Chapter 1: On Metrics and Spaces

1 The Idea of a Metric (Distance)

The core idea is a Metric, which is simply a formal way to measure distance between any two points in a set X.

1.1 Definition of a Metric (The Rules of Distance)

A function d that calculates distance between points x and y in a set X must obey three fundamental, intuitive rules:

1. **Non-negativity & Identity:** The distance between any two points is always greater than or equal to zero. The distance is zero only if the points are the exact same point.

   $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.

2. **Symmetry:** The distance from x to y is the same as the distance from y to x.

   $d(x, y) = d(y, x)$.

3. **Triangle Inequality:** Taking a shortcut is never longer than taking a path through another point z. (The shortest path between two points is a straight line).

   $d(x, y) \leq d(x, z) + d(z, y)$.

1.2 Metric Space

A Metric Space is simply the combination of a set of points (X) and the defined distance function (d) that measures distances between them, written as (X, d).

2 Examples of Metric Spaces

A metric can also be used to define limits and continuity of functions. Different types of spaces use different formulas for distance:

1. **Usual Metric on $\mathbb{R}$ (Real Numbers):** This is the standard, straight-line distance you learned in algebra.

   $d(x, y) = |x - y|$.

2. **Euclidean Metric (d_2) on $\mathbb{R}^k$ (k-Dimensional Space):** This is the familiar straight-line distance in 2D, 3D, and higher dimensions (Pythagorean theorem generalized).

   $d_2(x, y) = (\sum_{i=1}^{k} |x_i - y_i|^2)^{\frac{1}{2}}$.

3. **Manhattan Metric (d_1):** This is the "taxicab" distance, where you can only move along axes (like city blocks in Manhattan).

   $d_1(x, y) = \sum_{i=1}^{k} |x_i - y_i|$.

4. **p-Metric (d_p):** A general form that includes the Euclidean ($p = 2$) and Manhattan ($p = 1$) metrics.

   $d_p(x, y) = (\sum_{i=1}^{k} |x_i - y_i|^p)^{\frac{1}{p}}$ for $p \geq 1$.

5. **Maximum Metric (d_∞):** This is the largest single-coordinate difference, like the movement of a King on a chessboard (it's only limited by its longest move in one direction).

   $d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$.

6. **Discrete Metric:** In this space, the distance is either 1 (if the points are different) or 0 (if they are the same). It treats every point as equally "far" from every other point.

   $d(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$.

7. **Metrics on Continuous Functions ($C[a, b]$):**

   (a) d_1 (Area-based distance): The distance is the total area between the graphs of the two functions f and g over the interval $[a, b]$.

   $d_1(f, g) = \int_a^b |f(x) - g(x)|$.

   (b) d_∞ (Maximum difference distance): The distance is the single largest vertical difference between the graphs of f and g over the interval $[a, b]$.

   $d_\infty(f, g) = \sup_{a \leq x \leq b} |f(x) - g(x)|$.

3 Key Inequalities (Tools for Metrics)

These inequalities are crucial for proving the Triangle Inequality for general d_p metrics (Minkowski's inequality).

1. **Young's Inequality (Lemma 4):** A relationship between products and sums of powers for non-negative numbers x and y.

   If $x, y \geq 0$ and $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$.

2. **Hölder's Inequality:** A vector-based generalization of Young's inequality.

   Let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^k$. If $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then the sum of the products of their absolute components is less than or equal to the product of their p-norms and q-norms:

   $\sum_{i=1}^{k} |a_i|.|b_i|$ [cite_start]$\leq (\sum_{i=1}^{k} |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{k} |b_i|^q)^{\frac{1}{q}}$.

3. **Minkowski's Inequality (Theorem 5):** This is the formal name for the Triangle Inequality when using the d_p metric.

   For arbitrary vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^k$ and $p > 1$, the distance from $\mathbf{a}$ to $\mathbf{b}$ is less than or equal to the distance from $\mathbf{a}$ to $\mathbf{c}$ plus the distance from $\mathbf{c}$ to $\mathbf{b}$:

   $d_p(a, b) \leq d_p(a, c) + d_p(c, b)$.

4 Open and Closed Sets (Topology)

In a metric space (X, d), we use the distance function to define geometric shapes and properties.

4.1 Open Balls

The most basic shape is the Open Ball $B(x, r)$. It's the collection of all points y that are strictly less than a distance r away from a center point x.

$$B(x, r) = \{y \in X : d(x, y) < r\}$$

1. In $\mathbb{R}$ (usual metric): The open balls are just open intervals $B(x, r) = (x - r, x + r)$.

2. In a Discrete Metric Space: If the radius is $r = 1$, the open ball $B(x, 1)$ only contains the center point x itself, because any other point is exactly distance 1 away (and $1 \not< 1$).

4.2 Bounded Set

A subset A is bounded if you can completely enclose it within a single open ball $B(x, r)$.

4.3 Open Sets

A subset U of X is Open if for every point x inside U, you can find a small open ball centered at x that is completely contained within U.

4.4 Properties of Open Sets (Theorem 10)

1. The empty set ($\emptyset$) and the whole space (X) are open.
2. Any combination (union) of open sets is open.
3. The intersection of a finite number of open sets is open.
4. Note: The intersection of an infinite number of open sets is not necessarily open.
5. Every open ball is an open set.
6. A set is open if and only if you can write it as a union of open balls.
7. In a discrete metric space, every subset is open.

4.5 Closed Balls

The Closed Ball $\overline{B(x, r)}$ is similar to the open ball, but it includes all points exactly at distance r from x.

$$\overline{B(x, r)} = \{y \in X : d(x, y) \leq r\}$$

4.6 Closed Sets

A subset A is Closed if its complement, $X - A$ (all points not in A), is an open set.

4.7 Properties of Closed Sets (Theorem 15)

These properties are logically derived from the open set properties using De Morgan's laws.

1. The empty set ($\emptyset$) and the whole space (X) are closed.
2. The union of a finite number of closed sets is closed.
3. Any intersection of closed sets (even infinite) is closed.
4. A set A is closed if and only if every sequence of points in A that converges, converges to a point that is also in A.

5 Convergence of Sequences

5.1 Sequence Convergence

A sequence of points $\{x_n\}$ in a metric space (X, d) converges to a point $x \in X$ if, as you go further out in the sequence (as $n \to \infty$), the terms get arbitrarily close to x.

Formally: For any distance $\epsilon > 0$, you can find a point x_N in the sequence such that all points x_n after x_N are within distance ϵ of the limit point x.

- Notation: We write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
- Uniqueness: The limit of a sequence in a metric space is unique (it can only converge to one point).
- In $\mathbb{R}^k$: A sequence of vectors converges if and only if each of its component sequences converges in $\mathbb{R}$.

6 Continuity of Functions

Continuity relates the distances in the starting space to the distances in the ending space.

6.1 Function Continuity (Definition 16)

A function f from metric space (X, d_X) to (Y, d_Y) is continuous at a point x if small changes in the input produce small changes in the output.

- Formally: For any small distance ϵ in the output space (Y), you can find a corresponding distance δ in the input space (X) such that if an input y is within δ distance of x, its output $f(y)$ is within ϵ distance of $f(x)$.
- $d_X(x, y) < \delta \implies d_Y(f(x), f(y)) < \epsilon$.
- This also means that the image of the open ball $B(x, \delta)$ is contained within the open ball $B(f(x), \epsilon)$.

6.2 Special Cases:

- A constant function $f : X \to Y$ is always continuous.
- If the input space X uses the discrete metric, every function $f : X \to Y$ is continuous.

6.3 Properties of Continuous Functions (Theorems 17, 18)

- Composition: If you have two continuous functions, f and g, applying one after the other ($g \circ f$) is also a continuous function.
- Sequence Preservation: If a function f is continuous and an input sequence $\{x_n\}$ converges to x, then the output sequence $\{f(x_n)\}$ must converge to $f(x)$.
- Open Set Characterization: A function f is continuous if and only if the pre-image of every open set U in the output space Y (i.e., $f^{-1}(U)$) is an open set in the input space X.

6.4 Lipschitz Continuous Functions (Definition 19)

A function is Lipschitz continuous if its rate of change is absolutely bounded by a constant L (it's "nicely" continuous).

$d_Y(f(x), f(y)) \leq L \cdot d_X(x, y)$ for some constant $L \geq 0$.

6.5 Relationship to Continuity (Theorem 20):

- Every Lipschitz continuous function is automatically a continuous function.
- Differentiable Functions: If a function $f : [a, b] \to \mathbb{R}$ is differentiable and its derivative is bounded, then it is Lipschitz continuous on $[a, b]$.

7 Convergence of Functions

When dealing with sequences of functions $\{f_n\}$, we can define convergence in two main ways.

7.1 Pointwise Convergence

A sequence of functions $\{f_n(x)\}$ converges pointwise to $f(x)$ if, for every fixed point x in the domain, the sequence of numbers $\{f_n(x)\}$ converges to the number $f(x)$.

This means $|f_n(x) - f(x)|$ [cite_start]$\to 0$ as $n \to \infty$.

7.2 Uniform Convergence

A sequence of functions f_n converges uniformly to f on X if the rate of convergence is the same for all points x in X.

Formally: For any distance $\epsilon > 0$, you can find an N such that for all $n \geq N$, the maximum difference between $f_n(x)$ and $f(x)$ over the entire set X is less than ϵ.

This means $\sup_{x \in X} |f_n - f|$ [cite_start]$\to 0$ as $n \to \infty$.

7.3 Relationship:

If a sequence converges uniformly, it is guaranteed to converge pointwise.

7.4 Uniform Limit Theorem (Theorem 23)

If every function f_n in the sequence is continuous and the sequence converges uniformly to f, then the limit function f must also be continuous.

Caution: The pointwise limit of continuous functions is not necessarily continuous (for example, $f_n(x) = x^n$ on $[0, 1]$ converges pointwise to a discontinuous function).

8 Completeness

Completeness addresses the question of whether a space has "holes" or is "whole."

8.1 Cauchy Sequence (Definition 24)

A sequence $\{x_n\}$ is called a Cauchy sequence if the terms in the sequence eventually get arbitrarily close to each other.

Formally: For any $\epsilon > 0$, you can find an N such that the distance between any two terms x_m and x_n (where $m, n \geq N$) is less than ϵ.

8.2 Properties (Theorem 25):

- Every convergent sequence is a Cauchy sequence.
- Every Cauchy sequence is bounded.
- Caution: A Cauchy sequence does not have to be convergent (e.g., the sequence $x_n = 1/n$ in the space $X = (0, 2)$ is Cauchy but converges to 0, which is outside the space X).

8.3 Complete Metric Space (Definition 26)

A metric space (X, d) is called complete if every Cauchy sequence of points in X is guaranteed to converge to a point that is also in X. (A complete space is a space with "no holes" that could be the limit of one of its sequences.)

8.4 Completeness Results (Theorem 29, 30)

- If a Cauchy sequence has a convergent subsequence, then the full sequence also converges to the same limit.
- The Real Numbers $\mathbb{R}$ are a complete metric space.
- In $\mathbb{R}$: Every monotonic and bounded sequence converges.

- In $\mathbb{R}$: Every bounded sequence has a convergent subsequence (Bolzano-Weierstrass).
- The Euclidean Space $\mathbb{R}^k$ is complete with respect to its usual metric.
- The Space of Continuous Functions $C[a, b]$ is complete with respect to the d_∞ metric.
- A subset A of a complete metric space X is complete if and only if A is a closed set in X.

8.5 Contraction Mapping (Definition 31)

A function $f : X \to X$ is a contraction if it shrinks the distance between any two points by a factor of $L < 1$.

$d(f(x), f(y)) \leq L \cdot d(x, y)$, where $0 \leq L < 1$.

Every contraction is Lipschitz continuous, and therefore also continuous.

8.6 Banach's Fixed Point Theorem (Theorem 32)

This is a powerful result: If a function f is a contraction on a complete metric space X, then f has exactly one fixed point (a unique point x such that $f(x) = x$).

8.7 Completion of a Metric Space (Theorem 34)

For any metric space (X, d), you can always find a larger, complete metric space (X', d') that contains X in a way that preserves distances (via an isometry $f : X \to X'$). This new space X' is called the completion of X. (You can always "fill in the holes" of a metric space to make it complete.).