Application Questionnaire

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Question 1

Evaluate the limit:

$$\lim_{x \to a} \frac{x^x - a^a}{a^x - a^a}$$

Step 1: Identify the indeterminate form

Substitute $x = a$:

$$\frac{a^a - a^a}{a^a - a^a} = \frac{0}{0}$$

This is an indeterminate form, so we apply L'Hôpital's Rule.

Step 2: Differentiate numerator and denominator

Numerator: $f(x) = x^x$ Taking the natural logarithm,

$$\ln f(x) = x \ln x$$

Differentiate both sides,

$$\frac{f'(x)}{f(x)} = \ln x + 1 \implies f'(x) = x^x(\ln x + 1)$$

Denominator: $g(x) = a^x$

$$g'(x) = a^x \ln a$$

Step 3: Apply L'Hôpital's Rule

$$\lim_{x \to a} \frac{x^x - a^a}{a^x - a^a} = \lim_{x \to a} \frac{x^x(\ln x + 1)}{a^x \ln a} = \frac{a^a(\ln a + 1)}{a^a \ln a} = \frac{\ln a + 1}{\ln a}$$

Final answer

$$\boxed{\lim_{x \to a} \frac{x^x - a^a}{a^x - a^a} = \frac{\ln a + 1}{\ln a}}$$

Question 2

Define an orthonormal basis of vectors in $\mathbb{R}^n$.

Definition

An **orthonormal basis** of $\mathbb{R}^n$ is a set of n vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ such that:

- Each vector has unit length: $\|\mathbf{v}_i\| = 1$.
- The vectors are mutually orthogonal: $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$.
- The vectors span $\mathbb{R}^n$.

How many vectors are in the basis? Explain why.

There are exactly n vectors in the orthonormal basis because $\mathbb{R}^n$ is an n-dimensional vector space, so any basis must have n linearly independent vectors.

Separate the basis vectors into two sets A and B. Can vectors in A be linearly dependent on vectors in B? Explain why.

Suppose the basis vectors are divided into two disjoint sets A and B such that

$$A \cup B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\} \quad \text{and} \quad A \cap B = \emptyset.$$

Because the whole set is a basis (linearly independent spanning set), no vector in A can be written as a linear combination of vectors in B. This is because if there was such dependence, the total set would not be linearly independent, contradicting the fact that it is a basis.

Define the square matrix $P = [A \quad 0]$, where A is the matrix of basis vectors from set A and 0 is a zero matrix. Compute $P^\top P$.

Let the set A have k vectors, each of dimension n, so A is an $n \times k$ matrix. Let 0 be an $n \times (n-k)$ zero matrix. Then

$$P = [A \quad 0] \quad \text{is an } n \times n \text{ matrix.}$$

The transpose is

$$P^\top = \begin{bmatrix} A^\top \\ 0^\top \end{bmatrix} = \begin{bmatrix} A^\top \\ 0 \end{bmatrix}.$$

Now compute

$$P^\top P = \begin{bmatrix} A^\top \\ 0 \end{bmatrix} [A \quad 0] = \begin{bmatrix} A^\top A & A^\top 0 \\ 0 \cdot A & 0 \cdot 0 \end{bmatrix} = \begin{bmatrix} A^\top A & 0 \\ 0 & 0 \end{bmatrix}.$$

Since A consists of orthonormal vectors, $A^\top A = I_k$, the identity matrix of size k. Thus,

$$\boxed{P^\top P = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}.}$$

Question 3

Write the explicit formula for the gradient of

$$E[u] = \sum_{i=1}^{1} \sum_{j=0}^{1} \cos(u[i,j]) \sin(u[i-1,j])$$

with respect to u, where u is a 2×2 matrix indexed by 0 and 1 in both coordinates.

Step 1: Write out $E[u]$ explicitly

Note the indices: $i = 1$ only (since sum from 1 to 1) and $j = 0, 1$:

$$E[u] = \cos(u[1,0]) \sin(u[0,0]) + \cos(u[1,1]) \sin(u[0,1]).$$

Step 2: Compute gradient $\nabla_u E[u]$

The gradient is a matrix of partial derivatives:

$$\nabla_u E[u] = \left[\frac{\partial E}{\partial u[i,j]} \right]_{i,j=0,1}.$$

Calculate each partial derivative:

- For $u[0,0]$:
$$\frac{\partial E}{\partial u[0,0]} = \cos(u[1,0]) \cdot \frac{d}{du[0,0]} \sin(u[0,0]) = \cos(u[1,0]) \cos(u[0,0]).$$
- For $u[0,1]$:
$$\frac{\partial E}{\partial u[0,1]} = \cos(u[1,1]) \cdot \frac{d}{du[0,1]} \sin(u[0,1]) = \cos(u[1,1]) \cos(u[0,1]).$$
- For $u[1,0]$:
$$\frac{\partial E}{\partial u[1,0]} = \sin(u[0,0]) \cdot \frac{d}{du[1,0]} \cos(u[1,0]) = -\sin(u[0,0]) \sin(u[1,0]).$$
- For $u[1,1]$:
$$\frac{\partial E}{\partial u[1,1]} = \sin(u[0,1]) \cdot \frac{d}{du[1,1]} \cos(u[1,1]) = -\sin(u[0,1]) \sin(u[1,1]).$$

Step 3: Write the gradient matrix

$$\nabla_u E[u] = \begin{bmatrix} \cos(u[1,0])\cos(u[0,0]) & \cos(u[1,1])\cos(u[0,1]) \\ -\sin(u[0,0])\sin(u[1,0]) & -\sin(u[0,1])\sin(u[1,1]) \end{bmatrix}.$$

Question 4

Consider IID samples $x_1, \ldots, x_m$ that are Poisson distributed with mean λ.

(a) Write the probability density function $p(x)$ and compute the mean λ as expectation

The probability mass function (PMF) of Poisson distribution is:

$$p(x) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0, 1, 2, \ldots$$

Compute the mean $\mathbb{E}[X]$:

$$\mathbb{E}[X] = \sum_{x=0}^{\infty} x \cdot p(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!}.$$

Rewrite sum starting at $x = 1$ because $x = 0$ term is zero:

$$\mathbb{E}[X] = e^{-\lambda} \sum_{x=1}^{\infty} x \frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}.$$

Substitute $y = x - 1$:

$$= e^{-\lambda} \lambda \sum_{y=0}^{\infty} \frac{\lambda^y}{y!} = e^{-\lambda}\lambda e^{\lambda} = \lambda.$$

So,

$$\boxed{\mathbb{E}[X] = \lambda.}$$

(b) Maximum likelihood estimate for λ

Given IID samples $x_1, \ldots, x_m$, the likelihood function is:

$$L(\lambda) = \prod_{i=1}^{m} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} = e^{-m\lambda} \lambda^{\sum_{i=1}^{m} x_i} \prod_{i=1}^{m} \frac{1}{x_i!}.$$

Log-likelihood:

$$\ell(\lambda) = \log L(\lambda) = -m\lambda + \left(\sum_{i=1}^{m} x_i\right) \log \lambda - \sum_{i=1}^{m} \log(x_i!).$$

Find $\hat{\lambda}$ that maximizes $\ell(\lambda)$:

$$\frac{d\ell}{d\lambda} = -m + \frac{\sum x_i}{\lambda} = 0 \implies \hat{\lambda} = \frac{1}{m} \sum_{i=1}^{m} x_i.$$

Final answer:

$$\boxed{\hat{\lambda} = \frac{1}{m} \sum_{i=1}^{m} x_i,}$$

the sample mean.