UNIVERSITY OF ENGINEERING AND TECHNOLOGY
Department of Computer Science and Engineering

Probability and Random Variables
Fall 2018

Problem Set 3
Due: September 24, 2018

Note:

- These problems have been liberally copied from older versions of MIT's 6.041/6.431, Harvard's Stat 110, Al Drake's book and Gian Carlo Rota's notes. It is difficult to acknowledge each problem individually. There is no claim to originality in these problems, and the debt to all these resources is gratefully acknowledged. This applies to all problems in this course.
- You may be able to use "story proofs" in proving identities. We did an example of a "story proof" when we proved that for $0 \leq p \leq 1$

$$\sum_{n=0}^{k} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

1. How many ways are there to permute the letters in the word STATISTICS?
2. There are k people in a room. Assume each person's birthday is equally likely to be any of the 365 days of the year (we exclude February 29), and that people's birthdays are independent (we assume there are no twins in the room). What is the probability that two or more people in the group have the same birthday? Calculate the answer (using a calculator or a computer) when $k = 23$ and when $k = 50$ (your section size).
3. You are ordering two pizzas. A pizza can be small, medium, large, or extra large, with any combination of 8 possible toppings (getting no toppings is allowed, as is getting all 8). How many possibilities are there for your two pizzas?
4. Prove that
   (a)
   $$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^n y^{n-k}$$
   (b)
   $$\binom{m+n}{k} = \sum_{j=0}^{k} \binom{m}{j} \binom{n}{k-j}$$
   Hint: Consider a group of m women and n men.
   (c)
   $$\sum_{k=1}^{n} k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$$
5. Companies A, B, C, D, and E each send three delegates to a conference. A committee of four delegates, selected by lottery, is formed. Determine the probability that
   (a) Company A is not represented on the committee.

(b) Company A has exactly one representative on the committee.

(c) Neither company A nor company E is represented on the committee.

6. A parking lot contains 100 cars that all look quite nice from the outside. However, K of these cars happen to be lemons. The number K is known to lie in the range $\{0, 1, \ldots, 9\}$, with all values equally likely. We test drive 20 distinct cars chosen at random, and to our pleasant surprise, none of them turns out to be a lemon. Given this knowledge , give a formula for the probability that $K = 0$. (Do not need to find a numerical answer.)

7. Find the probability of winning in the following lottery: You choose 5 distinct integers in the range from 1 to 100. Then, the lottery operator chooses randomly 10 distinct integers in the same range (all outcomes being equally likely). You win if all of your 5 numbers are among those chosen by the operator.

8. Let $n \geq 2$ be an integer. Show that

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = n(n-1)2^{n-2}$$

9. Let N be the number of successes in n independent trials, and let p be the probability of success during each trial. Find the probability that

   (a) the first trial was a success, given that $N = 1$.

   (b) the first trial was a success , given that $N = 2$.

   (c) exactly two out of the first four trials were successes, given that $N = 6$

10. A candy factory has an endless supply of red , orange, yellow, green, blue, and violet jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each. One possible color distribution, for example, is a jar of 56 red, 22 yellow, and 22 green jelly beans. As a marketing gimmick, the factory guarantees that no two jars have the same color distribution. What is the maximum number of jars the factory can produce? Hint: Think of lining up the jelly beans, first placing the red ones , then the orange ones, etc. We also place 5 dividers to indicate where one color ends and another starts. The composition of jar can be determined from the location of the dividers.