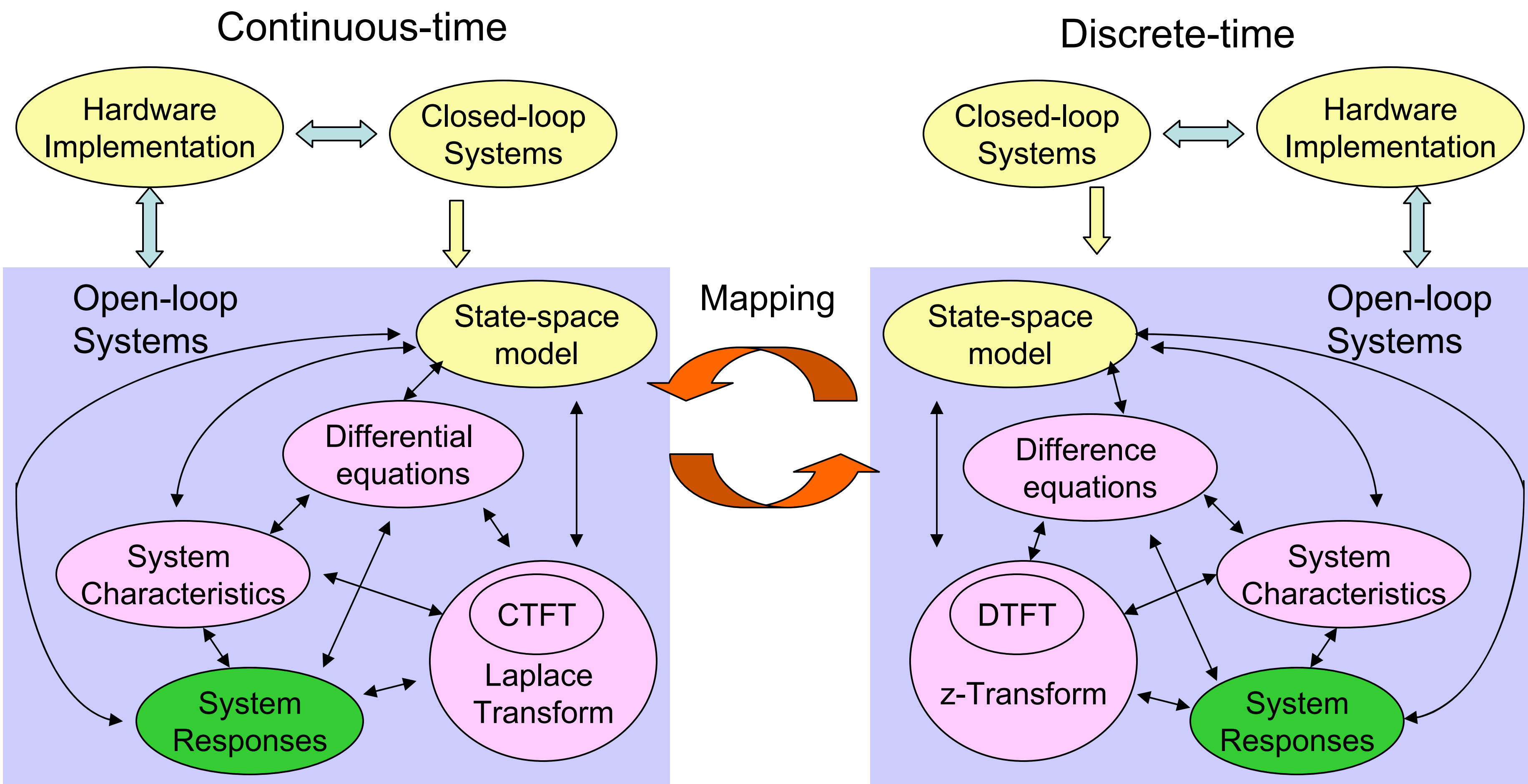
Chapter 5: Transient and Frequency responses of LTI systems

Spring 2009/10

Lecture: Tim Woo

Where we are



Will be covered if available	Done in 211	To be covered	In progress	Done

Expected Outcome

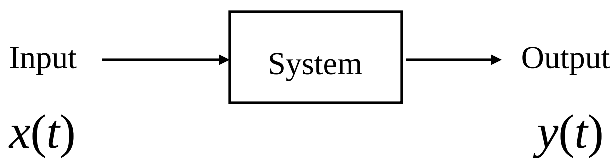
- In this chapter, you will be able to
  - Analysis the system performance with the use of transient and frequency response tools
    - Impulse response
    - Step response
    - Ramp response
    - Bode plot
  - Evaluate the transient and frequency responses of the first- and second-order continuous-time and discrete-time LTI systems
  - Understand the specification of filter in the filter design
  - Categories different linear-phase digital filters

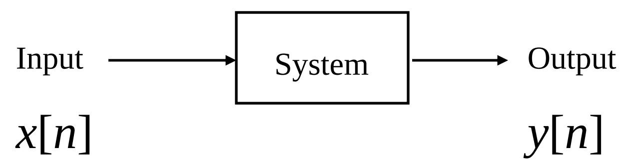
Outline

- Textbook:
  - Section 6.2 The Magnitude-phase representation of the frequency response of LTI systems
  - Section 6.3 Time-domain properties of an ideal frequency selective filters
  - Section 6.4 Time-domain and frequency-domain aspects of non-ideal filters
  - Section 6.5 First-order and Second-order Continuous-time systems
  - Section 6.6 First-order and Second-order Discrete-time systems
  - Section 6.7 Examples of time- and frequency domain analysis of systems
- Reference book
  - A. V. Oppenheim, et. al., ***Discrete-time Signal Processing***, 2nd edition, Prentice-Hall, 1999
  - Section 5.7.3 Causal FIR generalized linear phase system

Introduction

- Basically, any LTI system can be regarded as filtering system. Some signals are being suppressed or amplified. Meanwhile, they are also suffered from different amount of delays.





- Typically, we can analysis the system performance either in time-domain and/or frequency domain analytical tools.

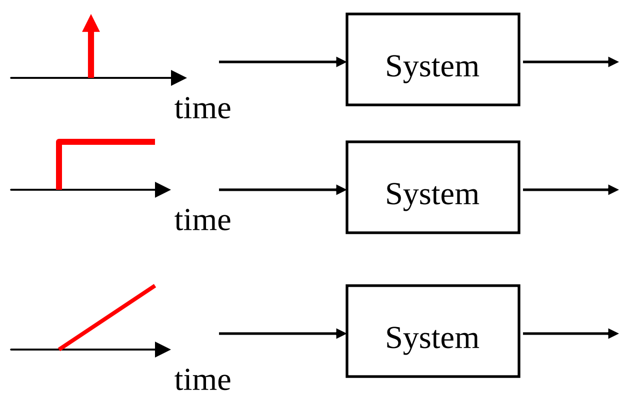
Time-domain analysis	Frequency domain analysis
Transient responses: -Impulse response -Step response -Ramp response	Frequency responses: -Magnitude and phase spectrum -Bode plot

Time-domain analysis: Transient response

- We refer to the output response as ‘Transient Response’, the process generated in going from the initial state to the final state.

- These responses will allow us to investigate the time domain characteristics of dynamic systems.

- Common test signals to produce the response include
  - Unit Impulse function
  - Unit Step function
  - Unit Ramp function



Impulse response

Step response

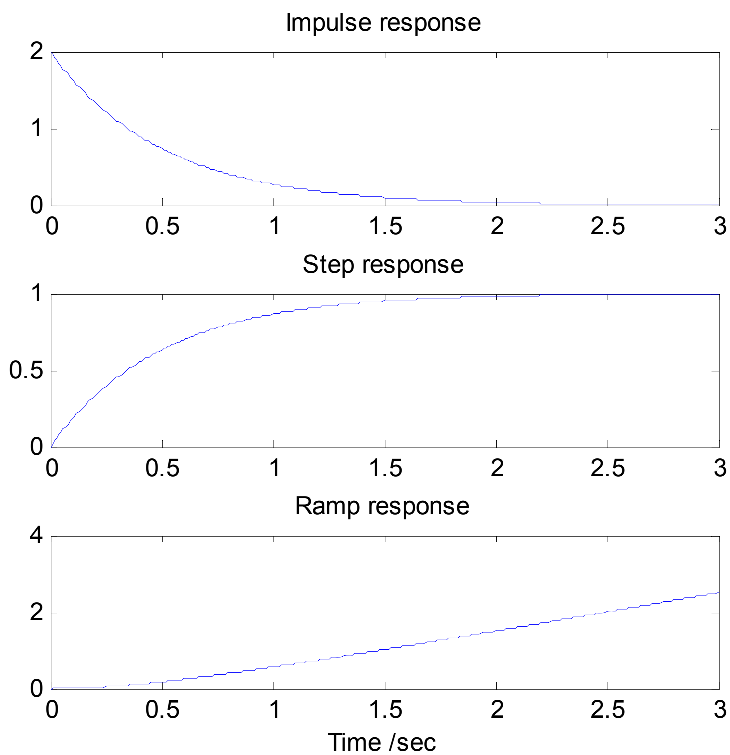
Ramp response

Time-domain analysis: Transient response

- These test signals are chosen because they are simple functions of time and can be easily implemented in an experimental setting.
- The different transient responses of the following systems are being simulated.
  - First-order continuous-time / discrete-time system
  - Second-order continuous-time / discrete-time system
  - Fifth-order continuous-time / discrete-time Butterworth filter

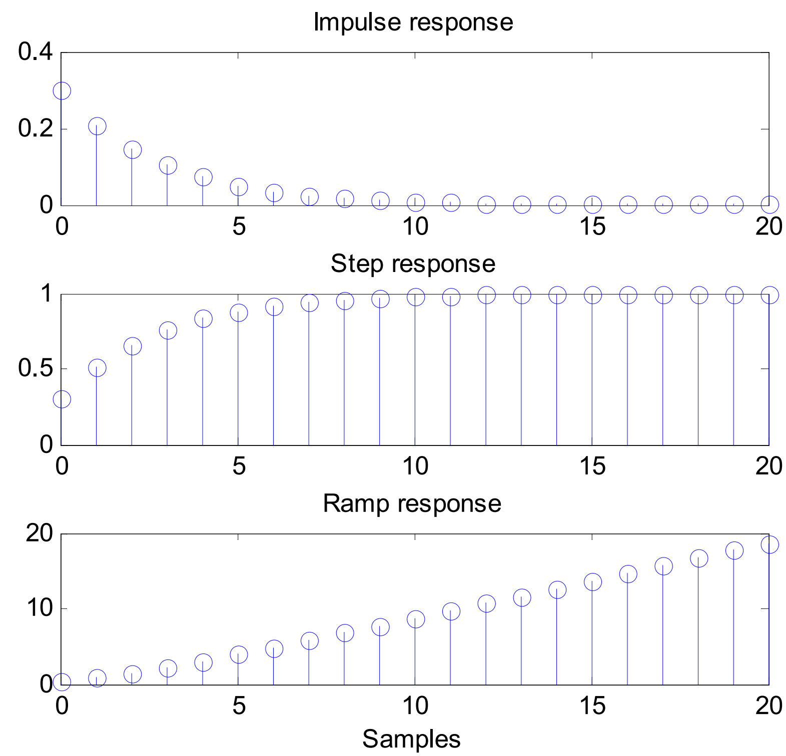
Time-domain analysis: Transient response

First-order continuous-time LTI system



$$H(j\omega)=\frac{2}{j\omega+2}$$

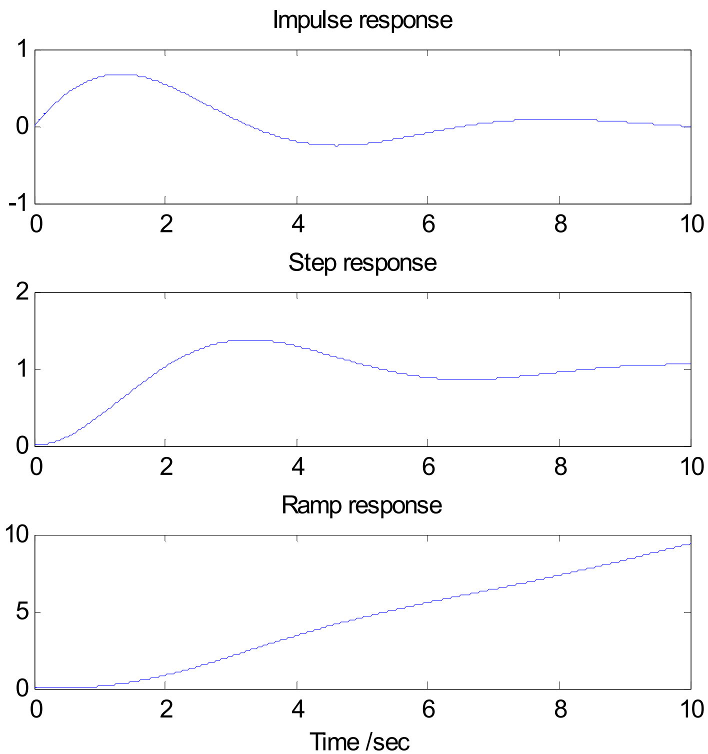
First-order discrete-time LTI system



$$H(e^{j\omega})=\frac{0.3}{1-0.7e^{-j\omega}}$$

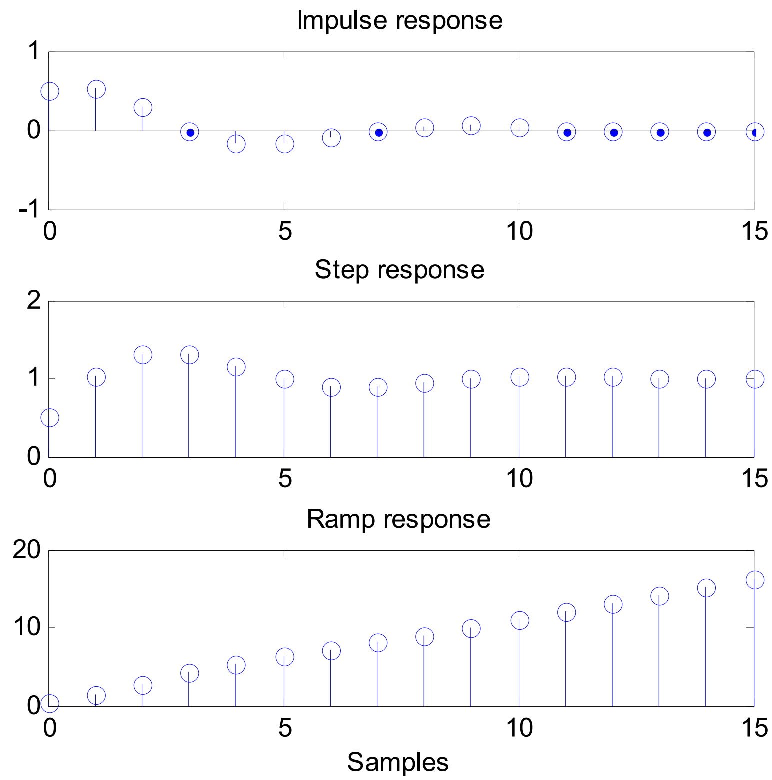
Time-domain analysis: Transient response

Second-order continuous-time LTI system



$$H(j\omega) = \frac{1}{(j\omega)^2 + 0.6(j\omega) + 1}$$

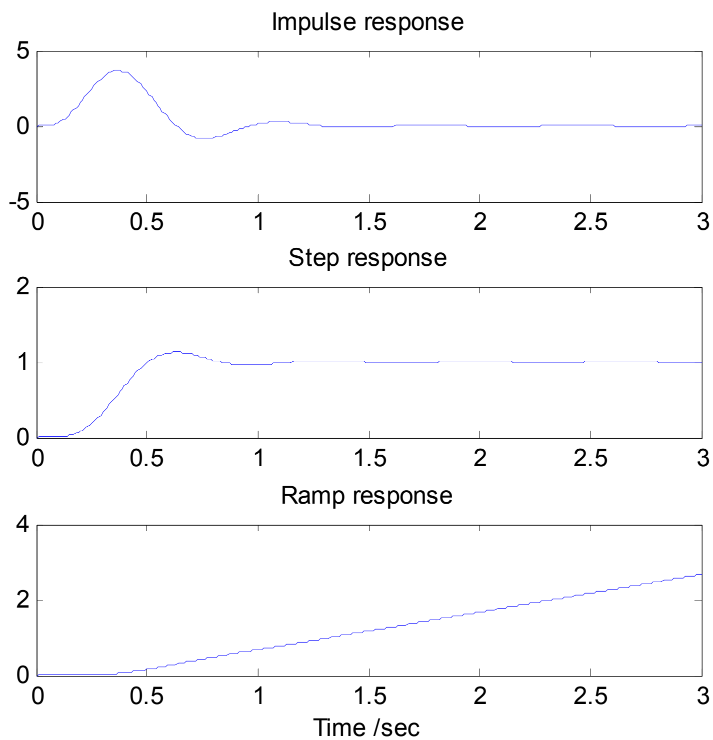
Second-order discrete-time LTI system



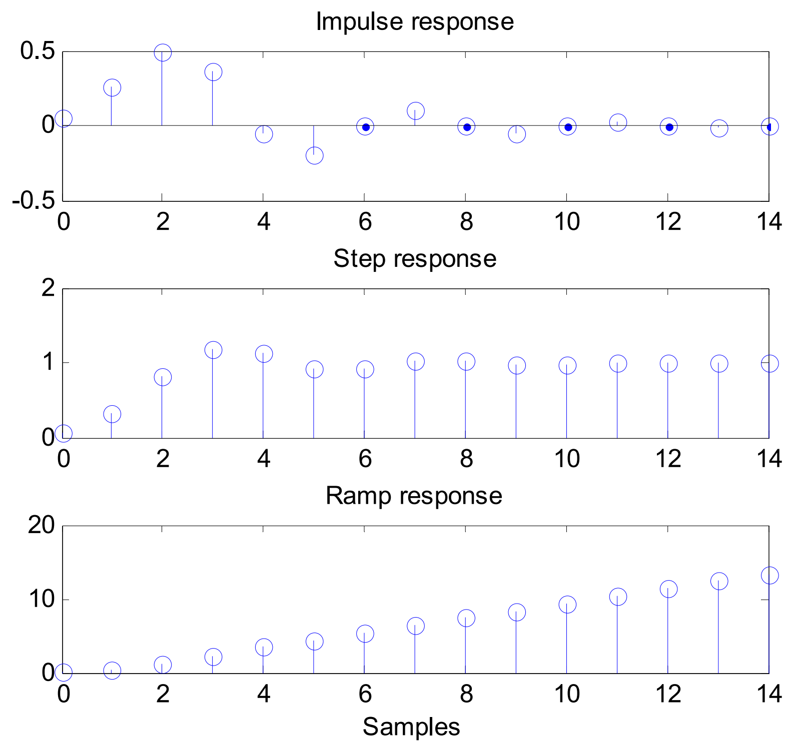
$$H(e^{j\omega}) = \frac{1 - 1.5\cos\left(\frac{\pi}{4}\right) + 0.75^2}{e^{-j2\omega} - 1.5\cos\left(\frac{\pi}{4}\right)e^{-j\omega} + 0.75^2}$$

Time-domain analysis: Transient response

A fifth-order analog Butterworth filter (BW of 10 rad/s)

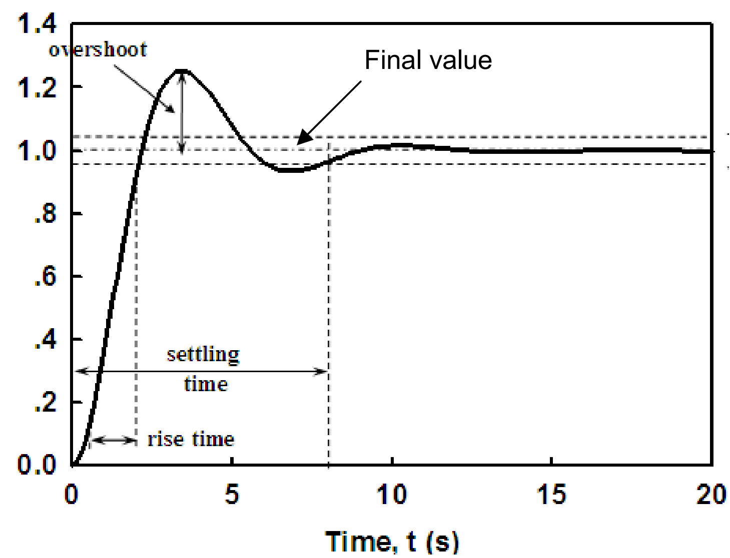


A fifth-order digital Butterworth filter (BW of 0.5π rad/s)



Time-domain analysis: Transient response

- Four measurements are often used to specify a system's performance: rise time, overshoot, settling time and steady-state error. Their definitions with respect to step response are provided below.
  - % *Overshoot* = (peak value - final value)/ (final value) x 100%
  - *Settling time* = time it takes the response to reach its final value (to within a certain percentage, typical value = 1% or 5%)
  - *Rise time* = time it takes the response to go from 10% to 90% of its final value
  - *Steady-state error* = % difference between the commanded position and the measured position after the transient has settled down.



The diagram illustrates the typical response of a 2^{nd} order continuous-time system and the corresponding characteristics.

Transient response of LTI systems

- After discussing the concept of transient response of LTI systems, we will address the mathematical analysis of
  – First-order and second-order continuous-time systems
  – First-order and second-order discrete-time systems

Section 6.5.1 First-order continuous-time systems

- The differential equation for a first-order system is often expressed in the form

$$\tau \frac{dy(t)}{dt} + y(t) = x(t)$$

- The corresponding impulse and frequency responses are

$$h(t) = \frac{1}{\tau} e^{-t/\tau} \xleftrightarrow{CTFT} H(j\omega) = \frac{1}{j\omega\tau + 1}$$

- The step response of the system is

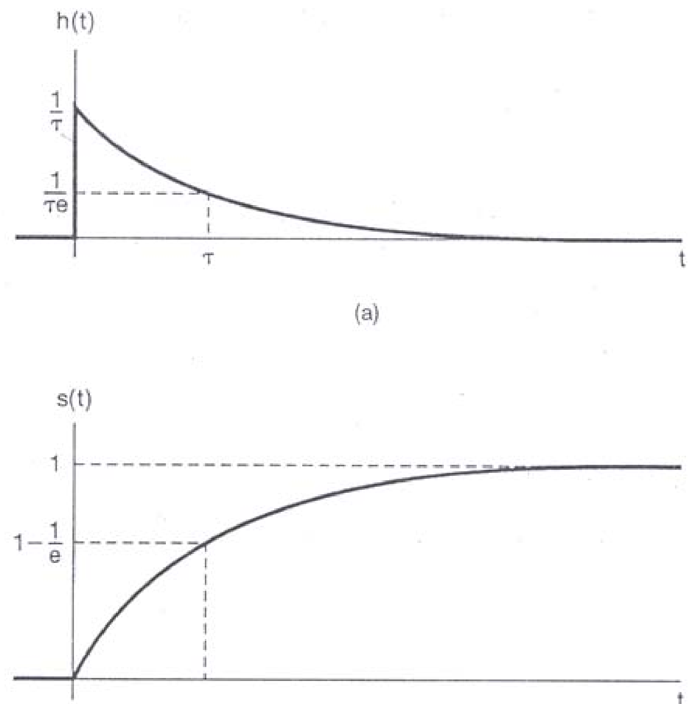


Figure 6.19 Continuous-time first-order system: (a) impulse response; (b) step response.

$$\begin{aligned} s(t) &= u(t) * h(t) \\ &= \left[1 - e^{-t/\tau}\right] u(t) \end{aligned} \xleftrightarrow{CTFT} \begin{aligned} S(j\omega) &= \left(\frac{1}{j\omega\tau} + \pi\delta(\omega)\right)\left(\frac{1}{j\omega\tau + 1}\right) \\ &= \left(\frac{1}{j\omega\tau}\right)\left(\frac{1}{j\omega\tau + 1}\right) + \left(\frac{\pi\delta(\omega)}{j\omega\tau + 1}\right) = \left[\frac{1}{j\omega\tau} + \pi\delta(\omega)\right] - \frac{1}{j\omega\tau + 1} \end{aligned}$$

Section 6.5.2 Second-order continuous-time systems

- The differential equation for a second-order system is

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

where ζ = damping ratio, ω_n = undamped natural frequency

- The frequency response for the second-order system is

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}$$

where

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \qquad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

Case 1: $0 < \zeta < 1: \quad c_1 = -\zeta\omega_n + j\omega_n\sqrt{1-\zeta^2}, \quad c_2 = -\zeta\omega_n - j\omega_n\sqrt{1-\zeta^2}$

Case 2: $\zeta = 1: \quad c_1 = c_2 = -\zeta\omega_n$

Case 3: $\zeta > 1: \quad c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} < 0, \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} < 0$

Section 6.5.2 Second-order continuous-time systems

Case 1: $0<\zeta<1$: $c_1=-\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}$, $c_2=-\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}$

The system function can be decomposed into two terms

$$H(j\omega)=\frac{\omega_n}{2\sqrt{\zeta^2-1}}\left(\frac{1}{j\omega-c_1}-\frac{1}{j\omega-c_2}\right)$$

The system is referred to as being under-damped.

The impulse response of the system is

$$h(t)=\frac{\omega_n}{2j\sqrt{1-\zeta^2}}\left[e^{\left(-\zeta\omega_n+j\omega_n\sqrt{1-\zeta^2}\right)t}-e^{\left(-\zeta\omega_n-j\omega_n\sqrt{1-\zeta^2}\right)t}\right]u(t)=\frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_n\sqrt{1-\zeta^2})t\right]u(t)$$

The step response of the system is

$$s(t)=h(t)*u(t)=\left\{1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_n\sqrt{1-\zeta^2})t+\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2})t\right]\right\}u(t)$$

Note: $$c_2e^{c_1t}-c_1e^{c_2t}=(a-jb)e^{(a+jb)t}-(a+jb)e^{(a-jb)t}=e^{at}\left[(a-jb)e^{jbt}-(a+jb)e^{-jbt}\right]$$
$$=e^{at}\left[a(e^{jbt}-e^{-jbt})-jb(e^{jbt}+e^{-jbt})\right]=j2e^{at}\left[a\sin(bt)-b\cos(bt)\right]$$

Section 6.5.2 Second-order continuous-time systems

Case 2: $\zeta = 1: \quad c_1 = c_2 = -\zeta\omega_n$

The impulse and frequency responses of the system are

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t) \xleftrightarrow{L} H(j\omega) = \frac{\omega_n^2}{(j\omega - \omega_n)^2}$$

The step response of the system is

$$s(t) = h(t) * u(t) = \left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right] u(t) \xleftrightarrow{L} S(j\omega) = \left[\frac{1}{j\omega} + \pi\delta(\omega)\right] - \frac{1}{j\omega - \omega_n} - \frac{\omega_n}{(j\omega - \omega_n)^2}$$

The system is referred to as being critical-damped.

Section 6.5.2 Second-order continuous-time systems

Case 3: $\zeta > 1: \quad c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} < 0, \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} < 0$

The impulse response of the system is

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(e^{c_1 t} - e^{c_2 t}\right)u(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}\left[e^{\left(\omega_n\sqrt{\zeta^2 - 1}\right)t} - e^{-\left(\omega_n\sqrt{\zeta^2 - 1}\right)t}\right]u(t)$$

The system is referred to as being over-damped.

The step response of the system is

$$s(t) = h(t) * u(t) = \left\{1 + \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left(\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right)\right\}u(t)$$

$$= \left\{1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{\zeta^2 - 1}}\left[\zeta\sinh(\omega_n\sqrt{\zeta^2 - 1})t + \sqrt{\zeta^2 - 1}\cosh(\omega_n\sqrt{\zeta^2 - 1})t\right]\right\}u(t)$$

Note:

$$c_2 e^{c_1 t} - c_1 e^{c_2 t} = (a - b)e^{(a+b)t} - (a + b)e^{(a-b)t} = e^{at}\left[(a - b)e^{bt} - (a + b)e^{-bt}\right]$$

$$= e^{at}\left[a(e^{bt} - e^{-bt}) - b(e^{bt} + e^{-bt})\right] = 2e^{at}\left[a\sinh(bt) - b\cosh(bt)\right]$$

Section 6.5.2 Second-order continuous-time systems

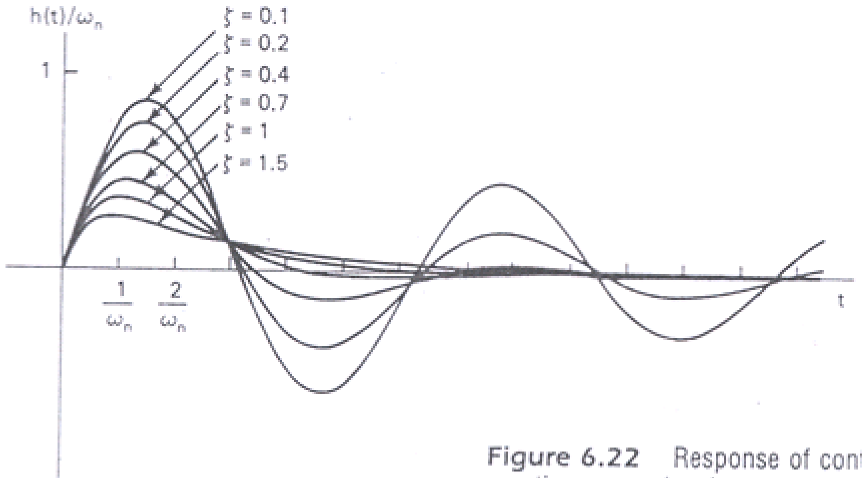
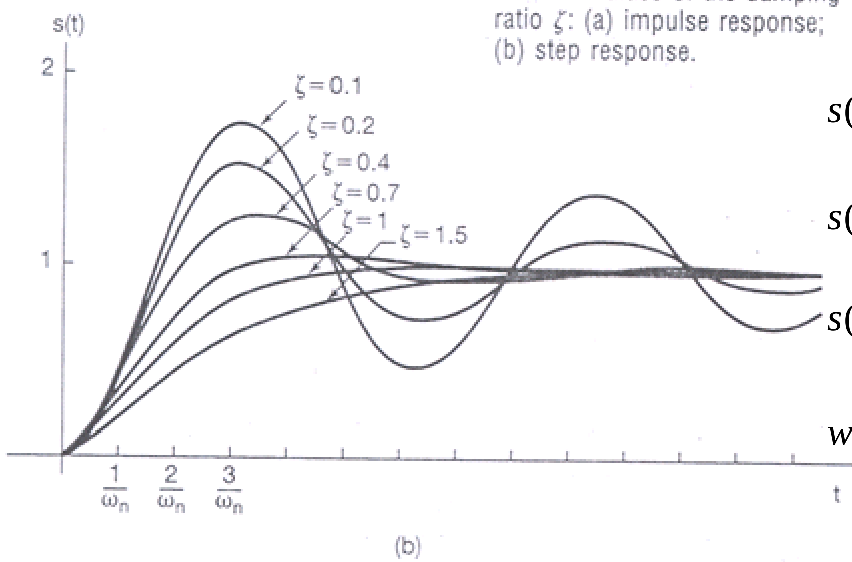


Figure 6.22 Response of continuous-time second-order systems with different values of the damping ratio ζ: (a) impulse response; (b) step response.



$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_n\sqrt{1-\zeta^2})t\right]u(t) \qquad 0<\zeta<1$$

$$h(t) = \omega_n^2 t e^{-\omega_n t}u(t) \qquad \zeta = 1$$

$$h(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2-1}}\left[e^{\left(\omega_n\sqrt{\zeta^2-1}\right)t} - e^{-\left(\omega_n\sqrt{\zeta^2-1}\right)t}\right]u(t) \qquad \zeta>1$$

$$s(t) = \left\{1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\zeta\sin(\omega_o t) + \sqrt{1-\zeta^2}\cos(\omega_o t)\right]\right\}u(t) \qquad 0<\zeta<1$$

$$s(t) = \left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right]u(t) \qquad \zeta = 1$$

$$s(t) = \left\{1 + \frac{e^{-\zeta\omega_n t}}{\sqrt{\zeta^2-1}}\left[\zeta\sinh(\omega_o t) + \sqrt{\zeta^2-1}\cosh(\omega_o t)\right]\right\}u(t) \qquad \zeta>1$$

$$where \qquad \omega_o = \omega_n\sqrt{\zeta^2-1}$$

Section 6.6.1 First-order discrete-time systems

- The difference equation for a first-order system is often expressed in the form

$$y[n]-ay[n-1]=x[n], \qquad |a|<1$$

- The corresponding impulse and frequency responses are

$$h[n]=a^n u[n] \xleftrightarrow{Z} H(e^{j\omega})=\frac{1}{1-ae^{-j\omega}}$$

- The step response of the system is

$$\begin{aligned} s[n] &= u[n]*h[n] \\ &= \frac{1-a^{n+1}}{1-a}u[n] \end{aligned} \quad \xleftrightarrow{Z} \quad \begin{aligned} S(e^{j\omega}) &= \left(\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)\right)\left(\frac{1}{1-ae^{-j\omega}}\right) \\ &= \left(\frac{1}{1-e^{-j\omega}}\right)\left(\frac{1}{1-ae^{-j\omega}}\right)+\left(\frac{1}{1-a}\right)\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k) \\ &= \frac{1}{1-a}\left(\frac{1}{1-e^{-j\omega}}+\sum_{k=-\infty}^{\infty}\pi\delta(\omega-2\pi k)-\frac{1}{1-ae^{-j\omega}}\right) \end{aligned}$$

Section 6.6.1 First-order discrete-time systems

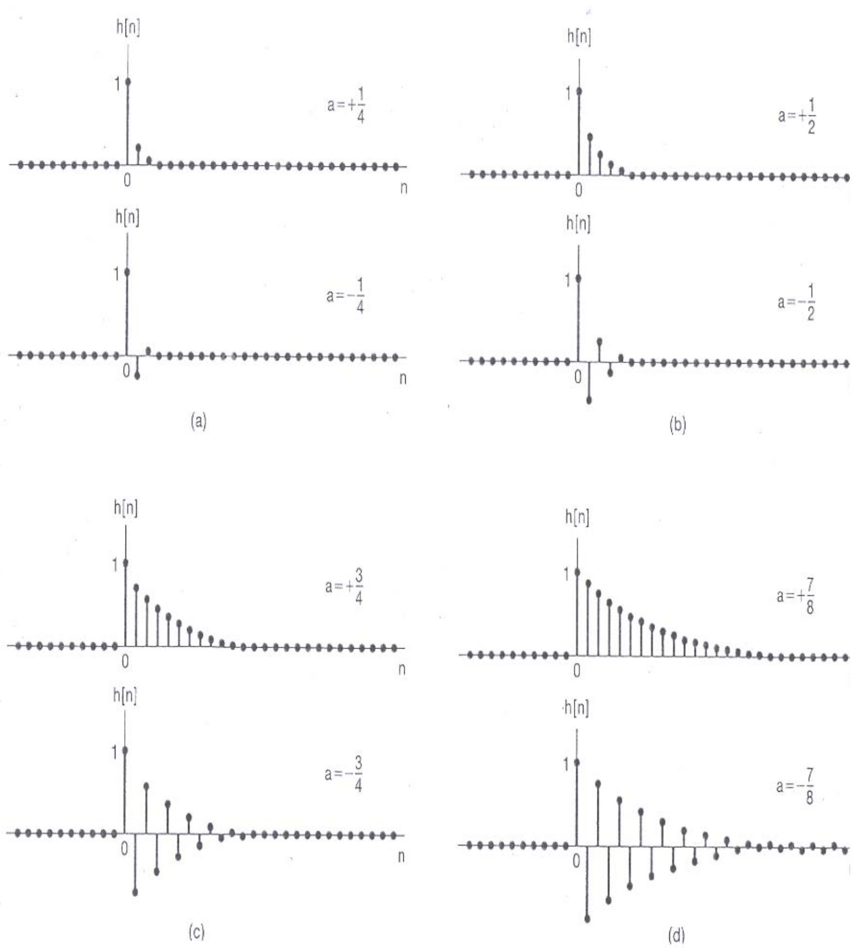


Figure 6.26 Impulse response $h[n] = a^n u[n]$ of a first-order system: (a) $a = \pm 1/4$; (b) $a = \pm 1/2$; (c) $a = \pm 3/4$; (d) $a = \pm 7/8$.

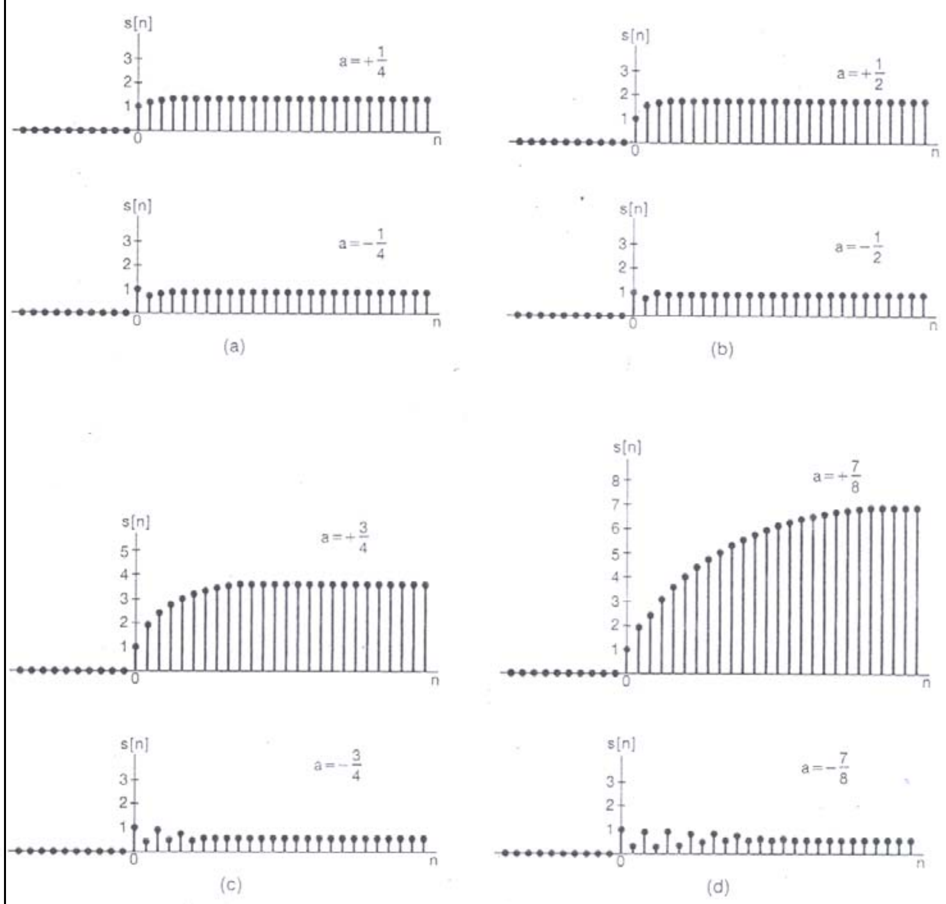


Figure 6.27 Step response $s[n]$ of a first-order system: (a) $a = \pm 1/4$; (b) $a = \pm 1/2$; (c) $a = \pm 3/4$; (d) $a = \pm 7/8$.

Section 6.6.2 Second-order discrete-time systems

- The difference equation for a second-order system is

$$y[n]-(2r\cos\theta)y[n-1]+r^2y[n-2]=x[n]$$

where $0<r<1$, $0\le\theta\le\pi$.

- The frequency response for the second-order system is

$$H(e^{j\omega})=\frac{1}{1-(2r\cos\theta)e^{-j\omega}+r^2e^{-j2\omega}}=\frac{1}{(1-re^{j\theta}e^{-j\omega})(1-re^{-j\theta}e^{-j\omega})}$$

Case 1: $\theta=0$ Critically damped system

Case 2: $\theta=\pi$

Case 3: $0<\theta<\pi$ Under-damped system

Case 4: The system has two distinct real roots.

Section 6.6.2 Second-order discrete-time systems

- Case 1: $\theta = 0$
- The frequency response becomes

$$H(e^{j\omega}) = \frac{1}{\left(1 - re^{-j\omega}\right)^2}$$

- The impulse response becomes

$$h[n] = (n+1)r^n u[n]$$

- The step response becomes

$$s[n] = h[n] * u[n]$$
$$= \left[\frac{1}{(r-1)^2} - \frac{r^{n+1}}{(r-1)^2} + \frac{(n+1)r^{n+1}}{r-1}\right]u[n]$$

- Case 2: $\theta = \pi$
- The frequency response becomes

$$H(e^{j\omega}) = \frac{1}{\left(1 + re^{-j\omega}\right)^2}$$

- The impulse response becomes

$$h[n] = (n+1)(-r)^n u[n]$$

- The step response becomes

$$s[n] = h[n] * u[n]$$
$$= \left[\frac{1}{(r+1)^2} - \frac{r(-r)^n}{(r+1)^2} + \frac{(n+1)r(-r)^n}{r+1}\right]u[n]$$

Section 6.6.2 Second-order discrete-time systems

- Case 3: $0 < \theta < \pi$

  The system function can be decomposed into two terms

$$H(e^{j\omega}) = \frac{1}{2j\sin\theta}\left[\frac{e^{j\theta}}{1-re^{j\theta}e^{-j\omega}} - \frac{e^{-j\theta}}{1-re^{-j\theta}e^{-j\omega}}\right]$$

  The impulse response of the system is

$$h[n] = \frac{1}{2j\sin\theta}\left[e^{j\theta}\left(re^{j\theta}\right)^n + e^{-j\theta}\left(re^{-j\theta}\right)^n\right]u[n] = r^n\,\frac{\sin[(n+1)\theta]}{\sin\theta}u[n]$$

  The step response of the system is

$$\begin{aligned} s[n] &= h[n] * u[n] \\ &= \frac{1}{2j\sin\theta}\left[e^{j\theta}\,\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}} - e^{-j\theta}\,\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right]u[n] \\ &= \left\{\frac{\sin\theta - r^{n+1}\sin[(n+2)\theta] + r^{n+2}\sin[(n+1)\theta]}{\sin\theta}\right\}u[n] \end{aligned}$$

Section 6.6.2 Second-order discrete-time systems

- Case 4: Consider the system has two distinct real roots. The system function becomes

$$H(e^{j\omega}) = \frac{1}{\left(1-d_1e^{-j\omega}\right)\left(1-d_2e^{-j\omega}\right)} = \frac{1}{d_1-d_2}\left[\frac{d_1}{1-d_1e^{-j\omega}} - \frac{d_2}{1-d_2e^{-j\omega}}\right]$$

- The impulse response of the system is $h[n] = \left(\frac{d_1^{n+1}-d_2^{n+1}}{d_1-d_2}\right)u[n]$

- The step response of the system is

$$s[n] = h[n]*u[n] = \frac{1}{d_1-d_2}\left[d_1\left(\frac{1-d_1^{n+1}}{1-d_1}\right) - d_2\left(\frac{1-d_2^{n+1}}{1-d_2}\right)\right]u[n]$$

- The system is over-damped when d_1 and d_2 are positive.

Section 6.6.2 Second-order discrete-time systems

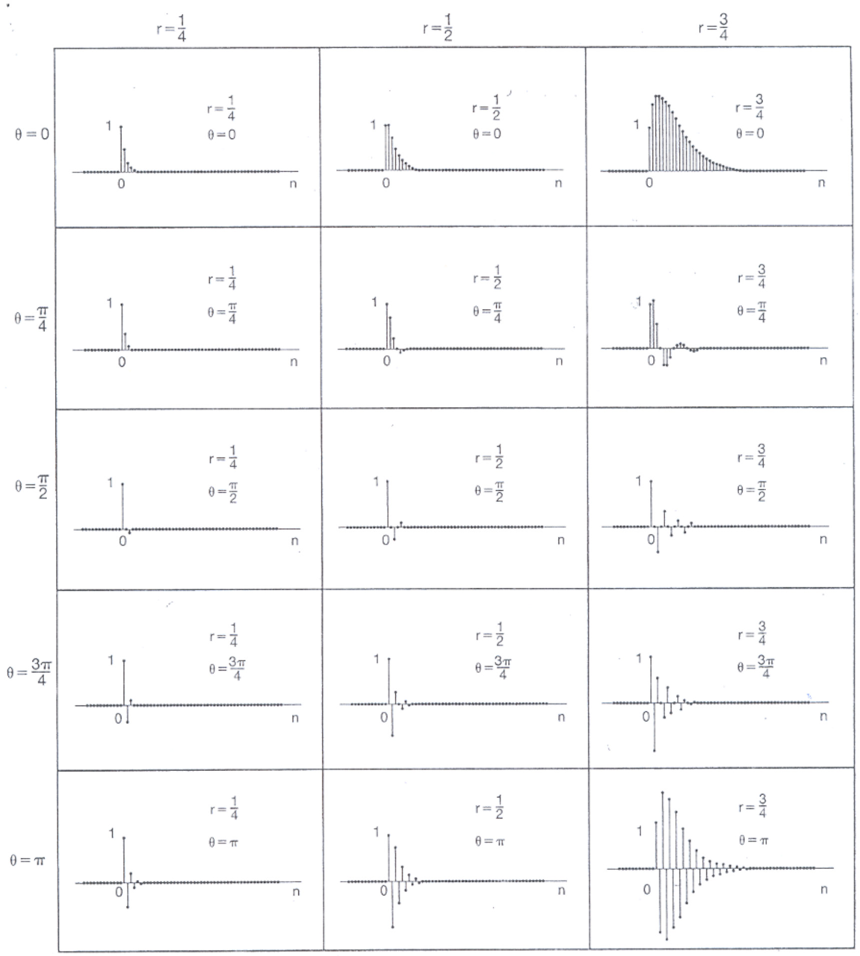
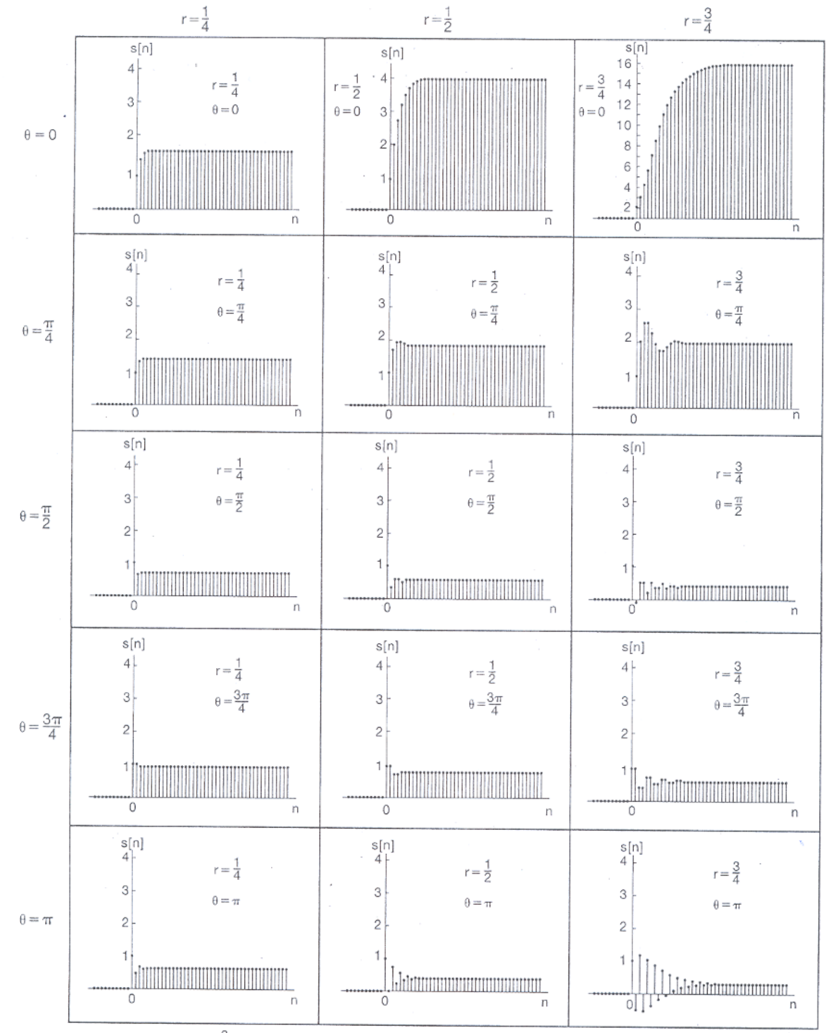


Figure 6.29 Impulse response of the second-order system of eq. (6.57) for a range of values of r and θ.



* Note: The plot for $r=\frac{3}{4}$, $\theta=0$ has a different scale from the others.

Figure 6.30 Step response of the second-order system of eq. (6.57) for a range of values of r and θ.

Frequency-domain analysis: Frequency response

- In the review, we addressed the advantages of Frequency domain analysis:
  1. Convolution in time
     Differentiation in time
     Difference in time
     } ➔ Plus, minus algebraic operations in frequency domain => easier!
  2. Filtering ⇨ easier to visualize in frequency domain
  3. Characteristics of speech/communication, etc. ⇨ easier to visualize in frequency domain
- Need trade-offs between time-domain and frequency-domain characteristics (e.g. implementing a filter) for system design analysis and implementation

Frequency-domain analysis: Frequency response

- From the convolution property, we have

Continuous-Time System response

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Discrete-Time system response

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

- With the use of magnitude-phase representation, the relationship between input signal, system function and output signal can be re-written as
  - Magnitude (It is also referred as Magnitude Response / Spectrum)

Continuous-Time System response

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

Discrete-Time system response

$$|Y(e^{j\omega})| = |H(e^{j\omega})||X(e^{j\omega})|$$

  - Phase (It is also referred as Phase Response / Spectrum)

Continuous-Time System response

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Discrete-Time system response

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$

Frequency-domain analysis: Frequency response

- Magnitude information:
  - It is also called Magnitude Spectrum / Magnitude Response
  - Magnitude: $|H(j\omega)|$ for continuous-time, $|H(e^{j\omega})|$ for discrete-time
  - It can be referred as gain of system.
  - Gain distortion affects the time-domain signal, and it can be important in signal amplification or rejection.

Frequency-domain analysis: Frequency response

- Phase information:
  - It is also called Phase Spectrum / Phase Response
  - Phase: $\angle H(j\omega)$ for continuous-time, $\angle H(e^{j\omega})$ for discrete-time
  - It can be referred as the amount of delay of system
  - Phase distortion affects time-domain signal, and it can be important (in image) or insignificant (in speech)
    - However, severe phase distortion is also important in speech signal $x(-t)$

A tape played backwards → $$\Im\{x(-t)\} = X(-j\omega) = |X(j\omega)|e^{-j\angle X(j\omega)}$$ ← Phase distortion

  - Meanwhile, the non-linearity of the phase response tells us how much phase distortion in the system.

Frequency-domain analysis: Frequency response

- For a stable LTI system has a rational system function, then its frequency response has the form,

Continuous-Time System response

$$H(j\omega) = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{M} a_k (j\omega)^k} = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} (j\omega - c_k)}{\prod_{k=0}^{N} (j\omega - d_k)}$$

Discrete-Time system response

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} a_k e^{-j\omega k}}{\sum_{k=0}^{M} b_k e^{-j\omega k}} = \frac{b_0}{a_0} \frac{\prod_{k=0}^{M} \left(1 - c_k e^{-j\omega k}\right)}{\prod_{k=0}^{N} \left(1 - d_k e^{-j\omega k}\right)}$$

Frequency-domain analysis: Frequency response

- The magnitude response of a rational system function is

Continuous-Time System response

$$|H(j\omega)| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=0}^{M} |j\omega - c_k|}{\prod_{k=0}^{N} |j\omega - d_k|}$$

Discrete-Time system response

$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=0}^{M} |1 - c_k e^{-j\omega k}|}{\prod_{k=0}^{N} |1 - d_k e^{-j\omega k}|}$$

- Sometimes it is convenient to consider the gain in dB scale, so

Continuous-Time System response

$$10\log_{10}|H(j\omega)|^2 = 20\log_{10}\left|\frac{b_0}{a_0}\right| + 20\sum_{k=0}^{M}\log_{10}|j\omega - c_k| - 20\sum_{k=0}^{N}\log_{10}|j\omega - d_k|$$

Discrete-Time system response

$$10\log_{10}|H(e^{j\omega})|^2 = 20\log_{10}\left|\frac{b_0}{a_0}\right| + 20\sum_{k=0}^{M}\log_{10}|1 - c_k e^{-j\omega k}| - 20\sum_{k=0}^{N}\log_{10}|1 - d_k e^{-j\omega k}|$$

Frequency-domain analysis: Frequency response

- The phase response of a rational system function is

  Continuous-Time System response

$$\angle H(j\omega) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=0}^{M} \angle\left(j\omega - c_k\right) - \sum_{k=0}^{N} \angle\left(j\omega - d_k\right)$$

  Discrete-Time system response

$$\angle H(e^{j\omega}) = \angle\left(\frac{b_0}{a_0}\right) + \sum_{k=0}^{M} \angle\left(1 - c_k e^{-j\omega k}\right) - \sum_{k=0}^{N} \angle\left(1 - d_k e^{-j\omega k}\right)$$

- Sometimes, $\angle H(j\omega)$ denotes as $\arg H(j\omega)$. So as $\arg H(e^{j\omega})$ for $\angle H(e^{j\omega})$
- Usually, two different format in phase response
  - Unwrapped phase: It is a continuous phase
  - Wrapped phase: It wraps the phase in the range from $-\pi$ to π.

Frequency-domain analysis: Frequency response

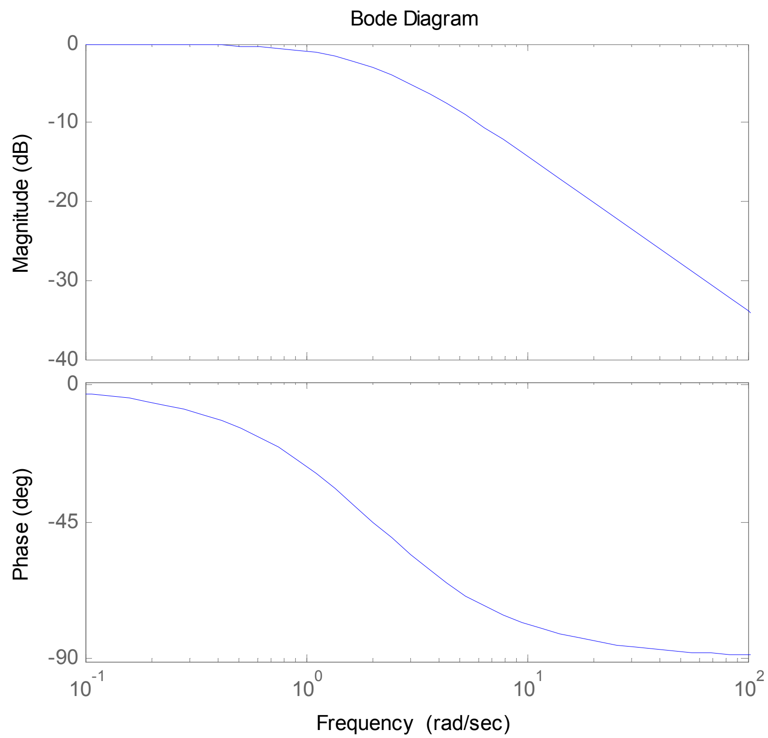
- A bode plot shows the frequency response of system function using
  - A logarithm frequency scale for magnitude spectrum
  - A wrapped / unwarpped phase spectrum

- If $h(t)$ (or $h[n]$) is real,
  - Magnitude spectrum $|H(j\omega)|$ (or $|H(e^{j\omega})|$) is even symmetry.
  - Phase spectrum $\angle H(j\omega)$ (or $\angle H(e^{j\omega})$) is odd symmetry.

- Therefore, the plot for positive ω is enough.

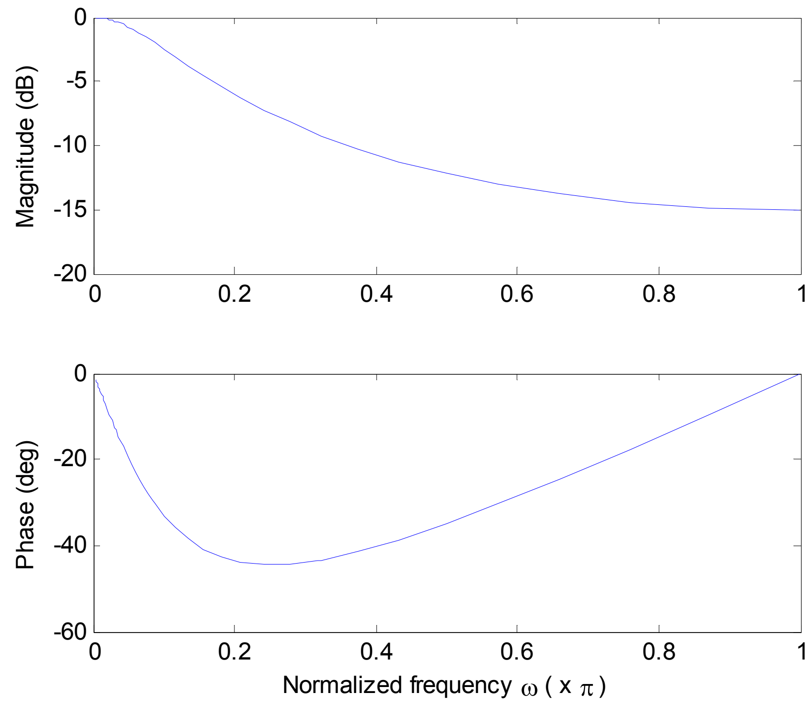
Frequency-domain analysis: Frequency response

First-order continuous-time LTI system



$$H(j\omega) = \frac{2}{j\omega + 2}$$

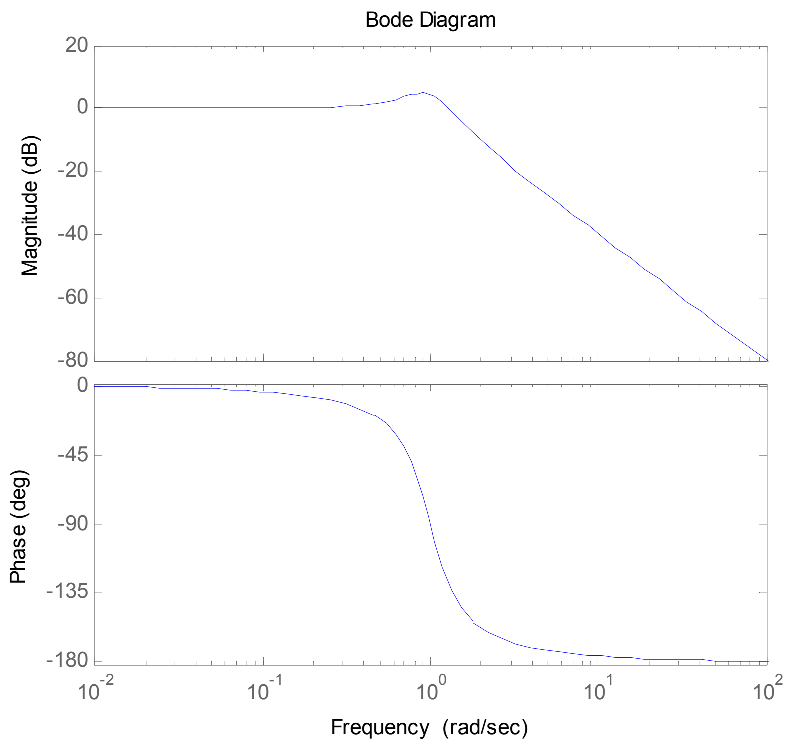
First-order discrete-time LTI system



$$H(e^{j\omega}) = \frac{0.3}{1 - 0.7e^{-j\omega}}$$

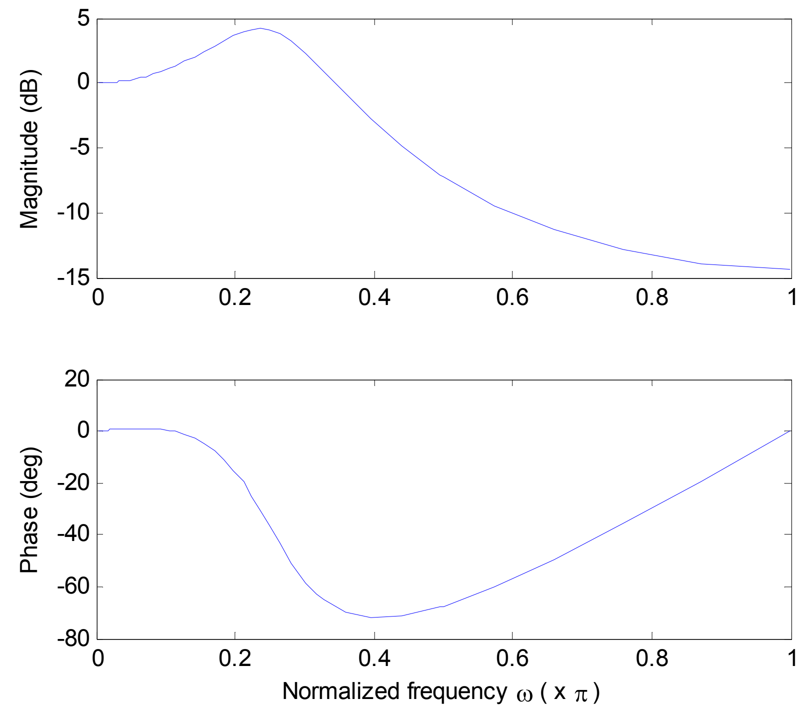
Frequency-domain analysis: Frequency response

Second-order continuous-time LTI system



$$H(j\omega)=\frac{1}{(j\omega)^2+0.6(j\omega)+1}$$

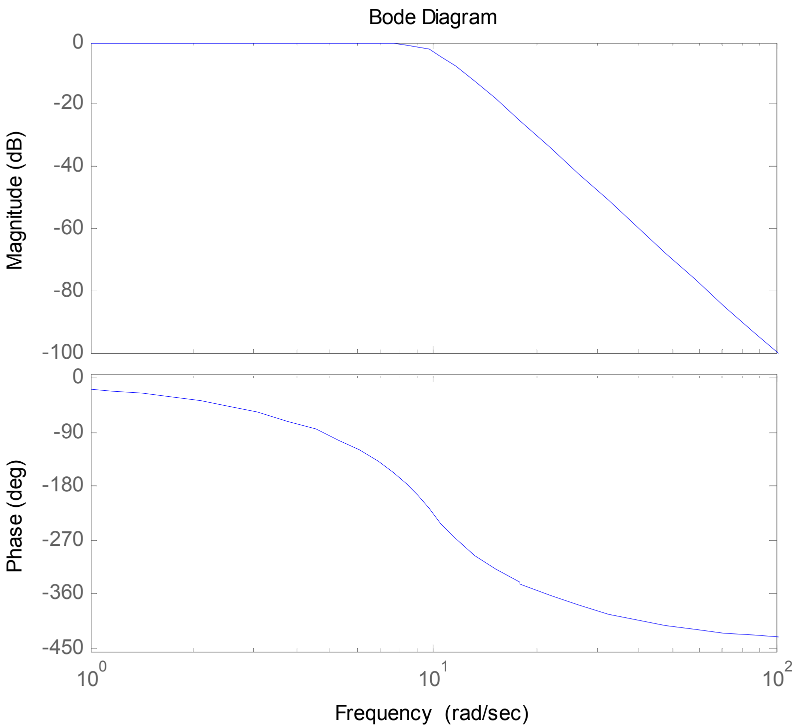
Second-order discrete-time LTI system



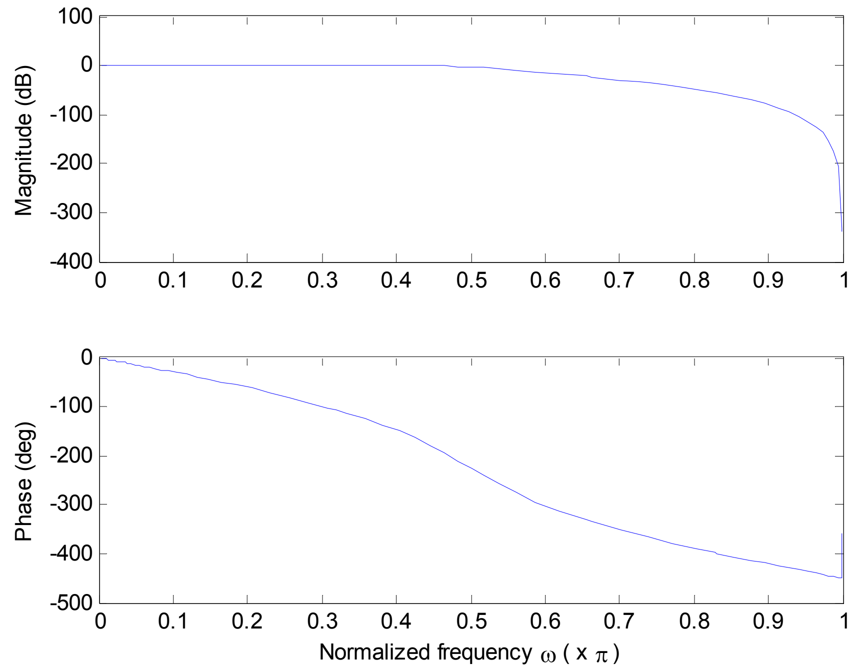
$$H(e^{j\omega})=\frac{1-1.5\cos\left(\frac{\pi}{4}\right)+0.75^2}{e^{-j2\omega}-1.5\cos\left(\frac{\pi}{4}\right)e^{-j\omega}+0.75^2}$$

Frequency-domain analysis: Frequency response

A fifth-order analog Butterworth filter (BW of 10 rad/s)



A fifth-order digital Butterworth filter (BW of 0.5π rad/s)



Frequency response of LTI systems

- After discussing the concept of frequency response of LTI systems, we will address the mathematical analysis of
  - First-order and second-order continuous-time systems
  - First-order and second-order discrete-time systems

Section 6.5.1 First-order continuous-time systems

- The frequency response of a typical first-order system is

$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

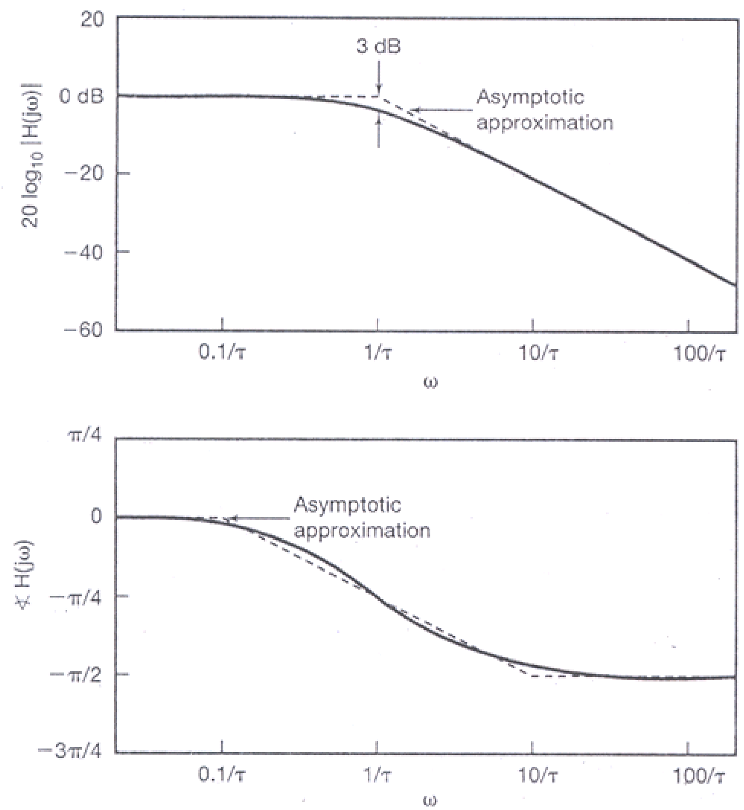


Figure 6.20 Bode plot for a continuous-time first-order system.

- The log magnitude of system is

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left[(\omega\tau)^2 + 1\right]$$

- The phase of system is

$$\angle H(j\omega) = -\tan^{-1}(\omega\tau)$$

Section 6.5.2 Second-order continuous-time systems

- The frequency response for the second-order system is

$$H(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}=\frac{\omega_n^2}{(\omega_n^2-\omega^2)+j(2\zeta\omega_n\omega)}$$

- The log magnitude of system is

$$20\log_{10}|H(j\omega)|=40\log_{10}\omega_n-10\log_{10}\left[(\omega_n^2-\omega^2)^2+(2\zeta\omega_n\omega)^2\right]$$

- The phase of system is

$$\angle H(j\omega)=-\tan^{-1}\left(\frac{2\zeta\omega_n\omega}{\omega_n^2-\omega^2}\right)$$

- $|H(j\omega)|$ has a maximum value at $\omega_{\max}=\omega_n\sqrt{1-2\zeta^2}$ with $|H(j\omega_{\max})|=\frac{1}{2\zeta\sqrt{1-\zeta^2}}$

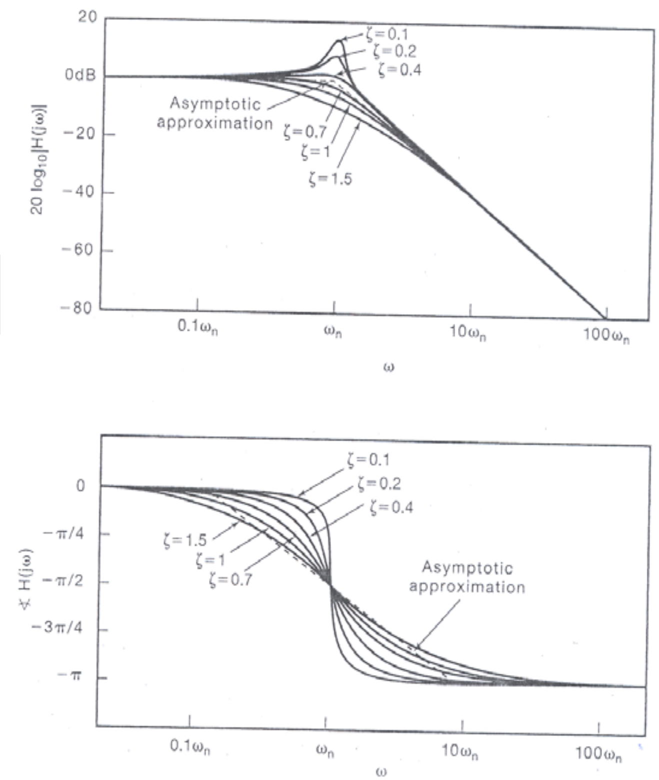
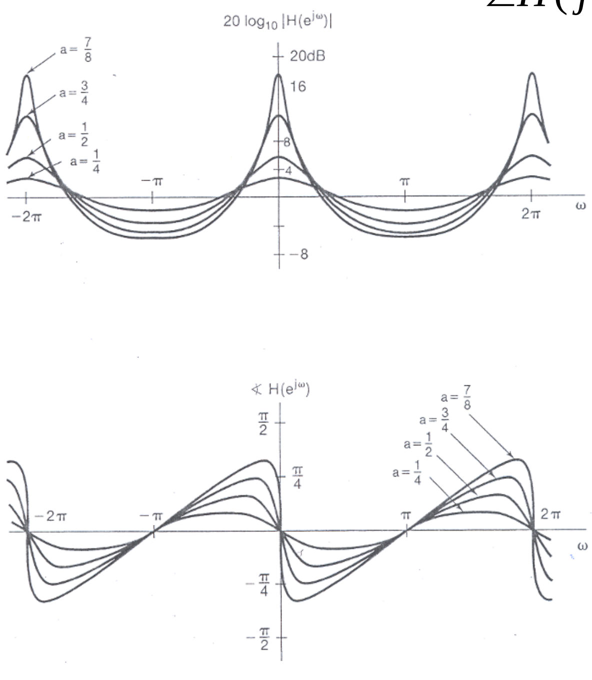


Figure 6.23 Bode plots for second-order systems with several different values of damping ratio ζ.

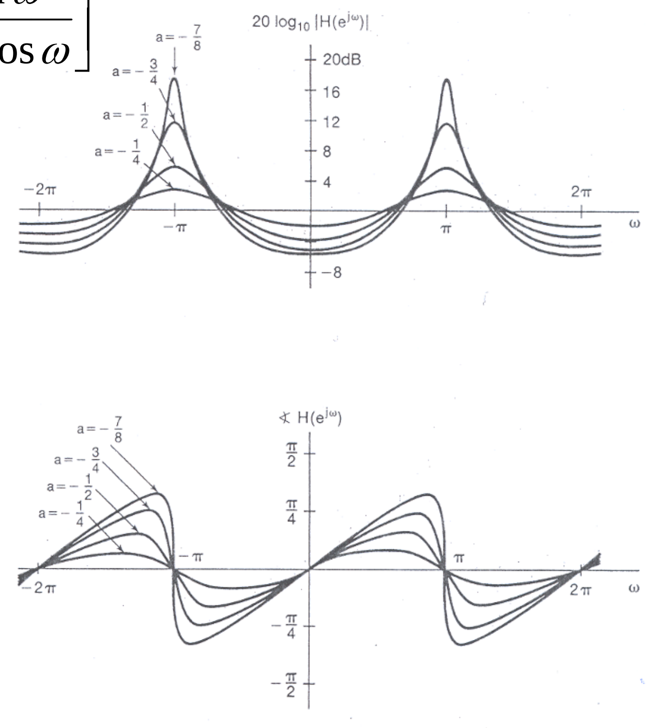
Section 6.6.1 First-order discrete-time systems

- The frequency response of first-order system is $H(e^{j\omega}) = \dfrac{1}{1 - ae^{-j\omega}}$
- The log magnitude of system is $20\log_{10}|H(j\omega)| = -20\log_{10}\left[1 + a^2 - 2a\cos\omega\right]$
- The phase of system is $\angle H(j\omega) = -\tan^{-1}\left[\dfrac{a\sin\omega}{1 - a\cos\omega}\right]$



(a)

Figure 6.28 Magnitude and phase of the frequency response of eq. (6.52) for a first-order system: (a) plots for several values of $a > 0$; (b) plots for several values of $a < 0$.



(b)

Figure 6.28 *Continued*

Section 6.6.2 Second-order discrete-time systems

- The frequency response for the second-order system is

$$H(e^{j\omega}) = \frac{1}{1-(2r\cos\theta)e^{-j\omega} + r^2 e^{-j2\omega}}$$

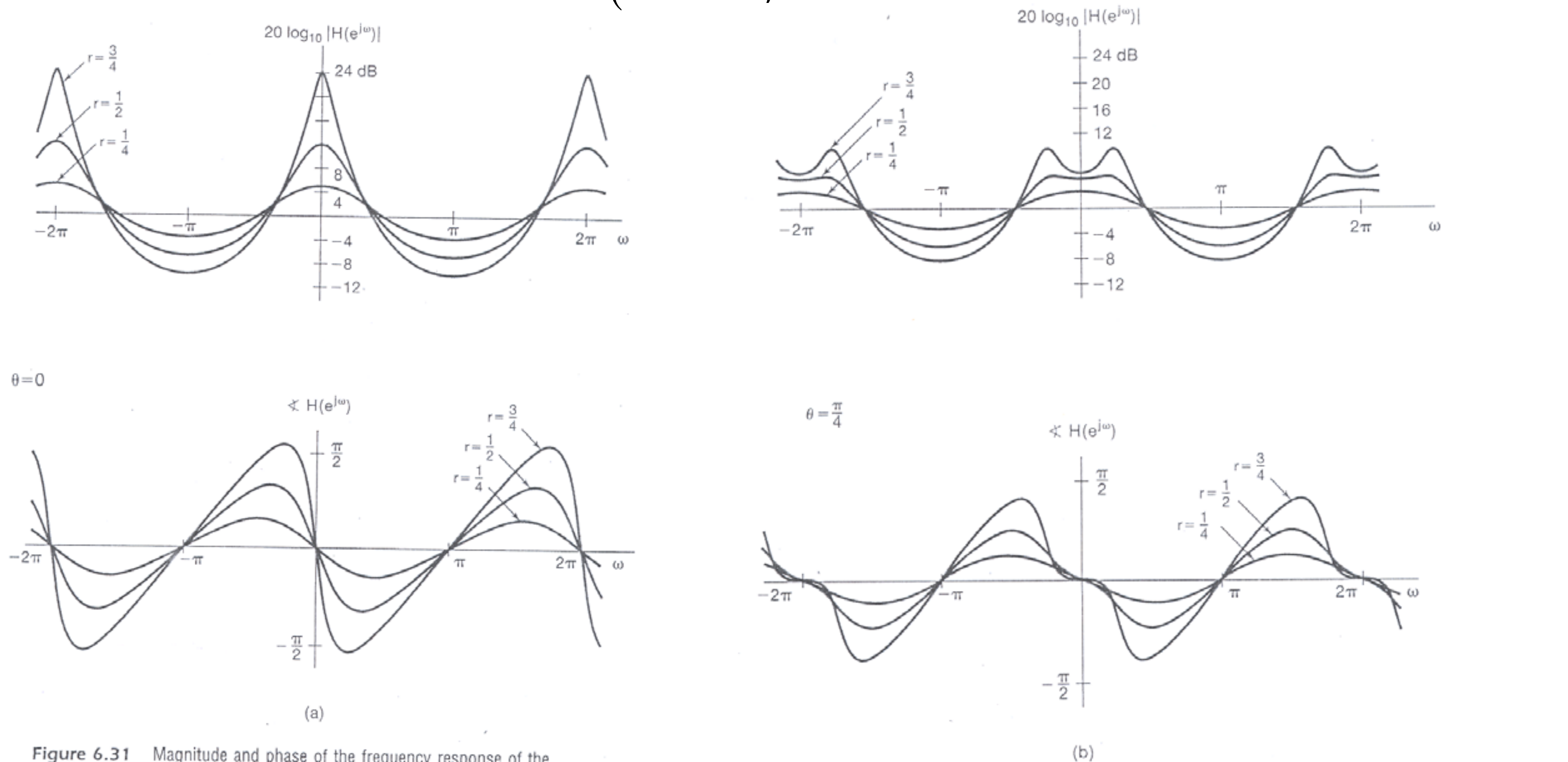


Figure 6.31 Magnitude and phase of the frequency response of the second-order system of eq. (6.57): (a) $\theta = 0$; (b) $\theta = \pi/4$; (c) $\theta = \pi/2$; (d) $\theta = 3\pi/4$; (e) $\theta = \pi$. Each plot contains curves corresponding to r = 1/4, 1/2, and 3/4.

Figure 6.31 *Continued*

Section 6.6.2 Second-order discrete-time systems

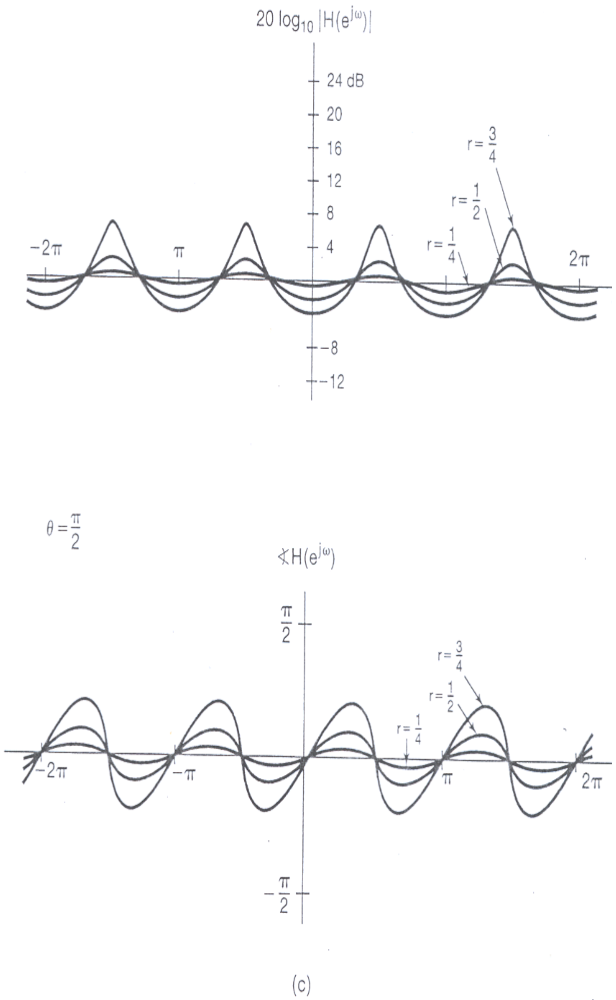


Figure 6.31 *Continued*

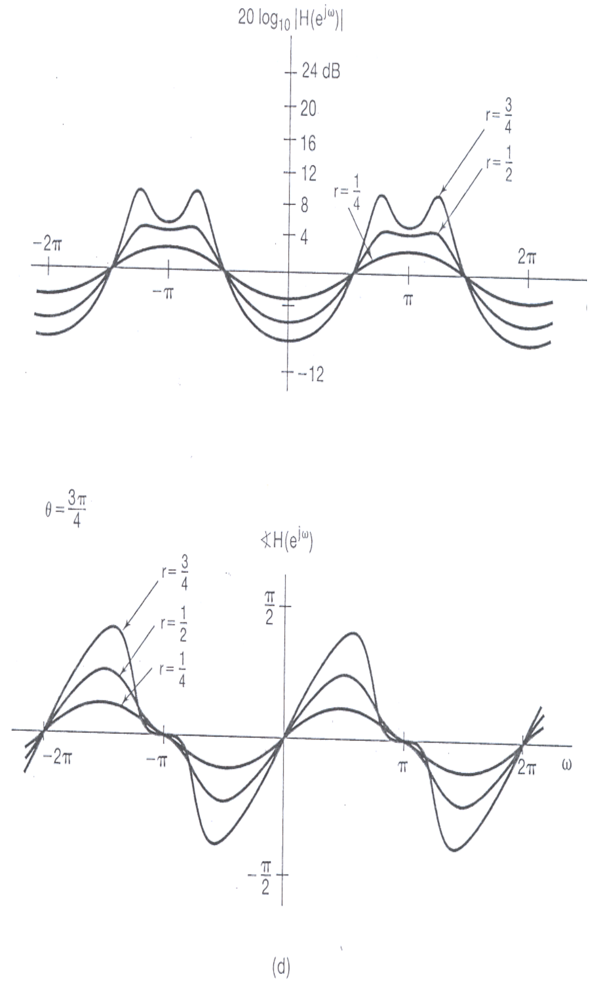


Figure 6.31 *Continued*

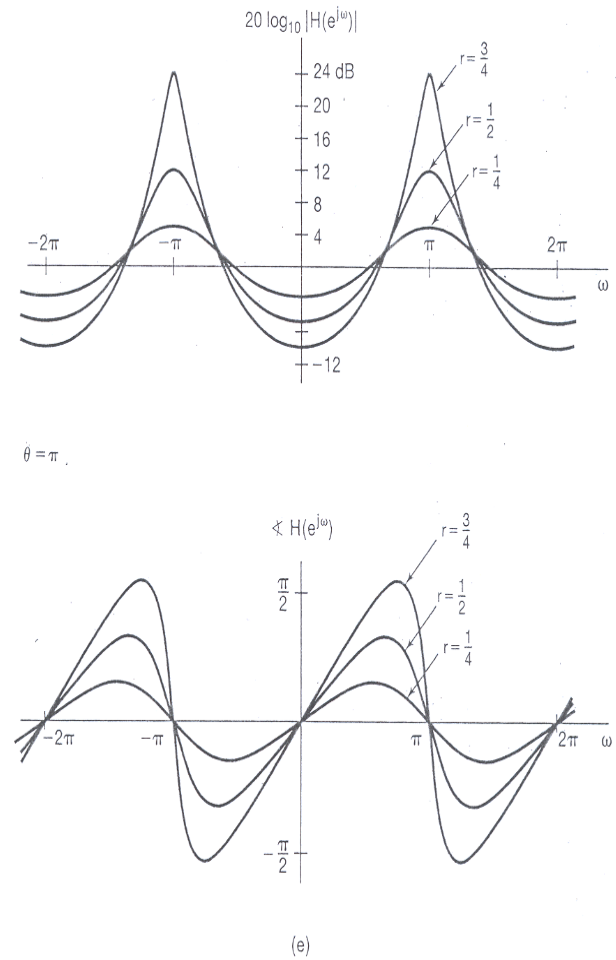


Figure 6.31 *Continued*

Section 6.5.3 Bode plots for rational Frequency responses

- Recall, the frequency response of a typical first-order continuous-time LTI system

$$H(j\omega) = \frac{1}{j\omega\tau + 1} = \frac{1}{\frac{j\omega}{1/\tau} + 1}$$

Numerical values

ω	$\lvert H(j\omega)\rvert$	$20\log\lvert H(j\omega)\rvert$	$\angle H(j\omega)$
0	1	0 dB	0
0.1 τ	≈ 1	0 dB	- 0.03 π
0.5 τ	≈ 0.893	- 0.97 dB	- 0.15 π
τ	≈ 0.707	- 3.01 dB	- 0.25 π
2 τ	≈ 0.447	- 6.99 dB	- 0.35 π
10 τ	≈ 0.1	- 20 dB	- 0.47 π
100 τ	≈ 0.01	- 40 dB	- 0.5 π
1000 τ	≈ 0.001	- 60 dB	- 0.5 π

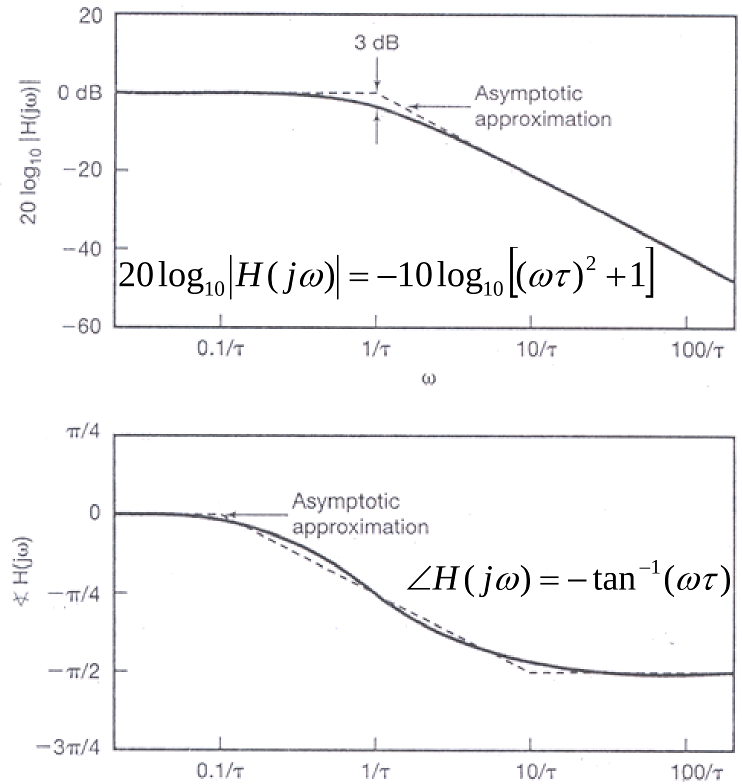


Figure 6.20 Bode plot for a continuous-time first-order system.

Section 6.5.3 Bode plots for rational Frequency responses

- Clearly, the actual curves are very difficult to plot, but we can identify some useful asymptotes for continuous-time LTI system.

- For the magnitude spectrum, we have two asymptotes:
  - A horizontal line
  - A straight line with a slope of -20 dB / decade
  - They meet at the point associated with $\omega = \frac{1}{\tau}$

- For the magnitude spectrum, we have three asymptotes:
  - Two horizontal lines
  - A straight line with a slope of -45° (or - π /4) / decade
  - They meet at the points associated with $\omega = \frac{0.1}{\tau}$ and $\omega = \frac{10}{\tau}$

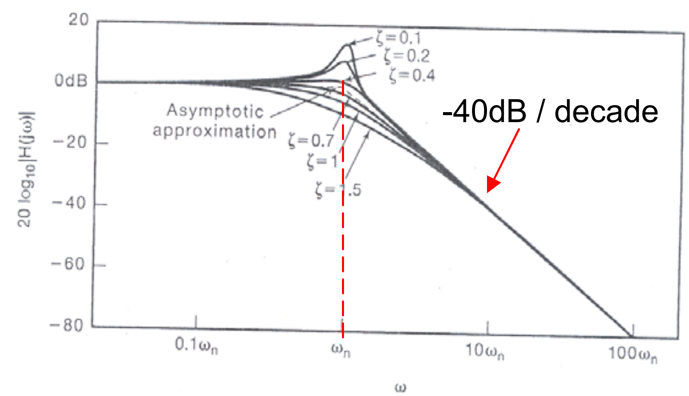
Section 6.5.3 Bode plots for rational Frequency responses

- Similarly, we can also identify some useful asymptotes for second-order continuous-time LTI system

$$H(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}=\frac{\omega_n^2}{(\omega_n^2-\omega^2)+j(2\zeta\omega_n\omega)}$$

Numerical values for $\zeta = 0.1$

ω	$\lvert H(j\omega)\rvert$	$20\log\lvert H(j\omega)\rvert$	$\angle H(j\omega)$
0	1	0 dB	0
0.1 ω_n	≈ 1.01	0.09 dB	- 0.01 π
0.5 ω_n	≈ 1.32	2.42 dB	- 0.04 π
$\omega_{\max}$	≈ 5.03	14.02 dB	- 0.47 π
ω_n	≈ 5	13.98 dB	- 0.50 π
2 ω_n	≈ 0.33	- 9.62 dB	- 0.96 π
10 ω_n	≈ 0.01	- 39.91 dB	- 0.99 π
100 ω_n	≈ 0.001	- 80 dB	- π
1000 ω_n	≈ 10^{-6}	- 120 dB	- π



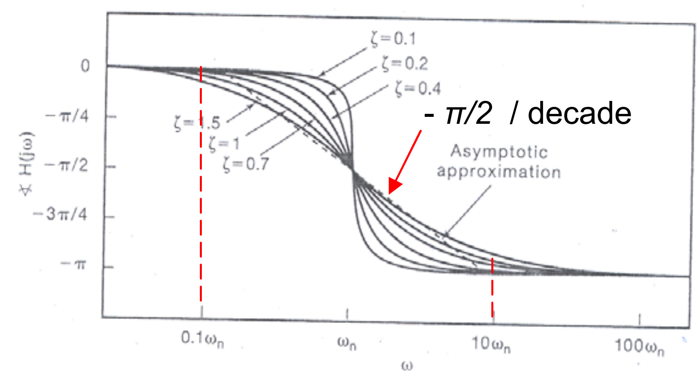


Figure 6.23 Bode plots for second-order systems with several different values of damping ratio ζ.

Section 6.5.3 Bode plots for rational Frequency responses

- Consider the frequency responses of the forms

$$\tilde{H}(j\omega) = 1 + j\omega\tau \qquad\qquad \tilde{H}(j\omega) = 1 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2$$

- The bode plots of these systems follow directly from Figures 6.20 and 6.23 and the from the fact that

$$20\log_{10}\left|\tilde{H}(j\omega)\right| = 20\log_{10}\left|\frac{1}{H(j\omega)}\right| = -20\log_{10}\left|H(j\omega)\right|$$

$$\angle\tilde{H}(j\omega) = \angle\frac{1}{H(j\omega)} = -\angle H(j\omega)$$

- For the magnitude and phase spectrum, the asymptotes are horizontal line(s) and straight line with positive slope. Their meeting points remain the same.

Section 6.5.3 Bode plots for rational Frequency responses

- Also, consider a system function that is a constant gain

$$\tilde{H}(j\omega) = K$$

- Its magnitude and phase spectrum are

$$20\log_{10}\left|\tilde{H}(j\omega)\right| = 20\log_{10}|K|$$

$$\angle\tilde{H}(j\omega) = \begin{cases} 0, & K > 0 \\ \pi, & K < 0 \end{cases}$$

Section 6.5.3 Bode plots for rational Frequency responses

- Example 6.5 Consider the frequency response

$$H(j\omega)=\frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)}$$
$$=\frac{1}{10}\left(\frac{1}{1+j\omega/10}\right)\left(\frac{1}{1+j\omega/100}\right)(1+j\omega)$$

DC gain : 1/10

poles: 10, 100

zero : 1

Solid line: actual Bode plot

Dash line: Bode plot is obtained by straight-line approximation

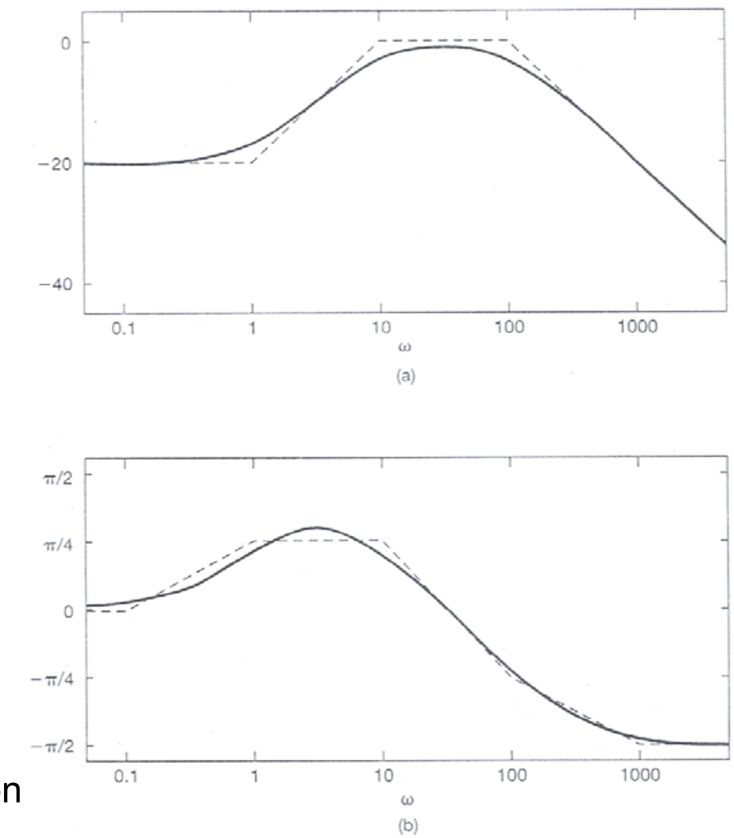


Figure 6.25 Bode plot for system function in Example 6.5: (a) magnitude; (b) phase.

Section 6.5.3 Bode plots for rational Frequency responses

- Example 6.5 (Cont'd) The details of straight-line approximation:

$$H(j\omega)=\frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)}=\frac{1}{10}\left(\frac{1}{1+j\omega/10}\right)\left(\frac{1}{1+j\omega/100}\right)(1+j\omega)$$

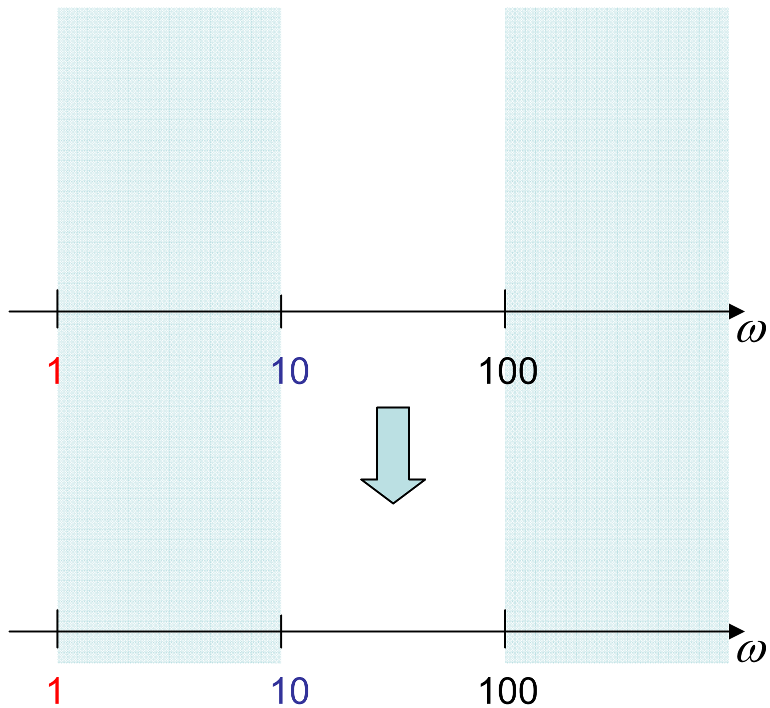
DC gain : 1/10

poles: 10, 100

zero : 1

Magnitude

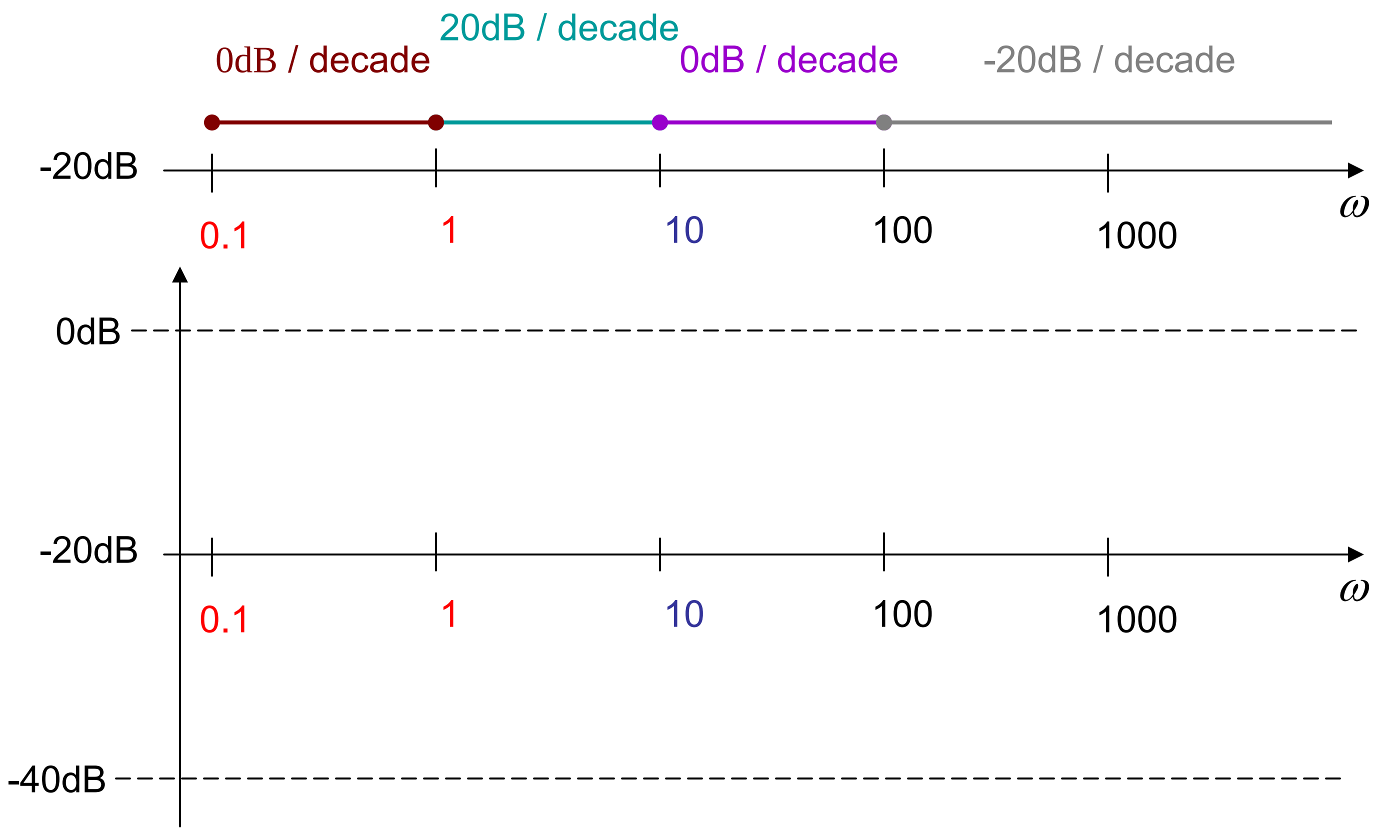
Magnitude



Section 6.5.3 Bode plots for rational Frequency responses

- Example 6.5 (Cont'd)

Magnitude



Section 6.5.3 Bode plots for rational Frequency responses

- Example 6.5 (Cont'd)

$$H(j\omega)=\frac{100(1+j\omega)}{(10+j\omega)(100+j\omega)}=\frac{1}{10}\left(\frac{1}{1+j\omega/10}\right)\left(\frac{1}{1+j\omega/100}\right)(1+j\omega)$$

DC gain : 1/10

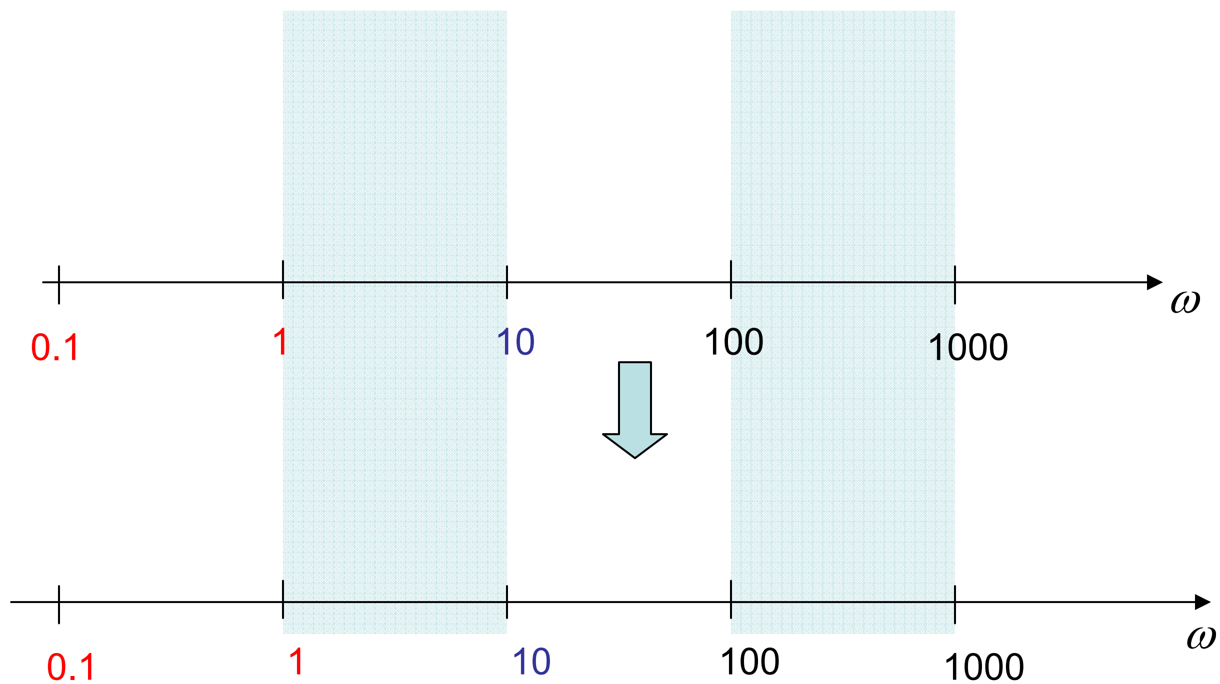
poles: 10, 100

zero : 1

Phase

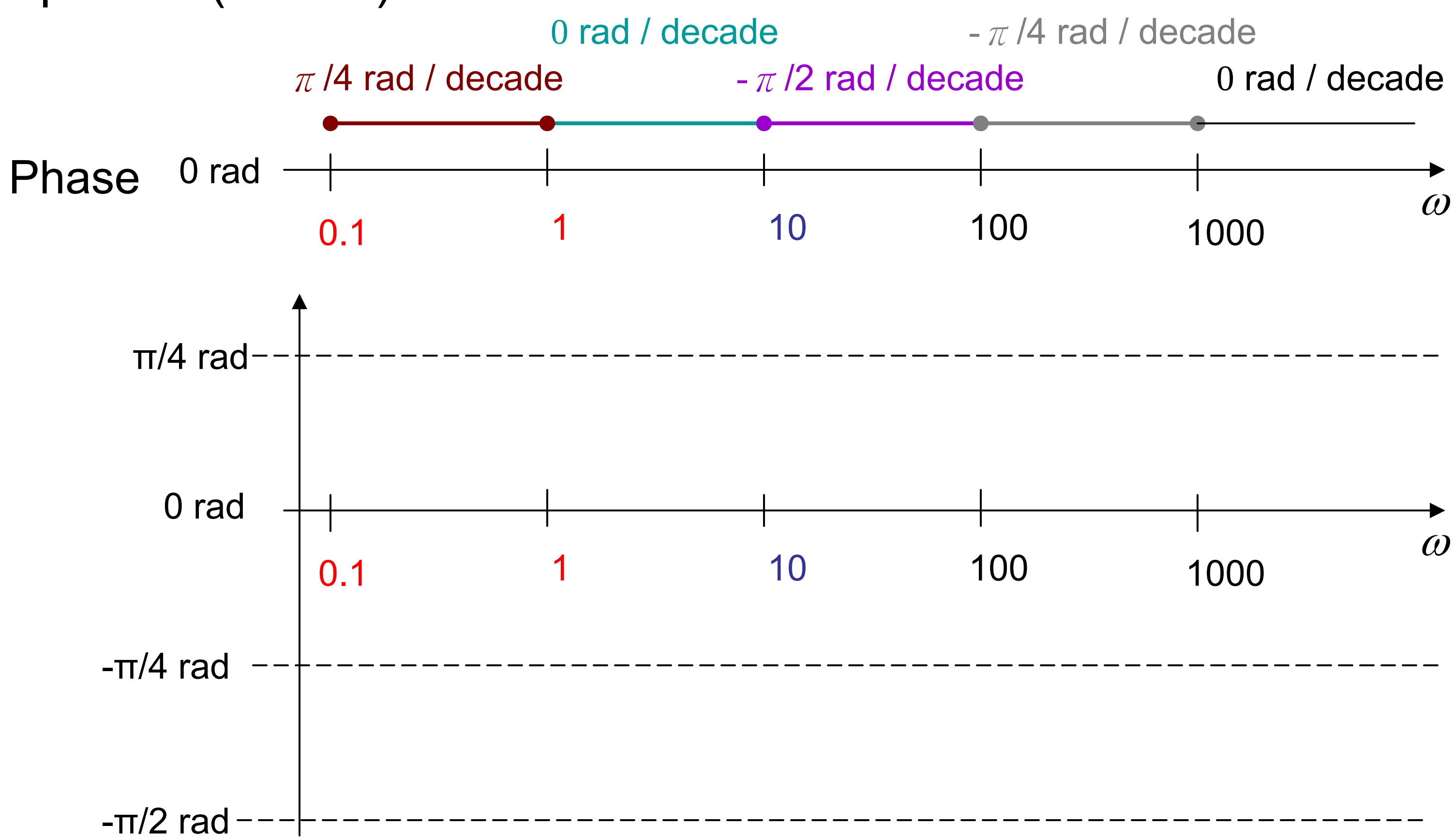
— — 0 rad / decade

Phase



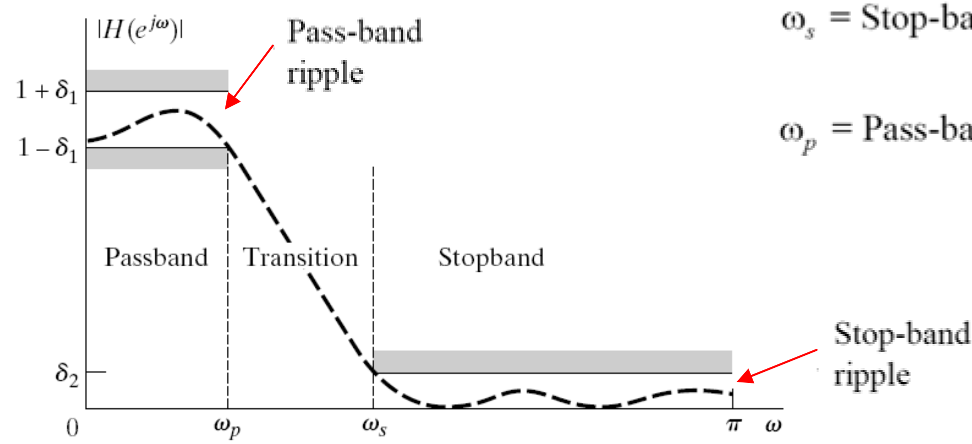
Section 6.5.3 Bode plots for rational Frequency responses

- Example 6.5 (Cont'd)



Section 6.4 Time- and Frequency- domain aspects of non-ideal filters

- The non-ideal filters have several advantages:
  - Easier to filter with gradual transition from pass-band to stop-band
  - Eliminate ringing / ripples in the step response
  - Causal filters for real-time operation
  - Easier and cheaper to implement with resisters, capacitors, op-amp



ω_s = Stop-band edge

ω_p = Pass-band edge

Section 6.4 Time- and Frequency- domain aspects of non-ideal filters

- Trade-off is observed between the width of the transition band (frequency-domain characteristic) and the settling time of the step response (time-domain characteristic):

Butterworth filter
- Longer transition band
- Shorter settling time

Elliptic filter
- Shorter transition band
- Longer settling time

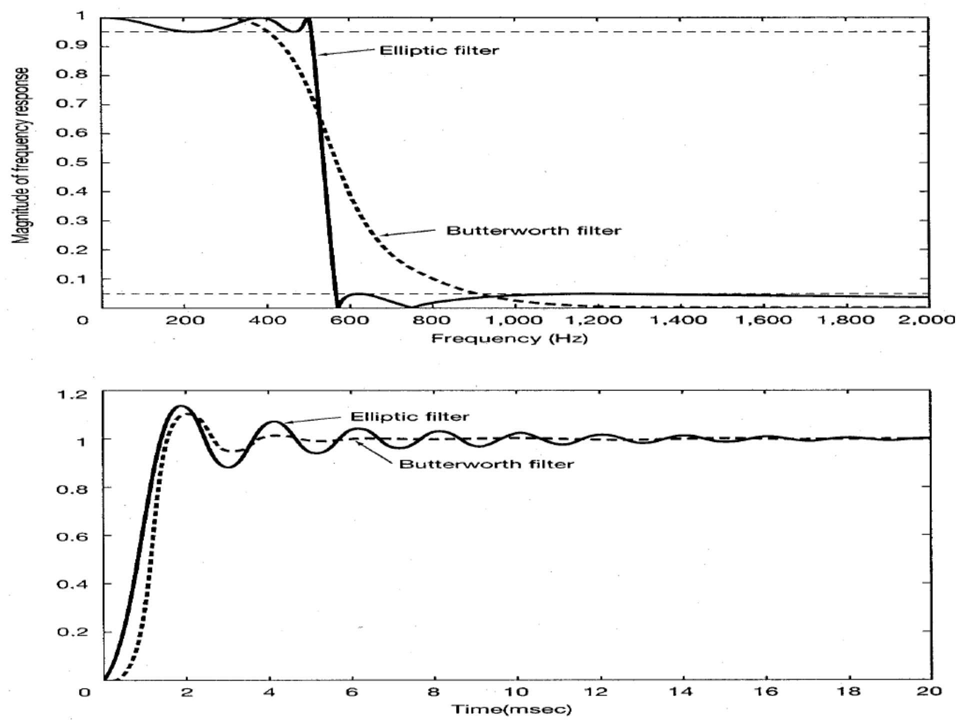


Figure 6.18 Example of a fifth-order Butterworth filter and a fifth-order elliptic filter designed to have the same passband and stopband ripple and the same cutoff frequency: (a) magnitudes of the frequency responses plotted versus frequency measured in Hertz; (b) step responses.

Section 6.2 Linear and non-linear phase systems

- A convenient measure of linearity of phase is group delay.

Continuous-Time System response

$$\tau(\omega) = grd[H(j\omega)] = -\frac{d}{d\omega}\{\arg[H(j\omega)]\}$$

Discrete-Time system response

$$\tau(\omega) = grd\left[H\left(e^{j\omega}\right)\right] = -\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\}$$

- The deviation of the group delay from a constant indicates the degree of non-linearity of phase.

- Consider a discrete-time system with a constant delay n_d

$$y[n] = Ax[n - n_d]$$

$$Y(e^{j\omega}) = AX(e^{j\omega})e^{-j\omega n_d} \Rightarrow \tau(\omega) = -\frac{d}{d\omega}\{-\omega n_d\} = n_d$$

$$H(e^{j\omega}) = Ae^{-j\omega n_d}$$

Section 6.2 Linear and non-linear phase systems

- Consider a continuous-time system with the system function

$$H(j\omega) = e^{-j\omega t_o}$$

  – Unity gain $|H(j\omega)|=1$
  – Linear phase: $\angle H(j\omega) = -\omega t_o$

$\Big\} \Rightarrow$

$$\boxed{\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ y(t) &= h(t) * x(t) \\ &= x(t-t_o) \end{aligned}}$$

- Non-linear phase: $\angle H_2(j\omega)$ is non-linear function of $\omega \rightarrow g(\omega)$
- Result:
  – Linear phase ⇔ simply a rigid shift in time, *no* distortion
  – Nonlinear phase ⇔ distortion as well as shift

Section 6.2 Linear and non-linear phase systems

- Let's consider an all-pass system
  - It passes all the frequencies with equal gain. i.e $|H(j\omega)| = K$ or $|H(e^{j\omega})| = K$
  - A simple but often used application of an all-pass filter is as a delay equalizer (phase compensator).

Continuous-time system

$$H(j\omega) = e^{-j\alpha\omega}$$ - Linear phase

$$|H(j\omega)| = 1$$

$$\angle H(j\omega) = -\alpha\omega$$

$$H(j\omega) = \frac{\alpha - j\omega}{\alpha + j\omega}$$ - Non-linear phase

$$|H(j\omega)| = \sqrt{\frac{\alpha^2 + \omega^2}{\alpha^2 + \omega^2}} = 1$$

$$\angle H(j\omega) = -2\tan^{-1}\frac{\omega}{\alpha}$$

Discrete-time system

$$H(e^{j\omega}) = e^{-j\omega n_d}$$ - Linear phase

$$|H(e^{j\omega})| = 1$$

$$\angle H(e^{j\omega}) = -\omega n_d$$

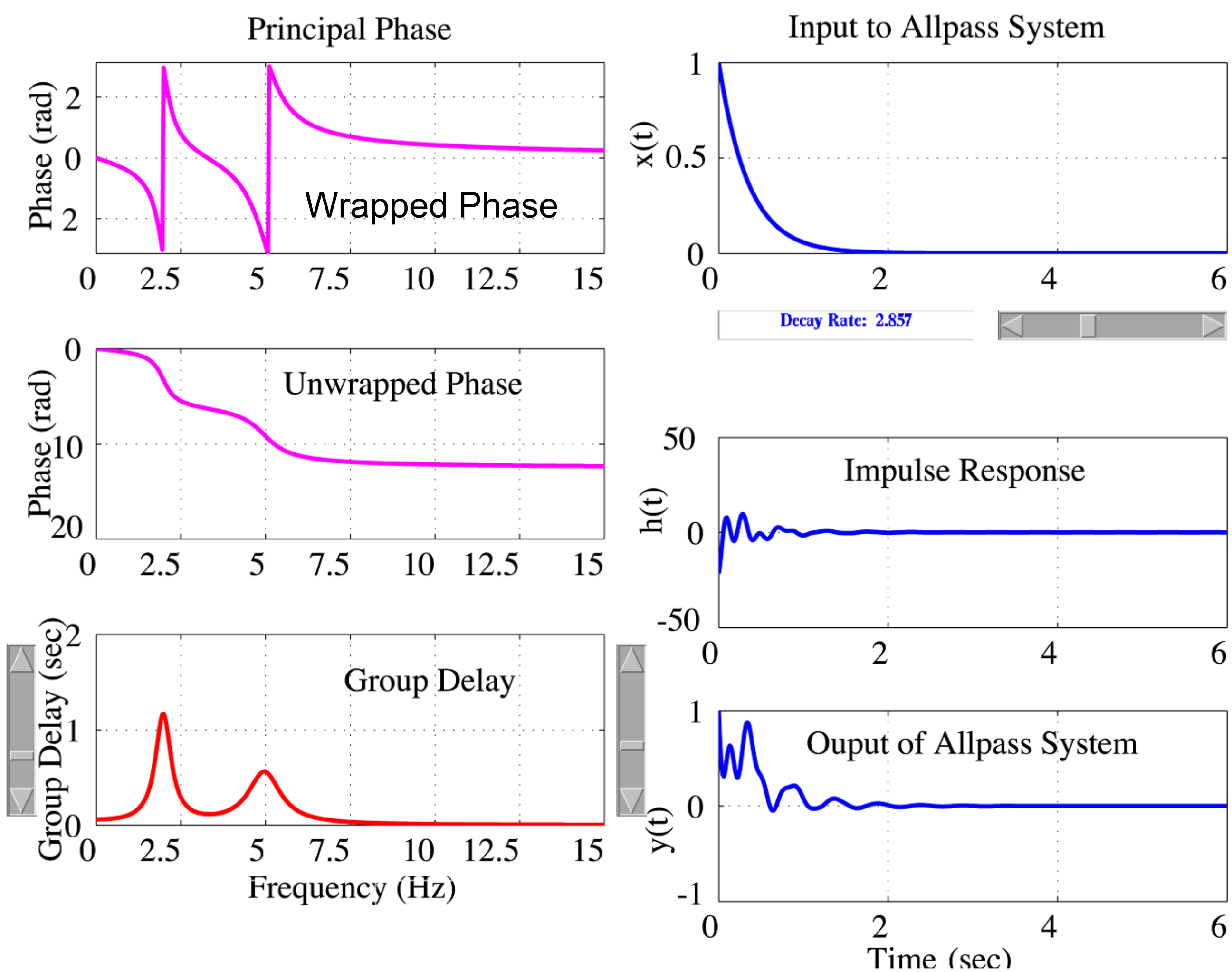
$$H(e^{j\omega}) = \frac{1 - \frac{1}{2}e^{j\omega}}{1 - \frac{1}{2}e^{-j\omega}}$$ - Non-linear phase

$$= \sqrt{\frac{\left(1 - \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}{\left(1 - \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} = 1$$

$$\angle H(e^{j\omega}) = -2\tan^{-1}\frac{\frac{1}{2}\sin\omega}{1 - \frac{1}{2}\cos\omega}$$

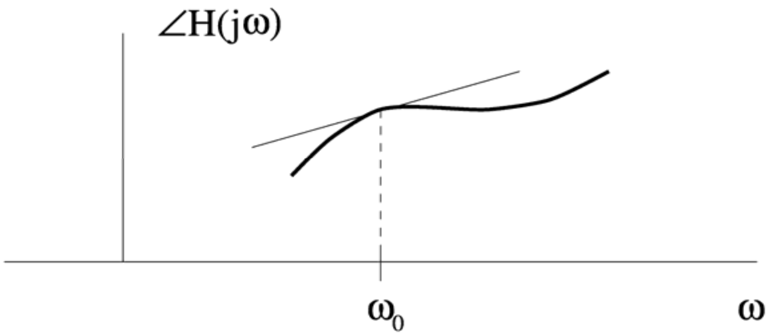
Section 6.2 Linear and non-linear phase systems

Impulse response and output of a continuous-time all-pass system with nonlinear phase



Section 6.2 Linear and non-linear phase systems

How do we think about signal delay when the phase is nonlinear?



For frequencies "near" ω_0

When the signal is narrow-band and concentrated near ω_0, $\angle H(j\omega) \sim$ linear with ω near ω_0, then $-\dfrac{d\angle H(j\omega)}{d\omega}$ instead of $-\dfrac{\angle H(j\omega)}{\omega}$ reflects the time delay.

$$\angle H(j\omega) \approx \angle H(j\omega_0) - \tau(\omega_0)(\omega - \omega_0) = \phi - \tau(\omega_0)\cdot\omega$$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H(j\omega)\} = \text{ Group Delay}$$

$$\Downarrow$$

For ω near ω_0

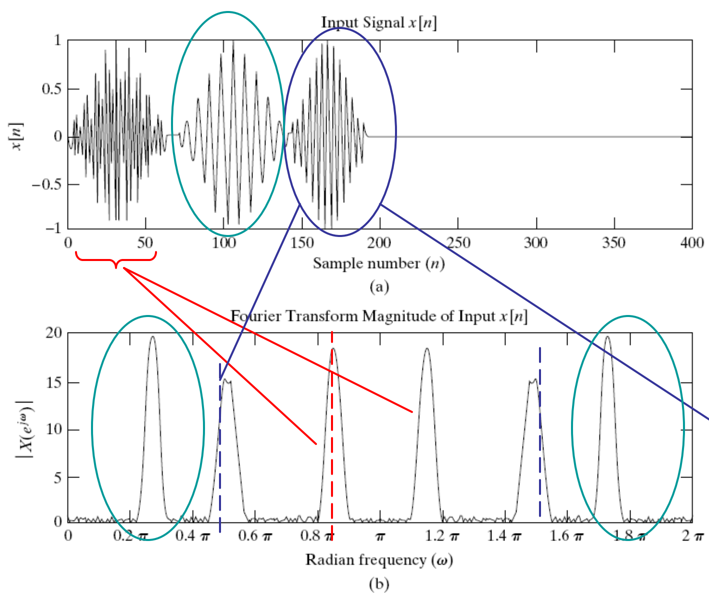
$$H(j\omega) \approx |H(j\omega_0)|e^{j\phi}e^{-j\tau(\omega_0)\omega}$$

$$\Rightarrow \quad e^{j\omega t} \longrightarrow \sim |H(j\omega)|e^{j\phi}e^{j\omega(t-\tau(\omega_0))}$$

Similar property can be applied onto discrete-time system

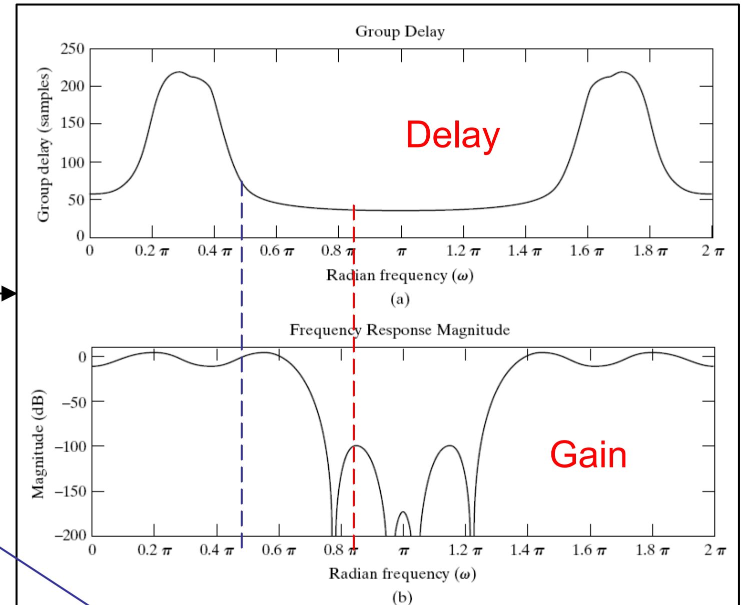
Section 6.2 Linear and non-linear phase systems

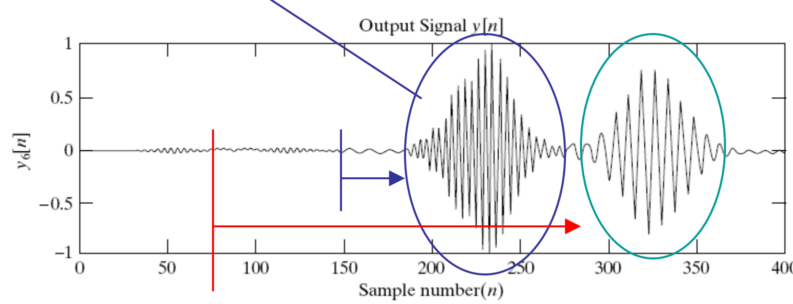
Example: Effects of Attenuation and Group Delay



Input signal

Frequency response of system





Output signal

at ω = 0.25 π, gain ~= -3dB, delay = 200 samples

at ω = 0.5 π, gain ~= 0dB, delay = 50 samples

at ω = 0.85 π, gain = -100dB, delay = 50 samples

Section 6.2 Linear and non-linear phase systems

- In the example, we demonstrate characteristic of filtering system:
  - It filtered out those un-desired frequency signals.
  - However, the remaining signals (signals in the passband) suffer from different delay spreading due to the non-linear phase response.

- In the followings, we will discuss some typical examples of discrete-time LTI linear-phase systems.

Reference book: 5.7.3 Causal FIR generalized linear phase system

- A causal M^{th} order FIR (Finite Impulse Response) systems have linear phase if they have impulse response a length of $(M+1)$ and satisfy either

$$H\left(e^{j\omega}\right)=A\left(e^{j\omega}\right)e^{-j(\alpha\omega-\beta)}$$

or

$$\tau(\omega)=grd\left[H\left(e^{j\omega}\right)\right]=-\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\}=-\frac{d}{d\omega}\left[-(\alpha\omega-\beta)\right]=\alpha$$

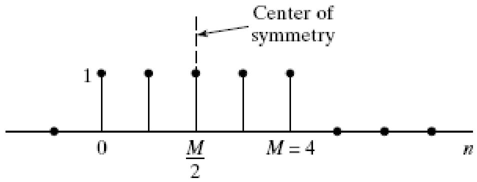
- In the deriving these expressions, it turns out that significantly different expressions results, depending on
  - the type of symmetry (even or odd symmetry) and
  - whether M is an even or odd integer.

- For this reason, it is generally useful to define four types of FIR linear-phase systems.

Reference book: 5.7.3 Causal FIR generalized linear phase system

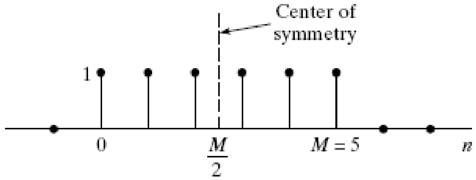
- There are 4 different types FIR linear-phase systems:
  - Type I FIR Linear-phase systems:

    $h[n] = h[M-n] \quad for \quad 0 \le n \le M$ with M is even.



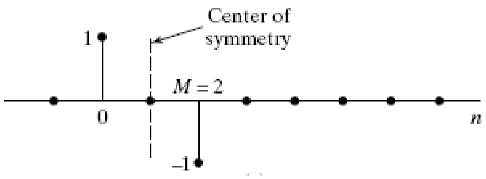
  - Type II FIR Linear-phase systems:

    $h[n] = h[M-n] \quad for \quad 0 \le n \le M$ with M is odd.



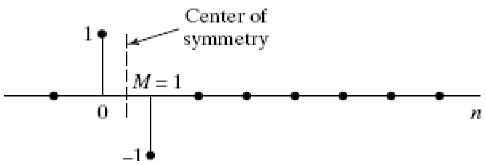
  - Type III FIR Linear-phase systems:

    $h[n] = -h[M-n] \quad for \quad 0 \le n \le M$ with M is even.



  - Type IV FIR Linear-phase systems:

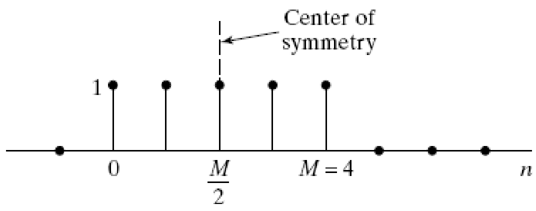
    $h[n] = -h[M-n] \quad for \quad 0 \le n \le M$ with M is odd.



Reference book: 5.7.3 Causal FIR generalized linear phase system

- Type I FIR filter with M is even,

$$h[n] = h[M-n] \qquad for \quad 0 \le n \le M$$



- Its frequency response becomes

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{M} h[n]e^{-j\omega n} = \sum_{n=0}^{M/2-1} h[n]e^{-j\omega n} + h[\tfrac{M}{2}]e^{-j\omega M/2} + \sum_{n=M/2+1}^{M} h[n]e^{-j\omega n}$$

$$= e^{-j\omega M/2} \sum_{k=0}^{M/2} a[k]\cos(\omega k)$$

where

$$a[0] = h[M/2]$$
$$a[k] = 2h[(M/2)-k]; \qquad for \quad k = 1,2,\ldots,M/2$$

Reference book: 5.7.3 Causal FIR generalized linear phase system

- The details of the expression of its frequency response:

$$H\left(e^{j\omega}\right)=\sum_{n=0}^{M}h[n]e^{-j\omega n}=\sum_{n=0}^{M/2-1}h[n]e^{-j\omega n}+h[\tfrac{M}{2}]e^{-j\omega M/2}+\sum_{n=M/2+1}^{M}h[n]e^{-j\omega n}$$

$$=e^{-j\omega M/2}\left(\sum_{n=0}^{M/2-1}h[n]e^{-j\omega(n-M/2)}+h[\tfrac{M}{2}]+\sum_{n=M/2+1}^{M}h[n]e^{-j\omega(n-M/2)}\right)$$

$$=e^{-j\omega M/2}\left(\sum_{n=0}^{M/2-1}h[n]e^{-j\omega(n-M/2)}+h[\tfrac{M}{2}]+\sum_{k=0}^{M/2-1}h[M-k]e^{-j\omega(M/2-k)}\right),\quad k=M-n$$

$$=e^{-j\omega M/2}\left(\sum_{n=0}^{M/2-1}h[n]e^{-j\omega(n-M/2)}+h[\tfrac{M}{2}]+\sum_{n=0}^{M/2-1}h[n]e^{j\omega(n-M/2)}\right)$$

$$=e^{-j\omega M/2}\left(\sum_{k=M/2}^{1}h[\tfrac{M}{2}-k]e^{-j\omega k}+h[\tfrac{M}{2}]+\sum_{k=M/2}^{1}h[\tfrac{M}{2}-k]e^{j\omega k}\right),\quad k=\tfrac{M}{2}-n$$

$$=e^{-j\omega M/2}\left(\sum_{k=M/2}^{1}2h[\tfrac{M}{2}-k]\cos\omega k+h[\tfrac{M}{2}]\right)$$

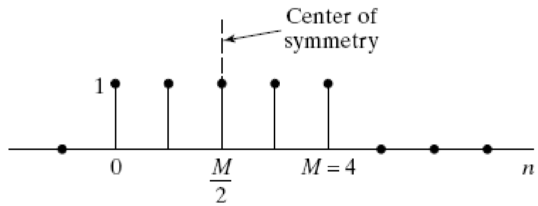
$$=e^{-j\omega M/2}\sum_{k=0}^{M/2}a[k]\cos(\omega k)\qquad a[0]=h[M/2]$$

$$a[k]=2h[(M/2)-k];\qquad for\quad k=1,2,\ldots,M/2$$

Reference book: 5.7.3 Causal FIR generalized linear phase system

- Example 5.17 Consider an impulse response

$$h[n]=\begin{cases}1, & 0\le n\le 4\\ 0, & otherwise\end{cases}$$



- The frequency response is

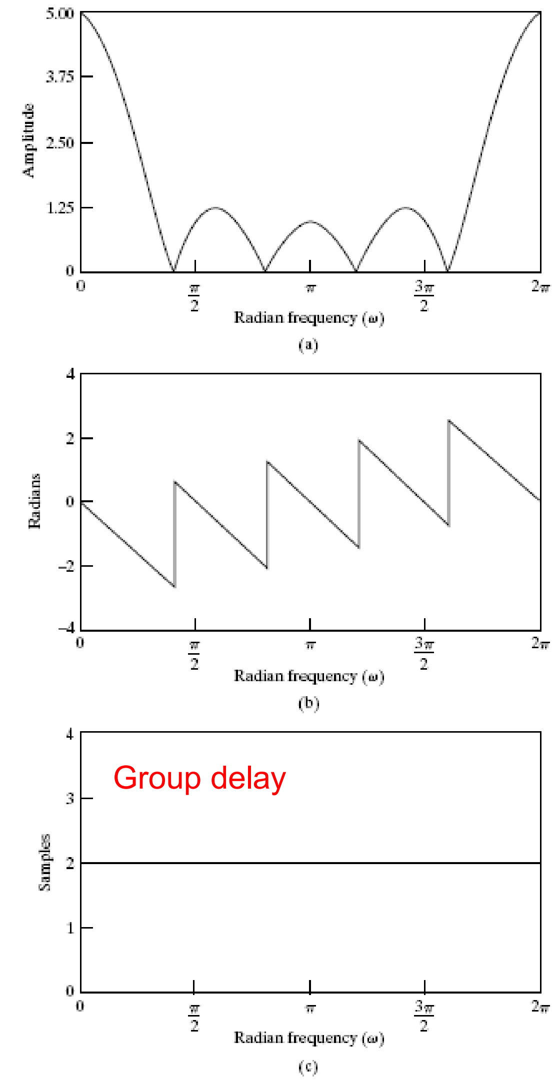
$$H\left(e^{j\omega}\right)=\sum_{n=0}^{4}(1)e^{-j\omega n}=\frac{1-e^{-j\omega 5}}{1-e^{-j\omega}}=$$

$$=$$

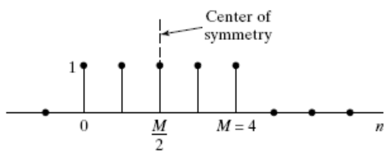
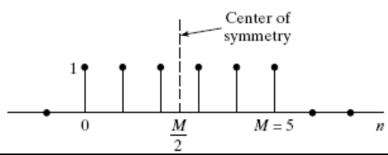
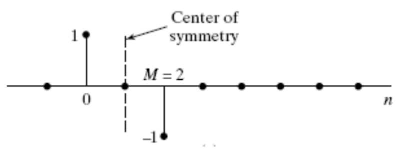
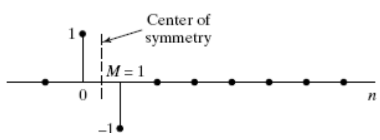
$$=$$

$$=$$

$$\tau(\omega)=grd\left[H\left(e^{j\omega}\right)\right]=-\frac{d}{d\omega}\left\{\arg\left[H\left(e^{j\omega}\right)\right]\right\}=$$



- Summary of the linear-phase systems

FIR Linear-phase system	Impulse response	Frequency response	Example
Type I	$h[n]=h[M-n]$ M is even	$H(e^{j\omega})=e^{-j\omega M/2}\left(\sum_{k=0}^{M/2}a[k]\cos(\omega k)\right)$ $a[0]=h[M/2]$ $a[k]=2h[(M/2)-k],\quad k=0,1,2,\ldots,M/2$	
Type II	$h[n]=h[M-n]$ M is odd	$H(e^{j\omega})=e^{-j\omega M/2}\left(\sum_{k=0}^{(M+1)/2}b[k]\cos(\omega(k-\tfrac{1}{2}))\right)$ $b[k]=2h[(M+1)/2-k],\quad k=0,1,2,\ldots,(M+1)/2$	
Type III	$h[n]=-h[M-n]$ M is even	$H(e^{j\omega})=je^{-j\omega M/2}\left(\sum_{k=0}^{M/2}c[k]\sin(\omega k)\right)$ $c[k]=2h[(M/2)-k],\quad k=0,1,2,\ldots,M/2$	
Type IV	$h[n]=-h[M-n]$ M is odd	$H(e^{j\omega})=je^{-j\omega M/2}\left(\sum_{k=0}^{(M+1)/2}d[k]\sin(\omega(k-\tfrac{1}{2}))\right)$ $d[k]=2h[(M+1)/2-k],\quad k=0,1,2,\ldots,(M+1)/2$	

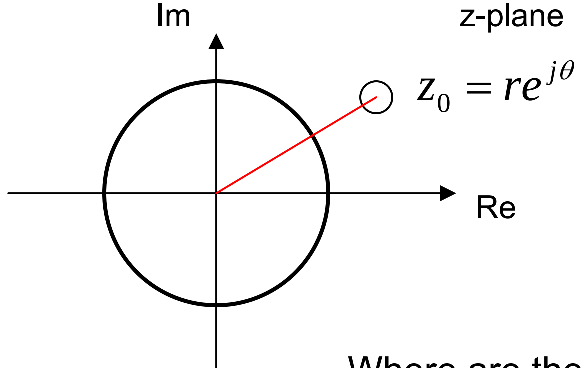
Reference book: 5.7.3 Location of zeros for causal FIR linear phase system

- In the symmetric cases (Type I and II), $h[n] = h[M-n]$

$$H(z) = \sum_{n=0}^{M} h[n]z^{-n} = \sum_{n=0}^{M} h[M-n]z^{-n} = \sum_{k=M}^{0} h[k]z^{k}z^{-M} = z^{-M}H(z^{-1})$$

- This implies that if $z_0 = re^{j\theta}$ is a zero of $H(z)$,

- $z_0^{-1} = r^{-1}e^{-j\theta}$ is also a zero of $H(z)$ for both M is even and M is odd.



Where are the poles?

- In addition, when $h[n]$ is real, the conjugates are also zeros.

Reference book: 5.7.3 Location of zeros for causal FIR linear phase system

- In the symmetric cases (Type I and II), $h[n] = h[M - n]$

$$H(z) = z^{-M} H(z^{-1})$$

- In particular, z = -1,

$$H(-1) = (-1)^{-M} H(-1)$$

- when M is odd (Type II FIR filter)

$\Rightarrow$ z = -1 must be a zero

- In summary:
  - The zeros of Type I and II FIR linear-phase systems are in reciprocal and conjugate pairs.
  - In addition, Type II FIR linear-phase system has a zero at z = -1

Reference book: 5.7.3 Location of zeros for causal FIR linear phase system

- Similarly, Type III and IV FIR IV linear-phase systems $h[n] = -h[M-n]$
- Their transfer functions are in the form of $H(z) = -z^{-M} H(z^{-1})$
- Then, we have
  - The zeros are also in reciprocal and conjugate pairs.
  - Type III and IV FIR linear-phase systems have a zero at z =1.
  - Type III FIR linear-phase system (M is even) has a zero at z = -1.

Reference book: 5.7.3 Location of zeros for causal FIR linear phase system

M is even

M is odd

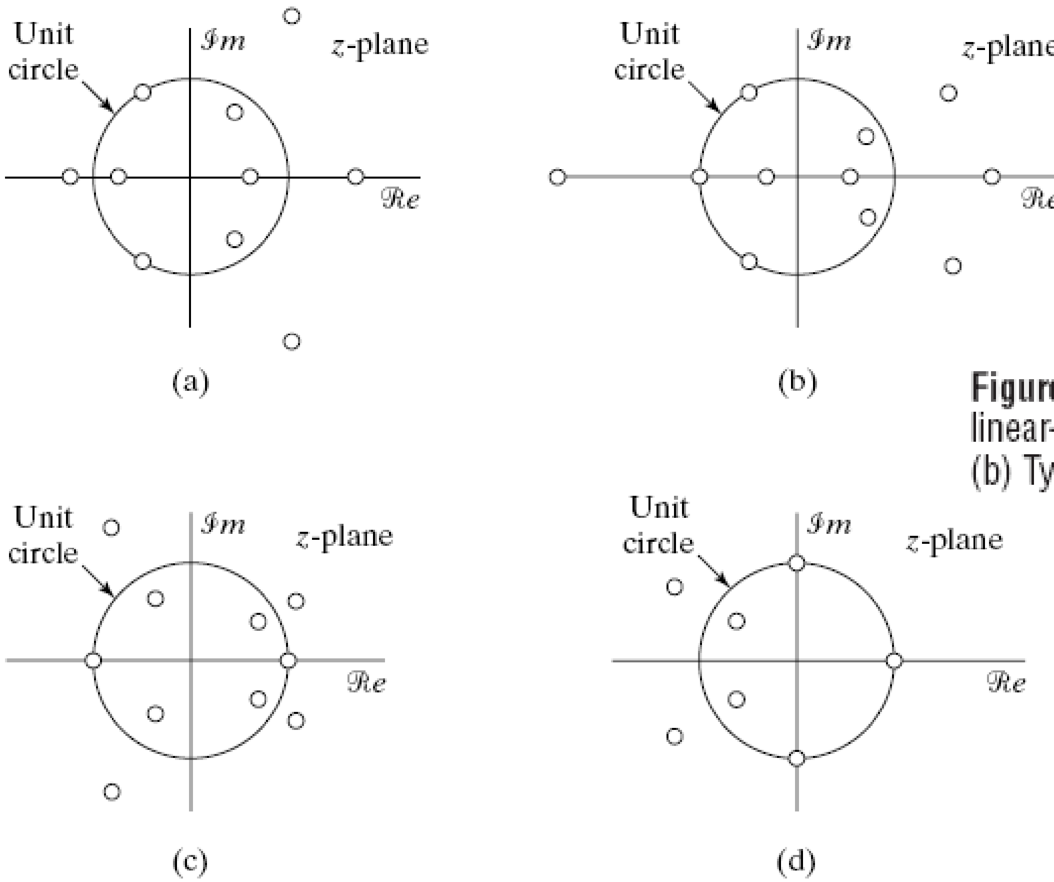


Figure 5.41 Typical plots of zeros for linear-phase systems. (a) Type I. (b) Type II. (c) Type III. (d) Type IV.

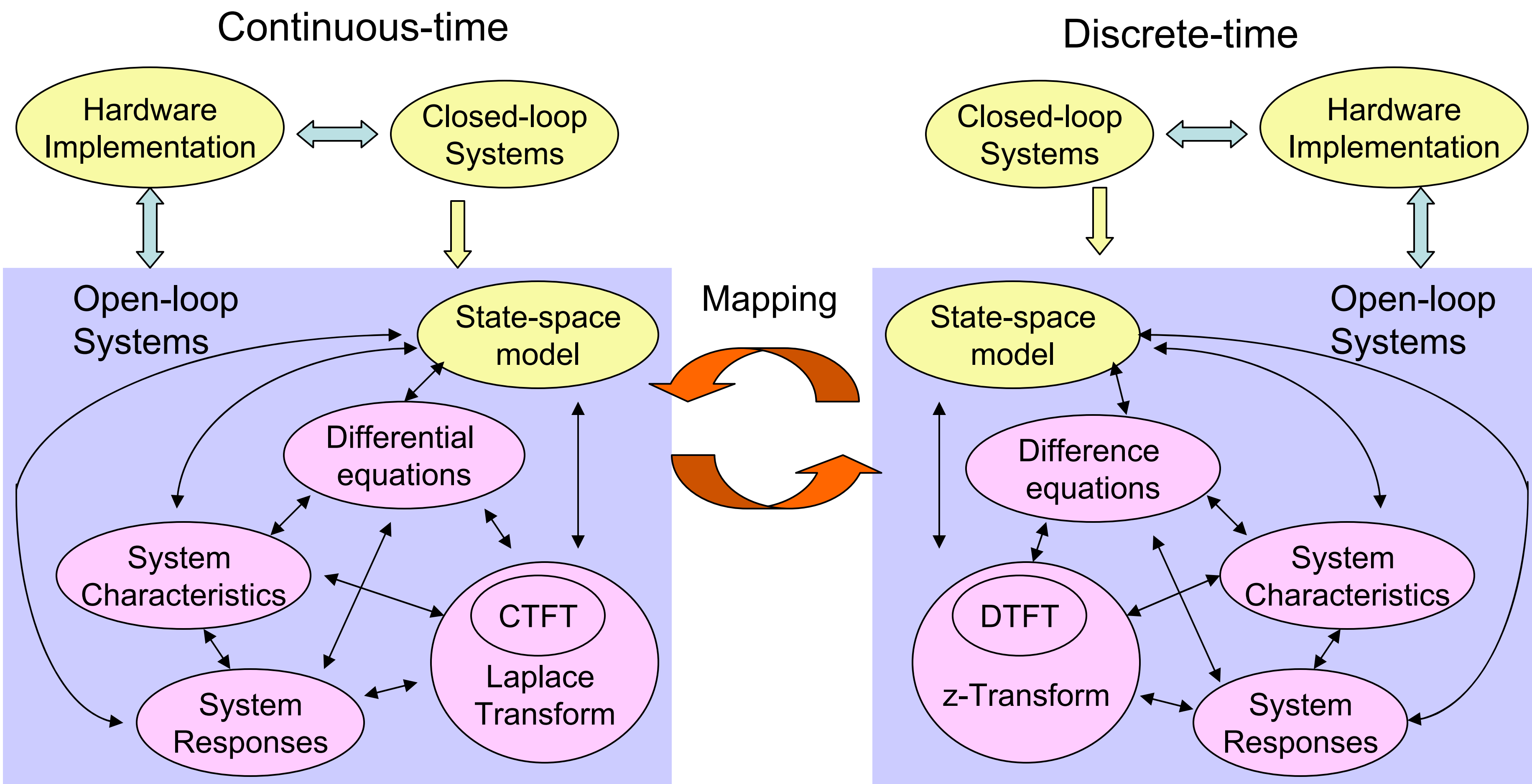
M is even

M is odd

Transient and Frequency response of LTI systems

- Readings
  - Textbook:
    - Section 6.2 The Magnitude-phase representation of the frequency response of LTI systems
    - Section 6.3 Time-domain properties of an ideal frequency selective filters
    - Section 6.4 Time-domain and frequency-domain aspects of non-ideal filters
    - Section 6.5 First-order and Second-order Continuous-time systems
    - Section 6.6 First-order and Second-order Discrete-time systems
    - Section 6.7 Examples of time- and frequency domain analysis of systems
  - Reference book
    - A. V. Oppenheim, et. al., ***Discrete-time Signal Processing***, 2nd edition, Prentice-Hall, 1999
    - Section 5.7.3 Causal FIR generalized linear phase system

Where we are



Will be covered if available	Done in 211	To be covered	In progress	Done