Numerical Methods for Partial differential equations

2016-CS-122

2016-CS-109

Agenda

- Numerical Techniques for Solving PDE
- Implementation, Scope and Importance of Numerical Methods
- Comparison of Different Methods

Problem Statement

- In Real world, there are many mathematical problems that cannot be solved analytically. Calculations are difficult and time consuming on each step. So, Numerical Methods provide us a Numerical approximation to that problem. These methods are useful because approximations can be made easily by using a computational machine.

Methodology

There are two methods we discussed so far for solving Partial differential equations

- Finite Difference Method
  - Forward Euler method (or Explicit Method)
  - Forward Euler method (or Implicit Method)
  - Crank–Nicolson method
- Finite Element Method
  - Galerkin method

Finite Difference Method

- The core idea of the finite-difference method is to replace continuous derivatives with so-called difference formulas that involve only the discrete values associated with positions on the mesh

Finite Element Method

- Finite Element Method techniques include
  1. Dividing the domain of the problem into a collection of subdomains, with each subdomain represented by a set of element equations to the original problem
  2. systematically recombining all sets of element equations into a global system of equations for the final calculation. The global system of equations has known solution techniques, and can be calculated from the initial values of the original problem to obtain a numerical answer.

Comparison Table

Property	Finite Difference Method	Finite Element Method
CALCULATIONS	Small amount of calculations in each step	Large amount of calculations in each step
MEMORY REQUIREMENT	No large memory requirement so no large matrices are formed	Large memory requirement for storage of stiffness matrices
ACCURACY	Requires more calculations and memory but highly accurate	Requires less calculations and memory but relatively low accuracy
DISCRETIZATION CRITERIA	Discretization is based upon the differential form of the PDE to be solved. Each derivative is replaced with an approximate difference formula	Discretization is based upon a piecewise representation of the solution in terms of specified basis functions.

Error Analysis

- Errors depend on mesh spacing and time step

  Consider a Taylor series expansion *φ*(*x*) about the point *xi*

$$\phi(x_i+\delta x)=\phi(x_i)+\delta x\left.\frac{\partial\phi}{\partial x}\right|_{x_i}+\frac{\delta x^2}{2}\left.\frac{\partial^2\phi}{\partial x^2}\right|_{x_i}+\frac{\delta x^3}{3!}\left.\frac{\partial^3\phi}{\partial x^3}\right|_{x_i}+\cdots$$

Solve for (*∂φ/∂x*)*xi*

$$\left.\frac{\partial\phi}{\partial x}\right|_{x_i}-\frac{\phi_{i+1}-\phi_i}{\Delta x}\approx\frac{\Delta x^2}{2}\left.\frac{\partial^2\phi}{\partial x^2}\right|_{\xi}$$

The term on the right hand side of above Equation is called the truncation error of the finite difference approximation. The “big O” notation can be used to express the dependence of the truncation error on the mesh spacing.

$$\boxed{\left.\frac{\partial\phi}{\partial x}\right|_{x_i}=\frac{\phi_{i+1}-\phi_i}{\Delta x}+\mathcal{O}(\Delta x)}$$

Results

Table 1a. Forward and Backward Euler's Method Compared To Exact Solution

t_n (time)	Y_n, forward Euler's approximation	Y_n, backward Euler's approximation	y_n, exact	$\|e_i\|$ error forward	$\|e_i\|$ error backward
0	1	1	1	0	0
0.1	0.9	0.9441	0.925795	0.025795	0.018305
0.2	0.853	0.916	0.889504	0.036504	0.026496
0.3	0.8374	0.9049	0.876191	0.03791	0.028709
0.4	0.8398	0.9039	0.876191	0.036391	0.027709
0.5	0.8517	0.9086	0.883728	0.032028	0.024872

Report Demo

Numerical Methods for Partial Differential Equations

Hamza Farooq, Muhammad Awais

2016-CS-122, 2016-CS-109

Introduction

Differential conditions are a standout amongst the most essential portrayals of the laws that utilize marvels including the development of bodies in a gravitational field or the development of populaces. Here we are going to explain numerical methods for parabolic partial differential equations.
When we turn in finding the solution of Partial differential equations in numerical analysis, we often mean approximation. Because computational machines (Rather mechanical or electrical) don't know the analytic method to solve a mathematical problem. So we have to utilize those numerical techniques in computers to get the closest approximation.
Here we are going to discuss numerical methods that involve partitioning the original problem into smaller parts of that problem. An initial condition is given and approximations are made consequently. After that rough formulation is made. We repeat the process until we get required accuracy.

Index Terms – **PDE:** Partial Differential Equation, **ODE:** Ordinary Differential Equation, **FDM:** Finite Difference Method, **FEM:** Finite Element Method, **CFL:** Courant–Friedrichs–Lewy, **DG:** Discontinuous Galerkin Method, **CG:** Continuous Galerkin Method

ORDINARY DIFFERENTIAL EQUATION

If all functions appearing in the equation depend only on one variable, we speak of an ordinary differential equation. Ordinary differential equations frequently describe the behavior of a system over time, e.g., the movement of an object depends on its velocity, and the velocity depends on the acceleration.

PARTIAL DIFFERENTIAL EQUATION

On the off chance that the capacities in the condition rely upon in excess of one variable and if the condition in this manner relies upon fractional subsidiaries, we discuss an incomplete differential condition.
A typical example is the potential equation of electrostatics. Given a domain R3, we consider

$$\frac{\partial^2 u}{\partial x_1^2}(x) + \frac{\partial^2 u}{\partial x_2^2}(x) + \frac{\partial^2 u}{\partial x_3^2}(x) = f(x) \ \textit{for all } x \in \Omega$$

Computers have only a finite amount of storage at their disposal, they are generally unable to represent the solution u as an element of the in finite-dimensional space exactly.

- Processing Limitation
- Memory Limitation
- Limited Clock Cycles
- Can't learn analytical methods
- Can only approximate problems

A genuinely basic discretization strategy is the technique for finite differences: we supplant the subsidiaries by difference remainders and supplant by a framework h with the end goal that the difference remainders in the framework focuses can be assessed utilizing just qualities in network focuses. For the situation of the potential condition, this prompts an arrangement of straight conditions that can be fathomed with a specific end goal to get a guess uh of u.

INTRODUCTION

Partial differential equations is the branch of numerical analysis that studies the numerical solution of partial differential equations (PDEs). It contains unknown functions containing more than one variable and their partial derivatives. PDEs are used to solve problems involving functions of more variables, and are either solved by hand, or computation is performed by computer.

A partial differential equation (PDE) for the function

$$u(x_1, \dots x_n)$$

is an equation of the form

$$f\left(x_1, \dots x_n, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_n}, \frac{\partial^2 u}{\partial x_1 \partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1 \partial x_n}, \dots\right) = 0$$

FINITE DIFFERENCE METHODS FOR PARABOLIC EQUATIONS

In mathematics, the methods for solving partial differential equations that involve approximating them with difference equations in which finite difference approximate the derivatives are called Finite difference methods.

Any Questions?