} \int_0^1 \sqrt{(1-x)^3} \, dx = \frac{2}{3}\left[-\frac{2}{5}(1-x)^{5/2}\right]_0^1 = \frac{4}{15}$

27. $u = x, du = dx; dv = \tan^2 x \, dx, v = \int \tan^2 x \, dx = \int \frac{\sin^2 x}{\cos^2 x} \, dx = \int \frac{1 - \cos^2 x}{\cos^2 x} \, dx = \int \frac{dx}{\cos^2 x} - \int dx$
$= \tan x - x$; $\int_0^{\pi/3} x \tan^2 x \, dx = [x(\tan x - x)]_0^{\pi/3} - \int_0^{\pi/3} (\tan x - x) \, dx = \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \left[\ln|\cos x| + \frac{x^2}{2}\right]_0^{\pi/3}$
$= \frac{\pi}{3}\left(\sqrt{3} - \frac{\pi}{3}\right) + \ln \frac{1}{2} + \frac{\pi^2}{18} = \frac{\pi\sqrt{3}}{3} - \ln 2 - \frac{\pi^2}{18}$

28. $u = \ln(x + x^2), du = \frac{(2x+1)\,dx}{x+x^2}$; $dv = dx, v = x$; $\int \ln(x + x^2) \, dx = x \ln(x + x^2) - \int \frac{2x+1}{x(x+1)} \cdot x \, dx$
$= x \ln(x + x^2) - \int \frac{(2x+1)\,dx}{x+1} = x \ln(x + x^2) - \int \frac{2(x+1) - 1}{x+1} \, dx = x \ln(x + x^2) - 2x + \ln|x+1| + C$

29. $\int \sin(\ln x) \, dx$; $\begin{bmatrix} u = \ln x \\ du = \frac{1}{x} \, dx \\ dx = e^{u} \, du \end{bmatrix} \rightarrow \int (\sin u) e^{u} \, du$. From Exercise 21, $\int (\sin u) e^{u} \, du = e^{u}\left(\frac{\sin u - \cos u}{2}\right) + C$
$= \frac{1}{2}[-x \cos(\ln x) + x \sin(\ln x)] + C$

30. $\int z(\ln z)^2\,dz;\ \begin{bmatrix} u = \ln z \\ du = \frac{1}{z}\,dz \\ dz = e^u\,du \end{bmatrix} \rightarrow \int e^u \cdot u^2 \cdot e^u\,du = \int e^{2u} \cdot u^2\,du;$

$$\begin{array}{lcl} & & e^{2u} \\ u^2 & \xrightarrow{(+)} & \frac{1}{2}e^{2u} \\ 2u & \xrightarrow{(-)} & \frac{1}{4}e^{2u} \\ 2 & \xrightarrow{(+)} & \frac{1}{8}e^{2u} \\ 0 & & \end{array}$$

$$\int u^2e^{2u}\,du = \frac{u^2}{2}e^{2u} - \frac{u}{2}e^{2u} + \frac{1}{4}e^{2u} + C = \frac{e^{2u}}{4}\left[2u^2 - 2u + 1\right] + C$$
$$= \frac{z^2}{4}\left[2(\ln z)^2 - 2\ln z + 1\right] + C$$

31. (a) $u = x,\ du = dx;\ dv = \sin x\,dx,\ v = -\cos x;$

$S_1 = \int_0^{\pi} x\sin x\,dx = [-x\cos x]_0^{\pi} + \int_0^{\pi}\cos x\,dx = \pi + [\sin x]_0^{\pi} = \pi$

(b) $S_2 = -\int_{\pi}^{2\pi} x\sin x\,dx = -\left[[-x\cos x]_{\pi}^{2\pi} + \int_{\pi}^{2\pi}\cos x\,dx\right] = -\left[-3\pi + [\sin x]_{\pi}^{2\pi}\right] = 3\pi$

(c) $S_3 = \int_{2\pi}^{3\pi} x\sin x\,dx = [-x\cos x]_{2\pi}^{3\pi} + \int_{2\pi}^{3\pi}\cos x\,dx = 5\pi + [\sin x]_{2\pi}^{3\pi} = 5\pi$

(d) $S_{n+1} = (-1)^{n+1}\int_{n\pi}^{(n+1)\pi} x\sin x\,dx = (-1)^{n+1}\left[[-x\cos x]_{n\pi}^{(n+1)\pi} + [\sin x]_{n\pi}^{(n+1)\pi}\right]$

$= (-1)^{n+1}\left[-(n+1)\pi(-1)^n + n\pi(-1)^{n+1}\right] + 0 = (2n+1)\pi$

32. (a) $u = x,\ du = dx;\ dv = \cos x\,dx,\ v = \sin x;$

$S_1 = -\int_{\pi/2}^{3\pi/2} x\cos x\,dx = -\left[[x\sin x]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2}\sin x\,dx\right] = -\left(-\frac{3\pi}{2} - \frac{\pi}{2}\right) - [\cos x]_{\pi/2}^{3\pi/2} = 2\pi$

(b) $S_2 = \int_{3\pi/2}^{5\pi/2} x\cos x\,dx = [x\sin x]_{3\pi/2}^{5\pi/2} - \int_{3\pi/2}^{5\pi/2}\sin x\,dx = \left[\frac{5\pi}{2} - \left(-\frac{3\pi}{2}\right)\right] - [\cos x]_{3\pi/2}^{5\pi/2} = 4\pi$

(c) $S_3 = -\int_{5\pi/2}^{7\pi/2} x\cos x\,dx = -\left[[x\sin x]_{5\pi/2}^{7\pi/2} - \int_{5\pi/2}^{7\pi/2}\sin x\,dx\right] = -\left(-\frac{7\pi}{2} - \frac{5\pi}{2}\right) - [\cos x]_{5\pi/2}^{7\pi/2} = 6\pi$

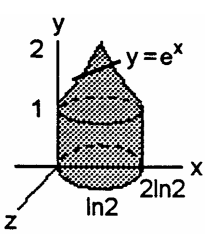
(d) $S_n = (-1)^n\int_{(2n-1)\pi/2}^{(2n+1)\pi/2} x\cos x\,dx = (-1)^n\left[[x\sin x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} - {}^{n}\int_{(2n-1)\pi/2}^{(2n+1)\pi/2}\sin x\,dx\right]$

$= (-1)^n\left[\frac{(2n+1)\pi}{2}(-1)^n - \frac{(2n-1)\pi}{2}(-1)^{n-1}\right] - [\cos x]_{(2n-1)\pi/2}^{(2n+1)\pi/2} = \frac{1}{2}(2n\pi + \pi + 2n\pi - \pi) = 2n\pi$

33. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)e^x\,dx = 2\pi\ln 2\int_0^{\ln 2} e^x\,dx - 2\pi\int_0^{\ln 2} xe^x\,dx$

$= (2\pi\ln 2)\left[e^x\right]_0^{\ln 2} - 2\pi\left([xe^x]_0^{\ln 2} - \int_0^{\ln 2} e^x\,dx\right)$

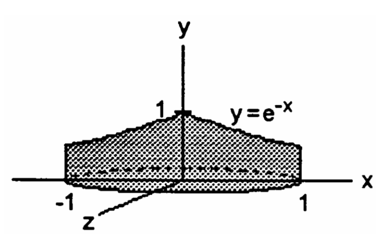
$= 2\pi\ln 2 - 2\pi\left(2\ln 2 - [e^x]_0^{\ln 2}\right) = -2\pi\ln 2 + 2\pi = 2\pi(1 - \ln 2)$




34. (a) $V = \int_0^1 2\pi xe^{-x}\,dx = 2\pi\left([-xe^{-x}]_0^1 + \int_0^1 e^{-x}\,dx\right)$

$= 2\pi\left(-\frac{1}{e} + [-e^{-x}]_0^1\right) = 2\pi\left(-\frac{1}{e} - \frac{1}{e} + 1\right)$

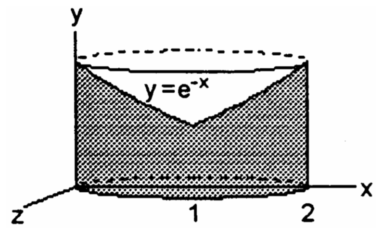
$= 2\pi - \frac{4\pi}{e}$



(b) $V = \int_0^1 2\pi(1-x)e^{-x}\,dx$; $u = 1-x$, $du = -dx$; $dv = e^{-x}\,dx$,

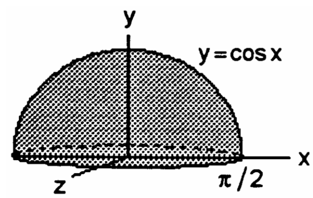
$v = -e^{-x}$; $V = 2\pi\left[\left[(1-x)\left(-e^{-x}\right)\right]_0^1 - \int_0^1 e^{-x}\,dx\right]$

$= 2\pi\left[[0-1(-1)] + \left[e^{-x}\right]_0^1\right] = 2\pi\left(1+\frac{1}{e}-1\right) = \frac{2\pi}{e}$




35. (a) $V = \int_0^{\pi/2} 2\pi x \cos x\,dx = 2\pi\left([x\sin x]_0^{\pi/2} - \int_0^{\pi/2}\sin x\,dx\right)$

$= 2\pi\left(\frac{\pi}{2} + [\cos x]_0^{\pi/2}\right) = 2\pi\left(\frac{\pi}{2}+0-1\right) = \pi(\pi-2)$




(b) $V = \int_0^{\pi/2} 2\pi\left(\frac{\pi}{2}-x\right)\cos x\,dx$; $u = \frac{\pi}{2}-x$, $du = -dx$; $dv = \cos x\,dx$, $v = \sin x$;

$V = 2\pi\left[\left(\frac{\pi}{2}-x\right)\sin x\right]_0^{\pi/2} + 2\pi\int_0^{\pi/2}\sin x\,dx = 0 + 2\pi[-\cos x]_0^{\pi/2} = 2\pi(0+1) = 2\pi$

36. (a) $V = \int_0^{\pi} 2\pi x(x\sin x)\,dx$;

$$\begin{array}{lcl} & & \sin x \\ x^2 & \xrightarrow{(+)} & -\cos x \\ 2x & \xrightarrow{(-)} & -\sin x \\ 2 & \xrightarrow{(+)} & \cos x \\ 0 & & \end{array}$$

$\Rightarrow V = 2\pi\int_0^{\pi} x^2\sin x\,dx = 2\pi\left[-x^2\cos x + 2x\sin x + 2\cos x\right]_0^{\pi} = 2\pi\left(\pi^2-4\right)$

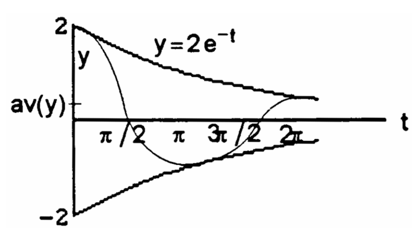
(b) $V = \int_0^{\pi} 2\pi(\pi-x)x\sin x\,dx = 2\pi^2\int_0^{\pi} x\sin x\,dx - 2\pi\int_0^{\pi} x^2\sin x\,dx = 2\pi^2\left[-x\cos x + \sin x\right]_0^{\pi} - \left(2\pi^3 - 8\pi\right)$

$= 8\pi$

37. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 2e^{-t}\cos t\,dt$

$= \frac{1}{\pi}\left[e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$

(see Exercise 22) $\Rightarrow$ $\text{av}(y) = \frac{1}{2\pi}\left(1-e^{-2\pi}\right)$

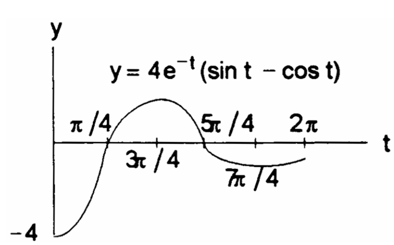



38. $\text{av}(y) = \frac{1}{2\pi}\int_0^{2\pi} 4e^{-t}(\sin t - \cos t)\,dt$

$= \frac{2}{\pi}\int_0^{2\pi} e^{-t}\sin t\,dt - \frac{2}{\pi}\int_0^{2\pi} e^{-t}\cos t\,dt$

$= \frac{2}{\pi}\left[e^{-t}\left(\frac{-\sin t - \cos t}{2}\right) - e^{-t}\left(\frac{\sin t - \cos t}{2}\right)\right]_0^{2\pi}$

$= \frac{2}{\pi}\left[-e^{-t}\sin t\right]_0^{2\pi} = 0$




39. $I = \int x^n\cos x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \cos x\,dx,\ v = \sin x]$

$\Rightarrow I = x^n\sin x - \int nx^{n-1}\sin x\,dx$

40. $I = \int x^n \sin x\,dx$; $[u = x^n,\ du = nx^{n-1}\,dx;\ dv = \sin x\,dx,\ v = -\cos x]$
$\Rightarrow I = -x^n \cos x + \int nx^{n-1} \cos x\,dx$

41. $I = \int x^n e^{ax}\,dx$; $\left[u = x^n,\ du = nx^{n-1}\,dx;\ dv = e^{ax}\,dx,\ v = \frac{1}{a}e^{ax}\right]$
$\Rightarrow I = \frac{x^n e^{ax}}{a} e^{ax} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx,\ a \neq 0$

42. $I = \int (\ln x)^n\,dx$; $\left[u = (\ln x)^n,\ du = \frac{n(\ln x)^{n-1}}{x}\,dx;\ dv = 1\,dx,\ v = x\right]$
$\Rightarrow I = x(\ln x)^n - \int n(\ln x)^{n-1}\,dx$

43. $\int \sin^{-1} x\,dx = x \sin^{-1} x - \int \sin y\,dy = x \sin^{-1} x + \cos y + C = x \sin^{-1} x + \cos(\sin^{-1} x) + C$

44. $\int \tan^{-1} x\,dx = x \tan^{-1} x - \int \tan y\,dy = x \tan^{-1} x + \ln|\cos y| + C = x \tan^{-1} x + \ln|\cos(\tan^{-1} x)| + C$

45. $\int \sec^{-1} x\,dx = x \sec^{-1} x - \int \sec y\,dy = x \sec^{-1} x - \ln|\sec y + \tan y| + C$
$= x \sec^{-1} x - \ln|\sec(\sec^{-1} x) + \tan(\sec^{-1} x)| + C = x \sec^{-1} x - \ln\left|x + \sqrt{x^2 - 1}\right| + C$

46. $\int \log_2 x\,dx = x \log_2 x - \int 2^y\,dy = x \log_2 x - \frac{2^y}{\ln 2} + C = x \log_2 x - \frac{x}{\ln 2} + C$

47. Yes, $\cos^{-1} x$ is the angle whose cosine is x which implies $\sin(\cos^{-1} x) = \sqrt{1 - x^2}$.

48. Yes, $\tan^{-1} x$ is the angle whose tangent is x which implies $\sec(\tan^{-1} x) = \sqrt{1 + x^2}$.

49. (a) $\int \sinh^{-1} x\,dx = x \sinh^{-1} x - \int \sinh y\,dy = x \sinh^{-1} x - \cosh y + C = x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C$;
check: $d[x \sinh^{-1} x - \cosh(\sinh^{-1} x) + C] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \sinh(\sinh^{-1} x)\frac{1}{\sqrt{1+x^2}}\right] dx$
$= \sinh^{-1} x\,dx$

(b) $\int \sinh^{-1} x\,dx = x \sinh^{-1} x - \int x\left(\frac{1}{\sqrt{1+x^2}}\right) dx = x \sinh^{-1} x - \frac{1}{2}\int (1 + x^2)^{-1/2} 2x\,dx$
$= x \sinh^{-1} x - (1 + x^2)^{1/2} + C$
check: $d\left[x \sinh^{-1} x - (1 + x^2)^{1/2} + C\right] = \left[\sinh^{-1} x + \frac{x}{\sqrt{1+x^2}} - \frac{x}{\sqrt{1+x^2}}\right] dx = \sinh^{-1} x\,dx$

50. (a) $\int \tanh^{-1} x\,dx = x \tanh^{-1} x - \int \tanh y\,dy = x \tanh^{-1} x - \ln|\cosh y| + C$
$= x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C$;
check: $d[x \tanh^{-1} x - \ln|\cosh(\tanh^{-1} x)| + C] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{\sinh(\tanh^{-1} x)}{\cosh(\tanh^{-1} x)}\frac{1}{1-x^2}\right] dx$
$= \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\,dx$

(b) $\int \tanh^{-1} x\,dx = x \tanh^{-1} x - \int \frac{x}{1-x^2}\,dx = x \tanh^{-1} x - \frac{1}{2}\int \frac{2x}{1-x^2}\,dx = x \tanh^{-1} x + \frac{1}{2}\ln|1 - x^2| + C$
check: $d\left[x \tanh^{-1} x + \frac{1}{2}\ln|1 - x^2| + C\right] = \left[\tanh^{-1} x + \frac{x}{1-x^2} - \frac{x}{1-x^2}\right] dx = \tanh^{-1} x\,dx$

## 8.3 INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

1. $\frac{5x - 13}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2} \Rightarrow 5x - 13 = A(x - 2) + B(x - 3) = (A + B)x - (2A + 3B)$
$\Rightarrow \left.\begin{array}{c} A + B = 5 \\ 2A + 3B = 13 \end{array}\right\} \Rightarrow -B = (10 - 13) \Rightarrow B = 3 \Rightarrow A = 2$; thus, $\frac{5x - 13}{(x-3)(x-2)} = \frac{2}{x-3} + \frac{3}{x-2}$

2. $\frac{5x-7}{x^2-3x+2} = \frac{5x-7}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1} \Rightarrow 5x-7 = A(x-1)+B(x-2) = (A+B)x-(A+2B)$
$\Rightarrow \left.\begin{array}{c} A+B=5 \\ A+2B=7 \end{array}\right\} \Rightarrow B=2 \Rightarrow A=3$; thus, $\frac{5x-7}{x^2-3x+2} = \frac{3}{x-2} + \frac{2}{x-1}$

3. $\frac{x+4}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow x+4 = A(x+1)+B = Ax+(A+B) \Rightarrow \left.\begin{array}{c} A=1 \\ A+B=4 \end{array}\right\} \Rightarrow A=1$ and $B=3$;
thus, $\frac{x+4}{(x+1)^2} = \frac{1}{x+1} + \frac{3}{(x+1)^2}$

4. $\frac{2x+2}{x^2-2x+1} = \frac{2x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 2x+2 = A(x-1)+B = Ax+(-A+B) \Rightarrow \left.\begin{array}{c} A=2 \\ -A+B=2 \end{array}\right\}$
$\Rightarrow A=2$ and $B=4$; thus, $\frac{2x+2}{x^2-2x+1} = \frac{2}{x-1} + \frac{4}{(x-1)^2}$

5. $\frac{z+1}{z^2(z-1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z-1} \Rightarrow z+1 = Az(z-1)+B(z-1)+Cz^2 \Rightarrow z+1 = (A+C)z^2+(-A+B)z-B$
$\Rightarrow \left.\begin{array}{c} A+C=0 \\ -A+B=1 \\ -B=1 \end{array}\right\} \Rightarrow B=-1 \Rightarrow A=-2 \Rightarrow C=2$; thus, $\frac{z+1}{z^2(z-1)} = \frac{-2}{z} + \frac{-1}{z^2} + \frac{2}{z-1}$

6. $\frac{z}{z^3-z^2-6z} = \frac{1}{z^2-z-6} = \frac{1}{(z-3)(z+2)} = \frac{A}{z-3} + \frac{B}{z+2} \Rightarrow 1 = A(z+2)+B(z-3) = (A+B)z+(2A-3B)$
$\Rightarrow \left.\begin{array}{c} A+B=0 \\ 2A-3B=1 \end{array}\right\} \Rightarrow -5B=1 \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5}$; thus, $\frac{z}{z^3-z^2-6z} = \frac{\frac{1}{5}}{z-3} + \frac{-\frac{1}{5}}{z+2}$

7. $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{5t+2}{t^2-5t+6}$ (after long division); $\frac{5t+2}{t^2-5t+6} = \frac{5t+2}{(t-3)(t-2)} = \frac{A}{t-3} + \frac{B}{t-2}$
$\Rightarrow 5t+2 = A(t-2)+B(t-3) = (A+B)t+(-2A-3B) \Rightarrow \left.\begin{array}{c} A+B=5 \\ -2A-3B=2 \end{array}\right\} \Rightarrow -B = (10+2) = 12$
$\Rightarrow B=-12 \Rightarrow A=17$; thus, $\frac{t^2+8}{t^2-5t+6} = 1 + \frac{17}{t-3} + \frac{-12}{t-2}$

8. $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^4+9t^2} = 1 + \frac{-9t^2+9}{t^2(t^2+9)}$ (after long division); $\frac{-9t^2+9}{t^2(t^2+9)} = \frac{A}{t} + \frac{B}{t^2} + \frac{Ct+D}{t^2+9}$
$\Rightarrow -9t^2+9 = At(t^2+9)+B(t^2+9)+(Ct+D)t^2 = (A+C)t^3+(B+D)t^2+9At+9B$
$\Rightarrow \left.\begin{array}{c} A+C=0 \\ B+D=-9 \\ 9A=0 \\ 9B=9 \end{array}\right\} \Rightarrow A=0 \Rightarrow C=0; B=1 \Rightarrow D=-10$; thus, $\frac{t^4+9}{t^4+9t^2} = 1 + \frac{1}{t^2} + \frac{-10}{t^2+9}$

9. $\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x} \Rightarrow 1 = A(1+x)+B(1-x); x=1 \Rightarrow A=\frac{1}{2}; x=-1 \Rightarrow B=\frac{1}{2}$;
$\int \frac{dx}{1-x^2} = \frac{1}{2}\int \frac{dx}{1-x} + \frac{1}{2}\int \frac{dx}{1+x} = \frac{1}{2}[\ln|1+x| - \ln|1-x|] + C$

10. $\frac{1}{x^2+2x} = \frac{A}{x} + \frac{B}{x+2} \Rightarrow 1 = A(x+2)+Bx; x=0 \Rightarrow A=\frac{1}{2}; x=-2 \Rightarrow B=-\frac{1}{2}$;
$\int \frac{dx}{x^2+2x} = \frac{1}{2}\int \frac{dx}{x} - \frac{1}{2}\int \frac{dx}{x+2} = \frac{1}{2}[\ln|x| - \ln|x+2|] + C$

11. $\frac{x+4}{x^2+5x-6} = \frac{A}{x+6} + \frac{B}{x-1} \Rightarrow x+4 = A(x-1)+B(x+6); x=1 \Rightarrow B=\frac{5}{7}; x=-6 \Rightarrow A=\frac{-2}{-7}=\frac{2}{7}$;
$\int \frac{x+4}{x^2+5x-6}\,dx = \frac{2}{7}\int \frac{dx}{x+6} + \frac{5}{7}\int \frac{dx}{x-1} = \frac{2}{7}\ln|x+6| + \frac{5}{7}\ln|x-1| + C = \frac{1}{7}\ln|(x+6)^2(x-1)^5| + C$

12. $\frac{2x+1}{x^2-7x+12} = \frac{A}{x-4} + \frac{B}{x-3} \Rightarrow 2x+1 = A(x-3)+B(x-4); x=3 \Rightarrow B=\frac{7}{-1}=-7; x=4 \Rightarrow A=\frac{9}{1}=9$;
$\int \frac{2x+1}{x^2-7x+12}\,dx = 9\int \frac{dx}{x-4} - 7\int \frac{dx}{x-3} = 9\ln|x-4| - 7\ln|x-3| + C = \ln\left|\frac{(x-4)^9}{(x-3)^7}\right| + C$

13. $\frac{y}{y^2-2y-3} = \frac{A}{y-3} + \frac{B}{y+1} \Rightarrow y = A(y+1) + B(y-3);\ y = -1 \Rightarrow B = \frac{-1}{-4} = \frac{1}{4};\ y = 3 \Rightarrow A = \frac{3}{4};$
$\int_4^8 \frac{y\,dy}{y^2-2y-3} = \frac{3}{4}\int_4^8 \frac{dy}{y-3} + \frac{1}{4}\int_4^8 \frac{dy}{y+1} = \left[\frac{3}{4}\ln|y-3| + \frac{1}{4}\ln|y+1|\right]_4^8 = \left(\frac{3}{4}\ln 5 + \frac{1}{4}\ln 9\right) - \left(\frac{3}{4}\ln 1 + \frac{1}{4}\ln 5\right)$
$= \frac{1}{2}\ln 5 + \frac{1}{2}\ln 3 = \frac{\ln 15}{2}$

14. $\frac{y+4}{y^2+y} = \frac{A}{y} + \frac{B}{y+1} \Rightarrow y+4 = A(y+1) + By;\ y = 0 \Rightarrow A = 4;\ y = -1 \Rightarrow B = \frac{3}{-1} = -3;$
$\int_{1/2}^1 \frac{y+4}{y^2+y}\,dy = 4\int_{1/2}^1 \frac{dy}{y} - 3\int_{1/2}^1 \frac{dy}{y+1} = [4\ln|y| - 3\ln|y+1|]_{1/2}^1 = (4\ln 1 - 3\ln 2) - \left(4\ln\frac{1}{2} - 3\ln\frac{3}{2}\right)$
$= \ln\frac{1}{8} - \ln\frac{1}{16} + \ln\frac{27}{8} = \ln\left(\frac{27}{8}\cdot\frac{1}{8}\cdot 16\right) = \ln\frac{27}{4}$

15. $\frac{1}{t^3+t^2-2t} = \frac{A}{t} + \frac{B}{t+2} + \frac{C}{t-1} \Rightarrow 1 = A(t+2)(t-1) + Bt(t-1) + Ct(t+2);\ t = 0 \Rightarrow A = -\frac{1}{2};\ t = -2$
$\Rightarrow B = \frac{1}{6};\ t = 1 \Rightarrow C = \frac{1}{3};\ \int \frac{dt}{t^3+t^2-2t} = -\frac{1}{2}\int \frac{dt}{t} + \frac{1}{6}\int \frac{dt}{t+2} + \frac{1}{3}\int \frac{dt}{t-1}$
$= -\frac{1}{2}\ln|t| + \frac{1}{6}\ln|t+2| + \frac{1}{3}\ln|t-1| + C$

16. $\frac{x+3}{2x^3-8x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-2} \Rightarrow \frac{1}{2}(x+3) = A(x+2)(x-2) + Bx(x-2) + Cx(x+2);\ x = 0 \Rightarrow A = \frac{3}{-8};\ x = -2$
$\Rightarrow B = \frac{1}{16};\ x = 2 \Rightarrow C = \frac{5}{16};\ \int \frac{x+3}{2x^3-8x}\,dx = -\frac{3}{8}\int \frac{dx}{x} + \frac{1}{16}\int \frac{dx}{x+2} + \frac{5}{16}\int \frac{dx}{x-2}$
$= -\frac{3}{8}\ln|x| + \frac{1}{16}\ln|x+2| + \frac{5}{16}\ln|x-2| + C = \frac{1}{16}\ln\left|\frac{(x-2)^5(x+2)}{x^6}\right| + C$

17. $\frac{x^3}{x^2+2x+1} = (x-2) + \frac{3x+2}{(x+1)^2}$ (after long division); $\frac{3x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \Rightarrow 3x+2 = A(x+1) + B$
$= Ax + (A+B) \Rightarrow A = 3,\ A+B = 2 \Rightarrow A = 3,\ B = -1;\ \int_0^1 \frac{x^3\,dx}{x^2+2x+1}$
$= \int_0^1 (x-2)\,dx + 3\int_0^1 \frac{dx}{x+1} - \int_0^1 \frac{dx}{(x+1)^2} = \left[\frac{x^2}{2} - 2x + 3\ln|x+1| + \frac{1}{x+1}\right]_0^1$
$= \left(\frac{1}{2} - 2 + 3\ln 2 + \frac{1}{2}\right) - (1) = 3\ln 2 - 2$

18. $\frac{x^3}{x^2-2x+1} = (x+2) + \frac{3x-2}{(x-1)^2}$ (after long division); $\frac{3x-2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} \Rightarrow 3x-2 = A(x-1) + B$
$= Ax + (-A+B) \Rightarrow A = 3,\ -A+B = -2 \Rightarrow A = 3,\ B = 1;\ \int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$
$= \int_{-1}^0 (x+2)\,dx + 3\int_{-1}^0 \frac{dx}{x-1} + \int_{-1}^0 \frac{dx}{(x-1)^2} = \left[\frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1}\right]_{-1}^0$
$= \left(0 + 0 + 3\ln 1 - \frac{1}{(-1)}\right) - \left(\frac{1}{2} - 2 + 3\ln 2 - \frac{1}{(-2)}\right) = 2 - 3\ln 2$

19. $\frac{1}{(x^2-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x+1)^2} + \frac{D}{(x-1)^2} \Rightarrow 1 = A(x+1)(x-1)^2 + B(x-1)(x+1)^2 + C(x-1)^2 + D(x+1)^2;$
$x = -1 \Rightarrow C = \frac{1}{4};\ x = 1 \Rightarrow D = \frac{1}{4}$; coefficient of $x^3 = A + B \Rightarrow A + B = 0$; constant $= A - B + C + D$
$\Rightarrow A - B + C + D = 1 \Rightarrow A - B = \frac{1}{2}$; thus, $A = \frac{1}{4} \Rightarrow B = -\frac{1}{4};\ \int \frac{dx}{(x^2-1)^2}$
$= \frac{1}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{4}\int \frac{dx}{(x+1)^2} + \frac{1}{4}\int \frac{dx}{(x-1)^2} = \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| - \frac{x}{2(x^2-1)} + C$

20. $\frac{x^2}{(x-1)(x^2+2x+1)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{(x+1)^2} \Rightarrow x^2 = A(x+1)^2 + B(x-1)(x+1) + C(x-1);\ x = -1$
$\Rightarrow C = -\frac{1}{2};\ x = 1 \Rightarrow A = \frac{1}{4}$; coefficient of $x^2 = A + B \Rightarrow A + B = 1 \Rightarrow B = \frac{3}{4};\ \int \frac{x^2\,dx}{(x-1)(x^2+2x+1)}$
$= \frac{1}{4}\int \frac{dx}{x-1} + \frac{3}{4}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{(x+1)^2} = \frac{1}{4}\ln|x-1| + \frac{3}{4}\ln|x+1| + \frac{1}{2(x+1)} + C$
$= \frac{\ln|(x-1)(x+1)^3|}{4} + \frac{1}{2(x+1)} + C$

21. $\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1} \Rightarrow 1 = A(x^2+1) + (Bx+C)(x+1);\ x = -1 \Rightarrow A = \frac{1}{2}$; coefficient of $x^2$
$= A + B \Rightarrow A + B = 0 \Rightarrow B = -\frac{1}{2}$; constant $= A + C \Rightarrow A + C = 1 \Rightarrow C = \frac{1}{2};\ \int_0^1 \frac{dx}{(x+1)(x^2+1)}$

$= \frac{1}{2}\int_0^1 \frac{dx}{x+1} + \frac{1}{2}\int_0^1 \frac{(-x+1)}{x^2+1}\,dx = \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1}x\right]_0^1$
$= \left(\frac{1}{2}\ln 2 - \frac{1}{4}\ln 2 + \frac{1}{2}\tan^{-1}1\right) - \left(\frac{1}{2}\ln 1 - \frac{1}{4}\ln 1 + \frac{1}{2}\tan^{-1}0\right) = \frac{1}{4}\ln 2 + \frac{1}{2}\left(\frac{\pi}{4}\right) = \frac{(\pi + 2\ln 2)}{8}$

22. $\frac{3t^2+t+4}{t^3+t} = \frac{A}{t} + \frac{Bt+C}{t^2+1} \Rightarrow 3t^2+t+4 = A(t^2+1) + (Bt+C)t$; $t = 0 \Rightarrow A = 4$; coefficient of $t^2$ $= A + B \Rightarrow A + B = 3 \Rightarrow B = -1$; coefficient of $t = C \Rightarrow C = 1$; $\int_1^{\sqrt{3}} \frac{3t^2+t+4}{t^3+1}\,dt$
$= 4\int_1^{\sqrt{3}} \frac{dt}{t} + \int_1^{\sqrt{3}} \frac{(-t+1)}{t^2+1}\,dt = \left[4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1}t\right]_1^{\sqrt{3}}$
$= \left(4\ln\sqrt{3} - \frac{1}{2}\ln 4 + \tan^{-1}\sqrt{3}\right) - \left(4\ln 1 - \frac{1}{2}\ln 2 + \tan^{-1}1\right) = 2\ln 3 - \ln 2 + \frac{\pi}{3} + \frac{1}{2}\ln 2 - \frac{\pi}{4}$
$= 2\ln 3 - \frac{1}{2}\ln 2 + \frac{\pi}{12} = \ln\left(\frac{9}{\sqrt{2}}\right) + \frac{\pi}{12}$

23. $\frac{y^2+2y+1}{(y^2+1)^2} = \frac{Ay+B}{y^2+1} + \frac{Cy+D}{(y^2+1)^2} \Rightarrow y^2+2y+1 = (Ay+B)(y^2+1) + Cy + D$
$= Ay^3 + By^2 + (A+C)y + (B+D) \Rightarrow A = 0, B = 1; A + C = 2 \Rightarrow C = 2; B + D = 1 \Rightarrow D = 0;$
$\int \frac{y^2+2y+1}{(y^2+1)^2}\,dy = \int \frac{1}{y^2+1}\,dy + 2\int \frac{y}{(y^2+1)^2}\,dy = \tan^{-1}y - \frac{1}{y^2+1} + C$

24. $\frac{8x^2+8x+2}{(4x^2+1)^2} = \frac{Ax+B}{4x^2+1} + \frac{Cx+D}{(4x^2+1)^2} \Rightarrow 8x^2+8x+2 = (Ax+B)(4x^2+1) + Cx + D$
$= 4Ax^3 + 4Bx^2 + (A+C)x + (B+D); A = 0, B = 2; A + C = 8 \Rightarrow C = 8; B + D = 2 \Rightarrow D = 0;$
$\int \frac{8x^2+8x+2}{(4x^2+1)^2}\,dx = 2\int \frac{dx}{4x^2+1} + 8\int \frac{x\,dx}{(4x^2+1)^2} = \tan^{-1}2x - \frac{1}{4x^2+1} + C$

25. $\frac{2s+2}{(s^2+1)(s-1)^3} = \frac{As+B}{s^2+1} + \frac{C}{s-1} + \frac{D}{(s-1)^2} + \frac{E}{(s-1)^3} \Rightarrow 2s+2$
$= (As+B)(s-1)^3 + C(s^2+1)(s-1)^2 + D(s^2+1)(s-1) + E(s^2+1)$
$= [As^4 + (-3A+B)s^3 + (3A-3B)s^2 + (-A+3B)s - B] + C(s^4 - 2s^3 + 2s^2 - 2s + 1) + D(s^3 - s^2 + s - 1)$
$\quad + E(s^2+1)$
$= (A+C)s^4 + (-3A+B-2C+D)s^3 + (3A-3B+2C-D+E)s^2 + (-A+3B-2C+D)s + (-B+C-D+E)$

$$\Rightarrow \left.\begin{array}{rl} A + C &= 0 \\ -3A + B - 2C + D &= 0 \\ 3A - 3B + 2C - D + E &= 0 \\ -A + 3B - 2C + D &= 2 \\ -B + C - D + E &= 2 \end{array}\right\} \text{summing all equations} \Rightarrow 2E = 4 \Rightarrow E = 2;$$

summing eqs (2) and (3) $\Rightarrow -2B + 2 = 0 \Rightarrow B = 1$; summing eqs (3) and (4) $\Rightarrow 2A + 2 = 2 \Rightarrow A = 0$; $C = 0$ from eq (1); then $-1 + 0 - D + 2 = 2$ from eq (5) $\Rightarrow D = -1$;
$\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds = \int \frac{ds}{s^2+1} - \int \frac{ds}{(s-1)^2} + 2\int \frac{ds}{(s-1)^3} = -(s-1)^{-2} + (s-1)^{-1} + \tan^{-1}s + C$

26. $\frac{s^4+81}{s(s^2+9)^2} = \frac{A}{s} + \frac{Bs+C}{s^2+9} + \frac{Ds+E}{(s^2+9)^2} \Rightarrow s^4 + 81 = A(s^2+9)^2 + (Bs+C)s(s^2+9) + (Ds+E)s$
$= A(s^4 + 18s^2 + 81) + (Bs^4 + Cs^3 + 9Bs^2 + 9Cs) + Ds^2 + Es$
$= (A+B)s^4 + Cs^3 + (18A + 9B + D)s^2 + (9C+E)s + 81A \Rightarrow 81A = 81$ or $A = 1$; $A + B = 1 \Rightarrow B = 0$; $C = 0$; $9C + E = 0 \Rightarrow E = 0$; $18A + 9B + D = 0 \Rightarrow D = -18$; $\int \frac{s^4+81}{s(s^2+9)^2}\,ds = \int \frac{ds}{s} - 18\int \frac{s\,ds}{(s^2+9)^2}$
$= \ln|s| + \frac{9}{(s^2+9)} + C$

27. $\frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2} = \frac{A\theta+B}{\theta^2+2\theta+2} + \frac{C\theta+D}{(\theta^2+2\theta+2)^2} \Rightarrow 2\theta^3 + 5\theta^2 + 8\theta + 4 = (A\theta+B)(\theta^2+2\theta+2) + C\theta + D$
$= A\theta^3 + (2A+B)\theta^2 + (2A+2B+C)\theta + (2B+D) \Rightarrow A = 2; 2A + B = 5 \Rightarrow B = 1; 2A + 2B + C = 8 \Rightarrow C = 2;$
$2B + D = 4 \Rightarrow D = 2$; $\int \frac{2\theta^3+5\theta^2+8\theta+4}{(\theta^2+2\theta+2)^2}\,d\theta = \int \frac{2\theta+1}{(\theta^2+2\theta+2)}\,d\theta + \int \frac{2\theta+2}{(\theta^2+2\theta+2)^2}\,d\theta$
$= \int \frac{2\theta+2}{\theta^2+2\theta+2}\,d\theta - \int \frac{d\theta}{\theta^2+2\theta+2} + \int \frac{d(\theta^2+2\theta+2)}{(\theta^2+2\theta+2)^2} = \int \frac{d(\theta^2+2\theta+2)}{\theta^2+2\theta+2} - \int \frac{d\theta}{(\theta+1)^2+1} - \frac{1}{\theta^2+2\theta+2}$

$= \frac{-1}{\theta^2+2\theta+2} + \ln(\theta^2+2\theta+2) - \tan^{-1}(\theta+1) + C$

28. $\frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3} = \frac{A\theta+B}{\theta^2+1} + \frac{C\theta+D}{(\theta^2+1)^2} + \frac{E\theta+F}{(\theta^2+1)^3} \Rightarrow \theta^4-4\theta^3+2\theta^2-3\theta+1$
$= (A\theta+B)(\theta^2+1)^2 + (C\theta+D)(\theta^2+1) + E\theta+F = (A\theta+B)(\theta^4+2\theta^2+1) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$
$= (A\theta^5+B\theta^4+2A\theta^3+2B\theta^2+A\theta+B) + (C\theta^3+D\theta^2+C\theta+D) + E\theta+F$
$= A\theta^5+B\theta^4+(2A+C)\theta^3+(2B+D)\theta^2+(A+C+E)\theta+(B+D+F) \Rightarrow A=0; B=1; 2A+C=-4$
$\Rightarrow C=-4; 2B+D=2 \Rightarrow D=0; A+C+E=-3 \Rightarrow E=1; B+D+F=1 \Rightarrow F=0;$
$\int \frac{\theta^4-4\theta^3+2\theta^2-3\theta+1}{(\theta^2+1)^3}\,d\theta = \int \frac{d\theta}{\theta^2+1} - 4\int \frac{\theta\,d\theta}{(\theta^2+1)^2} + \int \frac{\theta\,d\theta}{(\theta^2+1)^3} = \tan^{-1}\theta + 2(\theta^2+1)^{-1} - \frac{1}{4}(\theta^2+1)^{-2} + C$

29. $\frac{2x^3-2x^2+1}{x^2-x} = 2x + \frac{1}{x^2-x} = 2x + \frac{1}{x(x-1)}$; $\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1} \Rightarrow 1 = A(x-1)+Bx; x=0 \Rightarrow A=-1;$
$x=1 \Rightarrow B=1; \int \frac{2x^3-2x^2+1}{x^2-x} = \int 2x\,dx - \int \frac{dx}{x} + \int \frac{dx}{x-1} = x^2 - \ln|x| + \ln|x-1| + C = x^2 + \ln\left|\frac{x-1}{x}\right| + C$

30. $\frac{x^4}{x^2-1} = (x^2+1) + \frac{1}{x^2-1} = (x^2+1) + \frac{1}{(x+1)(x-1)}$; $\frac{1}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 1 = A(x-1)+B(x+1);$
$x=-1 \Rightarrow A=-\frac{1}{2}; x=1 \Rightarrow B=\frac{1}{2}; \int \frac{x^4}{x^2-1}\,dx = \int (x^2+1)\,dx - \frac{1}{2}\int \frac{dx}{x+1} + \frac{1}{2}\int \frac{dx}{x-1}$
$= \frac{1}{3}x^3 + x - \frac{1}{2}\ln|x+1| + \frac{1}{2}\ln|x-1| + C = \frac{x^3}{3} + x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

31. $\frac{9x^3-3x+1}{x^3-x^2} = 9 + \frac{9x^2-3x+1}{x^2(x-1)}$ (after long division); $\frac{9x^2-3x+1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}$
$\Rightarrow 9x^2-3x+1 = Ax(x-1)+B(x-1)+Cx^2; x=1 \Rightarrow C=7; x=0 \Rightarrow B=-1; A+C=9 \Rightarrow A=2;$
$\int \frac{9x^3-3x+1}{x^3-x^2}\,dx = \int 9\,dx + 2\int \frac{dx}{x} - \int \frac{dx}{x^2} + 7\int \frac{dx}{x-1} = 9x + 2\ln|x| + \frac{1}{x} + 7\ln|x-1| + C$

32. $\frac{16x^3}{4x^2-4x+1} = (4x+4) + \frac{12x-4}{4x^2-4x+1}$; $\frac{12x-4}{(2x-1)^2} = \frac{A}{2x-1} + \frac{B}{(2x-1)^2} \Rightarrow 12x-4 = A(2x-1)+B$
$\Rightarrow A=6; -A+B=-4 \Rightarrow B=2; \int \frac{16x^3}{4x^2-4x+1}\,dx = 4\int (x+1)\,dx + 6\int \frac{dx}{2x-1} + 2\int \frac{dx}{(2x-1)^2}$
$= 2(x+1)^2 + 3\ln|2x-1| - \frac{1}{2x-1} + C_1 = 2x^2+4x+3\ln|2x-1| - (2x-1)^{-1} + C$, where $C = 2 + C_1$

33. $\frac{y^4+y^2-1}{y^3+y} = y - \frac{1}{y(y^2+1)}$; $\frac{1}{y(y^2+1)} = \frac{A}{y} + \frac{By+C}{y^2+1} \Rightarrow 1 = A(y^2+1)+(By+C)y = (A+B)y^2+Cy+A$
$\Rightarrow A=1; A+B=0 \Rightarrow B=-1; C=0; \int \frac{y^4+y^2-1}{y^3+y}\,dy = \int y\,dy - \int \frac{dy}{y} + \int \frac{y\,dy}{y^2+1}$
$= \frac{y^2}{2} - \ln|y| + \frac{1}{2}\ln(1+y^2) + C$

34. $\frac{2y^4}{y^3-y^2+y-1} = 2y+2+\frac{2}{y^3-y^2+y-1}$; $\frac{2}{y^3-y^2+y-1} = \frac{2}{(y^2+1)(y-1)} = \frac{A}{y-1} + \frac{By+C}{y^2+1}$
$\Rightarrow 2 = A(y^2+1)+(By+C)(y-1) = (Ay^2+A)+(By^2+Cy-By-C) = (A+B)y^2+(-B+C)y+(A-C)$
$\Rightarrow A+B=0, -B+C=0$ or $C=B$, $A-C=A-B=2 \Rightarrow A=1, B=-1, C=-1;$
$\int \frac{2y^4}{y^3-y^2+y-1}\,dy = 2\int (y+1)\,dy + \int \frac{dy}{y-1} - \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= (y+1)^2 + \ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C_1 = y^2+2y+\ln|y-1| - \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C,$
where $C = C_1 + 1$

35. $\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = [e^t = y] \int \frac{dy}{y^2+3y+2} = \int \frac{dy}{y+1} - \int \frac{dy}{y+2} = \ln\left|\frac{y+1}{y+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

36. $\int \frac{e^{4t}+2e^{2t}-e^t}{e^{2t}+1}\,dt = \int \frac{e^{3t}+2e^t-1}{e^{2t}+1}e^t dt; \begin{bmatrix} y = e^t \\ dy = e^t\,dt \end{bmatrix} \to \int \frac{y^3+2y-1}{y^2+1}\,dy = \int \left(y + \frac{y-1}{y^2+1}\right)dy = \frac{y^2}{2} + \int \frac{y}{y^2+1}\,dy - \int \frac{dy}{y^2+1}$
$= \frac{y^2}{2} + \frac{1}{2}\ln(y^2+1) - \tan^{-1}y + C = \frac{1}{2}e^{2t} + \frac{1}{2}\ln(e^{2t}+1) - \tan^{-1}(e^t) + C$

37. $\int \frac{\cos y\, dy}{\sin^2 y + \sin y - 6}$; $[\sin y = t, \cos y\, dy = dt] \rightarrow \int \frac{dy}{t^2 + t - 6} = \frac{1}{5} \int \left(\frac{1}{t-2} - \frac{1}{t+3}\right) dt = \frac{1}{5} \ln\left|\frac{t-2}{t+3}\right| + C$
$= \frac{1}{5} \ln\left|\frac{\sin y - 2}{\sin y + 3}\right| + C$

38. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \rightarrow -\int \frac{dy}{y^2 + y - 2} = \frac{1}{3}\int \frac{dy}{y+2} - \frac{1}{3}\int \frac{dy}{y-1} = \frac{1}{3} \ln\left|\frac{y+2}{y-1}\right| + C = \frac{1}{3} \ln\left|\frac{\cos\theta + 2}{\cos\theta - 1}\right| + C$
$= \frac{1}{3} \ln\left|\frac{2 + \cos\theta}{1 - \cos\theta}\right| + C = -\frac{1}{3} \ln\left|\frac{\cos\theta - 1}{\cos\theta + 2}\right| + C$

39. $\int \frac{(x-2)^2 \tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\, dx = \int \frac{\tan^{-1}(2x)}{4x^2+1}\, dx - 3\int \frac{x}{(x-2)^2}\, dx$
$= \frac{1}{2} \int \tan^{-1}(2x)\, d\left(\tan^{-1}(2x)\right) - 3\int \frac{dx}{x-2} - 6\int \frac{dx}{(x-2)^2} = \frac{\left(\tan^{-1} 2x\right)^2}{4} - 3 \ln|x-2| + \frac{6}{x-2} + C$

40. $\int \frac{(x+1)^2 \tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\, dx = \int \frac{\tan^{-1}(3x)}{9x^2+1}\, dx + \int \frac{x}{(x+1)^2}\, dx$
$= \frac{1}{3} \int \tan^{-1}(3x)\, d\left(\tan^{-1}(3x)\right) + \int \frac{dx}{x+1} - \int \frac{dx}{(x+1)^2} = \frac{\left(\tan^{-1} 3x\right)^2}{6} + \ln|x+1| + \frac{1}{x+1} + C$

41. $(t^2 - 3t + 2)\frac{dx}{dt} = 1$; $x = \int \frac{dt}{t^2 - 3t + 2} = \int \frac{dt}{t-2} - \int \frac{dt}{t-1} = \ln\left|\frac{t-2}{t-1}\right| + C$; $\frac{t-2}{t-1} = Ce^x$; $t = 3$ and $x = 0$
$\Rightarrow \frac{1}{2} = C \Rightarrow \frac{t-2}{t-1} = \frac{1}{2}e^x \Rightarrow x = \ln\left|2\left(\frac{t-2}{t-1}\right)\right| = \ln|t-2| - \ln|t-1| + \ln 2$

42. $(3t^4 + 4t^2 + 1)\frac{dx}{dt} = 2\sqrt{3}$; $x = 2\sqrt{3}\int \frac{dt}{3t^4 + 4t^2 + 1} = \sqrt{3}\int \frac{dt}{t^2 + \frac{1}{3}} - \sqrt{3}\int \frac{dt}{t^2+1}$
$= 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1} t + C$; $t = 1$ and $x = \frac{-\pi\sqrt{3}}{4} \Rightarrow -\frac{\sqrt{3}\pi}{4} = \pi - \frac{\sqrt{3}}{4}\pi + C \Rightarrow C = -\pi$
$\Rightarrow x = 3\tan^{-1}\left(\sqrt{3}t\right) - \sqrt{3}\tan^{-1} t - \pi$

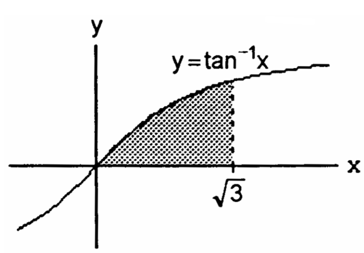
43. $(t^2 + 2t)\frac{dx}{dt} = 2x + 2$; $\frac{1}{2}\int \frac{dx}{x+1} = \int \frac{dt}{t^2 + 2t} \Rightarrow \frac{1}{2}\ln|x+1| = \frac{1}{2}\int \frac{dt}{t} - \frac{1}{2}\int \frac{dt}{t+2} \Rightarrow \ln|x+1| = \ln\left|\frac{t}{t+2}\right| + C$;
$t = 1$ and $x = 1 \Rightarrow \ln 2 = \ln\frac{1}{3} + C \Rightarrow C = \ln 2 + \ln 3 = \ln 6 \Rightarrow \ln|x+1| = \ln 6\left|\frac{t}{t+2}\right| \Rightarrow x + 1 = \frac{6t}{t+2}$
$\Rightarrow x = \frac{6t}{t+2} - 1, t > 0$

44. $(t+1)\frac{dx}{dt} = x^2 + 1 \Rightarrow \int \frac{dx}{x^2+1} = \int \frac{dt}{t+1} \Rightarrow \tan^{-1} x = \ln|t+1| + C$; $t = 0$ and $x = \frac{\pi}{4} \Rightarrow \tan^{-1}\frac{\pi}{4} = \ln|1| + C$
$\Rightarrow C = \tan^{-1}\frac{\pi}{4} = 1 \Rightarrow \tan^{-1} x = \ln|t+1| + 1 \Rightarrow x = \tan\left(\ln(t+1) + 1\right), t > -1$

45. $V = \pi\int_{0.5}^{2.5} y^2\, dx = \pi\int_{0.5}^{2.5} \frac{9}{3x - x^2}\, dx = 3\pi\left(\int_{0.5}^{2.5}\left(-\frac{1}{x-3} + \frac{1}{x}\right)\right) dx = \left[3\pi\ln\left|\frac{x}{x-3}\right|\right]_{0.5}^{2.5} = 3\pi\ln 25$

46. $V = 2\pi\int_0^1 xy\, dx = 2\pi\int_0^1 \frac{2x}{(x+1)(2-x)}\, dx = 4\pi\int_0^1\left(-\frac{1}{3}\left(\frac{1}{x+1}\right) + \frac{2}{3}\left(\frac{1}{2-x}\right)\right) dx$
$= \left[-\frac{4\pi}{3}\left(\ln|x+1| + 2\ln|2-x|\right)\right]_0^1 = \frac{4\pi}{3}(\ln 2)$

47. $A = \int_0^{\sqrt{3}} \tan^{-1} x\, dx = \left[x\tan^{-1} x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} \frac{x}{1+x^2}\, dx$
$= \frac{\pi\sqrt{3}}{3} - \left[\frac{1}{2}\ln\left(x^2+1\right)\right]_0^{\sqrt{3}} = \frac{\pi\sqrt{3}}{3} - \ln 2$;
$\overline{x} = \frac{1}{A}\int_0^{\sqrt{3}} x\tan^{-1} x\, dx$
$= \frac{1}{A}\left(\left[\frac{1}{2}x^2\tan^{-1} x\right]_0^{\sqrt{3}} - \frac{1}{2}\int_0^{\sqrt{3}} \frac{x^2}{1+x^2}\, dx\right)$
$= \frac{1}{A}\left[\frac{\pi}{2} - \left[\frac{1}{2}\left(x - \tan^{-1} x\right)\right]_0^{\sqrt{3}}\right]$
$= \frac{1}{A}\left(\frac{\pi}{2} - \frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) = \frac{1}{A}\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \cong 1.10$

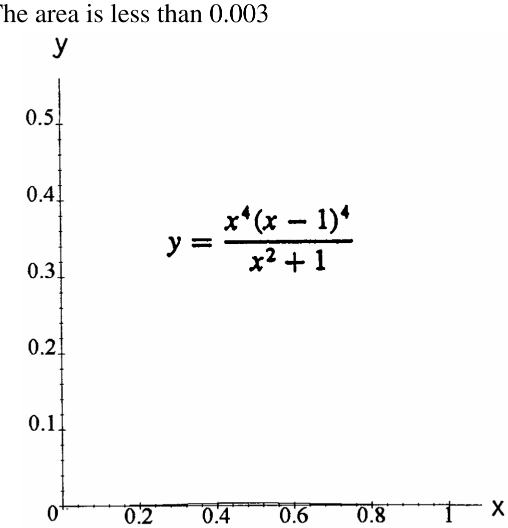


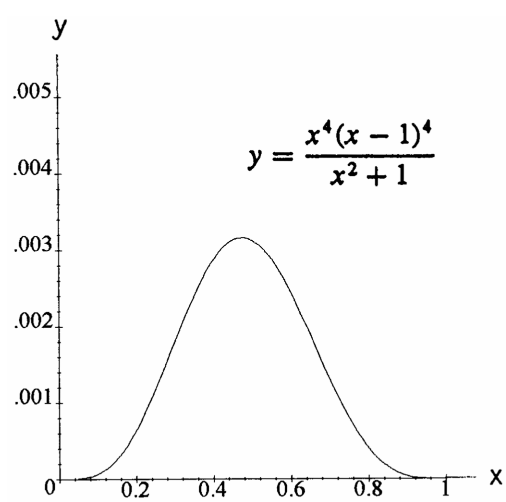
48. $A = \int_3^5 \frac{4x^2+13x-9}{x^3+2x^2-3x}\,dx = 3\int_3^5 \frac{dx}{x} - \int_3^5 \frac{dx}{x+3} + 2\int_3^5 \frac{dx}{x-1} = [3\ln|x| - \ln|x+3| + 2\ln|x-1|]_3^5 = \ln\frac{125}{9}$;
$\overline{x} = \frac{1}{A}\int_3^5 \frac{x(4x^2+13x-9)}{x^3+2x^2-3x}\,dx = \frac{1}{A}\left([4x]_3^5 + 3\int_3^5 \frac{dx}{x+3} + 2\int_3^5 \frac{dx}{x-1}\right) = \frac{1}{A}(8 + 11\ln 2 - 3\ln 6) \cong 3.90$

49. (a) $\frac{dx}{dt} = kx(N-x) \Rightarrow \int \frac{dx}{x(N-x)} = \int k\,dt \Rightarrow \frac{1}{N}\int \frac{dx}{x} + \frac{1}{N}\int \frac{dx}{N-x} = \int k\,dt \Rightarrow \frac{1}{N}\ln\left|\frac{x}{N-x}\right| = kt + C$;
$k = \frac{1}{250}$, $N = 1000$, $t = 0$ and $x = 2 \Rightarrow \frac{1}{1000}\ln\left|\frac{2}{998}\right| = C \Rightarrow \frac{1}{1000}\ln\left|\frac{x}{1000-x}\right| = \frac{t}{250} + \frac{1}{1000}\ln\left(\frac{1}{499}\right)$
$\Rightarrow \ln\left|\frac{499x}{1000-x}\right| = 4t \Rightarrow \frac{499x}{1000-x} = e^{4t} \Rightarrow 499x = e^{4t}(1000-x) \Rightarrow (499 + e^{4t})\,x = 1000e^{4t} \Rightarrow x = \frac{1000e^{4t}}{499+e^{4t}}$
(b) $x = \frac{1}{2}N = 500 \Rightarrow 500 = \frac{1000e^{4t}}{499+e^{4t}} \Rightarrow 500 \cdot 499 + 500e^{4t} = 1000e^{4t} \Rightarrow e^{4t} = 499 \Rightarrow t = \frac{1}{4}\ln 499 \approx 1.55$ days

50. $\frac{dx}{dt} = k(a-x)(b-x) \Rightarrow \frac{dx}{(a-x)(b-x)} = k\,dt$
(a) $a = b$: $\int \frac{dx}{(a-x)^2} = \int k\,dt \Rightarrow \frac{1}{a-x} = kt + C$; $t = 0$ and $x = 0 \Rightarrow \frac{1}{a} = C \Rightarrow \frac{1}{a-x} = kt + \frac{1}{a}$
$\Rightarrow \frac{1}{a-x} = \frac{akt+1}{a} \Rightarrow a - x = \frac{a}{akt+1} \Rightarrow x = a - \frac{a}{akt+1} = \frac{a^2kt}{akt+1}$
(b) $a \neq b$: $\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \Rightarrow \frac{1}{b-a}\int \frac{dx}{a-x} - \frac{1}{b-a}\int \frac{dx}{b-x} = \int k\,dt \Rightarrow \frac{1}{b-a}\ln\left|\frac{b-x}{a-x}\right| = kt + C$;
$t = 0$ and $x = 0 \Rightarrow \frac{1}{b-a}\ln\frac{b}{a} = C \Rightarrow \ln\left|\frac{b-x}{a-x}\right| = (b-a)kt + \ln\left(\frac{b}{a}\right) \Rightarrow \frac{b-x}{a-x} = \frac{b}{a}e^{(b-a)kt}$
$\Rightarrow x = \frac{ab\left[1 - e^{(b-a)kt}\right]}{a - be^{(b-a)kt}}$

51. (a) $\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx = \int_0^1 \left(x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{x^2+1}\right)dx = \frac{22}{7} - \pi$
(b) $\frac{\frac{22}{7}-\pi}{\pi} \cdot 100\% \cong 0.04\%$
(c) The area is less than 0.003




52. $P(x) = ax^2 + bx + c$, $P(0) = c = 1$ and $P'(0) = 0 \Rightarrow b = 0 \Rightarrow P(x) = ax^2 + 1$. Next,
$\frac{ax^2+1}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$; for the integral to be a rational function, we must have $A = 0$ and $D = 0$. Thus, $ax^2 + 1 = Bx(x-1)^2 + C(x-1)^2 + Ex^3 = (B+E)x^3 + (C-2B)x^2 + (B-2C)x + C$
$$\Rightarrow \left.\begin{array}{r} B+E=0 \\ C-2B=a \\ C=1 \end{array}\right\} \Rightarrow E = -B;\ x = 1 \Rightarrow a + 1 = E;\ \text{therefore, } 1 - 2B = a \Rightarrow 1 + 2E = a \Rightarrow 1 + 2(a+1) = a$$
$\Rightarrow a = -3$

## 8.4 TRIGONOMETRIC INTEGRALS

1. $\int_0^{\pi/2} \sin^5 x\,dx = \int_0^{\pi/2} (\sin^2 x)^2 \sin x\,dx = \int_0^{\pi/2} (1-\cos^2 x)^2 \sin x\,dx = \int_0^{\pi/2} (1 - 2\cos^2 x + \cos^4 x)\sin x\,dx$
$= \int_0^{\pi/2} \sin x\,dx - \int_0^{\pi/2} 2\cos^2 x \sin x\,dx + \int_0^{\pi/2} \cos^4 x \sin x\,dx = \left[-\cos x + 2\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\pi/2}$
$= (0) - \left(-1 + \frac{2}{3} - \frac{1}{5}\right) = \frac{8}{15}$

2. $\int_0^{\pi} \sin^5\left(\frac{x}{2}\right)dx$ (using Exercise 1) $= \int_0^{\pi} \sin\left(\frac{x}{2}\right)dx - \int_0^{\pi} 2\cos^2\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx + \int_0^{\pi} \cos^4\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right)dx$
$= \left[-2\cos\left(\frac{x}{2}\right) + \frac{4}{3}\cos^3\left(\frac{x}{2}\right) - \frac{2}{5}\cos^5\left(\frac{x}{2}\right)\right]_0^{\pi} = (0) - \left(-2 + \frac{4}{3} - \frac{2}{5}\right) = \frac{16}{15}$

3. $\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx = \int_{-\pi/2}^{\pi/2} (\cos^2 x)\cos x\,dx = \int_{-\pi/2}^{\pi/2} (1-\sin^2 x)\cos x\,dx = \int_{-\pi/2}^{\pi/2} \cos x\,dx - \int_{-\pi/2}^{\pi/2} \sin^2 x \cos x\,dx$
$= \left[\sin x - \frac{\sin^3 x}{3}\right]_{-\pi/2}^{\pi/2} = \left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) = \frac{4}{3}$

4. $\int_0^{\pi/6} 3\cos^5 3x\,dx = \int_0^{\pi/6} (\cos^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1-\sin^2 3x)^2 \cos 3x \cdot 3dx = \int_0^{\pi/6} (1 - 2\sin^2 3x + \sin^4 3x)\cos 3x \cdot 3dx$
$= \int_0^{\pi/6} \cos 3x \cdot 3dx - 2\int_0^{\pi/6} \sin^2 3x \cos 3x \cdot 3dx + \int_0^{\pi/6} \sin^4 3x \cos 3x \cdot 3dx = \left[\sin 3x - 2\frac{\sin^3 3x}{3} + \frac{\sin^5 3x}{5}\right]_0^{\pi/6}$
$= \left(1 - \frac{2}{3} + \frac{1}{5}\right) - (0) = \frac{8}{15}$

5. $\int_0^{\pi/2} \sin^7 y\,dy = \int_0^{\pi/2} \sin^6 y \sin y\,dy = \int_0^{\pi/2} (1-\cos^2 y)^3 \sin y\,dy = \int_0^{\pi/2} \sin y\,dy - 3\int_0^{\pi/2} \cos^2 y \sin y\,dy$
$+ 3\int_0^{\pi/2} \cos^4 y \sin y\,dy - \int_0^{\pi/2} \cos^6 y \sin y\,dy = \left[-\cos y + 3\frac{\cos^3 y}{3} - 3\frac{\cos^5 y}{5} + \frac{\cos^7 y}{7}\right]_0^{\pi/2} = (0) - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right) = \frac{16}{35}$

6. $\int_0^{\pi/2} 7\cos^7 t\,dt$ (using Exercise 5) $= 7\left[\int_0^{\pi/2} \cos t\,dt - 3\int_0^{\pi/2} \sin^2 t \cos t\,dt + 3\int_0^{\pi/2} \sin^4 t \cos t\,dt - \int_0^{\pi/2} \sin^6 t \cos t\,dt\right]$
$= 7\left[\sin t - 3\frac{\sin^3 t}{3} + 3\frac{\sin^5 t}{5} - \frac{\sin^7 t}{7}\right]_0^{\pi/2} = 7\left(1 - 1 + \frac{3}{5} - \frac{1}{7}\right) - 7(0) = \frac{16}{5}$

7. $\int_0^{\pi} 8\sin^4 x\,dx = 8\int_0^{\pi} \left(\frac{1-\cos 2x}{2}\right)^2 dx = 2\int_0^{\pi} (1 - 2\cos 2x + \cos^2 2x)dx = 2\int_0^{\pi} dx - 2\int_0^{\pi} \cos 2x \cdot 2dx + 2\int_0^{\pi} \frac{1+\cos 4x}{2}\,dx$
$= [2x - 2\sin 2x]_0^{\pi} + \int_0^{\pi} dx + \int_0^{\pi} \cos 4x\,dx = 2\pi + \left[x + \frac{1}{2}\sin 4x\right]_0^{\pi} = 2\pi + \pi = 3\pi$

8. $\int_0^{1} 8\cos^4 2\pi x\,dx = 8\int_0^{1} \left(\frac{1+\cos 4\pi x}{2}\right)^2 dx = 2\int_0^{1} (1 + 2\cos 4\pi x + \cos^2 4\pi x)dx = 2\int_0^{1} dx + 4\int_0^{1} \cos 4\pi x\,dx + 2\int_0^{1} \frac{1+\cos 8\pi x}{2}\,dx$
$= \left[2x + \frac{1}{\pi}\sin 4\pi x\right]_0^{1} + \int_0^{1} dx + \int_0^{1} \cos 8\pi x\,dx = 2 + \left[x + \frac{1}{8\pi}\sin 8\pi x\right]_0^{1} = 2 + 1 = 3$

9. $\int_{-\pi/4}^{\pi/4} 16\sin^2 x \cos^2 x\,dx = 16\int_{-\pi/4}^{\pi/4} \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)dx = 4\int_{-\pi/4}^{\pi/4} (1-\cos^2 2x)dx = 4\int_{-\pi/4}^{\pi/4} dx - 4\int_{-\pi/4}^{\pi/4} \left(\frac{1+\cos 4x}{2}\right)dx$
$= [4x]_{-\pi/4}^{\pi/4} - 2\int_{-\pi/4}^{\pi/4} dx - 2\int_{-\pi/4}^{\pi/4} \cos 4x\,dx = \pi + \pi - \left[2x + \frac{\sin 4x}{2}\right]_{-\pi/4}^{\pi/4} = 2\pi - \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi$

10. $\int_0^{\pi} 8\sin^4 y \cos^2 y\,dy = 8\int_0^{\pi} \left(\frac{1-\cos 2y}{2}\right)^2 \left(\frac{1+\cos 2y}{2}\right)dy = \int_0^{\pi} dy - \int_0^{\pi} \cos 2y\,dy - \int_0^{\pi} \cos^2 2y\,dy + \int_0^{\pi} \cos^3 2y\,dy$
$= \left[y - \frac{1}{2}\sin 2y\right]_0^{\pi} - \int_0^{\pi} \left(\frac{1+\cos 4y}{2}\right)dy + \int_0^{\pi} (1-\sin^2 2y)\cos 2y\,dy = \pi - \frac{1}{2}\int_0^{\pi} dy - \frac{1}{2}\int_0^{\pi} \cos 4y\,dy + \int_0^{\pi} \cos 2y\,dy$
$- \int_0^{\pi} \sin^2 2y \cos 2y\,dy = \pi + \left[-\frac{1}{2}y - \frac{1}{8}\sin 4y + \frac{1}{2}\sin 2y - \frac{1}{2}\cdot\frac{\sin^3 2y}{3}\right]_0^{\pi} = \pi - \frac{\pi}{2} = \frac{\pi}{2}$

11. $\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx = \int_0^{\pi/2} 35\sin^4 x\,(1-\sin^2 x)\cos x\,dx = 35\int_0^{\pi/2}\sin^4 x\cos x\,dx - 35\int_0^{\pi/2}\sin^6 x\cos x\,dx$
$= \left[35\frac{\sin^5 x}{5} - 35\frac{\sin^7 x}{7}\right]_0^{\pi/2} = (7-5)-(0) = 2$

12. $\int_0^{\pi}\cos^2 2x\sin 2x\,dx = \left[-\frac{1}{2}\frac{\cos^3 2x}{3}\right]_0^{\pi} = -\frac{1}{6}+\frac{1}{6} = 0$

13. $\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta = \left[8\left(-\frac{1}{2}\right)\frac{\cos^4 2\theta}{4}\right]_0^{\pi/4} = \left[-\cos^4 2\theta\right]_0^{\pi/4} = (0)-(-1) = 1$

14. $\int_0^{\pi/2}\sin^2 2\theta\cos^3 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta(1-\sin^2 2\theta)\cos 2\theta\,d\theta = \int_0^{\pi/2}\sin^2 2\theta\cos 2\theta\,d\theta - \int_0^{\pi/2}\sin^4 2\theta\cos 2\theta\,d\theta$
$= \left[\frac{1}{2}\cdot\frac{\sin^3 2\theta}{3} - \frac{1}{2}\cdot\frac{\sin^5 2\theta}{5}\right]_0^{\pi/2} = 0$

15. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx = \int_0^{2\pi}\left|\sin\frac{x}{2}\right|dx = \int_0^{2\pi}\sin\frac{x}{2}\,dx = \left[-2\cos\frac{x}{2}\right]_0^{2\pi} = 2+2 = 4$

16. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx = \int_0^{\pi}\sqrt{2}\,|\sin 2x\,|dx = \int_0^{\pi}\sqrt{2}\sin 2x\,dx = \left[-\sqrt{2}\cos 2x\right]_0^{\pi} = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

17. $\int_0^{\pi}\sqrt{1-\sin^2 t}\,dt = \int_0^{\pi}|\cos t\,|dt = \int_0^{\pi/2}\cos t\,dt - \int_{\pi/2}^{\pi}\cos t\,dt = [\sin t]_0^{\pi/2} - [\sin t]_{\pi/2}^{\pi} = 1-0-0+1 = 2$

18. $\int_0^{\pi}\sqrt{1-\cos^2\theta}\,d\theta = \int_0^{\pi}|\sin\theta\,|d\theta = \int_0^{\pi}\sin\theta\,d\theta = [-\cos\theta]_0^{\pi} = 1+1 = 2$

19. $\int_{-\pi/4}^{\pi/4}\sqrt{1+\tan^2 x}\,dx = \int_{-\pi/4}^{\pi/4}|\sec x\,|dx = \int_{-\pi/4}^{\pi/4}\sec x\,dx = [\ln|\sec x+\tan x\,|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln\left(\sqrt{2}-1\right)$
$= \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) = 2\ln\left(1+\sqrt{2}\right)$

20. $\int_{-\pi/4}^{\pi/4}\sqrt{\sec^2 x-1}\,dx = \int_{-\pi/4}^{\pi/4}|\tan x\,|dx = -\int_{-\pi/4}^{0}\tan x\,dx + \int_0^{\pi/4}\tan x\,dx = [-\ln|\sec x\,|]_{-\pi/4}^{0} + [-\ln|\sec x\,|]_0^{\pi/4}$
$= -\ln(1)+\ln\sqrt{2}+\ln\sqrt{2}-\ln(1) = 2\ln\sqrt{2} = \ln 2$

21. $\int_0^{\pi/2}\theta\sqrt{1-\cos 2\theta}\,d\theta = \int_0^{\pi/2}\theta\sqrt{2}\,|\sin\theta\,|\,d\theta = \sqrt{2}\int_0^{\pi/2}\theta\sin\theta\,d\theta = \sqrt{2}\,[-\theta\cos\theta+\sin\theta]_0^{\pi/2} = \sqrt{2}(1) = \sqrt{2}$

22. $\int_{-\pi}^{\pi}(1-\cos^2 t)^{3/2}\,dt = \int_{-\pi}^{\pi}(\sin^2 t)^{3/2}\,dt = \int_{-\pi}^{\pi}\left|\sin^3 t\right|\,dt = -\int_{-\pi}^{0}\sin^3 t\,dt + \int_0^{\pi}\sin^3 t\,dt = -\int_{-\pi}^{0}(1-\cos^2 t)\sin t\,dt$
$+ \int_0^{\pi}(1-\cos^2 t)\sin t\,dt = -\int_{-\pi}^{0}\sin t\,dt + \int_{-\pi}^{0}\cos^2 t\sin t\,dt + \int_0^{\pi}\sin t\,dt - \int_0^{\pi}\cos^2 t\sin t\,dt = \left[\cos t - \frac{\cos^3 t}{3}\right]_{-\pi}^{0}$
$+ \left[-\cos t + \frac{\cos^3 t}{3}\right]_0^{\pi} = \left(1-\frac{1}{3}+1-\frac{1}{3}\right) + \left(1-\frac{1}{3}+1-\frac{1}{3}\right) = \frac{8}{3}$

23. $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$; $u = \sec x$, $du = \sec x\tan x\,dx$, $dv = \sec^2 x\,dx$, $v = \tan x$;
$\int_{-\pi/3}^{0} 2\sec^3 x\,dx = [2\sec x\tan x]_{-\pi/3}^{0} - 2\int_{-\pi/3}^{0}\sec x\tan^2 x\,dx = 2\cdot 1\cdot 0 - 2\cdot 2\cdot\sqrt{3} - 2\int_{-\pi/3}^{0}\sec x\,(\sec^2 x-1)dx$
$= 4\sqrt{3} - 2\int_{-\pi/3}^{0}\sec^3 x\,dx + 2\int_{-\pi/3}^{0}\sec x\,dx$; $2\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 4\sqrt{3} + [2\ln|\sec x+\tan x|]_{-\pi/3}^{0}$
$2\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 4\sqrt{3} + 2\ln|1+0| - 2\ln|2-\sqrt{3}| = 4\sqrt{3} - 2\ln\left(2-\sqrt{3}\right)$
$\int_{-\pi/3}^{0} 2\sec^3 x\,dx = 2\sqrt{3} - \ln\left(2-\sqrt{3}\right)$

24. $\int e^x \sec^3(e^x) dx$; $u = \sec(e^x)$, $du = \sec(e^x)\tan(e^x)e^x dx$, $dv = \sec^2(e^x)e^x dx$, $v = \tan(e^x)$.

$$\int e^x \sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) - \int \sec(e^x)\tan^2(e^x)e^x dx$$
$$= \sec(e^x)\tan(e^x) - \int \sec(e^x)(\sec^2(e^x) - 1)e^x dx$$
$$= \sec(e^x)\tan(e^x) - \int \sec^3(e^x)e^x dx + \int \sec(e^x)e^x dx$$
$$2\int e^x \sec^3(e^x)\, dx = \sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right| + C$$
$$\int e^x \sec^3(e^x)\, dx = \tfrac{1}{2}\left(\sec(e^x)\tan(e^x) + \ln\left|\sec(e^x) + \tan(e^x)\right|\right) + C$$

25. $\int_0^{\pi/4} \sec^4\theta\, d\theta = \int_0^{\pi/4} (1 + \tan^2\theta)\sec^2\theta\, d\theta = \int_0^{\pi/4} \sec^2\theta\, d\theta + \int_0^{\pi/4} \tan^2\theta \sec^2\theta\, d\theta = \left[\tan\theta + \frac{\tan^3\theta}{3}\right]_0^{\pi/4}$
$= \left(1 + \frac{1}{3}\right) - (0) = \frac{4}{3}$

26. $\int_0^{\pi/12} 3\sec^4(3x)\, dx = \int_0^{\pi/12} (1 + \tan^2(3x))\sec^2(3x)3dx = \int_0^{\pi/} \sec^2(3x)3dx + \int_0^{\pi/12} \tan^2(3x)\sec^2(3x)3dx$
$= \left[\tan(3x) + \frac{\tan^3(3x)}{3}\right]_0^{\pi/12} = \left(1 + \frac{1}{3}\right) - (0) = \frac{4}{3}$

27. $\int_{\pi/4}^{\pi/2} \csc^4\theta\, d\theta = \int_{\pi/4}^{\pi/2} (1 + \cot^2\theta)\csc^2\theta\, d\theta = \int_{\pi/4}^{\pi/2} \csc^2\theta\, d\theta + \int_{\pi/4}^{\pi/2} \cot^2\theta \csc^2\theta\, d\theta = \left[-\cot\theta - \frac{\cot^3\theta}{3}\right]_{\pi/4}^{\pi/2}$
$= (0) - \left(-1 - \frac{1}{3}\right) = \frac{4}{3}$

28. $\int_{\pi/2}^{\pi} 3\csc^4\frac{\theta}{2}\, d\theta = 3\int_{\pi/2}^{\pi} \left(1 + \cot^2\frac{\theta}{2}\right)\csc^2\frac{\theta}{2}\, d\theta = 3\int_{\pi/2}^{\pi} \csc^2\frac{\theta}{2}\, d\theta + 3\int_{\pi/2}^{\pi} \cot^2\frac{\theta}{2}\csc^2\frac{\theta}{2}\, d\theta = \left[-6\cot\frac{\theta}{2} - 6\frac{\cot^3\frac{\theta}{2}}{3}\right]_{\pi/2}^{\pi}$
$= (-6\cdot 0 - 2\cdot 0) - (-6\cdot 1 - 2\cdot 1) = 8$

29. $\int_0^{\pi/4} 4\tan^3 x\, dx = 4\int_0^{\pi/4} (\sec^2 x - 1)\tan x\, dx = 4\int_0^{\pi/4} \sec^2 x \tan x\, dx - 4\int_0^{\pi/4} \tan x\, dx = \left[4\frac{\tan^2 x}{2} - 4\ln|\sec x|\right]_0^{\pi/4}$
$= 2(1) - 4\ln\sqrt{2} - 2\cdot 0 + 4\ln 1 = 2 - 2\ln 2$

30. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx = 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)\tan^2 x\, dx = 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} \tan^2 x\, dx$
$= 6\int_{-\pi/4}^{\pi/4} \sec^2 x \tan^2 x\, dx - 6\int_{-\pi/4}^{\pi/4} (\sec^2 x - 1)dx = \left[6\frac{\tan^3 x}{3}\right]_{-\pi/4}^{\pi/4} - 6\int_{-\pi/4}^{\pi/4} \sec^2 x\, dx + 6\int_{-\pi/4}^{\pi/4} dx$
$= 2(1 - (-1)) - [6\tan x]_{-\pi/4}^{\pi/4} + [6x]_{-\pi/4}^{\pi/4} = 4 - 6(1 - (-1)) + \frac{3\pi}{2} + \frac{3\pi}{2} = 3\pi - 8$

31. $\int_{\pi/6}^{\pi/3} \cot^3 x\, dx = \int_{\pi/6}^{\pi/3} (\csc^2 x - 1)\cot x\, dx = \int_{\pi/6}^{\pi/3} \csc^2 x \cot x\, dx - \int_{\pi/6}^{\pi/3} \cot x\, dx = \left[-\frac{\cot^2 x}{2} + \ln|\csc x|\right]_{\pi/6}^{\pi/3}$
$= -\frac{1}{2}\left(\frac{1}{3} - 3\right) + \left(\ln\frac{2}{\sqrt{3}} - \ln 2\right) = \frac{4}{3} - \ln\sqrt{3}$

32. $\int_{\pi/4}^{\pi/2} 8\cot^4 t\, dt = 8\int_{\pi/4}^{\pi/2} (\csc^2 t - 1)\cot^2 t\, dt = 8\int_{\pi/4}^{\pi/2} \csc^2 t \cot^2 t\, dt - 8\int_{\pi/4}^{\pi/2} \cot^2 t\, dt$
$= -8\left[-\frac{\cot^3 t}{3}\right]_{\pi/4}^{\pi/2} - 8\int_{\pi/4}^{\pi/2} (\csc^2 t - 1)dt = -\frac{8}{3}(0 - 1) + [8\cot t]_{\pi/4}^{\pi/2} + [8t]_{\pi/4}^{\pi/2} = \frac{8}{3} + 8(0 - 1) + 4\pi - 2\pi = 2\pi - \frac{16}{3}$

33. $\int_{-\pi}^{0} \sin 3x \cos 2x\, dx = \frac{1}{2}\int_{-\pi}^{0} (\sin x + \sin 5x)\, dx = \frac{1}{2}\left[-\cos x - \frac{1}{5}\cos 5x\right]_{-\pi}^{0} = \frac{1}{2}\left(-1 - \frac{1}{5} - 1 - \frac{1}{5}\right) = -\frac{6}{5}$

34. $\int_0^{\pi/2} \sin 2x \cos 3x\, dx = \frac{1}{2}\int_0^{\pi/2} (\sin(-x) + \sin 5x)\, dx = \frac{1}{2}\left[\cos(-x) - \frac{1}{5}\cos 5x\right]_0^{\pi/2} = \frac{1}{2}(0) - \frac{1}{2}\left(1 - \frac{1}{5}\right) = -\frac{2}{5}$

35. $\int_{-\pi}^{\pi} \sin 3x \sin 3x\, dx = \frac{1}{2}\int_{-\pi}^{\pi} (\cos 0 - \cos 6x)\, dx = \frac{1}{2}\int_{-\pi}^{\pi} dx - \frac{1}{2}\int_{-\pi}^{\pi} \cos 6x\, dx = \frac{1}{2}\left[x - \frac{1}{12}\sin 6x\right]_{-\pi}^{\pi} = \frac{\pi}{2} + \frac{\pi}{2} - 0 = \pi$

36. $\int_0^{\pi/2} \sin x \cos x\, dx = \frac{1}{2}\int_0^{\pi/2} (\sin 0 + \sin 2x)\, dx = \frac{1}{2}\int_0^{\pi/2} \sin 2x\, dx = -\frac{1}{4}[\cos 2x]_0^{\pi/2} = -\frac{1}{4}(-1-1) = \frac{1}{2}$

37. $\int_0^{\pi} \cos 3x \cos 4x\, dx = \frac{1}{2}\int_0^{\pi} (\cos(-x) + \cos 7x)\, dx = \frac{1}{2}\left[-\sin(-x) + \frac{1}{7}\sin 7x\right]_0^{\pi} = \frac{1}{2}(0) = 0$

38. $\int_{-\pi/2}^{\pi/2} \cos 7x \cos x\, dx = \frac{1}{2}\int_{-\pi/2}^{\pi/2} (\cos 6x + \cos 8x)\, dx = \frac{1}{2}\left[\frac{1}{6}\sin 6x + \frac{1}{8}\sin 8x\right]_{-\pi/2}^{\pi/2} = 0$

39. $x = t^{2/3} \Rightarrow t^2 = x^3;\ y = \frac{t^2}{2} \Rightarrow y = \frac{x^3}{2};\ 0 \le t \le 2 \Rightarrow 0 \le x \le 2^{2/3};$
$A = \int_0^{2^{2/3}} 2\pi\left(\frac{x^3}{2}\right)\sqrt{1+\frac{9}{4}x^4}\, dx;\ \begin{bmatrix} u = \frac{9}{4}x^4 \\ du = 9x^3 dx \end{bmatrix} \to \frac{\pi}{9}\int_0^{9(2^{2/3})} \sqrt{1+u}\, du = \left[\frac{\pi}{9}\cdot\frac{2}{3}(1+u)^{3/2}\right]_0^{9(2^{2/3})}$
$= \frac{2\pi}{27}\left[\left(1+9\left(2^{2/3}\right)\right)^{3/2} - 1\right]$

40. $y = \ln(\cos x);\ y' = \frac{-\sin x}{\cos x} = -\tan x;\ (y')^2 = \tan^2 x;\ \int_0^{\pi/3} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/3} |\sec x|\, dx = [\ln|\sec x + \tan x|]_0^{\pi/3}$
$= \ln\left(2+\sqrt{3}\right) - \ln(1+0) = \ln\left(2+\sqrt{3}\right)$

41. $y = \ln(\sec x);\ y' = \frac{\sec x \tan x}{\sec x} = \tan x; (y')^2 = \tan^2 x;\ \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} |\sec x|\, dx = [\ln|\sec x + \tan x|]_0^{\pi/4}$
$= \ln\left(\sqrt{2}+1\right) - \ln(0+1) = \ln\left(\sqrt{2}+1\right)$

42. $M = \int_{-\pi/4}^{\pi/4} \sec x\, dx = [\ln|\sec x + \tan x|]_{-\pi/4}^{\pi/4} = \ln\left(\sqrt{2}+1\right) - \ln\left|\sqrt{2}-1\right| = \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}$
$\bar{y} = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}\int_{-\pi/4}^{\pi/4} \frac{\sec^2 x}{2}\, dx = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}[\tan x]_{-\pi/4}^{\pi/4} = \frac{1}{2\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}(1-(-1)) = \frac{1}{\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}}$
$\Rightarrow (\bar{x}, \bar{y}) = \left(0, \left(\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)^{-1}\right)$

43. $V = \pi\int_0^{\pi} \sin^2 x\, dx = \pi\int_0^{\pi} \frac{1-\cos 2x}{2}\, dx = \frac{\pi}{2}\int_0^{\pi} dx - \frac{\pi}{2}\int_0^{\pi} \cos 2x\, dx = \frac{\pi}{2}[x]_0^{\pi} - \frac{\pi}{4}[\sin 2x]_0^{\pi} = \frac{\pi}{2}(\pi - 0) - \frac{\pi}{4}(0-0) = \frac{\pi^2}{2}$

44. $A = \int_0^{\pi} \sqrt{1+\cos 4x}\, dx = \int_0^{\pi} \sqrt{2}\,|\cos 2x|dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\, dx - \sqrt{2}\int_{\pi/4}^{3\pi/4} \cos 2x\, dx + \sqrt{2}\int_{3\pi/4}^{\pi} \cos 2x\, dx$
$= \frac{\sqrt{2}}{2}[\sin 2x]_0^{\pi/4} - \frac{\sqrt{2}}{2}[\sin 2x]_{\pi/4}^{3\pi/4} + \frac{\sqrt{2}}{2}[\sin 2x]_{3\pi/4}^{\pi} = \frac{\sqrt{2}}{2}(1-0) - \frac{\sqrt{2}}{2}(-1-1) + \frac{\sqrt{2}}{2}(0+1) = \sqrt{2}+\sqrt{2} = 2\sqrt{2}$

45. (a) $m^2 \ne n^2 \Rightarrow m+n \ne 0$ and $m-n \ne 0 \Rightarrow \int_k^{k+2\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_k^{k+2\pi} [\cos(m-n)x - \cos(m+n)x]dx$
$= \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right]_k^{k+2\pi}$
$= \frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)(k+2\pi)) - \frac{1}{m+n}\sin((m+n)(k+2\pi))\right) - \frac{1}{2}\left(\frac{1}{m-n}\sin((m-n)k) - \frac{1}{m+n}\sin((m+n)k)\right)$
$= \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k) - \frac{1}{2(m-n)}\sin((m-n)k) + \frac{1}{2(m+n)}\sin((m+n)k) = 0$
$\Rightarrow$ sin mx and sin nx are orthogonal.

(b) Same as part since $\frac{1}{2}\int_k^{k+2\pi} \cos 0\, dx = \pi$. $m^2 \ne n^2 \Rightarrow m+n \ne 0$ and $m-n \ne 0 \Rightarrow \int_k^{k+2\pi} \cos mx \cos nx\, dx$
$= \frac{1}{2}\int_k^{k+2\pi} [\cos(m-n)x + \cos(m+n)x]dx = \frac{1}{2}\left[\frac{1}{m-n}\sin(m-n)x + \frac{1}{m+n}\sin(m+n)x\right]_k^{k+2\pi}$
$= \frac{1}{2(m-n)}\sin((m-n)(k+2\pi)) + \frac{1}{2(m+n)}\sin((m+n)(k+2\pi)) - \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k)$
$= \frac{1}{2(m-n)}\sin((m-n)k) + \frac{1}{2(m+n)}\sin((m+n)k) - \frac{1}{2(m-n)}\sin((m-n)k) - \frac{1}{2(m+n)}\sin((m+n)k) = 0$

$\Rightarrow \cos mx$ and $\cos nx$ are orthogonal.

(c) Let $m = n \Rightarrow \sin mx \cos nx = \frac{1}{2}(\sin 0 + \sin((m+n)x))$ and $\frac{1}{2}\int_k^{k+2\pi} \sin 0\, dx = 0$ and $\frac{1}{2}\int_k^{k+2\pi} \sin((m+n)x)\, dx = 0$

$\Rightarrow \sin mx$ and $\cos nx$ are orthogonal if $m = n$.

Let $m \neq n$.

$\int_k^{k+2\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_k^{k+2\pi} [\sin(m-n)x + \sin(m+n)x]dx = \frac{1}{2}\left[-\frac{1}{m-n}\cos(m-n)x - \frac{1}{m+n}\cos(m+n)x\right]_k^{k+2\pi}$

$= -\frac{1}{2(m-n)}\cos((m-n)(k+2\pi)) - \frac{1}{2(m+n)}\cos((m+n)(k+2\pi)) + \frac{1}{2(m-n)}\cos((m-n)k) + \frac{1}{2(m+n)}\cos((m+n)k)$

$= -\frac{1}{2(m-n)}\cos((m-n)k) - \frac{1}{2(m+n)}\cos((m+n)k) + \frac{1}{2(m-n)}\cos((m-n)k) + \frac{1}{2(m+n)}\cos((m+n)k) = 0$

$\Rightarrow \sin mx$ and $\cos nx$ are orthogonal.

46. $\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\, dx = \sum_{n=1}^{N} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\, dx$. Since $\frac{1}{\pi}\int_{-\pi}^{\pi} \sin nx \sin mx\, dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases}$,

the sum on the right has only one nonzero term, namely $\frac{a_m}{\pi}\int_{-\pi}^{\pi} \sin mx \sin mx\, dx = a_m$.

## 8.5 TRIGONOMETRIC SUBSTITUTIONS

1. $y = 3\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dy = \frac{3\, d\theta}{\cos^2\theta},\ 9 + y^2 = 9\left(1 + \tan^2\theta\right) = \frac{9}{\cos^2\theta} \Rightarrow \frac{1}{\sqrt{9+y^2}} = \frac{|\cos\theta|}{3} = \frac{\cos\theta}{3}$

$\left(\text{because } \cos\theta > 0 \text{ when } -\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$;

$\int \frac{dy}{\sqrt{9+y^2}} = 3\int \frac{\cos\theta\, d\theta}{3\cos^2\theta} = \int \frac{d\theta}{\cos\theta} = \ln|\sec\theta + \tan\theta| + C' = \ln\left|\frac{\sqrt{9+y^2}}{3} + \frac{y}{3}\right| + C' = \ln\left|\sqrt{9+y^2} + y\right| + C$

2. $\int \frac{3\, dy}{\sqrt{1+9y^2}}$; $[3y = x] \rightarrow \int \frac{dx}{\sqrt{1+x^2}}$; $x = \tan t,\ -\frac{\pi}{2} < t < \frac{\pi}{2},\ dx = \frac{dt}{\cos^2 t},\ \sqrt{1+x^2} = \frac{1}{\cos t}$;

$\int \frac{dx}{\sqrt{1+x^2}} = \int \frac{dt}{\cos^2 t\left(\frac{1}{\cos t}\right)} = \ln|\sec t + \tan t| + C = \ln\left|\sqrt{x^2+1} + x\right| + C = \ln\left|\sqrt{1+9y^2} + 3y\right| + C$

3. $\int_{-2}^{2} \frac{dx}{4+x^2} = \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_{-2}^{2} = \frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}(-1) = \left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - \left(\frac{1}{2}\right)\left(-\frac{\pi}{4}\right) = \frac{\pi}{4}$

4. $\int_0^2 \frac{dx}{8+2x^2} = \frac{1}{2}\int_0^2 \frac{dx}{4+x^2} = \frac{1}{2}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2 = \frac{1}{2}\left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2}\tan^{-1}0\right) = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\pi}{16}$

5. $\int_0^{3/2} \frac{dx}{\sqrt{9-x^2}} = \left[\sin^{-1}\frac{x}{3}\right]_0^{3/2} = \sin^{-1}\frac{1}{2} - \sin^{-1}0 = \frac{\pi}{6} - 0 = \frac{\pi}{6}$

6. $\int_0^{1/2\sqrt{2}} \frac{2\, dx}{\sqrt{1-4x^2}}$; $[t = 2x] \rightarrow \int_0^{1/2\sqrt{2}} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1}t\right]_0^{1/\sqrt{2}} = \sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$

7. $t = 5\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = 5\cos\theta\, d\theta,\ \sqrt{25-t^2} = 5\cos\theta$;

$\int \sqrt{25-t^2}\, dt = \int (5\cos\theta)(5\cos\theta)\, d\theta = 25\int \cos^2\theta\, d\theta = 25\int \frac{1+\cos 2\theta}{2}\, d\theta = 25\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right) + C$

$= \frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\left[\sin^{-1}\left(\frac{t}{5}\right) + \left(\frac{t}{5}\right)\left(\frac{\sqrt{25-t^2}}{5}\right)\right] + C = \frac{25}{2}\sin^{-1}\left(\frac{t}{5}\right) + \frac{t\sqrt{25-t^2}}{2} + C$

8. $t = \frac{1}{3}\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\cos\theta\, d\theta,\ \sqrt{1-9t^2} = \cos\theta$;

$\int \sqrt{1-9t^2}\, dt = \frac{1}{3}\int (\cos\theta)(\cos\theta)\, d\theta = \frac{1}{3}\int \cos^2\theta\, d\theta = \frac{1}{6}(\theta + \sin\theta\cos\theta) + C = \frac{1}{6}\left[\sin^{-1}(3t) + 3t\sqrt{1-9t^2}\right] + C$

9. $x = \frac{7}{2}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{7}{2}\sec\theta\tan\theta\, d\theta,\ \sqrt{4x^2-49} = \sqrt{49\sec^2\theta - 49} = 7\tan\theta$;

$\int \frac{dx}{\sqrt{4x^2-49}} = \int \frac{\left(\frac{7}{2}\sec\theta\tan\theta\right)d\theta}{7\tan\theta} = \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{1}{2}\ln\left|\frac{2x}{7} + \frac{\sqrt{4x^2-49}}{7}\right| + C$

10. $x = \frac{3}{5}\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \frac{3}{5}\sec\theta\tan\theta\, d\theta,\ \sqrt{25x^2 - 9} = \sqrt{9\sec^2\theta - 9} = 3\tan\theta;$
$\int \frac{5\,dx}{\sqrt{25x^2-9}} = \int \frac{5\left(\frac{3}{5}\sec\theta\tan\theta\right)d\theta}{3\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{5x}{3} + \frac{\sqrt{25x^2-9}}{3}\right| + C$

11. $y = 7\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 7\sec\theta\tan\theta\, d\theta,\ \sqrt{y^2 - 49} = 7\tan\theta;$
$\int \frac{\sqrt{y^2-49}}{y}\,dy = \int \frac{(7\tan\theta)(7\sec\theta\tan\theta)\,d\theta}{7\sec\theta} = 7\int \tan^2\theta\, d\theta = 7\int (\sec^2\theta - 1)\, d\theta = 7(\tan\theta - \theta) + C$
$= 7\left[\frac{\sqrt{y^2-49}}{7} - \sec^{-1}\left(\frac{y}{7}\right)\right] + C$

12. $y = 5\sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dy = 5\sec\theta\tan\theta\, d\theta,\ \sqrt{y^2 - 25} = 5\tan\theta;$
$\int \frac{\sqrt{y^2-25}}{y^3}\,dy = \int \frac{(5\tan\theta)(5\sec\theta\tan\theta)\,d\theta}{125\sec^3\theta} = \frac{1}{5}\int \tan^2\theta\cos^2\theta\, d\theta = \frac{1}{5}\int \sin^2\theta\, d\theta = \frac{1}{10}\int (1 - \cos 2\theta)\, d\theta$
$= \frac{1}{10}(\theta - \sin\theta\cos\theta) + C = \frac{1}{10}\left[\sec^{-1}\left(\frac{y}{5}\right) - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + C = \left[\frac{\sec^{-1}\left(\frac{y}{5}\right)}{10} - \frac{\sqrt{y^2-25}}{2y^2}\right] + C$

13. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{dx}{x^2\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^2\theta\tan\theta} = \int \frac{d\theta}{\sec\theta} = \sin\theta + C = \frac{\sqrt{x^2-1}}{x} + C$

14. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\, d\theta,\ \sqrt{x^2 - 1} = \tan\theta;$
$\int \frac{2\,dx}{x^3\sqrt{x^2-1}} = \int \frac{2\tan\theta\sec\theta\, d\theta}{\sec^3\theta\tan\theta} = 2\int \cos^2\theta\, d\theta = 2\int \left(\frac{1+\cos 2\theta}{2}\right) d\theta = \theta + \sin\theta\cos\theta + C$
$= \theta + \tan\theta\cos^2\theta + C = \sec^{-1}x + \sqrt{x^2-1}\left(\frac{1}{x}\right)^2 + C = \sec^{-1}x + \frac{\sqrt{x^2-1}}{x^2} + C$

15. $x = 2\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{2\,d\theta}{\cos^2\theta},\ \sqrt{x^2+4} = \frac{2}{\cos\theta};$
$\int \frac{x^3\,dx}{\sqrt{x^2+4}} = \int \frac{(8\tan^3\theta)(\cos\theta)\,d\theta}{\cos^2\theta} = 8\int \frac{\sin^3\theta\, d\theta}{\cos^4\theta} = 8\int \frac{(\cos^2\theta - 1)(-\sin\theta)\,d\theta}{\cos^4\theta};$
$[t = \cos\theta] \ \rightarrow\ 8\int \frac{t^2-1}{t^4}\,dt = 8\int \left(\frac{1}{t^2} - \frac{1}{t^4}\right) dt = 8\left(-\frac{1}{t} + \frac{1}{3t^3}\right) + C = 8\left(-\sec\theta + \frac{\sec^3\theta}{3}\right) + C$
$= 8\left(-\frac{\sqrt{x^2+4}}{2} + \frac{(x^2+4)^{3/2}}{8\cdot 3}\right) + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$

16. $x = \tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \sec^2\theta\, d\theta,\ \sqrt{x^2+1} = \sec\theta;$
$\int \frac{dx}{x^2\sqrt{x^2+1}} = \int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta} = \int \frac{\cos\theta\, d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = \frac{-\sqrt{x^2+1}}{x} + C$

17. $w = 2\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 2\cos\theta\, d\theta,\ \sqrt{4 - w^2} = 2\cos\theta;$
$\int \frac{8\,dw}{w^2\sqrt{4-w^2}} = \int \frac{8\cdot 2\cos\theta\, d\theta}{4\sin^2\theta\cdot 2\cos\theta} = 2\int \frac{d\theta}{\sin^2\theta} = -2\cot\theta + C = \frac{-2\sqrt{4-w^2}}{w} + C$

18. $w = 3\sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dw = 3\cos\theta\, d\theta,\ \sqrt{9 - w^2} = 3\cos\theta;$
$\int \frac{\sqrt{9-w^2}}{w^2}\,dw = \int \frac{3\cos\theta\cdot 3\cos\theta\, d\theta}{9\sin^2\theta} = \int \cot^2\theta\, d\theta = \int \left(\frac{1-\sin^2\theta}{\sin^2\theta}\right) d\theta = \int (\csc^2\theta - 1)\, d\theta$
$= -\cot\theta - \theta + C = -\frac{\sqrt{9-w^2}}{w} - \sin^{-1}\left(\frac{w}{3}\right) + C$

19. $x = \sin\theta,\ 0 \le \theta \le \frac{\pi}{3},\ dx = \cos\theta\, d\theta,\ (1 - x^2)^{3/2} = \cos^3\theta;$
$\int_0^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}} = \int_0^{\pi/3} \frac{4\sin^2\theta\cos\theta\, d\theta}{\cos^3\theta} = 4\int_0^{\pi/3} \left(\frac{1-\cos^2\theta}{\cos^2\theta}\right) d\theta = 4\int_0^{\pi/3} (\sec^2\theta - 1)\, d\theta$
$= 4\left[\tan\theta - \theta\right]_0^{\pi/3} = 4\sqrt{3} - \frac{4\pi}{3}$

20. $x = 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6},\ dx = 2\cos\theta\,d\theta,\ (4-x^2)^{3/2} = 8\cos^3\theta;$
$\int_0^1 \frac{dx}{(4-x^2)^{3/2}} = \int_0^{\pi/6} \frac{2\cos\theta\,d\theta}{8\cos^3\theta} = \frac{1}{4}\int_0^{\pi/6} \frac{d\theta}{\cos^2\theta} = \frac{1}{4}\left[\tan\theta\right]_0^{\pi/6} = \frac{\sqrt{3}}{12} = \frac{1}{4\sqrt{3}}$

21. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ (x^2-1)^{3/2} = \tan^3\theta;$
$\int \frac{dx}{(x^2-1)^{3/2}} = \int \frac{\sec\theta\tan\theta\,d\theta}{\tan^3\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = -\frac{1}{\sin\theta} + C = -\frac{x}{\sqrt{x^2-1}} + C$

22. $x = \sec\theta,\ 0 < \theta < \frac{\pi}{2},\ dx = \sec\theta\tan\theta\,d\theta,\ (x^2-1)^{5/2} = \tan^5\theta;$
$\int \frac{x^2\,dx}{(x^2-1)^{5/2}} = \int \frac{\sec^2\theta\cdot\sec\theta\tan\theta\,d\theta}{\tan^5\theta} = \int \frac{\cos\theta}{\sin^4\theta}\,d\theta = -\frac{1}{3\sin^3\theta} + C = -\frac{x^3}{3(x^2-1)^{3/2}} + C$

23. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ (1-x^2)^{3/2} = \cos^3\theta;$
$\int \frac{(1-x^2)^{3/2}\,dx}{x^6} = \int \frac{\cos^3\theta\cdot\cos\theta\,d\theta}{\sin^6\theta} = \int \cot^4\theta\csc^2\theta\,d\theta = -\frac{\cot^5\theta}{5} + C = -\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5 + C$

24. $x = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \cos\theta\,d\theta,\ (1-x^2)^{1/2} = \cos\theta;$
$\int \frac{(1-x^2)^{1/2}\,dx}{x^4} = \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^4\theta} = \int \cot^2\theta\csc^2\theta\,d\theta = -\frac{\cot^3\theta}{3} + C = -\frac{1}{3}\left(\frac{\sqrt{1-x^2}}{x}\right)^3 + C$

25. $x = \frac{1}{2}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dx = \frac{1}{2}\sec^2\theta\,d\theta,\ (4x^2+1)^2 = \sec^4\theta;$
$\int \frac{8\,dx}{(4x^2+1)^2} = \int \frac{8\left(\frac{1}{2}\sec^2\theta\right)d\theta}{\sec^4\theta} = 4\int \cos^2\theta\,d\theta = 2(\theta + \sin\theta\cos\theta) + C = 2\tan^{-1}2x + \frac{4x}{(4x^2+1)} + C$

26. $t = \frac{1}{3}\tan\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dt = \frac{1}{3}\sec^2\theta\,d\theta,\ 9t^2+1 = \sec^2\theta;$
$\int \frac{6\,dt}{(9t^2+1)^2} = \int \frac{6\left(\frac{1}{3}\sec^2\theta\right)d\theta}{\sec^4\theta} = 2\int \cos^2\theta\,d\theta = \theta + \sin\theta\cos\theta + C = \tan^{-1}3t + \frac{3t}{(9t^2+1)} + C$

27. $v = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2},\ dv = \cos\theta\,d\theta,\ (1-v^2)^{5/2} = \cos^5\theta;$
$\int \frac{v^2\,dv}{(1-v^2)^{5/2}} = \int \frac{\sin^2\theta\cos\theta\,d\theta}{\cos^5\theta} = \int \tan^2\theta\sec^2\theta\,d\theta = \frac{\tan^3\theta}{3} + C = \frac{1}{3}\left(\frac{v}{\sqrt{1-v^2}}\right)^3 + C$

28. $r = \sin\theta,\ -\frac{\pi}{2} < \theta < \frac{\pi}{2};$
$\int \frac{(1-r^2)^{5/2}\,dr}{r^8} = \int \frac{\cos^5\theta\cdot\cos\theta\,d\theta}{\sin^8\theta} = \int \cot^6\theta\csc^2\theta\,d\theta = -\frac{\cot^7\theta}{7} + C = -\frac{1}{7}\left[\frac{\sqrt{1-r^2}}{r}\right]^7 + C$

29. Let $e^t = 3\tan\theta,\ t = \ln(3\tan\theta),\ \tan^{-1}\left(\frac{1}{3}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ \sqrt{e^{2t}+9} = \sqrt{9\tan^2\theta+9} = 3\sec\theta;$
$\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \frac{3\tan\theta\cdot\sec^2\theta\,d\theta}{\tan\theta\cdot 3\sec\theta} = \int_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)} \sec\theta\,d\theta = \left[\ln|\sec\theta+\tan\theta|\right]_{\tan^{-1}(1/3)}^{\tan^{-1}(4/3)}$
$= \ln\left(\frac{5}{3}+\frac{4}{3}\right) - \ln\left(\frac{\sqrt{10}}{3}+\frac{1}{3}\right) = \ln 9 - \ln\left(1+\sqrt{10}\right)$

30. Let $e^t = \tan\theta,\ t = \ln(\tan\theta),\ \tan^{-1}\left(\frac{3}{4}\right) \le \theta \le \tan^{-1}\left(\frac{4}{3}\right),\ dt = \frac{\sec^2\theta}{\tan\theta}\,d\theta,\ 1+e^{2t} = 1+\tan^2\theta = \sec^2\theta;$
$\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \frac{(\tan\theta)\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta}{\sec^3\theta} = \int_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} \cos\theta\,d\theta = \left[\sin\theta\right]_{\tan^{-1}(3/4)}^{\tan^{-1}(4/3)} = \frac{4}{5} - \frac{3}{5} = \frac{1}{5}$

31. $\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}};\ \left[u = 2\sqrt{t},\ du = \frac{1}{\sqrt{t}}\,dt\right] \to \int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2};\ u = \tan\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{4},\ du = \sec^2\theta\,d\theta,\ 1+u^2 = \sec^2\theta;$
$\int_{1/\sqrt{3}}^{1} \frac{2\,du}{1+u^2} = \int_{\pi/6}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{\sec^2\theta} = \left[2\theta\right]_{\pi/6}^{\pi/4} = 2\left(\frac{\pi}{4}-\frac{\pi}{6}\right) = \frac{\pi}{6}$

32. $y = e^{\tan\theta}$, $0 \le \theta \le \frac{\pi}{4}$, $dy = e^{\tan\theta}\sec^2\theta\, d\theta$, $\sqrt{1+(\ln y)^2} = \sqrt{1+\tan^2\theta} = \sec\theta$;
$\int_1^e \frac{dy}{y\sqrt{1+(\ln y)^2}} = \int_0^{\pi/4} \frac{e^{\tan\theta}\sec^2\theta}{e^{\tan\theta}\sec\theta}\, d\theta = \int_0^{\pi/4} \sec\theta\, d\theta = \left[\ln|\sec\theta+\tan\theta|\right]_0^{\pi/4} = \ln\left(1+\sqrt{2}\right)$

33. $x = \sec\theta$, $0 < \theta < \frac{\pi}{2}$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \tan\theta$;
$\int \frac{dx}{x\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec\theta\tan\theta} = \theta + C = \sec^{-1} x + C$

34. $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $1+x^2 = \sec^2\theta$;
$\int \frac{dx}{x^2+1} = \int \frac{\sec^2\theta\, d\theta}{\sec^2\theta} = \theta + C = \tan^{-1} x + C$

35. $x = \sec\theta$, $dx = \sec\theta\tan\theta\, d\theta$, $\sqrt{x^2-1} = \sqrt{\sec^2\theta-1} = \tan\theta$;
$\int \frac{x\, dx}{\sqrt{x^2-1}} = \int \frac{\sec\theta\cdot\sec\theta\tan\theta\, d\theta}{\tan\theta} = \int \sec^2\theta\, d\theta = \tan\theta + C = \sqrt{x^2-1} + C$

36. $x = \sin\theta$, $dx = \cos\theta\, d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$;
$\int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\cos\theta\, d\theta}{\cos\theta} = \theta + C = \sin^{-1} x + C$

37. $x\frac{dy}{dx} = \sqrt{x^2-4}$; $dy = \sqrt{x^2-4}\,\frac{dx}{x}$; $y = \int \frac{\sqrt{x^2-4}}{x}\, dx$; $\begin{bmatrix} x = 2\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 2\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-4} = 2\tan\theta \end{bmatrix}$
$\to\ y = \int \frac{(2\tan\theta)(2\sec\theta\tan\theta)\, d\theta}{2\sec\theta} = 2\int \tan^2\theta\, d\theta = 2\int(\sec^2\theta-1)\, d\theta = 2(\tan\theta-\theta) + C$
$= 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\left(\frac{x}{2}\right)\right] + C$; $x = 2$ and $y = 0 \Rightarrow 0 = 0 + C \Rightarrow C = 0 \Rightarrow y = 2\left[\frac{\sqrt{x^2-4}}{2} - \sec^{-1}\frac{x}{2}\right]$

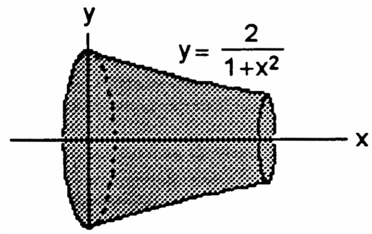
38. $\sqrt{x^2-9}\,\frac{dy}{dx} = 1$, $dy = \frac{dx}{\sqrt{x^2-9}}$; $y = \int \frac{dx}{\sqrt{x^2-9}}$; $\begin{bmatrix} x = 3\sec\theta,\ 0 < \theta < \frac{\pi}{2} \\ dx = 3\sec\theta\tan\theta\, d\theta \\ \sqrt{x^2-9} = 3\tan\theta \end{bmatrix}$ $\to\ y = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta}$
$= \int \sec\theta\, d\theta = \ln|\sec\theta+\tan\theta| + C = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right| + C$; $x = 5$ and $y = \ln 3 \Rightarrow \ln 3 = \ln 3 + C \Rightarrow C = 0$
$\Rightarrow\ y = \ln\left|\frac{x}{3} + \frac{\sqrt{x^2-9}}{3}\right|$

39. $(x^2+4)\frac{dy}{dx} = 3$, $dy = \frac{3\, dx}{x^2+4}$; $y = 3\int \frac{dx}{x^2+4} = \frac{3}{2}\tan^{-1}\frac{x}{2} + C$; $x = 2$ and $y = 0 \Rightarrow 0 = \frac{3}{2}\tan^{-1} 1 + C$
$\Rightarrow\ C = -\frac{3\pi}{8} \Rightarrow y = \frac{3}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{3\pi}{8}$

40. $(x^2+1)^2\frac{dy}{dx} = \sqrt{x^2+1}$, $dy = \frac{dx}{(x^2+1)^{3/2}}$; $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$, $(x^2+1)^{3/2} = \sec^3\theta$;
$y = \int \frac{\sec^2\theta\, d\theta}{\sec^3\theta} = \int \cos\theta\, d\theta = \sin\theta + C = \tan\theta\cos\theta + C = \frac{\tan\theta}{\sec\theta} + C = \frac{x}{\sqrt{x^2+1}} + C$; $x = 0$ and $y = 1$
$\Rightarrow\ 1 = 0 + C \Rightarrow y = \frac{x}{\sqrt{x^2+1}} + 1$

41. $A = \int_0^3 \frac{\sqrt{9-x^2}}{3}\, dx$; $x = 3\sin\theta$, $0 \le \theta \le \frac{\pi}{2}$, $dx = 3\cos\theta\, d\theta$, $\sqrt{9-x^2} = \sqrt{9-9\sin^2\theta} = 3\cos\theta$;
$A = \int_0^{\pi/2} \frac{3\cos\theta\cdot 3\cos\theta\, d\theta}{3} = 3\int_0^{\pi/2}\cos^2\theta\, d\theta = \frac{3}{2}\left[\theta + \sin\theta\cos\theta\right]_0^{\pi/2} = \frac{3\pi}{4}$

42. $V = \int_0^1 \pi \left(\frac{2}{1+x^2}\right)^2 dx = 4\pi \int_0^1 \frac{dx}{(x^2+1)^2}$;
$x = \tan\theta,\ dx = \sec^2\theta\, d\theta,\ x^2+1 = \sec^2\theta$;
$V = 4\pi \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = 4\pi \int_0^{\pi/4} \cos^2\theta\, d\theta$
$= 2\pi \int_0^{\pi/4} (1+\cos 2\theta)\, d\theta = 2\pi \left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \pi\left(\frac{\pi}{2}+1\right)$




43. $\int \frac{dx}{1-\sin x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{2z}{1+z^2}\right)} = \int \frac{2\,dz}{(1-z)^2} = \frac{2}{1-z} + C = \frac{2}{1-\tan\left(\frac{x}{2}\right)} + C$

44. $\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}+\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1+z^2+2z+1-z^2} = \int \frac{dz}{1+z} = \ln|1+z| + C$
$= \ln\left|\tan\left(\frac{x}{2}\right)+1\right| + C$

45. $\int_0^{\pi/2} \frac{dx}{1+\sin x} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1+\left(\frac{2z}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{(1+z)^2} = -\left[\frac{2}{1+z}\right]_0^1 = -(1-2) = 1$

46. $\int_{\pi/3}^{\pi/2} \frac{dx}{1-\cos x} = \int_{1/\sqrt{3}}^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int_{1/\sqrt{3}}^1 \frac{dz}{z^2} = \left[-\frac{1}{z}\right]_{1/\sqrt{3}}^1 = \sqrt{3}-1$

47. $\int_0^{\pi/2} \frac{d\theta}{2+\cos\theta} = \int_0^1 \frac{\left(\frac{2\,dz}{1+z^2}\right)}{2+\left(\frac{1-z^2}{1+z^2}\right)} = \int_0^1 \frac{2\,dz}{2+2z^2+1-z^2} = \int_0^1 \frac{2\,dz}{z^2+3} = \frac{2}{\sqrt{3}}\left[\tan^{-1}\frac{z}{\sqrt{3}}\right]_0^1 = \frac{2}{\sqrt{3}}\tan^{-1}\frac{1}{\sqrt{3}}$
$= \frac{\pi}{3\sqrt{3}} = \frac{\sqrt{3}\pi}{9}$

48. $\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\, d\theta}{\sin\theta\cos\theta+\sin\theta} = \int_1^{\sqrt{3}} \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{\left[\frac{2z\left(1-z^2\right)}{\left(1+z^2\right)^2}+\left(\frac{2z}{1+z^2}\right)\right]} = \int_1^{\sqrt{3}} \frac{2\left(1-z^2\right)dz}{2z-2z^3+2z+2z^3} = \int_1^{\sqrt{3}} \frac{1-z^2}{2z}\, dz$
$= \left[\frac{1}{2}\ln z - \frac{z^2}{4}\right]_1^{\sqrt{3}} = \left(\frac{1}{2}\ln\sqrt{3}-\frac{3}{4}\right) - \left(0-\frac{1}{4}\right) = \frac{\ln 3}{4} - \frac{1}{2} = \frac{1}{4}(\ln 3 - 2) = \frac{1}{2}\left(\ln\sqrt{3}-1\right)$

49. $\int \frac{dt}{\sin t-\cos t} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}-\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{2z-1+z^2} = \int \frac{2\,dz}{(z+1)^2-2} = \frac{1}{\sqrt{2}}\ln\left|\frac{z+1-\sqrt{2}}{z+1+\sqrt{2}}\right| + C$
$= \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{t}{2}\right)+1-\sqrt{2}}{\tan\left(\frac{t}{2}\right)+1+\sqrt{2}}\right| + C$

50. $\int \frac{\cos t\, dt}{1-\cos t} = \int \frac{\left(\frac{1-z^2}{1+z^2}\right)\left(\frac{2\,dz}{1+z^2}\right)}{1-\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)^2-\left(1+z^2\right)\left(1-z^2\right)} = \int \frac{2\left(1-z^2\right)dz}{\left(1+z^2\right)\left(1+z^2-1+z^2\right)}$
$= \int \frac{\left(1-z^2\right)dz}{\left(1+z^2\right)z^2} = \int \frac{dz}{z^2\left(1+z^2\right)} - \int \frac{dz}{1+z^2} = \int \frac{dz}{z^2} - 2\int \frac{dz}{z^2+1} = -\frac{1}{z} - 2\tan^{-1} z + C = -\cot\left(\frac{t}{2}\right) - t + C$

51. $\int \sec\theta\, d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{1-z^2}{1+z^2}\right)} = \int \frac{2\,dz}{1-z^2} = \int \frac{2\,dz}{(1+z)(1-z)} = \int \frac{dz}{1+z} + \int \frac{dz}{1-z}$
$= \ln|1+z| - \ln|1-z| + C = \ln\left|\frac{1+\tan\left(\frac{\theta}{2}\right)}{1-\tan\left(\frac{\theta}{2}\right)}\right| + C$

52. $\int \csc\theta\, d\theta = \int \frac{d\theta}{\sin\theta} = \int \frac{\left(\frac{2\,dz}{1+z^2}\right)}{\left(\frac{2z}{1+z^2}\right)} = \int \frac{dz}{z} = \ln|z| + C = \ln\left|\tan\frac{\theta}{2}\right| + C$

## 8.6 INTEGRAL TABLES AND COMPUTER ALGEBRA SYSTEMS

1. $\int \frac{dx}{x\sqrt{x-3}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\frac{x-3}{3}} + C$

   (We used FORMULA 13(a) with a = 1, b = 3)

2. $\int \frac{dx}{x\sqrt{x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{x+4}-\sqrt{4}}{\sqrt{x+4}+\sqrt{4}} \right| + C = \frac{1}{2} \ln \left| \frac{\sqrt{x+4}-2}{\sqrt{x+4}+2} \right| + C$

   (We used FORMULA 13(b) with a = 1, b = 4)

3. $\int \frac{x\,dx}{\sqrt{x-2}} = \int \frac{(x-2)\,dx}{\sqrt{x-2}} + 2\int \frac{dx}{\sqrt{x-2}} = \int \left(\sqrt{x-2}\right)^1 dx + 2\int \left(\sqrt{x-2}\right)^{-1} dx$

   $= \left(\frac{2}{1}\right) \frac{\left(\sqrt{x-2}\right)^3}{3} + 2\left(\frac{2}{1}\right) \frac{\left(\sqrt{x-2}\right)^1}{1} = \sqrt{x-2}\left[\frac{2(x-2)}{3} + 4\right] + C$

   (We used FORMULA 11 with a = 1, b = −2, n = 1 and a = 1, b = −2, n = −1)

4. $\int \frac{x\,dx}{(2x+3)^{3/2}} = \frac{1}{2} \int \frac{(2x+3)\,dx}{(2x+3)^{3/2}} - \frac{3}{2} \int \frac{dx}{(2x+3)^{3/2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2x+3}} - \frac{3}{2}\int \frac{dx}{\left(\sqrt{2x+3}\right)^3}$

   $= \frac{1}{2} \int \left(\sqrt{2x+3}\right)^{-1} dx - \frac{3}{2} \int \left(\sqrt{2x+3}\right)^{-3} dx = \left(\frac{1}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x+3}\right)^1}{1} - \left(\frac{3}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x+3}\right)^{-1}}{(-1)} + C$

   $= \frac{1}{2\sqrt{2x+3}}(2x+3+3) + C = \frac{(x+3)}{\sqrt{2x+3}} + C$

   (We used FORMULA 11 with a = 2, b = 3, n = −1 and a = 2, b = 3, n = −3)

5. $\int x\sqrt{2x-3}\,dx = \frac{1}{2} \int (2x-3)\sqrt{2x-3}\,dx + \frac{3}{2}\int \sqrt{2x-3}\,dx = \frac{1}{2} \int \left(\sqrt{2x-3}\right)^3 dx + \frac{3}{2} \int \left(\sqrt{2x-3}\right)^1 dx$

   $= \left(\frac{1}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x-3}\right)^5}{5} + \left(\frac{3}{2}\right)\left(\frac{2}{2}\right) \frac{\left(\sqrt{2x-3}\right)^3}{3} + C = \frac{(2x-3)^{3/2}}{2}\left[\frac{2x-3}{5} + 1\right] + C = \frac{(2x-3)^{3/2}(x+1)}{5} + C$

   (We used FORMULA 11 with a = 2, b = −3, n = 3 and a = 2, b = −3, n = 1)

6. $\int x(7x+5)^{3/2}\,dx = \frac{1}{7}\int (7x+5)(7x+5)^{3/2}\,dx - \frac{5}{7} \int (7x+5)^{3/2}\,dx = \frac{1}{7} \int \left(\sqrt{7x+5}\right)^5 dx - \frac{5}{7} \int \left(\sqrt{7x+5}\right)^3 dx$

   $= \left(\frac{1}{7}\right)\left(\frac{2}{7}\right) \frac{\left(\sqrt{7x+5}\right)^7}{7} - \left(\frac{5}{7}\right)\left(\frac{2}{7}\right) \frac{\left(\sqrt{7x+5}\right)^5}{5} + C = \left[\frac{(7x+5)^{5/2}}{49}\right]\left[\frac{2(7x+5)}{7} - 2\right] + C$

   $= \left[\frac{(7x+5)^{5/2}}{49}\right]\left(\frac{14x-4}{7}\right) + C$

   (We used FORMULA 11 with a = 7, b = 5, n = 5 and a = 7, b = 5, n = 3)

7. $\int \frac{\sqrt{9-4x}}{x^2}\,dx = -\frac{\sqrt{9-4x}}{x} + \frac{(-4)}{2} \int \frac{dx}{x\sqrt{9-4x}} + C$

   (We used FORMULA 14 with a = −4, b = 9)

   $= -\frac{\sqrt{9-4x}}{x} - 2\left(\frac{1}{\sqrt{9}}\right) \ln \left| \frac{\sqrt{9-4x}-\sqrt{9}}{\sqrt{9-4x}+\sqrt{9}} \right| + C$

   (We used FORMULA 13(b) with a = −4, b = 9)

   $= \frac{-\sqrt{9-4x}}{x} - \frac{2}{3} \ln \left| \frac{\sqrt{9-4x}-3}{\sqrt{9-4x}+3} \right| + C$

8. $\int \frac{dx}{x^2\sqrt{4x-9}} = -\frac{\sqrt{4x-9}}{(-9)x} + \frac{4}{18} \int \frac{dx}{x\sqrt{4x-9}} + C$

   (We used FORMULA 15 with a = 4, b = −9)

   $= \frac{\sqrt{4x-9}}{9x} + \left(\frac{2}{9}\right)\left(\frac{2}{\sqrt{9}}\right) \tan^{-1} \sqrt{\frac{4x-9}{9}} + C$

   (We used FORMULA 13(a) with a = 4, b = 9)

   $= \frac{\sqrt{4x-9}}{9x} + \frac{4}{27} \tan^{-1} \sqrt{\frac{4x-9}{9}} + C$

9. $\int x\sqrt{4x-x^2}\,dx = \int x\sqrt{2\cdot 2x - x^2}\,dx = \frac{(x+2)(2x-3\cdot 2)\sqrt{2\cdot 2\cdot x - x^2}}{6} + \frac{2^3}{2}\sin^{-1}\left(\frac{x-2}{2}\right) + C$
$= \frac{(x+2)(2x-6)\sqrt{4x-x^2}}{6} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C = \frac{(x+2)(x-3)\sqrt{4x-x^2}}{3} + 4\sin^{-1}\left(\frac{x-2}{2}\right) + C$
(We used FORMULA 51 with $a = 2$)

10. $\int \frac{\sqrt{x-x^2}}{x}\,dx = \int \frac{\sqrt{2\cdot\frac{1}{2}x - x^2}}{x}\,dx = \sqrt{2\cdot\frac{1}{2}x - x^2} + \frac{1}{2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + C = \sqrt{x-x^2} + \frac{1}{2}\sin^{-1}(2x-1) + C$
(We used FORMULA 52 with $a = \frac{1}{2}$)

11. $\int \frac{dx}{x\sqrt{7+x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 + x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2+x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7+x^2}}{x}\right| + C$
(We used FORMULA 26 with $a = \sqrt{7}$)

12. $\int \frac{dx}{x\sqrt{7-x^2}} = \int \frac{dx}{x\sqrt{\left(\sqrt{7}\right)^2 - x^2}} = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{\left(\sqrt{7}\right)^2-x^2}}{x}\right| + C = -\frac{1}{\sqrt{7}}\ln\left|\frac{\sqrt{7}+\sqrt{7-x^2}}{x}\right| + C$
(We used FORMULA 34 with $a = \sqrt{7}$)

13. $\int \frac{\sqrt{4-x^2}}{x}\,dx = \int \frac{\sqrt{2^2-x^2}}{x}\,dx = \sqrt{2^2-x^2} - 2\ln\left|\frac{2+\sqrt{2^2-x^2}}{x}\right| + C = \sqrt{4-x^2} - 2\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C$
(We used FORMULA 31 with $a = 2$)

14. $\int \frac{\sqrt{x^2-4}}{x}\,dx = \int \frac{\sqrt{x^2-2^2}}{x}\,dx = \sqrt{x^2-2^2} - 2\sec^{-1}\left|\frac{x}{2}\right| + C = \sqrt{x^2-4} - 2\sec^{-1}\left|\frac{x}{2}\right| + C$
(We used FORMULA 42 with $a = 2$)

15. $\int \sqrt{25-p^2}\,dp = \int \sqrt{5^2-p^2}\,dp = \frac{p}{2}\sqrt{5^2-p^2} + \frac{5^2}{2}\sin^{-1}\frac{p}{5} + C = \frac{p}{2}\sqrt{25-p^2} + \frac{25}{2}\sin^{-1}\frac{p}{5} + C$
(We used FORMULA 29 with $a = 5$)

16. $\int q^2\sqrt{25-q^2}\,dq = \int q^2\sqrt{5^2-q^2}\,dq = \frac{5^4}{8}\sin^{-1}\left(\frac{q}{5}\right) - \frac{1}{8}q\sqrt{5^2-q^2}\,(5^2-2q^2) + C$
$= \frac{625}{8}\sin^{-1}\left(\frac{q}{5}\right) - \frac{1}{8}q\sqrt{25-q^2}\,(25-2q^2) + C$
(We used FORMULA 30 with $a = 5$)

17. $\int \frac{r^2}{\sqrt{4-r^2}}\,dr = \int \frac{r^2}{\sqrt{2^2-r^2}}\,dr = \frac{2^2}{2}\sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{2^2-r^2} + C = 2\sin^{-1}\left(\frac{r}{2}\right) - \frac{1}{2}r\sqrt{4-r^2} + C$
(We used FORMULA 33 with $a = 2$)

18. $\int \frac{ds}{\sqrt{s^2-2}} = \int \frac{ds}{\sqrt{s^2-\left(\sqrt{2}\right)^2}} = \cosh^{-1}\frac{s}{\sqrt{2}} + C = \ln\left|s+\sqrt{s^2-\left(\sqrt{2}\right)^2}\right| + C = \ln\left|s+\sqrt{s^2-2}\right| + C$
(We used FORMULA 36 with $a = \sqrt{2}$)

19. $\int \frac{d\theta}{5+4\sin 2\theta} = \frac{-2}{2\sqrt{25-16}}\tan^{-1}\left[\sqrt{\frac{5-4}{5+4}}\tan\left(\frac{\pi}{4}-\frac{2\theta}{2}\right)\right] + C = -\frac{1}{3}\tan^{-1}\left[\frac{1}{3}\tan\left(\frac{\pi}{4}-\theta\right)\right] + C$
(We used FORMULA 70 with $b = 5$, $c = 4$, $a = 2$)

20. $\int \frac{d\theta}{4+5\sin 2\theta} = \frac{-1}{2\sqrt{25-16}}\ln\left|\frac{5+4\sin 2\theta+\sqrt{25-16}\cos 2\theta}{4+5\sin 2\theta}\right| + C = -\frac{1}{6}\ln\left|\frac{5+4\sin 2\theta+3\cos 2\theta}{4+5\sin 2\theta}\right| + C$
(We used FORMULA 71 with $a = 2$, $b = 4$, $c = 5$)

21. $\int e^{2t}\cos 3t\,dt = \frac{e^{2t}}{2^2+3^2}(2\cos 3t + 3\sin 3t) + C = \frac{e^{2t}}{13}(2\cos 3t + 3\sin 3t) + C$
(We used FORMULA 108 with a = 2, b = 3)

22. $\int e^{-3t}\sin 4t\,dt = \frac{e^{-3t}}{(-3)^2+4^2}(-3\sin 4t - 4\cos 4t) + C = \frac{e^{-3t}}{25}(-3\sin 4t - 4\cos 4t) + C$
(We used FORMULA 107 with a = −3, b = 4)

23. $\int x\cos^{-1}x\,dx = \int x^1\cos^{-1}x\,dx = \frac{x^{1+1}}{1+1}\cos^{-1}x + \frac{1}{1+1}\int\frac{x^{1+1}\,dx}{\sqrt{1-x^2}} = \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}}$
(We used FORMULA 100 with a = 1, n = 1)
$= \frac{x^2}{2}\cos^{-1}x + \frac{1}{2}\left(\frac{1}{2}\sin^{-1}x\right) - \frac{1}{2}\left(\frac{1}{2}x\sqrt{1-x^2}\right) + C = \frac{x^2}{2}\cos^{-1}x + \frac{1}{4}\sin^{-1}x - \frac{1}{4}x\sqrt{1-x^2} + C$
(We used FORMULA 33 with a = 1)

24. $\int x\tan^{-1}x\,dx = \int x^1\tan^{-1}(1x)\,dx = \frac{x^{1+1}}{1+1}\tan^{-1}(1x) - \frac{1}{1+1}\int\frac{x^{1+1}\,dx}{1+(1)^2x^2} = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{x^2\,dx}{1+x^2}$
(We used FORMULA 101 with a = 1, n = 1)
$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{1+x^2}\right)dx$ (after long division)
$= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int dx + \frac{1}{2}\int\frac{1}{1+x^2}dx = \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}x + \frac{1}{2}\tan^{-1}x + C = \frac{1}{2}\left((x^2+1)\tan^{-1}x - x\right) + C$

25. $\int\frac{ds}{(9-s^2)^2} = \int\frac{ds}{(3^3-s^2)^2} = \frac{s}{2\cdot 3^2\cdot(3^2-s^2)} + \frac{1}{4\cdot 3^3}\ln\left|\frac{s+3}{s-3}\right| + C$
(We used FORMULA 19 with a = 3)
$= \frac{s}{18(9-s^2)} + \frac{1}{108}\ln\left|\frac{s+3}{s-3}\right| + C$

26. $\int\frac{d\theta}{(2-\theta^2)^2} = \int\frac{d\theta}{\left[\left(\sqrt{2}\right)^2-\theta^2\right]^2} = \frac{\theta}{2\left(\sqrt{2}\right)^2\left[\left(\sqrt{2}\right)^2-\theta^2\right]} + \frac{1}{4\left(\sqrt{2}\right)^3}\ln\left|\frac{\theta+\sqrt{2}}{\theta-\sqrt{2}}\right| + C$
(We used FORMULA 19 with $a = \sqrt{2}$)
$= \frac{\theta}{4(2-\theta^2)} + \frac{1}{8\sqrt{2}}\ln\left|\frac{\theta+\sqrt{2}}{\theta-\sqrt{2}}\right| + C$

27. $\int\frac{\sqrt{4x+9}}{x^2}\,dx = -\frac{\sqrt{4x+9}}{x} + \frac{4}{2}\int\frac{dx}{x\sqrt{4x+9}}$
(We used FORMULA 14 with a = 4, b = 9)
$= -\frac{\sqrt{4x+9}}{x} + 2\left(\frac{1}{\sqrt{9}}\ln\left|\frac{\sqrt{4x+9}-\sqrt{9}}{\sqrt{4x+9}+\sqrt{9}}\right|\right) + C = -\frac{\sqrt{4x+9}}{x} + \frac{2}{3}\ln\left|\frac{\sqrt{4x+9}-3}{\sqrt{4x+9}+3}\right| + C$
(We used FORMULA 13(b) with a = 4, b = 9)

28. $\int\frac{\sqrt{9x-4}}{x^2}\,dx = -\frac{\sqrt{9x-4}}{x} + \frac{9}{2}\int\frac{dx}{x\sqrt{9x-4}} + C$
(We used FORMULA 14 with a = 9, b = −4)
$= -\frac{\sqrt{9x-4}}{x} + \frac{9}{2}\left(\frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{9x-4}{4}}\right) + C = -\frac{\sqrt{9x-4}}{x} + \frac{9}{2}\tan^{-1}\frac{\sqrt{9x-4}}{2} + C$
(We used FORMULA 13(a) with a = 9, b = 4)

29. $\int\frac{\sqrt{3t-4}}{t}\,dt = 2\sqrt{3t-4} + (-4)\int\frac{dt}{t\sqrt{3t-4}}$
(We used FORMULA 12 with a = 3, b = −4)
$= 2\sqrt{3t-4} - 4\left(\frac{2}{\sqrt{4}}\tan^{-1}\sqrt{\frac{3t-4}{4}}\right) + C = 2\sqrt{3t-4} - 4\tan^{-1}\frac{\sqrt{3t-4}}{2} + C$
(We used FORMULA 13(a) with a = 3, b = 4)

30. $\int \frac{\sqrt{3t+9}}{t}\,dt = 2\sqrt{3t+9} + 9\int \frac{dt}{t\sqrt{3t+9}}$

(We used FORMULA 12 with a = 3, b = 9)

$= 2\sqrt{3t+9} + 9\left(\frac{1}{\sqrt{9}}\ln\left|\frac{\sqrt{3t+9}-\sqrt{9}}{\sqrt{3t+9}+\sqrt{9}}\right|\right) + C = 2\sqrt{3t+9} + 3\ln\left|\frac{\sqrt{3t+9}-3}{\sqrt{3t+9}+3}\right| + C$

(We used FORMULA 13(b) with a = 3, b = 9)

31. $\int x^2 \tan^{-1} x\,dx = \frac{x^{2+1}}{2+1}\tan^{-1} x - \frac{1}{2+1}\int \frac{x^{2+1}}{1+x^2}\,dx = \frac{x^3}{3}\tan^{-1} x - \frac{1}{3}\int \frac{x^3}{1+x^2}\,dx$

(We used FORMULA 101 with a = 1, n = 2);

$\int \frac{x^3}{1+x^2}\,dx = \int x\,dx - \int \frac{x\,dx}{1+x^2} = \frac{x^2}{2} - \frac{1}{2}\ln(1+x^2) + C \Rightarrow \int x^2 \tan^{-1} x\,dx$

$= \frac{x^3}{3}\tan^{-1} x - \frac{x^2}{6} + \frac{1}{6}\ln(1+x^2) + C$

32. $\int \frac{\tan^{-1} x}{x^2}\,dx = \int x^{-2}\tan^{-1} x\,dx = \frac{x^{(-2+1)}}{(-2+1)}\tan^{-1} x - \frac{1}{(-2+1)}\int \frac{x^{(-2+1)}}{1+x^2}\,dx = \frac{x^{-1}}{(-1)}\tan^{-1} x + \int \frac{x^{-1}}{(1+x^2)}\,dx$

(We used FORMULA 101 with a = 1, n = −2);

$\int \frac{x^{-1}\,dx}{1+x^2} = \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x\,dx}{1+x^2} = \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

$\Rightarrow \int \frac{\tan^{-1} x}{x^2}\,dx = -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C$

33. $\int \sin 3x \cos 2x\,dx = -\frac{\cos 5x}{10} - \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with a = 3, b = 2)

34. $\int \sin 2x \cos 3x\,dx = -\frac{\cos 5x}{10} + \frac{\cos x}{2} + C$

(We used FORMULA 62(a) with a = 2, b = 3)

35. $\int 8\sin 4t \sin\frac{t}{2}\,dx = \frac{8}{7}\sin\left(\frac{7t}{2}\right) - \frac{8}{9}\sin\left(\frac{9t}{2}\right) + C = 8\left[\frac{\sin\left(\frac{7t}{2}\right)}{7} - \frac{\sin\left(\frac{9t}{2}\right)}{9}\right] + C$

(We used FORMULA 62(b) with a = 4, b = $\frac{1}{2}$)

36. $\int \sin\frac{t}{3}\sin\frac{t}{6}\,dt = 3\sin\left(\frac{t}{6}\right) - \sin\left(\frac{t}{2}\right) + C$

(We used FORMULA 62(b) with a = $\frac{1}{3}$, b = $\frac{1}{6}$)

37. $\int \cos\frac{\theta}{3}\cos\frac{\theta}{4}\,d\theta = 6\sin\left(\frac{\theta}{12}\right) + \frac{6}{7}\sin\left(\frac{7\theta}{12}\right) + C$

(We used FORMULA 62(c) with a = $\frac{1}{3}$, b = $\frac{1}{4}$)

38. $\int \cos\frac{\theta}{2}\cos 7\theta\,d\theta = \frac{1}{13}\sin\left(\frac{13\theta}{2}\right) + \frac{1}{15}\sin\left(\frac{15\theta}{2}\right) + C = \frac{\sin\left(\frac{13\theta}{2}\right)}{13} + \frac{\sin\left(\frac{15\theta}{2}\right)}{15} + C$

(We used FORMULA 62(c) with a = $\frac{1}{2}$, b = 7)

39. $\int \frac{x^3+x+1}{(x^2+1)^2}\,dx = \int \frac{x\,dx}{x^2+1} + \int \frac{dx}{(x^2+1)^2} = \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1} + \int \frac{dx}{(x^2+1)^2}$

$= \frac{1}{2}\ln(x^2+1) + \frac{x}{2(1+x^2)} + \frac{1}{2}\tan^{-1} x + C$

(For the second integral we used FORMULA 17 with a = 1)

40. $\int \frac{x^2+6x}{(x^2+3)^2}\,dx = \int \frac{dx}{x^2+3} + \int \frac{6x\,dx}{(x^2+3)^2} - \int \frac{3\,dx}{(x^2+3)^2} = \int \frac{dx}{x^2+\left(\sqrt{3}\right)^2} + 3\int \frac{d(x^2+3)}{(x^2+3)^2} - 3\int \frac{dx}{\left[x^2+\left(\sqrt{3}\right)^2\right]^2}$

$= \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{(x^2+3)} - 3\left(\frac{x}{2\left(\sqrt{3}\right)^2\left(\left(\sqrt{3}\right)^2+x^2\right)} + \frac{1}{2\left(\sqrt{3}\right)^3}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right) + C$

$\Big($For the first integral we used FORMULA 16 with $a = \sqrt{3}$; for the third integral we used FORMULA 17 with $a = \sqrt{3}\Big)$

$= \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right) - \frac{3}{x^2+3} - \frac{x}{2(x^2+3)} + C$

41. $\int \sin^{-1}\sqrt{x}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to 2\int u^1 \sin^{-1} u\,du = 2\left(\frac{u^{1+1}}{1+1}\sin^{-1} u - \frac{1}{1+1}\int \frac{u^{1+1}}{\sqrt{1-u^2}}\,du\right)$

$= u^2 \sin^{-1} u - \int \frac{u^2\,du}{\sqrt{1-u^2}}$

(We used FORMULA 99 with $a = 1$, $n = 1$)

$= u^2 \sin^{-1} u - \left(\frac{1}{2}\sin^{-1} u - \frac{1}{2}u\sqrt{1-u^2}\right) + C = \left(u^2 - \frac{1}{2}\right)\sin^{-1} u + \frac{1}{2}u\sqrt{1-u^2} + C$

(We used FORMULA 33 with $a = 1$)

$= \left(x - \frac{1}{2}\right)\sin^{-1}\sqrt{x} + \frac{1}{2}\sqrt{x - x^2} + C$

42. $\int \frac{\cos^{-1}\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{\cos^{-1} u}{u} \cdot 2u\,du = 2\int \cos^{-1} u\,du = 2\left(u\cos^{-1} u - \frac{1}{1}\sqrt{1-u^2}\right) + C$

(We used FORMULA 97 with $a = 1$)

$= 2\left(\sqrt{x}\cos^{-1}\sqrt{x} - \sqrt{1-x}\right) + C$

43. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{u \cdot 2u}{\sqrt{1-u^2}}\,du = 2\int \frac{u^2}{\sqrt{1-u^2}}\,du = 2\left(\frac{1}{2}\sin^{-1} u - \frac{1}{2}u\sqrt{1-u^2}\right) + C$

$= \sin^{-1} u - u\sqrt{1-u^2} + C$

(We used FORMULA 33 with $a = 1$)

$= \sin^{-1}\sqrt{x} - \sqrt{x}\sqrt{1-x} + C = \sin^{-1}\sqrt{x} - \sqrt{x - x^2} + C$

44. $\int \frac{\sqrt{2-x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u = \sqrt{x} \\ x = u^2 \\ dx = 2u\,du \end{bmatrix} \to \int \frac{\sqrt{2-u^2}}{u} \cdot 2u\,du = 2\int \sqrt{\left(\sqrt{2}\right)^2 - u^2}\,du$

$= 2\left[\frac{u}{2}\sqrt{\left(\sqrt{2}\right)^2 - u^2} + \frac{\left(\sqrt{2}\right)^2}{2}\sin^{-1}\left(\frac{u}{\sqrt{2}}\right)\right] + C = u\sqrt{2-u^2} + 2\sin^{-1}\left(\frac{u}{\sqrt{2}}\right) + C$

$\Big($We used FORMULA 29 with $a = \sqrt{2}\Big)$

$= \sqrt{2x - x^2} + 2\sin^{-1}\sqrt{\frac{x}{2}} + C$

45. $\int (\cot t)\sqrt{1 - \sin^2 t}\,dt = \int \frac{\sqrt{1-\sin^2 t}\,(\cos t)\,dt}{\sin t};\ \begin{bmatrix} u = \sin t \\ du = \cos t\,dt \end{bmatrix} \to \int \frac{\sqrt{1-u^2}\,du}{u}$

$= \sqrt{1-u^2} - \ln\left|\frac{1+\sqrt{1-u^2}}{u}\right| + C$

(We used FORMULA 31 with $a = 1$)

$= \sqrt{1-\sin^2 t} - \ln\left|\frac{1+\sqrt{1-\sin^2 t}}{\sin t}\right| + C$

46. $\int \frac{dt}{(\tan t)\sqrt{4-\sin^2 t}} = \int \frac{\cos t\, dt}{(\sin t)\sqrt{4-\sin^2 t}}$; $\begin{bmatrix} u = \sin t \\ du = \cos t\, dt \end{bmatrix} \to \int \frac{du}{u\sqrt{4-u^2}} = -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-u^2}}{u}\right| + C$
(We used FORMULA 34 with $a = 2$)
$= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-\sin^2 t}}{\sin t}\right| + C$

47. $\int \frac{dy}{y\sqrt{3+(\ln y)^2}}$; $\begin{bmatrix} u = \ln y \\ y = e^u \\ dy = e^u\, du \end{bmatrix} \to \int \frac{e^u\, du}{e^u\sqrt{3+u^2}} = \int \frac{du}{\sqrt{3+u^2}} = \ln\left|u+\sqrt{3+u^2}\right| + C$
$= \ln\left|\ln y + \sqrt{3+(\ln y)^2}\right| + C$
(We used FORMULA 20 with $a = \sqrt{3}$)

48. $\int \frac{\cos\theta\, d\theta}{\sqrt{5+\sin^2\theta}}$; $\begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \to \int \frac{du}{\sqrt{5+u^2}} = \ln\left|u+\sqrt{5+u^2}\right| + C = \ln\left|\sin\theta + \sqrt{5+\sin^2\theta}\right| + C$
(We used FORMULA 20 with $a = \sqrt{5}$)

49. $\int \frac{3\, dr}{\sqrt{9r^2-1}}$; $\begin{bmatrix} u = 3r \\ du = 3\, dr \end{bmatrix} \to \int \frac{du}{\sqrt{u^2-1}} = \ln\left|u+\sqrt{u^2-1}\right| + C = \ln\left|3r+\sqrt{9r^2-1}\right| + C$
(We used FORMULA 36 with $a = 1$)

50. $\int \frac{3\, dy}{\sqrt{1+9y^2}}$; $\begin{bmatrix} u = 3y \\ du = 3\, dy \end{bmatrix} \to \int \frac{du}{\sqrt{1+u^2}} = \ln\left|u+\sqrt{1+u^2}\right| + C = \ln\left|3y+\sqrt{1+9y^2}\right| + C$
(We used FORMULA 20 with $a = 1$)

51. $\int \cos^{-1}\sqrt{x}\, dx$; $\begin{bmatrix} t = \sqrt{x} \\ x = t^2 \\ dx = 2t\, dt \end{bmatrix} \to 2\int t\cos^{-1} t\, dt = 2\left(\frac{t^2}{2}\cos^{-1} t + \frac{1}{2}\int \frac{t^2}{\sqrt{1-t^2}}\, dt\right) = t^2\cos^{-1} t + \int \frac{t^2}{\sqrt{1-t^2}}\, dt$
(We used FORMULA 100 with $a = 1$, $n = 1$)
$= t^2\cos^{-1} t + \frac{1}{2}\sin^{-1} t - \frac{1}{2}t\sqrt{1-t^2} + C$
(We used FORMULA 33 with $a = 1$)
$= x\cos^{-1}\sqrt{x} + \frac{1}{2}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x}\sqrt{1-x} + C = x\cos^{-1}\sqrt{x} + \frac{1}{2}\sin^{-1}\sqrt{x} - \frac{1}{2}\sqrt{x-x^2} + C$

52. $\int \tan^{-1}\sqrt{y}\, dy$; $\begin{bmatrix} t = \sqrt{y} \\ y = t^2 \\ dy = 2t\, dt \end{bmatrix} \to 2\int t\tan^{-1} t\, dt = 2\left[\frac{t^2}{2}\tan^{-1} t - \frac{1}{2}\int \frac{t^2}{1+t^2}\, dt\right] = t^2\tan^{-1} t - \int \frac{t^2}{1+t^2}\, dt$
(We used FORMULA 101 with $n = 1$, $a = 1$)
$= t^2\tan^{-1} t - \int \frac{t^2+1}{t^2+1}\, dt + \int \frac{dt}{1+t^2} = t^2\tan^{-1} t - t + \tan^{-1} t + C = y\tan^{-1}\sqrt{y} + \tan^{-1}\sqrt{y} - \sqrt{y} + C$

53. $\int \sin^5 2x\, dx = -\frac{\sin^4 2x\cos 2x}{5\cdot 2} + \frac{5-1}{5}\int \sin^3 2x\, dx = -\frac{\sin^4 2x\cos 2x}{10} + \frac{4}{5}\left[-\frac{\sin^2 2x\cos 2x}{3\cdot 2} + \frac{3-1}{3}\int \sin 2x\, dx\right]$
(We used FORMULA 60 with $a = 2$, $n = 5$ and $a = 2$, $n = 3$)
$= -\frac{\sin^4 2x\cos 2x}{10} - \frac{2}{15}\sin^2 2x\cos 2x + \frac{8}{15}\left(-\frac{1}{2}\right)\cos 2x + C = -\frac{\sin^4 2x\cos 2x}{10} - \frac{2\sin^2 2x\cos 2x}{15} - \frac{4\cos 2x}{15} + C$

54. $\int \sin^5\frac{\theta}{2}\, d\theta = -\frac{\sin^4\frac{\theta}{2}\cos\frac{\theta}{2}}{5\cdot\frac{1}{2}} + \frac{5-1}{5}\int \sin^3\frac{\theta}{2}\, d\theta = -\frac{2}{5}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{4}{5}\left[-\frac{\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}}{3\cdot\frac{1}{2}} + \frac{3-1}{3}\int \sin\frac{\theta}{2}\, d\theta\right]$
(We used FORMULA 60 with $a = \frac{1}{2}$, $n = 5$ and $a = \frac{1}{2}$, $n = 3$)
$= -\frac{2}{5}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{8}{15}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} + \frac{8}{15}\left(-2\cos\frac{\theta}{2}\right) + C = -\frac{2}{5}\sin^4\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{8}{15}\sin^2\frac{\theta}{2}\cos\frac{\theta}{2} - \frac{16}{15}\cos\frac{\theta}{2} + C$

55. $\int 8\cos^4 2\pi t\, dt = 8\left(\frac{\cos^3 2\pi t \sin 2\pi t}{4\cdot 2\pi} + \frac{4-1}{4}\int \cos^2 2\pi t\, dt\right)$

(We used FORMULA 61 with $a = 2\pi$, $n = 4$)

$= \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + 6\left[\frac{t}{2} + \frac{\sin(2\cdot 2\pi\cdot t)}{4\cdot 2\pi}\right] + C$

(We used FORMULA 59 with $a = 2\pi$)

$= \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + 3t + \frac{3\sin 4\pi t}{4\pi} + C = \frac{\cos^3 2\pi t \sin 2\pi t}{\pi} + \frac{3\cos 2\pi t \sin 2\pi t}{2\pi} + 3t + C$

56. $\int 3\cos^5 3y\, dy = 3\left(\frac{\cos^4 3y \sin 3y}{5\cdot 3} + \frac{5-1}{5}\int \cos^3 3y\, dy\right)$

$= \frac{\cos^4 3y \sin 3y}{5} + \frac{12}{5}\left(\frac{\cos^2 3y \sin 3y}{3\cdot 3} + \frac{3-1}{3}\int \cos 3y\, dy\right)$

(We used FORMULA 61 with $a = 3$, $n = 5$ and $a = 3$, $n = 3$)

$= \frac{1}{5}\cos^4 3y \sin 3y + \frac{4}{15}\cos^2 3y \sin 3y + \frac{8}{15}\sin 3y + C$

57. $\int \sin^2 2\theta \cos^3 2\theta\, d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{2(2+3)} + \frac{3-1}{3+2}\int \sin^2 2\theta \cos 2\theta\, d\theta$

(We used FORMULA 69 with $a = 2$, $m = 3$, $n = 2$)

$= \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5}\int \sin^2 2\theta \cos 2\theta\, d\theta = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{2}{5}\left[\frac{1}{2}\int \sin^2 2\theta\, d(\sin 2\theta)\right] = \frac{\sin^3 2\theta \cos^2 2\theta}{10} + \frac{\sin^3 2\theta}{15} + C$

58. $\int 9\sin^3\theta \cos^{3/2}\theta\, d\theta = 9\left[-\frac{\sin^2\theta \cos^{5/2}\theta}{3+\left(\frac{3}{2}\right)} + \frac{3-1}{3+\left(\frac{3}{2}\right)}\int \sin\theta \cos^{3/2}\theta\, d\theta\right]$

$= -2\sin^2\theta \cos^{5/2}\theta + 4\int \cos^{3/2}\theta \sin\theta\, d\theta$

(We used FORMULA 68 with $a = 1$, $n = 3$, $m = \frac{3}{2}$)

$= -2\sin^2\theta \cos^{5/2}\theta - 4\int \cos^{3/2}\theta\, d(\cos\theta) = -2\sin^2\theta \cos^{5/2}\theta - 4\left(\frac{2}{5}\cos^{5/2}\theta\right) + C$

$= \left(-2\cos^{5/2}\theta\right)\left(\sin^2\theta + \frac{4}{5}\right) + C$

59. $\int 2\sin^2 t \sec^4 t\, dt = \int 2\sin^2 t \cos^{-4} t\, dt = 2\left(-\frac{\sin t \cos^{-3} t}{2-4} + \frac{2-1}{2-4}\int \cos^{-4} t\, dt\right)$

(We used FORMULA 68 with $a = 1$, $n = 2$, $m = -4$)

$= \sin t \cos^{-3} t - \int \cos^{-4} t\, dt = \sin t \cos^{-3} t - \int \sec^4 t\, dt = \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{4-1} + \frac{4-2}{4-1}\int \sec^2 t\, dt\right)$

(We used FORMULA 92 with $a = 1$, $n = 4$)

$= \sin t \cos^{-3} t - \left(\frac{\sec^2 t \tan t}{3}\right) - \frac{2}{3}\tan t + C = \frac{2}{3}\sec^2 t \tan t - \frac{2}{3}\tan t + C = \frac{2}{3}\tan t\left(\sec^2 t - 1\right) + C$

$= \frac{2}{3}\tan^3 t + C$

An easy way to find the integral using substitution:

$\int 2\sin^2 t \cos^{-4} t\, dt = \int 2\tan^2 t \sec^2 t\, dt = \int 2\tan^2 t\, d(\tan t) = \frac{2}{3}\tan^3 t + C$

60. $\int \csc^2 y \cos^5 y\, dy = \int \sin^{-2} y \cos^5 y\, dy = \frac{\left(\frac{1}{\sin y}\right)\cos^4 y}{5-2} + \frac{5-1}{5-2}\int \sin^{-2} y \cos^3 y\, dy$

$= \frac{\left(\frac{1}{\sin y}\right)\cos^4 y}{3} + \frac{4}{3}\left(\frac{\left(\frac{1}{\sin y}\right)\cos^2 y}{3-2} + \frac{3-1}{3-2}\int \sin^{-2} y \cos y\, dy\right)$

(We used FORMULA 69 with $n = -2$, $m = 5$, $a = 1$ and $n = -2$, $m = 3$, $a = 1$)

$= \frac{\left(\frac{1}{\sin y}\right)\cos^4 y}{3} + \frac{4}{3}\left(\frac{1}{\sin y}\right)\cos^2 y + \frac{8}{3}\int \sin^{-2} y\, d(\sin y) = \frac{\cos^4 y}{3\sin y} + \frac{4\cos^2 y}{3\sin y} - \frac{8}{3\sin y} + C$

61. $\int 4\tan^3 2x\, dx = 4\left(\frac{\tan^2 2x}{2\cdot 2} - \int \tan 2x\, dx\right) = \tan^2 2x - 4\int \tan 2x\, dx$

(We used FORMULA 86 with $n = 3$, $a = 2$)

$= \tan^2 2x - \frac{4}{2}\ln|\sec 2x| + C = \tan^2 2x - 2\ln|\sec 2x| + C$

62. $\int \tan^4\left(\frac{x}{2}\right) dx = \frac{\tan^3\left(\frac{x}{2}\right)}{\frac{1}{2}(4-1)} - \int \tan^2\left(\frac{x}{2}\right) dx = \frac{2}{3}\tan^3\left(\frac{x}{2}\right) - \int \tan^2\left(\frac{x}{2}\right) dx$

(We used FORMULA 86 with $n = 4$, $a = \frac{1}{2}$)

$= \frac{2}{3}\tan^3\frac{x}{2} - 2\tan\frac{x}{2} + x + C$

(We used FORMULA 84 with $a = \frac{1}{2}$)

63. $\int 8\cot^4 t\, dt = 8\left(-\frac{\cot^3 t}{3} - \int \cot^2 t\, dt\right)$

(We used FORMULA 87 with $a = 1$, $n = 4$)

$= 8\left(-\frac{1}{3}\cot^3 t + \cot t + t\right) + C$

(We used FORMULA 85 with $a = 1$)

64. $\int 4\cot^3 2t\, dt = 4\left[-\frac{\cot^2 2t}{2(3-1)} - \int \cot 2t\, dt\right] = -\cot^2 2t - 4\int \cot 2t\, dt$

(We used FORMULA 87 with $a = 2$, $n = 3$)

$= -\cot^2 2t - \frac{4}{2}\ln|\sin 2t| + C = -\cot^2 2t - 2\ln|\sin 2t| + C$

(We used FORMULA 83 with $a = 2$)

65. $\int 2\sec^3 \pi x\, dx = 2\left[\frac{\sec \pi x \tan \pi x}{\pi(3-1)} + \frac{3-2}{3-1}\int \sec \pi x\, dx\right]$

(We used FORMULA 92 with $n = 3$, $a = \pi$)

$= \frac{1}{\pi}\sec \pi x \tan \pi x + \frac{1}{\pi}\ln|\sec \pi x + \tan \pi x| + C$

(We used FORMULA 88 with $a = \pi$)

66. $\int \frac{1}{2}\csc^3\frac{x}{2}\, dx = \frac{1}{2}\left(-\frac{\csc\frac{x}{2}\cot\frac{x}{2}}{\frac{1}{2}(3-1)} + \frac{3-2}{3-1}\int \csc\frac{x}{2}\, dx\right)$

(We used FORMULA 93 with $a = \frac{1}{2}$, $n = 3$)

$= \frac{1}{2}\left[-\csc\frac{x}{2}\cot\frac{x}{2} - \ln\left|\csc\frac{x}{2} + \cot\frac{x}{2}\right|\right] + C = -\frac{1}{2}\csc\frac{x}{2}\cot\frac{x}{2} - \frac{1}{2}\ln\left|\csc\frac{x}{2} + \cot\frac{x}{2}\right| + C$

(We used FORMULA 89 with $a = \frac{1}{2}$)

67. $\int 3\sec^4 3x\, dx = 3\left[\frac{\sec^2 3x \tan 3x}{3(4-1)} + \frac{4-2}{4-1}\int \sec^2 3x\, dx\right]$

(We used FORMULA 92 with $n = 4$, $a = 3$)

$= \frac{\sec^2 3x \tan 3x}{3} + \frac{2}{3}\tan 3x + C$

(We used FORMULA 90 with $a = 3$)

68. $\int \csc^4\frac{\theta}{3}\, d\theta = -\frac{\csc^2\frac{\theta}{3}\cot\frac{\theta}{3}}{\frac{1}{3}(4-1)} + \frac{4-2}{4-1}\int \csc^2\frac{\theta}{3}\, d\theta$

(We used FORMULA 93 with $n = 4$, $a = \frac{1}{3}$)

$= -\csc^2\frac{\theta}{3}\cot\frac{\theta}{3} - \frac{2}{3}\cdot 3\cot\frac{\theta}{3} + C = -\csc^2\frac{\theta}{3}\cot\frac{\theta}{3} - 2\cot\frac{\theta}{3} + C$

(We used FORMULA 91 with $a = \frac{1}{3}$)

69. $\int \csc^5 x\, dx = -\frac{\csc^3 x \cot x}{5-1} + \frac{5-2}{5-1}\int \csc^3 x\, dx = -\frac{\csc^3 x \cot x}{4} + \frac{3}{4}\left(-\frac{\csc x \cot x}{3-1} + \frac{3-2}{3-1}\int \csc x\, dx\right)$

(We used FORMULA 93 with $n = 5$, $a = 1$ and $n = 3$, $a = 1$)

$= -\frac{1}{4}\csc^3 x \cot x - \frac{3}{8}\csc x \cot x - \frac{3}{8}\ln|\csc x + \cot x| + C$

(We used FORMULA 89 with $a = 1$)

70. $\int \sec^5 x\, dx = \frac{\sec^3 x \tan x}{5-1} + \frac{5-2}{5-1}\int \sec^3 x\, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\left(\frac{\sec x \tan x}{3-1} + \frac{3-2}{3-1}\int \sec x\, dx\right)$

(We used FORMULA 92 with $a = 1$, $n = 5$ and $a = 1$, $n = 3$)

$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{8}\sec x \tan x + \frac{3}{8}\ln|\sec x + \tan x| + C$
(We used FORMULA 88 with $a = 1$)

71. $\int 16x^3(\ln x)^2\,dx = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{2}{4}\int x^3 \ln x\,dx\right] = 16\left[\frac{x^4(\ln x)^2}{4} - \frac{1}{2}\left[\frac{x^4(\ln x)}{4} - \frac{1}{4}\int x^3\,dx\right]\right]$
(We used FORMULA 110 with $a = 1$, $n = 3$, $m = 2$ and $a = 1$, $n = 3$, $m = 1$)
$= 16\left(\frac{x^4(\ln x)^2}{4} - \frac{x^4(\ln x)}{8} + \frac{x^4}{32}\right) + C = 4x^4(\ln x)^2 - 2x^4\ln x + \frac{x^4}{2} + C$

72. $\int (\ln x)^3\,dx = \frac{x(\ln x)^3}{1} - \frac{3}{1}\int(\ln x)^2\,dx = x(\ln x)^3 - 3\left[\frac{x(\ln x)^2}{1} - \frac{2}{1}\int \ln x\,dx\right] = x(\ln x)^3 - 3x(\ln x)^2 + 6\left(\frac{x\ln x}{1} - \frac{1}{1}\int dx\right)$
$= x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$
(We used FORMULA 110 with $n = 0$, $a = 1$ and $m = 3, 2, 1$)

73. $\int xe^{3x}\,dx = \frac{e^{3x}}{3^2}(3x-1) + C = \frac{e^{3x}}{9}(3x-1) + C$
(We used FORMULA 104 with $a = 3$)

74. $\int xe^{-2x}\,dx = \frac{e^{-2x}}{(-2)^2}(-2x-1) + C = -\frac{e^{-2x}}{4}(2x+1) + C$
(We used FORMULA 104 with $a = -2$)

75. $\int x^3e^{x/2}\,dx = 2x^3e^{x/2} - 3\cdot 2\int x^2e^{x/2}\,dx = 2x^3e^{x/2} - 6\left(2x^2e^{x/2} - 2\cdot 2\int xe^{x/2}\,dx\right)$
$= 2x^3e^{x/2} - 12x^2e^{x/2} + 24\cdot 4e^{x/2}\left(\frac{x}{2} - 1\right) + C = 2x^3e^{x/2} - 12x^2e^{x/2} + 96e^{x/2}\left(\frac{x}{2} - 1\right) + C$
(We used FORMULA 105 with $a = \frac{1}{2}$ twice and FORMULA 104 with $a = \frac{1}{2}$)

76. $\int x^2e^{\pi x}\,dx = \frac{1}{\pi}x^2e^{\pi x} - \frac{2}{\pi}\int xe^{\pi x}\,dx$
(We used FORMULA 105 with $n = 2$, $a = \pi$)
$= \frac{1}{\pi}x^2e^{\pi x} - \frac{2}{\pi\cdot\pi^2}\cdot e^{\pi x}(\pi x - 1) + C = \frac{1}{\pi}x^2e^{\pi x} - \left(\frac{2e^{\pi x}}{\pi^3}\right)(\pi x - 1) + C$
(We used FORMULA 104 with $a = \pi$)

77. $\int x^2 2^x\,dx = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\int x2^x\,dx = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\left(\frac{x2^x}{\ln 2} - \frac{1}{\ln 2}\int 2^x\,dx\right) = \frac{x^2 2^x}{\ln 2} - \frac{2}{\ln 2}\left[\frac{x2^x}{\ln 2} - \frac{2^x}{(\ln 2)^2}\right] + C$
(We used FORMULA 106 with $a = 1$, $b = 2$, $n = 2$, $n = 1$)

78. $\int x^2 2^{-x}\,dx = \frac{x^2 2^{-x}}{-\ln 2} + \frac{2}{\ln 2}\int x2^{-x}\,dx = \frac{-x^2 2^{-x}}{\ln 2} + \frac{2}{\ln 2}\left(-\frac{x2^{-x}}{\ln 2} + \frac{1}{\ln 2}\int 2^{-x}\,dx\right)$
$= -\frac{x^2 2^{-x}}{\ln 2} + \frac{2}{\ln 2}\left[\frac{x2^{-x}}{-\ln 2} - \frac{2^{-x}}{(\ln 2)^2}\right] + C$
(We used FORMULA 106 with $a = -1$, $b = 2$, $n = 2$, $n = 1$)

79. $\int x\pi^x\,dx = \frac{x\pi^x}{\ln\pi} - \frac{1}{\ln\pi}\int \pi^x\,dx = \frac{x\pi^x}{\ln\pi} - \frac{1}{\ln\pi}\left(\frac{\pi^x}{\ln\pi}\right) + C = \frac{x\pi^x}{\ln\pi} - \frac{\pi^x}{(\ln\pi)^2} + C$
(We used FORMULA 106 with $n = 1$, $b = \pi$, $a = 1$)

80. $\int x2^{\sqrt{2}x}\,dx = \frac{x2^{\sqrt{2}x}}{\sqrt{2}\ln 2} - \frac{1}{\sqrt{2}\ln 2}\int 2^{\sqrt{2}x}\,dx = \frac{x2^{x\sqrt{2}}}{\sqrt{2}\ln 2} - \frac{2^{x\sqrt{2}}}{2(\ln 2)^2} + C$
(We used FORMULA 106 with $a = \sqrt{2}$, $b = 2$, $n = 1$)

81. $\int e^t\sec^3(e^t - 1)\,dt;\ \begin{bmatrix} x = e^t - 1 \\ dx = e^t\,dt \end{bmatrix} \to \int \sec^3 x\,dx = \frac{\sec x\tan x}{3-1} + \frac{3-2}{3-1}\int \sec x\,dx$
(We used FORMULA 92 with $a = 1$, $n = 3$)
$= \frac{\sec x\tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C = \frac{1}{2}\left[\sec(e^t - 1)\tan(e^t - 1) + \ln|\sec(e^t - 1) + \tan(e^t - 1)|\right] + C$

82. $\int \frac{\csc^3\sqrt{\theta}}{\sqrt{\theta}}\,d\theta;\ \begin{bmatrix} t=\sqrt{\theta} \\ \theta=t^2 \\ d\theta=2t\,dt \end{bmatrix} \to 2\int \csc^3 t\,dt = 2\left[-\frac{\csc t\cot t}{3-1}+\frac{3-2}{3-1}\int \csc t\,dt\right]$

(We used FORMULA 93 with a = 1, n = 3)

$= 2\left[-\frac{\csc t\cot t}{2}-\frac{1}{2}\ln|\csc t+\cot t|\right]+C = -\csc\sqrt{\theta}\cot\sqrt{\theta}-\ln\left|\csc\sqrt{\theta}+\cot\sqrt{\theta}\right|+C$

83. $\int_0^1 2\sqrt{x^2+1}\,dx;\ [x=\tan t] \to 2\int_0^{\pi/4}\sec t\cdot\sec^2 t\,dt = 2\int_0^{\pi/4}\sec^3 t\,dt = 2\left[\left[\frac{\sec t\cdot\tan t}{3-1}\right]_0^{\pi/4}+\frac{3-2}{3-1}\int_0^{\pi/4}\sec t\,dt\right]$

(We used FORMULA 92 with n = 3, a = 1)

$= [\sec t\cdot\tan t+\ln|\sec t+\tan t|]_0^{\pi/4} = \sqrt{2}+\ln\left(\sqrt{2}+1\right)$

84. $\int_0^{\sqrt{3}/2}\frac{dy}{(1-y^2)^{5/2}};\ [y=\sin x] \to \int_0^{\pi/3}\frac{\cos x\,dx}{\cos^5 x} = \int_0^{\pi/3}\sec^4 x\,dx = \left[\frac{\sec^2 x\tan x}{4-1}\right]_0^{\pi/3}+\frac{4-2}{4-1}\int_0^{\pi/3}\sec^2 x\,dx$

(We used FORMULA 92 with a = 1, n = 4)

$= \left[\frac{\sec^2 x\tan x}{3}+\frac{2}{3}\tan x\right]_0^{\pi/3} = \left(\frac{4}{3}\right)\sqrt{3}+\left(\frac{2}{3}\right)\sqrt{3} = 2\sqrt{3}$

85. $\int_1^2\frac{(r^2-1)^{3/2}}{r}\,dr;\ [r=\sec\theta] \to \int_0^{\pi/3}\frac{\tan^3\theta}{\sec\theta}(\sec\theta\tan\theta)\,d\theta = \int_0^{\pi/3}\tan^4\theta\,d\theta = \left[\frac{\tan^3\theta}{4-1}\right]_0^{\pi/3}-\int_0^{\pi/3}\tan^2\theta\,d\theta$

$= \left[\frac{\tan^3\theta}{3}-\tan\theta+\theta\right]_0^{\pi/3} = \frac{3\sqrt{3}}{3}-\sqrt{3}+\frac{\pi}{3} = \frac{\pi}{3}$

(We used FORMULA 86 with a = 1, n = 4 and FORMULA 84 with a = 1)

86. $\int_0^{1/\sqrt{3}}\frac{dt}{(t^2+1)^{7/2}};\ [t=\tan\theta] \to \int_0^{\pi/6}\frac{\sec^2\theta\,d\theta}{\sec^7\theta} = \int_0^{\pi/6}\cos^5\theta\,d\theta = \left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6}+\left(\frac{5-1}{5}\right)\int_0^{\pi/6}\cos^3\theta\,d\theta$

$= \left[\frac{\cos^4\theta\sin\theta}{5}\right]_0^{\pi/6}+\frac{4}{5}\left[\left[\frac{\cos^2\theta\sin\theta}{3}\right]_0^{\pi/6}+\left(\frac{3-1}{3}\right)\int_0^{\pi/6}\cos\theta\,d\theta\right]$

$= \left[\frac{\cos^4\theta\sin\theta}{5}+\frac{4}{15}\cos^2\theta\sin\theta+\frac{8}{15}\sin\theta\right]_0^{\pi/6}$

(We used FORMULA 61 with a = 1, n = 5 and a = 1, n = 3)

$= \frac{\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right)}{5}+\left(\frac{4}{15}\right)\left(\frac{\sqrt{3}}{2}\right)^2\left(\frac{1}{2}\right)+\left(\frac{8}{15}\right)\left(\frac{1}{2}\right) = \frac{9}{160}+\frac{1}{10}+\frac{4}{15} = \frac{3\cdot 9+48+32\cdot 4}{480} = \frac{203}{480}$

87. $\int \frac{1}{8}\sinh^5 3x\,dx = \frac{1}{8}\left(\frac{\sinh^4 3x\cosh 3x}{5\cdot 3}-\frac{5-1}{5}\int \sinh^3 3x\,dx\right)$

$= \frac{\sinh^4 3x\cosh 3x}{120}-\frac{1}{10}\left(\frac{\sinh 3x\cosh 3x}{3\cdot 3}-\frac{3-1}{3}\int \sinh 3x\,dx\right)$

(We used FORMULA 117 with a = 3, n = 5 and a = 3, n = 3)

$= \frac{\sinh^4 3x\cosh 3x}{120}-\frac{\sinh 3x\cosh 3x}{90}+\frac{2}{30}\left(\frac{1}{3}\cosh 3x\right)+C$

$= \frac{1}{120}\sinh^4 3x\cosh 3x-\frac{1}{90}\sinh 3x\cosh 3x+\frac{1}{45}\cosh 3x+C$

88. $\int \frac{\cosh^4\sqrt{x}}{\sqrt{x}}\,dx;\ \begin{bmatrix} u=\sqrt{x} \\ du=\frac{dx}{2\sqrt{x}} \end{bmatrix} \to 2\int \cosh^4 u\,du = 2\left(\frac{\cosh^3 u\sinh u}{4}+\frac{4-1}{4}\int \cosh^2 u\,du\right)$

$= \frac{\cosh^3 u\sinh u}{2}+\frac{3}{2}\left(\frac{\sinh 2u}{4}+\frac{u}{2}\right)+C$

(We used FORMULA 118 with a = 1, n = 4 and FORMULA 116 with a = 1)

$= \frac{1}{2}\cosh^3\sqrt{x}\sinh\sqrt{x}+\frac{3}{8}\sinh 2\sqrt{x}+\frac{3}{4}\sqrt{x}+C$

89. $\int x^2\cosh 3x\,dx = \frac{x^2}{3}\sinh 3x-\frac{2}{3}\int x\sinh 3x\,dx = \frac{x^2}{3}\sinh 3x-\frac{2}{3}\left(\frac{x}{3}\cosh 3x-\frac{1}{3}\int \cosh 3x\,dx\right)$

(We used FORMULA 122 with a = 3, n = 2 and FORMULA 121 with a = 3, n = 1)

$= \frac{x^2}{3} \sinh 3x - \frac{2x}{9} \cosh 3x + \frac{2}{27} \sinh 3x + C$

90. $\int x \sinh 5x \, dx = \frac{x}{5} \cosh 5x - \frac{1}{25} \sinh 5x + C$
(We used FORMULA 119 with $a = 5$)

91. $\int \operatorname{sech}^7 x \tanh x \, dx = -\frac{\operatorname{sech}^7 x}{7} + C$
(We used FORMULA 135 with $a = 1, n = 7$)

92. $\int \operatorname{csch}^3 2x \coth 2x \, dx = -\frac{\operatorname{csch}^3 2x}{3 \cdot 2} + C = -\frac{\operatorname{csch}^3 2x}{6} + C$
(We used FORMULA 136 with $a = 2, n = 3$)

93. $u = ax + b \Rightarrow x = \frac{u-b}{a} \Rightarrow dx = \frac{du}{a}$;
$\int \frac{x\,dx}{(ax+b)^2} = \int \frac{(u-b)}{au^2} \frac{du}{a} = \frac{1}{a^2} \int \left(\frac{1}{u} - \frac{b}{u^2}\right) du = \frac{1}{a^2}\left[\ln|u| + \frac{b}{u}\right] + C = \frac{1}{a^2}\left[\ln|ax+b| + \frac{b}{ax+b}\right] + C$

94. $x = a \tan\theta \Rightarrow a^2 + x^2 = a^2 \sec^2\theta \Rightarrow 2x\,dx = 2a^2 \sec^2\theta \tan\theta \Rightarrow dx = a\sec^2\theta\,d\theta$;
$\int \frac{dx}{(a^2+x^2)^2} = \int \frac{a\sec^2\theta}{(a^2\sec^2\theta)^2}\,d\theta = \frac{1}{a^3}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{2a^3}\int (1+\cos 2\theta)\,d\theta = \frac{1}{2a^3}\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$
$= \frac{1}{2a^3}(\theta + \sin\theta\cos\theta) + C = \frac{1}{2a^3}\left[\theta + \left(\frac{\sin\theta}{\cos\theta}\right)\cos^2\theta\right] + C = \frac{1}{2a^3}\left(\theta + \frac{\tan\theta}{1+\tan^2\theta}\right) + C$
$= \frac{1}{2a^3}\left[\tan^{-1}\frac{x}{a} + \frac{x}{a\left(1+\frac{x^2}{a^2}\right)}\right] + C = \frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\tan^{-1}\frac{x}{a} + C$

95. $x = a\sin\theta \Rightarrow a^2 - x^2 = a^2\cos^2\theta \Rightarrow -2x\,dx = -2a^2\cos\theta\sin\theta\,d\theta \Rightarrow dx = a\cos\theta\,d\theta$;
$\int \sqrt{a^2-x^2}\,dx = \int a\cos\theta(a\cos\theta)\,d\theta = a^2\int\cos^2\theta\,d\theta = \frac{a^2}{2}\int(1+\cos 2\theta)\,d\theta = \frac{a^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$
$= \frac{a^2}{2}(\theta + \cos\theta\sin\theta) + C = \frac{a^2}{2}\left(\theta + \sqrt{1-\sin^2\theta}\cdot\sin\theta\right) + C = \frac{a^2}{2}\left(\sin^{-1}\frac{x}{a} + \frac{\sqrt{a^2-x^2}}{a}\cdot\frac{x}{a}\right) + C$
$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

96. $x = a\sec\theta \Rightarrow x^2 - a^2 = a^2\tan^2\theta \Rightarrow 2x\,dx = 2a^2\tan\theta\sec^2\theta\,d\theta \Rightarrow dx = a\sec\theta\tan\theta\,d\theta$;
$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \int \frac{a\tan\theta\sec\theta\,d\theta}{(a^2\sec^2\theta)\,a\tan\theta} = \int \frac{d\theta}{a^2\sec\theta} = \frac{1}{a^2}\int\cos\theta\,d\theta = \frac{1}{a^2}\sin\theta + C = \frac{1}{a^2}\sqrt{1-\cos^2\theta} + C$
$= \left(\frac{1}{a^2}\right)\frac{\sqrt{\frac{1}{\cos^2\theta} - 1}}{\left(\frac{1}{\cos\theta}\right)} + C = \left(\frac{1}{a^2}\right)\frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} + C = \left(\frac{1}{a^2}\right)\frac{\sqrt{\frac{x^2}{a^2} - 1}}{\left(\frac{x}{a}\right)} + C = \frac{\sqrt{x^2-a^2}}{a^2 x} + C$

97. $\int x^n \sin ax\,dx = -\int x^n\left(\frac{1}{a}\right)d(\cos ax) = (\cos ax)\,x^n\left(-\frac{1}{a}\right) + \frac{1}{a}\int \cos ax \cdot nx^{n-1}\,dx$
$= -\frac{x^n}{a}\cos ax + \frac{n}{a}\int x^{n-1}\cos ax\,dx$
$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = x^n,\ v = -\frac{1}{a}\cos ax\right)$

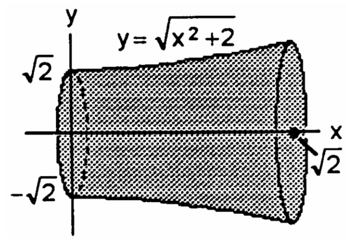
98. $\int x^n(\ln ax)^m\,dx = \int (\ln ax)^m\,d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}(\ln ax)^m}{n+1} - \int\left(\frac{x^{n+1}}{n+1}\right)m(\ln ax)^{m-1}\left(\frac{1}{x}\right)dx$
$= \frac{x^{n+1}(\ln ax)^m}{n+1} - \frac{m}{n+1}\int x^n(\ln ax)^{m-1}\,dx,\ n \ne -1$
$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = (\ln ax)^m,\ v = \frac{x^{n+1}}{n+1}\right)$

99. $\int x^n\sin^{-1}ax\,dx = \int \sin^{-1}ax\,d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1}\sin^{-1}ax - \int\left(\frac{x^{n+1}}{n+1}\right)\frac{a}{\sqrt{1-(ax)^2}}\,dx$
$= \frac{x^{n+1}}{n+1}\sin^{-1}ax - \frac{a}{n+1}\int\frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}},\ n \ne -1$

$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = \sin^{-1} ax,\ v = \frac{x^{n+1}}{n+1}\right)$

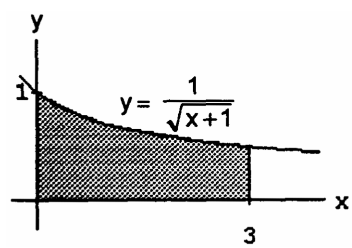
100. $\int x^n \tan^{-1} ax\,dx = \int \tan^{-1} ax\,d\left(\frac{x^{n+1}}{n+1}\right) = \frac{x^{n+1}}{n+1}\tan^{-1} ax - \int \left(\frac{x^{n+1}}{n+1}\right)\frac{a}{1+(ax)^2}\,dx$
$= \frac{x^{n+1}}{n+1}\tan^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{1+a^2x^2},\ n \neq -1$
$\left(\text{We used integration by parts } \int u\,dv = uv - \int v\,du \text{ with } u = \tan^{-1} ax,\ v = \frac{x^{n+1}}{n+1}\right)$

101. $S = \int_0^{\sqrt{2}} 2\pi y\sqrt{1+(y')^2}\,dx$
$= 2\pi \int_0^{\sqrt{2}} \sqrt{x^2+2}\sqrt{1+\frac{x^2}{x^2+2}}\,dx$
$= 2\sqrt{2}\pi \int_0^{\sqrt{2}} \sqrt{x^2+1}\,dx$
$= 2\sqrt{2}\pi \left[\frac{x\sqrt{x^2+1}}{2} + \frac{1}{2}\ln\left|x+\sqrt{x^2+1}\right|\right]_0^{\sqrt{2}}$
(We used FORMULA 21 with $a = 1$)
$= \sqrt{2}\pi\left[\sqrt{6} + \ln\left(\sqrt{2}+\sqrt{3}\right)\right] = 2\pi\sqrt{3} + \pi\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right)$



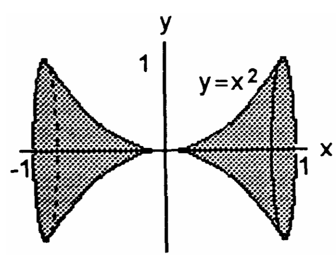

102. $L = \int_0^{\sqrt{3}/2} \sqrt{1+(2x)^2}\,dx = 2\int_0^{\sqrt{3}/2} \sqrt{\frac{1}{4}+x^2}\,dx = 2\left[\frac{x}{2}\sqrt{\frac{1}{4}+x^2} + \left(\frac{1}{4}\right)\left(\frac{1}{2}\right)\ln\left(x+\sqrt{\frac{1}{4}+x^2}\right)\right]_0^{\sqrt{3}/2}$
$\left(\text{We used FORMULA 2 with } a = \frac{1}{2}\right)$
$= \left[\frac{x}{2}\sqrt{1+4x^2} + \frac{1}{4}\ln\left(x+\frac{1}{2}\sqrt{1+4x^2}\right)\right]_0^{\sqrt{3}/2} = \frac{\sqrt{3}}{4}\sqrt{1+4\left(\frac{3}{4}\right)} + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2}+\frac{1}{2}\sqrt{1+4\left(\frac{3}{4}\right)}\right) - \frac{1}{4}\ln\frac{1}{2}$
$= \frac{\sqrt{3}}{4}(2) + \frac{1}{4}\ln\left(\frac{\sqrt{3}}{2}+1\right) + \frac{1}{4}\ln 2 = \frac{\sqrt{3}}{2} + \frac{1}{4}\ln\left(\sqrt{3}+2\right)$

103. $A = \int_0^3 \frac{dx}{\sqrt{x+1}} = \left[2\sqrt{x+1}\right]_0^3 = 2;\ \bar{x} = \frac{1}{A}\int_0^3 \frac{x\,dx}{\sqrt{x+1}}$
$= \frac{1}{A}\int_0^3 \sqrt{x+1}\,dx - \frac{1}{A}\int_0^3 \frac{dx}{\sqrt{x+1}}$
$= \frac{1}{2}\cdot\frac{2}{3}\left[(x+1)^{3/2}\right]_0^3 - 1 = \frac{4}{3};$
(We used FORMULA 11 with $a = 1, b = 1, n = 1$ and $a = 1, b = 1, n = -1$)
$\bar{y} = \frac{1}{2A}\int_0^3 \frac{dx}{x+1} = \frac{1}{4}\left[\ln(x+1)\right]_0^3 = \frac{1}{4}\ln 4 = \frac{1}{2}\ln 2 = \ln\sqrt{2}$

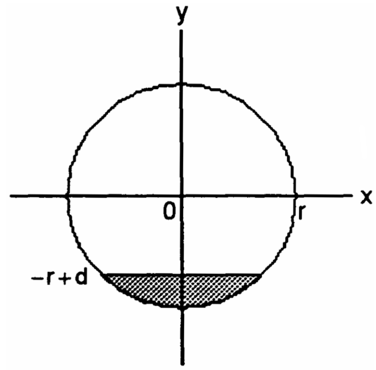



104. $M_y = \int_0^3 x\left(\frac{36}{2x+3}\right)dx = 18\int_0^3 \frac{2x+3}{2x+3}\,dx - 54\int_0^3 \frac{dx}{2x+3} = \left[18x - 27\ln|2x+3|\right]_0^3$
$= 18\cdot 3 - 27\ln 9 - (-27\ln 3) = 54 - 27\cdot 2\ln 3 + 27\ln 3 = 54 - 27\ln 3$

105. $S = 2\pi\int_{-1}^{1} x^2\sqrt{1+4x^2}\,dx;$
$\begin{bmatrix} u = 2x \\ du = 2\,dx \end{bmatrix} \to \frac{\pi}{4}\int_{-2}^{2} u^2\sqrt{1+u^2}\,du$
$= \frac{\pi}{4}\left[\frac{u}{8}(1+2u^2)\sqrt{1+u^2} - \frac{1}{8}\ln\left(u+\sqrt{1+u^2}\right)\right]_{-2}^{2}$
(We used FORMULA 22 with $a = 1$)
$= \frac{\pi}{4}\left[\frac{2}{8}(1+2\cdot 4)\sqrt{1+4} - \frac{1}{8}\ln\left(2+\sqrt{1+4}\right)\right.$
$\left. + \frac{2}{8}(1+2\cdot 4)\sqrt{1+4} + \frac{1}{8}\ln\left(-2+\sqrt{1+4}\right)\right]$
$= \frac{\pi}{4}\left[\frac{9}{2}\sqrt{5} - \frac{1}{8}\ln\left(\frac{2+\sqrt{5}}{-2+\sqrt{5}}\right)\right] \approx 7.62$



106. (a) The volume of the filled part equals the length of the tank times the area of the shaded region shown in the accompanying figure. Consider a layer of gasoline of thickness dy located at height y where $-r < y < -r + d$. The width of this layer is $2\sqrt{r^2 - y^2}$. Therefore, $A = 2\int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy$ and $V = L \cdot A = 2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy$




(b) $2L \int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy = 2L \left[ \frac{y\sqrt{r^2 - y^2}}{2} + \frac{r^2}{2} \sin^{-1} \frac{y}{r} \right]_{-r}^{-r+d}$

(We used FORMULA 29 with $a = r$)

$= 2L \left[ \frac{(d-r)}{2} \sqrt{2rd - d^2} + \frac{r^2}{2} \sin^{-1}\left(\frac{d-r}{r}\right) + \frac{r^2}{2}\left(\frac{\pi}{2}\right) \right] = 2L \left[ \left(\frac{d\text{-}r}{2}\right) \sqrt{2rd - d^2} + \left(\frac{r^2}{2}\right) \left(\sin^{-1}\left(\frac{d-r}{r}\right) + \frac{\pi}{2}\right) \right]$

107. The integrand $f(x) = \sqrt{x - x^2}$ is nonnegative, so the integral is maximized by integrating over the function's entire domain, which runs from $x = 0$ to $x = 1$

$\Rightarrow \int_0^1 \sqrt{x - x^2}\, dx = \int_0^1 \sqrt{2 \cdot \frac{1}{2} x - x^2}\, dx = \left[ \frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{2 \cdot \frac{1}{2} x - x^2} + \frac{\left(\frac{1}{2}\right)^2}{2} \sin^{-1}\left(\frac{x - \frac{1}{2}}{\frac{1}{2}}\right) \right]_0^1$

(We used FORMULA 48 with $a = \frac{1}{2}$)

$= \left[ \frac{\left(x - \frac{1}{2}\right)}{2} \sqrt{x - x^2} + \frac{1}{8} \sin^{-1}(2x - 1) \right]_0^1 = \frac{1}{8} \cdot \frac{\pi}{2} - \frac{1}{8}\left(-\frac{\pi}{2}\right) = \frac{\pi}{8}$

108. The integrand is maximized by integrating $g(x) = x\sqrt{2x - x^2}$ over the largest domain on which g is nonnegative, namely $[0, 2]$

$\Rightarrow \int_0^2 x\sqrt{2x - x^2}\, dx = \left[ \frac{(x+1)(2x-3)\sqrt{2x - x^2}}{6} + \frac{1}{2} \sin^{-1}(x - 1) \right]_0^2$

(We used FORMULA 51 with $a = 1$)

$= \frac{1}{2} \cdot \frac{\pi}{2} - \frac{1}{2}\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$

CAS EXPLORATIONS

109. Example CAS commands:

Maple:

```
q1 := Int( x*ln(x), x );                # (a)
q1 = value( q1 );
q2 := Int( x^2*ln(x), x );              # (b)
q2 = value( q2 );
q3 := Int( x^3*ln(x), x );              # (c)
q3 = value( q3 );
q4 := Int( x^4*ln(x), x );              # (d)
q4 = value( q4 );
q5 := Int( x^n*ln(x), x );              # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

110. Example CAS commands:

Maple:

```
q1 := Int( ln(x)/x, x );                      # (a)
q1 = value( q1 );
q2 := Int( ln(x)/x^2, x );                    # (b)
q2 = value( q2 );
q3 := Int( ln(x)/x^3, x );                    # (c)
q3 = value( q3 );
q4 := Int( ln(x)/x^4, x );                    # (d)
q4 = value( q4 );
q5 := Int( ln(x)/x^n, x );                    # (e)
q6 := value( q5 );
q7 := simplify(q6) assuming n::integer;
q5 = collect( factor(q7), ln(x) );
```

111. Example CAS commands:

Maple:

```
q := Int( sin(x)^n/(sin(x)^n+cos(x)^n), x=0..Pi/2 );     # (a)
q = value( q );
q1 := eval( q, n=1 ):                                     # (b)
q1 = value( q1 );
for N in [1,2,3,5,7] do
  q1 := eval( q, n=N );
  print( q1 = evalf(q1) );
end do:
qq1 := PDEtools[dchange]( x=Pi/2-u, q, [u] );           # (c)
qq2 := subs( u=x, qq1 );
qq3 := q + q = q + qq2;
qq4 := combine( qq3 );
qq5 := value( qq4 );
simplify( qq5/2 );
```

109-111. Example CAS commands:

Mathematica: (functions may vary)

In Mathematica, the natural log is denoted by Log rather than Ln, Log base 10 is Log[x,10]

Mathematica does not include an arbitrary constant when computing an indefinite integral,

```
Clear[x, f, n]
f[x_]:=Log[x] / x^n
Integrate[f[x], x]
```

For exercise 111, Mathematica cannot evaluate the integral with arbitrary n. It does evaluate the integral (value is $\pi/4$ in each case) for small values of n, but for large values of n, it identifies this integral as Indeterminate

109. (e) $\int x^n \ln x\, dx = \frac{x^{n+1}\ln x}{n+1} - \frac{1}{n+1}\int x^n\, dx,\ n \neq -1$

(We used FORMULA 110 with $a = 1$, $m = 1$)

$= \frac{x^{n+1}\ln x}{n+1} - \frac{x^{n+1}}{(n+1)^2} + C = \frac{x^{n+1}}{n+1}\left(\ln x - \frac{1}{n+1}\right) + C$

110. (e) $\int x^{-n} \ln x\, dx = \frac{x^{-n+1}\ln x}{-n+1} - \frac{1}{(-n)+1}\int x^{-n}\, dx,\ n \neq 1$

(We used FORMULA 110 with $a = 1$, $m = 1$, $n = -n$)

$$= \frac{x^{1-n}\ln x}{1-n} - \frac{1}{1-n}\left(\frac{x^{1-n}}{1-n}\right) + C = \frac{x^{1-n}}{1-n}\left(\ln x - \frac{1}{1-n}\right) + C$$

111. (a) Neither MAPLE nor MATHEMATICA can find this integral for arbitrary n.

(b) MAPLE and MATHEMATICA get stuck at about $n = 5$.

(c) Let $x = \frac{\pi}{2} - u \Rightarrow dx = -du$; $x = 0 \Rightarrow u = \frac{\pi}{2}$, $x = \frac{\pi}{2} \Rightarrow u = 0$;

$$I = \int_0^{\pi/2} \frac{\sin^n x\,dx}{\sin^n x+\cos^n x} = \int_{\pi/2}^0 \frac{-\sin^n\left(\frac{\pi}{2}-u\right)du}{\sin^n\left(\frac{\pi}{2}-u\right)+\cos^n\left(\frac{\pi}{2}-u\right)} = \int_0^{\pi/2} \frac{\cos^n u\,du}{\cos^n u+\sin^n u} = \int_0^{\pi/2} \frac{\cos^n x\,dx}{\cos^n x+\sin^n x}$$

$$\Rightarrow I + I = \int_0^{\pi/2} \left(\frac{\sin^n x+\cos^n x}{\sin^n x+\cos^n x}\right) dx = \int_0^{\pi/2} dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

## 8.7 NUMERICAL INTEGRATION

1. $\int_1^2 x\,dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;

$\sum mf(x_i) = 12 \Rightarrow T = \frac{1}{8}(12) = \frac{3}{2}$;

$f(x) = x \Rightarrow f'(x) = 1 \Rightarrow f'' = 0 \Rightarrow M = 0$

$\Rightarrow |E_T| = 0$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 1 | 1 | 1 | 1 |
| $x_1$ | 5/4 | 5/4 | 2 | 5/2 |
| $x_2$ | 3/2 | 3/2 | 2 | 3 |
| $x_3$ | 7/4 | 7/4 | 2 | 7/2 |
| $x_4$ | 2 | 2 | 1 | 2 |

(b) $\int_1^2 x\,dx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2} \Rightarrow |E_T| = \int_1^2 x\,dx - T = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;

$\sum mf(x_i) = 18 \Rightarrow S = \frac{1}{12}(18) = \frac{3}{2}$;

$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 1 | 1 | 1 | 1 |
| $x_1$ | 5/4 | 5/4 | 4 | 5 |
| $x_2$ | 3/2 | 3/2 | 2 | 3 |
| $x_3$ | 7/4 | 7/4 | 4 | 7 |
| $x_4$ | 2 | 2 | 1 | 2 |

(b) $\int_1^2 x\,dx = \frac{3}{2} \Rightarrow |E_S| = \int_1^2 x\,dx - S = \frac{3}{2} - \frac{3}{2} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

2. $\int_1^3 (2x-1)\,dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;

$\sum mf(x_i) = 24 \Rightarrow T = \frac{1}{4}(24) = 6$;

$f(x) = 2x - 1 \Rightarrow f'(x) = 2 \Rightarrow f'' = 0 \Rightarrow M = 0$

$\Rightarrow |E_T| = 0$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 1 | 1 | 1 | 1 |
| $x_1$ | 3/2 | 2 | 2 | 4 |
| $x_2$ | 2 | 3 | 2 | 6 |
| $x_3$ | 5/2 | 4 | 2 | 8 |
| $x_4$ | 3 | 5 | 1 | 5 |

(b) $\int_1^3 (2x-1)\,dx = \left[x^2 - x\right]_1^3 = (9-3) - (1-1) = 6 \Rightarrow |E_T| = \int_1^3 (2x-1)\,dx - T = 6 - 6 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;

$\sum mf(x_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;

$f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 1 | 1 | 1 | 1 |
| $x_1$ | 3/2 | 2 | 4 | 8 |
| $x_2$ | 2 | 3 | 2 | 6 |
| $x_3$ | 5/2 | 4 | 4 | 16 |
| $x_4$ | 3 | 5 | 1 | 5 |

(b) $\int_1^3 (2x-1)\,dx = 6 \Rightarrow |E_S| = \int_1^3 (2x-1)\,dx - S$

$= 6 - 6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

3. $\int_{-1}^1 (x^2+1)\,dx$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;

$\sum mf(x_i) = 11 \Rightarrow T = \frac{1}{4}(11) = 2.75$;

$f(x) = x^2 + 1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2 \Rightarrow M = 2$

$\Rightarrow |E_T| \le \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12}$ or 0.08333

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | −1 | 2 | 1 | 2 |
| $x_1$ | −1/2 | 5/4 | 2 | 5/2 |
| $x_2$ | 0 | 1 | 2 | 2 |
| $x_3$ | 1/2 | 5/4 | 2 | 5/2 |
| $x_4$ | 1 | 2 | 1 | 2 |

(b) $\int_{-1}^{1}(x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_{-1}^{1} = \left(\frac{1}{3}+1\right)-\left(-\frac{1}{3}-1\right)=\frac{8}{3} \Rightarrow E_T = \int_{-1}^{1}(x^2+1)\,dx - T = \frac{8}{3}-\frac{11}{4} = -\frac{1}{12}$
$\Rightarrow |E_T| = \left|-\frac{1}{12}\right| \approx 0.08333$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{8}{3}}\right) \times 100 \approx 3\%$

II. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(x_i) = 16 \Rightarrow S = \frac{1}{6}(16) = \frac{8}{3} = 2.66667$;
$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | $-1$ | 2 | 1 | 2 |
| $x_1$ | $-1/2$ | 5/4 | 4 | 5 |
| $x_2$ | 0 | 1 | 2 | 2 |
| $x_3$ | 1/2 | 5/4 | 4 | 5 |
| $x_4$ | 1 | 2 | 1 | 2 |

(b) $\int_{-1}^{1}(x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_{-1}^{1} = \frac{8}{3}$
$\Rightarrow |E_S| = \int_{-1}^{1}(x^2+1)\,dx - S = \frac{8}{3}-\frac{8}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

4. $\int_{-2}^{0}(x^2-1)\,dx$

I. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$
$\sum mf(x_i) = 3 \Rightarrow T = \frac{1}{4}(3) = \frac{3}{4}$;
$f(x) = x^2-1 \Rightarrow f'(x) = 2x \Rightarrow f''(x) = 2$
$\Rightarrow M = 2 \Rightarrow |E_T| \le \frac{0-(-2)}{12}\left(\frac{1}{2}\right)^2(2) = \frac{1}{12} = 0.08333$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | $-2$ | 3 | 1 | 3 |
| $x_1$ | $-3/2$ | 5/4 | 2 | 5/2 |
| $x_2$ | $-1$ | 0 | 2 | 0 |
| $x_3$ | $-1/2$ | $-3/4$ | 2 | $-3/2$ |
| $x_4$ | 0 | $-1$ | 1 | $-1$ |

(b) $\int_{-2}^{0}(x^2-1)\,dx = \left[\frac{x^3}{3}-x\right]_{-2}^{0} = 0-\left(-\frac{8}{3}+2\right) = \frac{2}{3} \Rightarrow E_T = \int_{-2}^{0}(x^2-1)\,dx - T = \frac{2}{3}-\frac{3}{4} = -\frac{1}{12}$
$\Rightarrow |E_T| = \frac{1}{12}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \left(\frac{\frac{1}{12}}{\frac{2}{3}}\right) \times 100 \approx 13\%$

II. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{0-(-2)}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$; $\sum mf(x_i) = 4 \Rightarrow S = \frac{1}{6}(4) = \frac{2}{3}$;
$f^{(3)}(x) = 0 \Rightarrow f^{(4)}(x) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | $-2$ | 3 | 1 | 3 |
| $x_1$ | $-3/2$ | 5/4 | 4 | 5 |
| $x_2$ | $-1$ | 0 | 2 | 0 |
| $x_3$ | $-1/2$ | $-3/4$ | 4 | $-3$ |
| $x_4$ | 0 | $-1$ | 1 | $-1$ |

(b) $\int_{-2}^{0}(x^2-1)\,dx = \frac{2}{3} \Rightarrow |E_S| = \int_{-2}^{0}(x^2-1)\,dx - S$
$= \frac{2}{3}-\frac{2}{3} = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

5. $\int_{0}^{2}(t^3+t)\,dt$

I. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(t_i) = 25 \Rightarrow T = \frac{1}{4}(25) = \frac{25}{4}$;
$f(t) = t^3+t \Rightarrow f'(t) = 3t^2+1 \Rightarrow f''(t) = 6t$
$\Rightarrow M = 12 = f''(2) \Rightarrow |E_T| \le \frac{2-0}{12}\left(\frac{1}{2}\right)^2(12) = \frac{1}{2}$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | 0 | 0 | 1 | 0 |
| $t_1$ | 1/2 | 5/8 | 2 | 5/4 |
| $t_2$ | 1 | 2 | 2 | 4 |
| $t_3$ | 3/2 | 39/8 | 2 | 39/4 |
| $t_4$ | 2 | 10 | 1 | 10 |

(b) $\int_{0}^{2}(t^3+t)\,dt = \left[\frac{t^4}{4}+\frac{t^2}{2}\right]_{0}^{2} = \left(\frac{2^4}{4}+\frac{2^2}{2}\right)-0 = 6 \Rightarrow |E_T| = \int_{0}^{2}(t^3+t)\,dt - T = 6-\frac{25}{4} = -\frac{1}{4} \Rightarrow |E_T| = \frac{1}{4}$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{\left|-\frac{1}{4}\right|}{6} \times 100 \approx 4\%$

II. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(t_i) = 36 \Rightarrow S = \frac{1}{6}(36) = 6$;
$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | 0 | 0 | 1 | 0 |
| $t_1$ | 1/2 | 5/8 | 4 | 5/2 |
| $t_2$ | 1 | 2 | 2 | 4 |
| $t_3$ | 3/2 | 39/8 | 4 | 39/2 |
| $t_4$ | 2 | 10 | 1 | 10 |

(b) $\int_{0}^{2}(t^3+t)\,dt = 6 \Rightarrow |E_S| = \int_{0}^{2}(t^3+t)\,dt - S$
$= 6-6 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

6. $\int_{-1}^{1} (t^3+1)\,dt$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | $-1$ | 0 | 1 | 0 |
| $t_1$ | $-1/2$ | 7/8 | 2 | 7/4 |
| $t_2$ | 0 | 1 | 2 | 2 |
| $t_3$ | 1/2 | 9/8 | 2 | 9/4 |
| $t_4$ | 1 | 2 | 1 | 2 |

I. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$; $\sum mf(t_i) = 8 \Rightarrow T = \frac{1}{4}(8) = 2$;
$f(t) = t^3 + 1 \Rightarrow f'(t) = 3t^2 \Rightarrow f''(t) = 6t$
$\Rightarrow M = 6 = f''(1) \Rightarrow |E_T| \le \frac{1-(-1)}{12}\left(\frac{1}{2}\right)^2(6) = \frac{1}{4}$

(b) $\int_{-1}^{1}(t^3+1)\,dt = \left[\frac{t^4}{4}+t\right]_{-1}^{1} = \left(\frac{1^4}{4}+1\right) - \left(\frac{(-1)^4}{4}+(-1)\right) = 2 \Rightarrow |E_T| = \int_{-1}^{1}(t^3+1)\,dt - T = 2-2 = 0$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = 0\%$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | $-1$ | 0 | 1 | 0 |
| $t_1$ | $-1/2$ | 7/8 | 4 | 7/2 |
| $t_2$ | 0 | 1 | 2 | 2 |
| $t_3$ | 1/2 | 9/8 | 4 | 9/2 |
| $t_4$ | 1 | 2 | 1 | 2 |

II. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = \frac{2}{4} = \frac{1}{2}$
$\Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$; $\sum mf(t_i) = 12 \Rightarrow S = \frac{1}{6}(12) = 2$;
$f^{(3)}(t) = 6 \Rightarrow f^{(4)}(t) = 0 \Rightarrow M = 0 \Rightarrow |E_S| = 0$

(b) $\int_{-1}^{1}(t^3+1)\,dt = 2 \Rightarrow |E_S| = \int_{-1}^{1}(t^3+1)\,dt - S$
$= 2-2 = 0$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = 0\%$

7. $\int_1^2 \frac{1}{s^2}\,ds$

| | $s_i$ | $f(s_i)$ | m | $mf(s_i)$ |
|---|---|---|---|---|
| $s_0$ | 1 | 1 | 1 | 1 |
| $s_1$ | 5/4 | 16/25 | 2 | 32/25 |
| $s_2$ | 3/2 | 4/9 | 2 | 8/9 |
| $s_3$ | 7/4 | 16/49 | 2 | 32/49 |
| $s_4$ | 2 | 1/4 | 1 | 1/4 |

I. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
$\sum mf(s_i) = \frac{179{,}573}{44{,}100} \Rightarrow T = \frac{1}{8}\left(\frac{179{,}573}{44{,}100}\right) = \frac{179{,}573}{352{,}800}$
$\approx 0.50899$; $f(s) = \frac{1}{s^2} \Rightarrow f'(s) = -\frac{2}{s^3}$
$\Rightarrow f''(s) = \frac{6}{s^4} \Rightarrow M = 6 = f''(1)$
$\Rightarrow |E_T| \le \frac{2-1}{12}\left(\frac{1}{4}\right)^2(6) = \frac{1}{32} = 0.03125$

(b) $\int_1^2 \frac{1}{s^2}\,ds = \int_1^2 s^{-2}\,ds = \left[-\frac{1}{s}\right]_1^2 = -\frac{1}{2} - \left(-\frac{1}{1}\right) = \frac{1}{2} \Rightarrow E_T = \int_1^2 \frac{1}{s^2}\,ds - T = \frac{1}{2} - 0.50899 = -0.00899$
$\Rightarrow |E_T| = 0.00899$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.00899}{0.5} \times 100 \approx 2\%$

| | $s_i$ | $f(s_i)$ | m | $mf(s_i)$ |
|---|---|---|---|---|
| $s_0$ | 1 | 1 | 1 | 1 |
| $s_1$ | 5/4 | 16/25 | 4 | 64/25 |
| $s_2$ | 3/2 | 4/9 | 2 | 8/9 |
| $s_3$ | 7/4 | 16/49 | 4 | 64/49 |
| $s_4$ | 2 | 1/4 | 1 | 1/4 |

II. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{2-1}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
$\sum mf(s_i) = \frac{264{,}821}{44{,}100} \Rightarrow S = \frac{1}{12}\left(\frac{264{,}821}{44{,}100}\right) = \frac{264{,}821}{529{,}200}$
$\approx 0.50042$; $f^{(3)}(s) = -\frac{24}{s^5} \Rightarrow f^{(4)}(s) = \frac{120}{s^6}$
$\Rightarrow M = 120 \Rightarrow |E_S| \le \left|\frac{2-1}{180}\right|\left(\frac{1}{4}\right)^4(120)$
$= \frac{1}{384} \approx 0.00260$

(b) $\int_1^2 \frac{1}{s^2}\,ds = \frac{1}{2} \Rightarrow E_S = \int_1^2 \frac{1}{s^2}\,ds - S = \frac{1}{2} - 0.50042 = -0.00042 \Rightarrow |E_S| = 0.00042$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.0004}{0.5} \times 100 \approx 0.08\%$

8. $\int_2^4 \frac{1}{(s-1)^2}\,ds$

| | $s_i$ | $f(s_i)$ | m | $mf(s_i)$ |
|---|---|---|---|---|
| $s_0$ | 2 | 1 | 1 | 1 |
| $s_1$ | 5/2 | 4/9 | 2 | 8/9 |
| $s_2$ | 3 | 1/4 | 2 | 1/2 |
| $s_3$ | 7/2 | 4/25 | 2 | 8/25 |
| $s_4$ | 4 | 1/9 | 1 | 1/9 |

I. (a) For $n=4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{2} = \frac{1}{4}$;
$\sum mf(s_i) = \frac{1269}{450}$
$\Rightarrow T = \frac{1}{4}\left(\frac{1269}{450}\right) = \frac{1269}{1800} = 0.70500$;
$f(s) = (s-1)^{-2} \Rightarrow f'(s) = -\frac{2}{(s-1)^3}$
$\Rightarrow f''(s) = \frac{6}{(s-1)^4} \Rightarrow M = 6$
$\Rightarrow |E_T| \le \frac{4-2}{12}\left(\frac{1}{2}\right)^2(6) = \frac{1}{4} = 0.25$

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \left[\frac{-1}{(s-1)}\right]_2^4 = \left(\frac{-1}{4-1}\right) - \left(\frac{-1}{2-1}\right) = \frac{2}{3} \Rightarrow E_T = \int_2^4 \frac{1}{(s-1)^2}\,ds - T = \frac{2}{3} - 0.705 \approx -0.03833$
$\Rightarrow |E_T| \approx 0.03833$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03833}{\left(\frac{2}{3}\right)} \times 100 \approx 6\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{4-2}{4} = \frac{1}{2} \Rightarrow \frac{\Delta x}{3} = \frac{1}{6}$;
$\sum mf(s_i) = \frac{1813}{450}$
$\Rightarrow S = \frac{1}{6}\left(\frac{1813}{450}\right) = \frac{1813}{2700} \approx 0.67148$;
$f^{(3)}(s) = \frac{-24}{(s-1)^5} \Rightarrow f^{(4)}(s) = \frac{120}{(s-1)^6} \Rightarrow M = 120$
$\Rightarrow |E_S| \le \frac{4-2}{180}\left(\frac{1}{2}\right)^4(120) = \frac{1}{12} \approx 0.08333$

| | $s_i$ | $f(s_i)$ | m | $mf(s_i)$ |
|---|---|---|---|---|
| $s_0$ | 2 | 1 | 1 | 1 |
| $s_1$ | 5/2 | 4/9 | 4 | 16/9 |
| $s_2$ | 3 | 1/4 | 2 | 1/2 |
| $s_3$ | 7/2 | 4/25 | 4 | 16/25 |
| $s_4$ | 4 | 1/9 | 1 | 1/9 |

(b) $\int_2^4 \frac{1}{(s-1)^2}\,ds = \frac{2}{3} \Rightarrow E_S = \int_2^4 \frac{1}{(s-1)^2}\,ds - S \approx \frac{2}{3} - 0.67148 = -0.00481 \Rightarrow |E_S| \approx 0.00481$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00481}{\left(\frac{2}{3}\right)} \times 100 \approx 1\%$

9. $\int_0^{\pi} \sin t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.8284$
$\Rightarrow T = \frac{\pi}{8}\left(2 + 2\sqrt{2}\right) \approx 1.89612$;
$f(t) = \sin t \Rightarrow f'(t) = \cos t \Rightarrow f''(t) = -\sin t$
$\Rightarrow M = 1 \Rightarrow |E_T| \le \frac{\pi-0}{12}\left(\frac{\pi}{4}\right)^2(1) = \frac{\pi^3}{192}$
$\approx 0.16149$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | 0 | 0 | 1 | 0 |
| $t_1$ | $\pi/4$ | $\sqrt{2}/2$ | 2 | $\sqrt{2}$ |
| $t_2$ | $\pi/2$ | 1 | 2 | 2 |
| $t_3$ | $3\pi/4$ | $\sqrt{2}/2$ | 2 | $\sqrt{2}$ |
| $t_4$ | $\pi$ | 0 | 1 | 0 |

(b) $\int_0^{\pi} \sin t\,dt = [-\cos t]_0^{\pi} = (-\cos \pi) - (-\cos 0) = 2 \Rightarrow |E_T| = \int_0^{\pi} \sin t\,dt - T \approx 2 - 1.89612 = 0.10388$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.10388}{2} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{\pi-0}{4} = \frac{\pi}{4} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{12}$;
$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.6569$
$\Rightarrow S = \frac{\pi}{12}\left(2 + 4\sqrt{2}\right) \approx 2.00456$;
$f^{(3)}(t) = -\cos t \Rightarrow f^{(4)}(t) = \sin t$
$\Rightarrow M = 1 \Rightarrow |E_S| \le \frac{\pi-0}{180}\left(\frac{\pi}{4}\right)^4(1) \approx 0.00664$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | 0 | 0 | 1 | 0 |
| $t_1$ | $\pi/4$ | $\sqrt{2}/2$ | 4 | $2\sqrt{2}$ |
| $t_2$ | $\pi/2$ | 1 | 2 | 2 |
| $t_3$ | $3\pi/4$ | $\sqrt{2}/2$ | 4 | $2\sqrt{2}$ |
| $t_4$ | $\pi$ | 0 | 1 | 0 |

(b) $\int_0^{\pi} \sin t\,dt = 2 \Rightarrow E_S = \int_0^{\pi} \sin t\,dt - S \approx 2 - 2.00456 = -0.00456 \Rightarrow |E_S| \approx 0.00456$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00456}{2} \times 100 \approx 0\%$

10. $\int_0^{1} \sin \pi t\,dt$

I. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{2} = \frac{1}{8}$;
$\sum mf(t_i) = 2 + 2\sqrt{2} \approx 4.828$
$\Rightarrow T = \frac{1}{8}\left(2 + 2\sqrt{2}\right) \approx 0.60355$; $f(t) = \sin \pi t$
$\Rightarrow f'(t) = \pi \cos \pi t$
$\Rightarrow f''(t) = -\pi^2 \sin \pi t \Rightarrow M = \pi^2$
$\Rightarrow |E_T| \le \frac{1-0}{12}\left(\frac{1}{4}\right)^2(\pi^2) \approx 0.05140$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | 0 | 0 | 1 | 0 |
| $t_1$ | 1/4 | $\sqrt{2}/2$ | 2 | $\sqrt{2}$ |
| $t_2$ | 1/2 | 1 | 2 | 2 |
| $t_3$ | 3/4 | $\sqrt{2}/2$ | 2 | $\sqrt{2}$ |
| $t_4$ | 1 | 0 | 1 | 0 |

(b) $\int_0^1 \sin \pi t\,dt = \left[-\frac{1}{\pi}\cos \pi t\right]_0^1 = \left(-\frac{1}{\pi}\cos \pi\right) - \left(-\frac{1}{\pi}\cos 0\right) = \frac{2}{\pi} \approx 0.63662 \Rightarrow |E_T| = \int_0^1 \sin \pi t\,dt - T$
$\approx \frac{2}{\pi} - 0.60355 = 0.03307$

(c) $\frac{|E_T|}{\text{True Value}} \times 100 = \frac{0.03307}{\left(\frac{2}{\pi}\right)} \times 100 \approx 5\%$

II. (a) For $n = 4$, $\Delta x = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} \Rightarrow \frac{\Delta x}{3} = \frac{1}{12}$;
$\sum mf(t_i) = 2 + 4\sqrt{2} \approx 7.65685$
$\Rightarrow S = \frac{1}{12}\left(2 + 4\sqrt{2}\right) \approx 0.63807$;
$f^{(3)}(t) = -\pi^3 \cos \pi t \Rightarrow f^{(4)}(t) = \pi^4 \sin \pi t$
$\Rightarrow M = \pi^4 \Rightarrow |E_S| \le \frac{1-0}{180}\left(\frac{1}{4}\right)^4(\pi^4) \approx 0.00211$

| | $t_i$ | $f(t_i)$ | m | $mf(t_i)$ |
|---|---|---|---|---|
| $t_0$ | 0 | 0 | 1 | 0 |
| $t_1$ | 1/4 | $\sqrt{2}/2$ | 4 | $2\sqrt{2}$ |
| $t_2$ | 1/2 | 1 | 2 | 2 |
| $t_3$ | 3/4 | $\sqrt{2}/2$ | 4 | $2\sqrt{2}$ |
| $t_4$ | 1 | 0 | 1 | 0 |

(b) $\int_0^1 \sin \pi t \, dt = \frac{2}{\pi} \approx 0.63662 \Rightarrow E_S = \int_0^1 \sin \pi t \, dt - S \approx \frac{2}{\pi} - 0.63807 = -0.00145 \Rightarrow |E_S| \approx 0.00145$

(c) $\frac{|E_S|}{\text{True Value}} \times 100 = \frac{0.00145}{\left(\frac{2}{\pi}\right)} \times 100 \approx 0\%$

11. (a) $n = 8 \Rightarrow \Delta x = \frac{1}{8} \Rightarrow \frac{\Delta x}{2} = \frac{1}{16}$;
$\sum mf(x_i) = 1(0.0) + 2(0.12402) + 2(0.24206) + 2(0.34763) + 2(0.43301) + 2(0.48789) + 2(0.49608)$
$+ 2(0.42361) + 1(0) = 5.1086 \Rightarrow T = \frac{1}{16}(5.1086) = 0.31929$

(b) $n = 8 \Rightarrow \Delta x = \frac{1}{8} \Rightarrow \frac{\Delta x}{3} = \frac{1}{24}$;
$\sum mf(x_i) = 1(0.0) + 4(0.12402) + 2(0.24206) + 4(0.34763) + 2(0.43301) + 4(0.48789) + 2(0.49608)$
$+ 4(0.42361) + 1(0) = 7.8749 \Rightarrow S = \frac{1}{24}(7.8749) = 0.32812$

(c) Let $u = 1 - x^2 \Rightarrow du = -2x \, dx \Rightarrow -\frac{1}{2} du = x \, dx$; $x = 0 \Rightarrow u = 1$, $x = 1 \Rightarrow u = 0$
$\int_0^1 x\sqrt{1-x^2} \, dx = \int_1^0 \sqrt{u}\left(-\frac{1}{2} du\right) = \frac{1}{2}\int_0^1 u^{1/2} \, du = \left[\frac{1}{2}\left(\frac{u^{3/2}}{\frac{3}{2}}\right)\right]_0^1 = \left[\frac{1}{3}u^{3/2}\right]_0^1 = \frac{1}{3}\left(\sqrt{1}\right)^3 - \frac{1}{3}\left(\sqrt{0}\right)^3 = \frac{1}{3}$;
$E_T = \int_0^1 x\sqrt{1-x^2} \, dx - T \approx \frac{1}{3} - 0.31929 = 0.01404$; $E_S = \int_0^1 x\sqrt{1-x^2} \, dx - S \approx \frac{1}{3} - 0.32812 = 0.00521$

12. (a) $n = 8 \Rightarrow \Delta x = \frac{3}{8} \Rightarrow \frac{\Delta x}{2} = \frac{3}{16}$;
$\sum mf(\theta_i) = 1(0) + 2(0.09334) + 2(0.18429) + 2(0.27075) + 2(0.35112) + 2(0.42443) + 2(0.49026)$
$+ 2(0.58466) + 1(0.6) = 5.3977 \Rightarrow T = \frac{3}{16}(5.3977) = 1.01207$

(b) $n = 8 \Rightarrow \Delta x = \frac{3}{8} \Rightarrow \frac{\Delta x}{3} = \frac{1}{8}$;
$\sum mf(\theta_i) = 1(0) + 4(0.09334) + 2(0.18429) + 4(0.27075) + 2(0.35112) + 4(0.42443) + 2(0.49026)$
$+ 4(0.58466) + 1(0.6) = 8.14406 \Rightarrow S = \frac{1}{8}(8.14406) = 1.01801$

(c) Let $u = 16 + \theta^2 \Rightarrow du = 2\theta \, d\theta \Rightarrow \frac{1}{2} du = \theta \, d\theta$; $\theta = 0 \Rightarrow u = 16$, $\theta = 3 \Rightarrow u = 16 + 3^2 = 25$
$\int_0^3 \frac{\theta}{\sqrt{16+\theta^2}} \, d\theta = \int_{16}^{25} \frac{1}{\sqrt{u}}\left(\frac{1}{2} du\right) = \frac{1}{2}\int_{16}^{25} u^{-1/2} \, du = \left[\frac{1}{2}\left(\frac{u^{1/2}}{\frac{1}{2}}\right)\right]_{16}^{25} = \sqrt{25} - \sqrt{16} = 1$;
$E_T = \int_0^3 \frac{\theta}{\sqrt{16+\theta^2}} \, d\theta - T \approx 1 - 1.01207 = -0.01207$; $E_S = \int_0^3 \frac{\theta}{\sqrt{16+\theta^2}} \, d\theta - S \approx 1 - 1.01801 = -0.01801$

13. (a) $n = 8 \Rightarrow \Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{16}$;
$\sum mf(t_i) = 1(0.0) + 2(0.99138) + 2(1.26906) + 2(1.05961) + 2(0.75) + 2(0.48821) + 2(0.28946) + 2(0.13429)$
$+ 1(0) = 9.96402 \Rightarrow T = \frac{\pi}{16}(9.96402) \approx 1.95643$

(b) $n = 8 \Rightarrow \Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$;
$\sum mf(t_i) = 1(0.0) + 4(0.99138) + 2(1.26906) + 4(1.05961) + 2(0.75) + 4(0.48821) + 2(0.28946) + 4(0.13429)$
$+ 1(0) = 15.311 \Rightarrow S \approx \frac{\pi}{24}(15.311) \approx 2.00421$

(c) Let $u = 2 + \sin t \Rightarrow du = \cos t \, dt$; $t = -\frac{\pi}{2} \Rightarrow u = 2 + \sin\left(-\frac{\pi}{2}\right) = 1$, $t = \frac{\pi}{2} \Rightarrow u = 2 + \sin\frac{\pi}{2} = 3$
$\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2} \, dt = \int_1^3 \frac{3}{u^2} \, du = 3\int_1^3 u^{-2} \, du = \left[3\left(\frac{u^{-1}}{-1}\right)\right]_1^3 = 3\left(-\frac{1}{3}\right) - 3\left(-\frac{1}{1}\right) = 2$;
$E_T = \int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2} \, dt - T \approx 2 - 1.95643 = 0.04357$; $E_S = \int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2+\sin t)^2} \, dt - S$
$\approx 2 - 2.00421 = -0.00421$

14. (a) $n = 8 \Rightarrow \Delta x = \frac{\pi}{32} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{64}$;
$\sum mf(y_i) = 1(2.0) + 2(1.51606) + 2(1.18237) + 2(0.93998) + 2(0.75402) + 2(0.60145) + 2(0.46364)$
$+ 2(0.31688) + 1(0) = 13.5488 \Rightarrow T \approx \frac{\pi}{64}(13.5488) = 0.66508$

(b) $n = 8 \Rightarrow \Delta x = \frac{\pi}{32} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{96}$;
$\sum mf(y_i) = 1(2.0) + 4(1.51606) + 2(1.18237) + 4(0.93988) + 2(0.75402) + 4(0.60145) + 2(0.46364)$
$+ 4(0.31688) + 1(0) = 20.29734 \Rightarrow S \approx \frac{\pi}{96}(20.29734) = 0.66423$

(c) Let $u = \cot y \Rightarrow du = -\csc^2 y\, dy$; $y = \frac{\pi}{4} \Rightarrow u = 1$, $y = \frac{\pi}{2} \Rightarrow u = 0$
$\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\, dy = \int_1^0 \sqrt{u}\,(-\,du) = \int_0^1 u^{1/2}\, du = \left[\frac{u^{3/2}}{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left(\sqrt{1}\right)^3 - \frac{2}{3}\left(\sqrt{0}\right)^3 = \frac{2}{3}$;
$E_T = \int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\, dy - T \approx \frac{2}{3} - 0.66508 = 0.00159$; $E_S = \int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\, dy - S$
$\approx \frac{2}{3} - 0.66423 = 0.00244$

15. (a) $M = 0$ (see Exercise 1): Then $n = 1 \Rightarrow \Delta x = 1 \Rightarrow |E_T| = \frac{1}{12}(1)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 1): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = \frac{1}{2} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{2}\right)^4(0) = 0 < 10^{-4}$

16. (a) $M = 0$ (see Exercise 2): Then $n = 1 \Rightarrow \Delta x = 2 \Rightarrow |E_T| = \frac{2}{12}(2)^2(0) = 0 < 10^{-4}$

(b) $M = 0$ (see Exercise 2): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

17. (a) $M = 2$ (see Exercise 3): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$
$\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 3): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

18. (a) $M = 2$ (see Exercise 4): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(2) = \frac{4}{3n^2} < 10^{-4} \Rightarrow n^2 > \frac{4}{3}(10^4) \Rightarrow n > \sqrt{\frac{4}{3}(10^4)}$
$\Rightarrow n > 115.4$, so let $n = 116$

(b) $M = 0$ (see Exercise 4): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

19. (a) $M = 12$ (see Exercise 5): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(12) = \frac{8}{n^2} < 10^{-4} \Rightarrow n^2 > 8(10^4) \Rightarrow n > \sqrt{8(10^4)}$
$\Rightarrow n > 282.8$, so let $n = 283$

(b) $M = 0$ (see Exercise 5): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

20. (a) $M = 6$ (see Exercise 6): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$
$= 200$, so let $n = 201$

(b) $M = 0$ (see Exercise 6): Then $n = 2$ (n must be even) $\Rightarrow \Delta x = 1 \Rightarrow |E_S| = \frac{2}{180}(1)^4(0) = 0 < 10^{-4}$

21. (a) $M = 6$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_T| \le \frac{1}{12}\left(\frac{1}{n}\right)^2(6) = \frac{1}{2n^2} < 10^{-4} \Rightarrow n^2 > \frac{1}{2}(10^4) \Rightarrow n > \sqrt{\frac{1}{2}(10^4)}$
$\Rightarrow n > 70.7$, so let $n = 71$

(b) $M = 120$ (see Exercise 7): Then $\Delta x = \frac{1}{n} \Rightarrow |E_S| = \frac{1}{180}\left(\frac{1}{n}\right)^4(120) = \frac{2}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{2}{3}(10^4)$
$\Rightarrow n > \sqrt[4]{\frac{2}{3}(10^4)} \Rightarrow n > 9.04$, so let $n = 10$ (n must be even)

22. (a) $M = 6$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(6) = \frac{4}{n^2} < 10^{-4} \Rightarrow n^2 > 4(10^4) \Rightarrow n > \sqrt{4(10^4)}$
$\Rightarrow n > 200$, so let $n = 201$

(b) $M = 120$ (see Exercise 8): Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(120) = \frac{64}{3n^4} < 10^{-4} \Rightarrow n^4 > \frac{64}{3}(10^4)$
$\Rightarrow n > \sqrt[4]{\frac{64}{3}(10^4)} \Rightarrow n > 21.5$, so let $n = 22$ (n must be even)

23. (a) $f(x) = \sqrt{x+1} \Rightarrow f'(x) = \frac{1}{2}(x+1)^{-1/2} \Rightarrow f''(x) = -\frac{1}{4}(x+1)^{-3/2} = -\frac{1}{4\left(\sqrt{x+1}\right)^3} \Rightarrow M = \frac{1}{4\left(\sqrt{1}\right)^3} = \frac{1}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{1}{4}\right) = \frac{9}{16n^2} < 10^{-4} \Rightarrow n^2 > \frac{9}{16}\left(10^4\right) \Rightarrow n > \sqrt{\frac{9}{16}\left(10^4\right)} \Rightarrow n > 75$,

so let $n = 76$

(b) $f^{(3)}(x) = \frac{3}{8}(x+1)^{-5/2} \Rightarrow f^{(4)}(x) = -\frac{15}{16}(x+1)^{-7/2} = -\frac{15}{16\left(\sqrt{x+1}\right)^7} \Rightarrow M = \frac{15}{16\left(\sqrt{1}\right)^7} = \frac{15}{16}$. Then $\Delta x = \frac{3}{n}$

$\Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{15}{16}\right) = \frac{3^5(15)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(15)\left(10^4\right)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(15)\left(10^4\right)}{16(180)}} \Rightarrow n > 10.6$, so let

$n = 12$ (n must be even)

24. (a) $f(x) = \frac{1}{\sqrt{x+1}} \Rightarrow f'(x) = -\frac{1}{2}(x+1)^{-3/2} \Rightarrow f''(x) = \frac{3}{4}(x+1)^{-5/2} = \frac{3}{4\left(\sqrt{x+1}\right)^5} \Rightarrow M = \frac{3}{4\left(\sqrt{1}\right)^5} = \frac{3}{4}$.

Then $\Delta x = \frac{3}{n} \Rightarrow |E_T| \le \frac{3}{12}\left(\frac{3}{n}\right)^2\left(\frac{3}{4}\right) = \frac{3^4}{48n^2} < 10^{-4} \Rightarrow n^2 > \frac{3^4\left(10^4\right)}{48} \Rightarrow n > \sqrt{\frac{3^4\left(10^4\right)}{48}} \Rightarrow n > 129.9$, so let

$n = 130$

(b) $f^{(3)}(x) = -\frac{15}{8}(x+1)^{-7/2} \Rightarrow f^{(4)}(x) = \frac{105}{16}(x+1)^{-9/2} = \frac{105}{16\left(\sqrt{x+1}\right)^9} \Rightarrow M = \frac{105}{16\left(\sqrt{1}\right)^9} = \frac{105}{16}$. Then $\Delta x = \frac{3}{n}$

$\Rightarrow |E_S| \le \frac{3}{180}\left(\frac{3}{n}\right)^4\left(\frac{105}{16}\right) = \frac{3^5(105)}{16(180)n^4} < 10^{-4} \Rightarrow n^4 > \frac{3^5(105)\left(10^4\right)}{16(180)} \Rightarrow n > \sqrt[4]{\frac{3^5(105)\left(10^4\right)}{16(180)}} \Rightarrow n > 17.25$, so

let $n = 18$ (n must be even)

25. (a) $f(x) = \sin(x+1) \Rightarrow f'(x) = \cos(x+1) \Rightarrow f''(x) = -\sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(1)$

$= \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8\left(10^4\right)}{12} \Rightarrow n > \sqrt{\frac{8\left(10^4\right)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = -\cos(x+1) \Rightarrow f^{(4)}(x) = \sin(x+1) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(1) = \frac{32}{180n^4} < 10^{-4}$

$\Rightarrow n^4 > \frac{32\left(10^4\right)}{180} \Rightarrow n > \sqrt[4]{\frac{32\left(10^4\right)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

26. (a) $f(x) = \cos(x+\pi) \Rightarrow f'(x) = -\sin(x+\pi) \Rightarrow f''(x) = -\cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n}$

$\Rightarrow |E_T| \le \frac{2}{12}\left(\frac{2}{n}\right)^2(1) = \frac{8}{12n^2} < 10^{-4} \Rightarrow n^2 > \frac{8\left(10^4\right)}{12} \Rightarrow n > \sqrt{\frac{8\left(10^4\right)}{12}} \Rightarrow n > 81.6$, so let $n = 82$

(b) $f^{(3)}(x) = \sin(x+\pi) \Rightarrow f^{(4)}(x) = \cos(x+\pi) \Rightarrow M = 1$. Then $\Delta x = \frac{2}{n} \Rightarrow |E_S| \le \frac{2}{180}\left(\frac{2}{n}\right)^4(1) = \frac{32}{180n^4} < 10^{-4}$

$\Rightarrow n^4 > \frac{32\left(10^4\right)}{180} \Rightarrow n > \sqrt[4]{\frac{32\left(10^4\right)}{180}} \Rightarrow n > 6.49$, so let $n = 8$ (n must be even)

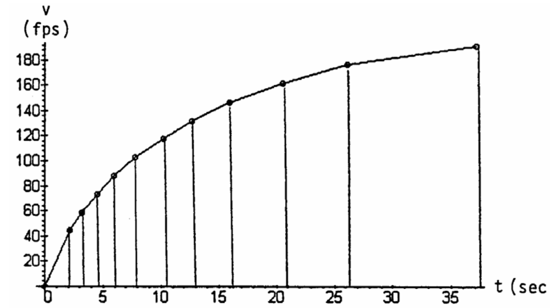
27. $\frac{5}{2}(6.0 + 2(8.2) + 2(9.1) \ldots + 2(12.7) + 13.0)(30) = 15{,}990 \text{ ft}^3$.

28. (a) Using Trapezoid Rule, $\Delta x = 200 \Rightarrow \frac{\Delta x}{2} = \frac{200}{2} = 100$; $\sum mf(x_i) = 13{,}180 \Rightarrow$ Area $\approx 100\,(13{,}180)$ $= 1{,}318{,}000 \text{ ft}^2$. Since the average depth $= 20$ ft we obtain Volume $\approx 20\,(\text{Area}) \approx 26{,}360{,}000 \text{ ft}^3$.

(b) Now, Number of fish $= \frac{\text{Volume}}{1000} = 26{,}360$ (to the nearest fish) $\Rightarrow$ Maximum to be caught $= 75\%$ of $26{,}360$ $= 19{,}770 \Rightarrow$ Number of licenses $= \frac{19{,}770}{20} = 988$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 0 | 0 | 1 | 0 |
| $x_1$ | 200 | 520 | 2 | 1040 |
| $x_2$ | 400 | 800 | 2 | 1600 |
| $x_3$ | 600 | 1000 | 2 | 2000 |
| $x_4$ | 800 | 1140 | 2 | 2280 |
| $x_5$ | 1000 | 1160 | 2 | 2320 |
| $x_6$ | 1200 | 1110 | 2 | 2220 |
| $x_7$ | 1400 | 860 | 2 | 1720 |
| $x_8$ | 1600 | 0 | 1 | 0 |

29. Use the conversion 30 mph = 44 fps (ft per sec) since time is measured in seconds. The distance traveled as the car accelerates from, say, 40 mph = 58.67 fps to 50 mph = 73.33 fps in $(4.5 - 3.2) = 1.3$ sec is the area of the trapezoid (see figure) associated with that time interval: $\frac{1}{2}(58.67 + 73.33)(1.3) = 85.8$ ft. The total distance traveled by the Ford Mustang Cobra is the sum of all these eleven trapezoids (using $\frac{\Delta t}{2}$ and the table below):




$s = (44)(1.1) + (102.67)(0.5) + (132)(0.65) + (161.33)(0.7) + (190.67)(0.95) + (220)(1.2) + (249.33)(1.25)$
$+ (278.67)(1.65) + (308)(2.3) + (337.33)(2.8) + (366.67)(5.45) = 5166.346 \text{ ft} \approx 0.9785 \text{ mi}$

| v (mph) | 0 | 30 | 40 | 50 | 60 | 70 | 80 | 90 | 100 | 110 | 120 | 130 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| v (fps) | 0 | 44 | 58.67 | 73.33 | 88 | 102.67 | 117.33 | 132 | 146.67 | 161.33 | 176 | 190.67 |
| t (sec) | 0 | 2.2 | 3.2 | 4.5 | 5.9 | 7.8 | 10.2 | 12.7 | 16 | 20.6 | 26.2 | 37.1 |
| Δt/2 | 0 | 1.1 | 0.5 | 0.65 | 0.7 | 0.95 | 1.2 | 1.25 | 1.65 | 2.3 | 2.8 | 5.45 |

30. Using Simpson's Rule, $\Delta x = \frac{b-a}{n} = \frac{24-0}{6} = \frac{24}{6} = 4$;
$\sum my_i = 350 \Rightarrow S = \frac{4}{3}(350) = \frac{1400}{3} \approx 466.7 \text{ in.}^2$

| | $x_i$ | $y_i$ | m | $my_i$ |
|---|---|---|---|---|
| $x_0$ | 0 | 0 | 1 | 0 |
| $x_1$ | 4 | 18.75 | 4 | 75 |
| $x_2$ | 8 | 24 | 2 | 48 |
| $x_3$ | 12 | 26 | 4 | 104 |
| $x_4$ | 16 | 24 | 2 | 48 |
| $x_5$ | 20 | 18.75 | 4 | 75 |
| $x_6$ | 24 | 0 | 1 | 0 |

31. Using Simpson's Rule, $\Delta x = 1 \Rightarrow \frac{\Delta x}{3} = \frac{1}{3}$;
$\sum my_i = 33.6 \Rightarrow$ Cross Section Area $\approx \frac{1}{3}(33.6)$
$= 11.2 \text{ ft}^2$. Let x be the length of the tank. Then the Volume V = (Cross Sectional Area) x = 11.2x.
Now 5000 lb of gasoline at 42 lb/ft$^3$
$\Rightarrow V = \frac{5000}{42} = 119.05 \text{ ft}^3$
$\Rightarrow 119.05 = 11.2x \Rightarrow x \approx 10.63$ ft

| | $x_i$ | $y_i$ | m | $my_i$ |
|---|---|---|---|---|
| $x_0$ | 0 | 1.5 | 1 | 1.5 |
| $x_1$ | 1 | 1.6 | 4 | 6.4 |
| $x_2$ | 2 | 1.8 | 2 | 3.6 |
| $x_3$ | 3 | 1.9 | 4 | 7.6 |
| $x_4$ | 4 | 2.0 | 2 | 4.0 |
| $x_5$ | 5 | 2.1 | 4 | 8.4 |
| $x_6$ | 6 | 2.1 | 1 | 2.1 |

32. $\frac{24}{2}[0.019 + 2(0.020) + 2(0.021) + \ldots + 2(0.031) + 0.035] = 4.2$ L

33. (a) $|E_S| \le \frac{b-a}{180}(\Delta x^4) M$; $n = 4 \Rightarrow \Delta x = \frac{\frac{\pi}{2}-0}{4} = \frac{\pi}{8}$; $\left|f^{(4)}\right| \le 1 \Rightarrow M = 1 \Rightarrow |E_S| \le \frac{\left(\frac{\pi}{2}-0\right)}{180}\left(\frac{\pi}{8}\right)^4(1) \approx 0.00021$

(b) $\Delta x = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$;
$\sum mf(x_i) = 10.47208705$
$\Rightarrow S = \frac{\pi}{24}(10.47208705) \approx 1.37079$

| | $x_i$ | $f(x_i)$ | m | $mf(x_{1i})$ |
|---|---|---|---|---|
| $x_0$ | 0 | 1 | 1 | 1 |
| $x_1$ | $\pi/8$ | 0.974495358 | 4 | 3.897981432 |
| $x_2$ | $\pi/4$ | 0.900316316 | 2 | 1.800632632 |
| $x_3$ | $3\pi/8$ | 0.784213303 | 4 | 3.136853212 |
| $x_4$ | $\pi/2$ | 0.636619772 | 1 | 0.636619772 |

(c) $\approx \left(\frac{0.00021}{1.37079}\right) \times 100 \approx 0.015\%$

34. (a) $\Delta x = \frac{b-a}{n} = \frac{1-0}{10} = 0.1 \Rightarrow \text{erf}(1) = \frac{2}{\sqrt{3}}\left(\frac{0.1}{3}\right)(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_9 + y_{10})$
$\frac{2}{30\sqrt{\pi}}(e^0 + 4e^{-0.01} + 2e^{-0.04} + 4e^{-0.09} + \ldots + 4e^{-0.81} + e^{-1}) \approx 0.843$

(b) $|E_S| \le \frac{1-0}{180}(0.1)^4(12) \approx 6.7 \times 10^{-6}$

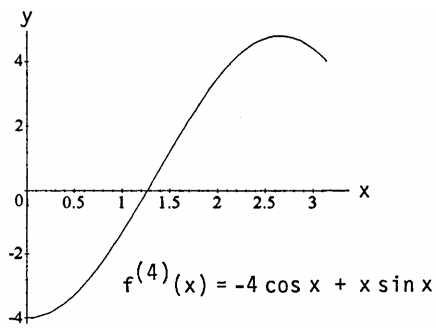
35. (a) $n = 10 \Rightarrow \Delta x = \frac{\pi - 0}{10} = \frac{\pi}{10} \Rightarrow \frac{\Delta x}{2} = \frac{\pi}{20}$;

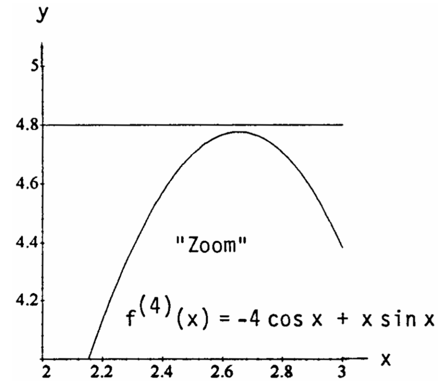
$\sum mf(x_i) = 1(0) + 2(0.09708) + 2(0.36932) + 2(0.76248) + 2(1.19513) + 2(1.57080) + 2(1.79270)$
$+ 2(1.77912) + 2(1.47727) + 2(0.87372) + 1(0) = 19.83524 \Rightarrow T = \frac{\pi}{20}(19.83524) = 3.11571$

(b) $\pi - 3.11571 \approx 0.02588$

(c) With $M = 3.11$, we get $|E_T| \le \frac{\pi}{12}\left(\frac{\pi}{10}\right)^2(3.11) = \frac{\pi^3}{1200}(3.11) < 0.08036$

36. (a) $f''(x) = 2\cos x - x\sin x \Rightarrow f^{(3)}(x) = -3\sin x - x\cos x \Rightarrow f^{(4)}(x) = -4\cos x + x\sin x$. From the graphs shown below, $|-4\cos x + x\sin x| < 4.8$ for $0 \le x \le \pi$.




(b) $n = 10 \Rightarrow \Delta x = \frac{\pi}{10} \Rightarrow |E_S| \le \frac{\pi}{180}\left(\frac{\pi}{10}\right)^4(4.8) \approx 0.00082$

(c) $\sum mf(x_i) = 1(0) + 4(0.09708) + 2(0.36932) + 4(0.76248) + 2(1.19513) + 4(1.57080) + 2(1.79270)$
$+ 4(1.77912) + 2(1.47727) + 4(0.87372) + 1(0) = 30.0016 \Rightarrow S = \frac{\pi}{30}(30.0016) = 3.14176$

(d) $|\pi - 3.14176| \approx 0.00017$

37. $T = \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n)$ where $\Delta x = \frac{b-a}{n}$ and f is continuous on [a, b]. So
$T = \frac{b-a}{n}\frac{(y_0 + y_1 + y_1 + y_2 + y_2 + \ldots + y_{n-1} + y_{n-1} + y_n)}{2} = \frac{b-a}{n}\left(\frac{f(x_0)+f(x_1)}{2} + \frac{f(x_1)+f(x_2)}{2} + \ldots + \frac{f(x_{n-1})+f(x_n)}{2}\right)$.
Since f is continuous on each interval $[x_{k-1}, x_k]$, and $\frac{f(x_{k-1})+f(x_k)}{2}$ is always between $f(x_{k-1})$ and $f(x_k)$, there is a point $c_k$ in $[x_{k-1}, x_k]$ with $f(c_k) = \frac{f(x_{k-1})+f(x_k)}{2}$; this is a consequence of the Intermediate Value Theorem. Thus our sum is $\sum_{k=1}^{n}\left(\frac{b-a}{n}\right)f(c_k)$ which has the form $\sum_{k=1}^{n}\Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{n}$ for all k. This is a Riemann Sum for f on [a, b].

38. $S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$ where n is even, $\Delta x = \frac{b-a}{n}$ and f is continuous on [a, b]. So
$S = \frac{b-a}{n}\left(\frac{y_0 + 4y_1 + y_2}{3} + \frac{y_2 + 4y_3 + y_4}{3} + \frac{y_4 + 4y_5 + y_6}{3} + \ldots + \frac{y_{n-2} + 4y_{n-1} + y_n}{3}\right)$
$= \frac{b-a}{\frac{n}{2}}\left(\frac{f(x_0) + 4f(x_1) + f(x_2)}{6} + \frac{f(x_2) + 4f(x_3) + f(x_4)}{6} + \frac{f(x_4) + 4f(x_5) + f(x_6)}{6} + \ldots + \frac{f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)}{6}\right)$
$\frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}$ is the average of the six values of the continuous function on the interval $[x_{2k}, x_{2k+2}]$, so it is between the minimum and maximum of f on this interval. By the Extreme Value Theorem for continuous functions, f takes on its maximum and minimum in this interval, so there are $x_a$ and $x_b$ with $x_{2k} \le x_a, x_b \le x_{2k+2}$ and $f(x_a) \le \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6} \le f(x_b)$. By the Intermediate Value Theorem, there is $c_k$ in $[x_{2k}, x_{2k+2}]$ with $f(c_k) = \frac{f(x_{2k}) + 4f(x_{2k+1}) + f(x_{2k+2})}{6}$. So our sum has the form $\sum_{k=1}^{n/2}\Delta x_k f(c_k)$ with $\Delta x_k = \frac{b-a}{(n/2)}$, a Riemann sum for f on [a, b].

Exercises 39-42 were done using a graphing calculator with $n = 50$

39. 1.08943
40. 1.37076
41. 0.82812
42. 51.05400

43. (a) $T_{10} \approx 1.983523538$
$T_{100} \approx 1.999835504$
$T_{1000} \approx 1.999998355$

(b)

| n | $\lvert E_T \rvert = 2 - T_n$ |
|---|---|
| 10 | $0.016476462 = 1.6476462 \times 10^{-2}$ |
| 100 | $1.64496 \times 10^{-4}$ |
| 1000 | $1.646 \times 10^{-6}$ |

(c) $\lvert E_{T_{10n}} \rvert \approx 10^{-2} \lvert E_{T_n} \rvert$

(d) $b - a = \pi$, $(\Delta x)^2 = \frac{\pi^2}{n^2}$, $M = 1$
$\lvert E_{T_n} \rvert \le \frac{\pi}{12}\left(\frac{\pi^2}{n^2}\right) = \frac{\pi^3}{12n^2}$
$\lvert E_{T_{10n}} \rvert \le \frac{\pi^3}{12(10n)^2} \le 10^{-2} \lvert E_{T_n} \rvert$

44. (a) $S_{10} \approx 2.000109517$
$S_{100} \approx 2.000000011$
$S_{1000} \approx 2.000000000$

(b)

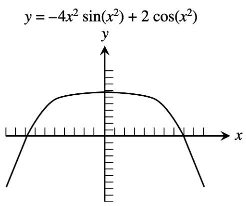
| n | $\lvert E_S \rvert = 2 - S_n$ |
|---|---|
| 10 | $1.09517 \times 10^{-4}$ |
| 100 | $1.1 \times 10^{-8}$ |
| 1000 | 0 |

(c) $\lvert E_{S_{10n}} \rvert \approx 10^{-4} \lvert E_{S_n} \rvert$

(d) $b - a = \pi$, $(\Delta x)^4 = \frac{\pi^4}{n^4}$, $M = 1$
$\lvert E_{S_n} \rvert \le \frac{\pi}{180}\left(\frac{\pi^4}{n^4}\right) = \frac{\pi^5}{180n^4}$
$\lvert E_{S_{10n}} \rvert \le \frac{\pi^5}{180(10n)^4} \le 10^{-4} \lvert E_{S_n} \rvert$

45. (a) $f'(x) = 2x\cos(x^2)$, $f''(x) = 2x \cdot (-2x)\sin(x^2) + 2\cos(x^2) = -4x^2\sin(x^2) + 2\cos(x^2)$

(b)




(c) The graph shows that $3 \le f''(x) \le 2$ so $\lvert f''(x) \rvert \le 3$ for $-1 \le x \le 1$.

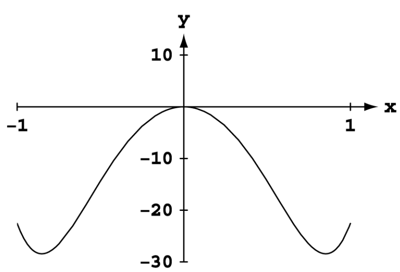
(d) $\lvert E_T \rvert \le \frac{1-(-1)}{12}(\Delta x)^2(3) = \frac{(\Delta x)^2}{2}$

(e) For $0 < \Delta x < 0.1$, $\lvert E_T \rvert \le \frac{(\Delta x)^2}{2} \le \frac{0.1^2}{2} = 0.005 < 0.01$

(f) $n \ge \frac{1-(-1)}{\Delta x} \ge \frac{2}{0.1} = 20$

46. (a) $f'''(x) = -4x^2 \cdot 2x\cos(x^2) - 8x\sin(x^2) - 4x\sin(x^2) = -8x^3\cos(x^2) - 12x\sin(x^2)$
$f^{(4)}(x) = -8x^3 \cdot 2x\sin(x^2) - 24x^2\cos(x^2) - 12x \cdot 2x\cos(x^2) - 12\sin(x^2) = (16x^4 - 12)\sin(x^2) - 48x^2\cos(x^2)$

(b)




(c) The graph shows that $-30 \le f^{(4)}(x) \le 0$ so $\left\lvert f^{(4)}(x) \right\rvert \le 30$ for $-1 \le x \le 1$.

(d) $\lvert E_S \rvert \le \frac{1-(-1)}{180}(\Delta x)^4(30) = \frac{(\Delta x)^4}{3}$

(e) For $0 < \Delta x < 0.4$, $\lvert E_S \rvert \le \frac{(\Delta x)^4}{3} \le \frac{0.4^2}{3} \approx 0.00853 < 0.01$

(f) $n \ge \frac{1-(-1)}{\Delta x} \ge \frac{2}{0.4} = 5$

47. (a) Using $d = \frac{C}{\pi}$, and $A = \pi\left(\frac{d}{2}\right)^2 = \frac{C^2}{4\pi}$ yields the following areas (in square inches, rounded to the nearest tenth):
2.3, 1.6, 1.5, 2.1, 3.2, 4.8, 7.0, 9.3, 10.7, 10.7, 9.3, 6.4, 3.2

(b) If $C(y)$ is the circumference as a function of y, then the area of a cross section is
$A(y) = \pi\left(\frac{C(y)/\pi}{2}\right)^2 = \frac{C^2(y)}{4\pi}$, and the volume is $\frac{1}{4\pi}\int_0^6 C^2(y)\,dy$.

(c) $\int_0^6 A(y)\,dy = \frac{1}{4\pi}\int_0^6 C^2(y)\,dy$
$\approx \frac{1}{4\pi}\left(\frac{6-0}{24}\right)[5.4^2 + 2(4.5^2 + 4.4^2 + 5.1^2 + 6.3^2 + 7.8^2 + 9.4^2 + 10.8^2 + 11.6^2 + 11.6^2 + 10.8^2 + 9.0^2) + 6.3^2]$
$\approx 34.7 \text{ in}^3$

(d) $V = \frac{1}{4\pi}\int_0^6 C^2(y)\,dy \approx \frac{1}{4\pi}\left(\frac{6-0}{36}\right)\Big[5.4^2 + 4(4.5^2) + 2(4.4^2) + 4(5.1^2) + 2(6.3^2) + 4(7.8^2) + 2(9.4^2) + 4(10.8^2)$
$+ 2(11.6^2) + 4(11.6^2) + 2(10.8^2) + 4(9.0^2) + 6.3^2\Big] = 34.792 \text{ in}^3$

by Simpson's Rule. The Simpson's Rule estimate should be more accurate than the trapezoid estimate. The error in the Simpson's estimate is proportional to $(\Delta y)^4 = 0.0625$ whereas the error in the trapezoid estimate is proportional to $(\Delta y)^2 = 0.25$, a larger number when $\Delta y = 0.5$ in.

48. (a) Displacement Volume $V \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$, $x_0 = 0$, $x_n = 10 - \Delta x$, $\Delta x = 2.54$, $n = 10 \Rightarrow \int_{x_0}^{x_n} A(x)\,dx \approx \frac{2.54}{3}\Big[0 + 4(1.07) + 2(3.84) + 4(7.82) + 2(12.20) + 4(15.18) + 2(16.14)$
$+ 4(14.00) + 2(9.21) + 4(3.24) + 0\Big] = \frac{2.54}{3}(248.02) = 209.99 \approx 210 \text{ ft}^3.$

(b) The weigth of water displaced is approximately $64 \cdot 120 = 13{,}440$ lb.

(c) The volume of a prism $= (2.54)(16.14) = 409.96 \approx 410 \text{ ft}^3$. Thus, the prismatic coefficient is $\frac{210 \text{ ft}^3}{410 \text{ ft}^3} \approx 0.51$.

49. (a) $a = 1, e = \frac{1}{2} \Rightarrow \text{Length} = 4\int_0^{\pi/2}\sqrt{1 - \frac{1}{4}\cos^2 t}\,dt$
$= 2\int_0^{\pi/2}\sqrt{4 - \cos^2 t}\,dt = \int_0^{\pi/2} f(t)\,dt$; use the
Trapezoid Rule with $n = 10 \Rightarrow \Delta t = \frac{b-a}{n} = \frac{\left(\frac{\pi}{2}\right)-0}{10}$
$= \frac{\pi}{20}$. $\int_0^{\pi/2}\sqrt{4 - \cos^2 t}\,dt \approx \sum_{n=0}^{10} mf(x_n) = 37.3686183$
$\Rightarrow T = \frac{\Delta t}{2}(37.3686183) = \frac{\pi}{40}(37.3686183)$
$= 2.934924419 \Rightarrow \text{Length} = 2(2.934924419)$
$\approx 5.870$

(b) $|f''(t)| < 1 \Rightarrow M = 1$
$\Rightarrow |E_T| \le \frac{b-a}{12}(\Delta t^2 M) \le \frac{\left(\frac{\pi}{2}\right)-0}{12}\left(\frac{\pi}{20}\right)^2 1 \le 0.0032$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 0 | 1.732050808 | 1 | 1.732050808 |
| $x_1$ | $\pi/20$ | 1.739100843 | 2 | 3.478201686 |
| $x_2$ | $\pi/10$ | 1.759400893 | 2 | 3.518801786 |
| $x_3$ | $3\pi/20$ | 1.790560631 | 2 | 3.581121262 |
| $x_4$ | $\pi/5$ | 1.82906848 | 1 | 3.658136959 |
| $x_5$ | $\pi/4$ | 1.870828693 | 1 | 3.741657387 |
| $x_6$ | $3\pi/10$ | 1.911676881 | 2 | 3.823353762 |
| $x_7$ | $7\pi/20$ | 1.947791731 | 2 | 3.895583461 |
| $x_8$ | $2\pi/5$ | 1.975982919 | 2 | 3.951965839 |
| $x_9$ | $9\pi/20$ | 1.993872679 | 2 | 3.987745357 |
| $x_{10}$ | $\pi/2$ | 2 | 1 | 2 |

50. $\Delta x = \frac{\pi-0}{8} = \frac{\pi}{8} \Rightarrow \frac{\Delta x}{3} = \frac{\pi}{24}$; $\sum mf(x_i) = 29.184807792$
$\Rightarrow S = \frac{\pi}{24}(29.18480779) \approx 3.82028$

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 0 | 1.414213562 | 1 | 1.414213562 |
| $x_1$ | $\pi/8$ | 1.361452677 | 4 | 5.445810706 |
| $x_2$ | $\pi/4$ | 1.224744871 | 2 | 2.449489743 |
| $x_3$ | $3\pi/8$ | 1.070722471 | 4 | 4.282889883 |
| $x_4$ | $\pi/2$ | 1 | 2 | 2 |
| $x_5$ | $5\pi/8$ | 1.070722471 | 4 | 4.282889883 |
| $x_6$ | $3\pi/4$ | 1.224744871 | 2 | 2.449489743 |
| $x_7$ | $7\pi/8$ | 1.361452677 | 4 | 5.445810706 |
| $x_8$ | $\pi$ | 1.414213562 | 1 | 1.414213562 |

51. The length of the curve $y = \sin\left(\frac{3\pi}{20}x\right)$ from 0 to 20 is: $L = \int_0^{20}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = \frac{3\pi}{20}\cos\left(\frac{3\pi}{20}x\right) \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right) \Rightarrow L = \int_0^{20}\sqrt{1 + \frac{9\pi^2}{400}\cos^2\left(\frac{3\pi}{20}x\right)}\,dx$. Using numerical integration we find $L \approx 21.07$ in

52. First, we'll find the length of the cosine curve: $L = \int_{-25}^{25}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$; $\frac{dy}{dx} = -\frac{25\pi}{50}\sin\left(\frac{\pi x}{50}\right)$
$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right) \Rightarrow L = \int_{-25}^{25}\sqrt{1 + \frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{50}\right)}\,dx$. Using a numerical integrator we find

$L \approx 73.1848$ ft. Surface area is: $A = \text{length} \cdot \text{width} \approx (73.1848)(300) = 21{,}955.44$ ft.
Cost $= 1.75A = (1.75)(21{,}955.44) = \$38{,}422.02$. Answers may vary slightly, depending on the numerical integration used.

53. $y = \sin x \Rightarrow \frac{dy}{dx} = \cos x \Rightarrow \left(\frac{dy}{dx}\right)^2 = \cos^2 x \Rightarrow S = \int_0^{\pi} 2\pi(\sin x)\sqrt{1+\cos^2 x}\,dx$; a numerical integration gives $S \approx 14.4$

54. $y = \frac{x^2}{4} \Rightarrow \frac{dy}{dx} = \frac{x}{2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{x^2}{4} \Rightarrow S = \int_0^2 2\pi\left(\frac{x^2}{4}\right)\sqrt{1+\frac{x^2}{4}}\,dx$; a numerical integration gives $S \approx 5.28$

55. $y = x + \sin 2x \Rightarrow \frac{dy}{dx} = 1 + 2\cos 2x \Rightarrow \left(\frac{dy}{dx}\right)^2 = (1+2\cos 2x)^2$; by symmetry of the graph we have that $S = 2\int_0^{2\pi/3} 2\pi(x+\sin 2x)\sqrt{1+(1+2\cos 2x)^2}\,dx$; a numerical integration gives $S \approx 54.9$

56. $y = \frac{x}{12}\sqrt{36-x^2} \Rightarrow \frac{dy}{dx} = \frac{\sqrt{36-x^2}}{12} + \frac{x}{12}\cdot\frac{1}{2}\,\frac{(-2x)}{\sqrt{36-x^2}} = \frac{\sqrt{36-x^2}}{12} - \frac{x^2}{12\sqrt{36-x^2}} = \frac{1}{12}\,\frac{(36-x^2-x^2)}{\sqrt{36-x^2}}$
$= \frac{1}{12}\,\frac{(36-2x^2)}{\sqrt{36-x^2}} = \frac{(18-x^2)}{6\sqrt{36-x^2}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(18-x^2)^2}{36(36-x^2)} \Rightarrow S = \int_0^6 \frac{2\pi\cdot x}{12}\sqrt{36-x^2}\sqrt{1+\frac{(18-x^2)^2}{36(36-x^2)}}\,dx$
$= \int_0^6 \frac{\pi x}{6}\sqrt{(36-x^2)+\left(\frac{18-x^2}{6}\right)^2}\,dx$; using numerical integration we get $S \approx 41.8$

57. A calculator or computer numerical integrator yields $\sin^{-1} 0.6 \approx 0.643501109$.

58. A calculator or computer numerical integrator yields $\pi \approx 3.1415929$.

## 8.8 IMPROPER INTEGRALS

1. $\int_0^{\infty} \frac{dx}{x^2+1} = \lim\limits_{b\to\infty} \int_0^b \frac{dx}{x^2+1} = \lim\limits_{b\to\infty} \left[\tan^{-1} x\right]_0^b = \lim\limits_{b\to\infty} \left(\tan^{-1} b - \tan^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

2. $\int_1^{\infty} \frac{dx}{x^{1.001}} = \lim\limits_{b\to\infty} \int_1^b \frac{dx}{x^{1.001}} = \lim\limits_{b\to\infty} \left[-1000x^{-0.001}\right]_1^b = \lim\limits_{b\to\infty} \left(\frac{-1000}{b^{0.001}} + 1000\right) = 1000$

3. $\int_0^1 \frac{dx}{\sqrt{x}} = \lim\limits_{b\to 0^+} \int_b^1 x^{-1/2}\,dx = \lim\limits_{b\to 0^+} \left[2x^{1/2}\right]_b^1 = \lim\limits_{b\to 0^+} \left(2 - 2\sqrt{b}\right) = 2 - 0 = 2$

4. $\int_0^4 \frac{dx}{\sqrt{4-x}} = \lim\limits_{b\to 4^-} \int_0^b (4-x)^{-1/2}dx = \lim\limits_{b\to 4^-} \left[-2\sqrt{4-b} - \left(-2\sqrt{4}\right)\right] = 0 + 4 = 4$

5. $\int_{-1}^1 \frac{dx}{x^{2/3}} = \int_{-1}^0 \frac{dx}{x^{2/3}} + \int_0^1 \frac{dx}{x^{2/3}} = \lim\limits_{b\to 0^-} \left[3x^{1/3}\right]_{-1}^b + \lim\limits_{c\to 0^+} \left[3x^{1/3}\right]_c^1$
$= \lim\limits_{b\to 0^-} \left[3b^{1/3} - 3(-1)^{1/3}\right] + \lim\limits_{c\to 0^+} \left[3(1)^{1/3} - 3c^{1/3}\right] = (0+3) + (3-0) = 6$

6. $\int_{-8}^1 \frac{dx}{x^{1/3}} = \int_{-8}^0 \frac{dx}{x^{1/3}} + \int_0^1 \frac{dx}{x^{1/3}} = \lim\limits_{b\to 0^-} \left[\frac{3}{2}x^{2/3}\right]_{-8}^b + \lim\limits_{c\to 0^+} \left[\frac{3}{2}x^{2/3}\right]_c^1$
$= \lim\limits_{b\to 0^-} \left[\frac{3}{2}b^{2/3} - \frac{3}{2}(-8)^{2/3}\right] + \lim\limits_{c\to 0^+} \left[\frac{3}{2}(1)^{2/3} - \frac{3}{2}c^{2/3}\right] = \left[0 - \frac{3}{2}(4)\right] + \left(\frac{3}{2} - 0\right) = -\frac{9}{2}$

7. $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim\limits_{b\to 1^-} \left[\sin^{-1} x\right]_0^b = \lim\limits_{b\to 1^-} \left(\sin^{-1} b - \sin^{-1} 0\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

8. $\int_0^1 \frac{dr}{r^{0.999}} = \lim\limits_{b\to 0^+} \left[1000r^{0.001}\right]_b^1 = \lim\limits_{b\to 0^+} \left(1000 - 1000b^{0.001}\right) = 1000 - 0 = 1000$

9. $\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1} = \int_{-\infty}^{-2} \frac{dx}{x-1} - \int_{-\infty}^{-2} \frac{dx}{x+1} = \lim\limits_{b \to -\infty} \left[\ln|x-1|\right]_b^{-2} - \lim\limits_{b \to -\infty} \left[\ln|x+1|\right]_b^{-2} = \lim\limits_{b \to -\infty} \left[\ln\left|\frac{x-1}{x+1}\right|\right]_b^{-2}$
$= \lim\limits_{b \to -\infty} \left(\ln\left|\frac{-3}{-1}\right| - \ln\left|\frac{b-1}{b+1}\right|\right) = \ln 3 - \ln\left(\lim\limits_{b \to -\infty} \frac{b-1}{b+1}\right) = \ln 3 - \ln 1 = \ln 3$

10. $\int_{-\infty}^{2} \frac{2\,dx}{x^2+4} = \lim\limits_{b \to -\infty} \left[\tan^{-1} \frac{x}{2}\right]_b^2 = \lim\limits_{b \to -\infty} \left(\tan^{-1} 1 - \tan^{-1} \frac{b}{2}\right) = \frac{\pi}{4} - \left(-\frac{\pi}{2}\right) = \frac{3\pi}{4}$

11. $\int_{2}^{\infty} \frac{2\,dv}{v^2-v} = \lim\limits_{b \to \infty} \left[2\ln\left|\frac{v-1}{v}\right|\right]_2^b = \lim\limits_{b \to \infty} \left(2\ln\left|\frac{b-1}{b}\right| - 2\ln\left|\frac{2-1}{2}\right|\right) = 2\ln(1) - 2\ln\left(\frac{1}{2}\right) = 0 + 2\ln 2 = \ln 4$

12. $\int_{2}^{\infty} \frac{2\,dt}{t^2-1} = \lim\limits_{b \to \infty} \left[\ln\left|\frac{t-1}{t+1}\right|\right]_2^b = \lim\limits_{b \to \infty} \left(\ln\left|\frac{b-1}{b+1}\right| - \ln\left|\frac{2-1}{2+1}\right|\right) = \ln(1) - \ln\left(\frac{1}{3}\right) = 0 + \ln 3 = \ln 3$

13. $\int_{-\infty}^{\infty} \frac{2x\,dx}{(x^2+1)^2} = \int_{-\infty}^{0} \frac{2x\,dx}{(x^2+1)^2} + \int_{0}^{\infty} \frac{2x\,dx}{(x^2+1)^2}$; $\begin{bmatrix} u = x^2+1 \\ du = 2x\,dx \end{bmatrix} \to \int_{\infty}^{1} \frac{du}{u^2} + \int_{1}^{\infty} \frac{du}{u^2} = \lim\limits_{b \to \infty} \left[-\frac{1}{u}\right]_b^1 + \lim\limits_{c \to \infty} \left[-\frac{1}{u}\right]_1^c$
$= \lim\limits_{b \to \infty} \left(-1 + \frac{1}{b}\right) + \lim\limits_{c \to \infty} \left[-\frac{1}{c} - (-1)\right] = (-1+0) + (0+1) = 0$

14. $\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}} = \int_{-\infty}^{0} \frac{x\,dx}{(x^2+4)^{3/2}} + \int_{0}^{\infty} \frac{x\,dx}{(x^2+4)^{3/2}}$; $\begin{bmatrix} u = x^2+4 \\ du = 2x\,dx \end{bmatrix} \to \int_{\infty}^{4} \frac{du}{2u^{3/2}} + \int_{4}^{\infty} \frac{du}{2u^{3/2}}$
$= \lim\limits_{b \to \infty} \left[-\frac{1}{\sqrt{u}}\right]_b^4 + \lim\limits_{c \to \infty} \left[-\frac{1}{\sqrt{u}}\right]_4^c = \lim\limits_{b \to \infty} \left(-\frac{1}{2} + \frac{1}{\sqrt{b}}\right) + \lim\limits_{c \to \infty} \left(-\frac{1}{\sqrt{c}} + \frac{1}{2}\right) = \left(-\frac{1}{2} + 0\right) + \left(0 + \frac{1}{2}\right) = 0$

15. $\int_{0}^{1} \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$; $\begin{bmatrix} u = \theta^2+2\theta \\ du = 2(\theta+1)\,d\theta \end{bmatrix} \to \int_{0}^{3} \frac{du}{2\sqrt{u}} = \lim\limits_{b \to 0^+} \int_{b}^{3} \frac{du}{2\sqrt{u}} = \lim\limits_{b \to 0^+} \left[\sqrt{u}\right]_b^3 = \lim\limits_{b \to 0^+} \left(\sqrt{3} - \sqrt{b}\right)$
$= \sqrt{3} - 0 = \sqrt{3}$

16. $\int_{0}^{2} \frac{s+1}{\sqrt{4-s^2}}\,ds = \frac{1}{2}\int_{0}^{2} \frac{2s\,ds}{\sqrt{4-s^2}} + \int_{0}^{2} \frac{ds}{\sqrt{4-s^2}}$; $\begin{bmatrix} u = 4-s^2 \\ du = -2s\,ds \end{bmatrix} \to -\frac{1}{2}\int_{4}^{0} \frac{du}{\sqrt{u}} + \lim\limits_{c \to 2^-} \int_{0}^{c} \frac{ds}{\sqrt{4-s^2}}$
$= \lim\limits_{b \to 0^+} \int_{b}^{4} \frac{du}{2\sqrt{u}} + \lim\limits_{c \to 2^-} \int_{0}^{c} \frac{ds}{\sqrt{4-s^2}} = \lim\limits_{b \to 0^+} \left[\sqrt{u}\right]_b^4 + \lim\limits_{c \to 2^-} \left[\sin^{-1} \frac{s}{2}\right]_0^c$
$= \lim\limits_{b \to 0^+} \left(2 - \sqrt{b}\right) + \lim\limits_{c \to 2^-} \left(\sin^{-1} \frac{c}{2} - \sin^{-1} 0\right) = (2-0) + \left(\frac{\pi}{2} - 0\right) = \frac{4+\pi}{2}$

17. $\int_{0}^{\infty} \frac{dx}{(1+x)\sqrt{x}}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int_{0}^{\infty} \frac{2\,du}{u^2+1} = \lim\limits_{b \to \infty} \int_{0}^{b} \frac{2\,du}{u^2+1} = \lim\limits_{b \to \infty} \left[2\tan^{-1} u\right]_0^b$
$= \lim\limits_{b \to \infty} \left(2\tan^{-1} b - 2\tan^{-1} 0\right) = 2\left(\frac{\pi}{2}\right) - 2(0) = \pi$

18. $\int_{1}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}} + \int_{2}^{\infty} \frac{dx}{x\sqrt{x^2-1}} = \lim\limits_{b \to 1^+} \int_{b}^{2} \frac{dx}{x\sqrt{x^2-1}} + \lim\limits_{c \to \infty} \int_{2}^{c} \frac{dx}{x\sqrt{x^2-1}}$
$= \lim\limits_{b \to 1^+} \left[\sec^{-1}|x|\right]_b^2 + \lim\limits_{c \to \infty} \left[\sec^{-1}|x|\right]_2^c = \lim\limits_{b \to 1^+} \left(\sec^{-1} 2 - \sec^{-1} b\right) + \lim\limits_{c \to \infty} \left(\sec^{-1} c - \sec^{-1} 2\right)$
$= \left(\frac{\pi}{3} - 0\right) + \left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \frac{\pi}{2}$

19. $\int_{0}^{\infty} \frac{dv}{(1+v^2)(1+\tan^{-1} v)} = \lim\limits_{b \to \infty} \left[\ln|1+\tan^{-1} v|\right]_0^b = \lim\limits_{b \to \infty} \left[\ln|1+\tan^{-1} b|\right] - \ln|1+\tan^{-1} 0|$
$= \ln\left(1+\frac{\pi}{2}\right) - \ln(1+0) = \ln\left(1+\frac{\pi}{2}\right)$

20. $\int_{0}^{\infty} \frac{16\tan^{-1} x}{1+x^2}\,dx = \lim\limits_{b \to \infty} \left[8\left(\tan^{-1} x\right)^2\right]_0^b = \lim\limits_{b \to \infty} \left[8\left(\tan^{-1} b\right)^2\right] - 8\left(\tan^{-1} 0\right)^2 = 8\left(\frac{\pi}{2}\right)^2 - 8(0) = 2\pi^2$

21. $\int_{-\infty}^{0} \theta e^{\theta}\, d\theta = \lim\limits_{b \to -\infty} \left[\theta e^{\theta} - e^{\theta}\right]_{b}^{0} = (0 \cdot e^{0} - e^{0}) - \lim\limits_{b \to -\infty} \left[be^{b} - e^{b}\right] = -1 - \lim\limits_{b \to -\infty} \left(\frac{b-1}{e^{-b}}\right)$
$= -1 - \lim\limits_{b \to -\infty} \left(\frac{1}{-e^{-b}}\right)$ (l'Hôpital's rule for $\frac{\infty}{\infty}$ form)
$= -1 - 0 = -1$

22. $\int_{0}^{\infty} 2e^{-\theta} \sin\theta\, d\theta = \lim\limits_{b \to \infty} \int_{0}^{b} 2e^{-\theta} \sin\theta\, d\theta$
$= \lim\limits_{b \to \infty} 2\left[\frac{e^{-\theta}}{1+1}(-\sin\theta - \cos\theta)\right]_{0}^{b}$ (FORMULA 107 with $a = -1$, $b = 1$)
$= \lim\limits_{b \to \infty} \frac{-2(\sin b + \cos b)}{2e^{b}} + \frac{2(\sin 0 + \cos 0)}{2e^{0}} = 0 + \frac{2(0+1)}{2} = 1$

23. $\int_{-\infty}^{0} e^{-|x|}\, dx = \int_{-\infty}^{0} e^{x}\, dx = \lim\limits_{b \to -\infty} \left[e^{x}\right]_{b}^{0} = \lim\limits_{b \to -\infty} (1 - e^{b}) = (1 - 0) = 1$

24. $\int_{-\infty}^{\infty} 2xe^{-x^{2}}\, dx = \int_{-\infty}^{0} 2xe^{-x^{2}}\, dx + \int_{0}^{\infty} 2xe^{-x^{2}}\, dx = \lim\limits_{b \to -\infty} \left[-e^{-x^{2}}\right]_{b}^{0} + \lim\limits_{c \to \infty} \left[-e^{-x^{2}}\right]_{0}^{c}$
$= \lim\limits_{b \to -\infty} \left[-1 - (-e^{-b^{2}})\right] + \lim\limits_{c \to \infty} \left[-e^{-c^{2}} - (-1)\right] = (-1 - 0) + (0 + 1) = 0$

25. $\int_{0}^{1} x \ln x\, dx = \lim\limits_{b \to 0^{+}} \left[\frac{x^{2}}{2} \ln x - \frac{x^{2}}{4}\right]_{b}^{1} = \left(\frac{1}{2} \ln 1 - \frac{1}{4}\right) - \lim\limits_{b \to 0^{+}} \left(\frac{b^{2}}{2} \ln b - \frac{b^{2}}{4}\right) = -\frac{1}{4} - \lim\limits_{b \to 0^{+}} \frac{\ln b}{\left(\frac{2}{b^{2}}\right)} + 0$
$= -\frac{1}{4} - \lim\limits_{b \to 0^{+}} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{4}{b^{3}}\right)} = -\frac{1}{4} + \lim\limits_{b \to 0^{+}} \left(\frac{b^{2}}{4}\right) = -\frac{1}{4} + 0 = -\frac{1}{4}$

26. $\int_{0}^{1} (-\ln x)\, dx = \lim\limits_{b \to 0^{+}} \left[x - x \ln x\right]_{b}^{1} = [1 - 1 \ln 1] - \lim\limits_{b \to 0^{+}} [b - b \ln b] = 1 - 0 + \lim\limits_{b \to 0^{+}} \frac{\ln b}{\left(\frac{1}{b}\right)} = 1 + \lim\limits_{b \to 0^{+}} \frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^{2}}\right)}$
$= 1 - \lim\limits_{b \to 0^{+}} b = 1 - 0 = 1$

27. $\int_{0}^{2} \frac{ds}{\sqrt{4 - s^{2}}} = \lim\limits_{b \to 2^{-}} \left[\sin^{-1} \frac{s}{2}\right]_{0}^{b} = \lim\limits_{b \to 2^{-}} \left(\sin^{-1} \frac{b}{2}\right) - \sin^{-1} 0 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

28. $\int_{0}^{1} \frac{4r\, dr}{\sqrt{1 - r^{4}}} = \lim\limits_{b \to 1^{-}} \left[2 \sin^{-1}(r^{2})\right]_{0}^{b} = \lim\limits_{b \to 1^{-}} \left[2 \sin^{-1}(b^{2})\right] - 2 \sin^{-1} 0 = 2 \cdot \frac{\pi}{2} - 0 = \pi$

29. $\int_{1}^{2} \frac{ds}{s\sqrt{s^{2} - 1}} = \lim\limits_{b \to 1^{+}} \left[\sec^{-1} s\right]_{b}^{2} = \sec^{-1} 2 - \lim\limits_{b \to 1^{+}} \sec^{-1} b = \frac{\pi}{3} - 0 = \frac{\pi}{3}$

30. $\int_{2}^{4} \frac{dt}{t\sqrt{t^{2} - 4}} = \lim\limits_{b \to 2^{+}} \left[\frac{1}{2} \sec^{-1} \frac{t}{2}\right]_{b}^{4} = \lim\limits_{b \to 2^{+}} \left[\left(\frac{1}{2} \sec^{-1} \frac{4}{2}\right) - \frac{1}{2} \sec^{-1}\left(\frac{b}{2}\right)\right] = \frac{1}{2}\left(\frac{\pi}{3}\right) - \frac{1}{2} \cdot 0 = \frac{\pi}{6}$

31. $\int_{-1}^{4} \frac{dx}{\sqrt{|x|}} = \lim\limits_{b \to 0^{-}} \int_{-1}^{b} \frac{dx}{\sqrt{-x}} + \lim\limits_{c \to 0^{+}} \int_{c}^{4} \frac{dx}{\sqrt{x}} = \lim\limits_{b \to 0^{-}} \left[-2\sqrt{-x}\right]_{-1}^{b} + \lim\limits_{c \to 0^{+}} \left[2\sqrt{x}\right]_{c}^{4}$
$= \lim\limits_{b \to 0^{-}} \left(-2\sqrt{-b}\right) - \left(-2\sqrt{-(-1)}\right) + 2\sqrt{4} - \lim\limits_{c \to 0^{+}} 2\sqrt{c} = 0 + 2 + 2 \cdot 2 - 0 = 6$

32. $\int_{0}^{2} \frac{dx}{\sqrt{|x - 1|}} = \int_{0}^{1} \frac{dx}{\sqrt{1 - x}} + \int_{1}^{2} \frac{dx}{\sqrt{x - 1}} = \lim\limits_{b \to 1^{-}} \left[-2\sqrt{1 - x}\right]_{0}^{b} + \lim\limits_{c \to 1^{+}} \left[2\sqrt{x - 1}\right]_{c}^{2}$
$= \lim\limits_{b \to 1^{-}} \left(-2\sqrt{1 - b}\right) - \left(-2\sqrt{1 - 0}\right) + 2\sqrt{2 - 1} - \lim\limits_{c \to 1^{+}} \left(2\sqrt{c - 1}\right) = 0 + 2 + 2 - 0 = 4$

33. $\int_{-1}^{\infty} \frac{d\theta}{\theta^{2} + 5\theta + 6} = \lim\limits_{b \to \infty} \left[\ln\left|\frac{\theta + 2}{\theta + 3}\right|\right]_{-1}^{b} = \lim\limits_{b \to \infty} \left[\ln\left|\frac{b + 2}{b + 3}\right|\right] - \ln\left|\frac{-1 + 2}{-1 + 3}\right| = 0 - \ln\left(\frac{1}{2}\right) = \ln 2$

34. $\int_0^\infty \frac{dx}{(x+1)(x^2+1)} = \lim_{b \to \infty} \left[\frac{1}{2}\ln|x+1| - \frac{1}{4}\ln(x^2+1) + \frac{1}{2}\tan^{-1} x\right]_0^b = \lim_{b \to \infty} \left[\frac{1}{2}\ln\left(\frac{x+1}{\sqrt{x^2+1}}\right) + \frac{1}{2}\tan^{-1} x\right]_0^b$
$= \lim_{b \to \infty} \left[\frac{1}{2}\ln\left(\frac{b+1}{\sqrt{b^2+1}}\right) + \frac{1}{2}\tan^{-1} b\right] - \left[\frac{1}{2}\ln\frac{1}{\sqrt{1}} + \frac{1}{2}\tan^{-1} 0\right] = \frac{1}{2}\ln 1 + \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\ln 1 - \frac{1}{2}\cdot 0 = \frac{\pi}{4}$

35. $\int_0^{\pi/2} \tan\theta\, d\theta = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln|\cos\theta|\right]_0^b = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln|\cos b|\right] + \ln 1 = \lim_{b \to \frac{\pi}{2}^-} \left[-\ln|\cos b|\right] = +\infty$, the integral diverges

36. $\int_0^{\pi/2} \cot\theta\, d\theta = \lim_{b \to 0^+} \left[\ln|\sin\theta|\right]_b^{\pi/2} = \ln 1 - \lim_{b \to 0^+} \left[\ln|\sin b|\right] = -\lim_{b \to 0^+} \left[\ln|\sin b|\right] = +\infty$, the integral diverges

37. $\int_0^\pi \frac{\sin\theta\, d\theta}{\sqrt{\pi-\theta}}$; $[\pi - \theta = x]$ $\to$ $-\int_\pi^0 \frac{\sin x\, dx}{\sqrt{x}} = \int_0^\pi \frac{\sin x\, dx}{\sqrt{x}}$. Since $0 \le \frac{\sin x}{\sqrt{x}} \le \frac{1}{\sqrt{x}}$ for all $0 \le x \le \pi$ and $\int_0^\pi \frac{dx}{\sqrt{x}}$ converges, then $\int_0^\pi \frac{\sin x}{\sqrt{x}}\, dx$ converges by the Direct Comparison Test.

38. $\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\, d\theta}{(\pi - 2\theta)^{1/3}}$; $\begin{bmatrix} x = \pi - 2\theta \\ \theta = \frac{\pi}{2} - \frac{x}{2} \\ d\theta = -\frac{dx}{2} \end{bmatrix}$ $\to$ $\int_{2\pi}^0 \frac{-\cos\left(\frac{\pi}{2} - \frac{x}{2}\right) dx}{2x^{1/3}} = \int_0^{2\pi} \frac{\sin\left(\frac{x}{2}\right) dx}{2x^{1/3}}$. Since $0 \le \frac{\sin\frac{x}{2}}{2x^{1/3}} \le \frac{1}{2x^{1/3}}$ for all $0 \le x \le 2\pi$ and $\int_0^{2\pi} \frac{dx}{2x^{1/3}}$ converges, then $\int_0^{2\pi} \frac{\sin\frac{x}{2}\, dx}{2x^{1/3}}$ converges by the Direct Comparison Test.

39. $\int_0^{\ln 2} x^{-2} e^{-1/x}\, dx$; $\left[\frac{1}{x} = y\right]$ $\to$ $\int_\infty^{1/\ln 2} \frac{y^2 e^{-y}\, dy}{-y^2} = \int_{1/\ln 2}^\infty e^{-y}\, dy = \lim_{b \to \infty} \left[-e^{-y}\right]_{1/\ln 2}^b = \lim_{b \to \infty} \left[-e^{-b}\right] - \left[-e^{-1/\ln 2}\right]$
$= 0 + e^{-1/\ln 2} = e^{-1/\ln 2}$, so the integral converges.

40. $\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$; $\left[y = \sqrt{x}\right]$ $\to$ $2\int_0^1 e^{-y}\, dy = 2 - \frac{2}{e}$, so the integral converges.

41. $\int_0^\pi \frac{dt}{\sqrt{t} + \sin t}$. Since for $0 \le t \le \pi$, $0 \le \frac{1}{\sqrt{t} + \sin t} \le \frac{1}{\sqrt{t}}$ and $\int_0^\pi \frac{dt}{\sqrt{t}}$ converges, then the original integral converges as well by the Direct Comparison Test.

42. $\int_0^1 \frac{dt}{t - \sin t}$; let $f(t) = \frac{1}{t - \sin t}$ and $g(t) = \frac{1}{t^3}$, then $\lim_{t \to 0} \frac{f(t)}{g(t)} = \lim_{t \to 0} \frac{t^3}{t - \sin t} = \lim_{t \to 0} \frac{3t^2}{1 - \cos t} = \lim_{t \to 0} \frac{6t}{\sin t}$
$= \lim_{t \to 0} \frac{6}{\cos t} = 6$. Now, $\int_0^1 \frac{dt}{t^3} = \lim_{b \to 0^+} \left[-\frac{1}{2t^2}\right]_b^1 = -\frac{1}{2} - \lim_{b \to 0^+} \left[-\frac{1}{2b^2}\right] = +\infty$, which diverges $\Rightarrow$ $\int_0^1 \frac{dt}{t - \sin t}$ diverges by the Limit Comparison Test.

43. $\int_0^2 \frac{dx}{1 - x^2} = \int_0^1 \frac{dx}{1 - x^2} + \int_1^2 \frac{dx}{1 - x^2}$ and $\int_0^1 \frac{dx}{1 - x^2} = \lim_{b \to 1^-} \left[\frac{1}{2}\ln\left|\frac{1+x}{1-x}\right|\right]_0^b = \lim_{b \to 1^-} \left[\frac{1}{2}\ln\left|\frac{1+b}{1-b}\right|\right] - 0 = \infty$, which diverges $\Rightarrow$ $\int_0^2 \frac{dx}{1 - x^2}$ diverges as well.

44. $\int_0^2 \frac{dx}{1 - x} = \int_0^1 \frac{dx}{1 - x} + \int_1^2 \frac{dx}{1 - x}$ and $\int_0^1 \frac{dx}{1 - x} = \lim_{b \to 1^-} \left[-\ln(1 - x)\right]_0^b = \lim_{b \to 1^-} \left[-\ln(1 - b)\right] - 0 = \infty$, which diverges $\Rightarrow$ $\int_0^2 \frac{dx}{1 - x}$ diverges as well.

45. $\int_{-1}^1 \ln|x|\, dx = \int_{-1}^0 \ln(-x)\, dx + \int_0^1 \ln x\, dx$; $\int_0^1 \ln x\, dx = \lim_{b \to 0^+} \left[x \ln x - x\right]_b^1 = [1 \cdot 0 - 1] - \lim_{b \to 0^+} [b \ln b - b]$
$= -1 - 0 = -1$; $\int_{-1}^0 \ln(-x)\, dx = -1$ $\Rightarrow$ $\int_{-1}^1 \ln|x|\, dx = -2$ converges.

46. $\int_{-1}^{1}(-x\ln|x|)\,dx = \int_{-1}^{0}[-x\ln(-x)]\,dx + \int_{0}^{1}(-x\ln x)\,dx = \lim_{b\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_b^1 - \lim_{c\to 0^+}\left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_c^1$
$= \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] - \lim_{b\to 0^+}\left[\frac{b^2}{2}\ln b - \frac{b^2}{4}\right] - \left[\frac{1}{2}\ln 1 - \frac{1}{4}\right] + \lim_{c\to 0^+}\left[\frac{c^2}{2}\ln c - \frac{c^2}{4}\right] = -\frac{1}{4} - 0 + \frac{1}{4} + 0 = 0 \Rightarrow$ the integral converges (see Exercise 25 for the limit calculations).

47. $\int_1^\infty \frac{dx}{1+x^3}$; $0 \le \frac{1}{x^3+1} \le \frac{1}{x^3}$ for $1 \le x < \infty$ and $\int_1^\infty \frac{dx}{x^3}$ converges $\Rightarrow \int_1^\infty \frac{dx}{1+x^3}$ converges by the Direct Comparison Test.

48. $\int_4^\infty \frac{dx}{\sqrt{x}-1}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x}-1}\right)}{\left(\frac{1}{\sqrt{x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x}-1} = \lim_{x\to\infty} \frac{1}{1-\frac{1}{\sqrt{x}}} = \frac{1}{1-0} = 1$ and $\int_4^\infty \frac{dx}{\sqrt{x}} = \lim_{b\to\infty}\left[2\sqrt{x}\right]_4^b = \infty$, which diverges $\Rightarrow \int_4^\infty \frac{dx}{\sqrt{x}-1}$ diverges by the Limit Comparison Test.

49. $\int_2^\infty \frac{dv}{\sqrt{v-1}}$; $\lim_{v\to\infty} \frac{\left(\frac{1}{\sqrt{v-1}}\right)}{\left(\frac{1}{\sqrt{v}}\right)} = \lim_{v\to\infty} \frac{\sqrt{v}}{\sqrt{v-1}} = \lim_{v\to\infty} \frac{1}{\sqrt{1-\frac{1}{v}}} = \frac{1}{\sqrt{1-0}} = 1$ and $\int_2^\infty \frac{dv}{\sqrt{v}} = \lim_{b\to\infty}\left[2\sqrt{v}\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{dv}{\sqrt{v-1}}$ diverges by the Limit Comparison Test.

50. $\int_0^\infty \frac{d\theta}{1+e^\theta}$; $0 \le \frac{1}{1+e^\theta} \le \frac{1}{e^\theta}$ for $0 \le \theta < \infty$ and $\int_0^\infty \frac{d\theta}{e^\theta} = \lim_{b\to\infty}\left[-e^{-\theta}\right]_0^b = \lim_{b\to\infty}\left(-e^{-b}+1\right) = 1$
$\Rightarrow \int_0^\infty \frac{d\theta}{e^\theta}$ converges $\Rightarrow \int_0^\infty \frac{d\theta}{1+e^\theta}$ converges by the Direct Comparison Test.

51. $\int_0^\infty \frac{dx}{\sqrt{x^6+1}} = \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{\sqrt{x^6+1}} < \int_0^1 \frac{dx}{\sqrt{x^6+1}} + \int_1^\infty \frac{dx}{x^3}$ and $\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty}\left[-\frac{1}{2x^2}\right]_1^b$
$= \lim_{b\to\infty}\left(-\frac{1}{2b^2}+\frac{1}{2}\right) = \frac{1}{2} \Rightarrow \int_0^\infty \frac{dx}{\sqrt{x^6+1}}$ converges by the Direct Comparison Test.

52. $\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{x^2-1}}\right)}{\left(\frac{1}{x}\right)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^2}}} = 1$; $\int_2^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\left[\ln b\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{dx}{\sqrt{x^2-1}}$ diverges by the Limit Comparison Test.

53. $\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$; $\lim_{x\to\infty} \frac{\left(\frac{\sqrt{x}}{x^2}\right)}{\left(\frac{\sqrt{x+1}}{x^2}\right)} = \lim_{x\to\infty} \frac{\sqrt{x}}{\sqrt{x+1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1+\frac{1}{x}}} = 1$; $\int_1^\infty \frac{\sqrt{x}}{x^2}\,dx = \int_1^\infty \frac{dx}{x^{3/2}}$
$= \lim_{b\to\infty}\left[-2x^{-1/2}\right]_1^b = \lim_{b\to\infty}\left(\frac{-2}{\sqrt{b}}+2\right) = 2 \Rightarrow \int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$ converges by the Limit Comparison Test.

54. $\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$; $\lim_{x\to\infty} \frac{\left(\frac{x}{\sqrt{x^4-1}}\right)}{\left(\frac{x}{\sqrt{x^4}}\right)} = \lim_{x\to\infty} \frac{\sqrt{x^4}}{\sqrt{x^4-1}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{1}{x^4}}} = 1$; $\int_2^\infty \frac{x\,dx}{\sqrt{x^4}} = \int_2^\infty \frac{dx}{x} = \lim_{b\to\infty}\left[\ln x\right]_2^b = \infty$, which diverges $\Rightarrow \int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$ diverges by the Limit Comparison Test.

55. $\int_\pi^\infty \frac{2+\cos x}{x}\,dx$; $0 < \frac{1}{x} \le \frac{2+\cos x}{x}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{dx}{x} = \lim_{b\to\infty}\left[\ln x\right]_\pi^b = \infty$, which diverges
$\Rightarrow \int_\pi^\infty \frac{2+\cos x}{x}\,dx$ diverges by the Direct Comparison Test.

56. $\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$; $0 \le \frac{1+\sin x}{x^2} \le \frac{2}{x^2}$ for $x \ge \pi$ and $\int_\pi^\infty \frac{2}{x^2}\,dx = \lim_{b\to\infty}\left[-\frac{2}{x}\right]_\pi^b = \lim_{b\to\infty}\left(-\frac{2}{b}+\frac{2}{\pi}\right) = \frac{2}{\pi}$
$\Rightarrow \int_\pi^\infty \frac{2\,dx}{x^2}$ converges $\Rightarrow \int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$ converges by the Direct Comparison Test.

57. $\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$; $\lim_{t\to\infty} \frac{t^{3/2}}{t^{3/2}-1} = 1$ and $\int_4^\infty \frac{2\,dt}{t^{3/2}} = \lim_{b\to\infty} \left[-4t^{-1/2}\right]_4^b = \lim_{b\to\infty} \left(\frac{-4}{\sqrt{b}} + 2\right) = 2 \Rightarrow \int_4^\infty \frac{2\,dt}{t^{3/2}}$ converges $\Rightarrow \int_4^\infty \frac{2\,dt}{t^{3/2}+1}$ converges by the Limit Comparison Test.

58. $\int_2^\infty \frac{dx}{\ln x}$; $0 < \frac{1}{x} < \frac{1}{\ln x}$ for $x > 2$ and $\int_2^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_2^\infty \frac{dx}{\ln x}$ diverges by the Direct Comparison Test.

59. $\int_1^\infty \frac{e^x}{x}\,dx$; $0 < \frac{1}{x} < \frac{e^x}{x}$ for $x > 1$ and $\int_1^\infty \frac{dx}{x}$ diverges $\Rightarrow \int_1^\infty \frac{e^x\,dx}{x}$ diverges by the Direct Comparison Test.

60. $\int_{e^e}^\infty \ln(\ln x)\,dx$; $[x = e^y] \to \int_e^\infty (\ln y)\,e^y\,dy$; $0 < \ln y < (\ln y)\,e^y$ for $y \ge e$ and $\int_e^\infty \ln y\,dy = \lim_{b\to\infty} [y\ln y - y]_e^b$ $= \infty$, which diverges $\Rightarrow \int_e^\infty \ln e^y\,dy$ diverges $\Rightarrow \int_{e^e}^\infty \ln(\ln x)\,dx$ diverges by the Direct Comparison Test.

61. $\int_1^\infty \frac{dx}{\sqrt{e^x - x}}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{\sqrt{e^x-x}}\right)}{\left(\frac{1}{\sqrt{e^x}}\right)} = \lim_{x\to\infty} \frac{\sqrt{e^x}}{\sqrt{e^x-x}} = \lim_{x\to\infty} \frac{1}{\sqrt{1-\frac{x}{e^x}}} = \frac{1}{\sqrt{1-0}} = 1$; $\int_1^\infty \frac{dx}{\sqrt{e^x}} = \int_1^\infty e^{-x/2}\,dx$ $= \lim_{b\to\infty} \left[-2e^{-x/2}\right]_1^b = \lim_{b\to\infty} \left(-2e^{-b/2} + 2e^{-1/2}\right) = \frac{2}{\sqrt{e}} \Rightarrow \int_1^\infty e^{-x/2}\,dx$ converges $\Rightarrow \int_1^\infty \frac{dx}{\sqrt{e^x-x}}$ converges by the Limit Comparison Test.

62. $\int_1^\infty \frac{dx}{e^x - 2^x}$; $\lim_{x\to\infty} \frac{\left(\frac{1}{e^x-2^x}\right)}{\left(\frac{1}{e^x}\right)} = \lim_{x\to\infty} \frac{e^x}{e^x-2^x} = \lim_{x\to\infty} \frac{1}{1-\left(\frac{2}{e}\right)^x} = \frac{1}{1-0} = 1$ and $\int_1^\infty \frac{dx}{e^x} = \lim_{b\to\infty} \left[-e^{-x}\right]_1^b$ $= \lim_{b\to\infty} \left(-e^{-b} + e^{-1}\right) = \frac{1}{e} \Rightarrow \int_1^\infty \frac{dx}{e^x}$ converges $\Rightarrow \int_1^\infty \frac{dx}{e^x-2^x}$ converges by the Limit Comparison Test.

63. $\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}} = 2\int_0^\infty \frac{dx}{\sqrt{x^4+1}}$; $\int_0^\infty \frac{dx}{\sqrt{x^4+1}} = \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{\sqrt{x^4+1}} < \int_0^1 \frac{dx}{\sqrt{x^4+1}} + \int_1^\infty \frac{dx}{x^2}$ and $\int_1^\infty \frac{dx}{x^2} = \lim_{b\to\infty} \left[-\frac{1}{x}\right]_1^b = \lim_{b\to\infty} \left(-\frac{1}{b} + 1\right) = 1 \Rightarrow \int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$ converges by the Direct Comparison Test.

64. $\int_{-\infty}^\infty \frac{dx}{e^x + e^{-x}} = 2\int_0^\infty \frac{dx}{e^x+e^{-x}}$; $0 < \frac{1}{e^x+e^{-x}} < \frac{1}{e^x}$ for $x > 0$; $\int_0^\infty \frac{dx}{e^x}$ converges $\Rightarrow 2\int_0^\infty \frac{dx}{e^x+e^{-x}}$ converges by the Direct Comparison Test.

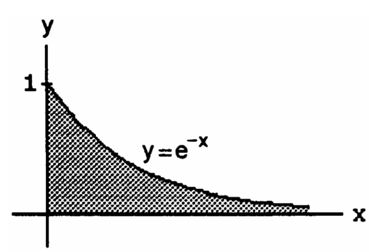
65. (a) $\int_1^2 \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_0^{\ln 2} \frac{dt}{t^p} = \lim_{b\to 0^+} \left[\frac{1}{-p+1}\,t^{1-p}\right]_b^{\ln 2} = \lim_{b\to 0^+} \frac{b^{1-p}}{p-1} + \frac{1}{1-p}(\ln 2)^{1-p}$

$\Rightarrow$ the integral converges for $p < 1$ and diverges for $p \ge 1$

(b) $\int_2^\infty \frac{dx}{x(\ln x)^p}$; $[t = \ln x] \to \int_{\ln 2}^\infty \frac{dt}{t^p}$ and this integral is essentially the same as in Exercise 65(a): it converges for $p > 1$ and diverges for $p \le 1$

66. $\int_0^\infty \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_0^b = \lim_{b\to\infty} \left[\ln(b^2+1)\right] - 0 = \lim_{b\to\infty} \ln(b^2+1) = \infty \Rightarrow$ the integral $\int_{-\infty}^\infty \frac{2x}{x^2+1}\,dx$ diverges. But $\lim_{b\to\infty} \int_{-b}^b \frac{2x\,dx}{x^2+1} = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_{-b}^b = \lim_{b\to\infty} \left[\ln(b^2+1) - \ln(b^2+1)\right] = \lim_{b\to\infty} \ln\left(\frac{b^2+1}{b^2+1}\right)$ $= \lim_{b\to\infty} (\ln 1) = 0$

67. $A = \int_0^\infty e^{-x}\,dx = \lim_{b\to\infty} \left[-e^{-x}\right]_0^b = \lim_{b\to\infty} \left(-e^{-b}\right) - \left(-e^{-0}\right)$ $= 0 + 1 = 1$



68. $\bar{x} = \frac{1}{A}\int_0^\infty xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^b = \lim_{b\to\infty}\left(-be^{-b} - e^{-b}\right) - \left(-0\cdot e^{-0} - e^{-0}\right) = 0 + 1 = 1;$
$\bar{y} = \frac{1}{2A}\int_0^\infty \left(e^{-x}\right)^2 dx = \frac{1}{2}\int_0^\infty e^{-2x}\,dx = \lim_{b\to\infty} \frac{1}{2}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \lim_{b\to\infty} \frac{1}{2}\left(-\frac{1}{2}e^{-2b}\right) - \frac{1}{2}\left(-\frac{1}{2}e^{-2\cdot 0}\right) = 0 + \frac{1}{4} = \frac{1}{4}$

69. $V = \int_0^\infty 2\pi xe^{-x}\,dx = 2\pi\int_0^\infty xe^{-x}\,dx = 2\pi \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_0^b = 2\pi\left[\lim_{b\to\infty}\left(-be^{-b} - e^{-b}\right) - 1\right] = 2\pi$

70. $V = \int_0^\infty \pi\left(e^{-x}\right)^2 dx = \pi\int_0^\infty e^{-2x}\,dx = \pi \lim_{b\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_0^b = \pi \lim_{b\to\infty}\left(-\frac{1}{2}e^{-2b} + \frac{1}{2}\right) = \frac{\pi}{2}$

71. $A = \int_0^{\pi/2}(\sec x - \tan x)\,dx = \lim_{b\to\frac{\pi}{2}^-}\left[\ln|\sec x + \tan x| - \ln|\sec x|\right]_0^b = \lim_{b\to\frac{\pi}{2}^-}\left(\ln\left|1 + \frac{\tan b}{\sec b}\right| - \ln|1+0|\right)$
$= \lim_{b\to\frac{\pi}{2}^-} \ln|1 + \sin b| = \ln 2$

72. (a) $V = \int_0^{\pi/2} \pi\sec^2 x\,dx - \int_0^{\pi/2}\pi\tan^2 x\,dx = \pi\int_0^{\pi/2}\left(\sec^2 x - \tan^2 x\right)dx = \int_0^{\pi/2}\pi\left[\sec^2 x - \left(\sec^2 x - 1\right)\right]dx$
$= \pi\int_0^{\pi/2} dx = \frac{\pi^2}{2}$

(b) $S_{\text{outer}} = \int_0^{\pi/2} 2\pi\sec x\sqrt{1 + \sec^2 x\tan^2 x}\,dx \ge \int_0^{\pi/2} 2\pi\sec x(\sec x\tan x)\,dx = \pi \lim_{b\to\frac{\pi}{2}^-}\left[\tan^2 x\right]_0^b$
$= \pi\left[\lim_{b\to\frac{\pi}{2}^-}\left[\tan^2 b\right] - 0\right] = \pi \lim_{b\to\frac{\pi}{2}^-}\left(\tan^2 b\right) = \infty \Rightarrow S_{\text{outer}}$ diverges; $S_{\text{inner}} = \int_0^{\pi/2} 2\pi\tan x\sqrt{1 + \sec^4 x}\,dx$
$\ge \int_0^{\pi/2} 2\pi\tan x\sec^2 x\,dx = \pi \lim_{b\to\frac{\pi}{2}^-}\left[\tan^2 x\right]_0^b = \pi\left[\lim_{b\to\frac{\pi}{2}^-}\left[\tan^2 b\right] - 0\right] = \pi \lim_{b\to\frac{\pi}{2}^-}\left(\tan^2 b\right) = \infty$
$\Rightarrow S_{\text{inner}}$ diverges

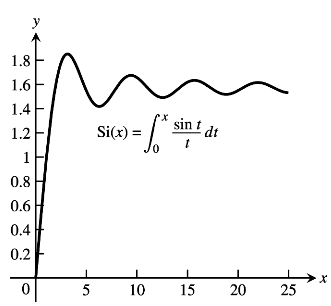
73. (a) $\int_3^\infty e^{-3x}\,dx = \lim_{b\to\infty}\left[-\frac{1}{3}e^{-3x}\right]_3^b = \lim_{b\to\infty}\left(-\frac{1}{3}e^{-3b}\right) - \left(-\frac{1}{3}e^{-3\cdot 3}\right) = 0 + \frac{1}{3}\cdot e^{-9} = \frac{1}{3}e^{-9}$
$\approx 0.0000411 < 0.000042$. Since $e^{-x^2} \le e^{-3x}$ for $x > 3$, then $\int_3^\infty e^{-x^2}\,dx < 0.000042$ and therefore $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error greater than 0.000042.

(b) $\int_0^3 e^{-x^2}\,dx \cong 0.88621$

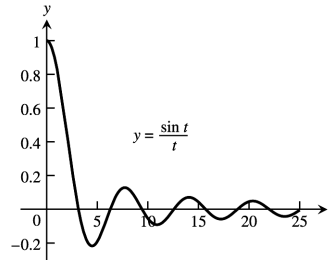
74. (a) $V = \int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi \lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^b = \pi\left[\lim_{b\to\infty}\left(-\frac{1}{b}\right) - \left(-\frac{1}{1}\right)\right] = \pi(0+1) = \pi$

(b) When you take the limit to $\infty$, you are no longer modeling the real world which is finite. The comparison step in the modeling process discussed in Section 4.2 relating the mathematical world to the real world fails to hold.

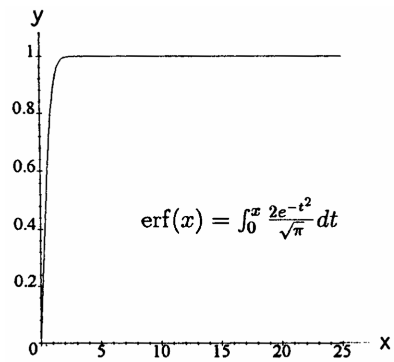
75. (a)



(b) > int((sin(t))/t, t=0..infinity); $\left(\text{answer is } \frac{\pi}{2}\right)$

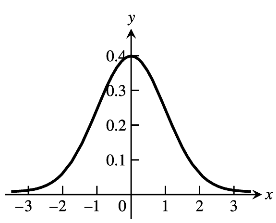



76. (a)




(b) > f:= 2*exp(−t^2)/sqrt(Pi);
> int(f, t=0..infinity); (answer is 1)

77. (a) $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$




f is increasing on $(-\infty, 0]$. f is decreasing on $[0, \infty)$. f has a local maximum at $(0, f(0)) = \left(0, \frac{1}{\sqrt{2\pi}}\right)$

(b) Maple commands:
>f: = exp(−x^2/2)(sqrt(2*pi);
>int(f, x = −1..1); $\approx 0.683$
>int(f, x = −2..2); $\approx 0.954$
>int(f, x = −3..3); $\approx 0.997$

(c) Part (b) suggests that as n increases, the integral approaches 1. We can take $\int_{-n}^{n} f(x)\,dx$ as close to 1 as we want by choosing $n > 1$ large enough. Also, we can make $\int_{n}^{\infty} f(x)\,dx$ and $\int_{-\infty}^{-n} f(x)\,dx$ as small as we want by choosing n large enough. This is because $0 < f(x) < e^{-x/2}$ for $x > 1$. (Likewise, $0 < f(x) < e^{x/2}$ for $x < -1$.)
Thus, $\int_{n}^{\infty} f(x)\,dx < \int_{n}^{\infty} e^{-x/2}dx$.
$\int_{n}^{\infty} e^{-x/2}dx = \lim\limits_{c\to\infty}\int_{n}^{c} e^{-x/2}dx = \lim\limits_{c\to\infty}\left[-2e^{-x/2}\right]_{n}^{c} = \lim\limits_{c\to\infty}\left[-2e^{-c/2} + 2e^{-n/2}\right] = 2e^{-n/2}$
As $n \to \infty$, $2e^{-n/2} \to 0$, for large enough n, $\int_{n}^{\infty} f(x)\,dx$ is as small as we want. Likewise for large enough n, $\int_{-\infty}^{-n} f(x)\,dx$ is as small as we want.

78. $\int_{3}^{\infty}\left(\frac{1}{x-2} - \frac{1}{x}\right)dx \neq \int_{3}^{\infty}\frac{dx}{x-2} - \int_{3}^{\infty}\frac{dx}{x}$, since the left hand integral converges but both of the right hand integrals diverge.

79. (a) The statement is true since $\int_{-\infty}^{b} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx$, $\int_{b}^{\infty} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx - \int_{a}^{b} f(x)\,dx$ and $\int_{a}^{b} f(x)\,dx$ exists since $f(x)$ is integrable on every interval $[a, b]$.

(b) $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_{a}^{b} f(x)\,dx - \int_{a}^{b} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx$
$= \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$

80. (a) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = -\int_{\infty}^{0} f(-u)\,du + \int_{0}^{\infty} f(x)\,dx$
$= \int_{0}^{\infty} f(-u)\,du + \int_{0}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx$, where $u = -x$

(b) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = -\int_{\infty}^{0} f(-u)\,du + \int_{0}^{\infty} f(x)\,dx$
$= \int_{0}^{\infty} -f(u)\,du + \int_{0}^{\infty} f(x)\,dx = -\int_{0}^{\infty} f(x)\,dx + \int_{0}^{\infty} f(x)\,dx = 0$, where $u = -x$

81. $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}} = \int_{-\infty}^{1} \frac{dx}{\sqrt{x^2+1}} + \int_{1}^{\infty} \frac{dx}{\sqrt{x^2+1}}$; $\int_{1}^{\infty} \frac{dx}{\sqrt{x^2+1}}$ diverges because $\lim\limits_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{\sqrt{x^2+1}}\right)}$
$= \lim\limits_{x \to \infty} \frac{\sqrt{x^2+1}}{x} = \lim\limits_{x \to \infty} \sqrt{1+\frac{1}{x^2}} = 1$ and $\int_{1}^{\infty} \frac{dx}{x}$ diverges; therefore, $\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}}$ diverges

82. $\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^6}}\,dx$ converges, since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{1+x^6}}\,dx = 2\int_{0}^{\infty} \frac{1}{\sqrt{1+x^6}}\,dx$ which was shown to converge in Exercise 51

83. $\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}} = \int_{-\infty}^{\infty} \frac{e^x\,dx}{e^{2x}+1}$; $\frac{e^x}{e^{2x}+1} = \frac{1}{e^x+e^{-x}} < \frac{1}{e^x}$ and $\int_{0}^{\infty} \frac{dx}{e^x} = \lim\limits_{c \to \infty} \left[-e^{-x}\right]_0^c = \lim\limits_{c \to \infty} \left(-e^{-c}+1\right) = 1$
$\Rightarrow \int_{-\infty}^{\infty} \frac{e^x\,dx}{e^{2x}+1} = 2\int_{0}^{\infty} \frac{dx}{e^x+e^{-x}}$ converges

84. $\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{x^2+1} = \int_{-\infty}^{-1} \frac{e^{-x}\,dx}{x^2+1} + \int_{-1}^{\infty} \frac{e^{-x}\,dx}{x^2+1}$; $\int_{-\infty}^{-1} \frac{e^{-x}\,dx}{x^2+1} = \int_{1}^{\infty} \frac{e^u\,du}{1+u^2}$, where $u = -x$, and since $\frac{e^u}{1+u^2} > \frac{1}{u}$ $(u > 1)$ and $\int_{1}^{\infty} \frac{du}{u}$ diverges, the integral $\int_{1}^{\infty} \frac{e^u\,du}{1+u^2}$ diverges $\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{x^2+1}$ diverges

85. $\int_{-\infty}^{\infty} e^{-|x|}\,dx = 2\int_{0}^{\infty} e^{-x}\,dx = 2\lim\limits_{b \to \infty} \int_{0}^{b} e^{-x}\,dx = -2\lim\limits_{b \to \infty} \left[e^{-x}\right]_0^b = 2$, so the integral converges.

86. $\int_{-\infty}^{\infty} \frac{dx}{(x+1)^2} = \int_{-\infty}^{-2} \frac{dx}{(x+1)^2} + \int_{-2}^{-1} \frac{dx}{(x+1)^2} + \int_{-1}^{2} \frac{dx}{(x+1)^2} + \int_{2}^{\infty} \frac{dx}{(x+1)^2}$;
$\lim\limits_{b \to -1^-} \int_{-2}^{b} \frac{dx}{(x+1)^2} = -\lim\limits_{b \to -1^-} \left[\frac{1}{x+1}\right]_{-2}^{b} = \infty$, which diverges $\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{(x+1)^2}$ diverges

87. $\int_{-\infty}^{\infty} \frac{|\sin x|+|\cos x|}{|x|+1}\,dx = 2\int_{0}^{\infty} \frac{|\sin x|+|\cos x|}{x+1}\,dx \geq 2\int_{0}^{\infty} \frac{\sin^2 x+\cos^2 x}{x+1}\,dx = 2\lim\limits_{b \to \infty} \int_{0}^{b} \frac{dx}{x+1}\,dx$
$= 2\lim\limits_{b \to \infty} \left[\ln|x+1|\right]_0^b = \infty$, which diverges $\Rightarrow \int_{-\infty}^{\infty} \frac{|\sin x|+|\cos x|}{|x|+1}\,dx$ diverges

88. $\int_{-\infty}^{\infty} \frac{x}{(x^2+1)(x^2+2)}\,dx = 0$ by Exercise 80(b) because the integrand is odd and the integral $\int_{0}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2)} \leq \int_{0}^{\infty} \frac{dx}{x^3}$ converges

89. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..exp(1);
fn_list := [seq( f(x,p), p=-2..2 )];
```

```
plot( fn_list, x=domain, y=-50..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p= -2","p = -1","p = 0","p = 1","p = 2"], title="#89 (Section 8.8)" );
q1 := Int( f(x,p), x=domain );
q2 := value( q1 );
q3 := simplify( q2 ) assuming p>-1;
q4 := simplify( q2 ) assuming p<-1;
q5 := value( eval( q1, p=-1 ) );
i1 := q1 = piecewise( p<-1, q4, p=-1, q5, p>-1, q3 );
```

90. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := exp(1)..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=exp(1)..10, y=0..100, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#90 (Section 8.8)" );
q6 := Int( f(x,p), x=domain );
q7 := value( q6 );
q8 := simplify( q7 ) assuming p>-1;
q9 := simplify( q7 ) assuming p<-1;
q10 := value( eval( q6, p=-1 ) );
i2 := q6 = piecewise( p<-1, q9, p=-1, q10, p>-1, q8 );
```

91. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(x);
domain := 0..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=0..10, y=-50..50, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9], thickness=[3,4,1,2,0],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#91 (Section 8.8)" );
q11 := Int( f(x,p), x=domain ):
q11 = lhs(i1+i2);
`` = rhs(i1+i2);
`` = piecewise( p<-1, q4+q9, p=-1, q5+q10, p>-1, q3+q8 );
`` = piecewise( p<-1, -infinity, p=-1, undefined, p>-1, infinity );
```

92. Example CAS commands:

Maple:

```
f := (x,p) -> x^p*ln(abs(x));
domain := -infinity..infinity;
fn_list := [seq( f(x,p), p=-2..2 )];
plot( fn_list, x=-4..4, y=-20..10, color=[red,blue,green,cyan,pink], linestyle=[1,3,4,7,9],
      legend=["p = -2","p = -1","p = 0","p = 1","p = 2"], title="#92 (Section 8.8)" );
q12 := Int( f(x,p), x=domain );
q12p := Int( f(x,p), x=0..infinity );
q12n := Int( f(x,p), x=-infinity..0 );
q12 = q12p + q12n;
`` = simplify( q12p+q12n );
```

89-92. Example CAS commands:

Mathematica: (functions and domains may vary)

```
Clear[x, f, p]
f[x_]:= x^p Log[Abs[x]]
int = Integrate[f[x], {x, e, 100)]
int /. p → 2.5
```

In order to plot the function, a value for p must be selected.

```
p = 3;
Plot[f[x], {x, 2.72, 10}]
```

**CHAPTER 8 PRACTICE EXERCISES**

1. $\int x\sqrt{4x^2-9}\,dx;\ \begin{bmatrix} u = 4x^2-9 \\ du = 8x\,dx \end{bmatrix} \rightarrow \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\cdot\frac{2}{3}u^{3/2}+C = \frac{1}{12}\left(4x^2-9\right)^{3/2}+C$

2. $\int 6x\sqrt{3x^2+5}\,dx;\ \begin{bmatrix} u = 3x^2+5 \\ du = 6x\,dx \end{bmatrix} \rightarrow \int \sqrt{u}\,du = \frac{2}{3}u^{3/2}+C = \frac{2}{3}\left(3x^2+5\right)^{3/2}+C$

3. $\int x(2x+1)^{1/2}\,dx;\ \begin{bmatrix} u = 2x+1 \\ du = 2\,dx \end{bmatrix} \rightarrow \frac{1}{2}\int\left(\frac{u-1}{2}\right)\sqrt{u}\,du = \frac{1}{4}\left(\int u^{3/2}\,du - \int u^{1/2}\,du\right) = \frac{1}{4}\left(\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right)+C$
$= \frac{(2x+1)^{5/2}}{10} - \frac{(2x+1)^{3/2}}{6} + C$

4. $\int \frac{x}{\sqrt{1-x}}\,dx;\ \begin{bmatrix} u = 1-x \\ du = -\,dx \end{bmatrix} \rightarrow -\int\frac{(1-u)}{\sqrt{u}}\,du = \int\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du = \frac{2}{3}u^{3/2}-2u^{1/2}+C$
$= \frac{2}{3}(1-x)^{3/2}-2(1-x)^{1/2}+C$

5. $\int \frac{x\,dx}{\sqrt{8x^2+1}};\ \begin{bmatrix} u = 8x^2+1 \\ du = 16x\,dx \end{bmatrix} \rightarrow \frac{1}{16}\int\frac{du}{\sqrt{u}} = \frac{1}{16}\cdot 2u^{1/2}+C = \frac{\sqrt{8x^2+1}}{8}+C$

6. $\int \frac{x\,dx}{\sqrt{9-4x^2}};\ \begin{bmatrix} u = 9-4x^2 \\ du = -8x\,dx \end{bmatrix} \rightarrow -\frac{1}{8}\int\frac{du}{\sqrt{u}} = -\frac{1}{8}\cdot 2u^{1/2}+C = -\frac{\sqrt{9-4x^2}}{4}+C$

7. $\int \frac{y\,dy}{25+y^2};\ \begin{bmatrix} u = 25+y^2 \\ du = 2y\,dy \end{bmatrix} \rightarrow \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln\left(25+y^2\right)+C$

8. $\int \frac{y^3\,dy}{4+y^4};\ \begin{bmatrix} u = 4+y^4 \\ du = 4y^3\,dy \end{bmatrix} \rightarrow \frac{1}{4}\int\frac{du}{u} = \frac{1}{4}\ln|u|+C = \frac{1}{4}\ln\left(4+y^4\right)+C$

9. $\int \frac{t^3\,dt}{\sqrt{9-4t^4}};\ \begin{bmatrix} u = 9-4t^4 \\ du = -16t^3\,dt \end{bmatrix} \rightarrow -\frac{1}{16}\int\frac{du}{\sqrt{u}} = -\frac{1}{16}\cdot 2u^{1/2}+C = -\frac{\sqrt{9-4t^4}}{8}+C$

10. $\int \frac{2t\,dt}{t^4+1};\ \begin{bmatrix} u = t^2 \\ du = 2t\,dt \end{bmatrix} \rightarrow \int\frac{du}{u^2+1} = \tan^{-1}u+C = \tan^{-1}t^2+C$

11. $\int z^{2/3}\left(z^{5/3}+1\right)^{2/3}dz;\ \begin{bmatrix} u = z^{5/3}+1 \\ du = \frac{5}{3}z^{2/3}\,dz \end{bmatrix} \rightarrow \frac{3}{5}\int u^{2/3}\,du = \frac{3}{5}\cdot\frac{3}{5}u^{5/3}+C = \frac{9}{25}\left(z^{5/3}+1\right)^{5/3}+C$

12. $\int z^{-1/5}\left(1+z^{4/5}\right)^{-1/2} dz$; $\begin{bmatrix} u = 1+z^{4/5} \\ du = \frac{4}{5} z^{-1/5}\, dz \end{bmatrix} \rightarrow \frac{5}{4}\int u^{-1/2}\, du = \frac{5}{4}\cdot 2\sqrt{u} + C = \frac{5}{2}\left(1+z^{4/5}\right)^{1/2} + C$

13. $\int \frac{\sin 2\theta\, d\theta}{(1-\cos 2\theta)^2}$; $\begin{bmatrix} u = 1-\cos 2\theta \\ du = 2\sin 2\theta\, d\theta \end{bmatrix} \rightarrow \frac{1}{2}\int \frac{du}{u^2} = -\frac{1}{2u} + C = -\frac{1}{2(1-\cos 2\theta)} + C$

14. $\int \frac{\cos\theta\, d\theta}{(1+\sin\theta)^{1/2}}$; $\begin{bmatrix} u = 1+\sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{du}{u^{1/2}} = 2u^{1/2} + C = 2\sqrt{1+\sin\theta} + C$

15. $\int \frac{\sin t\, dt}{3+4\cos t}$; $\begin{bmatrix} u = 3+4\cos t \\ du = -4\sin t\, dt \end{bmatrix} \rightarrow -\frac{1}{4}\int \frac{du}{u} = -\frac{1}{4}\ln|u| + C = -\frac{1}{4}\ln|3+4\cos t| + C$

16. $\int \frac{\cos 2t\, dt}{1+\sin 2t}$; $\begin{bmatrix} u = 1+\sin 2t \\ du = 2\cos 2t\, dt \end{bmatrix} \rightarrow \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|1+\sin 2t| + C$

17. $\int (\sin 2x)\, e^{\cos 2x}\, dx$; $\begin{bmatrix} u = \cos 2x \\ du = -2\sin 2x\, dx \end{bmatrix} \rightarrow -\frac{1}{2}\int e^u\, du = -\frac{1}{2}e^u + C = -\frac{1}{2}e^{\cos 2x} + C$

18. $\int (\sec x\tan x)\, e^{\sec x}\, dx$; $\begin{bmatrix} u = \sec x \\ du = \sec x\tan x\, dx \end{bmatrix} \rightarrow \int e^u\, du = e^u + C = e^{\sec x} + C$

19. $\int e^\theta \sin(e^\theta)\cos^2(e^\theta)\, d\theta$; $\begin{bmatrix} u = \cos\left(e^\theta\right) \\ du = -\sin\left(e^\theta\right)\cdot e^\theta\, d\theta \end{bmatrix} \rightarrow \int -u^2\, du = -\frac{1}{3}u^3 + C = -\frac{1}{3}\cos^3(e^\theta) + C$

20. $\int e^\theta \sec^2(e^\theta)\, d\theta$; $\begin{bmatrix} u = e^\theta \\ du = e^\theta\, d\theta \end{bmatrix} \rightarrow \int \sec^2 u\, du = \tan u + C = \tan(e^\theta) + C$

21. $\int 2^{x-1}\, dx = \frac{2^{x-1}}{\ln 2} + C$

22. $\int 5^{x\sqrt{2}}\, dx = \frac{1}{\sqrt{2}}\left(\frac{5^{x\sqrt{2}}}{\ln 5}\right) + C$

23. $\int \frac{dv}{v\ln v}$; $\begin{bmatrix} u = \ln v \\ du = \frac{1}{v}\, dv \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\ln v| + C$

24. $\int \frac{dv}{v(2+\ln v)}$; $\begin{bmatrix} u = 2+\ln v \\ du = \frac{1}{v}\, dv \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|2+\ln v| + C$

25. $\int \frac{dx}{(x^2+1)(2+\tan^{-1}x)}$; $\begin{bmatrix} u = 2+\tan^{-1}x \\ du = \frac{dx}{x^2+1} \end{bmatrix} \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|2+\tan^{-1}x| + C$

26. $\int \frac{\sin^{-1}x\, dx}{\sqrt{1-x^2}}$; $\begin{bmatrix} u = \sin^{-1}x \\ du = \frac{dx}{\sqrt{1-x^2}} \end{bmatrix} \rightarrow \int u\, du = \frac{1}{2}u^2 + C = \frac{1}{2}\left(\sin^{-1}x\right)^2 + C$

27. $\int \frac{2\, dx}{\sqrt{1-4x^2}}$; $\begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}(2x) + C$

28. $\int \frac{dx}{\sqrt{49-x^2}} = \frac{1}{7}\int \frac{dx}{\sqrt{1-\left(\frac{x}{7}\right)^2}}$; $\begin{bmatrix} u = \frac{x}{7} \\ du = \frac{1}{7}\, dx \end{bmatrix} \rightarrow \int \frac{du}{\sqrt{1-u^2}} = \sin^{-1}u + C = \sin^{-1}\left(\frac{x}{7}\right) + C$

29. $\int \frac{dt}{\sqrt{16-9t^2}} = \frac{1}{4}\int \frac{dt}{\sqrt{1-\left(\frac{3t}{4}\right)^2}}$ ; $\begin{bmatrix} u = \frac{3}{4}\,t \\ du = \frac{3}{4}\,dt \end{bmatrix} \rightarrow \frac{1}{3}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{3}\sin^{-1} u + C = \frac{1}{3}\sin^{-1}\left(\frac{3t}{4}\right) + C$

30. $\int \frac{dt}{\sqrt{9-4t^2}} = \frac{1}{3}\int \frac{dt}{\sqrt{1-\left(\frac{2t}{3}\right)^2}}$ ; $\begin{bmatrix} u = \frac{2}{3}\,t \\ du = \frac{2}{3}\,dt \end{bmatrix} \rightarrow \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}\left(\frac{2t}{3}\right) + C$

31. $\int \frac{dt}{9+t^2} = \frac{1}{9}\int \frac{dt}{1+\left(\frac{t}{3}\right)^2}$ ; $\begin{bmatrix} u = \frac{1}{3}\,t \\ du = \frac{1}{3}\,dt \end{bmatrix} \rightarrow \frac{1}{3}\int \frac{du}{1+u^2} = \frac{1}{3}\tan^{-1} u + C = \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$

32. $\int \frac{dt}{1+25t^2}$ ; $\begin{bmatrix} u = 5t \\ du = 5\,dt \end{bmatrix} \rightarrow \frac{1}{5}\int \frac{du}{1+u^2} = \frac{1}{5}\tan^{-1} u + C = \frac{1}{5}\tan^{-1}(5t) + C$

33. $\int \frac{4\,dx}{5x\sqrt{25x^2-16}} = \frac{4}{25}\int \frac{dx}{x\sqrt{x^2-\frac{16}{25}}} = \frac{1}{5}\sec^{-1}\left|\frac{5x}{4}\right| + C$

34. $\int \frac{6\,dx}{x\sqrt{4x^2-9}} = 3\int \frac{dx}{x\sqrt{x^2-\frac{9}{4}}} = 2\sec^{-1}\left|\frac{2x}{3}\right| + C$

35. $\int \frac{dx}{\sqrt{4x-x^2}} = \int \frac{d(x-2)}{\sqrt{4-(x-2)^2}} = \sin^{-1}\left(\frac{x-2}{2}\right) + C$

36. $\int \frac{dx}{\sqrt{4x-x^2-3}} = \int \frac{d(x-2)}{\sqrt{1-(x-2)^2}} = \sin^{-1}(x-2) + C$

37. $\int \frac{dy}{y^2-4y+8} = \int \frac{d(y-2)}{(y-2)^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{y-2}{2}\right) + C$

38. $\int \frac{dt}{t^2+4t+5} = \int \frac{d(t+2)}{(t+2)^2+1} = \tan^{-1}(t+2) + C$

39. $\int \frac{dx}{(x-1)\sqrt{x^2-2x}} = \int \frac{d(x-1)}{(x-1)\sqrt{(x-1)^2-1}} = \sec^{-1}|x-1| + C$

40. $\int \frac{dv}{(v+1)\sqrt{v^2+2v}} = \int \frac{d(v+1)}{(v+1)\sqrt{(v+1)^2-1}} = \sec^{-1}|v+1| + C$

41. $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

42. $\int \cos^2 3x\,dx = \int \frac{1+\cos 6x}{2}\,dx = \frac{x}{2} + \frac{\sin 6x}{12} + C$

43. $\int \sin^3 \frac{\theta}{2}\,d\theta = \int \left(1-\cos^2 \frac{\theta}{2}\right)\left(\sin \frac{\theta}{2}\right) d\theta$; $\begin{bmatrix} u = \cos \frac{\theta}{2} \\ du = -\frac{1}{2}\sin \frac{\theta}{2}\,d\theta \end{bmatrix} \rightarrow -2\int \left(1-u^2\right) du = \frac{2u^3}{3} - 2u + C$
$= \frac{2}{3}\cos^3 \frac{\theta}{2} - 2\cos \frac{\theta}{2} + C$

44. $\int \sin^3 \theta \cos^2 \theta\,d\theta = \int (1-\cos^2 \theta)(\sin \theta)(\cos^2 \theta)\,d\theta$; $\begin{bmatrix} u = \cos \theta \\ du = -\sin \theta\,d\theta \end{bmatrix} \rightarrow -\int (1-u^2)\,u^2\,du = \int (u^4-u^2)\,du$
$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{\cos^5 \theta}{5} - \frac{\cos^3 \theta}{3} + C$

45. $\int \tan^3 2t\, dt = \int (\tan 2t)(\sec^2 2t - 1)\, dt = \int \tan 2t \sec^2 2t\, dt - \int \tan 2t\, dt;\ \begin{bmatrix} u = 2t \\ du = 2\, dt \end{bmatrix}$
$\rightarrow \frac{1}{2}\int \tan u \sec^2 u\, du - \frac{1}{2}\int \tan u\, du = \frac{1}{4}\tan^2 u + \frac{1}{2}\ln|\cos u| + C = \frac{1}{4}\tan^2 2t + \frac{1}{2}\ln|\cos 2t| + C$
$= \frac{1}{4}\tan^2 2t - \frac{1}{2}\ln|\sec 2t| + C$

46. $\int 6\sec^4 t\, dt = 6\int (\tan^2 t + 1)(\sec^2 t)\, dt;\ \begin{bmatrix} u = \tan t \\ du = \sec^2 t\, dt \end{bmatrix} \rightarrow 6\int (u^2 + 1)\, du = 2u^3 + 6u + C$
$= 2\tan^3 t + 6\tan t + C$

47. $\int \frac{dx}{2\sin x \cos x} = \int \frac{dx}{\sin 2x} = \int \csc 2x\, dx = -\frac{1}{2}\ln|\csc 2x + \cot 2x| + C$

48. $\int \frac{2\, dx}{\cos^2 x - \sin^2 x} = \int \frac{2\, dx}{\cos 2x};\ \begin{bmatrix} u = 2x \\ du = 2\, dx \end{bmatrix} \rightarrow \int \frac{du}{\cos u} = \int \sec u\, du = \ln|\sec u + \tan u| + C$
$= \ln|\sec 2x + \tan 2x| + C$

49. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y - 1}\, dy = \int_{\pi/4}^{\pi/2} \cot y\, dy = [\ln|\sin y|]_{\pi/4}^{\pi/2} = \ln 1 - \ln \frac{1}{\sqrt{2}} = \ln\sqrt{2}$

50. $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t + 1}\, dt = \int_{\pi/4}^{3\pi/4} \csc t\, dt = [-\ln|\csc t + \cot t|]_{\pi/4}^{3\pi/4} = -\ln\left|\csc\frac{3\pi}{4} + \cot\frac{3\pi}{4}\right| + \ln\left|\csc\frac{\pi}{4} + \cot\frac{\pi}{4}\right|$
$= -\ln\left|\sqrt{2} - 1\right| + \ln\left|\sqrt{2} + 1\right| = \ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| = \ln\left|\frac{\left(\sqrt{2}+1\right)\left(\sqrt{2}+1\right)}{2-1}\right| = \ln\left(3 + 2\sqrt{2}\right)$

51. $\int_0^{\pi} \sqrt{1 - \cos^2 2x}\, dx = \int_0^{\pi} |\sin 2x|\, dx = \int_0^{\pi/2} \sin 2x\, dx - \int_{\pi/2}^{\pi} \sin 2x\, dx = -\left[\frac{\cos 2x}{2}\right]_0^{\pi/2} + \left[\frac{\cos 2x}{2}\right]_{\pi/2}^{\pi}$
$= -\left(-\frac{1}{2} - \frac{1}{2}\right) + \left[\frac{1}{2} - \left(-\frac{1}{2}\right)\right] = 2$

52. $\int_0^{2\pi} \sqrt{1 - \sin^2 \frac{x}{2}}\, dx = \int_0^{2\pi} \left|\cos\frac{x}{2}\right| dx = \int_0^{\pi} \cos\frac{x}{2}\, dx - \int_{\pi}^{2\pi} \cos\frac{x}{2}\, dx = \left[2\sin\frac{x}{2}\right]_0^{\pi} - \left[2\sin\frac{x}{2}\right]_{\pi}^{2\pi}$
$= (2 - 0) - (0 - 2) = 4$

53. $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos 2t}\, dt = \sqrt{2}\int_{-\pi/2}^{\pi/2} |\sin t|\, dt = 2\sqrt{2}\int_0^{\pi/2} \sin t\, dt = \left[-2\sqrt{2}\cos t\right]_0^{\pi/2} = 2\sqrt{2}\,[0 - (-1)] = 2\sqrt{2}$

54. $\int_{\pi}^{2\pi} \sqrt{1 + \cos 2t}\, dt = \sqrt{2}\int_{\pi}^{2\pi} |\cos t|\, dt = -\sqrt{2}\int_{\pi}^{3\pi/2} \cos t\, dt + \sqrt{2}\int_{3\pi/2}^{2\pi} \cos t\, dt$
$= -\sqrt{2}\,[\sin t]_{\pi}^{3\pi/2} + \sqrt{2}\,[\sin t]_{3\pi/2}^{2\pi} = -\sqrt{2}(-1 - 0) + \sqrt{2}\,[0 - (-1)] = 2\sqrt{2}$

55. $\int \frac{x^2\, dx}{x^2 + 4} = x - \int \frac{4\, dx}{x^2 + 4} = x - 2\tan^{-1}\left(\frac{x}{2}\right) + C$

56. $\int \frac{x^3\, dx}{9 + x^2} = \int \left[\frac{x(x^2 + 9) - 9x}{x^2 + 9}\right] dx = \int \left(x - \frac{9x}{x^2 + 9}\right) dx = \frac{x^2}{2} - \frac{9}{2}\ln(9 + x^2) + C$

57. $\int \frac{4x^2 + 3}{2x - 1}\, dx = \int \left[(2x + 1) + \frac{4}{2x - 1}\right] dx = x + x^2 + 2\ln|2x - 1| + C$

58. $\int \frac{2x\, dx}{x - 4} = \int \left(2 + \frac{8}{x - 4}\right) dx = 2x + 8\ln|x - 4| + C$

59. $\int \frac{2y - 1}{y^2 + 4}\, dy = \int \frac{2y\, dy}{y^2 + 4} - \int \frac{dy}{y^2 + 4} = \ln(y^2 + 4) - \frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) + C$

60. $\int \frac{y+4}{y^2+1}\, dy = \int \frac{y\, dy}{y^2+1} + 4\int \frac{dy}{y^2+1} = \frac{1}{2}\ln(y^2+1) + 4\tan^{-1} y + C$

61. $\int \frac{t+2}{\sqrt{4-t^2}}\, dt = \int \frac{t\, dt}{\sqrt{4-t^2}} + 2\int \frac{dt}{\sqrt{4-t^2}} = -\sqrt{4-t^2} + 2\sin^{-1}\left(\frac{t}{2}\right) + C$

62. $\int \frac{2t^2+\sqrt{1-t^2}}{t\sqrt{1-t^2}}\, dt = \int \frac{2t\, dt}{\sqrt{1-t^2}} + \int \frac{dt}{t} = -2\sqrt{1-t^2} + \ln|t| + C$

63. $\int \frac{\tan x\, dx}{\tan x + \sec x} = \int \frac{\sin x\, dx}{\sin x + 1} = \int \frac{(\sin x)(1-\sin x)}{1-\sin^2 x}\, dx = \int \frac{\sin x - 1 + \cos^2 x}{\cos^2 x}\, dx$
$= -\int \frac{d(\cos x)}{\cos^2 x} - \int \frac{dx}{\cos^2 x} + \int dx = \frac{1}{\cos x} - \tan x + x + C = x - \tan x + \sec x + C$

64. $\int \frac{\cot x\, dx}{\cot x + \csc x} = \int \frac{\cos x\, dx}{\cos x + 1} = \int \frac{(\cos x)(1-\cos x)}{1-\cos^2 x}\, dx = \int \frac{\cos x - 1 + \sin^2 x}{\sin^2 x}\, dx$
$= \int \frac{d(\sin x)}{\sin^2 x} - \int \frac{dx}{\sin^2 x} + \int dx = -\frac{1}{\sin x} + \cot x + x + C = x + \cot x - \csc x + C$

65. $\int \sec(5-3x)\, dx;\ \begin{bmatrix} y = 5-3x \\ dy = -3\, dx \end{bmatrix} \to \int \sec y \cdot \left(-\frac{dy}{3}\right) = -\frac{1}{3}\int \sec y\, dy = -\frac{1}{3}\ln|\sec y + \tan y| + C$
$= -\frac{1}{3}\ln|\sec(5-3x) + \tan(5-3x)| + C$

66. $\int x\csc(x^2+3)\, dx = \frac{1}{2}\int \csc(x^2+3)\, d(x^2+3) = -\frac{1}{2}\ln|\csc(x^2+3) + \cot(x^2+3)| + C$

67. $\int \cot\left(\frac{x}{4}\right) dx = 4\int \cot\left(\frac{x}{4}\right) d\left(\frac{x}{4}\right) = 4\ln\left|\sin\left(\frac{x}{4}\right)\right| + C$

68. $\int \tan(2x-7)\, dx = \frac{1}{2}\int \tan(2x-7)\, d(2x-7) = -\frac{1}{2}\ln|\cos(2x-7)| + C = \frac{1}{2}\ln|\sec(2x-7)| + C$

69. $\int x\sqrt{1-x}\, dx;\ \begin{bmatrix} u = 1-x \\ du = -dx \end{bmatrix} \to -\int (1-u)\sqrt{u}\, du = \int \left(u^{3/2} - u^{1/2}\right) du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$
$= \frac{2}{5}(1-x)^{5/2} - \frac{2}{3}(1-x)^{3/2} + C = -2\left[\frac{\left(\sqrt{1-x}\right)^3}{3} - \frac{\left(\sqrt{1-x}\right)^5}{5}\right] + C$

70. $\int 3x\sqrt{2x+1}\, dx;\ \begin{bmatrix} u = 2x+1 \\ du = 2\, dx \end{bmatrix} \to \int 3\left(\frac{u-1}{2}\right)\sqrt{u}\cdot\frac{1}{2}\, du = \frac{3}{4}\int \left(u^{3/2} - u^{1/2}\right) du = \frac{3}{4}\cdot\frac{2}{5}u^{5/2} - \frac{3}{4}\cdot\frac{2}{3}u^{3/2} + C$
$= \frac{3}{10}(2x+1)^{5/2} - \frac{1}{2}(2x+1)^{3/2} + C = \frac{3\left(\sqrt{2x+1}\right)^5}{10} - \frac{\left(\sqrt{2x+1}\right)^3}{2} + C$

71. $\int \sqrt{z^2+1}\, dz;\ \begin{bmatrix} z = \tan\theta \\ dz = \sec^2\theta\, d\theta \end{bmatrix} \to \int \sqrt{\tan^2\theta + 1}\cdot\sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$
$= \frac{\sec\theta\tan\theta}{3-1} + \frac{3-2}{3-1}\int \sec\theta\, d\theta$ (FORMULA 92)
$= \frac{\sin\theta}{2\cos^2\theta} + \frac{1}{2}\ln|\sec\theta + \tan\theta| + C = \frac{z\sqrt{z^2+1}}{2} + \frac{1}{2}\ln\left|z + \sqrt{1+z^2}\right| + C$

72. $\int (16+z^2)^{-3/2}\, dz;\ \begin{bmatrix} z = 4\tan\theta \\ dz = 4\sec^2\theta\, d\theta \end{bmatrix} \to \int \frac{4\sec^2\theta\, d\theta}{64\sec^3\theta\, d\theta} = \frac{1}{16}\int \cos\theta\, d\theta = \frac{1}{16}\sin\theta + C = \frac{z}{16\sqrt{16+z^2}} + C$
$= \frac{z}{16(16+z^2)^{1/2}} + C$

73. $\int \frac{dy}{\sqrt{25+y^2}} = \frac{1}{5}\int \frac{dy}{\sqrt{1+\left(\frac{y}{5}\right)^2}} = \int \frac{du}{\sqrt{1+u^2}}$, $\left[u = \frac{y}{5}\right]$; $\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}} = \int \sec\theta\, d\theta$
$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\sqrt{1+u^2}+u\right| + C_1 = \ln\left|\sqrt{1+\left(\frac{y}{5}\right)^2}+\frac{y}{5}\right| + C_1 = \ln\left|\frac{\sqrt{25+y^2}+y}{5}\right| + C_1$
$= \ln\left|y+\sqrt{25+y^2}\right| + C$

74. $\int \frac{dy}{\sqrt{25+9y^2}} = \frac{1}{5}\int \frac{dy}{\sqrt{1+\left(\frac{3y}{5}\right)^2}} = \frac{1}{3}\int \frac{du}{\sqrt{1+u^2}} = \frac{1}{3}\ln\left|\sqrt{1+u^2}+u\right| + C_1$ from Exercise 73
$\rightarrow \frac{1}{3}\ln\left|\sqrt{25+9y^2}+3y\right| + C$

75. $\int \frac{dx}{x^2\sqrt{1-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{\cos\theta\, d\theta}{\sin^2\theta\cos\theta} = \int \csc^2\theta\, d\theta = -\cot\theta + C = \frac{-\sqrt{1-x^2}}{x} + C$

76. $\int \frac{x^3\, dx}{\sqrt{1-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{\sin^3\theta\cos\theta\, d\theta}{\cos\theta} = \int \sin^3\theta\, d\theta = \int (1-\cos^2\theta)(\sin\theta)\, d\theta$;
$[u = \cos\theta] \rightarrow -\int (1-u^2)\, du = -u + \frac{u^3}{3} + C = -\cos\theta + \frac{1}{3}\cos^3\theta = -\sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{3/2} + C$
Note: Ans $\equiv \frac{-x^2\sqrt{1-x^2}}{3} - \frac{2}{3}\sqrt{1-x^2} + C$ by another method

77. $\int \frac{x^2\, dx}{\sqrt{1-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{\sin^2\theta\cos\theta\, d\theta}{\cos\theta} = \int \sin^2\theta\, d\theta = \int \frac{1-\cos 2\theta}{2}\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$
$= \frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta = \frac{\sin^{-1}x}{2} - \frac{x\sqrt{1-x^2}}{2} + C$

78. $\int \sqrt{4-x^2}\, dx$; $\begin{bmatrix} x = 2\sin\theta \\ dx = 2\cos\theta\, d\theta \end{bmatrix} \rightarrow \int 2\cos\theta \cdot 2\cos\theta\, d\theta = 2\int (1+\cos 2\theta)\, d\theta = 2\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$
$= 2\theta + 2\sin\theta\cos\theta + C = 2\sin^{-1}\left(\frac{x}{2}\right) + x\sqrt{1-\left(\frac{x}{2}\right)^2} + C = 2\sin^{-1}\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$

79. $\int \frac{dx}{\sqrt{x^2-9}}$; $\begin{bmatrix} x = 3\sec\theta \\ dx = 3\sec\theta\tan\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta - 9}} = \int \frac{3\sec\theta\tan\theta\, d\theta}{3\tan\theta} = \int \sec\theta\, d\theta$
$= \ln|\sec\theta + \tan\theta| + C_1 = \ln\left|\frac{x}{3} + \sqrt{\left(\frac{x}{3}\right)^2 - 1}\right| + C_1 = \ln\left|\frac{x+\sqrt{x^2-9}}{3}\right| + C_1 = \ln\left|x+\sqrt{x^2-9}\right| + C$

80. $\int \frac{12\, dx}{(x^2-1)^{3/2}}$; $\begin{bmatrix} x = \sec\theta \\ dx = \sec\theta\tan\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{12\sec\theta\tan\theta\, d\theta}{\tan^3\theta} = \int \frac{12\cos\theta\, d\theta}{\sin^2\theta}$; $\begin{bmatrix} u = \sin\theta \\ du = \cos\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{12\, du}{u^2}$
$= -\frac{12}{u} + C = -\frac{12}{\sin\theta} + C = -\frac{12x}{\sqrt{x^2-1}} + C$

81. $\int \frac{\sqrt{w^2-1}}{w}\, dw$; $\begin{bmatrix} w = \sec\theta \\ dw = \sec\theta\tan\theta\, d\theta \end{bmatrix} \rightarrow \int \left(\frac{\tan\theta}{\sec\theta}\right) \cdot \sec\theta\tan\theta\, d\theta = \int \tan^2\theta\, d\theta = \int (\sec^2\theta - 1)\, d\theta$
$= \tan\theta - \theta + C = \sqrt{w^2-1} - \sec^{-1}w + C$

82. $\int \frac{\sqrt{z^2-16}}{z}\, dz$; $\begin{bmatrix} z = 4\sec\theta \\ dz = 4\sec\theta\tan\theta\, d\theta \end{bmatrix} \rightarrow \int \frac{4\tan\theta \cdot 4\sec\theta\tan\theta\, d\theta}{4\sec\theta} = 4\int \tan^2\theta\, d\theta = 4(\tan\theta - \theta) + C$
$= \sqrt{z^2-16} - 4\sec^{-1}\left(\frac{z}{4}\right) + C$

83. $u = \ln(x+1)$, $du = \frac{dx}{x+1}$; $dv = dx$, $v = x$;
$\int \ln(x+1)\, dx = x\ln(x+1) - \int \frac{x}{x+1}\, dx = x\ln(x+1) - \int dx + \int \frac{dx}{x+1} = x\ln(x+1) - x + \ln(x+1) + C_1$
$= (x+1)\ln(x+1) - x + C_1 = (x+1)\ln(x+1) - (x+1) + C$, where $C = C_1 + 1$

84. $u = \ln x,\ du = \frac{dx}{x};\ dv = x^2\,dx,\ v = \frac{1}{3}x^3;$

$\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \int \frac{1}{3}x^3\left(\frac{1}{x}\right)dx = \frac{x^3}{3}\ln x - \frac{x^3}{9} + C$

85. $u = \tan^{-1} 3x,\ du = \frac{3\,dx}{1+9x^2};\ dv = dx,\ v = x;$

$\int \tan^{-1} 3x\,dx = x\tan^{-1} 3x - \int \frac{3x\,dx}{1+9x^2};\ \begin{bmatrix} y = 1+9x^2 \\ dy = 18x\,dx \end{bmatrix} \to x\tan^{-1} 3x - \frac{1}{6}\int \frac{dy}{y}$

$= x\tan^{-1}(3x) - \frac{1}{6}\ln(1+9x^2) + C$

86. $u = \cos^{-1}\left(\frac{x}{2}\right),\ du = \frac{-dx}{\sqrt{4-x^2}};\ dv = dx,\ v = x;$

$\int \cos^{-1}\left(\frac{x}{2}\right)dx = x\cos^{-1}\left(\frac{x}{2}\right) + \int \frac{x\,dx}{\sqrt{4-x^2}};\ \begin{bmatrix} y = 4-x^2 \\ dy = -2x\,dx \end{bmatrix} \to x\cos^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\int \frac{dy}{\sqrt{y}}$

$= x\cos^{-1}\left(\frac{x}{2}\right) - \sqrt{4-x^2} + C = x\cos^{-1}\left(\frac{x}{2}\right) - 2\sqrt{1-\left(\frac{x}{2}\right)^2} + C$

87.
$$\begin{array}{lcl} & & e^x \\ (x+1)^2 & \xrightarrow{(+)} & e^x \\ 2(x+1) & \xrightarrow{(-)} & e^x \\ 2 & \xrightarrow{(+)} & e^x \\ 0 & & \end{array}$$
$\Rightarrow \int (x+1)^2 e^x\,dx = [(x+1)^2 - 2(x+1) + 2]\,e^x + C$

88.
$$\begin{array}{lcl} & & \sin(1-x) \\ x^2 & \xrightarrow{(+)} & \cos(1-x) \\ 2x & \xrightarrow{(-)} & -\sin(1-x) \\ 2 & \xrightarrow{(+)} & -\cos(1-x) \\ 0 & & \end{array}$$
$\Rightarrow \int x^2 \sin(1-x)\,dx = x^2\cos(1-x) + 2x\sin(1-x) - 2\cos(1-x) + C$

89. $u = \cos 2x,\ du = -2\sin 2x\,dx;\ dv = e^x\,dx,\ v = e^x;$

$I = \int e^x \cos 2x\,dx = e^x\cos 2x + 2\int e^x \sin 2x\,dx;$

$u = \sin 2x,\ du = 2\cos 2x\,dx;\ dv = e^x\,dx,\ v = e^x;$

$I = e^x\cos 2x + 2\left[e^x \sin 2x - 2\int e^x \cos 2x\,dx\right] = e^x\cos 2x + 2e^x\sin 2x - 4I \Rightarrow I = \frac{e^x\cos 2x}{5} + \frac{2e^x\sin 2x}{5} + C$

90. $u = \sin 3x,\ du = 3\cos 3x\,dx;\ dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x};$

$I = \int e^{-2x}\sin 3x\,dx = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\int e^{-2x}\cos 3x\,dx;$

$u = \cos 3x,\ du = -3\sin 3x\,dx;\ dv = e^{-2x}\,dx,\ v = -\frac{1}{2}e^{-2x};$

$I = -\frac{1}{2}e^{-2x}\sin 3x + \frac{3}{2}\left[-\frac{1}{2}e^{-2x}\cos 3x - \frac{3}{2}\int e^{-2x}\sin 3x\,dx\right] = -\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x - \frac{9}{4}I$

$\Rightarrow I = \frac{4}{13}\left(-\frac{1}{2}e^{-2x}\sin 3x - \frac{3}{4}e^{-2x}\cos 3x\right) + C = -\frac{2}{13}e^{-2x}\sin 3x - \frac{3}{13}e^{-2x}\cos 3x + C$

91. $\int \frac{x\,dx}{x^2-3x+2} = \int \frac{2\,dx}{x-2} - \int \frac{dx}{x-1} = 2\ln|x-2| - \ln|x-1| + C$

92. $\int \frac{x\,dx}{x^2+4x+3} = \frac{3}{2}\int \frac{dx}{x+3} - \frac{1}{2}\int \frac{dx}{x+1} = \frac{3}{2}\ln|x+3| - \frac{1}{2}\ln|x+1| + C$

93. $\int \frac{dx}{x(x+1)^2} = \int \left(\frac{1}{x} - \frac{1}{x+1} + \frac{-1}{(x+1)^2}\right) dx = \ln|x| - \ln|x+1| + \frac{1}{x+1} + C$

94. $\int \frac{x+1}{x^2(x-1)}\, dx = \int \left(\frac{2}{x-1} - \frac{2}{x} - \frac{1}{x^2}\right) dx = 2\ln\left|\frac{x-1}{x}\right| + \frac{1}{x} + C = -2\ln|x| + \frac{1}{x} + 2\ln|x-1| + C$

95. $\int \frac{\sin\theta\, d\theta}{\cos^2\theta + \cos\theta - 2}$; $[\cos\theta = y] \rightarrow -\int \frac{dy}{y^2+y-2} = -\frac{1}{3}\int \frac{dy}{y-1} + \frac{1}{3}\int \frac{dy}{y+2} = \frac{1}{3}\ln\left|\frac{y+2}{y-1}\right| + C$
$= \frac{1}{3}\ln\left|\frac{\cos\theta+2}{\cos\theta-1}\right| + C = -\frac{1}{3}\ln\left|\frac{\cos\theta-1}{\cos\theta+2}\right| + C$

96. $\int \frac{\cos\theta\, d\theta}{\sin^2\theta + \sin\theta - 6}$; $[\sin\theta = x] \rightarrow \int \frac{dx}{x^2+x-6} = \frac{1}{5}\int \frac{dx}{x-2} - \frac{1}{5}\int \frac{dx}{x+3} = \frac{1}{5}\ln\left|\frac{\sin\theta-2}{\sin\theta+3}\right| + C$

97. $\int \frac{3x^2+4x+4}{x^3+x}\, dx = \int \frac{4}{x}\, dx - \int \frac{x-4}{x^2+1}\, dx = 4\ln|x| - \frac{1}{2}\ln(x^2+1) + 4\tan^{-1}x + C$

98. $\int \frac{4x\, dx}{x^3+4x} = \int \frac{4\, dx}{x^2+4} = 2\tan^{-1}\left(\frac{x}{2}\right) + C$

99. $\int \frac{(v+3)\, dv}{2v^3-8v} = \frac{1}{2}\int \left(-\frac{3}{4v} + \frac{5}{8(v-2)} + \frac{1}{8(v+2)}\right) dv = -\frac{3}{8}\ln|v| + \frac{5}{16}\ln|v-2| + \frac{1}{16}\ln|v+2| + C$
$= \frac{1}{16}\ln\left|\frac{(v-2)^5(v+2)}{v^6}\right| + C$

100. $\int \frac{(3v-7)\, dv}{(v-1)(v-2)(v-3)} = \int \frac{(-2)\, dv}{v-1} + \int \frac{dv}{v-2} + \int \frac{dv}{v-3} = \ln\left|\frac{(v-2)(v-3)}{(v-1)^2}\right| + C$

101. $\int \frac{dt}{t^4+4t^2+3} = \frac{1}{2}\int \frac{dt}{t^2+1} - \frac{1}{2}\int \frac{dt}{t^2+3} = \frac{1}{2}\tan^{-1}t - \frac{1}{2\sqrt{3}}\tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + C = \frac{1}{2}\tan^{-1}t - \frac{\sqrt{3}}{6}\tan^{-1}\frac{t}{\sqrt{3}} + C$

102. $\int \frac{t\, dt}{t^4-t^2-2} = \frac{1}{3}\int \frac{t\, dt}{t^2-2} - \frac{1}{3}\int \frac{t\, dt}{t^2+1} = \frac{1}{6}\ln|t^2-2| - \frac{1}{6}\ln(t^2+1) + C$

103. $\int \frac{x^3+x^2}{x^2+x-2}\, dx = \int \left(x + \frac{2x}{x^2+x-2}\right) dx = \int x\, dx + \frac{2}{3}\int \frac{dx}{x-1} + \frac{4}{3}\int \frac{dx}{x+2}$
$= \frac{x^2}{2} + \frac{4}{3}\ln|x+2| + \frac{2}{3}\ln|x-1| + C$

104. $\int \frac{x^3+1}{x^3-x}\, dx = \int \left(1 + \frac{x+1}{x^3-x}\right) dx = \int \left[1 + \frac{1}{x(x-1)}\right] dx = \int dx + \int \frac{dx}{x-1} - \int \frac{dx}{x} = x + \ln|x-1| - \ln|x| + C$

105. $\int \frac{x^3+4x^2}{x^2+4x+3}\, dx = \int \left(x - \frac{3x}{x^2+4x+3}\right) dx = \int x\, dx + \frac{3}{2}\int \frac{dx}{x+1} - \frac{9}{2}\int \frac{dx}{x+3}$
$= \frac{x^2}{2} - \frac{9}{2}\ln|x+3| + \frac{3}{2}\ln|x+1| + C$

106. $\int \frac{2x^3+x^2-21x+24}{x^2+2x-8}\, dx = \int \left[(2x-3) + \frac{x}{x^2+2x-8}\right] dx = \int (2x-3)\, dx + \frac{1}{3}\int \frac{dx}{x-2} + \frac{2}{3}\int \frac{dx}{x+4}$
$= x^2 - 3x + \frac{2}{3}\ln|x+4| + \frac{1}{3}\ln|x-2| + C$

107. $\int \frac{dx}{x\left(3\sqrt{x+1}\right)}$; $\begin{bmatrix} u = \sqrt{x+1} \\ du = \frac{dx}{2\sqrt{x+1}} \\ dx = 2u\, du \end{bmatrix} \rightarrow \frac{2}{3}\int \frac{u\, du}{(u^2-1)\, u} = \frac{1}{3}\int \frac{du}{u-1} - \frac{1}{3}\int \frac{du}{u+1} = \frac{1}{3}\ln|u-1| - \frac{1}{3}\ln|u+1| + C$
$= \frac{1}{3}\ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C$

108. $\int \frac{dx}{x(1+\sqrt[3]{x})}$ ; $\begin{bmatrix} u = \sqrt[3]{x} \\ du = \frac{dx}{3x^{2/3}} \\ dx = 3u^2\,du \end{bmatrix}$ $\rightarrow \int \frac{3u^2\,du}{u^3(1+u)} = 3\int \frac{du}{u(1+u)} = 3\ln\left|\frac{u}{u+1}\right| + C = 3\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right| + C$

109. $\int \frac{ds}{e^s - 1}$ ; $\begin{bmatrix} u = e^s - 1 \\ du = e^s\,ds \\ ds = \frac{du}{u+1} \end{bmatrix}$ $\rightarrow \int \frac{du}{u(u+1)} = -\int \frac{du}{u+1} + \int \frac{du}{u} = \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{e^s-1}{e^s}\right| + C = \ln|1 - e^{-s}| + C$

110. $\int \frac{ds}{\sqrt{e^s+1}}$ ; $\begin{bmatrix} u = \sqrt{e^s+1} \\ du = \frac{e^s\,ds}{2\sqrt{e^s+1}} \\ ds = \frac{2u\,du}{u^2-1} \end{bmatrix}$ $\rightarrow \int \frac{2u\,du}{u(u^2-1)} = 2\int \frac{du}{(u+1)(u-1)} = \int \frac{du}{u-1} - \int \frac{du}{u+1} = \ln\left|\frac{u-1}{u+1}\right| + C$
$= \ln\left|\frac{\sqrt{e^s+1}-1}{\sqrt{e^s+1}+1}\right| + C$

111. (a) $\int \frac{y\,dy}{\sqrt{16-y^2}} = -\frac{1}{2}\int \frac{d(16-y^2)}{\sqrt{16-y^2}} = -\sqrt{16-y^2} + C$

(b) $\int \frac{y\,dy}{\sqrt{16-y^2}}$ ; $[y = 4\sin x] \rightarrow 4\int \frac{\sin x\cos x\,dx}{\cos x} = -4\cos x + C = -\frac{4\sqrt{16-y^2}}{4} + C = -\sqrt{16-y^2} + C$

112. (a) $\int \frac{x\,dx}{\sqrt{4+x^2}} = \frac{1}{2}\int \frac{d(4+x^2)}{\sqrt{4+x^2}} = \sqrt{4+x^2} + C$

(b) $\int \frac{x\,dx}{\sqrt{4+x^2}}$ ; $[x = 2\tan y] \rightarrow \int \frac{2\tan y \cdot 2\sec^2 y\,dy}{2\sec y} = 2\int \sec y\tan y\,dy = 2\sec y + C = \sqrt{4+x^2} + C$

113. (a) $\int \frac{x\,dx}{4-x^2} = -\frac{1}{2}\int \frac{d(4-x^2)}{4-x^2} = -\frac{1}{2}\ln|4-x^2| + C$

(b) $\int \frac{x\,dx}{4-x^2}$ ; $[x = 2\sin\theta] \rightarrow \int \frac{2\sin\theta \cdot 2\cos\theta\,d\theta}{4\cos^2\theta} = \int \tan\theta\,d\theta = -\ln|\cos\theta| + C = -\ln\left(\frac{\sqrt{4-x^2}}{2}\right) + C$
$= -\frac{1}{2}\ln|4-x^2| + C$

114. (a) $\int \frac{t\,dt}{\sqrt{4t^2-1}} = \frac{1}{8}\int \frac{d(4t^2-1)}{\sqrt{4t^2-1}} = \frac{1}{4}\sqrt{4t^2-1} + C$

(b) $\int \frac{t\,dt}{\sqrt{4t^2-1}}$ ; $\left[t = \frac{1}{2}\sec\theta\right] \rightarrow \int \frac{\frac{1}{2}\sec\theta\tan\theta \cdot \frac{1}{2}\sec\theta\,d\theta}{\tan\theta} = \frac{1}{4}\int \sec^2\theta\,d\theta = \frac{\tan\theta}{4} + C = \frac{\sqrt{4t^2-1}}{4} + C$

115. $\int \frac{x\,dx}{9-x^2}$ ; $\begin{bmatrix} u = 9-x^2 \\ du = -2x\,dx \end{bmatrix}$ $\rightarrow -\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln|u| + C = \ln\frac{1}{\sqrt{u}} + C = \ln\frac{1}{\sqrt{9-x^2}} + C$

116. $\int \frac{dx}{x(9-x^2)} = \frac{1}{9}\int \frac{dx}{x} + \frac{1}{18}\int \frac{dx}{3-x} - \frac{1}{18}\int \frac{dx}{3+x} = \frac{1}{9}\ln|x| - \frac{1}{18}\ln|3-x| - \frac{1}{18}\ln|3+x| + C$
$= \frac{1}{9}\ln|x| - \frac{1}{18}\ln|9-x^2| + C$

117. $\int \frac{dx}{9-x^2} = \frac{1}{6}\int \frac{dx}{3-x} + \frac{1}{6}\int \frac{dx}{3+x} = -\frac{1}{6}\ln|3-x| + \frac{1}{6}\ln|3+x| + C = \frac{1}{6}\ln\left|\frac{x+3}{x-3}\right| + C$

118. $\int \frac{dx}{\sqrt{9-x^2}}$ ; $\begin{bmatrix} x = 3\sin\theta \\ dx = 3\cos\theta\,d\theta \end{bmatrix}$ $\rightarrow \int \frac{3\cos\theta}{3\cos\theta}\,d\theta = \int d\theta = \theta + C = \sin^{-1}\frac{x}{3} + C$

119. $\int \sin^3 x\cos^4 x\,dx = \int \cos^4 x(1-\cos^2 x)\sin x\,dx = \int \cos^4 x\sin x\,dx - \int \cos^6 x\sin x\,dx = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

120. $\int \cos^5 x\sin^5 x\,dx = \int \sin^5 x\cos^4 x\cos x\,dx = \int \sin^5 x\,(1-\sin^2 x)^2\cos x\,dx$
$= \int \sin^5 x\cos x\,dx - 2\int \sin^7 x\cos x\,dx + \int \sin^9 x\cos x\,dx = \frac{\sin^6 x}{6} - \frac{2\sin^8 x}{8} + \frac{\sin^{10} x}{10} + C$

121. $\int \tan^4 x \sec^2 x\, dx = \frac{\tan^5 x}{5} + C$

122. $\int \tan^3 x \sec^3 x\, dx = \int \left(\sec^2 x - 1\right) \sec^2 x \cdot \sec x \cdot \tan x\, dx = \int \sec^4 x \cdot \sec x \cdot \tan x\, dx - \int \sec^2 x \cdot \sec x \cdot \tan x\, dx$
$= \frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + C$

123. $\int \sin 5\theta \cos 6\theta\, d\theta = \frac{1}{2}\int (\sin(-\theta) + \sin(11\theta))\, d\theta = \frac{1}{2}\int \sin(-\theta)\, d\theta + \frac{1}{2}\int \sin(11\theta)\, d\theta = \frac{1}{2}\cos(-\theta) - \frac{1}{22}\cos 11\theta + C$
$= \frac{1}{2}\cos\theta - \frac{1}{22}\cos 11\theta + C$

124. $\int \cos 3\theta \cos 3\theta\, d\theta = \frac{1}{2}\int \left(\cos 0 + \cos 6\theta\right) d\theta = \frac{1}{2}\int d\theta + \frac{1}{2}\int \cos 6\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{12}\sin 6\theta + C$

125. $\int \sqrt{1 + \cos\left(\frac{t}{2}\right)}\, dt = \int \sqrt{2}\left|\cos\ \frac{t}{4}\right| dt = 4\sqrt{2}\left|\sin\ \frac{t}{4}\right| + C$

126. $\int e^t \sqrt{\tan^2 e^t + 1}\, dt = \int |\sec e^t|\, e^t\, dt = \ln|\sec e^t + \tan e^t| + C$

127. $|E_s| \le \frac{3-1}{180}(\triangle x)^4 M$ where $\triangle x = \frac{3-1}{n} = \frac{2}{n}$; $f(x) = \frac{1}{x} = x^{-1} \Rightarrow f'(x) = -x^{-2} \Rightarrow f''(x) = 2x^{-3} \Rightarrow f'''(x) = -6x^{-4}$
$\Rightarrow f^{(4)}(x) = 24x^{-5}$ which is decreasing on $[1,3] \Rightarrow$ maximum of $f^{(4)}(x)$ on $[1,3]$ is $f^{(4)}(1) = 24 \Rightarrow M = 24$. Then
$|E_s| \le 0.0001 \Rightarrow \left(\frac{3-1}{180}\right)\left(\frac{2}{n}\right)^4(24) \le 0.0001 \Rightarrow \left(\frac{768}{180}\right)\left(\frac{1}{n^4}\right) \le 0.0001 \Rightarrow \frac{1}{n^4} \le (0.0001)\left(\frac{180}{768}\right) \Rightarrow n^4 \ge 10{,}000\left(\frac{768}{180}\right)$
$\Rightarrow n \ge 14.37 \Rightarrow n \ge 16$ (n must be even)

128. $|E_T| \le \frac{1-0}{12}(\triangle x)^2 M$ where $\triangle x = \frac{1-0}{n} = \frac{1}{n}$; $0 \le f''(x) \le 8 \Rightarrow M = 8$. Then $|E_T| \le 10^{-3} \Rightarrow \frac{1}{12}\left(\frac{1}{n}\right)^2(8) \le 10^{-3}$
$\Rightarrow \frac{2}{3n^2} \le 10^{-3} \Rightarrow \frac{3n^2}{2} \ge 1000 \Rightarrow n^2 \ge \frac{2000}{3} \Rightarrow n \ge 25.82 \Rightarrow n \ge 26$

129. $\triangle x = \frac{b-a}{n} = \frac{\pi - 0}{6} = \frac{\pi}{6} \Rightarrow \frac{\triangle x}{2} = \frac{\pi}{12}$;
$\sum_{i=0}^{6} mf(x_i) = 12 \Rightarrow T = \left(\frac{\pi}{12}\right)(12) = \pi$;

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 0 | 0 | 1 | 0 |
| $x_1$ | $\pi/6$ | 1/2 | 2 | 1 |
| $x_2$ | $\pi/3$ | 3/2 | 2 | 3 |
| $x_3$ | $\pi/2$ | 2 | 2 | 4 |
| $x_4$ | $2\pi/3$ | 3/2 | 2 | 3 |
| $x_5$ | $5\pi/6$ | 1/2 | 2 | 1 |
| $x_6$ | $\pi$ | 0 | 1 | 0 |

$\sum_{i=0}^{6} mf(x_i) = 18$ and $\frac{\triangle x}{3} = \frac{\pi}{18} \Rightarrow$
$S = \left(\frac{\pi}{18}\right)(18) = \pi$.

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 0 | 0 | 1 | 0 |
| $x_1$ | $\pi/6$ | 1/2 | 4 | 2 |
| $x_2$ | $\pi/3$ | 3/2 | 2 | 3 |
| $x_3$ | $\pi/2$ | 2 | 4 | 8 |
| $x_4$ | $2\pi/3$ | 3/2 | 2 | 3 |
| $x_5$ | $5\pi/6$ | 1/2 | 4 | 2 |
| $x_6$ | $\pi$ | 0 | 1 | 0 |

130. $\left|f^{(4)}(x)\right| \le 3 \Rightarrow M = 3$; $\triangle x = \frac{2-1}{n} = \frac{1}{n}$. Hence $|E_s| \le 10^{-5} \Rightarrow \left(\frac{2-1}{180}\right)\left(\frac{1}{n}\right)^4(3) \le 10^{-5} \Rightarrow \frac{1}{60n^4} \le 10^{-5} \Rightarrow n^4 \ge \frac{10^5}{60}$
$\Rightarrow n \ge 6.38 \Rightarrow n \ge 8$ (n must be even)

131. $y_{av} = \frac{1}{365-0}\int_0^{365}\left[37\sin\left(\frac{2\pi}{365}(x-101)\right) + 25\right] dx = \frac{1}{365}\left[-37\left(\frac{365}{2\pi}\cos\left(\frac{2\pi}{365}(x-101)\right) + 25x\right)\right]_0^{365}$
$= \frac{1}{365}\left[\left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(365-101)\right] + 25(365)\right) - \left(-37\left(\frac{365}{2\pi}\right)\cos\left[\frac{2\pi}{365}(0-101)\right] + 25(0)\right)\right]$
$= -\frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(264)\right) + 25 + \frac{37}{2\pi}\cos\left(\frac{2\pi}{365}(-101)\right) = -\frac{37}{2\pi}\left(\cos\left(\frac{2\pi}{365}(264)\right) - \cos\left(\frac{2\pi}{365}(-101)\right)\right) + 25$

$\approx -\frac{37}{2\pi}(0.16705 - 0.16705) + 25 = 25°\,F$

132. $\text{av}(C_v) = \frac{1}{675-20}\int_{20}^{675}[8.27 + 10^{-5}(26T - 1.87T^2)]\,dT = \frac{1}{655}\left[8.27T + \frac{13}{10^5}T^2 - \frac{0.62333}{10^5}T^3\right]_{20}^{675}$
$\approx \frac{1}{655}[(5582.25 + 59.23125 - 1917.03194) - (165.4 + 0.052 - 0.04987)] \approx 5.434;$
$8.27 + 10^{-5}(26T - 1.87T^2) = 5.434 \Rightarrow 1.87T^2 - 26T - 283{,}600 = 0 \Rightarrow T \approx \frac{26+\sqrt{676+4(1.87)(283{,}600)}}{2(1.87)}$
$\approx 396.45°\,C$

133. (a) Each interval is 5 min $= \frac{1}{12}$ hour.
$\frac{1}{24}[2.5 + 2(2.4) + 2(2.3) + \ldots + 2(2.4) + 2.3] = \frac{29}{12} \approx 2.42$ gal
(b) $(60\text{ mph})\left(\frac{12}{29}\text{ hours/gal}\right) \approx 24.83$ mi/gal

134. Using the Simpson's rule, $\triangle x = 15 \Rightarrow \frac{\triangle x}{3} = 5;$
$\sum mf(x_i) = 1211.8 \Rightarrow \text{Area} \approx (1211.8)(5) = 6059\text{ ft}^2;$
The cost is $\text{Area} \cdot (\$2.10/\text{ft}^2) \approx (6059\text{ ft}^2)(\$2.10/\text{ft}^2)$
$= \$12{,}723.90 \Rightarrow$ the job cannot be done for \$11,000.

| | $x_i$ | $f(x_i)$ | m | $mf(x_i)$ |
|---|---|---|---|---|
| $x_0$ | 0 | 0 | 1 | 0 |
| $x_1$ | 15 | 36 | 4 | 144 |
| $x_2$ | 30 | 54 | 2 | 108 |
| $x_3$ | 45 | 51 | 4 | 204 |
| $x_4$ | 60 | 49.5 | 2 | 99 |
| $x_5$ | 75 | 54 | 4 | 216 |
| $x_6$ | 90 | 64.4 | 2 | 128.8 |
| $x_7$ | 105 | 67.5 | 4 | 270 |
| $x_8$ | 120 | 42 | 1 | 42 |

135. $\int_0^3 \frac{dx}{\sqrt{9-x^2}} = \lim_{b\to 3^-}\int_0^b \frac{dx}{\sqrt{9-x^2}} = \lim_{b\to 3^-}\left[\sin^{-1}\left(\frac{x}{3}\right)\right]_0^b = \lim_{b\to 3^-}\sin^{-1}\left(\frac{b}{3}\right) - \sin^{-1}\left(\frac{0}{3}\right) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

136. $\int_0^1 \ln x\,dx = \lim_{b\to 0^+}[x\ln x - x]_b^1 = (1\cdot\ln 1 - 1) - \lim_{b\to 0^+}[b\ln b - b] = -1 - \lim_{b\to 0^+}\frac{\ln b}{\left(\frac{1}{b}\right)} = -1 - \lim_{b\to 0^+}\frac{\left(\frac{1}{b}\right)}{\left(-\frac{1}{b^2}\right)}$
$= -1 + 0 = -1$

137. $\int_{-1}^1 \frac{dy}{y^{2/3}} = \int_{-1}^0 \frac{dy}{y^{2/3}} + \int_0^1 \frac{dy}{y^{2/3}} = 2\int_0^1 \frac{dy}{y^{2/3}} = 2\cdot 3\lim_{b\to 0^+}\left[y^{1/3}\right]_b^1 = 6\left(1 - \lim_{b\to 0^+} b^{1/3}\right) = 6$

138. $\int_{-2}^\infty \frac{d\theta}{(\theta+1)^{3/5}} = \int_{-2}^{-1}\frac{d\theta}{(\theta+1)^{3/5}} + \int_{-1}^2 \frac{d\theta}{(\theta+1)^{3/5}} + \int_2^\infty \frac{d\theta}{(\theta+1)^{3/5}}$ converges if each integral converges, but
$\lim_{\theta\to\infty}\frac{\theta^{3/5}}{(\theta+1)^{3/5}} = 1$ and $\int_2^\infty \frac{d\theta}{\theta^{3/5}}$ diverges $\Rightarrow \int_{-2}^\infty \frac{d\theta}{(\theta+1)^{3/5}}$ diverges

139. $\int_3^\infty \frac{2\,du}{u^2-2u} = \int_3^\infty \frac{du}{u-2} - \int_3^\infty \frac{du}{u} = \lim_{b\to\infty}\left[\ln\left|\frac{u-2}{u}\right|\right]_3^b = \lim_{b\to\infty}\left[\ln\left|\frac{b-2}{b}\right|\right] - \ln\left|\frac{3-2}{3}\right| = 0 - \ln\left(\frac{1}{3}\right) = \ln 3$

140. $\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv = \int_1^\infty \left(\frac{1}{v} + \frac{1}{v^2} - \frac{4}{4v-1}\right)dv = \lim_{b\to\infty}\left[\ln v - \frac{1}{v} - \ln(4v-1)\right]_1^b$
$= \lim_{b\to\infty}\left[\ln\left(\frac{b}{4b-1}\right) - \frac{1}{b}\right] - (\ln 1 - 1 - \ln 3) = \ln\frac{1}{4} + 1 + \ln 3 = 1 + \ln\frac{3}{4}$

141. $\int_0^\infty x^2e^{-x}\,dx = \lim_{b\to\infty}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^b = \lim_{b\to\infty}\left(-b^2e^{-b} - 2be^{-b} - 2e^{-b}\right) - (-2) = 0 + 2 = 2$

142. $\int_{-\infty}^0 xe^{3x}\,dx = \lim_{b\to-\infty}\left[\frac{x}{3}e^{3x} - \frac{1}{9}e^{3x}\right]_b^0 = -\frac{1}{9} - \lim_{b\to-\infty}\left(\frac{b}{3}e^{3b} - \frac{1}{9}e^{3b}\right) = -\frac{1}{9} - 0 = -\frac{1}{9}$

143. $\int_{-\infty}^\infty \frac{dx}{4x^2+9} = 2\int_0^\infty \frac{dx}{4x^2+9} = \frac{1}{2}\int_0^\infty \frac{dx}{x^2+\frac{9}{4}} = \frac{1}{2}\lim_{b\to\infty}\left[\frac{2}{3}\tan^{-1}\left(\frac{2x}{3}\right)\right]_0^b = \frac{1}{2}\lim_{b\to\infty}\left[\frac{2}{3}\tan^{-1}\left(\frac{2b}{3}\right)\right] - \frac{1}{3}\tan^{-1}(0)$

$= \frac{1}{2}\left(\frac{2}{3}\cdot\frac{\pi}{2}\right) - 0 = \frac{\pi}{6}$

144. $\int_{-\infty}^{\infty} \frac{4\,dx}{x^2+16} = 2\int_0^{\infty} \frac{4\,dx}{x^2+16} = 2 \lim\limits_{b\to\infty} \left[\tan^{-1}\left(\frac{x}{4}\right)\right]_0^b = 2\left(\lim\limits_{b\to\infty} \left[\tan^{-1}\left(\frac{b}{4}\right)\right] - \tan^{-1}(0)\right) = 2\left(\frac{\pi}{2}\right) - 0 = \pi$

145. $\lim\limits_{\theta\to\infty} \frac{\theta}{\sqrt{\theta^2+1}} = 1$ and $\int_6^{\infty} \frac{d\theta}{\theta}$ diverges $\Rightarrow \int_6^{\infty} \frac{d\theta}{\sqrt{\theta^2+1}}$ diverges

146. $I = \int_0^{\infty} e^{-u}\cos u\,du = \lim\limits_{b\to\infty} \left[-e^{-u}\cos u\right]_0^b - \int_0^{\infty} e^{-u}\sin u\,du = 1 + \lim\limits_{b\to\infty} \left[e^{-u}\sin u\right]_0^b - \int_0^{\infty} (e^{-u})\cos u\,du$
$\Rightarrow I = 1 + 0 - I \Rightarrow 2I = 1 \Rightarrow I = \frac{1}{2}$ converges

147. $\int_1^{\infty} \frac{\ln z}{z}\,dz = \int_1^{e} \frac{\ln z}{z}\,dz + \int_e^{\infty} \frac{\ln z}{z}\,dz = \left[\frac{(\ln z)^2}{2}\right]_1^e + \lim\limits_{b\to\infty} \left[\frac{(\ln z)^2}{2}\right]_e^b = \left(\frac{1^2}{2} - 0\right) + \lim\limits_{b\to\infty} \left[\frac{(\ln b)^2}{2} - \frac{1}{2}\right]$
$= \infty \Rightarrow$ diverges

148. $0 < \frac{e^{-t}}{\sqrt{t}} \le e^{-t}$ for $t \ge 1$ and $\int_1^{\infty} e^{-t}\,dt$ converges $\Rightarrow \int_1^{\infty} \frac{e^{-t}}{\sqrt{t}}\,dt$ converges

149. $\int_{-\infty}^{\infty} \frac{2\,dx}{e^x+e^{-x}} = 2\int_0^{\infty} \frac{2\,dx}{e^x+e^{-x}} < \int_0^{\infty} \frac{4\,dx}{e^x}$ converges $\Rightarrow \int_{-\infty}^{\infty} \frac{2\,dx}{e^x+e^{-x}}$ converges

150. $\int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)} = \int_{-\infty}^{-1} \frac{dx}{x^2(1+e^x)} + \int_{-1}^{0} \frac{dx}{x^2(1+e^x)} + \int_0^{1} \frac{dx}{x^2(1+e^x)} + \int_1^{\infty} \frac{dx}{x^2(1+e^x)}$;
$\lim\limits_{x\to 0} \frac{\left(\frac{1}{x^2}\right)}{\left[\frac{1}{x^2(1+e^x)}\right]} = \lim\limits_{x\to 0} \frac{x^2(1+e^x)}{x^2} = \lim\limits_{x\to 0} (1+e^x) = 2$ and $\int_0^1 \frac{dx}{x^2}$ diverges $\Rightarrow \int_0^1 \frac{dx}{x^2(1+e^x)}$ diverges
$\Rightarrow \int_{-\infty}^{\infty} \frac{dx}{x^2(1+e^x)}$ diverges

151. $\int \frac{x\,dx}{1+\sqrt{x}}$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{u^2\cdot 2u\,du}{1+u} = \int \left(2u^2 - 2u + 2 - \frac{2}{1+u}\right) du = \frac{2}{3}u^3 - u^2 + 2u - 2\ln|1+u| + C$
$= \frac{2x^{3/2}}{3} - x + 2\sqrt{x} - 2\ln\left(1+\sqrt{x}\right) + C$

152. $\int \frac{x^3+2}{4-x^2}\,dx = -\int \left(x + \frac{4x+2}{x^2-4}\right) dx = -\int x\,dx - \frac{3}{2}\int \frac{dx}{x+2} - \frac{5}{2}\int \frac{dx}{x-2} = -\frac{x^2}{2} - \frac{3}{2}\ln|x+2| - \frac{5}{2}\ln|x-2| + C$

153. $\int \frac{dx}{x(x^2+1)^2}$; $\begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\,d\theta \end{bmatrix} \to \int \frac{\sec^2\theta\,d\theta}{\tan\theta\sec^4\theta} = \int \frac{\cos^3\theta\,d\theta}{\sin\theta} = \int \left(\frac{1-\sin^2\theta}{\sin\theta}\right) d(\sin\theta)$
$= \ln|\sin\theta| - \frac{1}{2}\sin^2\theta + C = \ln\left|\frac{x}{\sqrt{x^2+1}}\right| - \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)^2 + C$

154. $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$; $\begin{bmatrix} u = \sqrt{x} \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{\cos u\cdot 2u\,du}{u} = 2\int \cos u\,du = 2\sin u + C = 2\sin\sqrt{x} + C$

155. $\int \frac{dx}{\sqrt{-2x-x^2}} = \int \frac{d(x+1)}{\sqrt{1-(x+1)^2}} = \sin^{-1}(x+1) + C$

156. $\int \frac{(t-1)\,dt}{\sqrt{t^2-2t}}$, $\begin{bmatrix} u = t^2-2t \\ du = (2t-2)\,dt = 2(t-1)\,dt \end{bmatrix} \to \frac{1}{2}\int \frac{du}{\sqrt{u}} = \sqrt{u} + C = \sqrt{t^2-2t} + C$

157. $\int \frac{du}{\sqrt{1+u^2}}$; $[u = \tan\theta] \to \int \frac{\sec^2\theta\,d\theta}{\sec\theta} = \ln|\sec\theta + \tan\theta| + C = \ln\left|\sqrt{1+u^2} + u\right| + C$

158. $\int e^t \cos e^t\, dt = \sin e^t + C$

159. $\int \frac{2-\cos x+\sin x}{\sin^2 x}\, dx = \int 2\csc^2 x\, dx - \int \frac{\cos x\, dx}{\sin^2 x} + \int \csc x\, dx = -2\cot x + \frac{1}{\sin x} - \ln|\csc x + \cot x| + C$
$= -2\cot x + \csc x - \ln|\csc x + \cot x| + C$

160. $\int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta = \int \frac{1-\cos^2\theta}{\cos^2\theta}\, d\theta = \int \sec^2\theta\, d\theta - \int d\theta = \tan\theta - \theta + C$

161. $\int \frac{9\, dv}{81-v^4} = \frac{1}{2}\int \frac{dv}{v^2+9} + \frac{1}{12}\int \frac{dv}{3-v} + \frac{1}{12}\int \frac{dv}{3+v} = \frac{1}{12}\ln\left|\frac{3+v}{3-v}\right| + \frac{1}{6}\tan^{-1}\frac{v}{3} + C$

162. $\int \frac{\cos x\, dx}{1+\sin^2 x} = \int \frac{d(\sin x)}{1+\sin^2 x} = \tan^{-1}(\sin x) + C$

163.
$$\begin{array}{lcl} & & \cos(2\theta+1) \\ \theta & \xrightarrow{(+)} & \frac{1}{2}\sin(2\theta+1) \\ 1 & \xrightarrow{(-)} & -\frac{1}{4}\cos(2\theta+1) \\ 0 & & \end{array}$$
$\Rightarrow \int \theta\cos(2\theta+1)\, d\theta = \frac{\theta}{2}\sin(2\theta+1) + \frac{1}{4}\cos(2\theta+1) + C$

164. $\int_2^\infty \frac{dx}{(x-1)^2} = \lim_{b\to\infty}\left[\frac{1}{1-x}\right]_2^b = \lim_{b\to\infty}\left[\frac{1}{1-b} - (-1)\right] = 0 + 1 = 1$

165. $\int \frac{x^3\, dx}{x^2-2x+1} = \int \left(x + 2 + \frac{3x-2}{x^2-2x+1}\right) dx = \int (x+2)\, dx + 3\int \frac{dx}{x-1} + \int \frac{dx}{(x-1)^2}$
$= \frac{x^2}{2} + 2x + 3\ln|x-1| - \frac{1}{x-1} + C$

166. $\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}}$; $\begin{bmatrix} x = 1+\sqrt{\theta} \\ dx = \frac{d\theta}{2\sqrt{\theta}} \\ d\theta = 2(x-1)\, dx \end{bmatrix} \to \int \frac{2(x-1)\, dx}{\sqrt{x}} = 2\int \sqrt{x}\, dx - 2\int \frac{dx}{\sqrt{x}} = \frac{4}{3}x^{3/2} - 4x^{1/2} + C$
$= \frac{4}{3}\left(1+\sqrt{\theta}\right)^{3/2} - 4\left(1+\sqrt{\theta}\right)^{1/2} + C = 4\left[\frac{\left(\sqrt{1+\sqrt{\theta}}\right)^3}{3} - \sqrt{1+\sqrt{\theta}}\right] + C$

167. $\int \frac{2\sin\sqrt{x}\, dx}{\sqrt{x}\sec\sqrt{x}}$; $\begin{bmatrix} y = \sqrt{x} \\ dy = \frac{dx}{2\sqrt{x}} \end{bmatrix} \to \int \frac{2\sin y \cdot 2y\, dy}{y\sec y} = \int 2\sin 2y\, dy = -\cos(2y) + C = -\cos\left(2\sqrt{x}\right) + C$

168. $\int \frac{x^5\, dx}{x^4-16} = \int \left(x + \frac{16x}{x^4-16}\right) dx = \frac{x^2}{2} + \int \left(\frac{2x}{x^2-4} - \frac{2x}{x^2+4}\right) dx = \frac{x^2}{2} + \ln\left|\frac{x^2-4}{x^2+4}\right| + C$

169. $\int \frac{dy}{\sin y\cos y} = \int \frac{2\, dy}{\sin 2y} = \int 2\csc(2y)\, dy = -\ln|\csc(2y) + \cot(2y)| + C$

170. $\int \frac{d\theta}{\theta^2-2\theta+4} = \int \frac{d\theta}{(\theta-1)^2+3} = \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{\theta-1}{\sqrt{3}}\right) + C$

171. $\int \frac{\tan x}{\cos^2 x}\, dx = \int \tan x\sec^2 x\, dx = \int \tan x \cdot d(\tan x) = \frac{1}{2}\tan^2 x + C$

172. $\int \frac{dr}{(r+1)\sqrt{r^2+2r}} = \int \frac{d(r+1)}{(r+1)\sqrt{(r+1)^2-1}} = \sec^{-1}|r+1| + C$

173. $\int \frac{(r+2)\,dr}{\sqrt{-r^2-4r}} = \int \frac{(r+2)\,dr}{\sqrt{4-(r+2)^2}}$ ; $\begin{bmatrix} u = 4-(r+2)^2 \\ du = -2(r+2)\,dr \end{bmatrix} \rightarrow -\int \frac{du}{2\sqrt{u}} = -\sqrt{u} + C = -\sqrt{4-(r+2)^2} + C$

174. $\int \frac{y\,dy}{4+y^4} = \frac{1}{2}\int \frac{d(y^2)}{4+(y^2)^2} = \frac{1}{4}\tan^{-1}\left(\frac{y^2}{2}\right) + C$

175. $\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2} = -\frac{1}{2}\int \frac{d(1+\cos 2\theta)}{(1+\cos 2\theta)^2} = \frac{1}{2(1+\cos 2\theta)} + C = \frac{1}{4}\sec^2\theta + C$

176. $\int \frac{dx}{(x^2-1)^2} = \int \frac{dx}{(1-x^2)^2} = \frac{x}{2(1-x^2)} + \frac{1}{4}\ln\left|\frac{x+1}{x-1}\right| + C$ (FORMULA 19)

177. $\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx = -\sqrt{2}\int_{\pi/4}^{\pi/2} \cos 2x\,dx = \left[-\frac{\sqrt{2}}{2}\sin 2x\right]_{\pi/4}^{\pi/2} = \frac{\sqrt{2}}{2}$

178. $\int (15)^{2x+1}\,dx = \frac{1}{2}\int (15)^{2x+1}\,d(2x+1) = \frac{1}{2}\left(\frac{15^{2x+1}}{\ln 15}\right) + C$

179. $\int \frac{x\,dx}{\sqrt{2-x}}$ ; $\begin{bmatrix} y = 2-x \\ dy = -dx \end{bmatrix} \rightarrow -\int \frac{(2-y)\,dy}{\sqrt{y}} = \frac{2}{3}y^{3/2} - 4y^{1/2} + C = \frac{2}{3}(2-x)^{3/2} - 4(2-x)^{1/2} + C$
$= 2\left[\frac{\left(\sqrt{2-x}\right)^3}{3} - 2\sqrt{2-x}\right] + C$

180. $\int \frac{\sqrt{1-v^2}}{v^2}\,dv$; $[v = \sin\theta] \rightarrow \int \frac{\cos\theta\cdot\cos\theta\,d\theta}{\sin^2\theta} = \int \frac{(1-\sin^2\theta)\,d\theta}{\sin^2\theta} = \int \csc^2\theta\,d\theta - \int d\theta = \cot\theta - \theta + C$
$= -\sin^{-1}v - \frac{\sqrt{1-v^2}}{v} + C$

181. $\int \frac{dy}{y^2-2y+2} = \int \frac{d(y-1)}{(y-1)^2+1} = \tan^{-1}(y-1) + C$

182. $\int \ln\sqrt{x-1}\,dx$; $\begin{bmatrix} y = \sqrt{x-1} \\ dy = \frac{dx}{2\sqrt{x-1}} \end{bmatrix} \rightarrow \int \ln y \cdot 2y\,dy$; $u = \ln y$, $du = \frac{dy}{y}$ ; $dv = 2y\,dy$, $v = y^2$
$\Rightarrow \int 2y\ln y\,dy = y^2\ln y - \int y\,dy = y^2\ln y - \frac{1}{2}y^2 + C = (x-1)\ln\sqrt{x-1} - \frac{1}{2}(x-1) + C_1$
$= \frac{1}{2}[(x-1)\ln|x-1| - x] + \left(C_1 + \frac{1}{2}\right) = \frac{1}{2}[x\ln|x-1| - x - \ln|x-1|] + C$

183. $\int \theta^2\tan(\theta^3)\,d\theta = \frac{1}{3}\int \tan(\theta^3)\,d(\theta^3) = \frac{1}{3}\ln|\sec\theta^3| + C$

184. $\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}} = \frac{1}{2}\int \frac{d(x^2+1)}{\sqrt{9-(x^2+1)^2}} = \frac{1}{2}\sin^{-1}\left(\frac{x^2+1}{3}\right) + C$

185. $\int \frac{z+1}{z^2(z^2+4)}\,dz = \frac{1}{4}\int \left(\frac{1}{z} + \frac{1}{z^2} - \frac{z+1}{z^2+4}\right)dz = \frac{1}{4}\ln|z| - \frac{1}{4z} - \frac{1}{8}\ln(z^2+4) - \frac{1}{8}\tan^{-1}\frac{z}{2} + C$

186. $\int x^3 e^{x^2}\,dx = \frac{1}{2}\int x^2 e^{x^2}\,d(x^2) = \frac{1}{2}\left(x^2e^{x^2} - e^{x^2}\right) + C = \frac{(x^2-1)e^{x^2}}{2} + C$

187. $\int \frac{t\,dt}{\sqrt{9-4t^2}} = -\frac{1}{8}\int \frac{d(9-4t^2)}{\sqrt{9-4t^2}} = -\frac{1}{4}\sqrt{9-4t^2} + C$

188. $\int_0^{\pi/10} \sqrt{1+\cos 5\theta}\,d\theta = \sqrt{2}\int_0^{\pi/10} \cos\left(\frac{5\theta}{2}\right)d\theta = \frac{2\sqrt{2}}{5}\left[\sin\left(\frac{5\theta}{2}\right)\right]_0^{\pi/10} = \frac{2\sqrt{2}}{5}\left(\sin\frac{\pi}{4} - 0\right) = \frac{2}{5}$

189. $\int \frac{\cot\theta\, d\theta}{1+\sin^2\theta} = \int \frac{\cos\theta\, d\theta}{(\sin\theta)(1+\sin^2\theta)}$ ; $\left[\begin{array}{c} x = \sin\theta \\ dx = \cos\theta\, d\theta \end{array}\right] \rightarrow \int \frac{dx}{x(1+x^2)} = \int \frac{dx}{x} - \int \frac{x\, dx}{x^2+1}$
$= \ln|\sin\theta| - \frac{1}{2}\ln(1+\sin^2\theta) + C$

190. $u = \tan^{-1} x$, $du = \frac{dx}{1+x^2}$ ; $dv = \frac{dx}{x^2}$ , $v = -\frac{1}{x}$ ;
$\int \frac{\tan^{-1} x\, dx}{x^2} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x(1+x^2)} = -\frac{1}{x}\tan^{-1} x + \int \frac{dx}{x} - \int \frac{x\, dx}{1+x^2}$
$= -\frac{1}{x}\tan^{-1} x + \ln|x| - \frac{1}{2}\ln(1+x^2) + C = -\frac{\tan^{-1} x}{x} + \ln|x| - \ln\sqrt{1+x^2} + C$

191. $\int \frac{\tan\sqrt{y}\, dy}{2\sqrt{y}}$ ; $\left[\sqrt{y} = x\right] \rightarrow \int \frac{\tan x \cdot 2x\, dx}{2x} = \ln|\sec x| + C = \ln\left|\sec\sqrt{y}\right| + C$

192. $\int \frac{e^t\, dt}{e^{2t}+3e^t+2}$ ; $[e^t = x] \rightarrow \int \frac{dx}{(x+1)(x+2)} = \int \frac{dx}{x+1} - \int \frac{dx}{x+2} = \ln|x+1| - \ln|x+2| + C$
$= \ln\left|\frac{x+1}{x+2}\right| + C = \ln\left(\frac{e^t+1}{e^t+2}\right) + C$

193. $\int \frac{\theta^2\, d\theta}{4-\theta^2} = \int \left(-1 + \frac{4}{4-\theta^2}\right) d\theta = -\int d\theta - \int \frac{d\theta}{\theta-2} + \int \frac{d\theta}{\theta+2} = -\theta - \ln|\theta-2| + \ln|\theta+2| + C$
$= -\theta + \ln\left|\frac{\theta+2}{\theta-2}\right| + C$

194. $\int \frac{1-\cos 2x}{1+\cos 2x}\, dx = \int \tan^2 x\, dx = \int (\sec^2 x - 1)\, dx = \tan x - x + C$

195. $\int \frac{\cos(\sin^{-1} x)\, dx}{\sqrt{1-x^2}}$ ; $\left[\begin{array}{c} u = \sin^{-1} x \\ du = \frac{dx}{\sqrt{1-x^2}} \end{array}\right] \rightarrow \int \cos u\, du = \sin u + C = \sin(\sin^{-1} x) + C = x + C$

196. $\int \frac{\cos x\, dx}{\sin^3 x - \sin x} = -\int \frac{\cos x\, dx}{(\sin x)(1-\sin^2 x)} = -\int \frac{\cos x\, dx}{(\sin x)(\cos^2 x)} = -\int \frac{2\, dx}{\sin 2x} = -2\int \csc 2x\, dx$
$= \ln|\csc(2x) + \cot(2x)| + C$

197. $\int \sin\frac{x}{2}\cos\frac{x}{2}\, dx = \int \frac{1}{2}\sin\left(\frac{x}{2}+\frac{x}{2}\right) dx = \frac{1}{2}\int \sin x\, dx = -\frac{1}{2}\cos x + C$

198. $\int \frac{x^2-x+2}{(x^2+2)^2}\, dx = \int \frac{dx}{x^2+2} - \int \frac{x\, dx}{(x^2+2)^2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2}(x^2+2)^{-1} + C$
$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{2(x^2+2)} + C$

199. $\int \frac{e^t\, dt}{1+e^t} = \ln(1+e^t) + C$

200. $\int \tan^3 t\, dt = \int (\tan t)(\sec^2 t - 1)\, dt = \frac{\tan^2 t}{2} - \int \tan t\, dt = \frac{\tan^2 t}{2} - \ln|\sec t| + C$

201. $\int_1^\infty \frac{\ln y\, dy}{y^3}$ ; $\left[\begin{array}{c} x = \ln y \\ dx = \frac{dy}{y} \\ dy = e^x\, dx \end{array}\right] \rightarrow \int_0^\infty \frac{x \cdot e^x}{e^{3x}}\, dx = \int_0^\infty x e^{-2x}\, dx = \lim_{b\to\infty}\left[-\frac{x}{2}e^{-2x} - \frac{1}{4}e^{-2x}\right]_0^b$
$= \lim_{b\to\infty}\left(\frac{-b}{2e^{2b}} - \frac{1}{4e^{2b}}\right) - \left(0 - \frac{1}{4}\right) = \frac{1}{4}$

202. $\int \frac{3+\sec^2 x+\sin x}{\tan x}\, dx = 3\int \cot x\, dx + \int \frac{\sec^2 x\, dx}{\tan x} + \int \cos x\, dx = 3\ln|\sin x| + \ln|\tan x| + \sin x + C$

203. $\int \frac{\cot v\, dv}{\ln(\sin v)} = \int \frac{\cos v\, dv}{(\sin v)\ln(\sin v)}$ ; $\left[\begin{array}{c} u = \ln(\sin v) \\ du = \frac{\cos v\, dv}{\sin v} \end{array}\right] \rightarrow \int \frac{du}{u} = \ln|u| + C = \ln|\ln(\sin v)| + C$

204. $\int \frac{dx}{(2x-1)\sqrt{x^2-x}} = \int \frac{2\,dx}{(2x-1)\sqrt{4x^2-4x}} = \int \frac{2\,dx}{(2x-1)\sqrt{(2x-1)^2-1}}$ ; $\begin{bmatrix} u = 2x-1 \\ du = 2\,dx \end{bmatrix} \rightarrow \int \frac{du}{u\sqrt{u^2-1}}$
$= \sec^{-1}|u| + C = \sec^{-1}|2x-1| + C$

205. $\int e^{\ln\sqrt{x}}\,dx = \int \sqrt{x}\,dx = \frac{2}{3}x^{3/2} + C$

206. $\int e^\theta\sqrt{3+4e^\theta}\,d\theta$; $\begin{bmatrix} u = 4e^\theta \\ du = 4e^\theta\,d\theta \end{bmatrix} \rightarrow \frac{1}{4}\int\sqrt{3+u}\,du = \frac{1}{4}\cdot\frac{2}{3}(3+u)^{3/2} + C = \frac{1}{6}\left(3+4e^\theta\right)^{3/2} + C$

207. $\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2}$ ; $\begin{bmatrix} u = \cos 5t \\ du = -5\sin 5t\,dt \end{bmatrix} \rightarrow -\frac{1}{5}\int\frac{du}{1+u^2} = -\frac{1}{5}\tan^{-1}u + C = -\frac{1}{5}\tan^{-1}(\cos 5t) + C$

208. $\int \frac{dv}{\sqrt{e^{2v}-1}}$ ; $\begin{bmatrix} x = e^v \\ dx = e^v\,dv \end{bmatrix} \rightarrow \int\frac{dx}{x\sqrt{x^2-1}} = \sec^{-1}x + C = \sec^{-1}(e^v) + C$

209. $\int (27)^{3\theta+1}\,d\theta = \frac{1}{3}\int(27)^{3\theta+1}\,d(3\theta+1) = \frac{1}{3\ln 27}(27)^{3\theta+1} + C = \frac{1}{3}\left(\frac{27^{3\theta+1}}{\ln 27}\right) + C$

210.
$$\begin{array}{lcl} & & \sin x \\ x^5 & \xrightarrow{(+)} & -\cos x \\ 5x^4 & \xrightarrow{(-)} & -\sin x \\ 20x^3 & \xrightarrow{(+)} & \cos x \\ 60x^2 & \xrightarrow{(-)} & \sin x \\ 120x & \xrightarrow{(+)} & -\cos x \\ 120 & \xrightarrow{(-)} & -\sin x \\ 0 & & \end{array}$$
$\Rightarrow \int x^5\sin x\,dx = -x^5\cos x + 5x^4\sin x + 20x^3\cos x - 60x^2\sin x - 120x\cos x$
$+ 120\sin x + C$

211. $\int \frac{dr}{1+\sqrt{r}}$ ; $\begin{bmatrix} u = \sqrt{r} \\ du = \frac{dr}{2\sqrt{r}} \end{bmatrix} \rightarrow \int\frac{2u\,du}{1+u} = \int\left(2 - \frac{2}{1+u}\right)du = 2u - 2\ln|1+u| + C = 2\sqrt{r} - 2\ln\left(1+\sqrt{r}\right) + C$

212. $\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx = \int\frac{d\left(x^4-10x^2+9\right)}{x^4-10x^2+9} = \ln|x^4-10x^2+9| + C$

213. $\int \frac{8\,dy}{y^3(y+2)} = \int\frac{dy}{y} - \int\frac{2\,dy}{y^2} + \int\frac{4\,dy}{y^3} - \int\frac{dy}{(y+2)} = \ln\left|\frac{y}{y+2}\right| + \frac{2}{y} - \frac{2}{y^2} + C$

214. $\int \frac{(t+1)\,dt}{(t^2+2t)^{2/3}}$ ; $\begin{bmatrix} u = t^2+2t \\ du = 2(t+1)\,dt \end{bmatrix} \rightarrow \frac{1}{2}\int\frac{du}{u^{2/3}} = \frac{1}{2}\cdot 3u^{1/3} + C = \frac{3}{2}(t^2+2t)^{1/3} + C$

215. $\int \frac{8\,dm}{m\sqrt{49m^2-4}} = \frac{8}{7}\int\frac{dm}{m\sqrt{m^2-\left(\frac{2}{7}\right)^2}} = 4\sec^{-1}\left|\frac{7m}{2}\right| + C$

216. $\int \frac{dt}{t(1+\ln t)\sqrt{(\ln t)(2+\ln t)}}$ ; $\begin{bmatrix} u = \ln t \\ du = \frac{dt}{t} \end{bmatrix} \rightarrow \int\frac{du}{(1+u)\sqrt{u(2+u)}} = \int\frac{du}{(u+1)\sqrt{(u+1)^2-1}}$
$= \sec^{-1}|u+1| + C = \sec^{-1}|\ln t + 1| + C$

217. If $u = \int_0^x \sqrt{1+(t-1)^4}\, dt$ and $dv = 3(x-1)^2\, dx$, then $du = \sqrt{1+(x-1)^4}\, dx$, and $v = (x-1)^3$ so integration by parts $\Rightarrow \int_0^1 3(x-1)^2 \left[\int_0^x \sqrt{1+(t-1)^4}\, dt\right] dx = \left[(x-1)^3 \int_0^x \sqrt{1+(t-1)^4}\, dt\right]_0^1$
$- \int_0^1 (x-1)^3 \sqrt{1+(x-1)^4}\, dx = \left[-\frac{1}{6}\left(1+(x-1)^4\right)^{3/2}\right]_0^1 = \frac{\sqrt{8}-1}{6}$

218. $\frac{4v^3+v-1}{v^2(v-1)(v^2+1)} = \frac{A}{v} + \frac{B}{v^2} + \frac{C}{v-1} + \frac{Dv+E}{v^2+1} \Rightarrow 4v^3+v-1$
$= Av(v-1)(v^2+1) + B(v-1)(v^2+1) + Cv^2(v^2+1) + (Dv+E)(v^2)(v-1)$
$v = 0$: $-1 = -B \Rightarrow B = 1$;
$v = 1$: $4 = 2C \Rightarrow C = 2$;
coefficient of $v^4$: $0 = A + C + D \Rightarrow A + D = -2$;
coefficient of $v^3$: $4 = -A + B + E - D$
coefficient of $v^2$: $0 = A - B + C - E \Rightarrow C - D = 4 \Rightarrow D = -2$ (summing with previous equation);
coefficient of v: $1 = -A + B \Rightarrow A = 0$;
in summary: $A = 0, B = 1, C = 2, D = -2$ and $E = 1$
$\Rightarrow \int_2^\infty \frac{4v^3+v-1}{v^2(v-1)(v^2+1)}\, dv = \lim_{b\to\infty} \int_2^b \left(\frac{2}{v-1} + v^{-2} + \frac{1}{1+v^2} - \frac{2v}{1+v^2}\right) dv$
$= \lim_{b\to\infty} \left[\ln(v-1)^2 - \frac{1}{v} + \tan^{-1} v - \ln(1+v^2)\right]_2^b$
$= \lim_{b\to\infty} \left[\ln\left(\frac{(b-1)^2}{1+b^2}\right) - \frac{1}{b} + \tan^{-1} b\right] - \left(\ln 1 - \frac{1}{2} + \tan^{-1} 2 - \ln 5\right) = \left(0 - 0 + \frac{\pi}{2}\right) - \left(0 - \frac{1}{2} + \tan^{-1} 2 - \ln 5\right)$
$= \frac{\pi}{2} + \ln(5) + \frac{1}{2} - \tan^{-1} 2$

219. $u = f(x)$, $du = f'(x)\, dx$; $dv = dx$, $v = x$;
$\int_{\pi/2}^{3\pi/2} f(x)\, dx = [x\, f(x)]_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} x f'(x)\, dx = \left[\frac{3\pi}{2} f\left(\frac{3\pi}{2}\right) - \frac{\pi}{2} f\left(\frac{\pi}{2}\right)\right] - \int_{\pi/2}^{3\pi/2} \cos x\, dx$
$= \left(\frac{3\pi b}{2} - \frac{\pi a}{2}\right) - [\sin x]_{\pi/2}^{3\pi/2} = \frac{\pi}{2}(3b-a) - [(-1)-1] = \frac{\pi}{2}(3b-a) + 2$

220. $\int_0^a \frac{dx}{1+x^2} = [\tan^{-1} x]_0^a = \tan^{-1} a$; $\int_a^\infty \frac{dx}{1+x^2} = \lim_{b\to\infty} [\tan^{-1} x]_a^b = \lim_{b\to\infty} (\tan^{-1} b - \tan^{-1} a) = \frac{\pi}{2} - \tan^{-1} a$;
therefore, $\tan^{-1} a = \frac{\pi}{2} - \tan^{-1} a \Rightarrow \tan^{-1} a = \frac{\pi}{4} \Rightarrow a = 1$ since $a > 0$.

## CHAPTER 8 ADDITIONAL AND ADVANCED EXERCISES

1. $u = (\sin^{-1} x)^2$, $du = \frac{2\sin^{-1} x\, dx}{\sqrt{1-x^2}}$; $dv = dx$, $v = x$;
$\int (\sin^{-1} x)^2\, dx = x(\sin^{-1} x)^2 - \int \frac{2x \sin^{-1} x\, dx}{\sqrt{1-x^2}}$;
$u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = -\frac{2x\, dx}{\sqrt{1-x^2}}$, $v = 2\sqrt{1-x^2}$;
$-\int \frac{2x \sin^{-1} x\, dx}{\sqrt{1-x^2}} = 2(\sin^{-1} x)\sqrt{1-x^2} - \int 2\, dx = 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C$; therefore
$\int (\sin^{-1} x)^2\, dx = x(\sin^{-1} x)^2 + 2(\sin^{-1} x)\sqrt{1-x^2} - 2x + C$

2. $\frac{1}{x} = \frac{1}{x}$,
$\frac{1}{x(x+1)} = \frac{1}{x} - \frac{1}{x+1}$,
$\frac{1}{x(x+1)(x+2)} = \frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)}$,
$\frac{1}{x(x+1)(x+2)(x+3)} = \frac{1}{6x} - \frac{1}{2(x+1)} + \frac{1}{2(x+2)} - \frac{1}{6(x+3)}$,
$\frac{1}{x(x+1)(x+2)(x+3)(x+4)} = \frac{1}{24x} - \frac{1}{6(x+1)} + \frac{1}{4(x+2)} - \frac{1}{6(x+3)} + \frac{1}{24(x+4)} \Rightarrow$ the following pattern:
$\frac{1}{x(x+1)(x+2)\cdots(x+m)} = \sum_{k=0}^{m} \frac{(-1)^k}{(k!)(m-k)!(x+k)}$; therefore $\int \frac{dx}{x(x+1)(x+2)\cdots(x+m)}$

$= \sum_{k=0}^{m} \left[\frac{(-1)^k}{(k!)(m-k)!} \ln|x+k|\right] + C$

3. $u = \sin^{-1} x$, $du = \frac{dx}{\sqrt{1-x^2}}$; $dv = x\,dx$, $v = \frac{x^2}{2}$;
$\int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{x^2\,dx}{2\sqrt{1-x^2}}$; $\begin{bmatrix} x = \sin\theta \\ dx = \cos\theta\,d\theta \end{bmatrix} \rightarrow \int x\sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \int \frac{\sin^2\theta\cos\theta\,d\theta}{2\cos\theta}$
$= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \sin^2\theta\,d\theta = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right) + C = \frac{x^2}{2}\sin^{-1} x + \frac{\sin\theta\cos\theta - \theta}{4} + C$
$= \frac{x^2}{2}\sin^{-1} x + \frac{x\sqrt{1-x^2} - \sin^{-1} x}{4} + C$

4. $\int \sin^{-1}\sqrt{y}\,dy$; $\begin{bmatrix} z = \sqrt{y} \\ dz = \frac{dy}{2\sqrt{y}} \end{bmatrix} \rightarrow \int 2z\sin^{-1} z\,dz$; from Exercise 3, $\int z\sin^{-1} z\,dz$
$= \frac{z^2\sin^{-1} z}{2} + \frac{z\sqrt{1-z^2} - \sin^{-1} z}{4} + C \Rightarrow \int \sin^{-1}\sqrt{y}\,dy = y\sin^{-1}\sqrt{y} + \frac{\sqrt{y}\sqrt{1-y} - \sin^{-1}\sqrt{y}}{2} + C$
$= y\sin^{-1}\sqrt{y} + \frac{\sqrt{y-y^2}}{2} - \frac{\sin^{-1}\sqrt{y}}{2} + C$

5. $\int \frac{d\theta}{1-\tan^2\theta} = \int \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta}\,d\theta = \int \frac{1+\cos 2\theta}{2\cos 2\theta}\,d\theta = \frac{1}{2}\int (\sec 2\theta + 1)\,d\theta = \frac{\ln|\sec 2\theta + \tan 2\theta| + 2\theta}{4} + C$

6. $u = \ln\left(\sqrt{x} + \sqrt{1+x}\right)$, $du = \left(\frac{dx}{\sqrt{x}+\sqrt{1+x}}\right)\left(\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{1+x}}\right) = \frac{dx}{2\sqrt{x}\sqrt{1+x}}$; $dv = dx$, $v = x$;
$\int \ln\left(\sqrt{x}+\sqrt{1+x}\right)dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{1}{2}\int \frac{x\,dx}{\sqrt{x}\sqrt{1+x}}$; $\frac{1}{2}\int \frac{x\,dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 - \frac{1}{4}}}$;
$\begin{bmatrix} x + \frac{1}{2} = \frac{1}{2}\sec\theta \\ dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta \end{bmatrix} \rightarrow \frac{1}{4}\int \frac{(\sec\theta - 1)\cdot\sec\theta\tan\theta\,d\theta}{\left(\frac{1}{2}\tan\theta\right)} = \frac{1}{2}\int (\sec^2\theta - \sec\theta)\,d\theta$
$= \frac{\tan\theta - \ln|\sec\theta + \tan\theta|}{2} + C = \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{2} + C$
$\Rightarrow \int \ln\left(\sqrt{x}+\sqrt{1+x}\right)dx = x\ln\left(\sqrt{x}+\sqrt{1+x}\right) - \frac{2\sqrt{x^2+x} - \ln\left|2x+1+2\sqrt{x^2+x}\right|}{4} + C$

7. $\int \frac{dt}{t - \sqrt{1-t^2}}$; $\begin{bmatrix} t = \sin\theta \\ dt = \cos\theta\,d\theta \end{bmatrix} \rightarrow \int \frac{\cos\theta\,d\theta}{\sin\theta - \cos\theta} = \int \frac{d\theta}{\tan\theta - 1}$; $\begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\,d\theta \\ d\theta = \frac{du}{u^2+1} \end{bmatrix} \rightarrow \int \frac{du}{(u-1)(u^2+1)}$
$= \frac{1}{2}\int \frac{du}{u-1} - \frac{1}{2}\int \frac{du}{u^2+1} - \frac{1}{2}\int \frac{u\,du}{u^2+1} = \frac{1}{2}\ln\left|\frac{u-1}{\sqrt{u^2+1}}\right| - \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\ln\left|\frac{\tan\theta - 1}{\sec\theta}\right| - \frac{1}{2}\theta + C$
$= \frac{1}{2}\ln\left(t - \sqrt{1-t^2}\right) - \frac{1}{2}\sin^{-1} t + C$

8. $\int \frac{(2e^{2x} - e^x)\,dx}{\sqrt{3e^{2x} - 6e^x - 1}}$; $\begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \rightarrow \int \frac{(2u-1)\,du}{\sqrt{3u^2 - 6u - 1}} = \frac{1}{\sqrt{3}}\int \frac{(2u-1)\,du}{\sqrt{(u-1)^2 - \frac{4}{3}}}$;
$\begin{bmatrix} u - 1 = \frac{2}{\sqrt{3}}\sec\theta \\ du = \frac{2}{\sqrt{3}}\sec\theta\tan\theta\,d\theta \end{bmatrix} \rightarrow \frac{1}{\sqrt{3}}\int \left(\frac{4}{\sqrt{3}}\sec\theta + 1\right)(\sec\theta)\,d\theta = \frac{4}{3}\int \sec^2\theta\,d\theta + \frac{1}{\sqrt{3}}\int \sec\theta\,d\theta$
$= \frac{4}{3}\tan\theta + \frac{1}{\sqrt{3}}\ln|\sec\theta + \tan\theta| + C_1 = \frac{4}{3}\cdot\sqrt{\frac{3}{4}(u-1)^2 - 1} + \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}}{2}(u-1) + \sqrt{\frac{3}{4}(u-1)^2 - 1}\right| + C_1$
$= \frac{2}{3}\sqrt{3u^2 - 6u - 1} + \frac{1}{\sqrt{3}}\ln\left|u - 1 + \sqrt{(u-1)^2 - \frac{4}{3}}\right| + \left(C_1 + \frac{1}{\sqrt{3}}\ln\frac{\sqrt{3}}{2}\right)$
$= \frac{1}{\sqrt{3}}\left[2\sqrt{e^{2x} - 2e^x - \frac{1}{3}} + \ln\left|e^x - 1 + \sqrt{e^{2x} - 2e^x - \frac{1}{3}}\right|\right] + C$

9. $\int \frac{1}{x^4+4}\,dx = \int \frac{1}{(x^2+2)^2 - 4x^2}\,dx = \int \frac{1}{(x^2+2x+2)(x^2-2x+2)}\,dx$
$= \frac{1}{16}\int \left[\frac{2x+2}{x^2+2x+2} + \frac{2}{(x+1)^2+1} - \frac{2x-2}{x^2-2x+2} + \frac{2}{(x-1)^2+1}\right]dx$

$= \frac{1}{16} \ln \left| \frac{x^2+2x+2}{x^2-2x+2} \right| + \frac{1}{8} \left[ \tan^{-1}(x+1) + \tan^{-1}(x-1) \right] + C$

10. $\int \frac{1}{x^6-1}\, dx = \frac{1}{6} \int \left( \frac{1}{x-1} - \frac{1}{x+1} + \frac{x-2}{x^2-x+1} - \frac{x+2}{x^2+x+1} \right) dx$
$= \frac{1}{6} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{12} \int \left[ \frac{2x-1}{x^2-x+1} - \frac{3}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{2x+1}{x^2+x+1} - \frac{3}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}} \right] dx$
$= \frac{1}{6} \ln \left| \frac{x-1}{x+1} \right| + \frac{1}{12} \left[ \ln \left| \frac{x^2-x+1}{x^2+x+1} \right| - 2\sqrt{3} \tan^{-1}\left( \frac{2x-1}{\sqrt{3}} \right) - 2\sqrt{3} \tan^{-1}\left( \frac{2x+1}{\sqrt{3}} \right) \right] + C$

11. $\lim\limits_{x \to \infty} \int_{-x}^{x} \sin t\, dt = \lim\limits_{x \to \infty} \left[ -\cos t \right]_{-x}^{x} = \lim\limits_{x \to \infty} \left[ -\cos x + \cos(-x) \right] = \lim\limits_{x \to \infty} (-\cos x + \cos x) = \lim\limits_{x \to \infty} 0 = 0$

12. $\lim\limits_{x \to 0^+} \int_x^1 \frac{\cos t}{t^2}\, dt$; $\lim\limits_{t \to 0^+} \frac{\left(\frac{1}{t^2}\right)}{\left(\frac{\cos t}{t^2}\right)} = \lim\limits_{t \to 0^+} \frac{1}{\cos t} = 1 \Rightarrow \lim\limits_{x \to 0^+} \int_x^1 \frac{\cos t}{t^2}\, dt$ diverges since $\int_0^1 \frac{dt}{t^2}$ diverges; thus
$\lim\limits_{x \to 0^+} x \int_x^1 \frac{\cos t}{t^2}\, dt$ is an indeterminate $0 \cdot \infty$ form and we apply l'Hôpital's rule:
$\lim\limits_{x \to 0^+} x \int_x^1 \frac{\cos t}{t^2}\, dt = \lim\limits_{x \to 0^+} \frac{-\int_1^x \frac{\cos t}{t^2}\, dt}{\frac{1}{x}} = \lim\limits_{x \to 0^+} \frac{-\left(\frac{\cos x}{x^2}\right)}{\left(-\frac{1}{x^2}\right)} = \lim\limits_{x \to 0^+} \cos x = 1$

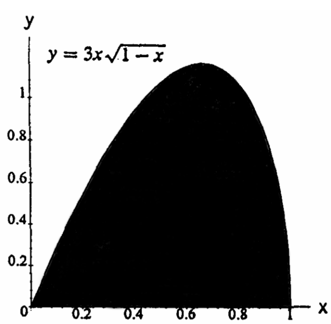
13. $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \ln \sqrt[n]{1+\frac{k}{n}} = \lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \ln\left(1 + k\left(\frac{1}{n}\right)\right)\left(\frac{1}{n}\right) = \int_0^1 \ln(1+x)\, dx$; $\left[ \begin{array}{c} u = 1+x,\ du = dx \\ x = 0 \Rightarrow u = 1,\ x = 1 \Rightarrow u = 2 \end{array} \right]$
$\to \int_1^2 \ln u\, du = \left[ u \ln u - u \right]_1^2 = (2 \ln 2 - 2) - (\ln 1 - 1) = 2 \ln 2 - 1 = \ln 4 - 1$

14. $\lim\limits_{n \to \infty} \sum\limits_{k=0}^{n-1} \frac{1}{\sqrt{n^2-k^2}} = \lim\limits_{n \to \infty} \sum\limits_{k=0}^{n-1} \left( \frac{n}{\sqrt{n^2-k^2}} \right) \left( \frac{1}{n} \right) = \lim\limits_{n \to \infty} \sum\limits_{k=0}^{n-1} \left( \frac{1}{\sqrt{1 - \left[ k\left(\frac{1}{n}\right) \right]^2}} \right) \left( \frac{1}{n} \right)$
$= \int_0^1 \frac{1}{\sqrt{1-x^2}}\, dx = \left[ \sin^{-1} x \right]_0^1 = \frac{\pi}{2}$

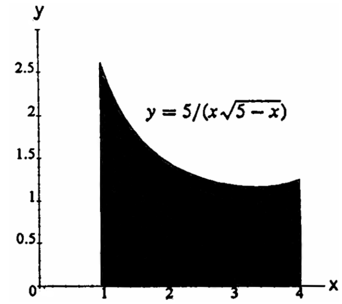
15. $\frac{dy}{dx} = \sqrt{\cos 2x} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \cos 2x = 2\cos^2 x$; $L = \int_0^{\pi/4} \sqrt{1 + \left(\sqrt{\cos 2t}\right)^2}\, dt = \sqrt{2} \int_0^{\pi/4} \sqrt{\cos^2 t}\, dt$
$= \sqrt{2} \left[ \sin t \right]_0^{\pi/4} = 1$

16. $\frac{dy}{dx} = \frac{-2x}{1-x^2} \Rightarrow 1 + \left(\frac{dy}{dx}\right)^2 = \frac{\left(1-x^2\right)^2 + 4x^2}{\left(1-x^2\right)^2} = \frac{1+2x^2+x^4}{\left(1-x^2\right)^2} = \left(\frac{1+x^2}{1-x^2}\right)^2$; $L = \int_0^{1/2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$
$= \int_0^{1/2} \left( \frac{1+x^2}{1-x^2} \right) dx = \int_0^{1/2} \left( -1 + \frac{2}{1-x^2} \right) dx = \int_0^{1/2} \left( -1 + \frac{1}{1+x} + \frac{1}{1-x} \right) dx = \left[ -x + \ln \left| \frac{1+x}{1-x} \right| \right]_0^{1/2}$
$= \left( -\frac{1}{2} + \ln 3 \right) - (0 + \ln 1) = \ln 3 - \frac{1}{2}$

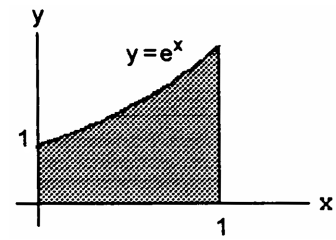
17. $V = \int_a^b 2\pi \left( \begin{smallmatrix} \text{shell} \\ \text{radius} \end{smallmatrix} \right) \left( \begin{smallmatrix} \text{shell} \\ \text{height} \end{smallmatrix} \right) dx = \int_0^1 2\pi xy\, dx$
$= 6\pi \int_0^1 x^2 \sqrt{1-x}\, dx$; $\left[ \begin{array}{c} u = 1 - x \\ du = -dx \\ x^2 = (1-u)^2 \end{array} \right]$
$\to -6\pi \int_1^0 (1-u)^2 \sqrt{u}\, du$
$= -6\pi \int_1^0 \left( u^{1/2} - 2u^{3/2} + u^{5/2} \right) du$
$= -6\pi \left[ \frac{2}{3} u^{3/2} - \frac{4}{5} u^{5/2} + \frac{2}{7} u^{7/2} \right]_1^0 = 6\pi \left( \frac{2}{3} - \frac{4}{5} + \frac{2}{7} \right)$
$= 6\pi \left( \frac{70 - 84 + 30}{105} \right) = 6\pi \left( \frac{16}{105} \right) = \frac{32\pi}{35}$



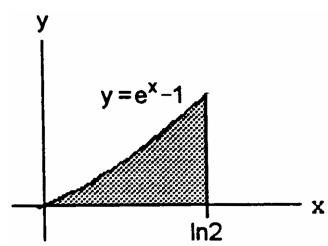
18. $V = \int_a^b \pi y^2\, dx = \pi \int_1^4 \frac{25\, dx}{x^2(5-x)}$
$= \pi \int_1^4 \left(\frac{dx}{x} + \frac{5\, dx}{x^2} + \frac{dx}{5-x}\right)$
$= \pi \left[\ln\left|\frac{x}{5-x}\right| - \frac{5}{x}\right]_1^4 = \pi\left(\ln 4 - \frac{5}{4}\right) - \pi\left(\ln \frac{1}{4} - 5\right)$
$= \frac{15\pi}{4} + 2\pi \ln 4$



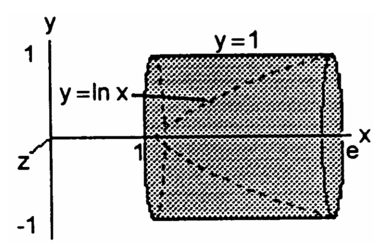

19. $V = \int_a^b 2\pi \left(\begin{smallmatrix}\text{shell}\\ \text{radius}\end{smallmatrix}\right)\left(\begin{smallmatrix}\text{shell}\\ \text{height}\end{smallmatrix}\right) dx = \int_0^1 2\pi x e^x\, dx$
$= 2\pi\left[xe^x - e^x\right]_0^1 = 2\pi$



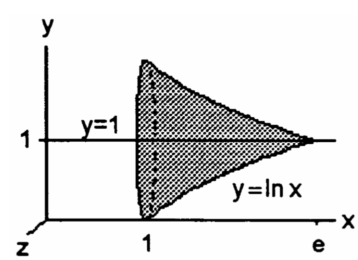

20. $V = \int_0^{\ln 2} 2\pi(\ln 2 - x)\left(e^x - 1\right) dx$
$= 2\pi \int_0^{\ln 2} \left[(\ln 2)\, e^x - \ln 2 - xe^x + x\right] dx$
$= 2\pi \left[(\ln 2)\, e^x - (\ln 2)x - xe^x + e^x + \frac{x^2}{2}\right]_0^{\ln 2}$
$= 2\pi \left[2 \ln 2 - (\ln 2)^2 - 2 \ln 2 + 2 + \frac{(\ln 2)^2}{2}\right] - 2\pi(\ln 2 + 1)$
$= 2\pi \left[-\frac{(\ln 2)^2}{2} - \ln 2 + 1\right]$




21. (a) $V = \int_1^e \pi\left[1 - (\ln x)^2\right] dx$
$= \pi\left[x - x(\ln x)^2\right]_1^e + 2\pi \int_1^e \ln x\, dx$
(FORMULA 110)
$= \pi\left[x - x(\ln x)^2 + 2(x \ln x - x)\right]_1^e$
$= \pi\left[-x - x(\ln x)^2 + 2x \ln x\right]_1^e$
$= \pi\left[-e - e + 2e - (-1)\right] = \pi$




(b) $V = \int_1^e \pi(1 - \ln x)^2\, dx = \pi \int_1^e \left[1 - 2 \ln x + (\ln x)^2\right] dx$
$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2\right]_1^e - 2\pi \int_1^e \ln x\, dx$
$= \pi\left[x - 2(x \ln x - x) + x(\ln x)^2 - 2(x \ln x - x)\right]_1^e$
$= \pi\left[5x - 4x \ln x + x(\ln x)^2\right]_1^e$
$= \pi\left[(5e - 4e + e) - (5)\right] = \pi(2e - 5)$




22. (a) $V = \pi \int_0^1 \left[(e^y)^2 - 1\right] dy = \pi \int_0^1 (e^{2y} - 1)\, dy = \pi\left[\frac{e^{2y}}{2} - y\right]_0^1 = \pi\left[\frac{e^2}{2} - 1 - \left(\frac{1}{2}\right)\right] = \frac{\pi\,(e^2 - 3)}{2}$

(b) $V = \pi \int_0^1 (e^y - 1)^2\, dy = \pi \int_0^1 (e^{2y} - 2e^y + 1)\, dy = \pi\left[\frac{e^{2y}}{2} - 2e^y + y\right]_0^1 = \pi\left[\left(\frac{e^2}{2} - 2e + 1\right) - \left(\frac{1}{2} - 2\right)\right]$
$= \pi\left(\frac{e^2}{2} - 2e + \frac{5}{2}\right) = \frac{\pi\,(e^2 - 4e + 5)}{2}$

23. (a) $\lim_{x \to 0^+} x \ln x = 0 \;\Rightarrow\; \lim_{x \to 0^+} f(x) = 0 = f(0) \;\Rightarrow\;$ f is continuous

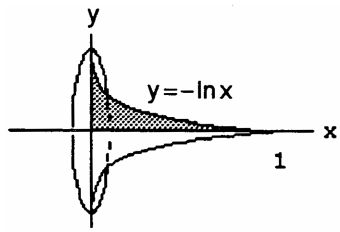
(b) $V = \int_0^2 \pi x^2 (\ln x)^2\, dx;\ \begin{bmatrix} u = (\ln x)^2 \\ du = (2\ln x)\frac{dx}{x} \\ dv = x^2 dx \\ v = \frac{x^3}{3} \end{bmatrix} \rightarrow \pi\left(\lim_{b\to 0^+}\left[\frac{x^3}{3}(\ln x)^2\right]_b^2 - \int_0^2 \left(\frac{x^3}{3}\right)(2\ln x)\frac{dx}{x}\right)$

$= \pi\left[\left(\frac{8}{3}\right)(\ln 2)^2 - \left(\frac{2}{3}\right)\lim_{b\to 0^+}\left[\frac{x^3}{3}\ln x - \frac{x^3}{9}\right]_b^2\right] = \pi\left[\frac{8(\ln 2)^2}{3} - \frac{16(\ln 2)}{9} + \frac{16}{27}\right]$

24. $V = \int_0^1 \pi(-\ln x)^2\, dx$

$= \pi\left(\lim_{b\to 0^+}\left[x(\ln x)^2\right]_b^1 - 2\int_0^1 \ln x\, dx\right)$

$= -2\pi \lim_{b\to 0^+}\left[x\ln x - x\right]_b^1 = 2\pi$




25. $M = \int_1^e \ln x\, dx = [x\ln x - x]_1^e = (e - e) - (0 - 1) = 1;$

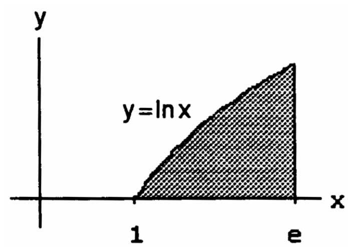
$M_x = \int_1^e (\ln x)\left(\frac{\ln x}{2}\right) dx = \frac{1}{2}\int_1^e (\ln x)^2\, dx$

$= \frac{1}{2}\left(\left[x(\ln x)^2\right]_1^e - 2\int_1^e \ln x\, dx\right) = \frac{1}{2}(e-2);$

$M_y = \int_1^e x\ln x\, dx = \left[\frac{x^2 \ln x}{2}\right]_1^e - \frac{1}{2}\int_1^e x\, dx$

$= \frac{1}{2}\left[x^2\ln x - \frac{x^2}{2}\right]_1^e = \frac{1}{2}\left[\left(e^2 - \frac{e^2}{2}\right) + \frac{1}{2}\right] = \frac{1}{4}(e^2+1);$

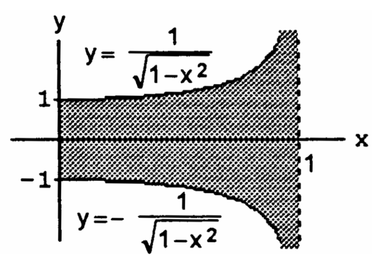
therefore, $\bar{x} = \frac{M_y}{M} = \frac{e^2+1}{4}$ and $\bar{y} = \frac{M_x}{M} = \frac{e-2}{2}$




26. $M = \int_0^1 \frac{2\, dx}{\sqrt{1-x^2}} = 2\left[\sin^{-1} x\right]_0^1 = \pi;$

$M_y = \int_0^1 \frac{2x\, dx}{\sqrt{1-x^2}} = 2\left[-\sqrt{1-x^2}\right]_0^1 = 2;$

therefore, $\bar{x} = \frac{M_y}{M} = \frac{2}{\pi}$ and $\bar{y} = 0$ by symmetry




27. $L = \int_1^e \sqrt{1+\frac{1}{x^2}}\, dx = \int_1^e \frac{\sqrt{x^2+1}}{x}\, dx;\ \begin{bmatrix} x = \tan\theta \\ dx = \sec^2\theta\, d\theta \end{bmatrix} \rightarrow L = \int_{\pi/4}^{\tan^{-1} e} \frac{\sec\theta \cdot \sec^2\theta\, d\theta}{\tan\theta}$

$= \int_{\pi/4}^{\tan^{-1} e} \frac{(\sec\theta)(\tan^2\theta + 1)}{\tan\theta}\, d\theta = \int_{\pi/4}^{\tan^{-1} e} (\tan\theta\sec\theta + \csc\theta)\, d\theta = \left[\sec\theta - \ln|\csc\theta + \cot\theta|\right]_{\pi/4}^{\tan^{-1} e}$

$= \left(\sqrt{1+e^2} - \ln\left|\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right|\right) - \left[\sqrt{2} - \ln\left(1+\sqrt{2}\right)\right] = \sqrt{1+e^2} - \ln\left(\frac{\sqrt{1+e^2}}{e} + \frac{1}{e}\right) - \sqrt{2} + \ln\left(1+\sqrt{2}\right)$

28. $y = \ln x \Rightarrow 1 + \left(\frac{dx}{dy}\right)^2 = 1 + x^2 \Rightarrow S = 2\pi\int_c^d x\sqrt{1+x^2}\, dy \Rightarrow S = 2\pi\int_0^1 e^y\sqrt{1+e^{2y}}\, dy;\ \begin{bmatrix} u = e^y \\ du = e^y\, dy \end{bmatrix}$

$\rightarrow S = 2\pi\int_1^e \sqrt{1+u^2}\, du;\ \begin{bmatrix} u = \tan\theta \\ du = \sec^2\theta\, d\theta \end{bmatrix} \rightarrow 2\pi\int_{\pi/4}^{\tan^{-1} e} \sec\theta \cdot \sec^2\theta\, d\theta$

$= 2\pi\left(\frac{1}{2}\right)\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right]_{\pi/4}^{\tan^{-1} e} = \pi\left[\left(\sqrt{1+e^2}\right)e + \ln\left|\sqrt{1+e^2} + e\right|\right] - \pi\left[\sqrt{2}\cdot 1 + \ln\left(\sqrt{2}+1\right)\right]$

$= \pi\left[e\sqrt{1+e^2} + \ln\left(\frac{\sqrt{1+e^2}+e}{\sqrt{2}+1}\right) - \sqrt{2}\right]$

29. $L = 4\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx;\ x^{2/3} + y^{2/3} = 1 \Rightarrow y = \left(1 - x^{2/3}\right)^{3/2} \Rightarrow \frac{dy}{dx} = -\frac{3}{2}\left(1 - x^{2/3}\right)^{1/2}\left(x^{-1/3}\right)\left(\frac{2}{3}\right)$

$$\Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{1-x^{2/3}}{x^{2/3}} \Rightarrow L = 4\int_0^1 \sqrt{1+\left(\frac{1-x^{2/3}}{x^{2/3}}\right)}\,dx = 4\int_0^1 \frac{dx}{x^{1/3}} = 6\left[x^{2/3}\right]_0^1 = 6$$

30. $S = 2\pi \int_{-1}^{1} f(x)\sqrt{1+[f'(x)]^2}\,dx$; $f(x) = \left(1-x^{2/3}\right)^{3/2} \Rightarrow [f'(x)]^2 + 1 = \frac{1}{x^{2/3}} \Rightarrow S = 2\pi \int_{-1}^{1} \left(1-x^{2/3}\right)^{3/2} \cdot \frac{dx}{\sqrt{x^{2/3}}}$

$$= 4\pi\int_0^1 \left(1-x^{2/3}\right)^{3/2}\left(\frac{1}{x^{1/3}}\right) dx;\ \begin{bmatrix} u = x^{2/3} \\ du = \frac{2}{3}\frac{dx}{x^{1/3}} \end{bmatrix} \rightarrow 4 \cdot \frac{3}{2}\pi \int_0^1 (1-u)^{3/2}\,du = -6\pi\int_0^1 (1-u)^{3/2}\,d(1-u)$$

$$= -6\pi \cdot \frac{2}{5}\left[(1-u)^{5/2}\right]_0^1 = \frac{12\pi}{5}$$

31. $\left(\frac{dy}{dx}\right)^2 = \frac{1}{4x} \Rightarrow \frac{dy}{dx} = \frac{\pm 1}{2\sqrt{x}} \Rightarrow y = \sqrt{x}$ or $y = -\sqrt{x}$, $0 \le x \le 4$

32. The integral $\int_{-1}^{1}\sqrt{1-x^2}\,dx$ is the area enclosed by the x-axis and the semicircle $y = \sqrt{1-x^2}$. This area is half the circle's area, or $\frac{\pi}{2}$ and multiplying by 2 gives $\pi$. The length of the circular arc $y = \sqrt{1-x^2}$ from $x = -1$ to $x = 1$ is $L = \int_{-1}^{1}\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int_{-1}^{1}\sqrt{1+\left(\frac{-x}{\sqrt{1-x^2}}\right)^2}\,dx = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} = \frac{1}{2}(2\pi) = \pi$ since L is half the circle's circumference. In conclusion, $2\int_{-1}^{1}\sqrt{1-x^2}\,dx = \int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}}$.

33. (b) $\int_{-\infty}^{\infty} e^{(x-e^x)}\,dx = \int_{-\infty}^{\infty} e^{(-e^x)}e^x\,dx$

$$= \lim_{a\to-\infty}\int_a^0 e^{(-e^x)}e^x\,dx + \lim_{b\to+\infty}\int_0^b e^{(-e^x)}e^x\,dx;$$

$$\begin{bmatrix} u = e^x \\ du = e^x\,dx \end{bmatrix} \rightarrow$$

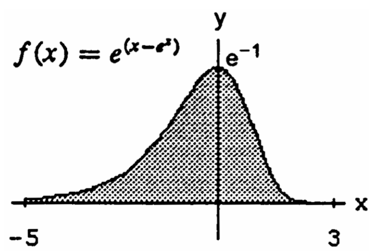
$$\lim_{a\to-\infty}\int_{e^a}^{1} e^{-u}\,du + \lim_{b\to+\infty}\int_1^{e^b} e^{-u}\,du$$

$$= \lim_{a\to-\infty}\left[-e^{-u}\right]_{e^a}^{1} + \lim_{b\to-\infty}\left[-e^{-u}\right]_1^{e^b}$$

$$= \lim_{a\to-\infty}\left[-\frac{1}{e} + e^{-(e^a)}\right] + \lim_{b\to+\infty}\left[-e^{-(e^b)} + \frac{1}{e}\right]$$

$$= \left(-\frac{1}{e} + e^0\right) + \left(0 + \frac{1}{e}\right) = 1$$

(a)




34. $u = \frac{1}{1+y}$, $du = -\frac{dy}{(1+y)^2}$; $dv = ny^{n-1}\,dy$, $v = y^n$;

$\lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy = \lim_{n\to\infty}\left(\left[\frac{y^n}{1+y}\right]_0^1 + \int_0^1 \frac{y^n}{1+y^2}\,dy\right) = \frac{1}{2} + \lim_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy$. Now, $0 \le \frac{y^n}{1+y^2} \le y^n$

$$\Rightarrow 0 \le \lim_{n\to\infty}\int_0^1 \frac{y^n}{1+y^2}\,dy \le \lim_{n\to\infty}\int_0^1 y^n\,dy = \lim_{n\to\infty}\left[\frac{y^{n+1}}{n+1}\right]_0^1 = \lim_{n\to\infty}\frac{1}{n+1} = 0 \Rightarrow \lim_{n\to\infty}\int_0^1 \frac{ny^{n-1}}{1+y}\,dy$$

$$= \frac{1}{2} + 0 = \frac{1}{2}$$

35. $u = x^2 - a^2 \Rightarrow du = 2x\,dx$;

$$\int x\left(\sqrt{x^2-a^2}\right)^n dx = \frac{1}{2}\int\left(\sqrt{u}\right)^n du = \frac{1}{2}\int u^{n/2}\,du = \frac{1}{2}\left(\frac{u^{n/2+1}}{\frac{n}{2}+1}\right) + C,\ n \ne -2$$

$$= \frac{u^{(n+2)/2}}{n+2} + C = \frac{\left(\sqrt{u}\right)^{n+2}}{n+2} + C = \frac{\left(\sqrt{x^2-a^2}\right)^{n+2}}{n+2} + C$$

36. $\frac{\pi}{6} = \sin^{-1}\frac{1}{2} = \left[\sin^{-1}\frac{x}{2}\right]_0^1 = \int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} \frac{du}{\sqrt{4-u^2}}$

$$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{u}{2}\right]_0^{\sqrt{2}} = \frac{1}{\sqrt{2}}\sin^{-1}\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}\left(\frac{\pi}{4}\right) = \frac{\pi\sqrt{2}}{8}$$

37. $\int_1^{\infty}\left(\frac{ax}{x^2+1}-\frac{1}{2x}\right)dx = \lim\limits_{b\to\infty}\int_1^{b}\left(\frac{ax}{x^2+1}-\frac{1}{2x}\right)dx = \lim\limits_{b\to\infty}\left[\frac{a}{2}\ln\left(x^2+1\right)-\frac{1}{2}\ln x\right]_1^b = \lim\limits_{b\to\infty}\left[\frac{1}{2}\ln\frac{\left(x^2+1\right)^a}{x}\right]_1^b$
$= \lim\limits_{b\to\infty}\frac{1}{2}\left[\ln\frac{\left(b^2+1\right)^a}{b}-\ln 2^a\right]$; $\lim\limits_{b\to\infty}\frac{\left(b^2+1\right)^a}{b} > \lim\limits_{b\to\infty}\frac{b^{2a}}{b} = \lim\limits_{b\to\infty} b^{2\left(a-\frac{1}{2}\right)} = \infty$ if $a > \frac{1}{2} \Rightarrow$ the improper integral diverges if $a > \frac{1}{2}$; for $a = \frac{1}{2}$: $\lim\limits_{b\to\infty}\frac{\sqrt{b^2+1}}{b} = \lim\limits_{b\to\infty}\sqrt{1+\frac{1}{b^2}} = 1 \Rightarrow \lim\limits_{b\to\infty}\frac{1}{2}\left[\ln\frac{\left(b^2+1\right)^{1/2}}{b}-\ln 2^{1/2}\right]$
$= \frac{1}{2}\left(\ln 1-\frac{1}{2}\ln 2\right) = -\frac{\ln 2}{4}$; if $a < \frac{1}{2}$: $0 \le \lim\limits_{b\to\infty}\frac{\left(b^2+1\right)^a}{b} < \lim\limits_{b\to\infty}\frac{(b+1)^{2a}}{b+1} = \lim\limits_{b\to\infty}(b+1)^{2a-1} = 0$
$\Rightarrow \lim\limits_{b\to\infty}\ln\frac{\left(b^2+1\right)^a}{b} = -\infty \Rightarrow$ the improper integral diverges if $a < \frac{1}{2}$; in summary, the improper integral $\int_1^{\infty}\left(\frac{ax}{x^2+1}-\frac{1}{2x}\right)dx$ converges only when $a = \frac{1}{2}$ and has the value $-\frac{\ln 2}{4}$

38. $G(x) = \lim\limits_{b\to\infty}\int_0^b e^{-xt}\,dt = \lim\limits_{b\to\infty}\left[-\frac{1}{x}e^{-xt}\right]_0^b = \lim\limits_{b\to\infty}\left(\frac{1-e^{-xb}}{x}\right) = \frac{1-0}{x} = \frac{1}{x}$ if $x > 0 \Rightarrow xG(x) = x\left(\frac{1}{x}\right)$
$= 1$ if $x > 0$

39. $A = \int_1^{\infty}\frac{dx}{x^p}$ converges if $p > 1$ and diverges if $p \le 1$. Thus, $p \le 1$ for infinite area. The volume of the solid of revolution about the x-axis is $V = \int_1^{\infty}\pi\left(\frac{1}{x^p}\right)^2 dx = \pi\int_1^{\infty}\frac{dx}{x^{2p}}$ which converges if $2p > 1$ and diverges if $2p \le 1$. Thus we want $p > \frac{1}{2}$ for finite volume. In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $\frac{1}{2} < p \le 1$.

40. The area is given by the integral $A = \int_0^1\frac{dx}{x^p}$;
$p = 1$: $A = \lim\limits_{b\to 0^+}\left[\ln x\right]_b^1 = -\lim\limits_{b\to 0^+}\ln b = \infty$, diverges;
$p > 1$: $A = \lim\limits_{b\to 0^+}\left[x^{1-p}\right]_b^1 = 1-\lim\limits_{b\to 0^+}b^{1-p} = -\infty$, diverges;
$p < 1$: $A = \lim\limits_{b\to 0^+}\left[x^{1-p}\right]_b^1 = 1-\lim\limits_{b\to 0^+}b^{1-p} = 1-0$, converges; thus, $p \ge 1$ for infinite area.

The volume of the solid of revolution about the x-axis is $V_x = \pi\int_0^1\frac{dx}{x^{2p}}$ which converges if $2p < 1$ or $p < \frac{1}{2}$, and diverges if $p \ge \frac{1}{2}$. Thus, $V_x$ is infinite whenever the area is infinite $(p \ge 1)$.

The volume of the solid of revolution about the y-axis is $V_y = \pi\int_1^{\infty}[R(y)]^2\,dy = \pi\int_1^{\infty}\frac{dy}{y^{2/p}}$ which converges if $\frac{2}{p} > 1 \Leftrightarrow p < 2$ (see Exercise 39). In conclusion, the curve $y = x^{-p}$ gives infinite area and finite volume for values of p satisfying $1 \le p < 2$, as described above.

41.

| | | |
|---|---|---|
| $e^{2x}$ | $(+)$ | $\cos 3x$ |
| $2e^{2x}$ | $(-)$ | $\frac{1}{3}\sin 3x$ |
| $4e^{2x}$ | $(+)$ | $-\frac{1}{9}\cos 3x$ |

$I = \frac{e^{2x}}{3}\sin 3x + \frac{2e^{2x}}{9}\cos 3x - \frac{4}{9}I \Rightarrow \frac{13}{9}I = \frac{e^{2x}}{9}(3\sin 3x + 2\cos 3x) \Rightarrow I = \frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$

42.

| | | |
|---|---|---|
| $e^{3x}$ | $(+)$ | $\sin 4x$ |
| $3e^{3x}$ | $(-)$ | $-\frac{1}{4}\cos 4x$ |
| $9e^{3x}$ | $(+)$ | $-\frac{1}{16}\sin 4x$ |

$I = -\frac{e^{3x}}{4}\cos 4x + \frac{3e^{3x}}{16}\sin 4x - \frac{9}{16}I \Rightarrow \frac{25}{16}I = \frac{e^{3x}}{16}(3\sin 4x - 4\cos 4x) \Rightarrow I = \frac{e^{3x}}{25}(3\sin 4x - 4\cos 4x) + C$

43.

| | | |
|---|---|---|
| $\sin 3x$ | $(+)$ | $\sin x$ |
| $3\cos 3x$ | $(-)$ | $-\cos x$ |
| $-9\sin 3x$ | $(+)$ | $-\sin x$ |

$I = -\sin 3x \cos x + 3\cos 3x \sin x + 9I \Rightarrow -8I = -\sin 3x \cos x + 3\cos 3x \sin x$
$\Rightarrow I = \frac{\sin 3x \cos x - 3\cos 3x \sin x}{8} + C$

44.

| | | |
|---|---|---|
| $\cos 5x$ | $(+)$ | $\sin 4x$ |
| $-\sin 5x$ | $(-)$ | $-\frac{1}{4}\cos 4x$ |
| $-25\cos 5x$ | $(+)$ | $-\frac{1}{16}\sin 4$ |

$I = -\frac{1}{4}\cos 5x \cos 4x - \frac{5}{16}\sin 5x \sin 4x + \frac{25}{16} I \Rightarrow -\frac{9}{16} I = -\frac{1}{4}\cos 5x \cos 4x - \frac{5}{16}\sin 5x \sin 4x$
$\Rightarrow I = \frac{1}{9}(4\cos 5x \cos 4x + 5\sin 5x \sin 4x) + C$

45.

| | | |
|---|---|---|
| $e^{ax}$ | $(+)$ | $\sin bx$ |
| $ae^{ax}$ | $(-)$ | $-\frac{1}{b}\cos bx$ |
| $a^2e^{ax}$ | $(+)$ | $-\frac{1}{b^2}\sin bx$ |

$I = -\frac{e^{ax}}{b}\cos bx + \frac{ae^{ax}}{b^2}\sin bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a\sin bx - b\cos bx)$
$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + C$

46.

| | | |
|---|---|---|
| $e^{ax}$ | $(+)$ | $\cos bx$ |
| $ae^{ax}$ | $(-)$ | $\frac{1}{b}\sin bx$ |
| $a^2e^{ax}$ | $(+)$ | $-\frac{1}{b^2}\cos bx$ |

$I = \frac{e^{ax}}{b}\sin bx + \frac{ae^{ax}}{b^2}\cos bx - \frac{a^2}{b^2} I \Rightarrow \left(\frac{a^2+b^2}{b^2}\right) I = \frac{e^{ax}}{b^2}(a\cos bx + b\sin bx)$
$\Rightarrow I = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx) + C$

47.

| | | |
|---|---|---|
| $\ln(ax)$ | $(+)$ | $1$ |
| $\frac{1}{x}$ | $(+)$ | $x$ |

$I = x\ln(ax) - \int \left(\frac{1}{x}\right) x\,dx = x\ln(ax) - x + C$

48.

| | | |
|---|---|---|
| $\ln(ax)$ | $(+)$ | $x^2$ |
| $\frac{1}{x}$ | $(+)$ | $\frac{1}{3}x^3$ |

$I = \frac{1}{3}x^3\ln(ax) - \int \left(\frac{1}{x}\right)\left(\frac{x^3}{3}\right) dx = \frac{1}{3}x^3\ln(ax) - \frac{1}{9}x^3 + C$

49. (a) $\Gamma(1) = \int_0^\infty e^{-t}\,dt = \lim_{b\to\infty} \int_0^b e^{-t}\,dt = \lim_{b\to\infty} \left[-e^{-t}\right]_0^b = \lim_{b\to\infty} \left[-\frac{1}{e^b} - (-1)\right] = 0 + 1 = 1$

(b) $u = t^x$, $du = xt^{x-1}\,dt$; $dv = e^{-t}\,dt$, $v = -e^{-t}$; $x =$ fixed positive real
$\Rightarrow \Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt = \lim_{b\to\infty} \left[-t^x e^{-t}\right]_0^b + x\int_0^\infty t^{x-1}e^{-t}\,dt = \lim_{b\to\infty} \left(-\frac{b^x}{e^b} + 0^x e^0\right) + x\Gamma(x) = x\Gamma(x)$

(c) $\Gamma(n+1) = n\Gamma(n) = n!$:

$n = 0$: $\Gamma(0+1) = \Gamma(1) = 0!$;

$n = k$: Assume $\Gamma(k+1) = k!$ for some $k > 0$;

$n = k+1$: $\Gamma(k+1+1) = (k+1)\,\Gamma(k+1)$ from part (b)

$= (k+1)k!$ induction hypothesis

$= (k+1)!$ definition of factorial

Thus, $\Gamma(n+1) = n\Gamma(n) = n!$ for every positive integer n.

50. (a) $\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}$ and $n\Gamma(n) = n!$ $\Rightarrow$ $n! \approx n\left(\frac{n}{e}\right)^n \sqrt{\frac{2\pi}{n}} = \left(\frac{n}{e}\right)^n \sqrt{2n\pi}$

(b)

| n | $\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$ | calculator |
|---|---|---|
| 10 | 3598695.619 | 3628800 |
| 20 | $2.4227868 \times 10^{18}$ | $2.432902 \times 10^{18}$ |
| 30 | $2.6451710 \times 10^{32}$ | $2.652528 \times 10^{32}$ |
| 40 | $8.1421726 \times 10^{47}$ | $8.1591528 \times 10^{47}$ |
| 50 | $3.0363446 \times 10^{64}$ | $3.0414093 \times 10^{64}$ |
| 60 | $8.3094383 \times 10^{81}$ | $8.3209871 \times 10^{81}$ |

(c)

| n | $\left(\frac{n}{e}\right)^n \sqrt{2n\pi}$ | $\left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/12n}$ | calculator |
|---|---|---|---|
| 10 | 3598695.619 | 3628810.051 | 3628800 |

**NOTES:**