UNIVERSITY OF ENGINEERING AND TECHNOLOGY
Department of Computer Science and Engineering

Probability and Random Variables
Fall 2018

Problem Set 4
Due: October 1, 2018

Note:

- These problems have been liberally copied from older versions of MIT's 6.041/6.431, Harvard's Stat 110, Al Drake's book and Gian Carlo Rota's notes. It is difficult to acknowledge each problem individually. There is no claim to originality in these problems, and the debt to all these resources is gratefully acknowledged. This applies to all problems in this course.

1. A and B are playing a chess match. The match goes on until the first player wins. If no one wins for 10 matches, the match is considered a draw. A wins a match with a probability of 0.3 and B wins a match with probability 0.2, and a game is drawn with probability 0.5. The games are independent.
   (a) What is the probability that A wins the match?
   (b) Find the PMF of the duration of the match.

2. The probability that any particular bult will burn out during its Kth month of use is given by the PMF for K,
   $$p_K(k) = \frac{1}{5}\left(\frac{4}{5}\right)^{k-1} \qquad \text{k } = 1, 2, 3, \ldots$$
   Four bulbs are life-tested simultaneously. Determine the probability that
   (a) None of the four bulbs fails during its first month of use.
   (b) Exactly two bulbs have failed by the end of the first month.
   (c) Exactly one bulb fails during each of the first three months.
   (d) Exactly one bulb has failed by the end of the second month, and exactly two bulbs are still working at the start of the fifth month.

3. Discrete random variable X is described by the PMF $p_X(x)$. Before an experiment is performed, we are required to guess a value d. After an experimental value of X is obtained, we shall be paid an amount $A - B(X - d)^2$ dollars.
   (a) What value of d should we use to maximize the expected value of our financial gain?
   (b) Determine the value of A such that the expected value of the gain is zero dollars.

4. The PMF for discrete random variable K is defined to be
   $$p_K(k) = \begin{cases} C(1-P)^{k-1} & \text{if } k = 1, 2, 3, \ldots, \text{and } 0 < P < 1 \\ 0 & \text{for all other values of k} \end{cases}$$
   (a) Determine the value of C.
   (b) Let n be a positive integer. Determine the probability that an experimental value of K will be greater than n.
   (c) Given that an experimental value of random variable K is greater than integer n, what is the conditional probability that it is also larger than $2n$?

(d) What is the probability that an experimental value of K is equal to an integer multiple of 3?

5. In San Francisco, a drunk leaves a bar and every 10 seconds staggers either one yard down the street with probability 3/4 or one yard up the street with probability 1/4.
   (a) Where is the drunk after one minute? after two minutes?
   (b) What is his most likely location in each case? How is the most likely location varying in time?
   (c) Computer the average position of the drunk after one minute? after two minutes?
   (d) How is the average position changing in time?

6. James Bernoulli, proposed the following dice game. The player pays one dollar and throws a single die. He then throws a set of n dice, where n is the number shown by the first die. The total number of dots shown by the n dice is then used to determine the payoff.

   If the number is less than 12, he loses the bet, if the number equals 12 his dollar is returned, while if the number exceeds 12 he receives two dollars. Find the expected number of dots shown by the n dice. Is the game favourable to the player?

7. Let X and Y be independent random variables that take values in the set $\{1, 2, 3\}$. Let $V = 2X+2Y$, and $W = X-Y$. For parts (b) to (f) of this problem, assume that X and Y are uniformly distributed on $\{1, 2, 3\}$.
   (a) Assume that $P(X = k)$ and $P(Y = k)$ are positive for any $k \in \{1, 2, 3\}$. Can V and W be independent? Explain. (No calculations needed.)
   (b) Find and plot $p_V(v)$. Also, determine $E[V]$ and $var(V)$.
   (c) Find and show in a diagram $p_{V,W}(v, w)$.
   (d) Find $E[V|W > 0]$.
   (e) Find the conditional variance of W given the event $V = 8$.
   (f) Find and plot the conditional PMF $p_{X|V}(x|v)$, for all values.

8. Chuck will go shopping for probability books for K hours. Here, K is a random variable and is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops. We are told that

$$PN|K(n|k) = \frac{1}{k}, \qquad \text{for } n = 1, \ldots, k.$$

   (a) Find the joint PMF of K and N.
   (b) Find the marginal PMF of N.
   (c) Find the conditional PMF of K given that $N = 2$.
   (d) We are now told that he bought at least 2 but no more than 3 books. Find the conditional mean and variance of K, given this piece of information.
   (e) The cost of each book is a random variable with mean 3. What is the expected value of his total expenditure? Hint: Condition on events $N = 1, \ldots, N = 4$ and use the total expectation theorem.

9. Professor May B. Right often makes mistakes in her science class. She answers each of her students' questions incorrectly with probability 1/4, independently of other questions. In each lecture May is asked 1, 2 , or 3 questions with equal probability 1/3.
   (a) What is the probability that May gives wrong answers to all the questions she is, asked in a given lecture?

(b) Given that May gave wrong answers to all the questions she was asked in a given lecture, what is the probability that she was asked three questions?

(c) Let X and Y be the number of questions May is asked and the number of questions she answers correctly in a lecture , respectively. What is the mean and variance of X and of Y?

(d) Give a clearly labeled sketch of the joint PMF $P_{X,Y}(x, y)$.

(e) To encourage questions in May's class, May's college adopts an unusual incentive policy and offers a bonus of $10X + 20Y$ dollars to May. What is the expected value and the variance of the bonus.

(f) May's semester has 20 lectures . Let Z be the total number of questions she answers wrong in a semester. What is the mean and variance of Z?

(g) Determined to improve her reputation , May decides to teach an additional 20-lecture class in her specialty (math), where she answers questions incorrectly with probability 1/10 rather than 1/4. What is the expected number of questions that she will answer wrong in a randomly chosen lecture (math or science)?

(h) Given that May has a perfect semester (no wrong answers) in exactly one of her two classes , what is the conditional probability that this class is the science class?

10. Oscar, ecstatic that you've helped him find his lost dog, now seeks your assistance on another problem. At his workplace, the first thing Oscar does every morning is to go to the supply room and pick up either one, two, or three pens (he is equally likely to do any of these three events). If he receives three pens, he does not return to the supply room again that day. If he receives only one or two pens, he will make one additional trip to the supply room, where he again has an equally likely opportunity to pick up one, two, or three pens. Note: The number of pens taken in one trip will not affect the number of pens taken in any other trip. Evaluate:

(a) $P(A)$, where $P(A)$ is the probability that Oscar ends up with a total of three pens on any particular day.

(b) $P(B|A)$, where $P(B|A)$ is the conditional probability that he visited the supply room twice today, given that it is a day in which he received a total of three pens.

(c) $E[N]$ and $E[N|C]$, where $E[N]$ is the unconditional expectation of N, the total number of pens Oscar receives on any given day, and $E[N|C]$ is the conditional expectation of N given C (C is the event that (N $>$ 3)).

(d) $\sigma_{N|C}$, where $\sigma_{N|C}$ is the conditional standard deviation of the total number of pens Oscar receives on a particular day, given that C again is the event that $(N > 3)$.

(e) $P(D)$, where $P(D)$ is the probability he receives more than three pens on each of the next 16 days.

(f) $\sigma_{M|D}$, where $\sigma_{M|D}$ is the conditional standard deviation of the total number of pens he gets in the next 16 days given that he receives more than three pens on each of those days.