UNIVERSITY OF ENGINEERING AND TECHNOLOGY

Department of Computer Science and Engineering

Probability and Random Variables

Fall 2018

Problem Set 1
Due: September 10, 2018

Note: These problems have been liberally copied from older versions of MIT's 6.041/6.431, Harvard's Stat 110, Al Drake's book and Gian Carlo Rota's notes. It is difficult to acknowledge each problem individually. There is no claim to originality in these problems, and the debt to all these resources is gratefully acknowledged. This applies to all problems in this course.

1. Use Venn diagrams, axioms of algebra of events, or anything else to determine which of the following are valid relations in algebra of events for arbitrary events A, B, and C. We will denote $A \cup B$ by $A + B$ and $A \cap B$ as AB.
   (a) $(A + B + C)^c = A^c + B^c + C^c$
   (b) $AB + AB^c + A^cB = (A^cB^c)^c$

2. We are given that$P(A) = 0.55$, $P(B^c) = 0.35$, and $P(A \cup B) = 0.75$. Determine
   (a) $P(B)$
   (b) $P(A \cap B)$

3. Explain the answers to the following questions. Note here A^c is denoted by A'.
   (a) If events A and B are mutually exclusive and collectively exhaustive, then are A' and B' mutually exclusive and collectively exhaustive?
   (b) If events A and B are mutually exclusive but not collectively exhaustive, then are A' and B' collectively exhaustive?

4. Anne and Bob each have a deck of playing cards. Each flips over a randomly selected card. Assume that all pairs of cards are equally likely to be drawn. Determine the probability that:
   (a) At least one card is an ace
   (b) The two cards are of the same suit
   (c) Neither card is an ace
   (d) Neither card is a diamond or a club

5. Alice and bob each chose at random a number between zero and one according to the uniform probability law. Consider the following events:

$$\begin{aligned}
A &= \{\text{The sum of the two numbers is less than 1}\} \\
B &= \{\text{The magnitude of the difference between the two numbers is greater than } 1/3\} \\
C &= \{\text{At least one of the numbers is greater than } 1/4\} \\
D &= \{\text{The sum of two numbers is 1}\} \\
E &= \{\text{Alice's number is greater than } 1/4\}
\end{aligned}$$

   Find the following probabilities: $P(A)$, $P(B)$, $P(C)$, $P(B \cap C)$, $P(D)$, $P(B \cap D)$

6. Show that

$$P(AB' + A'B) = P(A) + P(B) - 2P(AB)$$

Note that the left hand side is the probability that exactly one of the events A or B will occur.

7. Let D_1 be the event that "exactly one of the events A, B and C occur". Express D_1 in terms of $P(A)$, $P(B)$, $P(C)$, $P(AB)$, $P(AC)$, $P(BC)$, and $P(ABC)$.

8. Let A and B be two events. Consider the two inequalities:

   - $P(A) + P(B) \geq P(A \cup B) \geq \max\{P(A), P(B)\}$
   - $\min\{P(A), P(B)\} \geq P(A \cap B) \geq P(A) + P(B) - 1$

   (a) Prove the above two inequalities.

   (b) Under what conditions on A and B are these inequalities satisfied as equalities?

   Note that you can use Venn diagrams to guide your thinking. However, you are required to argue mathematically using algebra of sets or axioms of probability theory.

9. Mary and Tom park their cars in an ampty parking lot that consists of N parking spaces in a row, where $N \geq 4$. Assume that each possible pair of parking locations is equally likely. Calculate the probability that the parking spaces they select are exactly 4 apart i.e. exactly three empty spaces between them.

10. For three tosses of a fair coin, determine:

    (a) The probability of the sequence HHH.

    (b) The probability of the sequence HTH

    (c) The probability of two heads and a tail.

    (d) The probability of "More heads than tails".

    (e) The conditional probability of "More heads than tails given at least one tail".

    (f) The conditional probability of "More heads than tails given less than two tails".