PROBLEM SET 3

Q.No.1

STATISTICS

There are 3 S's, 3 T's, 2 I's and 1 C.
Total Ten places ~~could~~ can be ~~permuted~~ arranged as

$$\binom{10}{3}\binom{7}{3}\binom{4}{2}\binom{2}{1} \quad \text{or} \quad \frac{10!}{3!\,3!\,2!} = 50400$$

Q.No.2

K people in room and all 365 days are equally likely so,

Total No. of outcomes = 365^K

Since K can be large, counting of birthday matches of any two, three --- K number of ~~people birthday~~ will be difficult so we count No birthday matches and will use compliment theorem.

$$P(\text{No b.d Match}) = \frac{365 \cdot 364 \cdot 363 \cdots (365-K+1)}{365^K}$$

$$P(\text{Atleast two b.d Matches}) = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365-K+1)}{365^K}$$

Q.No.3

8 different ~~Combinations~~ subsets of toppings are 2^8 and four different sizes so choices for one pizza are 4×2^8.

For one pizza $\Rightarrow 4 \times 2^8$
$= 2^{10}$

Multiplication Rule can give us No. of possible outcomes with order. If we want to calculate No. of possible pizza combination without order then we need to divide the No. of possible outcomes in which pizza are Not same by two and then add with No. of possible outcomes in which pizza are same.

$$\frac{2^{10} \times (2^{10}-1)}{2} + 2^{10} \text{ possible outcomes.}$$

$\Rightarrow$ You can start thinking by counting No. of outcomes of 2 Die Rolls ~~and~~ pairs where order doesnot matter. $\frac{6 \times 5}{2} + 6 = 21$ possibilities.

Q.No.4

$$(x+y)^n = \sum_{K=0}^{n} \binom{n}{k} x^n y^{n-k}$$

$$(x+y)(x+y) = x^2+y^2+2xy = \binom{2}{2}x^2y^0 + \binom{2}{1}xy + \binom{2}{0}x^0y^2$$

$$(x+y)^3 = x^3+3x^2y+3xy^2+y^3 = \binom{3}{3}x^3y^0 + \binom{3}{2}x^2y + \binom{3}{1}xy^2 + \binom{3}{0}x^0y^3$$

$$\vdots$$

$$(x+y)^n = \sum_{K=0}^{n} \binom{n}{K} x^K y^{n-K}.$$

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(b) $\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j}$

This is a famous relationship b/w binomial co-efficients also known as Vendermonde's identity.

<u>Story proof:-</u>

L.H.S ⇒ We have m No. of men, and n No. of Women. We can choose a committee of size k as $\binom{m+n}{k}$

is Equivalent to

R.H.S ⇒ Making all possible combinations of j No. of men and k-j No. of women for a committee of size k. j varies from 0 to k.

(c) $\sum_{k=1}^{n} k\binom{n}{k}^2 = n\binom{2n-1}{n-1}$

L.H.S ⇒ $\sum_{k=1}^{n} k\binom{n}{k}\binom{n}{k}$

$= \sum_{k=1}^{n} k\binom{n}{k}\binom{n}{n-k}$ $\left(\because \binom{n}{k} = \binom{n}{n-k}\right)$

<u>Story proof:-</u>

L.H.S ⇒ We have n No. of women and n No. of men, total size of 2n. Choosing k No. of women and (n-k) No. of men for a committee of size n and then selecting one women as committee Head is $k\binom{n}{k}\binom{n}{n-k}$

is Equivalent to

R.H.S ⇒ First choosing a committee head from n No. of women and then choosing rest of $(n-1)$ members of Committee from $(2n-1)$ choices

Q.No.5

Total 15 delegates and we have to choose 4 out of them so

Total No. of outcomes $= \binom{15}{4}$

(a) If A is not represented on committe then we have remaining 12 delegates for choosing 4 out of them

favourable outcomes $= \binom{12}{4}$

$P(A \text{ not Represented}) = \binom{12}{4} / \binom{15}{4}$

(b) Choosing 1 out of 3 delegates of company A and rest of 3 from 12 delegates of B, C, D & E

favourable outcomes $= \binom{3}{1}\binom{12}{3}$

$P(A \text{ has one Representative}) = \binom{3}{1}\binom{12}{3} / \binom{15}{4}$

(c) Choosing 4 members from delegates of B, C, D

favourable outcomes $= \binom{9}{4}$

$$P(A \,\&\, E \text{ not Represented} = \frac{\binom{9}{4}}{\binom{15}{4}}$$

Q.No.6 K is the No. of lemon cars in 100.
$K = \{0, 1, 2, 3, \dots, 9\}$ Every outcome is equally likely with probability $1/10$.

$$P(K=0 / 20 \text{ cars test Nice}) = \frac{P(20 \text{ cars test Nice} \mid K=0) \cdot P(K=0)}{P(20 \text{ cars test Nice})}$$

$P(20 \text{ cars test Nice} / K=0) = 1$ ($\because$ If there is no lemon car then all 20 cars will be Nice)

$P(K=0) = 1/10$

$$P(20 \text{ cars test Nice}) = \frac{\binom{100-K}{20}}{\binom{100}{20}} \quad K = \{0, 1, 2, \dots 9\}$$

or

$$= \frac{\sum_{K=0}^{9} \binom{100-K}{20}}{\binom{100}{20}}$$

So

$$P(K=0 / 20 \text{ cars test Nice}) = \frac{1 \cdot (1/10)}{\frac{\sum_{K=0}^{9} \binom{100-K}{20}}{\binom{100}{20}}}$$

Q.No.7

Lottery operator chooses 10 out of 100 and we choose 5 out of 100.

Total No. of outcomes $= \binom{100}{10} \binom{100}{5}$

We need to choose 5 digits out of those 10 which were choosen by lottery operator to win. So

No. of favourable outcomes $= \binom{100}{10}\binom{10}{5}$

So

$$P(\text{Win}) = \frac{\binom{100}{10}\binom{10}{5}}{\binom{100}{10}\binom{100}{5}} = \frac{\binom{10}{5}}{\binom{100}{5}}$$

Q.No.8 $\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$

~~First We need a story proof~~

Story Proof:-

L.H.S $\Rightarrow$ Choosing a committee of k people from n-choices and then selecting one chairman and one vice-chairman out of those k choosen members is $k(k-1)\binom{n}{k}$. $\sum_{k=2}^{n}$ means committe can be of atleast 2 members.

is equivalent to

R.H.S $\Rightarrow$ choosing a chairman from n-choices then a vice chairman from $(n-1)$ choices and then choosing any subset of remaining $n-2$ choices for rest of committee members is $n(n-1)2^{n-2}$

Q. No. 9

(a) $P(N=1) = \binom{n}{1} p^1 (1-p)^{n-1}$

$= n \cdot p\,(1-p)^{n-1}$

FS $\Rightarrow$ First ~~success~~ trial was success.

$$P(FS/N=1) = \frac{P(FS \cap N=1)}{P(N=1)}$$

$P(FS \cap N=1) = 1 \cdot p\,(1-p)^{n-1}$ (∵

So,

$$P(FS/N=1) = \frac{p(1-p)^{n-1}}{n \cdot p\,(1-p)^{n-1}}$$

$$= \frac{1}{n}$$

(b)

$$P(FS \mid N=2) = \frac{P(FS \cap N=2)}{P(N=2)}$$

$P(N=2) = \binom{n}{2} p^2 (1-p)^{n-2}$

$P(FS \cap N=2) = 1 \cdot \binom{n-1}{1} p^2 (1-p)^{n-2}$

(We have only 1 option to place first success and $(n-1)$ places to place 2nd success.)

$$P(FS \mid N=2) = \frac{\binom{n-1}{1} p^2 (1-p)^{n-2}}{\binom{n}{2} p^2 (1-p)^{n-2}} = \frac{n-1}{\frac{n!}{2!\,(n-2)!}}$$

$$= \frac{2}{n}$$

(G)

Two S out of First Four trials $\Leftarrow$ 2S4T

$$P(2S4T \mid N=6) = \frac{P(2S4T \cap N=6)}{P(N=6)}$$

$$P(N=6) = \binom{n}{6} p^6 (1-p)^{n-6}$$

$$P(2S4T \cap N=6) = \binom{4}{2}\binom{n-4}{4} p^6 (1-p)^{n-6}$$

(2 success out of first 4 trials $\binom{4}{2}$, rest of 4 success can be placed anywhere on $(n-4)$ places $\binom{n-4}{4}$)

$$P(2S4T \mid N=6) = \frac{\binom{4}{2}\binom{n-4}{4}}{\binom{n}{6}}$$

Q.No.20

There are 100 places for jelly beans and 5 places for dividers. The color distribution can be determined from the location of dividers so

Max. No. of jars = $\binom{105}{5}$