# Problem Set 4

## Q. No. 1

$P(A) = 0.3$, $P(B) = 0.2$, $P(D) = 0.5$

**(a)**

$P(A \text{ wins}) = P(A \text{ wins } 1^{st} \text{ game}) + P(1^{st} \text{ Draw } \& \text{ } 2^{nd} \text{ won by } A) + P(\text{first 2 Draw } \& \text{ } 3^{rd} \text{ Won by } A) + \cdots$
$+ P(\text{first 9 Draw } \& \text{ } 10^{th} \text{ won by } A)$

$= (0.3) + (0.5)(0.3) + (0.5)^2 \, 0.3 + \cdots + (0.5)^9 (0.3)$

$= 0.3(1 + 0.5 + (0.5)^2 + (0.5)^3 + \cdots + (0.5)^9)$

**(b)** Let $K$ be the Match Duration

$P_K(K=1) = P(1^{st} \text{ game won by A or B})$
$= 0.3 + 0.2$

$P_K(K=2) = P(1^{st} \text{ Draw } \& \text{ } 2^{nd} \text{ game won by A or B})$
$= (0.5)(0.3+0.2)$

$P_K(K=3) = (0.5)^2(0.3+0.2)$

$\vdots$

$P_K(K=10) = P(9 \text{ games Draw } \& \text{ Last won by A or B or Draw})$
$= (0.5)^9(0.5+0.3+0.2)$
$= (0.5)^9$

$$P_K(K=k) = \begin{cases} (0.5)^{k+1} & \text{for } k = 1, 2, \ldots, 9 \\ (0.5)^9 & \text{for } k = 10 \\ 0 & \text{for else} \end{cases}$$

**Q. No. 2** $P_K(K=i) \to$ Probability of any bulb burning in $i^{th}$ Month

$P_K(K=1) = \frac{1}{5}$ , $q_K(K=1) = \frac{4}{5}$

$P_K(K=2) = \frac{4}{25}$ , $q_K(K=2) = \frac{21}{25}$

$P_K(K=3) = \frac{16}{125}$ , $q_K(K=3) = \frac{109}{125}$

$P_K(K=4) = \frac{64}{625}$ , $q_K(K=4) = \frac{561}{625}$

**(a)** $P(\text{No bulb burnt}) = \left(\frac{4}{5}\right)^4$

**(b)** $P(\text{Exactly 2 bulb burnt}) = \binom{4}{2}\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^2$

**(c)** P(Exactly 1 bulb for each of first 3 Months)

$= P(\text{1 bulb in } 1^{st} \text{ Month}) \times P(\text{1 bulb in } 2^{nd} \text{ Month}) \times P(\text{1 Bulb in } 3^{rd} \text{ M})$

$= \left[\binom{4}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^3\right] \times \left[\binom{3}{1}\left(\frac{4}{25}\right)\left(\frac{21}{25}\right)^2\right] \times \left[\binom{2}{1}\left(\frac{16}{125}\right)\left(\frac{109}{125}\right)\right]$

**(d)** P(1 bulb burnt in first 2 Months $\cap$ 2 bulbs working in fifth)

= P(1 bulb burnt in first 2 M $\cap$ 1 bulb burnt in $3^{rd}$ & $4^{th}$ M)

= P(1 bulb burnt in first 2 M) ( 1 bulb burnt in $3^{rd}$ & $4^{th}$ M

P(1 bulb burnt in first 2 M) = P(1 burnt in $1^{st}$ & No bulb burnt in $2^{nd}$)
+ P(No burnt in $1^{st}$ & 1 burnt in $2^{nd}$)

$= \binom{4}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^3\left(\frac{21}{25}\right)^3 + \left(\frac{4}{5}\right)^4\binom{4}{1}\left(\frac{4}{25}\right)\left(\frac{21}{25}\right)^3$

$P(1 \text{ bulb burnt in } 3^{rd} \& 4^{th} M) = P(1 \text{ burnt in } 3^{rd} \& \text{ No burnt in } 4^{th}) + P(\text{No burnt in } 3^{rd} \& 1 \text{ burnt in } 4^{th})$

$$= \binom{3}{1} \frac{16}{125}\left(\frac{109}{125}\right)^2\left(\frac{561}{625}\right)^2 + \left(\frac{109}{125}\right)^3\binom{3}{1}\left(\frac{64}{625}\right)\left(\frac{561}{625}\right)^2$$

$$P(D) = \left[\binom{4}{1}\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^3\left(\frac{21}{25}\right)^3 + \left(\frac{4}{5}\right)^4\binom{4}{1}\left(\frac{4}{25}\right)\left(\frac{21}{25}\right)^3\right]$$
$$\times\left[\binom{3}{1}\left(\frac{16}{125}\right)\left(\frac{109}{125}\right)^2\left(\frac{561}{625}\right)^2 + \left(\frac{109}{125}\right)^3\binom{3}{1}\left(\frac{64}{625}\right)\left(\frac{561}{625}\right)^2\right]$$

## Q.No.3

$$G = A - B(x-d)^2$$

$$E[G] = E[A] - E[B]\,E[x^2 - 2dx + d^2]$$
$$= A - B\left(E[x^2] - 2dE[x] + d^2\right) \quad \left(\because E[a] = a \text{ where } a \text{ is a Constant}\right)$$
$$= A - BE[x^2] + 2d\,E[x] - d^2B$$

To Maximize $E[G]$ we take derivative of $E[G]$ and put it equal to zero.

$$\frac{\partial E[G]}{\partial d} = 0$$

$$2B(E[x]) - 2dB = 0$$

$$\Rightarrow d = E[x]$$

# Q.No.4

(a) We know that sum of all probabilities is equal to 1.

$$\sum_{k=1}^{\infty} C\,(1-P)^{k-1} = 1$$

$$C \sum_{k=1}^{\infty} (1-P)^{k-1} = 1$$

$$C \sum_{k=0}^{\infty} (1-P)^{k} = 1$$

$$C \, \frac{1}{1-(1-P)} = 1$$

$$C = P$$

(b)

$$P(k>n) = C\,(1-P)^n + C\,(1-P)^{n+1} + C\,(1-P)^{n+2} + \cdots$$

$$= C\,(1-P)^n\left[\,1 + (1-P) + (1-P)^2 + \cdots\,\right]$$

$$= C\,(1-P)^n \cdot \frac{1}{1-(1-P)}$$

$$= \frac{C\,(1-P)^n}{P} = \frac{P\,(1-P)^n}{P} \quad (\because C = P)$$

$$= (1-P)^n$$

(c) $$P(K>2n \mid K>n) = \frac{P(K>2n \cap K>n)}{P(K>n)}$$

$$= \frac{P(K>2n)}{P(K>n)}$$

$$= \frac{(1-p)^{2n}}{(1-p)^{n}}$$

$$= (1-p)^{n}$$

(d) $$P(K=3,6,9,12,\ldots) = c(1-p)^2 + c(1-p)^5 + c(1-p)^8 + \ldots$$

$$= c(1-p)^2\left[1 + (1-p)^3 + (1-p)^6 + (1-p)^9 + \ldots\right]$$

$$= c(1-p)^2\left[1 + \left[(1-p)^3\right]^1 + \left[(1-p)^3\right]^2 + \left[(1-p)^3\right]^3 + \ldots\right]$$

$$= \frac{c(1-p)^2}{1-(1-p)^3}$$

**Q.No.5** one Minutes includes six ~~minute~~ steps.
K represents the position of Drunk.

(a) $p(K=6) = P(\text{All steps up}) = \left(\frac{3}{4}\right)^6$

$p(K=4) = p(5 \text{ up } 1 \text{ down}) = \binom{6}{5}\left(\frac{3}{4}\right)^5\left(\frac{1}{4}\right)$

$p(K=2) = p(4 \text{ up } 2 \text{ down}) = \binom{6}{4}\left(\frac{3}{4}\right)^4\left(\frac{1}{4}\right)^2$

$p(K=0) = p(3 \text{ up } 3 \text{ down}) = \binom{6}{3}\left(\frac{3}{4}\right)^3\left(\frac{1}{4}\right)^3$

$p(K=-2) = p(2 \text{ up } 4 \text{ down}) = \binom{6}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^4$

$p(K=-4) = p(1 \text{ up } 5 \text{ down}) = \binom{6}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^5$

$p(K=6) = p(6 \text{ down}) = \left(\frac{1}{4}\right)^6$

Similar for 2 minutes (with 12 steps)

(b) Most ~~place~~ likely place is the one with highest probability in part (a).

⇒ After calculating all probabilities w.r.t 1 & 2 minutes, the behaviour can be observed by seeing the plots of PMF of both time frames.

(c)

$$E[K] = [6 \times p(K=6)] + [4 \times p(K=4)] + [2 \times p(K=2)] + [0 \times p(K=0)]$$
$$+ [-2 \times p(K=-2)] + [-4 \times p(K=-4)] + [-6 \times p(K=-6)]$$

Similarly Average for 2 minutes.

(d) Average position change w.r.t time can be seen by answers of part (c).

**Q.No.6**

Let $N$ be the number shown on initial die

$X_i$ = number shown on $i^{th}$ die

If $N=n \Rightarrow X = X_1 + X_2 + X_3 + \cdots + X_n$

$$E[X] = E[X_1] + E[X_2] + E[X_3] + \cdots + E[X_n]$$

for every possible outcome $i$ we know that

$$E[X_i] = \frac{1+2+3+4+5+6}{6} = \frac{21}{6}$$

$$E[X] = \frac{21}{6} n$$

$$\text{Average money} = \sum_x x \, p_X(x)$$
$$= \sum_x x \, P(\{X = x\})$$
$$= \sum_x x \sum_n P(\{X = x \text{ and } N = n\})$$

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$$= \sum_x x \sum_n P(\{X=x\} \mid N=n)\, P(N=n)$$

$$= \sum_n \left[ \sum_x x\, P(\{X=x\} \mid N=n) \right] P(N=n)$$

$$= \sum_{n=1}^{6} \frac{21}{6} n \frac{1}{\cancel{1}6}$$

$$= \sum_{n=1}^{6} \frac{21}{36} n$$

$$= \frac{21}{36} \sum_{n=}^{6} n$$

$$= \frac{21^2}{36} = 12.25$$

## Q. No. 7

$V = 2X + 2Y$ , $W = X - Y$

**(a)** V & W are not independent. Info. of W gives us information of V

Given $W=0$ gives $X=Y$ so V can be only 4, 8, 12.

**(b)** $$p_V(v) = \begin{cases} 1/9 & \text{for } V = 4, 12 \\ 2/9 & \text{for } V = 6, 10 \\ 3/9 & \text{for } V = 8 \\ 0 & \text{else} \end{cases}$$

values of V

| Y \ X | 1 | 2 | 3 |
|---|---|---|---|
| 3 | 8 | 10 | 12 |
| 2 | 6 | 8 | 10 |
| 1 | 4 | 6 | 8 |

$$E[V] = \sum_v v\, p_V(v)$$

$$= 4 \cdot \frac{1}{9} + 6 \cdot \frac{2}{9} + 8 \cdot \frac{3}{9} + 10 \cdot \frac{2}{9} + 12 \cdot \frac{1}{9}$$

$$E[V] = 8$$

$$E[V^2] = 4^2 \cdot \frac{1}{9} + 6^2 \cdot \frac{2}{9} + 8^2 \frac{3}{9} + 10^2 \cdot \frac{2}{9} + 12^2 \cdot \frac{1}{9}$$

$$= 69.3$$

$$\mathrm{Var}[V] = E[V^2] - E[V]^2$$
$$= 69.3 - 64$$
$$= 5.3$$

(c)
$$P_{V,W}(v,w) = \begin{cases} 1/9 & \text{for } Y,X = 1,2,3 \\ 0 & \text{otherwise} \end{cases}$$

| | | |
|---|---|---|
| 8,-2 | 10,-1 | 12,0 |
| 6,-1 | 8,0 | 10,1 |
| 4,0 | 6,1 | 8,2 |

(d) $E[V|W>0] = \sum_v v \cdot p_{V|W}(v|W>0)$

$$p_{V|W}(v|W>0) = \begin{cases} 1/3 & \text{for } v = 6,8,10 \\ 0 & \text{otherwise} \end{cases}$$

$$E[V|W>0] = 6\cdot\frac{1}{3} + 8\cdot\frac{1}{3} + 10\cdot\frac{1}{3}$$

(e) $E[W|V=8] = \sum_v w \cdot p_{W|V}(w|V=8)$

$$p_{W|V}(w|V=8) = \begin{cases} 1/3 & \text{for } w = 0,2,-2 \\ 0 & \text{otherwise} \end{cases}$$

$$E[W|V=8] = 0\cdot\frac{1}{3} + 2\cdot\frac{1}{3} + (-2)\cdot\frac{1}{3}$$
$$= 0$$
$$E[W^2|V=8] = 0\cdot\frac{1}{3} + 2\cdot\frac{1}{3} + 4\cdot\frac{1}{3} = \frac{8}{3}$$
$$\mathrm{Var}[W|V=8] = E[W^2|V=8] - E[W|V=8]^2$$
$$= \frac{8}{3} - 0 = \frac{8}{3}$$

(f) find probabilities of every possible value of X conditioned on every value of V.

**Q.No. 8**

$$P_K = \begin{cases} \frac{1}{4} & \text{if } K = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$

$$P_{N|K}(n|K) = \begin{cases} \frac{1}{K} & \text{for } n = 1, 2, \dots, K \\ 0 & \text{otherwise} \end{cases}$$

(a)

$$P_{N,K}(n,K) = P_{N|K}(n|K) \cdot P_K(K)$$

$$= \begin{cases} \frac{1}{4K} & \text{for } K = 1, 2, 3, 4 \\ & n = 1, \dots, K \\ 0 & \text{otherwise} \end{cases}$$

(b) Marginal P.M.F $P_N(n)$

$$P_N(n) = \sum_K P_{N,K}(n,K)$$

$$= \sum_K \frac{1}{4K} \quad \Rightarrow \text{Here } K = n, \dots, 4$$

For $n = 1$

$$P_N(n=1) = P_{N,K}(1,1) + P_{N,K}(1,2) + P_{N,K}(1,3) + P_{N,K}(1,4)$$

$$= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16}$$

$$= \frac{25}{48}$$

$$P_N(n=2) = P_{N,K}(2,2) + P_{N,K}(2,3) + P_{N,K}(2,4)$$

$$= \frac{13}{48}$$

$$P_N(n=3) = \frac{7}{48}$$

$$P_N(n=4) = \frac{3}{48}$$

(c)

Conditional P.M.F $P_{K|N}(k/2)$

$$P_{K|N}(k|2) = \frac{P_{N,K}(2,k)}{P_N(n=2)} \begin{matrix}\rightarrow \text{Part (a)} \\ \rightarrow \text{part (b)}\end{matrix}$$

As $n=2 \Rightarrow k$ starts from 2

For $k=2$

$$P_{N,K}(2,2) = \frac{1}{8} = \frac{6}{48}$$

$$P_N(n=2) = \frac{13}{48}$$

$$P_{K|N}(k=2/n=2) = \frac{6/48}{13/48} = \frac{6}{13}$$

Similarly

$$P_{K|N}(k=3/n=2) = \frac{4}{13}$$

$$P_{K|N}(k=4/n=2) = \frac{3}{13}$$

(d)

$P_{X,Y}(x,y) \Rightarrow$

| $Y$ \ $X$ | 1 | 2 | 3 |
|---|---|---|---|
| 3 | 0 | 0 | $\frac{9}{64}$ |
| 2 | 0 | $3/16$ | $9/64$ |
| 1 | $\frac{3}{4}\ \frac{1}{3}$ | $2/16$ | $\frac{3}{64}$ |
| 0 | $\frac{1}{4}\ \frac{1}{3}$ | $\frac{1}{16}\times\frac{1}{3}$ | $\frac{1}{3}\times\frac{1}{64}$ |

$P_{X,Y}(X=3, Y=3)$
$= p(3R|3Q)\cdot p(3Q)$
$= \left(\frac{3}{4}\right)^3 \cdot \frac{1}{3}$

(e)

$B = 10X + 20Y$

$E[B] = 10\,E[X] + 20\,E[Y]$

$= 10\times(2) + 20\times(1.5)$

(f) $Z = 20Y$ (Y is for No. of Q's Answered wrong in one lecture)

$E[Z] = 20E[Y]$

$Var[Z] = 20^2 Var[Y]$

(g) We have Y divided in two Cases

$P_Y(y) = P_{Y|Science}(y|Science)\cdot p(Science) + P_{Y|Math}(y|Math)\cdot p(Math)$

$= \frac{1}{2} P_{Y|Science}(y|Science) + \frac{1}{2} P_{Y|Math}(y|Math)$

$P_{Y|S}(y|S) \Rightarrow$ Same as part c, but rember Now y means No. of Q's wrong.

$P_{Y|M}(y|M) = p(AW|1Q)\cdot p(1Q) + p(AW|2Q)\cdot p(2Q) + p(AW|3Q)\cdot p(3Q)$

$= \left(\frac{1}{10}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{10}\right)^2\left(\frac{1}{3}\right) + \left(\frac{1}{10}\right)^3\cdot\left(\frac{1}{3}\right)$

Similarly find for $y = 2, 1, 0$

At the end Calculate $E[Y]$

## Q. No. 9

(a) All Wrong ⇒ All W or AW, 1 Question Asked = 1Q.

$$P(\text{All } W) = P(\text{All } W|1Q)\cdot P(1Q) + P(\text{All } W|2Q)\cdot P(2Q) + P(\text{All } W|3Q)\cdot P(3Q)$$

$$= \frac{1}{3}\left(\frac{1}{4}\right) + \frac{1}{3}\left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3\frac{1}{3} = \frac{7}{64}$$

(b)

$$P(3Q|\text{All } W) = \frac{P(\text{All } W|3Q)\cdot P(3Q)}{P(\text{All } W)}$$

$$= \frac{\left(\frac{1}{4}\right)^3 \frac{1}{3}}{\frac{7}{64}} = \frac{1}{21}$$

(c)

$$E[X] = \sum_x x\cdot P_X(x)$$

$$= 1\cdot\frac{1}{3} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{3}$$

$$= 2$$

$$E[X^2] = 1^2\cdot\frac{1}{3} + 2^2\cdot\frac{1}{3} + 3^2\cdot\frac{1}{3}$$

$$= \frac{14}{3}$$

$$\text{Var}[X] = E[X^2] - E[X]^2$$

$$= \frac{14}{3} - 4 = \frac{2}{3}$$

$$E[Y] = \sum_y y\cdot P_Y(y)$$

$$P_Y(y=0) = P(AW|1Q)\cdot P(1Q) + P(AW|2Q)\cdot P(2Q) + P(AW|3Q)\cdot P(3Q)$$

$$= \frac{7}{64} \text{ from part (a)}$$

1R ⇒ 1 Right answer

$$p_Y(y=1) = p(1R\mid 1Q)\cdot p(1Q) + p(1R\mid 2Q)\cdot p(2Q) + p(1R\mid 3Q)\cdot p(3Q)$$

$$= \left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \binom{2}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{3}\right) + \binom{3}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right)$$

$$= \frac{27}{64}$$

$$p_Y(y=2) = 0 + \left(\frac{3}{4}\right)^2\left(\frac{1}{3}\right) + \binom{3}{1}\left(\frac{3}{4}\right)^2\frac{1}{4}\cdot\frac{1}{3}$$

If 1Q asked, 0 probability of having 2R answers.

$$= \frac{21}{64}$$

$$p_Y(y=3) = 0 + 0 + \left(\frac{3}{4}\right)^3\cdot\frac{1}{3}$$

$$= \frac{9}{64}$$

$$E[Y] = 0\cdot\frac{7}{64} + 1\cdot\frac{27}{64} + 2\cdot\frac{21}{64} + 3\cdot\frac{9}{64}$$

$$= 1.5$$

$$E[Y^2] = 0\cdot\frac{7}{64} + 1\cdot\frac{27}{64} + 4\cdot\frac{21}{64} + 9\cdot\frac{9}{64}$$

$$= 3$$

$$Var[Y] = E[Y^2] - E[Y]^2$$

$$= 3 - 2.25$$

$$= 0.75$$

$$= \frac{3}{4}$$

## Q.No.10

$V_i \Rightarrow i$th visit, $P_j \Rightarrow j$ no. of Pens

(a)

$$P(3 \text{ pens}) = P(V_1P_1)\cdot P(V_2\cdot P_2) + P(V_1P_2)\cdot P(V_2\cdot P_1) + P(V_1P_3)$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \left(\frac{1}{3}\right)\left(\frac{1}{3}\right) + \frac{1}{3}$$

$$= \frac{5}{9}$$

(b)

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$$P(A\cap B) = P(3 \text{ pens } \& \text{ } 2 \text{ visits})$$

$$= P(V_1P_1)\cdot P(V_2P_2) + P(V_1P_2)\cdot P(V_2P_1)$$

$$= \frac{2}{9}$$

$$P(B|A) = \frac{2/9}{5/9} = \frac{2}{5}$$

(c) $N \Rightarrow$ No. of Pens recieved. $n = 2, 3, 4, 5$

$$p(2) = P(V_1P_1 \cap V_2P_1) = \frac{1}{9}$$

$$p(3) = P(V_1P_1 \cap V_2P_2) + P(V_1P_2 \cap V_2P_1) + P(V_1P_3) = \frac{5}{9}$$

$$p(4) = P(V_1P_2 \cap V_2P_2) + P(V_1P_1 \cap V_2P_3) = \frac{2}{9}$$

$$p(5) = P(V_1P_2 \cap V_2P_3) = \frac{1}{9}$$

$$E[N] = 2\cdot\frac{1}{9} + 3\cdot\frac{5}{9} + 4\cdot\frac{2}{9} + 5\cdot\frac{1}{9}$$

$$= \frac{10}{3}$$

$$E[N|C] = \sum_n n \cdot p_{N|C}(n/N>3)$$

$$p_{N|C}(N=4|N>3) = \frac{2}{3}$$

$$p_{N|C}(N=5|N>3) = \frac{1}{3}$$

$$E[N|C] = 0 + 0 + 4 \cdot \frac{2}{3} + 5 \cdot \frac{1}{3}$$

$$= 4 \cdot 3 = \frac{13}{3}$$

**(d)** $\sigma^2(N|C) = E[N^2|C] + E[N|C]^2$

$$E[N^2|C] = 4^2 \cdot \frac{2}{3} + 5^2 \cdot \frac{1}{3} = 19$$

$$\sigma^2(N|C) = 19 - \frac{169}{9}$$

$$= \frac{2}{9}$$

**(e)** $P(D) = [P(\text{Pens} > 3)]^{16}$

$$P(\text{Pens} > 3) = p_N(N > 3) = p_N(4) + p_N(5)$$

$$= \frac{3}{9}$$

$$P(D) = \left(\frac{3}{9}\right)^{16} = \left(\frac{1}{3}\right)^{16}$$