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Linear Algebra (2-4) M

Sub topics

1) determinants
2) Inverse
3) Rank

} Matrix algebra

4) Solution of System of linear Equations.
5) Eigen values, Eigen vectors & cayley Hamilton theorem.

Determinants:-

Matrix:- Matrix is represented by [ ], ( ), || ||

↑ Nor

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{vmatrix}_{m\times n}$$

(or)

$A = (a_{ij})_{m\times n}$) $A = (a_{ij})$ where $1 \le i \le m$ & $1 \le j \le n$.

Note:-

① $A = (a_{ij})_{m\times n}$

where $a_{ij} = i+j+2ij$
$= i-j+2$
$= i^2+j+3$
$\vdots$

$A = (a_{ij})_{4\times 3}$

$a_{ij} = i+4$ $\quad a_{42} = 4+4 = 8$

2) 2^{nd} order determinant

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then the expression $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ is called 2^{nd} order determinant of $A_{2\times2}$. It is denoted by $|A|$ (or) $\det(A)$.

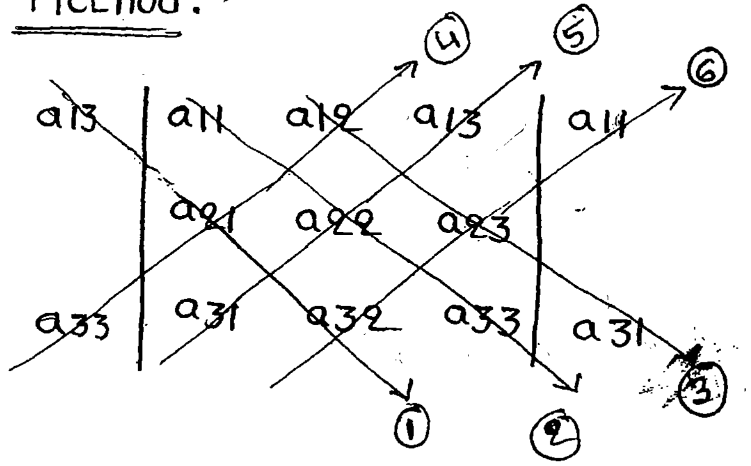
Here the unique no "$\boxed{a_{11}a_{22} - a_{12}a_{21}}$" is called Value (or) ~~Expression~~ Expansion of 2^{nd} order det $(A_{2\times2})$

3) 3^{rd} order determinant:-

$$|A| = \begin{array}{c} \begin{array}{ccc} + & - & + \end{array} \\ \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \end{array}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

2^{nd} Method.



$$|A| = (1 + 2 + 3) - (4 + 5 + 6)$$

4) 4^{th} order determinant:-

$$|A| = \begin{vmatrix} \overset{+}{a_{11}} & \overset{-}{a_{12}} & \overset{+}{a_{13}} & \overset{-}{a_{14}} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \leftarrow$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} +$$

$$a_{13}\begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14}\begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix}$$

Elementary operations:-

→ Interchanging of two rows. $\boxed{R_i \longleftrightarrow R_j}$

→ ultiply the Any row , column by a consta

$\boxed{R_i \longrightarrow KR_i}$ $(K \neq 0)$

→ Multiply by a constant and add with the othe rows. $\boxed{R_j \longrightarrow R_j + KR_i}$

These operations are used for the Matric & determinants.

|A| A

$\downarrow$

$R_j \to R_j + KR_1$ $R_j \to R_j + KR_1$ $R_j \to lR_j + KR_1$

Properties of determinants :-

1) $\boxed{|A_{n\times n}\; B_{n\times n}| = |A_{n\times n}|\,|B_{n\times n}|}$

> $\boxed{|A^K| = |A|^K}$

> $\begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 7 & -5 \end{vmatrix} = 0.$ If the Elements of the rows (or) columns of any one zero then total determinant value is zero.

> $\begin{vmatrix} 1 & 2 & 3 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{vmatrix} = 0,\quad \begin{vmatrix} 1 & 2 & 3 \\ 5 & 10 & 15 \\ 3 & 6 & 9 \end{vmatrix} = 0 \quad \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 0$

$\Downarrow$

$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{vmatrix} = 0. \qquad 3\begin{vmatrix} 1 & 2 & 3 \\ 5 & 10 & 15 \\ 1 & 2 & 3 \end{vmatrix} = 0.$

→ Any 16 consecutiv no of the 4th order and higher order are zero's.

→ when the two rows are same (or) two rows are proportional then the determinant is zero.

> $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 27$ then $\begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix} = 27$

* Interchanging rows by columns and vice versa the value of the determinant does not change

$|A| = |A'|$

(6) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = 27 \Rightarrow \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 0 \end{vmatrix} = (-1)(27)$

$= -27$

$\begin{vmatrix} 7 & 8 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = (-1)(-1)(27) = 27$ $\quad (-1)^K 27$

→ For interchanging the rows by odd no of times then the value is Negative of the previous value

→ Interchanging rows by even no of times then determinant value is same as the previous val[ue]

(7) $\begin{vmatrix} 1 & 2 & 3 \\ 4K & 5K & 6K \\ 7 & 8 & 0 \end{vmatrix} = K \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{vmatrix} = (27)(K)$

$\begin{bmatrix} 1 & 2 & 3 \\ 4K & 5K & 6K \\ 7 & 8 & 0 \end{bmatrix} \neq K \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$\begin{bmatrix} K & 2K & 3K \\ 4K & 5K & 6K \\ 7K & 8K & 0 \end{bmatrix} = K \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{bmatrix}$

$\boxed{|KA_{n\times n}| = K^n |A_n}$

→ $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 7 & 5 \\ 0 & 0 & 8 \end{bmatrix}$ upper triangle

→ $A = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 8 & 0 \\ 9 & 10 & 5 \end{bmatrix}$ Lower triangle

→ $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ diagonal Matrix $\quad A = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$ scalar Matrix

→ $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ unit Matrix

→ $\begin{vmatrix} a & b & c \\ 0 & d & c \\ 0 & 0 & f \end{vmatrix} = adf \qquad \begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & c & f \end{vmatrix} = acf.$

$\begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc, \qquad \begin{vmatrix} b & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & b \end{vmatrix} = b^3$

problems:-

The total no of terms in the expansion of nth order determinant is

a) n b) n^2 c) n^n d) None

Sol $A = (a_{11})_{1\times1} \longrightarrow |A| = |a_{11}| = a_{11} \longrightarrow 1!$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} \longrightarrow 2!$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad |A| = 6 = 3!$

$\boxed{A = (a_{ij})_{n\times n} \longrightarrow |A_{n\times n}| = \longrightarrow n!}$

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The value of 3rd order determinant is

$$\begin{vmatrix} 1 & 3 & 2 \\ 4 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix}$$ a) 24 b) −24 c) 32 d) 36

$= 1(3-1) - 3(12-2) + 2(4-2)$

$= 2 - 30 + 2(2) = 4 + 2 - 30 = -24$

→ The determinant of a matrix $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 100 & 1 & 0 & 0 \\ 100 & 200 & 1 & 0 \\ 100 & 200 & 300 & 1 \end{bmatrix}$

Sol It is Lower triangular Matrix and determinant is product of the all the diagonal Elements.

$(1)(1)(1)(1) = 1$

→ The determinant of a matrix given below is

$$\begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & 0 & 1 \end{bmatrix}$$ b) −1 c) 0 d) None

$= 0 - 1 \begin{bmatrix} 0 & 1 & 0 \\ -1 & 1 & 1 \\ 1 & -2 & 0 \end{bmatrix} = -1[0(2) - 1(-1)]$

$= -1(1) = -1$

→ If $\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ then which of the follow is a factor of Δ

a) a+b b) a−b c) abc d) a+b+c

$= 1(b^2a - c^2a) - 1(a^2b - c^2b) + 1(a^2c - b^2c)$

$= a(b^2 - c^2) - b(a^2 - c^2) + c(a^2 - b^2)$

$= a(b+c)(b-c) - b(a+c)(a-c) + c(a+b)(a-b)$

$R_1 \rightarrow R_1 - R_2$

$$\Delta = \begin{vmatrix} 0 & a-b & -c(a-b) \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$\Delta = (a-b)\begin{vmatrix} 0 & \cancel{a-b}\ 1 & -c \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

Hence from this $(a-b)$ is a factor of Δ

→ A square matrix B is symmetric if

a) $B^{-1} = B^T$ b) $B = B^{-1}$ c) $B^T = -B$ d) $B^T = B$

Symmetric Matrix :-

It will be applicable for the square Matrix only

$\boxed{A^T = A}$

$A = (a_{ij})_{n\times n}$ then the Matrix is symmetric if and only if $\boxed{a_{ij} = a_{ji}}$

Eg:- $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & 3 \\ 5 & 3 & 6 \end{bmatrix}$, $A = \begin{bmatrix} 2-i & 4 & 5+i \\ 4 & 0 & 2+i \\ 5+i & 2+i & 3 \end{bmatrix}$

Skew symmetric:-

$A_{n\times n}$ $\boxed{A^T = -A}$

$A = (a_{ij})_{n\times n} \longrightarrow a_{ij} = -a_{ji} \ \forall \ i, j$

$\boxed{a_{ij} = -a_{ji}}$ $j = i$

$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0$

$$A = (a_{ij})_{n\times n} \longrightarrow a_{ij} = \begin{cases} 0, & \forall \ i=j \\ -a_{ji}, & \forall \ i \neq j \end{cases}$$

eg:- $$A = \begin{bmatrix} 0 & 4i & -3+i \\ -4i & 0 & 6 \\ 3-i & -6 & 0 \end{bmatrix}$$

orthogonal Matrix:-

$A_{n\times n} \longrightarrow \boxed{AA^T = A^TA = I}$

$\rightarrow \boxed{AB = BA}$ compotative property of Multiplicatio

$\boxed{A^T = A^{-1}}$

Eg:- $$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \quad A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ If $A = (a_{ij})$ is defined by $a_{ij} = i+j \ \forall \ i, j$ when $1 \le i \le m$ & $1 \le j \le n$ then find the sum of the all the Elements of the Matrix.

Sol

Given $a_{ij} = i+j$

$$A = \begin{bmatrix} \cdots & \cdots & & \cdots & \cdots \\ 2+1 & 2+2 & 2+3 & \cdots & 2+n \\ \vdots & & & & \\ m+1 & m+2 & \cdots & \cdots & m+n \end{bmatrix}_{m\times n} \begin{matrix} (\ ) \\ (\ ) \\ \\ (\ ) \end{matrix}$$

$2, 3, 4, 5 \cdots (n+1)$

$$\boxed{S_n = \frac{n}{2}(a+l)} = n/2\,(2+n+1) = \frac{n(n+3)}{2}$$

In the same manner all rows sum is calculated at last sum of all rows is findout.

$$S_m = \frac{m}{2}(a+l)$$

II Method:- It will be split into two equal matrices

$$A = \underbrace{\begin{bmatrix} 1 & 1 & \cdots \\ 2 & 2 & \\ 3 & 3 & \\ \vdots & \vdots & \\ m & m & \end{bmatrix}}_{n} + \left.\begin{bmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \\ 1 & \vdots & \cdots & n \\ \vdots & \vdots & \cdots & n \\ & & \cdots & n \end{bmatrix}\right\} m$$

$$\text{Sum} = n\left(\frac{m(m+1)}{2}\right) + m\left(\frac{n(n+1)}{2}\right)$$

$$\boxed{\text{Sum.} = \frac{mn}{2}(m+n+2)}$$

008 → If p is a 2×2 real orthogonal matrix, $\bar{x}$ is a real vector $[x_1\ x_2]^T$ with length $||\bar{x}|| = \sqrt{x_1^2 + x_2^2}$ then which one of the following statements is correct

a) $||p\bar{x}|| \leq ||\bar{x}||$ where atleast one vector satisfying $\|Px\| < \|x\|$

b) $\|P\bar{x}\| \geq \|\bar{x}\|$ " " " " $\|P\bar{x}\| > \|\bar{x}\|$

c) $\|P\bar{x}\| = \|x\| \ \forall$ vector $\bar{x}$

d) No relation can be established b/w $\|P\bar{x}\|$ & $\|\bar{x}$

Sol Let $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\times 2}$, Given $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{2\times 1}$

Then $P\bar{x} = \begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix}_{2\times 1}$

$\|P\bar{x}\| = \sqrt{(x_1\cos\theta - x_2\sin\theta)^2 + (x_1\sin\theta + x_2\cos\theta)^2}$ $\left(\because \bar{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \|\bar{x}\| = \sqrt{x_1^2 +} \right.$

$= \sqrt{x_1^2\cos^2\theta + x_2^2\sin^2\theta + x_1^2\sin^2\theta + x_2^2\cos^2\theta}$

$\|P\bar{x}\| = \sqrt{x_1^2 + x_2^2}$

$= \|\bar{x}\|$

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→ In the matrices $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ & $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ commute under multiplications.

(a) always b) never c) If $a\cos\theta \neq b\sin\theta$

d) If $a=b$ (or) $\theta = n\pi, n \in I$

Sol Let $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$B = \begin{bmatrix} a & 0 \\ \end{bmatrix}$

$$AB = \begin{bmatrix} a\cos\theta & -b\sin\theta \\ a\sin\theta & b\cos\theta \end{bmatrix}$$

$$BA = \begin{bmatrix} a\cos\theta & -a\sin\theta \\ b\sin\theta & b\cos\theta \end{bmatrix}$$

when $a=b$ then $\boxed{AB = BA}$ (or) when $\theta = n\pi$ it will be satisfied.

Inverse of a Square Matrix:-

Minor & co-factor of an Element:-

If $A = (a_{ij})_{n\times n}$ then

(i) The minor of a_{ij} is $M_{ij} = (n-1)^{th}$ order determinant which remains after deleting corresponding i^{th} row & j^{th} column in $A_{n\times n}$

→ The co-factor of a_{ij} is $\boxed{A_{ij} = (-1)^{i+j} M_{ij}}$

Eg:- $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 7 & 6 \\ 0 & -3 & 5 \end{bmatrix}$ $a_{23} = 6 \rightarrow M_{23} = \begin{vmatrix} 1 & 2 \\ 0 & -3 \end{vmatrix}$

$= -3$

$A_{23} = (-1)^{2+3} M_{23} = (-1)(-3) = 3$

$a_{31} = 0$ $M_{31} = \begin{vmatrix} 2 & 3 \\ 7 & 6 \end{vmatrix} = -9$

$A_{31} = (-1)^{3+1} M_{31} = (1)(-9) = -9$

Cofactor Matrix:-

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the

Matrix $B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ is called the co-fac

Matrix of $A_{3\times3}$.

Adjoint Matrix:-
If $B_{n\times n}$ is a cofactor matrix of $A_{n\times n}$ then

$$\boxed{adj(A) = B^T}$$

Singular Matrix:- $\boxed{|A_{n\times n}| = 0}$

→ Non singular Matrix $\boxed{|A_{n\times n}| \neq 0}$

Inverse (or) Reciprocal of $A_{n\times n}$

If $AB = BA = I_n$ for $|A_{n\times n}| \neq 0$ & $|B'_{n\times n}| \neq 0$

then B is called inverse of $A_{n\times n}$.

$$\boxed{B = A^{-1} \quad \& \quad AA^{-1} = A^{-1}A = I_n}$$

Note 1:-

→ If $|A_{n\times n}| \neq 0$ then $\boxed{A^{-1} = \dfrac{adj(A)}{|A|}}$

→ If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $A^{-1} = \dfrac{adj(A)}{|A|} = \dfrac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{(ad-bc)}$

→ $\boxed{A \cdot adj(A) = adj(A) \cdot A = |A| I_n}$ for any $A_{n\times n}$

→ If $AB = I_n$ for $|A_{n\times n}| \neq 0$ & $|B_{n\times n}| \neq 0$ then BF

→ Every odd order Skew Symmetric Matrix is always a Singular

Problems:-

- If $A = \begin{bmatrix} 3+2i & -i \\ i & 3-2i \end{bmatrix}$ then find A^{-1}

$|A| = 13-1 = 12 \neq 0$

$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{12}\begin{bmatrix} 3-2i & i \\ -i & 3-2i \end{bmatrix}$

- If $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & +2 \end{bmatrix}$ then find A^{-1}

$|A| = 1\,(2+3) - 1\,(6-2) = 5-4 = 1$

$A^{-1} = \frac{adj(A)}{|A|}$

$A_{11} = 1\,(2+3) = 5$

$A_{12} = 0$

$A_{13} = -1(6-2) = -1(4) = -4$

$A_{21} = -2(3) = -6$

$A_{22} = 1\,(2+2) = 4$

$A_{23} = 1\,(3) = 3$

$A_{31} = 2(1) = 2$

$A_{32} = -3\,(-1+2)$
$= -3(1) = -3$

$A_{33} = 2\,(1) = 2$

$A^{-1} = \frac{adj(A)}{|A|} = \begin{bmatrix} 5 & 0 & -4 \\ -6 & 4 & 3 \\ 2 & -3 & 2 \end{bmatrix}$

$A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 2 & 3 & 2 \end{bmatrix}$

II method:-

$\begin{bmatrix} 1 & 3 & 0 & 1 \\ -1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 2 \\ 1 & 3 & 0 & 1 \end{bmatrix}$ — R1, R2, R3

$$A^{-1} = \frac{\text{adj}A}{|A|} \Rightarrow \frac{\begin{bmatrix} 2+3 & -3 & 1 \\ -2-4 & 2+2 & -2+1 \\ 6-2 & -3 & 1-0 \end{bmatrix}}{1}$$

$$\Rightarrow \begin{bmatrix} 6 & -3 & 1 \\ -6 & 4 & -1 \\ 4 & -3 & 1 \end{bmatrix}$$

→ If A, B, C, D, E, F, G are Non Singular Matrices of Same order Such that $GFEBDEA = I_n$ then find B^{-1}

Sol Given

$$(GFCBDEA)^{-1} = (I_n)^{-1} \qquad \boxed{(AB)^{-1} = B^{-1}A^{-1}}$$

$$A^{-1}E^{-1}D^{-1}B^{-1}C^{-1}F^{-1}G^{-1} = I \qquad \boxed{\because I^{-1} = I}$$

To Eliminate A^{-1} multiply both sides by A.

$$\Rightarrow A(A^{-1}E^{-1}D^{-1}B^{-1}C^{-1}F^{-1}G^{-1})G = AIG$$

$$\Rightarrow E(E^{-1}D^{-1}B^{-1}C^{-1}F^{-1})F = EAGF$$

$$\Rightarrow D(D^{-1}B^{-1}C^{-1})C = DEAGFC$$

$$\Rightarrow \boxed{B^{-1} = DEAGFC}$$

II Method

$$GFCBDEA = I$$

$$B^{-1} = DEAGFC$$

2009 $G^{-1} = FCBDEA$

For a Matrix $M = \begin{bmatrix} 3/5 & -4/5 \\ x & 3/5 \end{bmatrix}$ The transpose of the Matrix is equal to inverse of Matrix then find value of 'x'

$$M^T = M^{-1}$$

$$M^{-1} = \frac{adj(M)}{|M|}$$

$$|M| = 9/25 + 4/5x$$

This will be compare with the real orthogonal Matrix $= \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$$\boxed{MM^T = I = M^TM}$$

$$\frac{1}{5}\begin{bmatrix} 3 & -4 \\ 5x & 3 \end{bmatrix} \frac{1}{5}\begin{bmatrix} 3 & 5x \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{1}{25}\begin{bmatrix} 9-16 & 15x-12 \\ 15x-12 & 25x^2+9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\frac{15x-12}{25} = 0$$

$$\boxed{x = 12/15 = 4/5.}$$

2010
If A, B are two non zero Matrices of same order such that $AB = O_{n\times n}$ then

(a) $|A| \neq 0$ & $|B| = 0$

b) $|A| = 0$ & $|B| \neq 0$

c) $|A| \neq 0$ & $|B| \neq$ ...

d) $|A| = 0$ & $|B| = 0$

consider $AB = 0$

$$|AB| = |0|$$
$$|A|\,|B| = 0$$

(a) $|A| \neq 0$ & $|B| = 0$ consider $AB = 0$

$\downarrow$

A^{-1} exists

$$A^{-1}(AB) = A^{-1}(0)$$
$$IB = 0$$
$$B = 0.$$

Not satisfied.

(d) is correct option

$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$ $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$

$B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0$ $|A| = 0$

$|B| = 0.$

→ If $A = (a_{ij})$ is defined by $a_{ij} = i^2 - j^2 \;\forall\; i, j$ where $1 \leq i, j \leq n$ then find A^{-1} for $n = 5$

Sol

Note:- Every odd order skew symmetric Matrix is singular.

Given $A = (a_{ij})_{5\times 5}$

where $a_{ij} = i^2 - j^2 = -a_{ji}$

$\Rightarrow A_{5\times 5}$ is a skew symmetric Matrix

But odd order skew symmetric matrix is always singular.

$\therefore A^{-1}$ does not Exist.

→ If A is Non-singular matrix of order n then find (i) $|adj(A)| =$

(ii) $adj(adj(A)) =$

1) $|adj(adj A)| =$

Sol Given $A_{n\times n} \Rightarrow |A_{n\times n}| \neq 0$

$\Rightarrow A^{-1}$ Exists

$$A^{-1} = \frac{adj(A)}{|A|}$$

$$\boxed{adj(A) = |A| A^{-1}}$$

) $|adj(A)| =$

consider $adj(A) = |A| A^{-1}$

$\Rightarrow |adj(A)| = ||A| A^{-1}|$

$\Rightarrow |adj(A)| = |A|^n |A^{-1}|$

$|adj(A)| = |A|^n |A|^{-1}$

$$\boxed{|adj(A)| = |A|^{n-1}}$$

i) $adj(\underbrace{adj(A)}_{B}) = adj(B)$

$= |B| B^{-1}$

$= |adj(A)| (adj(A))^{-1}$

$= |A|^{n-1} (|A| A^{-1})^{-1}$

$= |A|^{n-1} |A|^{-1} (A^{-1})^{-1}$

$= A . |A|^{n-2}$

ii) $|adj(adj(A))| = |A . |A|^{n-2}|$

$= |A|^{n-1} |A|^{n-2}$

$$\boxed{|adj(adj A)| = |A|^{(n-1)^2}}$$

$|adj(adj(adj(A)))| = |A|^{(n-1)^3}$

Rank of a Matrix :-

Sub Matrix :- If a Matrix 'B' is obtained from a given Matrix by deleting only rows (or) only columns (or) both rows and columns. then the Matrix 'B' is called a Submatrix of 'A')

Ex :-

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}_{3\times4}$$

$B = [6]_{1\times1}$ $(R_1, R_3, C_1, C_3, C_4)$ —— (12)

$C = [9 \ \ 10 \ \ 11 \ \ 12]_{1\times4}$ (R_1, R_2)

$D = \begin{bmatrix} 1 & 4 \\ 9 & 12 \end{bmatrix}_{2\times2}$ (R_2, C_2, C_3) —— C1

$E = \begin{bmatrix} 2 & 3 & 4 \\ 6 & 7 & 8 \\ 10 & 11 & 12 \end{bmatrix}_{3\times3}$ (C_1) —— (4)

→ The Submatrix may be Square (or) may not be a Square.

→ If 3x4 Matrix contain higher order Submatrix is the minimum no in the matrix

3x3 Matrix obtained is the Maximum No in the order of the Matrix

Minor of a Matrix:-

The determinant of Every Square Submatrix of a given Matrix is called minor of a Matrix

Ex:- $A_{3\times4} \to B_{1\times1} \to |B|$ — 1^{st} order minor — (12)

$\quad D_{2\times2} \to |D|$ — 2^{nd} order " — (18)

$\quad E_{3\times3} \to |E|$ — 3^{rd} order — (4)

If atleast one of the highest order square Submatrix of given order is Nonsingular then the order of that Square Submatrix is the Rank of a Matrix.

(or)

If atleast one of the highest order minor of a given matrix A is $\neq 0$ then the order of that minor is called rank of Matrix and it is denoted by $\rho(A)$ (or) 'r'

Note: Rank of a Null Matrix "0"

1) Find the Rank of Matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}_{3\times(4)}$

$= \begin{vmatrix} 2 & 1 & 0 \\ 4 & 3 & 0 \\ 0 & 2 & -8 \end{vmatrix}$

$= -8(6-4) = -8(2) = -16 \neq 0$

$\rho(H_{3\times4}) = 3.$

2) find the Rank of Matrix $A = \begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}_{3\times}$

$$= \begin{vmatrix} 0 & 1 & 2 \\ 4 & 0 & 2 \\ 2 & 1 & 3 \end{vmatrix}$$

$$= -1(12-4)+2(4)$$

$$= -1(8)+8 = 8+8=0$$

$$\begin{vmatrix} 1 & 2 & -2 \\ 0 & 2 & 6 \\ 1 & 3 & 1 \end{vmatrix} = (1)(2-18)-2(-6)-2(-2)$$

$$= -16+12+4=0$$

$$\begin{vmatrix} 0 & 2 & -2 \\ 4 & 2 & 6 \\ 2 & 3 & 1 \end{vmatrix} = -2(4-12)-2(12-4)$$

$$= -2(-8)-2(8) = 0.$$

$$\begin{vmatrix} 0 & 1 & -2 \\ 4 & 0 & 6 \\ 2 & 1 & 1 \end{vmatrix} = -1(4-12)-2(4) \quad -8+8=0$$

$$\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = -4 \neq 0.$$

$$\boxed{\rho(A)=2}$$

3) find the Rank of matrix $A = \begin{bmatrix} 8 & 16 & 24 & 32 \\ 1 & 2 & 3 & 4 \\ 4 & 8 & 12 & 16 \end{bmatrix}$

$$= 32 \begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{vmatrix}$$

$\rho(A) \neq 3$

$\rho(A) \neq 2$

$\rho(A)=1$ Rank of Matrix $=1$

Note:-

- Rank of a Matrix is unique
- $\rho(O_{m\times n}) = 0$
- If $A \neq 0$ then $\rho(A) \geq 1$
- If $A_{m\times n} \neq 0$ then $\rho(A_{m\times n}) = \min\{m,n\}$
- If $|A_{n\times n}| \neq 0$ then $\rho(A_{n\times n}) = n$
- If $|A_{n\times n}| = 0$ then $\rho(A_{n\times n}) < n$
- $\rho(I_n) = n$

  Rank of unit matrix is its order
- $\rho(AB) \leq \min\{\rho(A), \rho(B)\}$
- $\rho(A+B) \leq \rho(A) + \rho(B)$
- $\rho(A-B) \geq \rho(A) - \rho(B)$
- If $\rho(A_{n\times n}) = n$ then $\rho(\mathrm{adj}(A)) = n$
- If $\rho(A_{n\times n}) = (n-1)$ then $\rho(\mathrm{adj}(A)) = 1$
- If $\rho(A_{n\times n}) \leq (n-2)$ then $\rho(\mathrm{adj}(A)) = 0$.

Echelon form:- A matrix $A_{m\times n}$ is said to be in Echelon form if it satisfies

(i) The No of zero's before a first Non-zero No in any row is less than No of

Such zeros in the Next Row.

→ The zero rows must be below the Non-zero row's

Note:-
If $A_{m\times n}$ in an Echelon form then $\rho(A_{m\times n}) = r$ of non-zero row's in an Echelon form. It is also same as the No of independent vectors Row's & column's.

Ex(1):- $A = \begin{bmatrix} 0 & 2 & 0 & -9 \\ 0 & 0 & 5 & 10 \\ 0 & 0 & 0 & 7 \end{bmatrix}_{3\times 4}$ $\rho(A) = 3.$

2) $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 9 & 10 \\ 0 & 0 & 5 \end{bmatrix}$ $\rho(A) = 3$

No of Non-zero row's $= 3.$

3) $A = \begin{bmatrix} 0 & 7 & 9 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 2$

4) $A = \begin{bmatrix} 0 & 7 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\rho(A) = 1$

Every upper triangle is in Echelon form but Every Echelon form is not a upper triangle form

5) $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{pmatrix}$ $\rho(A) = 3$

6) $A = \begin{pmatrix} 7 & 0 & 0 \\ 0 & 8 & 0 \end{pmatrix}$ $\rho(A) = 2$

If $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\rho(A)=1$

$A_{m\times n}$ Apply only Row operations.

$\downarrow$

Echelon form

$\downarrow$

$\rho(A_{m\times n})$ = No of Non-zero rows in an Echelon form.

Ex(1):

$$A = \begin{bmatrix} 1 & 2 & 3 & 7 & 8 & -9 \\ 4 & 5 & 11 & 12 & 17 & 19 \\ 20 & 25 & 26 & 12 & 11 & 10 \\ 5 & 4 & -3 & -2 & -1 & 10 \\ 4 & 2 & 3 & -7 & -8 & -10 \end{bmatrix}$$

Find the rank of Matrix A

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 2 & -3 & 4 & 8 \\ -1 & 2 & 1 & 7 \end{bmatrix}_{3\times}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 4 & 7 \end{bmatrix}$$

$\rho(A_{3\times3}) = 0$

Linearly independent (or) dependent vectors.—

If x_1 and x_2 are two vectors

$x_1 = k x_2$
$x_2 = l x_1$ } then both x_1 and x_2 are dependent vectors.

Rank of a matrix = No of independent vectors

Method:- If $x_1, x_2 ---- x_n$ are Row vectors of Same order m and

(i) If $\rho(A)$ = No of given vectors $= n$
then $x_1, x_2 --- x_n$ are L. Independent Vectors.

(ii) If $\rho(A) \neq$ no of given vectors $= n$
then $x_1, x_2 ---- x_n$ are L. Dependent vecto

where $A = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times m}$ $x_i = [x_1, x_2 --- x_m]$

① check whether the following vectors are linearly independent (or) linearly dependent

① (i) $x_1 = [1\ 2\ 3]$, $x_2 = [2\ 4\ 6]$

$x_2 = [2\ 4\ 6]$

$x_1 = 1/2\,[1\ 2\ 3]$

x_2 — or. $\longrightarrow$ $x_1 = 1/2\, x_2$

x_1, x_2 are Linearly dependent vectors.

Method - II

$$A = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$$

$\rho(A) = 1 \neq$ No of given vectors $= 2$

$\therefore x_1, x_2$ are linearly dependent vectors.

) $x_1 = [1\ 2\ 3]$, $x_2 = [2, 4, 7]$

Method - I:-

$x_2 = [2\ 4\ 7]$

$x_2 \neq k x_1$ (or) $x_1 \neq l x_2$

then x_1, x_2 are Linearly independent vectors.

Method - II:-

$$A = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} & & 3 \\ 2 & 4 & 6 \end{bmatrix}_{2\times 3}$$

$\rho(A) = 2 =$ No of given vectors $= 2$

$\therefore x_1, x_2$ are L.I vectors.

ii) $x_1 = [1\ 7\ 9]$, $x_2 = [7\ 5\ 4]$, $x_3 = [2\ 5\ 6]$

$x_4 = [7\ 5\ 2]$

$$A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 & 7 & 9 \\ 7 & 5 & 4 \\ 2 & 5 & 6 \\ 7 & 5 & 2 \end{bmatrix}$$

$\rho(A)_{4\times 3} \leq 3 \neq$ No of given vectors. $= 4$

$\therefore x_1, x_2, x_3, x_4$ are Linearly dependent vectors.

~~If $\rho(A) = 4$~~

2) If $x = [x_1\ x_2\ x_3\ ---\ x_n]^T$ is a Non-zero n tuple vector then $\rho(xx^T)$

a) 1 b) 0 c) n d) n

Sol

$$\boxed{\begin{array}{l}\rho(AB) \leq \min\{\rho(A), \rho(B)\} \\ \rho(A) = \rho(A^T)\end{array}}$$

Given $x = \begin{pmatrix} x_1 \\ x_2 \\ | \\ x_n \end{pmatrix}_{n\times 1} \neq 0$

$\rho(x_{n\times 1}) \leq 1$

$$\boxed{\therefore \rho(x_{n\times 1}) = 1 \quad \& \ \rho(x^T_{1\times n}) = 1}$$

$$\rho((xx^T)_{n\times n}) \leq \min\{\rho(x), \rho(x^T)\}$$

$$= \min\{1, 1\} = 1$$

$\rho(xx^T) \leq 1$

$\boxed{\rho(xx^T) = 1}$ $(x \neq 0 \Rightarrow x^T \neq 0 \Rightarrow xx^T \neq 0)$

$$x = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} (3\ 0\ 0) = x^T$$

$$xx^T = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3} \neq 0$$

⇒ If $A = (a_{ij})$ is defined by $a_{ij} = i \cdot j \;\; \forall \; i, j$ where $1 \le i, j \le n$ then $\rho(A)$ ——

a) 0 b) 1 c) n d) None.

Sol

$$A = \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 2 & 4 & 6 & \cdots & 2n \\ 3 & 6 & 9 & \cdots & 3n \\ \vdots & & & & \\ n & 2n & 3n & \cdots & n^2 \end{bmatrix}_{n \times n}$$

$R_2 - 2R_1, \; R_3 - 3R_1$ — —

$$= \begin{bmatrix} 1 & 2 & 3 & \cdots & n \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\boxed{\rho(A) = 1}$$

⇒ If A is singular matrix of order 'n' then $\rho(A \operatorname{adj}(A))$

Sol Given $A_{n \times n} \Longrightarrow |A_{n \times n}| = 0$

But $A \cdot \operatorname{Adj}(A) = \operatorname{adj}(A) \cdot A = |A| I_n$

$$A \cdot \operatorname{Adj}(A) = |A| I_n$$

$$\Longrightarrow A \cdot \operatorname{Adj}(A) = 0 \cdot I_n$$

$$\Longrightarrow \boxed{A \cdot \operatorname{Adj}(A) = 0}$$

For Non Singular $\underline{|A \cdot (Adj(A))| = ...|}$

5) If A is $m \times n$ full rank matrix with $m > n$, I is unit matrix & $A^+ = (A^T A^{-1}) A^T$ then which of the following is false

(a) $(AA^+)^2 = AA^+$ b) $A^+ A = I$ c) $AA^+A = A$

~~d)~~ $AA^+A = A^+$

Sol $[\cancel{A^+ = (A^T A)^{-1} A^T = (A^{-1})}(A^T)^{-1} A^T]$

This property is used for product of two Non-Singular with the same order

Given $A_{m\times n} \Rightarrow \rho(A_{m\times n}) = n$

$$\Rightarrow \rho(A^T_{n\times m}) = n$$

$$\Rightarrow \rho((A^T A)_{n\times n}) = n$$

$\Rightarrow$ Then the Matrix is Non Singular

$$|A^T A| \neq 0$$

$$\Rightarrow (A^T A)^{-1} \text{ Exist}$$

$$\Rightarrow (A^T A)^{-1} A^T A = I$$

$$\Rightarrow \boxed{A^+ A = I} \quad (\because A^+ = (A^T A)^{-1} A^T)$$

Q7 If $q_1, q_2 ---- q_m$ are vectors of same order 'n'
with $m<n$, the set of vectors is linearly dependent
set and Q is a matrix with given vector as columns
then $\rho(Q)$ ——

a) m b) n c) $\leq m$ ~~d) m~~

Given $Q = [q_1\ q_2 ------ q_m]_{n\times m}$

$$q_1 = \begin{bmatrix} x_1 \\ x_2 \\ | \\ x_n \end{bmatrix}$$

$$\rho(Q_{n\times m}) \leq \min\{m, n\}$$

But $m<n$

$$\rho(Q_{n\times m}) \leq m \begin{cases} = m \\ < m \end{cases}$$

But $q_1, q_2 ---- q_m$ are Linearly dependent
vectors $\boxed{\rho(Q_{n\times m}) < m}$

Q8 If the rank of a 5×6 Matrix Q is 4 then
which of the following statement is correct

a) QQ^T will be invertible

b) Q^TQ will be invertible

c) Q will have 4 Linearly independent rows
& 5 L.I columns

∴ Q will have 4 L.I rows & (4) L.I columns.

Given that $Q_{5\times6} \Rightarrow \rho(Q_{5\times6}) = 4$

$\rho(Q^T_{6\times5}) = 4$

(a) $Q_{5\times6}\, Q^T_{6\times5} \Rightarrow (QQ^T)_{5\times5}$

$\rho(QQ^T) \leq \min\{\rho(Q), \rho(Q^T)\}$

$= \min(4, 4)$

$= 4$

$\boxed{\rho(QQ^T_{5\times5}) \leq 4} \neq$ order of Matrix

$|QQ^T| = 0 \quad (QQ^T)^{-1}$ does not Exist

(b) $Q^T_{6\times5}\, Q_{5\times6} \Rightarrow (Q^TQ)_{6\times6}$.

$|Q^TQ| = 0 \quad (Q^TQ)^{-1} \rightarrow$ does not Exist.

$\rho(A_{m\times n}) = r$ then it will have (r) independent rows and (r) independent ~~then~~ columns. Hence it will be obtained

Solution of System of linear Equations:-

Non homogenous System:-

consider

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots\ a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\ a_{2n}x_n &= b_2 \\ &\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\ a_{mn}x_n &= b_n \end{aligned}\right\}$$

m- Equations in n - unknowns.

$$\boxed{AX = B}$$

where $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$

$\downarrow$

co-efficient Matrix

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n \times 1}$ $\longrightarrow$ Variable Matrix (vector)

(or)

unknown Matrix (vector)

(or)

Solution Matrix (Vector)

$B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}_{m \times 1}$ constant Matrix (vector)

<u>Augmented Matrix</u>:- $[A|B] = \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]_{m \times (n+1)}$

<u>Step 1</u>:- consider $AX = B$

2) consider $[A|B]$ & reduce it into Echelon form by applying only row operations.

3) find rank of A, $\rho(A|B)$ & no of variables

4) (i) If $\rho(A) = \rho(A|B) = n$ then $AX = B$

consistent and has unique solution

(ii) If $\rho(A) = \rho(A|B) < n$ then $Ax = B$ is consistent and has infinite no of solutions.

(iii) If $\rho(A) \neq \rho(A|B)$ then $Ax = B$ is inconsistent and has No Solution.

5) If the Solution Exist then rewrite the System of Equations from the Echelon form of $(A|B)$ matrix and find the solution by backward Substitution

Note(1):-

$$Ax = B$$
$$\downarrow$$
$$A_{n \times n}$$

- $|A| \neq 0$
  - only unique solution
- $|A| = 0$
  - No Solution
  - Many Solution

Homogenous System:-

consider

definition:- If $B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m \times 1} = 0$ vector in $Ax = B$

then $Ax = B$ is called homogenous system and it is denoted by $Ax = 0$

Note:-

→ Every $Ax = 0$ is always consistent

$$X = \begin{bmatrix} x_1 \\ x_2 \\ | \\ | \\ x_n \end{bmatrix}_{n\times 1} = \begin{bmatrix} 0 \\ 0 \\ | \\ | \\ | \end{bmatrix} = 0$$ is always a solution of Every $AX=0$ and it is called trivial Solution (or) unique Solution (or) zero Solution (or) finite Solution

) If any solution Exist other than trivial solution for $AX=0$ then Such a Solution is called Non-trivial (or) Non-zero (or) infinitely many Solutions.

Method:-

1) consider $AX=0$

2) consider Matrix 'A' and reduced it into its Echelon form by applying on row operations.

3) Find $\rho(A)$ & n = no of variables

4) (i) If $\rho(A)=n$ then $AX=0$ will have only trivial solutions

(ii) If $\rho(A)\neq n$ then $AX=0$ will have trivial & also Non-trivial Solutions

5) If the Non trivial Solution Exist then rewrite the system of Equations from the Echelon form of matrix 'A' and find the

Solution by the backward substitution

<u>Note:-</u>

$$AX = 0$$
$$\downarrow$$
$$A_{n\times n}$$

$|A| \neq 0$ $\downarrow$ only unique Solution

$|A| = 0$ Trivial Solution & Non-trivial Solution.

2) If $\rho(A) = r$ & n = no of variables in $AX=0$ then $AX=0$ will have $(n-r)$ independent solutions.

2004
① For what values of x, y, z satisfy the following system of linear Equations

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 12 \end{bmatrix}$$

×a) $x = 12, y = 3, z = -4$

b) $x = 6, y = 6, z = -4$

c) $x = 12, y = 3, z = 4$

d) $x = 6, y = 6, z = 4$

<u>Method I:-</u>

$$x + 2y + 3z = 6$$

(a) $12 + 6 - 12 = 6$

$$x + 3y + 4z = 8$$

$$12 + 9 - 16 \neq 8$$

b) $6 + 12 - 12 = 6$

$$6 + 18 - 16 = 8$$

$$2x + 2y + 3z = 12$$

Method (2):-

$$(A|B) = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 1 & 3 & 4 & 8 \\ 2 & 2 & 3 & 12 \end{array}\right]$$

$R_2 \rightarrow R_2 - R_1 \quad R_3 \rightarrow R_3 - 2R_1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & -2 & -3 & 0 \end{array}\right]$$

$R_3 \longrightarrow R_3 + 2R_2$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

$\rho(A) = 3$

$\rho(AB) = 3. \qquad n = 3.$

But $\rho(A) = \rho(A|B) = 3 = n$

Hence from this unique solution is obtained.

$$x + 2y + 3z = 6$$
$$y + z = 2$$
$$-z = 4$$

$+z = -4$

$y = 2 - z = 6$

$x = 6 - 2y - 3z = 6 - \cancel{12} + \cancel{12} = 6$

$\cancel{x}\ \cancel{y}\ z = -4$

$y = 6 \qquad x = 6$

2) If A 3x3 Matrix has rank 2 then $AX = 0$ has

(a) only trivial Solution

~~b)~~ One indipendent Solutions

c) Two independent Solutions

d) Three " "

Sol

$$AX = 0$$
$$\downarrow$$
$$A\ 3\times 3$$
$$\downarrow$$
$$r = \rho(A\ 3\times 3) = 2 \neq n = 3$$
$$\downarrow$$
$$(n-r) = (3-2) = 1 \text{ independent solution}$$

3) For what values of K, the following values of linear Equations have a unique Solution

$x+y+z=3$

$x+2y+3z=4$

$x+4y+Kz=6.$

The condition for unique Solution $AX=B$ is

$\rho(A) = \rho(B) = n$ (or) $|A| \neq 0$

$$|A| \neq 0 \quad \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & K \end{vmatrix} \neq 0$$

$$(4+2K+3) - (K+2+12) \neq 0$$

$$K-7 \neq 0. \text{ (or) } K \neq 7$$

For all values of K except $K=7$ will have unique Solution

) If A is $m \times n$ matrix with $A^2 = I$ and A is real then $AX = B$ has

a) only 2 - Solutions

b) unique Solution

c) Many solution

d) No Solution

Sol Given $AX = B$

$\downarrow$

$A_{n \times n}$

real $\quad\quad$ $A^2 = I$

$|A| \neq 0$

$\downarrow$

unique Solution

$$A^2 = I \quad\quad \Longrightarrow \quad\quad |A^2| = 1$$

$$|A^2| = |I| \quad\quad\quad A = \pm 1 \neq 0$$

5) For what values of λ and μ the system has infinite no of solutions where the system is

$$x + y + z = 6$$

$$x + 4y + 6z = 20$$

$$x + 4y + \lambda z = \mu$$

a) $\lambda = 6, \ \mu = 20$

b) $\lambda = 6, \ \mu \neq 20$

c) $\lambda \neq 6, \ \mu = 20$

d) $\lambda \neq 6, \ \mu \neq 20$

The condition for many solutions of

$$\rho(A) = \rho(A|B) < n-3$$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 4 & 6 & 20 \\ 1 & 4 & \lambda & \mu \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 3 & \lambda-1 & \mu-6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 3 & 5 & 14 \\ 0 & 0 & \lambda-6 & \mu-20 \end{array}\right]$$

for $\lambda = 6$, $\mu = 20$ the system will have infinitely many solution

⑥ If $AX = B$ represents a system of linear Equatio where A is $m \times n$ & $B \to m \times 1$ then which of the following statement statement is false

a) system has a solution If and only if $\rho(A) = \rho(A|B)$

b) If $m < n$ & 'B' is a zero vector then the system has many solutions.

c) The system has a trivial solution when $m = n$, $\rho(A) = n$ & B is a zero vector

~~d)~~ If $m = n$ & $B \neq 0$ then the system have unique Solution

<u>sol</u>

$$AX = 0$$
$$\downarrow$$
$$A_{m\times n}$$
$$\downarrow$$
$$\rho(A_{m\times n}) < m < n$$

$\rho(A_{m\times n}) < n$

$\downarrow$

Non-trivial Solution

) $AX = 0$

$\downarrow$

$A_{n\times n}$

$\downarrow$

$\rho(A_{n\times n}) = n$

$\downarrow$

only trivial Solution

(d) $AX = B$

$\downarrow$

$A_{n\times n}$

$\downarrow$

$\downarrow$

unique Solution

7) If the system $x+y+z=0$, $(\lambda-1)y+(\lambda-1)z=0$ (or) $(\lambda-1)^2 z = 0$ has 2 L.I Solutions then $\lambda =$ ____

~~a) 1~~ b) -1 c) ± 1 d) None.

<u>Sol</u> Given $x+y+z=0$

$(\lambda-1)y+(\lambda-1)z =$

$(\lambda^2-1)z = 0$

$n-r=2$

$3-r=2$

$\boxed{r=1}$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & \lambda-1 & \lambda-1 \\ 0 & 0 & (\lambda-1)^2 \end{bmatrix}$$

It will be in Echelon form and given $r=1$ and for this two rows are Equal to zero

$\rho(A)=1$ $\lambda-1=0 \Rightarrow \lambda=1$

Eigen Values & Eigen vectors, Cayley – Hamilton theorem

Eigen Matrix, Eigen polynomial, Eigen Equation & Eigen Values:-

If $A = (a_{ij})_{n\times n}$, λ is a Scalar & I_n is unit Matrix then (i) The matrix $A-\lambda I$ is called characteristic matrix of $A_{n\times n}$

(ii) The determinant of '$A-\lambda I$' i.e $|A-\lambda I|$ is called characteristic polynomial of $A_{n\times n}$

(iii) The Equation $|A-\lambda I|=0$ is called characteristic Equation of $A_{n\times n}$

iv) The Roots of characteristic Equation of $A_{n\times n}$ is called Eigen values or characteristic roots of $A_{n\times n}$

Ex(1): $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$

(i) $|A-\lambda I| = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{bmatrix}$

(ii) $|A-\lambda I|$ (or) $\begin{vmatrix} 4-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$ (or) $\lambda^2 - 8\lambda + 12$

(iii) $|A-\lambda I| = 0$ (or) $\lambda^2 - 8\lambda + 12 = 0$

$\lambda = 2, 6$ are Eigen values of 'A'

Eigen vector:

A Non zero common vector $X_{n\times 1}$ is said to be an Eigen vector of matrix $A_{n\times n}$. If $\boxed{AX=\lambda X}$ for some Scalar λ

Ex(i) $A=\begin{pmatrix}4&2\\2&4\end{pmatrix}_{2\times 2}$, $X\begin{pmatrix}1\\1\end{pmatrix}\neq 0$

$$AX=\begin{bmatrix}4&2\\2&4\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}6\\6\end{bmatrix}=6\begin{bmatrix}1\\1\end{bmatrix}=\lambda X$$

$X\begin{bmatrix}1\\1\end{bmatrix}$ is called as an Eigen vector

2) $A=\begin{bmatrix}4&2\\2&4\end{bmatrix}_{2\times 2}$ $X=\begin{bmatrix}1\\2\end{bmatrix}\neq 0 \Rightarrow AX=\begin{bmatrix}4&2\\2&4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}$

$$=\begin{bmatrix}8\\10\end{bmatrix}$$

$$=2\begin{bmatrix}4\\5\end{bmatrix}\neq\lambda X$$

$AX\neq\lambda X$. Hence $X=\begin{bmatrix}1\\2\end{bmatrix}$ is Not an Eigen Vector

Note(i): $AX=\lambda X$.

$$(AX-\lambda X)=0$$

$$\boxed{(A-\lambda I)X=0}$$

Trivial

Non-trivial solution

↓

Eigen vector

① $\boxed{|A-\lambda I|=0}$

2) $\boxed{(A-\lambda I)X=0}$

1) bind the Eigen values & Eigen vectors of a matrix $A=\begin{bmatrix}3&4\\4&-3\end{bmatrix}$

Sol) Eigen values $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}3-\lambda & 4\\4 & -3-\lambda\end{vmatrix}=0$

$$(3-\lambda)(-3-\lambda)-16=0$$

$$-9-\cancel{3\lambda}+\cancel{3\lambda}+\lambda^2-16=0$$

$$\lambda^2-25=0 \implies \lambda^2=25 \implies \lambda=\pm 5 \text{ are}$$

Eigen values of $A_{2\times 2}$.

2) Eigen vector $(A-\lambda I)X=0$

$$\begin{bmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow (1)$$

<u>case (i):-</u> put $\lambda=5$ in (1)

$$\begin{bmatrix} -2 & 4 \\ 4 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2x_1+4x_2=0$$

$$4x_1-8x_2=0$$

$$\boxed{x_1=2x_2}$$

Let $x_2=K$ where K is an orbitary constant

Then $x_1=2K$

$$X=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 2K \\ K \end{bmatrix}=K\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad (K\neq 0)$$

In this corresponding one Eigen values there are infinite no of solutions are Exist.

$$X_1=\begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ for } K=1$$

$\hookrightarrow$ It is simplest Eigen vector

<u>case (ii):-</u> put $\lambda=-5$ in (1)

$$\begin{bmatrix} 8 & 4 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{matrix} 8x_1+4x_2=0 \\ 4x_1+2x_2=0 \end{matrix}$$

$\Rightarrow 2x_1 \mp x_2 = 0$

Let $x_1 = k_1$ where k_1 is an arbitrary constant

Then $x_2 = -2k_1$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ -2k_1 \end{bmatrix} = k_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} \quad (k_1 \neq 0)$$

$X_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ for $k_1 = 1$

$\hookrightarrow$ Simplest Eigen vector

<u>properties of Eigen values:-</u>

1) If $\lambda_1, \lambda_2 ---- \lambda_n$ are Eigen values of $A_{n\times n}$

then (i) $\lambda_1 + \lambda_2 + ---- + \lambda_n = \text{trace}(A_{n\times n})$

(ii) If $\boxed{\lambda_1 \cdot \lambda_2 ----- \lambda_n = |A_{n\times n}|}$

2) If λ is an Eigen value of $A_{n\times n}$ and $k \neq 0$ then

(i) λ^m is an Eigen value of A^m.

(ii) $k\lambda$ is an Eigen value of kA

(iii) $\lambda \pm k$ is an Eigen value of $A \pm kI$

(iv) $a_0 + a_1\lambda + a_2\lambda^2$ is an Eigen value of

$\boxed{a_0 I + a_1 A + a_2 A^2}$

3) If λ is an Eigen value of non singular Matrix $|A_{n\times n}| \neq 0$ then

(i) $\frac{1}{\lambda}$ is an Eigen value of A^{-1}

(ii) $\frac{|A|}{\lambda}$ " " $\text{adj}(A)$

(4) Eigen values of a diagonal, scalar, unit, lower triangle (or) upper triangle of a Matrix are just it diagonal Elements.

Ex:- (i) $A = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $\lambda = 2, 7, 9$

(ii) $A = \begin{bmatrix} 7 & 0 & 0 \\ 8 & 10 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, $\lambda = 7, 10, 9$

(iii) If $A = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 9 & \\ 0 & 0 & 10 \end{bmatrix}$, $\lambda = 8, 9, 10$

iv) If $A = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, $\lambda = 4, 4, 4$

v) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\lambda = 1, 1, 1$

5) Eigen values of A and A^T are same

6) If λ is an Eigen value of an orthogonal Matrix $A_{n \times n}$ then $\frac{1}{\lambda}$ is also another Eigen value of same Matrix 'A'

7) If $a+ib$ (or $a+\sqrt{b}$) is an Eigen value of real Matrix $A_{n \times n}$ then $a-ib$ (or $a-\sqrt{b}$) is als Another Eigen value of Matrix $A_{n \times n}$

8) If the Eigen values of a real symmetric Mat are always real $A^T = A$

- Eigen values of a real skew symmetric Matrix either $\lambda = 0$ (or) purely imaginary

1) The Eigen values of a 3x3 Matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$ are

a) 3, 7, 8 b) 2, 2, 14 ~~c)~~ 3, 0, 15 d) 1, -4, 9

Sol

$$\lambda_1 + \lambda_2 + \lambda_3 = tr(A) = 18$$

$$\lambda_1 \cdot \lambda_2 \cdot \lambda_3 = |A| = 0$$

$$\begin{bmatrix} 2 & -4 & 3 \\ -6 & 7 & -4 \\ 8 & -6 & 2 \end{bmatrix}$$ $R_2 + 3R_1$

$R_3 - 4R_1$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 10 & -10 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -4 & 3 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

2005

**

2) The Eigen values of Matrix $S = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$ are 5, 1. what are the Eigen values of $S.S = S^2$?

a) 1, 5 b) 2, 10 ~~c)~~ 1, 25 d) None

S — λ 1, 5

S^2 — $\lambda^2 = 1^2, 5^2 = 1, 25$

3) For a Matrix $A = \begin{bmatrix} 2 & -2 & 3 \\ -2 & -1 & 6 \\ 1 & 2 & 0 \end{bmatrix}$ one of the Eigen value is 3 then find other two Eigen values

a) 2, 5 b) 2, 5 c) 5, 5 d) 5, 5

$$\lambda_1+\lambda_2+\lambda_3 = tr(A) = 1$$

$$3+\lambda_2+\lambda_3 = 1$$

$$\lambda_2+\lambda_3 = -2$$

4) If a real Matrix $A = (a_{ij})_{n\times n}$ is defined as follo

$a_{ij} = \begin{cases} 1, & \forall i=j \\ 0, & \text{otherwise} \end{cases}$ then find the Sum of all Eigen values.

a) $\frac{n+1}{2}$ b) $\frac{n(n-1)}{2}$ ~~c) n~~ d) $\frac{n(n+1)}{2}$

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

$$\lambda = 1, 1, 1 \cdots\cdots 1$$

$$\boxed{\text{Sum} = tr(A) = n}$$

5) If A 3x3 Matrix has Eigen values 1, 2, -1 the

a) $tr(A^2+A^{-1}-adj(A)) =$

b) $|A^2+A^{-1}-adj(A)| =$

Sol: Here Eigen values 1, 2, -1

$$A \rightarrow \lambda$$

$$A^2 \longrightarrow \lambda^2$$

$$A^{-1} \longrightarrow 1/\lambda$$

$$adj(A) = \frac{|A|}{}$$

$A^2+A^{-1}-adj(A) \longrightarrow \lambda^2+\frac{1}{\lambda}-\frac{|A|}{\lambda}$

$$\begin{matrix} 1 \\ 2 \\ -1 \end{matrix} \Big\rangle \; \lambda^2+\frac{1}{\lambda}-\frac{|A|}{\lambda} \; \Big\langle \begin{matrix} 1^2+\frac{1}{1}-\frac{(-2)}{1}=4 \\ 2^2+1/2-\frac{(-2)}{2} = 11/2 \\ (-1)^2+\frac{1}{(-1)}-\frac{(-2)}{(-1)} = (-2) \end{matrix}$$

$tr(A^2+A^{-1}-adj(A)) = 4+11/2-2 = 2+11/2 = 15/2$

$|A^2+A^{-1}-adj(A)| = (4)(11/2)(-2) = -44$

Properties of Eigen vectors:-

① If $\lambda_1 \lambda_2 ---- \lambda_n$ are the different values of Eigen vectors of $A_{n\times n}$ then $x_1, x_2 --- x_n$ are Linearly independent Eigen vectors.

2) If same eigen values of $A_{n\times n}$ then the Eigen ~~the~~ Eigen vectors may (or) may not be independent

ii) If one of Eigen value of Matrix 'λ' of matrix $A_{n\times n}$ is repeated m times then the Eigen values corresponding to that repeated Eigen values λ which are linearly independent and which are given by the No of independent vector then

$$P = n-r = \begin{pmatrix} \text{no of variables in} \\ \text{Eigen Vector} \end{pmatrix} - \rho(A-\lambda I)$$

where $1 \leq P \leq m$

Eg:- (1) $A_{2\times 2}$ $\downarrow$ λ_1, λ_2

$m = 2$

$P = n - r \quad 1 \leq P \leq 2$

- $\lambda_1 \neq \lambda_2$ $\rightarrow$ X_1, X_2 $\rightarrow$ L.I
- $\lambda_1 = \lambda_2$
  - X_1 $\rightarrow$ L.I
  - X_1, X_2 $\rightarrow$ L.I

(2) $A_{3\times 3}$ $\downarrow$ $\lambda_1, \lambda_2, \lambda_3$

$m = 3$

$P = n - r$

$\downarrow$

$1 \leq P \leq 3.$

- $\lambda_1 \neq \lambda_2 \neq \lambda_3$ $\rightarrow$ X_1, X_2, X_3 L.I
- $\lambda_1 = \lambda_2 = \lambda_3$
  - X_1 $\rightarrow$ L.I
  - X_1, X_2 $\rightarrow$ L.I.
  - X_1, X_2, X_3 $\rightarrow$ L.I
- $\lambda_1 \neq \lambda_2 =$
  - X_1, X_2 $\rightarrow$ L.I (or) L.D
  - X_1, X_2 $\rightarrow$ L. (or) L.

(1) <u>problems</u>

(1) which of the following is not an Eigen vectors a Some 2x2 Matrix

a) $[1 \ 2]^T$ b) $[0 \ 7]^T$ c) $[8 \ 0]^T$ ~~d)~~ $[0 \ 0]^T$

The zero vector cannot be an Eigen vector of any 2x2 Matrix

2) The No of Linearly independent Eigen vectors of a Matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ is

a) 0 ~~b)~~ 1 c) 2 d) ∞

$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ It is a upper triangle Matrix

$\lambda = 2, 2$

The characteristic Equation $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix}$

$\lambda = 2$ in this $A - \lambda I = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} \Rightarrow r = \rho(A - \lambda I) = 1$

$n = 2$ $\quad p = (n - r) = 2 - 1 = 1$

$\rightarrow$ one independent vector

Which of the following is an Eigen vector of Matrix $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

a) $[0\ 8\ 2]^T$ b) $[0\ 7\ -2]^T$ c) $[0\ 0\ 9]^T$ ~~d)~~ $[8\ 2\ 0]^T$

$(A - \lambda I)X = 0$

$$\begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \\ \\ 0 \end{bmatrix}$$

$\boxed{\lambda = 0, 0, 0}$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \boxed{x_3 = 0}$$

Let $x_1 = k_1$

$x_2 = k_2$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ 0 \end{bmatrix}$$

(or) $\begin{bmatrix} k_1 \\ 0 \\ 0 \end{bmatrix}$ (or) $\begin{bmatrix} 0 \\ k_2 \\ 0 \end{bmatrix}$

Every Non zero column vector with the Last Eigen value is (0) then it will be a Eigen Vector.

2006

(4) The Eigen Values and Eigen vectors of a 2x2 Matrix are given by 8, 4 & $\begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}$ then the Matrix 'A' is

(a) $\begin{bmatrix}8 & 2\\2 & 4\end{bmatrix}$ b) $\begin{bmatrix}10 & 2\\2 & 2\end{bmatrix}$ ~~c)~~ $\begin{bmatrix}6 & 2\\2 & 6\end{bmatrix}$ d) $\begin{bmatrix}4 & 8\\8 & 4\end{bmatrix}$

$A_{2\times2}$: $\lambda_1 = 8 \longrightarrow X_1 = \begin{bmatrix}1\\1\end{bmatrix}$

$\lambda_2 = 4 \longrightarrow X_2 = \begin{bmatrix}1\\-1\end{bmatrix}$

$\lambda_1 + \lambda_2 = 12$

$\lambda_1 \cdot \lambda_2 = 32 = |A|$

(1) $AX = \lambda X$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = 8 \begin{bmatrix} \\ \end{bmatrix} \Rightarrow \begin{matrix}a+b=8\\c+d=8\end{matrix}$

(2) $AX = \lambda X$

$\begin{bmatrix}a & b\\c & d\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = 4\begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow \begin{matrix}a-b=4\\c-d=-4\end{matrix}$

$a+\cancel{b}=8$
$a-\cancel{b}=4$

$2a=12$

$\boxed{a=6}$

$b=2$

$c+d=8$
$c-d=-4$

$2c=4$

$\boxed{c=2}$

Cayley – Hamilton theorem :-

Statement:- Every Square Matrix 'A' Satisfies its own characteristic Equation.

Eg:- If $\lambda^2 - 8\lambda + 12 = 0$ is a characteristic

Equation of $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$ then by cayley - Hamilton theorem we have $A^2 - 8A + 12I = 0$.

1) +ve powers of $A_{n\times n}$:-

(1) $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \Longrightarrow \lambda^2 - 8\lambda + 12 = 0$

$$A^2 - 8A + 12I = 0$$

$$\boxed{A^2 = 8A - 12I}$$

$$A \cdot A^2 = A(8A - 12I)$$

$$\boxed{A^3 = 8A^2 - 12A}$$

Similarly $A^4 = 8A^3 - 12A^2$

$\vdots$

2) -ve powers of $A_{n\times n}$:-

(1) If $A = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix} \Longrightarrow \lambda^2 - 8\lambda + 12 = 0$

$$A^2 - 8A + 12I = 0$$

$$A^{-1}(A^2 - 8A + 12I) = 0$$

$$A - 8I + 12A^{-1} = 0$$

$$\boxed{A^{-1} = 1/12\,(8I - A)}$$

$$A^{-1} \cdot A^{-1} = 1/12\, A^{-1}(8I - A)$$

$$A^{-2} = \frac{1}{12}\left(8A^{-1} - I\right)$$

Note - I :- If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the characteristic Equation of $A_{2\times2}$ is $\lambda^2 - (a+d)\lambda + |A| = 0$

① If $A = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$ then find A^9

$$\lambda^2 - (2-2)\lambda - 5 = 0$$
$$\lambda^2 - 5 = 0$$
$$\lambda^2 = 5 \longrightarrow A^2 - 5I = 0$$
$$\boxed{A^2 = 5I}$$

$A \cdot A = A^2$
$A^2 \cdot A^2 = A^4$
$A^4 \cdot A^4 = A^8$
$A^8 \cdot A = A^9$

$$A^9 = A \cdot A^8 = A(A^2)^4 = A(5I)^4 = A(625I)$$
$$= 625\,AI$$
$$\boxed{A^9 = 625A}$$

2008

2) The characteristic Equation of a 3×3 Matrix P is defined as $\alpha_p(\lambda) = |\lambda I - P| = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$ then find P^{-1}.

Sol:

$$P_{3\times3} = \lambda^3 + \lambda^2 + 2\lambda + 1 = 0$$
$$P^3 + P^2 + 2P + I = 0$$
$$P^{-1}(P^3 + P^2 + 2P + I) = P^{-1}(0)$$
$$P^2 + P^1 + 2I + P^{-1} = 0$$
$$P^{-1} = -(P^2 + P + 2I)$$

2007, 2008

3) If $A = \begin{bmatrix} -3 & 2 \\ -1 & 0 \end{bmatrix}$ then find A^9

~~a) $511A + 510I$~~ b) $309A + 307A$ c) $154A + 155I$

d) $e^9 A$.

$\lambda^2 - (-3\lambda) + 2 = 0$

$\lambda^2 + 3\lambda + 2 = 0$

$A^2 + 3A + 2I = 0$

$\boxed{A^2 = -3A - 2I}$

$A^3 = A(-3A - 2I) = -3A^2 - 2A = -3(-3A - 2I) - 2A$

$A^4 = -3A^3 - 2A^2 \Rightarrow A^5 =$ $= 9A + 6I - 2A$

$= 7A + 6I$

$= -3(7A + 6I) - 2(-3A - 2I)$

$= -21A - 18I + 6A + 4I$

$A^4 = -14A - 14I$

$A^5 = A(-14A - 14I) = -14A^2 - 14A$

$= -14(-3A - 2I) - 14A$

$= 42A + 28I - 14A$

$= 28A + 28I$

$14 \times 3 = 42$, $42 - 14 = 8$

$= 28A^2 + 28A$

$= 28(-3A - 2I) + 28A$

$= -84A - 56I + 28A$

$28 \times 3 = 84$, $84 - 28 = 56$

$A^6 = -56A - 56I$

$A^7 = -56A^2 - 56A$

$= -56(-3A - 2I) - 56A$

$= 168A + 112I - 56A$

$= 112A + 112I$

$A^8 = 112A^2 + 112A$

$56 \times 3 = 168$, $56 \times 2 = 112$, $168 - 56 = 112$

$A^8 = 112\,(-3A - 2I) + 112A$

$= -336A - 224I + 112A$

$A^8 = -224I - 224A$

$A^9 = -224\,(-3A - 2I) - 224A$

$= 672A + 448I - 224A$

$= \cancel{511A}\ \ 448A + 448I$

$112 \times 3 = 336$, 67, 22, 1

$112 \times 2 = 224$

$224 \times 3 = 672$

$224 \times 2 = 448$

4) Given that $A = \begin{bmatrix} -5 & -3 \\ 2 & 0 \end{bmatrix}$ & $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then the value of A^3 is

a) $15A + 21I$

b) $17A + 15I$

c) $17A + 21I$

d) $19A + 30I$

$\lambda^2 - (-5+0)\lambda + 6 = 0$

$A^2 + 5A + 6I = 0$

$A^2 = -5A - 6I$

$A^3 = -5A^2 - 6A$

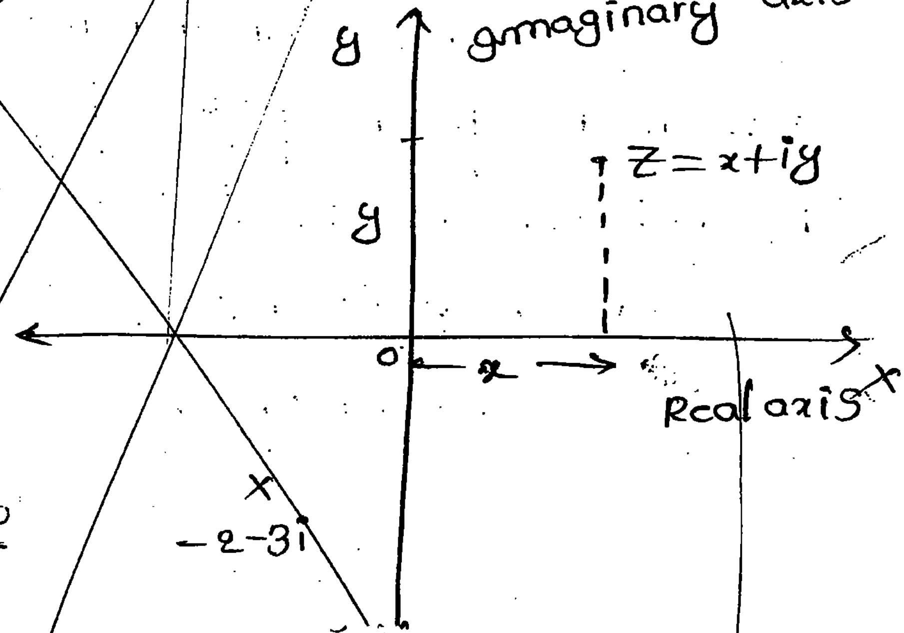
$A^3 = -5\,(-5A - 6I) - 6A$

$\boxed{A^3 = 19A + 30I}$

Complex Variables

Complex No:- A No of the form $z = x+iy$ is called a complex No. where x, y are real Numbers and $i=\sqrt{-1}$. x is called real part and y is called imaginary part.

Representation of complex No in the plane:-



Conjugate of x+iy

conjugate of $x+iy$ is

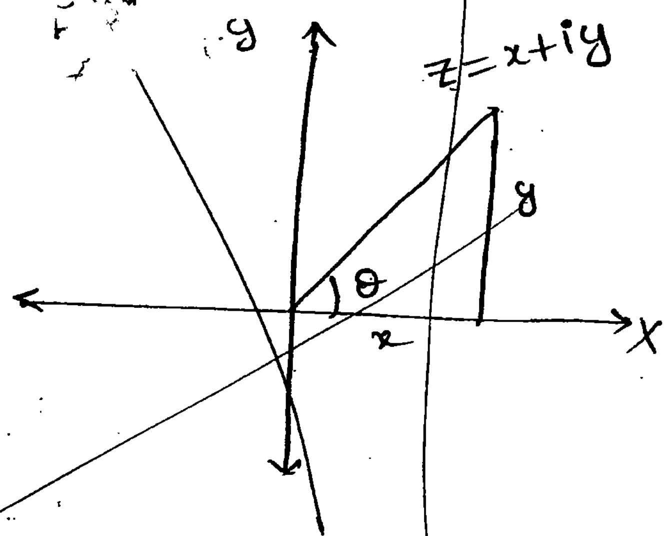
$\bar{z} = x-iy$

Modulus of complex No

Modulus (x,y) is

$|z| = \sqrt{x^2+y^2}$

Polar form of a complex No:-



$z = x+iy$

$\frac{x}{r} = \cos\theta, \quad \frac{y}{r} = \sin\theta$

$x = r\cos\theta, \quad y = r\sin\theta$

$z = r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$

$|z| = |re^{i\theta}| = |r||e^{i\theta}| = |r||\cos\theta + i\sin\theta|$

18.1 — 29

Complex Variables

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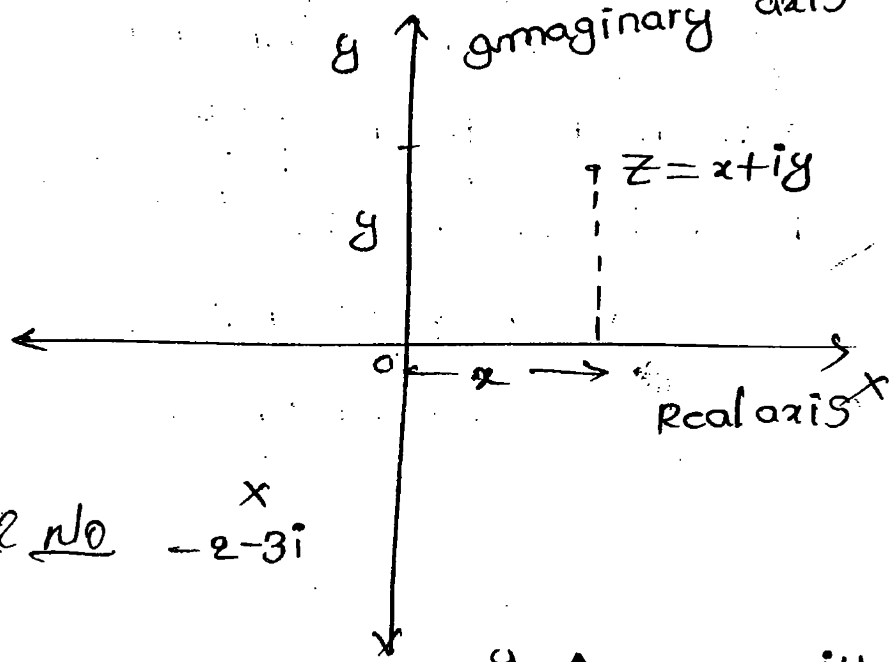
x is called real part and y is called imaginary part

Representation of Complex No in the plane:-



Conjugate of $x+iy$

conjugate of $x+iy$ is

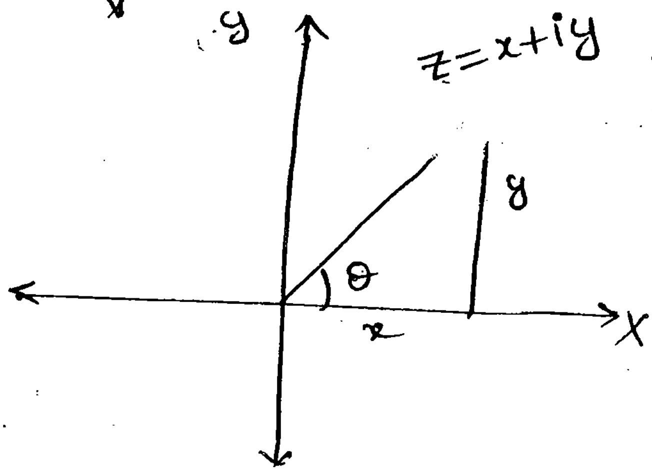
$\bar{z} = x - iy$

Modulus of Complex No $-2-3i$

Modulus (x, y) is

$|z| = \sqrt{x^2+y^2}$

Polar form of a complex No:-



$z = x+iy$

$\frac{x}{r} = \cos\theta, \quad \frac{y}{r} = \sin\theta$

$x = r\cos\theta, \quad y = r\sin\theta$

$z = r\cos\theta + ir\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$z = re^{i\theta}$

$|z| = |re^{i\theta}| = |r||e^{i\theta}| = |r||\cos\theta + i\sin\theta|$

$= |r| \sqrt{\cos^2\theta + \sin^2\theta} = r$

$\tan\theta = y/x$

$$\boxed{\theta = \tan^{-1}(y/x)}$$

→ If $\theta = 0$ then $z = r(\cos\theta + i\sin\theta) \Rightarrow z = r(1)$

→ If $\theta = \pi/2$ then the No is purely imaginary

* $x^x = ?$ where $x = \sqrt{-1}$

Sol

$i = \sqrt{-1}$

$$i^i = (\cos\pi/2 + i\sin\pi/2)^i = (e^{i\pi/2})^i = e^{-\pi/2}$$

* what is the value of the Expression

$$\frac{-5+10i}{3+4i} = \frac{(-5+10i)(3-4i)}{25} = \frac{-15+20i+30i+4}{25}$$

$$= \frac{25+50i}{25} = (1+2i)$$

* what is the modulus of

$$\frac{3+4i}{1-2i} = \left|\frac{3+4i}{1-2i}\right| = \frac{|3+4i|}{|1-2i|} = \frac{\sqrt{9+16}}{\sqrt{1+4}}$$

$$= \sqrt{\frac{25}{5}} = \sqrt{5}.$$

* If $z = x+iy$ then $|e^{iz}|$

$$|e^{i(x+iy)}| = |e^{ix-y}| = |e^{ix}||e^{-y}|$$

$$= e^{-y}|\cos x + i\sin x|$$

$$= e^{-y}\sqrt{\cos^2 x + \sin^2 x}$$

$$\boxed{|e^{iz}| = e^{-y}}$$

? If $z = \sqrt{3}/2 + i/2$ then find z^4

a) $\sqrt{3}+i/2$ ~~b)~~ $-1/2 + i\sqrt{3}/2$ c) $\sqrt{3}/2 - i/2$ d) $-\frac{\sqrt{3}}{2} - i/2$

$$z = \sqrt{3}/2 + i/2 = e^{i\theta}$$

$$= e^{i(\pi/6)}$$

$$z^4 = e^{4i\pi/6} = e^{2i\pi/3} = e^{i(2\pi/3)}$$

$$= \cos(2\pi/3) + i\sin(2\pi/3)$$

$$= \cos(\pi - \pi/3) + i\sin(\pi - \pi/3)$$

$$= -\cos(\pi/3) + i\sin(\pi/3)$$

$$= -1/2 + i\sqrt{3}/2$$

<u>omplex</u> <u>function</u>:-

→ corresponding to Each complex variable z in the region 'R' there corresponds an another complex variable $\omega = f(z)$ in the region R' of W plane then 'W' is called a complex function

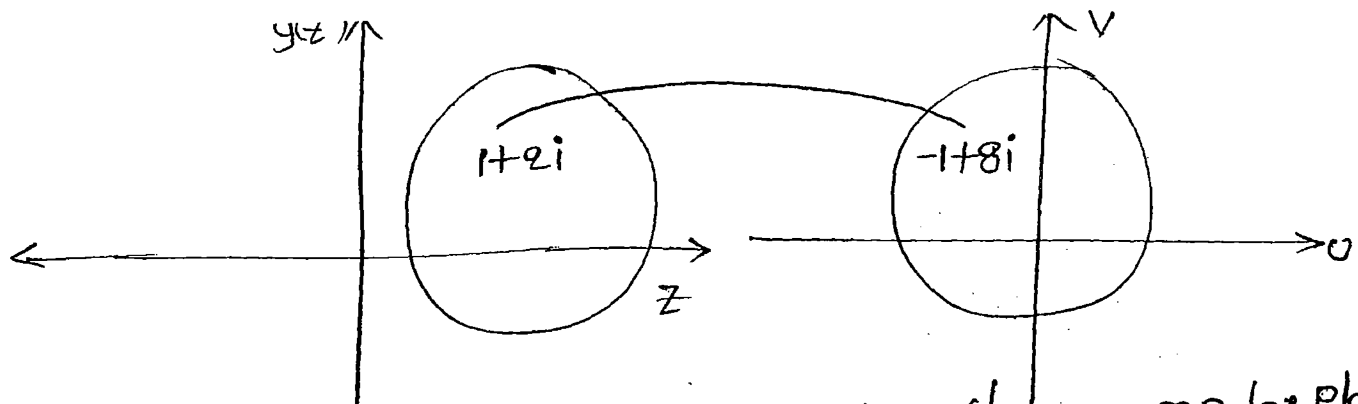
The image of $(1+2i)$ under this function z^2+2z lies in the second quadrant of W plane.

$$f(z) = z^2 + 2z \qquad z = 1+2i$$

$$= (1+2i)^2 + 2(1+2i)$$

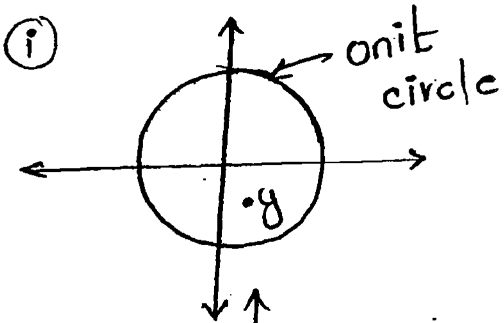
$$= 1 - 4 + 2 + i(4+4)$$

$$\boxed{f(z) = -1+8i}$$

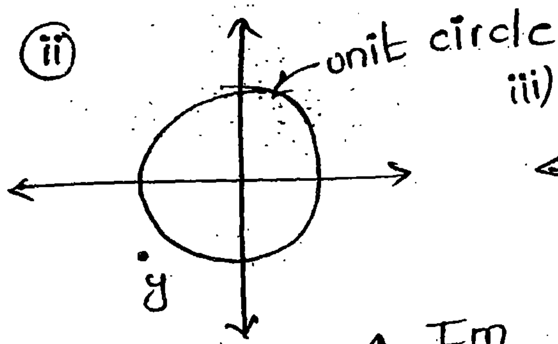


1) A point z has been plotted in the complex plan as shown in the fig. then the plot of the complex No $y = 1/z$ is

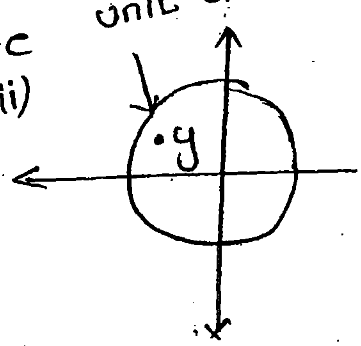
(i)



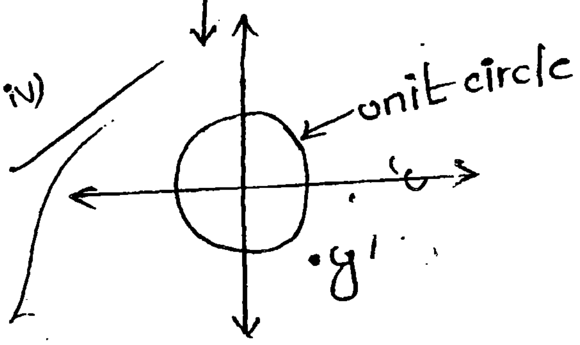
(ii)

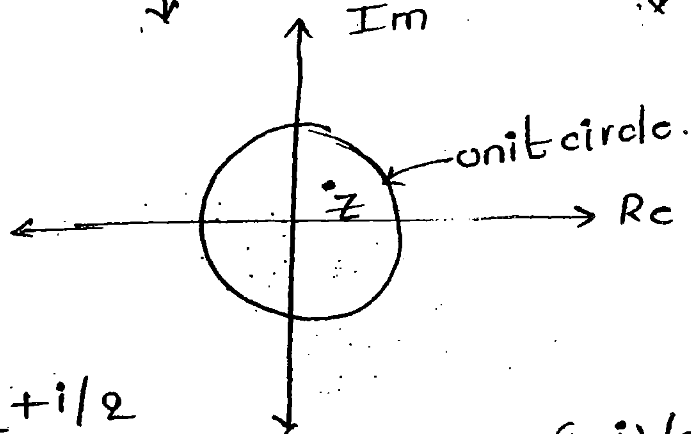


iii)



iv)





Sol consider $z = 1/2 + i/2$

$$y = 1/z = \frac{1}{(1/2 + i/2)} \, \frac{(1/2 - i/2)}{(1/2 - i/2)} = \frac{(1-i)/2}{\frac{1}{4} + 1/4}$$

$$y = \frac{(1-i)}{\cancel{2}} \times \frac{\cancel{2}}{1} \Longrightarrow y = 1-i \qquad |y| = \sqrt{1+1} = \sqrt{2}$$

II method:- $z = x + iy$

where $x, y > 0$

$$y = \frac{1}{z} = \frac{1}{x+iy} = \frac{(x-iy)}{(x+iy)(x-iy)} = \frac{(x-iy)}{x^2+y^2}$$

$$y = \frac{x}{x^2+y^2} - i\frac{y}{(x^2+y^2)}$$

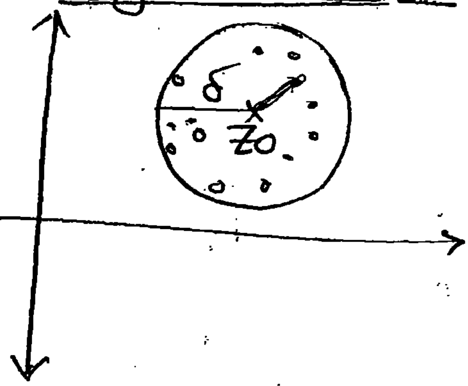
Given condition $|z|<1$

$$y = 1/z \implies z = 1/y$$

$$|1/y| = |z| < 1 \qquad |1/y| < 1 \implies 1 < |y| \implies \boxed{|y|>1}$$

Neighbourhood of a point:-

The set of all points lies inside the circle with centre z_0 & some radius is called neighbourhood of a point z_0



$|z-z_0| \leq \delta$ represents set of all points which lies within and on the circle with centre z_0 and radius 'δ'

Analytic function

A complex function $f(z)$ is said to be Analytic at a point z_0. If $f'(z)$ Exist not only if the point z_0 but also some neighbourhood of point z_0

Note:- A function $f(z)$ is said to be Analytic in the region if $f'(z)$ exist at every point of the region.

Singularity:- A point at which the function is not Analytic is called Singularity

Ex:- $f(z) = \frac{z+1}{z^2+9}$

$z^2+9=0 \implies z = \pm 3i$

$3i, -3i$ are singularities.

2) $f(z) = (z-2)^{1/2}$

$f'(z) = 1/2\,(z-2)^{-1/2}$

$z=2$ is singularity

Entire function:- A function which is Analytic throughout the finite complex plane is called as an Entire function.

Eg: (1) $a_0 + a_1 z + a_2 z^2 + \cdots$

$\sin z, \cos z, e^z$ ---

$f(z) = u(x,y) + iv(x,y)$ is Analytic

It will satisfies the following conditions

① $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$

2) $\underbrace{\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}}$

cauchy's Riemann Equations.

Cauchy Riemann Equation in polar form:-

$$f(z) = u(r,\theta) + iv(r,\theta)$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \;\&\; \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}}$$

Construction of Analytic function:-

$f(z) = u(x,y) + iv(x,y)$ is Analytic function

If Real part is given (u is given)

$$\boxed{f(z) = \int \frac{du}{dx}(z,0)\,dz - i\int \frac{du}{dy}(z,0)\,dz + c}$$

$$\boxed{v = -\int \frac{du}{dy}\,dx + \int (\text{terms of } \tfrac{du}{\partial x} \text{ without } x)\,dy + c}$$

take 'y' as constant

If Imaginary part is given (v is given)

$$\boxed{f(z) = \int \frac{dv}{dy}(z,0)\,dz + i\int \frac{dv}{\partial x}(z,0)\,dz + c}$$

$$\boxed{u = \int \frac{dv}{dy}\,dx - \int (\text{terms of } \tfrac{dv}{dx} \text{ without } x)\,dy + c}$$

take 'y' as constant

Note: If $f(z) = u + iv$ is Analytic then 'u' is called harmonic conjugate of 'v' and 'v' is called harmonic conjugate of 'u'.

1) If $u = 3x^2 - 3y^2$ then find 'v' so that $f(z) = u + iv$ is Analytic.

a) $3y^2 - 3x^2 + k$ b) $6x - 6y + k$ c) $6y - 6x + k$ ~~d)~~ $6xy + k$

Sol: $f(z)$ is Analytic $\Rightarrow$ $u_x = v_y$ & $u_y = -v_x$

$u_x = v_y$

$6x = 6x$

$-6y = -6y$

* If $V = e^{x}\sin y$ then find harmonic conjugate of V'
So that $f(z) = u + iv$ is Analytic

a) $e^{-x}\sin y$ b) $e^{-x}\cos y$ ~~c)~~ $e^{x}\cos y$ d) None

$V = e^{x}\sin y$ $u_y = -V_x$

$u_x = V_y$ $-\frac{dv}{dx} = -e^{x}\sin y$

$\frac{dv}{dy} = e^{x}\cos y$ $\frac{du}{dy} = -e^{x}\sin y$

* If $f(x+iy) = x^3 - 3xy^2 + i\phi(x,y)$ where $f(x+iy)$ is Analytic then find $\phi(x,y)$

a) $y^3 - 3x^2y$ ~~b)~~ $3x^2y - y^3$ c) $x^4 - 3x^3y$ d) $xy^3 - y^2$

Sol

$f(x+iy) = i(3x^2y - y^3) + x^3 - 3xy^2$

$\frac{du}{dx} = \frac{dV}{dy}$ $\frac{du}{dy} = -\frac{dV}{dx}$

$3x^2 - 3y^2 = 3x^2 - 3y^2$ $-6xy = -6xy$

* $f(z) = u + iv$ is Analytic and $u = \log(x^2+y^2)$
then find $v = ?$

Sol

$V = -\int \frac{du}{dy}dx + \int (\text{terms of } \frac{du}{\partial x} \text{ without } x)\, dy + c$

$\frac{du}{dy} = \frac{(2y)}{(x^2+y^2)}$ $\frac{du}{\partial x} = \frac{2x}{x^2+y^2}$

$V = -\int \frac{2y}{x^2+y^2}dx + \int 0\, dy + c$

$y = \text{cons}$

$V = -2y\int \frac{1}{x^2+y^2}dx + c$

$$\boxed{\int \frac{1}{(x^2+a^2)} dx = \frac{1}{a} \tan^{-1}(x/a)}$$

$$V = -\cancel{2y}\left(\frac{1}{\cancel{y}}\right) \tan^{-1}(x/y)+c = -2\tan^{-1}(x/y)+c$$

Find $f(z) = u+iv$ if $u = x^3-3xy^2$ where $f(z)$ is Analytic

Sol $\frac{du}{dx} = 3x^2-3y^2 \qquad \frac{du}{dy} = -6xy$

$\frac{du}{dx}(z,0) = 3z^2 \qquad \frac{du}{dy}(z,0) = 0$

$f(z) = \int \frac{du}{dx}(z,0)\, dz - i\int \frac{du}{dy}(z,0)\, dz + c$

$= \int 3z^2 dz - i\int 0 + c$

$= \cancel{3}(z^3/\cancel{3}) + c = z^3+c$

If $v = e^x(\cos y+\sin y)$ then find the Analytic function $f(z) = u+iv$

Sol $\frac{\partial v}{\partial x} = e^x(\cos y+\sin y) \qquad \frac{\partial v}{\partial x}\Big|_{(z,0)} = e^z(\cos 0 + \sin 0) = e^z$

$\frac{dv}{dy} = e^x(-\sin y+\cos y) \qquad \frac{dv}{dy}\Big|_{(z,0)} = e^z(1)$

$f(z) = \int \frac{dv}{dy}(z,0)\, dz + i\int \frac{dv}{dx}(z,0)\, dz + c$

$= \int e^z dz + i\int e^z + c$

$= e^z + ie^z + c = e^z(1+i)+c$

* $f(z) = (x+ay)+i(bx+cy)$ is Analytic then

(a) $c=1, b=2, a=2$ b) $c=1, a=-1, b=2$

(c) $c=1, b=1, a=2$ ~~d)~~ $c=1, a=-1, b=1$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$$

$$\boxed{1 = c} \qquad \boxed{a = -b.}$$

* find 'p' so that $f(z) = r^2\cos 2\theta + i r^2 \sin p\theta$ is Analytic

$$u = r^2\cos 2\theta$$

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$$

$$2r\cos 2\theta = \frac{1}{r}(r^2 p \cancel{\sin p\theta} \cos p\theta)$$

$$2r\cos 2\theta = rp \cancel{\sin p\theta} \cos p\theta$$

$$\boxed{p=2}$$

$$-2r^2\sin 2\theta = -r(2r \sin p$$

$$\boxed{p=2}$$

* $f(z) = 1/2 \log(x^2+y^2) + i\tan^{-1}(y/x)$ is Not Analytic at a) $(1,1)$ b) $(1,0)$ ~~c) (0,0)~~ d) $(0,1)$

$$u = 1/2 \log(x^2+y^2) \qquad v = \tan^{-1}(y/x)$$

The functions are not defined at (0,0) then it will not Exist at (0,0)

* If $f(z) = e^{-x}(\cos y - i\sin y)$ is a Analytic

$$f(z) = e^{-x}\cos y - ie^{-x}\sin y$$

$$u(x) = e^{-x}\cos y \qquad \frac{\partial u}{\partial y} = -e^{-x}\sin y$$

$$\frac{\partial u}{\partial x} = -e^{-x}\cos y$$

$$\frac{\partial v}{\partial x} = +e^{-x}\sin y \qquad \frac{\partial v}{\partial y} = -e^{-x}\cos y$$

$u_x = v_y$ and $u_y = -v_x$ All conditions are satisfied

Hence all conditions are satisfied. Hence the given function is Entire function.

- If $f(z) = u - iv$ is Analytic then which of the following is true

1) $iu_x + v_x = u_y + iv_y$ ii) $iu_x + v_x = -u_y - iv_y$

~~iii)~~ $u_x + iv_x = -iu_y + v_y$ iv) $u_x + iv_y = iu_y - v_y$

$$u_x + iv_x = -iu_y + v_y$$

Equating real & imaginary terms.

$$u_x = v_y \quad \& \quad v_x = -u_y.$$

Complex Integration:-

Evaluation of integral of a complex function along a continous curve c is called a complex intrigation and is def... d by $\int_c f(z)dz$ in

$$\int_c f(z)\, dz = \int_c (u+iv)\,(dx+idy)$$

$$= \int_c (udx - vdy) + i\int(udy + vdx)$$

* $\int_0^{2\pi} z^2 \sin 4z \, dz$

$$\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$v_1 = \int v, \quad v_2 = \int v_1 \; ; \quad v_3 = \int v_2$$

$$u' = d(u) \; ; \quad u^2 = d(u_1) \; ; \quad u''' = d(u_2)$$

$$\int_0^{2\pi} z^2 \sin 4z \, dz = \left\{ (z^2)\left[\frac{-\cos 4z}{4}\right] - (2z)\left[\frac{-\sin 4z}{16}\right] + (+2)\left[\frac{+\cos 4z}{64}\right] \right\}_0^{2\pi}$$

$$= \left[(2\pi)^2 (-1/4) - (4\pi)(0) + \frac{1}{32}(1) - \frac{1}{32} \right]$$

$$= \frac{-4\pi^2}{4} = -\pi^2$$

* $$\int_1^{1+i\pi} e^z \, dz = \left((e^z)\right)_1^{1+i\pi} = e^{1+i\pi} - e^1$$

$$= e^1 \cdot e^{i\pi} - e = e\left[\cos\pi + i\sin\pi\right] - e$$

$$= e\left[-1+0\right] - e = -e - e = -2e$$

* $$\int_0^{1+i} (x^2 - iy)\, dz \quad \text{along } y = x^2$$

Sol

$$\int_0^{1+i} (x^2 - iy)\, dz \qquad y = x^2, \quad dy = 2x\,dx$$

$$= \int_0^{1+i} (x^2 - iy)(dx + i\,dy)$$

$$= \int_0^{1+i} (x^2 - ix^2)(dx + i\,2x\,dx)$$

$$= \int_{0+i0}^{1+i} x^2(1-i)\, dx\,(1+i2x) = \int_0^1 (1-i)x^2\, dx\,(1+2ix)$$

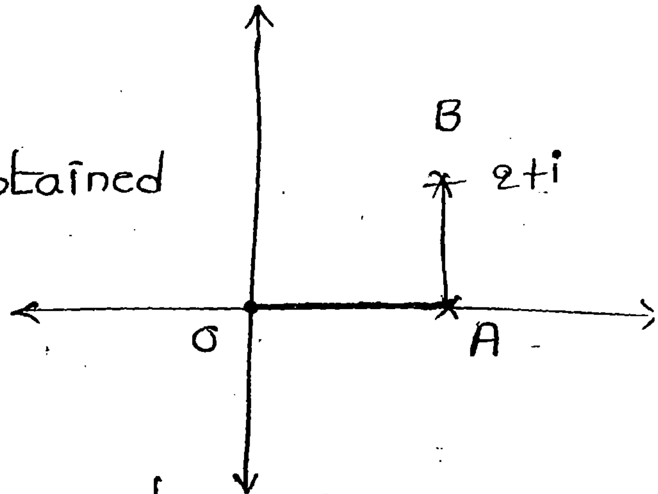
$$= (1-i)\left[\frac{(x^3)}{3} + 2i\frac{x^4}{4}\right]_0^1$$

$$= (1-i)\left[1/3 + i/2\right]$$

$$= 1/3 - i/3 + i/2 + 1/2$$

$$= 5/6 + i(1/2 - 1/3) \qquad 5/6 + i(1/6)$$

Evaluate $\int_0^{2+i} (\bar{z})^2 dz$ along the real axis '2' and then to $2+i$



Note:- If 'c' is the curve obtained by combining the c_1, c_2, c_3 then

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

$$\int_0^{2+i} (\bar{z})^2 dz = \int_{OA} (\bar{z})^2 dz + \int_{AB} (\bar{z})^2 dz$$

Along OA $\quad y=0 \longrightarrow dy=0$

$$\int_{OA} (\bar{z})^2 dz = \int (x-iy)^2 (dx+idy)$$

$$= \int (x-0)^2 (dx+0) = \int_0^2 (x^2)\, dx = \frac{x^3}{3}\Big]_0^2 = 8/3$$

Along AB:-

$$\int_{AB} (\bar{z})^2 dz = \int_{AB} (x-iy)^2 (dx+idy)$$

$$x=2 \Rightarrow dx=0 \quad (y \to 0 \text{ to } 1)$$

$$\int_{AB} (\bar{z})^2 dz = \int_{AB} (x-iy)^2 (dx+idy)$$

$$= \int_{AB} (2-iy)^2 (idy)$$

$$= \int^1 (2i+y)^2 dy$$

$$= \int_0^1 [-4+y^2+4i]\,dy = -4+4i \int_0^1 -4y+\frac{y^3}{3}+4iy$$

$$= [-4+1/3+4i]$$

$$= \left[\frac{-12+1}{3}+4i\right] =$$

$$= \int_0^1 (4-y^2-4iy)\,i\,dy = i\,[4y - y^3/3 - 4iy^2/2]$$

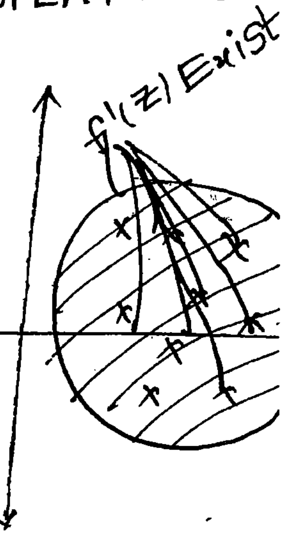
$$= i\,[4-1/3-2i] = \frac{11i}{3}+2$$

$$\int_0^{2+i} (\bar{z})^2\,dz = 8/3 + 11i/3 + 2 = \frac{14+11i}{3}$$

* Evaluate $\int_c (\bar{z})^2\,dz$ where 'c' is OABC

cauchy's integral theorem:-

Let $f(z)$ be an Analytic function within & on a closed curve 'c' then $\boxed{\int_c f(z)\,dz = 0}$



cauchy's integral formula

Let $f(z)$ be an Analytic function within and on a closed curve 'c' and 'a' is Any point inside the curve 'c' then $\int_c \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$

derivative of caushy's integral formula

$$\int_c \frac{f(z)}{(z-a)}\,dz = 2\pi i f(a)$$

differentiate w.r.to a

$$\int_C \frac{f(z)}{(z-a)^2}\,dz = 2\pi i f'(a)$$

differentiate w.r.to 'a'

$$\int_C \frac{2f(z)}{(z-a)^3}\,dz = 2\pi i f''(a)$$

$$\boxed{\int_C \frac{f(z)}{(z-a)^3}\,dz = \frac{2\pi i}{2!} f''(a)}$$

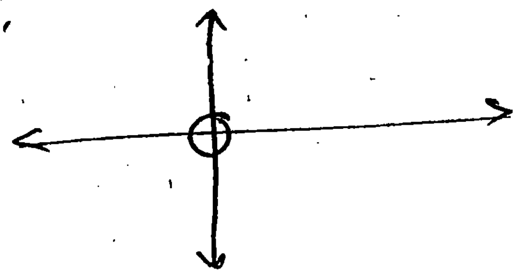
$$\boxed{\int_C \frac{f(z)}{(z-a)^n}\,dz = \frac{2\pi i}{(n-1)!} f^{n-1}(a)}$$

- find $\int_C \frac{z^2-z+1}{z-1}\,dz$ where 'c' is $|z|=1$

$$= 2\pi i f(1) = 2\pi i(1-1+1)$$

$$= 2\pi i$$

- $\int_C \frac{dz}{z^2 e^z}$ where 'c' is the simple closed curve around the origin



$$\int_C \frac{e^{-z}\,dz}{(z-0)^2} = 2\pi i f'(a)$$

$$= \frac{2\pi i f'(0)}{(1!)} = -2\pi i e^{-0} = -2\pi i$$

- $\int_C \frac{e^{2z}}{(z-1)(z-2)}\,dz$ where 'c' is $|z|=3$

$$\int_C \frac{e^{2z}}{(z-1)(z-2)} = \int_C e^{2z}\left(\frac{1}{(z-2)} - \frac{1}{(z-1)}\right)$$

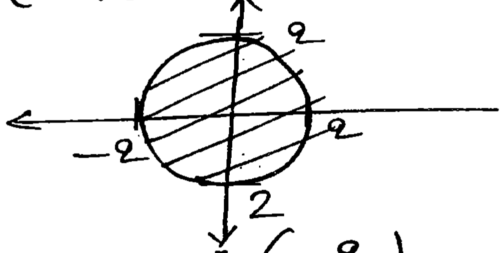
$$= \int_c e^{2z}\left[\frac{1}{(z-2)} \quad \frac{1}{(z-1)}\right] dz$$

$$= \int_c \frac{e^{2z}}{(z-2)}\, dz - \int_c \frac{e^{2z}}{(z-1)}\, dz$$

$$= 2\pi i f(2) - 2\pi i f(1)$$

$$= 2\pi i\, e^4 - 2\pi i\, e^2 = 2\pi i (e^4 e^2)$$

* find $\int_c \frac{e^{2z}}{(z-1)(z-3)}\, dz$ where (c) is $|z| = 2$

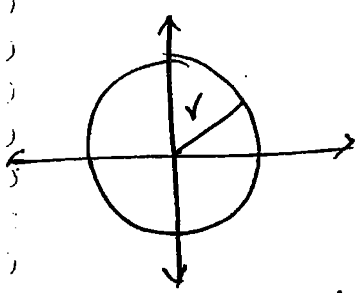


$$= \int_c \frac{\frac{e^{2z}}{(z-3)}}{(z-1)}\, dz = 2\pi i f(z)$$

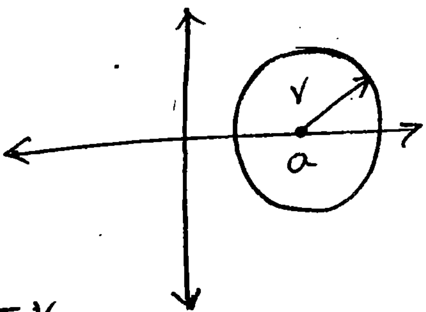
$$= 2\pi i f(1) = 2\pi i \left(\frac{e^2}{-2}\right)$$

$$\boxed{\int_c \frac{\frac{e^{2z}}{(z-3)}}{(z-1)}\, dz = -\pi i (e^2)}$$

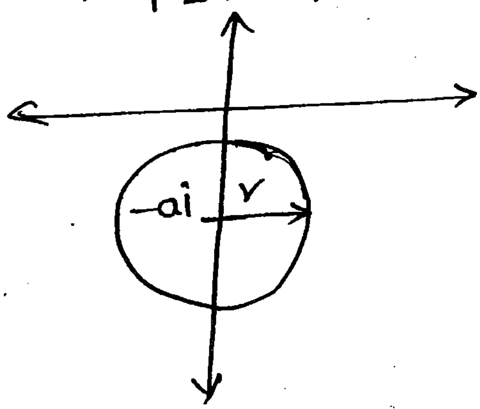
① $|z| = r$



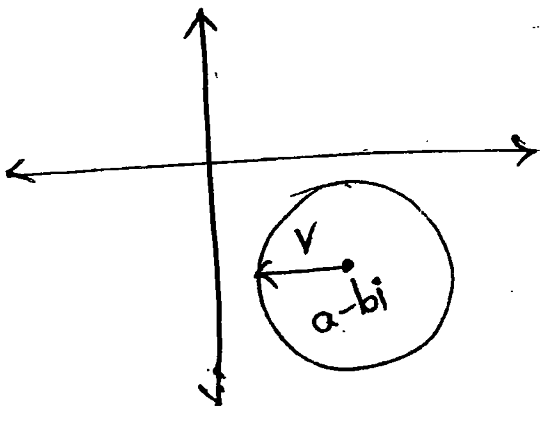
(ii) $|z-a| = r$



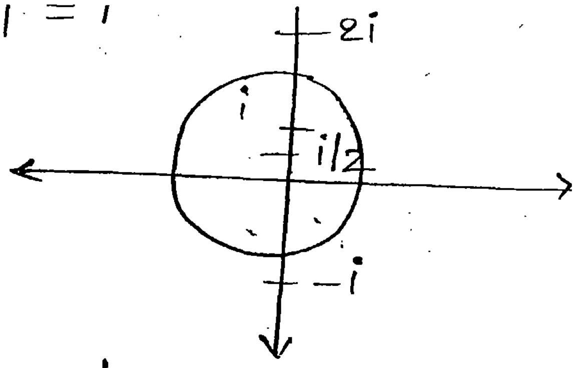
3) $|z+ai| = r$



4) $|z-a+bi| = r$



$\int_c \frac{1}{(1+z^2)} dz$ where c is $|z - i/2| = 1$



$\int_c \frac{1}{(z+i)(z-i)} dz = \int \frac{\frac{1}{(z+i)}}{(z-i)} dz$

$= 2\pi i f(i) = \cancel{2}\pi \cancel{i} \frac{1}{\cancel{2} \cancel{i}} = \pi$

find $f(2)$ and $f(3)$ where $f(a) = \int_c \frac{2z^2 - z - 2}{(z-a)} dz$

(c) $|z| = 2.5$

$f(z)$

$f(3) = \int_c \left[\frac{(2z^2 - z - 2)}{(z-3)}\right] dz = 0.$

$f(2) = \int_c \frac{(2z^2 - z - 2)}{(z-2)} dz = 2\pi i f(2)$

$= 2\pi i (8 - 2 - 2)$

$= 2\pi i (4) = 8\pi i$

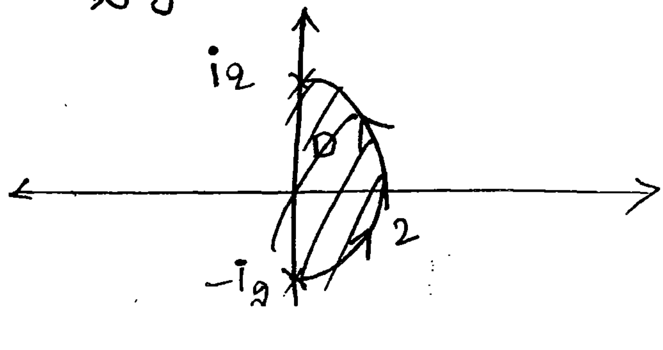
** find $\int_c f(z) dz$ along the unit circle where

$f(z) = \frac{\cos z}{z}$ $\quad |z| = 1$

$\int_c \frac{\cos z}{(z-0)} dz = 2\pi i f(0)$

$= 2\pi i (\cos 0) = 2\pi i$

* If the semicircular ~~contowrity~~ contour D of radius (2) is shown in the fig. Then the value of integral



$$= \oint_D \frac{1}{(s^2-1)} ds = \oint_D \frac{1}{(s+1)(s-1)} ds = \oint_D \frac{\frac{1}{(s+1)}}{(s-1)} ds$$

$$= 2\pi i f(1) = 2\pi i (1/2)$$

$$= \underline{\underline{\pi i}}$$

* $\int_c \frac{z^2+8}{0.5z-1.5j} dz$ where 'c' is $x^2+y^2=16$.

a) $-2\pi j$ b) $2\pi j$ c) $4\pi j$ ~~d)~~ $-4\pi j$

$$\frac{1}{0.5}\int_c \frac{(z^2+8)}{(z-3j)} dz$$

$$2\int_c \frac{(z^2+8)}{(z-3j)} dz = 2(2\pi j) f(3j)$$

$$= 4\pi j(-9+8) = -4\pi j$$

* $f(z) = c_0 + c_1 z^{-1}$ then find the value of

$\oint_c \frac{1+f(z)}{z} dz$ $\quad |z|=1$

$$\oint_c \left(\frac{1+c_0+\frac{c_1}{z}}{z}\right) dz = \oint_c \frac{z+c_0 z+c_1}{z^2}$$

$$= 2\pi i f'(0)$$

$$= 2\pi i (1+c_0)$$

* $\oint_c \frac{\cos 2\pi z}{(2z-1)(z-3)} dz$ where 'c' is unit circle

$$\frac{1}{2}\oint_c \frac{\cos 2\pi z}{(z-0.5)(z-3)} = \frac{1}{2}\frac{\frac{\cos 2\pi z}{(z-3)}}{\quad} = 0.5(2\pi i) f(0$$

$$= \pi i \frac{\cos(\pi)}{-2.5}$$

$$= \pi i \frac{1}{2.5} = \frac{\pi i}{2.5} = 0.4\pi i$$

$\int_c \frac{\sin 2z}{(z-\pi/6)^3} \, dz$ where $|z|=1$

$\pi/6 < 1$

$$\oint_c \frac{\sin 2z}{(z-\pi/6)^3} = \frac{2\pi i f''(\pi/6)}{2!}$$

$f(z) = \sin 2z$

$f'(z) = \sin 2z$

$f''(z) = 2\cos 2z$

$f''(\pi/6) = 2\cos(\pi/3) = 2(1/2)$

$$\oint_c 2\pi i \, (1) = \frac{2\pi i}{2!} = \pi i$$

* $\int \frac{(z+4)}{z^2+2z+5} \, dz$ where 'c' is $|z|=1$

z^2+2z+5

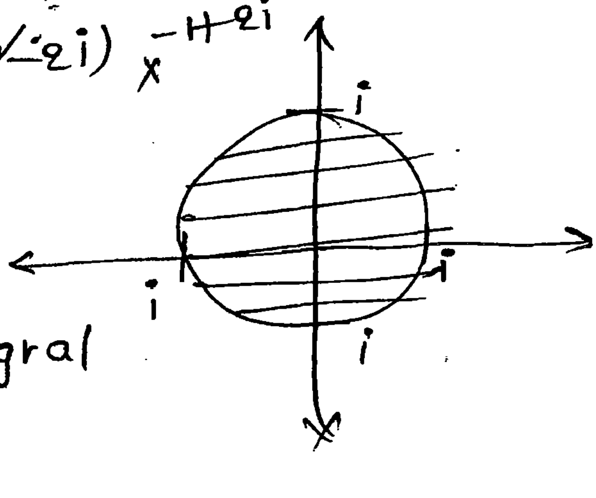
$$\int_c \frac{(z+4)}{(z+1-2i)(z+1+2i)} \qquad = \frac{-2 \pm \sqrt{4-20}}{2}$$

$$= \frac{-2 \pm 4i}{2} = -1+2i, \; = -1-2i$$

$$\int_c \frac{\frac{(z+4)}{(z+1+2i)}}{(z+1-2i)} \neq \cancel{2\pi i f(1-2i)} \quad -1+2i$$

$$= 2\pi i$$

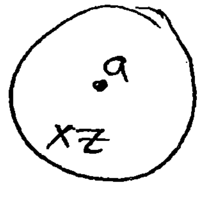
Hence the total function is Analytic. By the integral theorem



$$\int_C \frac{z+4}{z^2+2z+5} = 0$$

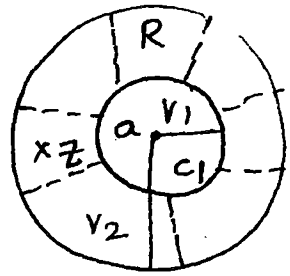
Taylor's theorem:-

Let $f(z)$ is an Analytic function inside the circle 'c' with centre 'a' then for Any point 'z' inside the circle 'c' we have



$$\boxed{f(z) = f(a) + (z-a) f'(a) + \frac{(z-a)^2}{2!} f''(a) + \cdots}$$

Laurent's Series:-



Let $f(z)$ is an Analytic function in a ring shaped region bounded by two concentric circles c_1 and c_2 of radius r_1 and r_2 respectively with centre as 'a' then for any point 'z' inside the Region 'R' we have

$$\boxed{f(z) = a_0 + a_1 (z-a) + a_2 (z-a)^2 + \cdots + a_{-1} (z-a)^{-1} + a_{-2} (z-a)^{-2} + a_{-3} (z-a)^{-3} + \cdots}$$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{Analytical part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} \left(\frac{1}{(z-a)^n}\right)}_{\text{principal part}}$$

A point at which the function is zero is called zero of the function.

Eg:- $f(z) = \frac{(z^2+4)}{z^2-9}$

$z^2+4=0 \implies z=-2i, +2i$ are zero's

isolated Singularity:-

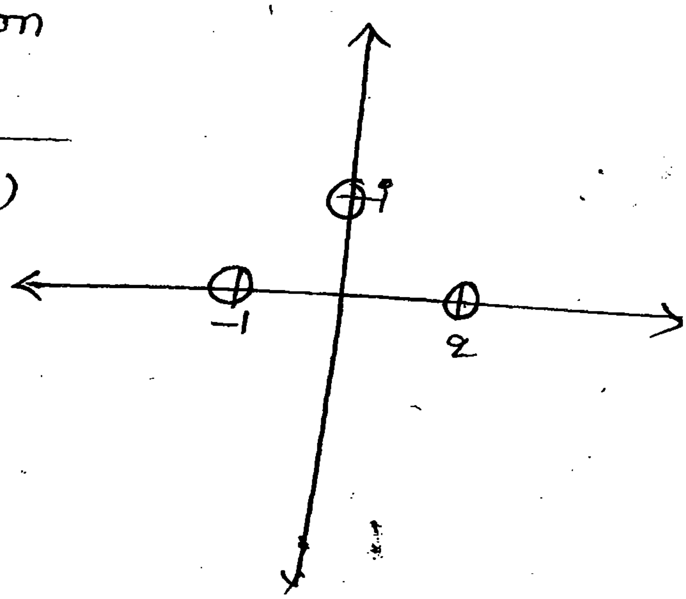
The Singularities are said to be isolated

A Singularity z_0 is said to be an isolated singularity if there exist some Neighbourhood at the point z_0, which does not contain any other singularity of the function

$f(z) = \frac{z^2}{(z-i)(z-2)(z+1)}$

$1=$

$z=i, 2, -1$ are called as isolated singularities



Removable Singularities:-

$f(z) = \frac{\sin z}{z}$

$= \frac{z - z^3/3! + z^5/5! + \cdots}{z}$

$f(z) = 1 - z^2/3! + z^4/5!$

In the Expansion of the function by Laurent Series if it contains only the terms of Analytical part then the Singularity is called removable singularity

In the above

$z=0$ is Removable singularity.

<u>pole of order 1</u>:-

If we Expand the function in the form of Laurent Series if it contains the terms till $\frac{1}{(z-a)}$ only then $z=a$ is called pole of order (1)

i.e $f(z)= a_0+a_1(z-a)+a_2(z-a)^2+ \cdots + a_{-1}\frac{1}{(z}$
$+ \cdots$

<u>pole of order 2</u>:-

In the Expansion of the function if r

not $f(z)= a_0+a_1(z-a)+ \cdots a_{-1}\frac{1}{(z-a)} + a_{-1}\frac{1}{(z-a)}$ then

$z=a$ is called pole of order 2

<u>Pole of order n</u>:-

$f(z) = a_0+a_1(z-a)+a_2(z-a)^2+ \cdots + a_{-1}\frac{1}{(z-a)}+$

$a_{-2}\frac{1}{(z-a)^2} + a_{-3}\frac{1}{(z-a)^3} + \cdots a_{-n}\frac{1}{(z-a)^n}$

<u>Ex</u>:-

$f(z)= \frac{e^{z^2}}{z^3}$

$= \frac{1+z^2+(z^2)^2/2!+ \cdots}{z^3}$

$= 1/z^3 + 1/z + z/2! + \cdots$

$z=0$ pole of order 0

Ex: $f(z)=\dfrac{1-\cos z}{z^3}$

$$= \frac{1-\left[1-z^2/2! + \frac{z^4}{4!} + \cdots\right]}{z^3}$$

$$= \frac{1}{z\,2!} - \frac{z}{4!} + \cdots$$

$z=0$ is a pole of order 1 $\quad$ $z=0$ is a removable singularity

Essential Singularity:-

In the Expansion of the function if it contains infinite no of terms in the principle part then the singularity is called Essential singularity

$f(z) = e^{\frac{1}{(z+2)}}$

$$= 1+\frac{1}{(z+2)} + \frac{1}{(z+2)^2} + \frac{1}{(z+2)^3} + \cdots$$

$z=-2$ is called essential singularity.

isolated essential singularity:-

The limit points of zero's is called isolated essential singularity

Ex: $f(z) = \sin\left(\frac{1}{z-a}\right)$

Zeros: $\sin\left(\frac{1}{(z-a)}\right) = 0 = \sin(n\pi)$

$$\frac{1}{(z-a)} = n\pi \implies (z-a) = \frac{1}{n\pi}$$

$$z = a+\frac{1}{n\pi}$$

Zero's: $a+1/\pi,\ a+1/2\pi,\ a+1/3\pi \cdots$

as $n \to \infty$

$$z = a + \frac{1}{\infty\pi} = a$$

$z=a$ is called ~~Non~~ isolated essential singularity

__Non isolated essential singularity:-__

The limit point of poles is called Non-isolated essential singularity

__Ex:__ $$f(z) = \frac{1}{\sin\left(\frac{1}{z-a}\right)}$$

poles: $\sin\left(\frac{1}{z-a}\right) = 0 = \sin(n\pi)$

$$\frac{1}{(z-a)} = n\pi \Rightarrow (z-a) = \frac{1}{n\pi}$$

$$z = a + \frac{1}{n\pi} \quad \forall\, n = 1, 2, 3 \ldots$$

poles: $z = a + \frac{1}{\pi},\ a + \frac{1}{2\pi},\ a + 1/3\pi \ldots$

as $n \to \infty$

$$z = a + \frac{1}{n\pi}$$

$z = a + \frac{1}{\infty} = a$ is called as Non isolated essential singularity

* $$f(z) = \frac{e^z}{(z-3)^2(z^2+4)}$$

$z = 3$ is pole of order 2

$z^2+4=0 \Rightarrow z = \pm 2i$ are called as simple poles

ii) $f(z) = \frac{\cdot}{\sin z - \cos z}$ at $z = \pi/4$

At $z = \pi/4$ $\sin\pi/4 - \cos\pi/4 = 0$.

$z = \pi/4$ is a poleof order 1

iii) $f(z) = \frac{1-e^{2z}}{z^4}$ at $z = 0$

$$f(z) = \frac{1-\left(1+(2z)+\frac{(2z)^2}{2!}+ - - \right)}{z^4}$$

$$= \frac{-2z - 2z^4 + -}{z^4}$$

$$= -2/z^3 - 2 + ——$$

Hence from the above Expansion it is a pole of order 3.

iv) $f(z) = \frac{z-1}{z^2+1} = \frac{(z-1)}{(z+i)(z-i)}$

$z = \pm i$ are the singolarities of the given function

v) $f(z) = \frac{z^2-1}{(z+1)(z-1)^3} = \frac{\cancel{(z+1)}(z-1)}{\cancel{(z+1)}(z-1)^3} = \frac{1}{(z-1)^2}$

$z = -1$ is the Removable singularity

$z = 1$ is the pole of order (2)

vi) $f(z) = \frac{1}{1-e^z}$ at $z = 2\pi i$

$$f(z) = \frac{1}{\cancel{1}-\cancel{1}-z-z^2/2! - z^3/3! - ——}$$

$$f(z) = \frac{1}{-z-z^2/2! - z^3/3!}$$

$1-e^{2\pi i} = 1-[\cos 2\pi + i\sin 2\pi] = 1-[1+0] = 0$

Then $z=2\pi i$ is the pole of order (1)

$$f(z) = \frac{1}{z(e^z-1)} \text{ at } z=0$$

$$f(z) = \frac{1}{z(\cancel{1}+z+\frac{z^2}{2!}+\cdots-\cancel{1})}$$

$$f(z) = \frac{1}{(z^2+z^3/3!+\cdots)}$$

Hence $z=0$ is a pole of order (2)

* $f(z) = \sin\left(\frac{1}{1-z}\right)$

$$f(z) = \frac{1}{(1-z)} - \frac{1}{\frac{(1-z)^3}{3!}} + \cdots$$

Then it is isolated singularity.

Zero's: $\sin\left(\frac{1}{1-z}\right) = 0$

$$\frac{1}{(1-z)} = n\pi$$

$$(1-z) = \frac{1}{n\pi}$$

$$z = 1-\frac{1}{n\pi}$$

Zero's: $z = 1-\frac{1}{\pi},\ 1-1/2\pi,\ 1-1/3\pi \cdots$

as $n\to\infty$

$z = 1-\frac{1}{\infty\pi} = 1 \longrightarrow$ It is a isolated essential singularity

Residue:-

The co-efficient of $\frac{1}{(z-a)}$ in the Laurent's series expansion of the function $f(z)$ about the point $z=a$ is called Residue of the function at $z=a$

If $f(z)$ is a complex function with $z=a$, as pole of order 1 then

$$\boxed{\text{Residue of } f(z) \text{ at } z=a = \lim_{z\to a} (z-a)f(z)}$$

If $z=a$ is a pole of order 2

[Then Residue of the function $f(z)$] at $z=a$

$$= \lim_{z\to a} \frac{d}{dz}\left[(z-a)^2 f(z)\right]$$

If $z=a$ is pole of order (n)

$$\left[\text{Residue of } f(z)\right] = \frac{1}{(n-1)!} \lim_{z\to a} \frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]$$

$$f(z) = \frac{e^{2z}}{(z-1)^3} \text{ at } z=1$$

$$[\text{Res}]_{z=1} = \frac{1}{2!} \lim_{z\to 1} \frac{d^2}{dz^2}\left[\cancel{(z-1)^3} \frac{e^{2z}}{\cancel{(z-1)^3}}\right]$$

$$= 1/2 \lim_{z\to 1} 4e^{2z} = 2e^2$$

* $f(z) = \frac{1}{(z^2+1)^2}$ at $z=i$

$z=i$ is the pole of order 2

$$z^2+1=(z+i)(z-i)$$

$$f(z)=\frac{1}{(z-i)^2(z+i)^2}$$

$$[Rs]_{z=i}=\underset{z\to i}{Lt}\ \frac{d}{dz}\left[\cancel{(z-i)^2}\ \frac{1}{(z+i)^2\cancel{(z-i)^2}}\right]$$

$$=\underset{z\to i}{Lt}\ \frac{-2}{(z+i)^3}=\frac{-2}{(i+i)^3}=\frac{-2}{8i^3}=-i/4$$

* $$\frac{1-2z}{z(z-1)(z-2)}$$

$$[Rs]_{z=0}=\underset{z\to 0}{Lt}\ \frac{\cancel{(z-0)}(1-2z)}{\cancel{z}(z-1)(z-2)}=1/2$$

$$[Rs]_{z=1}=\underset{z\to 1}{Lt}\ \frac{\cancel{(z-1)}(1-2z)}{z\cancel{(z-1)}(z-2)}=1$$

$$[Rs]_{z=2}=\underset{z\to 2}{Lt}\ \frac{\cancel{(z-2)}(1-2z)}{z(z-1)\cancel{(z-2)}}=-3/2$$

* $$\frac{1}{(z+2)^2(z-2)^2}\ \text{at}\ z=2$$

$$\underset{z\to i}{Lt}\ \frac{d}{dx}\left[\cancel{(z-2)^2}\ \frac{1}{(z+2)^2\cancel{(z-2)^2}}\right]$$

$$\underset{z\to i}{Lt}\ \frac{-2}{(z+2)^3}=-1/32$$

* $f(z) = e^{\frac{1}{z-a}}$ at $z=a$.

$$1 + \frac{1}{(z-a)} + \frac{1}{(z-a)^2}\frac{1}{2!} + \text{------}$$

$$\left[\text{Residue of } e^{\frac{1}{z-a}}\right]_{\text{at } z=a} =$$

co-efficient of $\frac{1}{(z-a)}$ in the expansion of $e^{\frac{1}{z-a}}$

$= 1$

* $f(z) = e^{z/z-2}$ at $z=2$

find $(z-2)_1 + \frac{z}{(z-2)} + \frac{z^2}{(z-a)^2}\frac{1}{2!} + \text{--}$

$(z-2) + z(z-2)$

Should not Expand

Not in Laurent's form

$$e^{\frac{z}{z-2}} = e^{\frac{z-2+2}{z-2}} = e^{1+\frac{2}{(z-2)}}$$

$$e^{\frac{z}{z-2}} = e^1 \cdot e^{2/z-2}$$

$$= e\left[1 + \frac{2}{(z-2)} + \frac{(2)^2}{(z-2)^2}\frac{1}{2!} + \text{------}\right]$$

$$\left[\text{Residue of } e^{\frac{z}{z-2}}\right]_{z=2} = 2e$$

co-efficient of $\frac{1}{(z-2)}$ in Expansion $e^{\frac{z}{z-2}}$

* $f(z) = (z-3)\sin\frac{1}{(z+2)}$ at $z=-2$

$$(z+2) = u$$

$$z = u-2$$

$$(z-3)\sin\frac{1}{(z+2)} = (u-2-3)\sin(1/u)$$

$$= (u-5)\sin(1/u)$$

$$= [u-5]\left[\frac{1}{u} - \frac{1}{u^3}(1/3!) + \frac{1}{u^5}\frac{1}{5!} + \cdots\right]$$

$$= 1 - 5/u - \frac{1}{u^2 3!} + \frac{5}{3!}\frac{1}{u^3} + \cdots (z-3)\sin\frac{1}{(z}$$

$$= 1 - 5/u - \frac{1}{u^2 3!} + \frac{5}{3!}\frac{1}{u^3} \cdots$$

$$\boxed{[\mathrm{Res}]_{z=-2} = -5.}$$

* $\frac{z - \sin z}{z}$ at $z=0$ 3

$$\frac{1}{z^2} - \frac{\sin z}{z^3} = \frac{z - \sin z}{z^3}$$

$$= \frac{z - \left[z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right]}{z^3}$$

$$= \frac{1}{3!} - \frac{z^2}{5!} + \cdots$$

Residue $=0$

* $f(z) = \frac{1+e^z}{\sin z + z\cos z}$ at $z=0$

$$= 1 + 1 + 1/z + 1/z^2 \ 1/2! + \frac{1}{z^3} \ \frac{1}{3!}$$

$z = 0$ is pole of order 1

$$\text{Res}_{z=0} \ \underset{z \to 0}{Lt} \ \frac{(z-0)(1+e^z)}{\sin z + z\cos z} \qquad (0/0)$$

$$\underset{z \to 0}{Lt} \ \frac{1 + e^z + z e^z}{\cos z + \cos z - z\sin z}$$

$$= \frac{1+1+0}{1+1-0} = 1$$

* $\int_c \frac{z^2}{(z-2)(z-1)^3} \, dz \qquad |z| = 1$

Let $f(z)$ be an Analytic function with in and on a closed curve 'c' except at a finite no of point then

$\int_c f(z) dz = 2\pi i$ (Sum of Residue of $f(z)$ - - -) at each of it pole which lies inside the curve

$\int_c \frac{1+e^z}{\sin z + z\cos z} \, dz$ at $|z| = 1$ is $2\pi i$

$\int_c e^{1/z} \, dz$ where 'c' is the simple closed curve along the origin

$z = 0$ is singularity

$1 + 1/z + 1/z^2 \; 1/2!$

$[Res]_{z=0} = 1$ is $2\pi i$

**

$$\int_c \frac{e^z}{(z^2+1)}\,dz \qquad |z| = 2$$

$(z+i)(z-i)$ $\quad z = i, -i$ are simple poles.

$|z| = 2$

$$\underset{z \to i}{Lt} \; \cancel{(z-i)} \frac{e^z}{\cancel{(z-i)}(z+i)}$$

$$\underset{z=i}{at} = \frac{e^i}{2i} = \underset{z=-i}{at} \quad \frac{-e^{-i}}{2i}$$

$$2\pi i \left[\frac{e^i}{2i} - \frac{e^{-i}}{2i}\right]$$

$$= \left(\frac{e^i - e^{-i}}{2i}\right) 2\pi i = \pi(e^i \; {-i}]$$

**

$$\int_c \left[\frac{z^2+1}{z(2z+1)}\right] dz \qquad |z| = 1$$

$z = 0, -1/2 \in |z| = 1$ simple

$$[Res]_{z=0} = \underset{z \to 0}{Lt} \frac{(z-0)(z^2+1)}{z(2z+1)} = 0$$

$$[Res]_{z=-1/2} = \underset{z \to -1/2}{Lt} \frac{(z+1/2)(z^2+1)}{2z(z+1/2)}$$

$= \frac{5/4}{-1} = -5/4$

$= 2\pi i\,[1-5/4] = i\pi/2$

$\int_c \tan z\, dz$ where $|z|=2$

$$\int_c \frac{\sin z}{\cos z}\, dz$$

$\cos z = 0$

$z = \pi/2,\ -\pi/2,\ 3\pi/2,\ -3\pi/2 \ \ldots$

only $z = \pi/2,\ -\pi/2 \in |z|=2$

poles of order 1

$$(\mathrm{Res})_{z=\pi/2} = \lim_{z\to\pi/2} (z-\pi/2)\frac{\sin z}{\cos z}$$

$$= \lim_{z\to\pi/2} \frac{(\sin z) + (z-\pi/2)\cos z}{-\sin z}$$

$$= \frac{1+0}{-1} = -1$$

$z = -\pi/2$

$\int_c \tan z\, dz = 2\pi i\,\{-2\} = -4\pi i$

$$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)}\, dx$$

1) $f(x)$ — (2)

2) $F(x) \neq 0$

$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} dx = 2\pi i$ { Sum of Residues of the function $\frac{f(Z)}{F(Z)}$ inside the upper half pl

Integral

$\int_{-\infty}^{\infty} \frac{fx}{Fx} dx$ is the Rational function of (x)

Satisfying

① F(x) degree should be minimum '2' units more than the degree of f(x)

2) F(x) should not be zero for any real value then

$\int_{-\infty}^{\infty} \frac{f(x)}{F(x)} = 2\pi i$ { Sum of Residues of the function $\frac{f(Z)}{F(Z)}$ at Each of its poles which lies in the upper half plane

3

$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$

$\frac{f(Z)}{F(Z)} = \frac{1}{Z^2+1}$ The poles are $Z = \pm i$

only $Z = i \in$ upper half plane

$[Res]_{Z=i} = \lim_{Z \to i} \cancel{(Z-i)} \frac{1}{\cancel{(Z-i)}(Z+i)}$

$= \frac{1}{2i}$

$$\int_{-\infty} \frac{1}{1+x^2}\,dx = 2\pi\,[1/2i] = \pi.$$

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} \qquad x = \pm i$$

$$\mathrm{Res}_{x=i} = \frac{d}{dx}\,\cancel{(x-i)^2}\,\frac{1}{(x+i)^2\cancel{(x-i)^2}}$$

$$= \frac{d}{dx}\left(\frac{1}{x+i}\right)^2$$

$$x = i \Longrightarrow \quad \frac{-1}{(i+i)^3} = \frac{-1}{(2i)^3} = -1/8i = i/8$$

$$= 2\pi i\,(i/8) = -\pi/4$$

Expand $f(z) = \dfrac{1}{(z-1)(z-2)}$ in the region $|z|<1$

(2) $1<|z|<2$ 3) $|z|>2$

$$[1+Q(z)]^{-1} = 1 - Q(\;) + \{Q(z)]^2 + \text{----} \quad z$$

$$\{1-Q(z)]^{-1} = 1 + Q(z) + (Q(z))^2 + \text{----}$$

(1) $|z|<1$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$\frac{1}{-2(1-z/2)} = -1/2\,(1-z/2)^{-1}$$

$$= -1/2\,[\,1 + (z/2) + (z/2)^2 + \text{--}\,]$$

$$= \frac{1}{-2(1-z/2)} - \frac{1}{-(1-z)} = -1/2\,[1-z/2]^{-1} + (1-z)^{-1}$$

$|z|<1 \quad \therefore\ |z/2|<1$

$$= -1/2\,[1+z/2+(z/2)^2+\cdots] + [1+z+z^2+\cdots]$$

$$= 1/2 + 3z/4 + 7z^2/8 + \cdots$$

(2) $\quad 1<|z|<2 \qquad |z|<2$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{-2[1-z/2]} - \frac{1}{z[1-1/z]}$$

$$= -\frac{1}{2}[1-z/2]^{-1} - 1/z\,[1-1/z]^{-1}$$

$$= -1/2\,[1+z/2+\frac{z^2}{2^2}+\cdots] - 1/z\,[1+1/z+1/z^2\cdots$$

$$f(z) = \frac{1}{(z-2)} - \frac{1}{(z-1)}$$

$$= \frac{1}{z(1-2/z)} - \frac{1}{z(1-1/z)}$$

$$= 1/z\,[1-2/z]^{-1} - 1/z\,[1-1/z]^{-1}$$

$$= 1/z\,[1 + 2/z + (2/z)^2 + \cdots] - 1/z\,[1 + 1/z + 1/z^2 + \cdots]$$

$$= 1/z^2 + 3/z^3 + \cdots$$

* $\dfrac{e^{2z}}{(z-1)^3}$ at $z = 1$

Let $z - 1 = u$

$z = u + 1$

$$\frac{e^{2z}}{(z-1)^3} = \frac{e^{2u}\, e^2}{u^3}$$

$$= \frac{e^2}{u^3}\left[1 + 2u + \frac{(2u)^2}{2!} + \frac{(2u)^3}{3!} + \cdots\right]$$

$$= e^2\left[\frac{1}{u^3} + \frac{2}{u^2} + \frac{2}{u} + \frac{4}{3} + \frac{2}{3}u\right] \cdots$$

$$= e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{4}{3} + 2/3\,(z-1)\right]$$

Laplace transformation

$$I[f(t)] = \int_{-\infty}^{\infty} f(t)\, k(s,t)\, dt = \overline{f(s)}$$

$k(s,t) \longrightarrow$ Kerner of I.T

①

2) $k(s,t) = e^{ist}$ (or) $e^{-ist} \longrightarrow$ fourier transform

$$L[f(t)] = \int_{-\infty}^{\infty} e^{-st} f(t)\, dt = \overline{f(s)}$$

$\hookrightarrow$ Bilatural L.T

If $f(t)$ is defined $\forall\, t \geq 0$

$$L[f(t)] = \int_{0}^{\infty} e^{-st} f(t)\, dt \longrightarrow \text{unilatural L.T}$$

s may be real (or) complex $\Rightarrow e^{-st}$ is the Kernel of L.T

Laplace of $f(t)$ Exist if $\int_0^{\infty} e^{-st} f(t)\, dt$ is convergent i.e finite.

Necessary condition for Existency:

$$\int_0^{\infty} e^{-st} f(t)\, dt \longrightarrow \text{convergent i.e finite}$$

Sufficient condition for Existency

(i) $f(t)$ is piecewise continous (or) sectionally continous.

→ f(t) is a function of Exponential order

i.e $\boxed{\lim_{t\to\infty} f(t)e^{-st} \longrightarrow 0 \text{ (or) finite.}}$

Properties:-

Linearity property:-

$$L[af(t) \pm bg(t)] = aL\{f(t)\} + bL\{g(t)\}$$

Laplace transform of Some Elementary functions:-

1) $L\{1\} = 1/s \; ; \; s>0$

2) $L[e^{at}] = \frac{1}{s-a} : s>a$

$L[e^{-at}] = \frac{1}{s+a} : s>-a$

4) $L[\sin at] = \frac{a}{s^2+a^2} : s>0$

5) $L[\cos at\} = \frac{s}{s^2+a^2} : s>0$

6) $L[\sinh at\} = \frac{a}{s^2-a^2} : s>|a|$

7) $L\{\cosh at\} = \frac{s}{s^2-a^2} : (s>|a|)$

8) $L\{t^n\} = \begin{cases} \frac{n!}{s^{n+1}} & : n\in Z^+ \\ \frac{\Gamma(n+1)}{s^{n+1}} & : n\notin Z^+ \end{cases}$

$s>0$

$n>-1$

Gamma function:-

1) $\Gamma(n+1) = n\Gamma(n) = n(n-1)(n-2)(n-3) \text{———}$ for +ve values of 'n'

2) $\Gamma(n+1) = n! \; : \; n\in Z^+$

3) $\Gamma(1) = 1$ & $\Gamma(1/2) = \sqrt{\pi}$

4) $\Gamma(n) = \frac{\Gamma(n+1)}{n}$ for n is -ve

5) $\Gamma 0, \Gamma -1, \Gamma -2, \Gamma -3$ --- are undefined.

Frequency shifting of Laplace transform:-

If $L\{f(t)\} = \bar{F}(s)$

① $L\{e^{at}f(t)\} = \bar{F}(s-a)$

2) $L\{e^{-at}f(t)\} = \bar{F}(s+a)$

Second shifting theorem of L.T:-

If $L\{f(t)\} = \bar{F}(s)$ and $G(t) = \begin{cases} f(t-a) & : t \geq a \\ 0 & : t < a. \end{cases}$

$$L\{G(t)\} = e^{-as}\bar{F}(s)$$

Scaling property:-

① $L\{f(at)\} = 1/a \, \bar{F}(s/a)$

2) $L\{f(t/a)\} = a\bar{F}(as)$

Multiplication by t^n $(n \in z^+)$

① $L\{tf(t)\} = -\frac{d}{ds}(\bar{F}(s))$

2) $L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\bar{F}(s))$

division by t^n $(n \in z^+)$

① $L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} \bar{F}(s)\,ds$

2) $L\left\{\frac{f(t)}{t^n}\right\} = \int_s^{\infty}\int_s^{\infty} \cdots \bar{F}(s)\,(ds)^n$

L.T of derivatives

① $L\{f'(t)\} = s\bar{F}(s) - f(0)$

2) $L\{f^{n}(t)\} = s^{n}\bar{F}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{n-1}(0)$

L.T of integrals:-

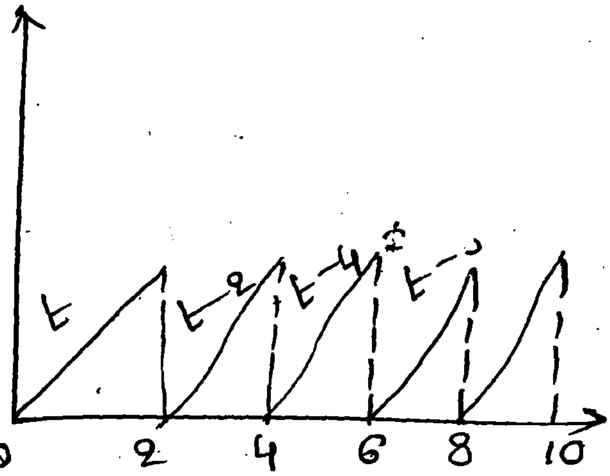
$$L\left[\int_0^t f(t)\,dt\right] = \frac{1}{s}\bar{F}(s)$$

$$L\left[\int_0^t \int_0^t \dots f(t)\,(dt)^n\right] = \frac{1}{s^n}\bar{F}(s)$$

L.T of periodic functions:-

If $f(t) = t \quad 0<t<2$ (periodic function)

then $f(t) = ?\ (t-6)$



If $f(t)$ is a periodic fn with period of T

$f(t+T) = f(t), \ \forall t$

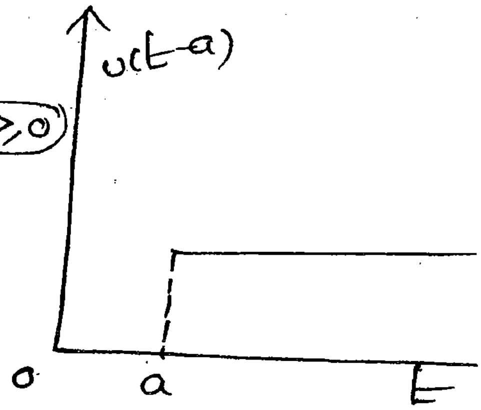
then
$$\boxed{L\{f(t)\} = \frac{\int_0^T e^{-st} f(t)\,dt}{1-e^{-sT}}}$$

$$\frac{\int_0^T e^{-st} f(t)\,dt}{(1-e^{-Ts})} = (1+e^{-Ts}+e^{-2Ts}+\dots)$$

Unit Step function:- (Heaviside's function)

It is denoted by $u(t-a)$ (or) $H(t-a)$

$$u(t-a) = \begin{cases} 1 \; ; & t \geq a \\ 0 \; ; & t < a \end{cases} \quad (a \geq 0)$$



* ① $L\{u(t-a)\} = e^{-as}/s$.

* 2) $L\{u(t)\} = 1/s$.

* (3) $L\{f(t-a)\,u(t-a)\} = e^{-as}\,L\{f(t)\}$

4) $\boxed{L\{f(t)\,u(t-a)\} = e^{-as}\,L\{f(t+a)\}}$ $f(t); \; t \geq a$; $0 : t < a$

* 5) $f(t) = \begin{cases} f_1(t) \; ; & 0 \leq t < a \\ f_2(t) \; ; & t \geq a \end{cases}$

$f(t) = f_1(t) + [f_2(t) - f_1(t)]\,u(t-a)$, $0 \leq t < a$

$= \cancel{f_1(t)} + [f_2(t) - \cancel{f_1(t)}]\,(1)$

$= f_2(t)$ $t \geq a$

(6) $f(t) =$ $f_1(t) : \; 0 \leq t \leq a_1$

$f_2(t) : \; a_1 \leq t \leq a_2$

$f_3(t) : \; a_2 \leq t < a_3$

|

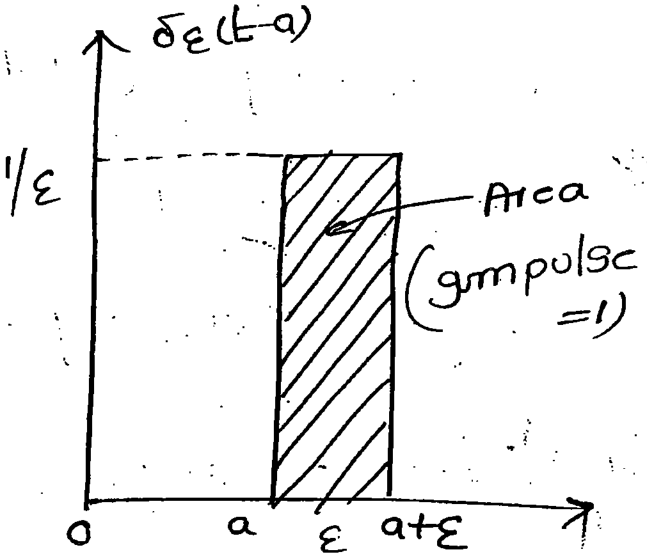
$f_n(t) =$ $t \geq a_{n-1}$

$$\boxed{f(t) = f_1(t) + [f_2(t) - f_1(t)]\, u(t-a_1) + [f_3(t) - f_2(t)]\, u(t-a_2) + \cdots + [f_n - f_{n-1}]\, u(t-a_{n-1})}$$

Unit impulse / Dirac's delta function

$$\delta(t-a) = \lim_{\varepsilon \to 0} \left[\delta_\varepsilon(t-a)\right]$$

$$= \lim_{\varepsilon \to 0} \begin{cases} 1/\varepsilon & : \quad a \le t \le a+\varepsilon \\ 0 & : \quad \text{otherwise.} \end{cases}$$



$$= \begin{cases} \infty \,; & t = a \\ 0 \,; & t \ne a \end{cases} \quad \text{such that}$$

$$\boxed{\int_0^\infty \delta(t-a)\, dt = 1}$$

1) $L[\delta(t-a)] = e^{-as}$

2) $L[\delta(t)] = 1$

3) $L\{f(t)\, \delta(t-a)\} = f(a) e^{-as}$

4) $L\{u'(t-a)\} = L[\delta(t-a)]$.

5) $\int_{-\infty}^{\infty} f(t)\delta(t-a)\, dt = f(a)$

Every discontinuous function will be ... in the unit step function and then converted in time domain

* $L[t^{7/2} + t^{-1/2} + t^{3}\}$

$$= \frac{\Gamma(7/2+1)}{s^{7/2+1}} + \frac{\Gamma(-1/2+1)}{s^{-1/2+1}} + \frac{\Gamma(3+1)}{s^4}$$

$$= \frac{7/2 \cdot 5/2 \cdot 3/2 \cdot 1/2\, \Gamma(1/2)}{s^{9/2}} + \frac{\Gamma(1/2)}{s^{1/2}} + \frac{6}{s^4}$$

$$= \frac{105 \cdot \sqrt{\pi}}{s^{9/2}} + \sqrt{\pi/s} + 6/s^4$$

$$\Gamma(-1/2) = \frac{\Gamma(-1/2+1)}{-1/2} = -2\,\Gamma(1/2) = -2\sqrt{\pi}$$

$$\Gamma(-3/2) = \frac{\Gamma(-3/2+1)}{-3/2} = -2/3\,\Gamma(-1/2) = 4/3\sqrt{\pi}$$

$$\boxed{\begin{aligned} \Gamma(-1/2) &= -2\sqrt{\pi} \\ \Gamma(-3/2) &= 4/3\sqrt{\pi} \end{aligned}}$$

* $L\{1/t\} = L[t^{-1}]$ does not Exist

* $L\{1/t^2\} =$ does not Exist

* $L\left[\left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2\right] = L(t + 1/t - 2)$ ← does not Exist

* $L\left[\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3\right] = L[\, t^{3/2} + t^{-3/2} + 3t^{-1/2} + 3t^{1}$

$= 1/2 \; L\{2\sin 3t \cos 5t\}$

$= 1/2 \; L\{\sin(3t+5t) + \sin(-2t)\}$

$= 1/2 \; L\{\sin(8t) + \sin(-2t)\}$

$= 1/2 \; L\{\sin(8t) - \sin(2t)\}$

$= 1/2 \left[\frac{8}{s^2+64} - \frac{2}{s^2+4}\right]$ GL

$= 1/2 \left[\frac{8s^2 - 2s^2 + 32 - 128}{(s^2+64)(s^2+4)}\right]$

$= 1/2 \left[\frac{6s^2 - 96}{(s^2+64)(s^2+4)}\right]$

$L\{\cos t \cos 2 \cdot \cos 3t\}$

$= 1/2 L\{2\cos 2t \cos 3t \cos t\}$

$= 1/2 L\{[\cos(t+2t) + \cos(-t)] \cos 3t\}$

$= 1/2 [(\cos 3t + \cos t] \cos 3t]$

$= 1/2 L[(\cos^2 3t + \cos t \cos 3t]$

$= 1/4 \; L\{1 + \cos 6t + \cos 4t + \cos 2t\}$

$$= 1/4 \left[1/s + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4} \right]$$

** $2\sin^2 t = 1 - \cos 2t$

$2\cos^2 t = 1 + \cos 2t$

$2\sinh^2 t = \cosh 2t - 1$

$2\cosh^2 t = \cosh 2t + 1$

$\sin 3t = 3\sin t - 4\sin^3$
$\cos 3t = 4\cos^3 t - 3\cos t$
$\sinh 3t = 3\sinh t + 4\sinh^3 t$
$\cosh 3t = 4\cosh^3 t - 3\cosh t$

*** $L\{\sin^3 2t\}$

$$= L\left\{ \frac{3\sin 2t - \sin 6t}{4} \right\}$$

$$= 1/4 \left[\frac{3 \times 2}{s^2+4} - \frac{6}{s^2+36} \right]$$

$$= \frac{6}{4} \left[\frac{(s^2+36) - (s^2+4)}{(s^2+4)(s^2+36)} \right]$$

$$= 6/4 \left[\frac{32}{(s^2+4)(s^2+36)} \right]$$

** $L[\sin(\omega t + \alpha)]$

$$= L[\sin \omega t \cos\alpha + \cos \omega t \sin\alpha]$$

$$= \left[\cos\alpha \cdot \frac{\omega}{s^2+\omega^2} + \sin\alpha \cdot \frac{s}{s^2+\omega^2} \right]$$

$$= \frac{1}{s^2+\omega^2} \left(\omega\cos\alpha + s\sin\alpha \right)$$

7) $L\{\sin\sqrt{t}\}$

$$\left[\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right]$$

$L\{\sin(t^{1/2})\}$

$L\left[1 - \cancel{\frac{(\sqrt{t})^2}{2!}} \cancel{+} \cancel{\frac{(\sqrt{t})^4}{4!}}\right.$

$= L\left[\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} + \dots\right]$

$= L\left[\sqrt{t} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} + \dots\right]$

$= \frac{\Gamma 1/2+1}{s^{3/2}} - \frac{\Gamma 3/2+1}{3!\, s^{5/2}} + \frac{\Gamma 5/2+1}{5!\, s^{7/2}} + \dots$

$= \frac{1/2\sqrt{\pi}}{s^{3/2}} - \frac{3/2 \cdot 1/2 \cdot \sqrt{\pi}}{3!\, s^{5/2}} + \frac{5/2 \cdot 3/2 \cdot 1/2\sqrt{\pi}}{5!\, s^{7/2}} + \dots$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \frac{1}{4s} + \frac{1}{2!\,(4s)^2} - \frac{1}{3!\,(4s)^3} + \dots\right]$

$= \frac{\sqrt{\pi}}{2s^{3/2}}\, e^{-1/4s} \quad (s>0)$

**

$L[e^{t^2}]$

$L\left[t^2 + \frac{(t^2)^2}{2!} + \frac{(t^2)^3}{3!} + \frac{(t^2)^4}{4!} + \dots\right]$

$L\left[t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \frac{t^8}{4!} + \dots\right.$

$$= \frac{\sqrt{2+1}}{\frac{2!\, s^3}{1!}} + \frac{\sqrt{4+1}}{\frac{4!\, s^5}{2!}} + \frac{\sqrt{6+1}}{\frac{6!\, s^7}{3!}} + \cdots$$

$$= \frac{(\cancel{2})(1)}{\cancel{2}\, s^3} + \frac{(\cancel{4})(\cancel{3})(\cancel{2})}{4!\, s^5} + \frac{(\cancel{6})(\cancel{5})}{s^7} \cdots$$

$$= 1/s + \frac{2!}{s^3} + \frac{4!}{2!\, s^5} + \frac{6!}{s^7 . 3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{n!\, s^{2n+1}} \longrightarrow \text{does not Exist}$$

$\longrightarrow$ divergent series.

** $L[\sinh at \sin at]$

$$= L\left\{ \frac{e^{at} - e^{-at}}{2} \sin at \right]$$

$$= 1/2\, L\left(e^{at}\sin at - e^{-at}\sin at\right)$$

$$= 1/2\left[\frac{a}{(s-a)^2 + a^2} - \frac{a}{(s+a)^2 + a^2}\right]$$

$$= 1/2\left[\frac{a}{(s^2 + a^2 - 2as + a^2)} - \frac{a}{(s^2 + a^2 + 2as + a^2}\right]$$

$$= 1/2\left[\frac{a}{s^2 + 2a^2 - 2as} - \frac{a}{s^2 + 2a^2 + 2as}\right]$$

$$= \frac{a}{\cancel{2}}\left(\frac{\overset{2}{\cancel{4}}as}{(s^2 + 2a^2 - 2as)(s^2 + 2a^2 + 2as)}\right)$$

$$= \frac{2a^2 s}{(s^2 + 2a^2)^2 - (2as)^2} = \frac{2a^2 s}{s^4 + 4a^4}$$

$$L\left[e^{-3t}(3\sin 4t + 5\cos 4t)\right]$$

$$= L\left[3\,e^{-3t}\sin 4t + 5\,e^{-3t}\cos 4t\right]$$

$$= \cancel{L}\left\{3\,\frac{4}{(s+3)^2+16} + \frac{5(s+3)}{(s+3)^2+16}\right]$$

$$= \frac{5s+27}{(s^2+6s+25)}$$

** $L\{f(t)\} = \bar{F}(s)$ Modulation property.

$$L\{f(t)\cos at\}$$

$$L\{f(t)\cos at\} = L\left\{f(t)\left[\frac{e^{iat}+e^{-iat}}{2}\right]\right\}$$

$$= 1/2\left[\bar{F}(s-ia) + \bar{F}(s+ia)\right]$$

* $L[f(t)] = \dfrac{e^{-1/s}}{s.} = \bar{F}(s)$

$$L\left\{\int_0^t e^{-3t} f(3t)\,dt\right]$$

It will involved integral property, shifting and Scaling property.

$$= 1/s\; L\{e^{-3t} f(3t)\}$$

$$= 1/s\left\{L\{f(3t)\}\right]_{s\to s+3} = 1/s\left[\frac{1}{\cancel{3}}.\;\frac{e^{-1/s/3}}{s/\cancel{3}}\right]_{s\to s+3}$$

$$= 1/s \cdot \left[\frac{e^{\cdots}(s+3)}{(s+3)} \right]$$

** $f(t) = \begin{cases} (t-2)^2 & ; t \geq 2 \\ 0 & ; t < 2 \end{cases}$

$f(t) = (t-2)^2 u(t-2)$

$F(s) = ?$

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$= \int_2^{\infty} \cancel{e^{-2s}} (t-2)^2\, dt$$

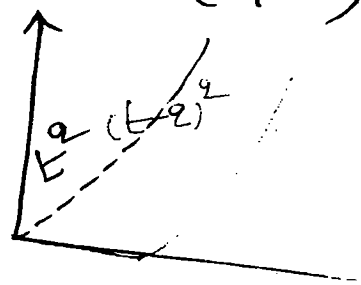
$$= e^{-2s} \frac{(2)}{s^3}.$$

$L[f(t)] =$

$= e^{-2s} L\{(t+\cancel{2}) - \cancel{2})$

$= e^{-2s} L(t^2)$

$= e^{-2s} (2/s^3)$



** $f(t) = \begin{cases} t^2 ; & t \geq 2 \\ 0 ; & t < 2 \end{cases}$

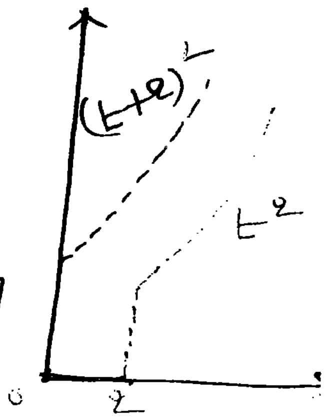
$f(t) = t^2 u(t-2)$

$$L[f(t) u(t-a)] = e^{-as} L[f(t+a)]$$

$$L[f(t)] = e^{-2s} L[\{(t+2)^2]$$

$$= e^{-2s} L[t^2 + 4t + 4]$$

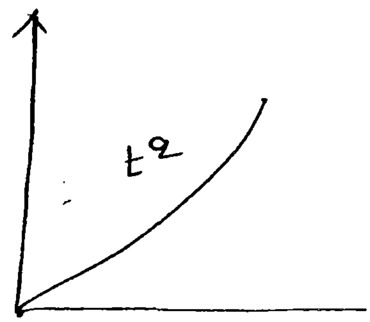
$$= e^{-2s} [2/s^3 + 4/s^2 + 4/s].$$



* $f(t) = \begin{cases} t^2 ; & t \geq 0 \\ 0 ; & t < 0. \end{cases}$

$f(t) = t^2 u(t)$

$$\boxed{F(s) = 2/s^3}$$

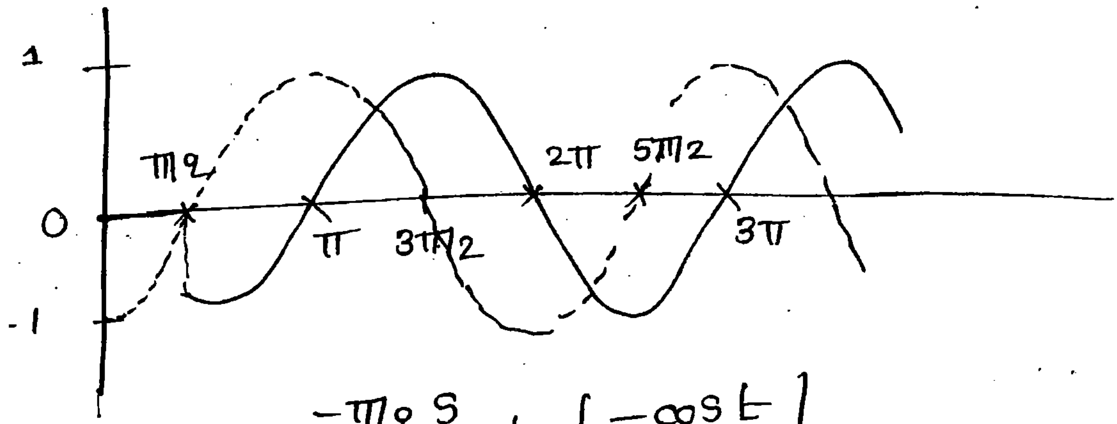


* $L[4\sin(t-3)\,u(t-3)]$

$= e^{-3s} L\{4\sin(t-\cancel{3}+\cancel{3})\}$

$= e^{-3s} L\{4\sin t\}$

$= 4e^{-3s}\left(\frac{1}{s^2+1}\right)$



$= e^{-\pi/2\, s} L[-\cos t]$

$= e^{-\pi/2\, s}\left(\frac{-s}{s^2+1}\right)$

$f(t) = -\sin t\; u(t-\pi/2)$

$L[f(t)\} = -e^{-\pi/2\, s} L[\sin(t+\pi/2)]$

$= -e^{-\pi/2\, s}\left(\frac{s}{s^2+1}\right)$

* $f(t) = \begin{cases} \sin t \; ; \; t \geq \pi \\ \cos t \; ; 0 < t < \pi \end{cases}$

~~$f(t) = \sin t$~~ $u(t)$

$f(t) = \cos t + [\sin t - \cos t]\,u(t-\pi)$

$$L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s} L[\underbrace{\sin(t+\pi)}_{-\sin t} - \underbrace{\cos(t+\pi)}_{-\cos t}]$$

$$= \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{-1}{s^2+1} + \frac{s}{s^2+1}\right]$$

* $f(t) = |t+1| + |t-1|$; $t \geq 0$

$|x| = x$; $x \geq$
$= -x$; $x <$

$$f(t) = |t+1| + |t-1|$$

$$f(t) = |t-1| + (|t+1| - |(t-1)|)\, u(t)$$

$$f(t) = \begin{cases} (t+1) - (t-1) & 0<t<1 \\ t+1+t-1 & t \geq 1 \end{cases}$$

$$f(t) = \begin{cases} 2 \;; & 0<t<1 \\ 2t \;; & t \geq 1 \end{cases}$$

$$f(t) = 2\, u(t) + 2t\, u(t-1)$$

$$f(t) = 2 + [2t-2]\, u(t-1)$$

$$F(s) = 2/s + e^{-s} L[2(t+1)-2]$$

$$= 2/s + e^{-s} L[2t]$$

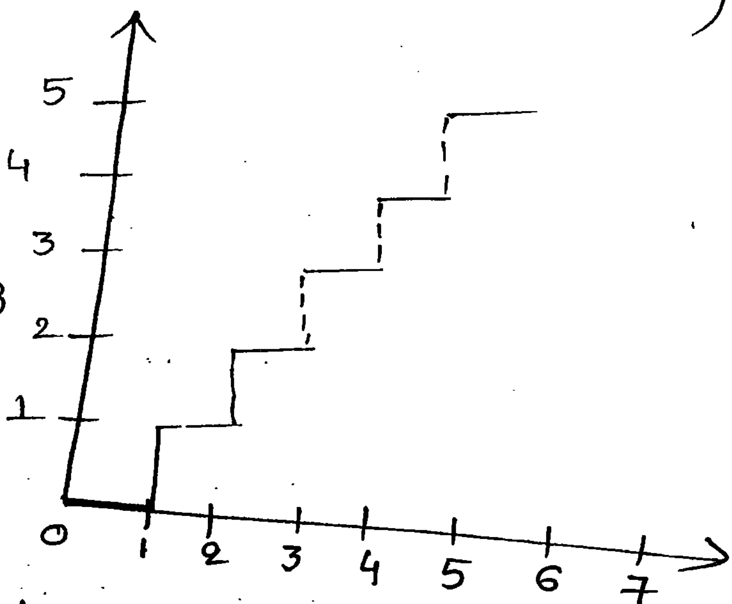
$$= 2/s + e^{-s}(2)(1/s^2)$$

$$F(s) = 2/s\left[1 + \frac{e^{-s}}{s}\right]$$

Staircase/ Step/ Bracket function. – (Integral function)

$L\{[t]\} =$

$$[t] = \begin{cases} 0 & 0 \le t < 1 \\ 1 & 1 \le t < 2 \\ 2 & 2 \le t < 3 \\ \vdots & \end{cases}$$



It is an infinite function.

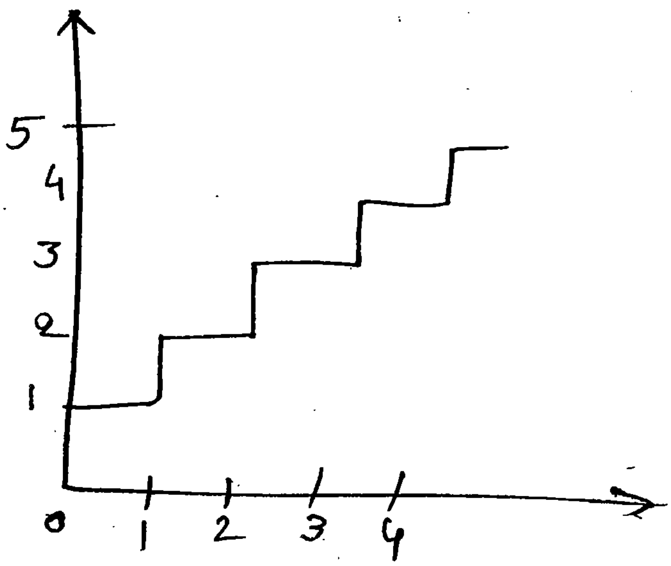
$$f(t) = u(t-1) + u(t-2) + u(t-3) + \cdots$$

$$F(s) = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} + \cdots$$

$$= 1/s \left(e^{-s} + e^{-2s} + e^{-3s} + e^{-4s} + \cdots\right)$$

$$= \frac{e^{-s}}{s}\left(1-e^{-s}\right)^{-1} = \frac{e^{-s}}{s(1-e^{-s})}$$

$$[t] = \begin{cases} 1 & 0 < t \le 1 \\ 2 & 1 < t \le 2 \\ 3 & 2 < t \le 3. \\ \vdots & \end{cases}$$

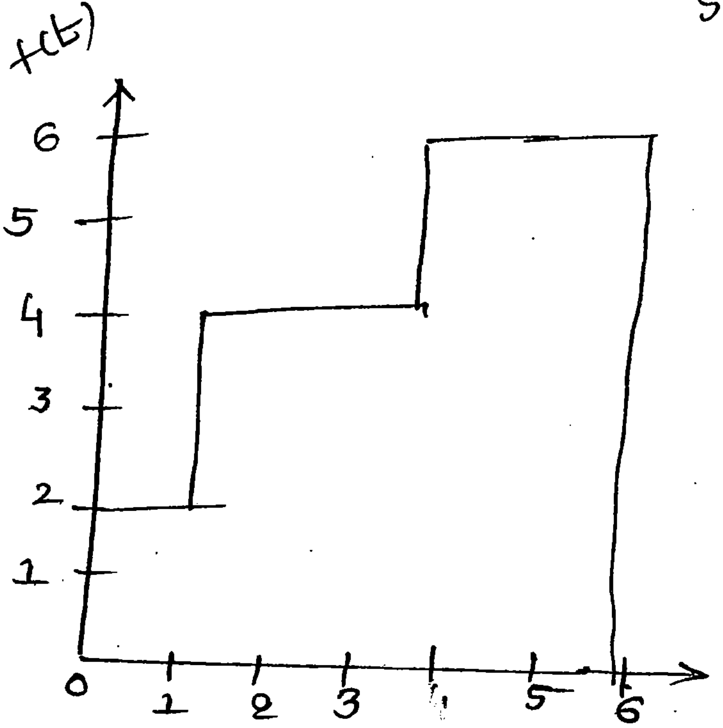


$$F(t) = u(t) + u(t-1) + u(t-2) + u(t-3) + \text{——}$$

$$F(S) = 1/s + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} + \text{——}$$

$$= 1/s \, (1+e^{-s}+e^{-2s}+ \text{——})$$

$$= 1/s \, (1-e^{-s})^{-1} = \frac{+1}{s\,(1-e^{-s})}$$



$$f(t) = 2u(t) + 2u(t-1) + 2u(t-4) - 6u(t-6)$$

$$F(S) = 2/s + \frac{2e^{-s}}{s} + \frac{2e^{-4s}}{s} - \frac{6e^{-6s}}{s}.$$

* $L[t^2 \sin at]$

$$= (-1)^2 \frac{d^2}{ds^2}\left[\frac{a}{s^2+a^2}\right]$$

$$= \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= -2a\left(\frac{d}{ds}\,\frac{s}{(s^2+a^2)^2}\right)$$

$$= \frac{(-2a)\left[(s^2+a^2)^2(1) - s \cdot 2(s^2+a^2)\, 2s\right]}{(s^2+a^2)^4}$$

$$= (-2a)(s^2+a^2)\left[\frac{s^2+a^2-4s^2}{(s^2+a^2)^4}\right]$$

$$= \frac{(-2a)\left[a^2-3s^2\right]}{(s^2+a^2)^3}$$

$\div$ $L[t\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$

$$= \frac{s^2-a^2}{(s^2+a^2)^2}$$

* $L[t^6 e^{-3t}]$

$$= (-1)^6 \frac{d^6}{ds^6}\left[\frac{1}{s+3}\right]$$

$$= (1)\frac{d^5}{ds^5}\left[\frac{1}{(s+3)^2}\right]$$

$L[t^6] = \frac{6!}{s^7}$

first shifting property

$L[t^6 e^{-3t}] = \frac{6!}{(s+3)^6}$

* $L[t^3 \sin at]$

$L[t^3] = \frac{3!}{(s^4)}$

$$L[t^3 \sin at] = \frac{3!\, a}{(s^2+a^2)^4}$$

* $L[te^{-3t}\cos 2t]$

consider $L(t\cos 2t) = \dfrac{s^2-4}{(s^2+4)^2}$

First shifting property

$$L[te^{-3t}\cos 2t] = \frac{(s+3)^2-4}{[(s+3)^2+4]^2}$$

* $L\left[\dfrac{t^{n-1}}{1-e^{-t}}\right]$

$$= L[t^{n-1}(1-e^{-t})^{-1}]$$

$$= L[t^{n-1}](1+e^{-t}+e^{-2t}+e^{-3t}+ \text{---})$$

$$L\left[\sum_{k=0}^{\infty} e^{-kt}\, t^{n-1}\right]$$

$\downarrow$

first shifting property

$$\sum_{k=0}^{\infty} \frac{\Gamma n}{(s+k)^n}$$

* $L\left[\frac{\sin at}{t}\right]$

$$= \int_s^\infty \frac{a}{(s^2+a^2)}\, ds = \tan^{-1}(s/a)\Big|_s^\infty$$

$$= \tan^{-1}(\infty) - \tan^{-1}(s/a)$$

$$= \pi/2 - \tan^{-1}(s/a) \quad \text{(or)}$$

$$\cot^{-1}(s/a) \quad \text{(or)} \quad \tan^{-1}(a/s)$$

* $L\left[\frac{\cos at}{t}\right] \longrightarrow$ undefined does not Exist

$$\int_s^\infty \frac{s}{(s^2+a^2)}\, ds = 1/2\left[\log(s^2+a^2)\right]_s^\infty$$

Similarly $L\left[\frac{e^{-at}}{t}\right] \longrightarrow$ does not Exists

$L[1/t] \longrightarrow$ does not Exists.

* $L\left[\frac{e^{-at}-e^{-bt}}{t}\right]$

$$= \int_s^\infty \left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

Indetermination form

$$\left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{s+b}\right)\right]_s^\infty$$

$$= \log\left[\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right]_s$$

$$= -\log\left(\frac{1+a/s}{1+b/s}\right)$$

$$= \log\left(\frac{s+b}{s+a}\right)$$

* $L\left[\frac{\cos at - \cos bt}{t}\right]$

$$\int_s^{\infty} \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$$

$$\frac{1}{2}\log\left[\frac{s^2+b^2}{s^2+a^2}\right]$$

* $L\left[\frac{e^{-at}+e^{-bt}}{t}\right] \longrightarrow$ undefined doesnot Exist

* $L\left[\frac{\cos at + \cos bt}{t}\right] \longrightarrow$ doesnot Exist

* $L\left[\frac{\sin^2 t}{t}\right]$

$$L\left[\frac{1-\cos 2t}{2t}\right]$$

$$= \frac{1}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= \frac{1}{2}\log\left[\frac{s^2+2^2}{s^2}\right]$$

$L\left[\frac{\cos t}{t}\right] \rightarrow$ does not Exists

$1/2\, L\left[\frac{\sin t \sin 4t}{t}\right]$

$\left(\frac{\sin 5t}{t} - \frac{\sin 3t}{t}\right) \rightarrow$ Exist

$L[\sin\sqrt{t}]$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} e^{-1/4s}.$$

$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

$$f(t) = \sin\sqrt{t}$$

$$f'(t) = \frac{\cos\sqrt{t}}{\sqrt{t}}$$

$$L[f'(t)] = s\,L f(t) - f(0)$$

$$L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = \frac{s\sqrt{\pi}\, e^{-1/4s}}{2s^{3/2}} - (0)$$

$$= \sqrt{\pi/s}\; e^{-1/4s}.$$

* $L[f(t)] = \frac{e^{-1/s}}{s}$ then

find $L\left[\int_0^t e^{-3t} f(3t)\, dt\right] = ?$

Integral

$= 1/s \; L\left[e^{-3t} f(3t) dt \right]$

$= 1/s \; L\left[L\left(f(3t) \right) \right]_{s \to s+3}$

$= 1/s \left[1/3 \; \dfrac{e^{-3/s}}{s/3} \right]_{s \to s+3}$

$= 1/s \; \dfrac{e^{-3/s+3}}{(s+3)}$

** $L\left[\int_0^t e^{-t} \dfrac{\sin t}{t} dt \right]$

$= \frac{1}{s} L\left[e^{-t} \dfrac{\sin t}{t} dt \right]$

$= 1/s \; L\left[\left[L \left(\dfrac{\sin t}{t} \right) \right] \right]_{s \to s+1}$

$= 1/s \left[\cot^{-1} s \right]_{s \to s+1}$

$= 1/s \; \cot^{-1}(s+1)$

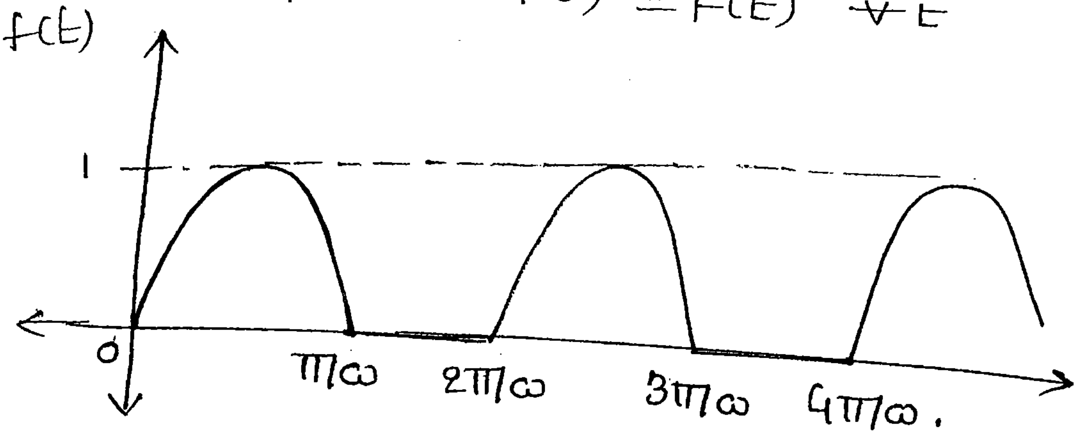
** $L\left[e^{-t} \int_0^t \dfrac{\sin t}{t} dt \right]$

$= \dfrac{1}{s+1} \; \cot^{-1}(s+1)$

*

$$f(t) = \begin{cases} \sin \omega t & 0 \le t < \pi/\omega \\ 0 & \pi/\omega \le t \le 2\pi/\omega \end{cases}$$

$f(t+2\pi/\omega) = f(t) \quad \forall t$



$$F(s) = \frac{\int_0^{\pi/\omega} e^{-st} \sin\omega t \, dt}{(1-e^{-s2\pi/\omega})}$$

$$\boxed{\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]}$$

$$\boxed{\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]}$$

$$= \frac{\frac{e^{-st}}{s^2+\omega^2}\left[(-s)(0) - \omega(-1)\right] - \frac{1}{s^2+\omega^2}(-s/0) - \omega(t)]}{[1^2 - e^{-2s\pi/\omega}]}$$

$$= \frac{\omega}{s^2+\omega^2} \frac{\cancel{1+e^{-\pi/\omega s}}}{(1-e^{-s\pi/\omega})\cancel{(1+e^{-s\pi/\omega})}}$$

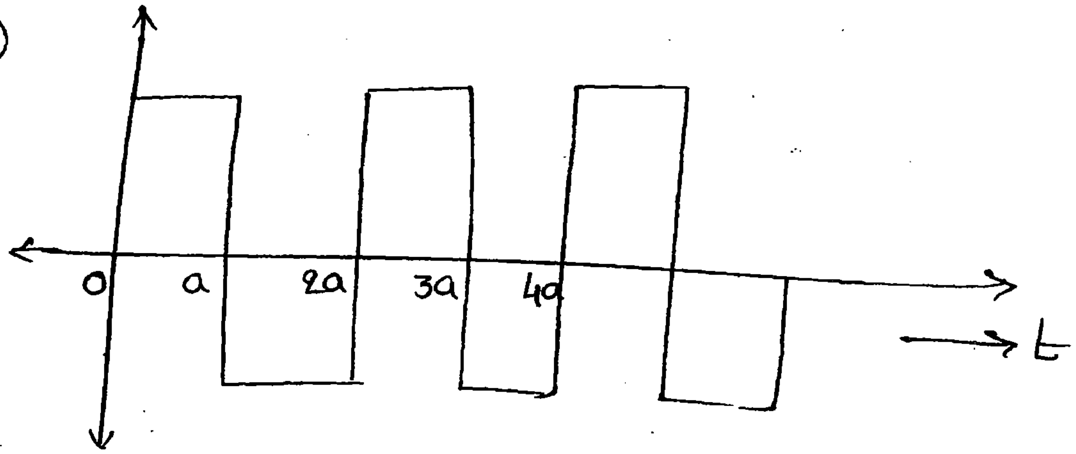
$$= \frac{\omega}{s^2+\omega^2\,(1-e^{-s\pi/\omega})}$$

$*$ $f(t) = |\sin\omega t|$

$$= \frac{\omega}{s^2+\omega^2}\left[\frac{(1+e^{-\pi/\omega s})}{1-e^{-s\pi/\omega}}\right.$$

$$= \frac{\omega}{s^2+\omega^2}\;\frac{\left(e^{-\pi s/2\omega}+e^{-\pi s/2\omega}\right)}{\left(e^{s\pi/2\omega}-e^{-s\pi/2\omega}\right)}$$

$$= \frac{\omega}{s^2+\omega^2}\;\coth\left(\pi s/2\omega\right)$$

① 

① $1/s \tanh(as)$

2) $1/2 \coth(as/2)$

3) $1/2 \tanh(as/2)$

4) $1/2 \coth(as)$

$$\frac{\int_0^a e^{-st}\,dt + \int_a^{2a} -e^{-st}\,dt}{(1-e^{-2as})}$$

$$\frac{\left(\frac{e^{-st}}{-s}\right)_0^a + \left(\frac{e^{-st}}{s}\right)_a^{2a}}{(1-e^{-2as})}$$

$$= \frac{\left[\frac{-e^{-as}}{s} + \frac{1}{s} + \frac{e^{-2as}}{s} \quad \frac{-e^{-as}}{s} \right]}{(1-e^{-2as})},$$

$$= 1/s \left[\frac{(1-e^{-as})^{\cancel{2}}}{\cancel{(1-e^{-as})}\ (1+e^{-as})} \right]$$

$$= 1/s\ \frac{e^{-as/2}}{e^{-as/2}} \left[\frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \right]$$

$$= 1/s \tanh (as/2)$$

* $L[\,t^4\, \delta(t-2)]$

$$= e^{-2s}(2)^4 = 16e^{-2s}.$$

<u>Evaluation of improper function</u>

$$\int_0^\infty \frac{\sin t}{t}\, dt$$

$$\int \frac{t - t^3 + t^5 - t^7 - - - -}{t}$$

$$\int 1 - t^2 + t^4 - t^6$$

$$= t - t^3 + t^5 - t^7$$

$\infty - \infty + \infty - \infty$ Not defined

* $L\left(\frac{\sin t}{t}\right) = \cot^{-1}(s)$

$$\int_0^\infty e^{-st}\, \frac{\sin t}{t}\, dt = \cot^{-1}(s)$$

for $S=0$

$$= \int_0^\infty \frac{\sin t}{t} = \cot^{-1}(0) = \pi/2$$

$$\boxed{\int_{-\infty}^{\infty} \frac{\sin t}{t} = 2\int_0^\infty \frac{\sin t}{t}}$$

** $\int_0^\infty t e^{-3t} \cos t \, dt$

$$= \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=3} = \left(\frac{9-1}{(9+1)^2}\right) = 8/100.$$

* $\int_0^\infty \frac{e^{-3t} - e^{-2t}}{t} dt$

$$= \int_0^\infty \left(\frac{1}{(S+3)} - \frac{1}{(S+2)}\right)$$

$$= \log(S+3) - \log(S+2)$$

$$= \log\left(\frac{S+3}{S+2}\right) \quad S=0$$

$$= \log(2/3)$$

$$L\left[\frac{e^{-3t} - e^{-2t}}{t}\right] \text{at } S=$$

$$\log\left(\frac{S+2}{S+3}\right)\Big|_{S=}$$

$$\log(2/3)$$

** $\int_0^\infty \frac{\sin t}{t} dt$

$$\int_{x=0}^{t} \int_{t=0}^{\infty} \frac{e^{-t} \sin x}{x} dx\, dt$$

$$= \int_{t=0}^{\infty} e^{-t} \left[\int_{x=0}^{t} \frac{\sin x}{x} dx\right] dt$$

$L\left[\int_0^t \frac{\sin x}{x}\,dx\right]_{S=1}$

$$\left[1/s \cot^{-1}(s)\right]_{S=1} = \cot^{-1}(1) = \pi/4$$

$\int_0^\infty \sin t\,dt \longrightarrow$ divergent does not Exist

$$\left(-\cos t\right)_0^\infty \longrightarrow 1$$

sint should be convergent.

$L\int_0^\infty te^{3t}\cos t\,dt \longrightarrow$ divergent does not Exist

$-\int_0^\infty \sin t\left(\delta\left(t-\pi/2\right)\right)dt$

$$\sin \pi/2 = 1$$

$\int_0^\infty \sin t\ \sigma(t-\pi/4)\,dt$

$$\int_0^{\pi/4}(0) + \int_{\pi/4}^{\infty} \sin t\,dt \longrightarrow \text{divergent.}$$

<u>Inverse</u> <u>Laplace</u> <u>transformation</u>:-

If $L(f(t)) = f(s) = L^{-1}[f(s)]$

<u>Linear</u>:-

$$L^{-1}\left[a\overline{f(s)} \pm b\,\overline{g(s)}\right]$$

$$= aL^{-1}\left[\overline{f(s)}\right] \pm b\,L^{-1}\left[\overline{g(s)}\right]$$

→ $L^{-1}[1] = \delta(t)$

→ $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

→ $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

→ $L^{-1}\left[\frac{1}{s^2+a^2}\right] = 1/a \sin at$

→ $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

→ $L^{-1}\left[\frac{s}{s^2-a^2}\right] = 1/a \sinh at$

→ $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \cosh at$

$$L^{-1}[1/s^n] = \begin{cases} \frac{t^{n-1}}{(n-1)!} & n \in z^+ \\ \frac{t^{n-1}}{\Gamma_n} & n \in z^+ \end{cases}$$

$n > 0$

$\overline{n}$

F.S theorem of I.L.T:-

① $L^{-1}\{f^{-1}(s-a)\} = e^{at} L^{-1}[\bar{f}(s)]$

2) $L^{-1}[\bar{f}(s+a)] = e^{-at} L^{-1}[\bar{f}(s)]$

S.S theorem of I.L.T:-

If $L^{-1}[\bar{f}(s)] = f(t)$

$L^{-1}[e^{-as} \bar{f}(s)] = f(t-a)\, u(t-a)$

Scaling property

① $L^{-1}[\bar{F}(as)] = 1/a \, f(t/a)$

2) $L^{-1}[\bar{F}(s/a)] = a f(at)$

Multiplication by s^n $(n \in Z^+)$

$L^{-1}[s\bar{F}(s)] = f'(t)$ provided $f(0)=0$

$L^{-1}[s^n F(s)] = f^n(t)$ provided $f(0)=f'(0)$

$f^{n-1}(0) = 0.$

division by s^n $(n \in Z^+)$

$$L^{-1}\left[\frac{1}{s}\bar{F}(s)\right] = \int_0^t f(t)\,dt$$

$$L^{-1}\left[\frac{1}{s^n}\bar{F}(s)\right] = \int_0^t \int_0^t \cdots\cdots f(t)\,(dt)^n$$

I.L.T of derivatives:-

$$L^{-1}\left[\frac{d}{ds}F(s)\right] = (-t)\,L^{-1}(\bar{F}(s))$$

$$L^{-1}\left[\frac{d^n}{ds^n}\bar{F}(s)\right] = (-t)^n\,L^{-1}[\bar{F}(s)]$$

I.L.T of integral:-

$$L^{-1}\left[\int_s^\infty \bar{F}(s)\,ds\right] = \frac{1}{t}L^{-1}(\bar{F}(s))$$

convolution theorem:-

$$L^{-1}[\bar{F}(s)\cdot\bar{g}(s)] = f(t) * g(t)$$

$$= \int_0^t f(x)\,g(t-x)\,dx$$

Initial value theorem | Final value theorem:-

If $L[f(t)] = \bar{F}(S)$

① $\underset{t\to 0}{Lt}\, f(t) = \underset{S\to\infty}{Lt}\, S\bar{F}(S)$ — Initial value theorem

2) $\underset{t\to\infty}{Lt}\, f(t) = \underset{S\to 0}{Lt}\, S\bar{F}(S)$ — final value theorem.

[If poles on left side only F.V.T is valid]

condition:- [$F(S)$ is zero $\underset{t\to\infty}{Lt}\, e^{at} \to \infty$]

The final value theorem is applicable if the poles of $\bar{F}(S)$ lie in the left plane

$\rightarrow L^{-1}\left[\frac{1}{\sqrt{2S+3}}\right]$

$$= \frac{1}{\sqrt{2}} L^{-1}\left\{\frac{1}{(S+3/2)^{1/2}}\right\}$$

F.S theorem

$$= \frac{1}{\sqrt{2}} e^{-3/2t} L^{-1}\left[1/S^{1/2}\right]$$

$$= \frac{1}{\sqrt{2}} e^{-3/2t} \frac{t^{1/2-1}}{\sqrt{1/2}} = \frac{e^{-3/2t}}{\sqrt{2\pi t}}$$

$$L^{-1}\left[\frac{1}{(aS+b)^n}\right] = \frac{1}{a^n} e^{-b/at} \frac{t^{n-1}}{\sqrt{n}}$$

$$L^{-1}\left\{\frac{3S+5}{4S^2-9} - \frac{2S-1}{16S^2+9}\right\}$$

$$L^{-1}\left[\frac{}{4s^2-9} \quad \frac{}{16s^2+9}\right]$$

$$= \frac{1}{4} L^{-1}\left[\frac{3s}{s^2-9/4} + \frac{5}{s^2-9/4}\right] - \frac{1}{16} L^{-1}\left[\frac{2s}{s^2+9/16} - \frac{1}{s^2+9/16}\right]$$

$$= \frac{1}{4}\left[3\cosh 3/2t + 5\ 1/3/2 \sinh 3/2t\right.$$

$$- \frac{1}{16}\left[2\cos 3/4t - 1/3/4 \sin 3/4 t\right]$$

$$L^{-1}\left[\frac{s-3+3}{(s-3)^2}\right]$$

$$= e^{3t} L^{-1}\left[\frac{s+3}{s^2}\right]$$

$$= e^{3t} L^{-1}\left[1/s + 3/s^2\right] = e^{3t}(1+3t)$$

* $$L^{-1}\left[\frac{s^2-4+4}{(s-2)^2}\right]$$

$$L^{-1}\left[\frac{[(s-2)+2]^2}{(s-2)^2}\right]$$

Fourier Series transform

$$= e^{2t} L^{-1}\left[\frac{(s+2)^2}{s^2}\right]$$

$$= e^{2t} L^{-1}\left[\frac{s^2+4s+4}{s^2}\right]$$

$$= e^{2t} L^{-1}\left[1 + 4/s + 4/s^2\right]$$

$$= e^{2t}\left[\delta(t) + 4(t) + 4\right]$$

$$\rightarrow \quad L^{-1}\left[\frac{3s+2}{s^2-4s+3}\right]$$

$$= L^{-1}\left[\frac{(3s+2)}{(s-3)(s-1)}\right]$$

$$= L^{-1}\left[\frac{11/2}{(s-3)} - \frac{5/2}{(s-1)}\right]$$

$$= 11/2\, e^{3t} - 5/2\, e^{t}$$

** $L^{-1}\left[\frac{2s+1}{s^2+s+1}\right]$

$$= L^{-1}\left(\frac{2(s+1/2)-1+1}{(s+1/2)^2+1-1/4}\right)$$

fourier series transform

$$= e^{-t/2}\, L^{-1}\left[\frac{2s}{s^2+3/4}\right]$$

$$= e^{-t/2}\left[2\cos \sqrt{3}/2\, t\right]$$

** $L^{-1}\left[\frac{2s+3}{s^2-5s+3}\right]$

$$L^{-1}\left(\frac{2(s-5/2)+5+3}{(s-5/2)^2-25/4+3}\right)$$

$$= e^{5/2t}\, L^{-1}\left[\frac{2(s)+8}{s^2-13/4}\right]$$

$$= e^{5/2t}\left(2\cosh \sqrt{13}/2 + 8\frac{1}{\sqrt{13}/2}\sinh \sqrt{13}/2\, t\right)$$

$$L^{-1}\left[\frac{\quad}{(s+3)(s^2+9)}\right]$$

$$L^{-1}\left[\frac{-3/18}{(s+3)}+\frac{(Bs+c)}{s^2+9}\right]$$

co-efficient of s^2 $\quad -3/18+B=0$

$$B=3/18$$

$$\frac{-27}{18}+3c=0$$

$$c=9/18=1/2$$

$$L^{-1}\left(\frac{-3/18}{(s+3)}+\frac{3/18\,s+1/2}{(s^2+9)}\right]$$

$$=-3/18\,e^{-3t}+3/18\cos 3t+1/2\cdot 1/3\sin 3t$$

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$$

$$=\int_0^t\int_0^t\int_0^t \sin t\,(dt)^3$$

$$=\int_0^t\int_0^t(-\cos t)_0^t\,(dt)^2$$

$$=\int_0^t\int_0^t\left[-\cos t+1\right](dt)^2$$

$$=\int_0^t\left[-\sin t+t\right]_0^t dt$$

$$=\int_0^t\left[-\sin t+t\right]dt$$

$$= \left(\cos t + t^2/2\right)_0^t$$

$$= \cos t + t^2/2 - 1$$

* $L^{-1}\left[\dfrac{s}{(s^2+a^2)^2}\right]$

$$\bar{F}(s) = \frac{1}{s^2+a^2}$$

$$\frac{d}{ds}\bar{F}(s) = \frac{-2s}{(s^2+a^2)}$$

$$L^{-1}\left[\frac{d}{ds}\bar{F}(s)\right] = -t\, L^{-1}\left[\bar{F}(s)\right]$$

$$L^{-1}\left[\frac{2(s)}{(s^2+a^2)^2}\right] = -t \cdot 1/a \sin at$$

$$= \frac{t \sin at}{2a}$$

* $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)^2}\right]$

$$L^{-1}\left[s \cdot \frac{s}{(s^2+a^2)^2}\right]$$

$$\frac{d}{dt}\left[t\, \frac{\sin at}{2a}\right]$$

$$= 1/2a\left[ta\cos at + \sin at\right]$$

* $L^{-1}\left[\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right]$

$$L^{-1}\left[\frac{s}{s^2+a^2} - \frac{s}{(s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}\left[\cos at - \cos bt\right]$$

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$$

$$= \frac{1}{b^2-a^2} L^{-1}\left[\frac{s^2+a^2-a^2}{s^2+a^2} - \frac{s^2}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2} L^{-1}\left[\left(1-\frac{a^2}{s^2+a^2}\right) - \left(1-\frac{b^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{b^2-a^2}\left[-a\sin at + b\sin bt\right]$$

*

$$L^{-1}\left[\frac{s}{(s^2+4)(9-s^2)}\right]$$

$$= L^{-1}\left[\frac{s}{s^2+4} - \frac{s}{s^2-9}\right]$$

$$= \frac{1}{13}\left[\cos 2t - \cosh 3t\right]$$

*

$$L^{-1}\left[\frac{s}{s^4+4a^4}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+2a^2-2as)(s^2+2a^2+2as)}\right]$$

$$L^{-1}\left[\frac{s}{(s^2+2a^2)^2-(2as)^2}\right]$$

$$= \frac{1}{4a} L^{-1}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$$

$$= \frac{1}{4a} L^{-1}\left[\frac{1}{(s-a)^2+a^2} - \frac{1}{(s+a)^2+a^2}\right]$$

$$= \frac{1}{4a}\left(\frac{e^{at}\sin at}{a} - \frac{e^{-at}\sin at}{a}\right)$$

$$= \frac{1}{4a^2}\sin at\left(e^{at}-e^{-at}\right)$$

$$= \frac{1}{\cancel{4}_{2}a^2}\sin at\ \cancel{2}\sinh at$$

$$= \frac{1}{2a^2}\sin at\ \sinh at$$

* $L^{-1}\left[\frac{b^2}{(s^2+a^2)^2}\right\}$

$$= b^2 L^{-1}\left[\frac{1}{(s^2+b^2)^2}\right]$$

$$= b^2 L^{-1}\left[\frac{1}{s}\cdot\frac{s}{(s^2+b^2)^2}\right]$$

$$= \int_0^t \frac{t\sin bt}{2b}\,dt.$$

* $L^{-1}\left[e^{-3s}\int_s^{\infty}\bar{F}(3s+5)\,ds\right] = ?$

$$L^{-1}\left\{\bar{F}(3s+5)\right\} = L^{-1}\left\{\bar{F}\left(3(s+5/3)\right)\right\}$$

$$= e^{-5/3t}\cdot 1/3\cdot t/3\cdot\frac{\sin 3(t/3)}{3}$$

$$= e^{-5/3t}\ t\ \frac{\sin t}{\ }$$

$$L^{-1}\left[\int_s \bar{f}(3s+5)\,ds\right] = \frac{1}{t}\cdot e^{-5/3\,t}\ \frac{t\sin t}{27}$$

$$L^{-1}\left[e^{-3s}\int_s^{\infty}\bar{f}(3s+5)\,ds\right]$$

$$= \frac{e^{-5/3\,(t-3)}\sin(t-3)\,u(t-3)}{27}$$

$$L^{-1}\left[\frac{2+5s}{s^4e^{4s}}\right]$$

$$= L^{-1}\left[e^{-4s}\left[2/s^4 + 5/s^3\right]\right]$$

$$= \frac{2t^3}{3!} + \frac{5t^2}{2!}$$

$$= \left(\frac{2(t-4)^3}{3!} + \frac{5(t-4)^2}{2!}\right]u(t-4)$$

* $$L^{-1}\left[\frac{1}{s(1-e^{-s})}\right]$$

$$L^{-1}\left(\frac{1}{s(1-e^{-s})}\right] = L^{-1}\left[\frac{1}{s}(1+e^{-s}+e^{-2s}+\cdots\right.$$

$$= L^{-1}\left[\sum_{k=0}^{\infty}\frac{e^{-ks}}{s}\right]$$

$$= \sum_{k=0}^{\infty} u(t-k)$$

* $$L^{-1}\left[\tan^{-1}(2/s^2)\right]$$

$$\bar{f}(s) = \tan^{-1}(2/s^2)$$

$$= \frac{1}{1+(2/s^2)^2}\left[-4/s^3\right]$$

$$= \frac{s^4}{s^4+4} - 4/s^3$$

$$= \frac{-4s}{s^4+4}$$

$$L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right] = -tL^{-1}\left(\bar{f}(s)\right)$$

$$L^{-1}\left[\frac{-4s}{s^4+4}\right] = -t\,L^{-1}\left[\tan^{-1}(2/s^2)\right]$$

$$s^4+4 = (s^2+2)^2 - (2s)^2$$

$$(s^2+2+2s)\ (s^2+2-2s)$$

$$(s+1)^2+1 \qquad (s-1)^2+1$$

$$= 4\cdot\frac{1}{2(1)^2}\ \frac{\sin t \sinh t}{t} = L^{-1}\left[\tan^{-1}(2/s^2)\right]$$

$$L^{-1}\left[\frac{1}{s}\cot^{-1}(s)\right]$$

$$L^{-1}\left[\frac{d}{ds}\bar{f}(s)\right] = -t\left[L^{-1}\left(\bar{f}(s)\right)\right]$$

$$L^{-1}\left[\frac{1}{(s^2+1)}\right] = t\ L^{-1}\left[\bar{f}(s)\right]$$

$$\frac{\sin t}{t} = L^{-1}\left[\cot^{-1}(s)\right]$$

$$= \sin t\ (1/t)$$

$$= 1/t\ (-\cos t)$$

$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+2}\right)\right]$

$$= \int_0^t \left(\frac{e^{-2t}-e^{-3t}}{t}\right) dt$$

$$L[f(t)] = \frac{s-1}{s^2-2s+2}$$

then $\underset{t\to\infty}{Lt} f(t) = ?$

Sol

poles are $1 \pm i$.

Ans 0.

* $$L[f(t)] = \frac{s-1}{s^2+2s+2}$$

poles are $-1 \pm i$

Ans 0

* If $L[f(t)] = \frac{s}{s^2+4}$

then $\underset{t\to\infty}{Lt} f(t) = ?$

a) 1
b) 0
c) $-1 \le f(t) \le 1 \longleftrightarrow$ divergent (oscillate finitely.
d) None

Improper definite integrals:- (calculus)

The integral of the form.

① $\int_{-\infty}^{b} f(x)\,dx$ 2) $\int_{a}^{\infty} f(x)\,dx$ 3) $\int_{-\infty}^{\infty} f(x)\,dx$

The above integrals are called as improper definite integrals.

* $\boxed{y^2(a+x) = x^2(a-x)}$

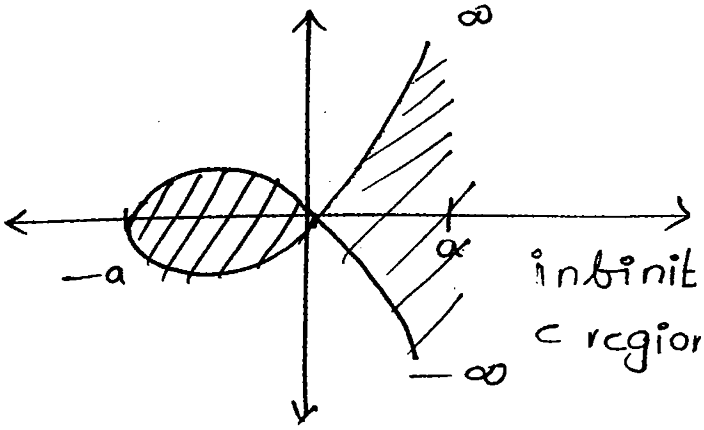
substitute $y=-y$ then there is No change in this. Hence it is symmetrical about the x-axis

$$y = \pm x\sqrt{\frac{(a-x)}{(a+x)}}$$

when $x=0,\ y=0$

$x=a,\ y=0$

$x=-a,\ y=\pm\infty$.



Geometrically: Area of infinite region

Improper definite integrals are Evaluated by using the limiting process.

1). $\int_{-\infty}^{b} f(x)\,dx = \underset{p\to-\infty}{Lt} \int_{p}^{b} f(x)\,dx$ = finite then it is said to be convergent

$= \pm\infty$ then it is said to be divergent

$$x\,dx = \left|e^x\right|_{-\infty}^{0} = e^0 - e^{-\infty} = 1 - 0 = 1$$

$\therefore$ is converges to $\underline{\underline{1}}$

$$\frac{1}{1+x^2}\,dx = \left|\tan^{-1}x\right|_{-\infty}^{0} = \tan^{-1}(0) - \tan^{-1}(-\infty)$$

$$= 0 + \pi/2 = \pi/2 \text{ (convergent}$$

converges to $\pi/2$ //

$$\int_{-\infty}^{1}\left(\frac{1}{x^2}\right)dx = -\left|1/x\right|_{-\infty}^{1} = -1 + \frac{1}{-\infty} = -1 + 0 = \underline{-1}$$

(convergent

$$\int_{-\infty}^{1}\frac{1}{x\sqrt{x^2-1}}\,dx = \left|\sec^{-1}(x)\right|_{-\infty}^{1} = \sec^{-1}(1) - \sec^{-1}(-\infty)$$

$$\sec^{-1}(-\infty) = y$$

$$(-\infty) = \sec(y)$$

$$\cos y = \frac{1}{-\infty} = 0$$

$$y = \cos^{-1}(0) = \pi/2$$

$$= \sec^{-1}(1) - \sec^{-1}(-\infty)$$

$$= 0 - \pi/2 = \underline{\underline{-\pi/2}} \text{ (convergent)}$$

$$\int_0^1 \frac{1}{1-x}\,dx = -\left|\log(1-x)\right|_0^1 = -\log(0) + \log(1)$$

$$= \infty \text{ (}\underline{\text{divergent}}\text{)}$$

$$\int_a^\infty f(x)\,dx = \lim_{P\to\infty}\int_a^P f(x)\,dx = \text{finite it is said}$$

to be convergent $= \pm\infty$ then it is said to be divergent

1) $\int_0^{\infty} x\sin x\,dx = \left(-x\cos x+\sin x\right)_0^{\infty}$

$= \left[-\infty\cos\infty+\sin(\infty)\right] - (0+0) = \underline{\underline{-\infty}}$ (d

2) $\int_1^{\infty} \frac{1}{1+x^2}\,dx = \left|\tan^{-1}x\right|_1^{\infty} = \tan^{-1}(\infty) - \tan^{-1}(1)$

$= \pi/2 - \pi/4 = \underline{\underline{\pi/4}}$ (conv

3) $\int_0^{\infty} \left(\frac{1}{x^2}\right) dx = -\left|1/x\right|_0^{\infty} = \frac{1}{\infty}+\frac{1}{0} = \underline{\underline{\infty}}$ (divergent)

4) $\underset{a\to\infty}{Lt} \int_1^{a} x^{-4}\,dx = -\left|\frac{1}{3x^3}\right|_1^{\infty} = \underline{\underline{1/3}}$ (convergent)

5) $\int_0^{\infty} xe^{-x}\,dx = \left|-xe^{-x}+e^{-x}\right|_0^{\infty}$

$= \left[-\infty(\infty)+e^{-\infty}\right] - (0-1)$

$= 0+1 = 1$ (convergent)

$\underset{x\to\infty}{Lt}\; xe^{-x} = (x/e^{x})$

$\underset{x\to\infty}{Lt}\; \frac{f'(x)}{g'(x)} = \frac{1}{e^{x}} = \frac{1}{e^{\infty}} = 1/\infty = 0$

<u>I Method:-</u>

$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$

$= l_1 + l_2$ (convergent) (finite) otherwise it is diverges. $(\infty + l_2)$

$\left(\frac{1}{x^2}\right) dx = -\left|1/x\right|_{-\infty}^{\infty} = -\left|\frac{1}{\infty} + \frac{1}{\infty}\right|$

$= -(0+0)$

$= \int_{-\infty}^{0} \left(\frac{1}{x^2}\right) dx + \int_{0}^{\infty} (1/x^2) dx$

$= -\left|1/x\right|_{-\infty}^{0} - \left|1/x\right|_{0}^{\infty}$

$= -|\infty| + |-\infty| = -\infty - \infty$

$= -\infty$ (divergent).

(ग)

$\int_{-\infty}^{\infty} (1/x^2)\, dx = 2\int_{0}^{\infty} \left(\frac{1}{x^2}\right) dx = -\left|2/x\right|_{0}^{\infty}$

$= \infty$ (diverges)

$\int_{-1}^{1} \left(\frac{1}{x^2}\right) dx = 2\int_{0}^{1} (1/x^2)\, dx = \left|-2/x\right|_{0}^{1}$

$= \infty$ (diverges)

$\int_{\infty}^{\infty} \frac{1}{1+x^2}\, dx = \left|\tan^{-1} x\right|_{-\infty}^{\infty} = (\pi/2 + \pi/2)$

$=$ (convergent)

10

The value of $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\, dx$

a) 0 b) 1/2 c) 1 d) 2

) $\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-x^2/8}\, dx$

3) $\int_{0}^{\infty} \sqrt{x}\; e^{-\sqrt{x}}\, dx$

4) $\int_0^{\infty} e^{-x^2}\,dx$

5) $\int_0^{\infty} e^{-\sqrt{x}}\,dx$

Gamma function:-

Definition:- $\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\,dx \quad (n>0)$ is called gamma function.

$\Gamma 1 = 1$

$\Gamma 2 = 1$

$\Gamma(3) = (3-1)! = 2! = 2$

$\Gamma(4) = (4-1)! = (3!) = 6$

Replacing n by (n+1) then $\Gamma(n+1) = n\,\Gamma n = n!$

$\Gamma 1/2 = \sqrt{\pi}$

$$\Gamma 1/2 = \int_0^{\infty} e^{-x} x^{-1/2}\,dx$$

$$= x^{-1/2}(-e^{-x}) + \int (-1/2)\, x^{-3/2} e^{-x}$$

$$\Gamma(n) = \frac{1}{n}\Gamma(n+1) \Rightarrow \Gamma(-1/2) = -2\,\Gamma(1/2) = -2\sqrt{\pi}$$

$$\boxed{\Gamma 3/2 = 1/2\,\Gamma 1/2 = 1/2\sqrt{\pi}}$$

(1) $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$

It is Even function

$$\frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-x^2/2}\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x^2/2}\, dx$$

put $x^2/2 = t$

$x^2 = 2t$

$x = \sqrt{2t}$

$dx = \frac{1}{\sqrt{2}} t^{-1/2} dt$

$$= \frac{1}{\cancel{\sqrt{2}}} \frac{\cancel{\sqrt{2}}}{\sqrt{\pi}} \int_0^\infty e^{-t} t^{-1/2}\, dt$$

$n-1 = -1/2$

$n = 1/2$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1/2) = \frac{\cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}} = 1$$

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-x^2/8} dt \quad \text{put } x^2/8 = t$$

$x^2 = 8t$

$2x\,dx = 8dt$

$x = 2\sqrt{2t}$

$dx = \sqrt{2}\, t^{-1/2} dt$

$$= \frac{1}{\cancel{\sqrt{2}}\sqrt{\pi}} \times \cancel{\sqrt{2}} \int_0^\infty e^{-t} t^{-1/2}\, dt$$

$n-1 = -1/2$

$n = 1/2$

$$= \frac{1}{\sqrt{\pi}} \Gamma(1/2) = \sqrt{\pi}/\pi = 1$$

3) $\int_0^\infty \sqrt{x}\, e^{-\sqrt{x}}\, dx$

put $x = t^2$

$dx = 2t\,dt$

$$= 2\int_0^\infty t^2 e^{-t}.dt \qquad n-1=2$$
$$n=3$$

$$= 2\Gamma(3) = 2\times 2 = 4$$

④ $$\int_0^\infty e^{-x^2}dx = 1/2 \int_0^\infty e^{-t}\, t^{-1/2}\, dt \qquad n-1 = -1/2$$
$$\boxed{n=1}$$

$$= 1/2\, \Gamma 1/2$$

$$\boxed{\int_0^\infty e^{-x^2}dx = 1/2\sqrt{\pi}}$$

5) $$\int_0^\infty e^{-\sqrt{x}}\,dx$$

put $x = t^2$

$dx = 2t\,dt$

$$\int_0^\infty e^{-t^2}(2t\,dt) = 2\int_0^\infty e^{-t^2}\, t\, dt \qquad \begin{array}{l} n-1=1 \\ n=2 \end{array}$$

$$= 2\Gamma 2 = 2(1) = 2$$

⑥ $$\int_0^\infty 2^{-9x^2}dx$$

$$= \int_0^\infty e^{-9x^2 \log 2}\,dx$$

$\left(\cancel{9x^2 = t}\right.$

$18x\,dx = dt$

$\left. dx = \frac{1}{18\sqrt{t}}\,dt\right)$

$$\boxed{\begin{array}{l} e^{a\log f(x)} = f(x)^a \\ e^{f(x)\log a} = a^{f(x)} \end{array}}$$

put $9x^2 \log 2 = t$

$$x^2 = \frac{t}{9\log 2}$$

$$x = \frac{t}{3\sqrt{\log 2}}$$

$$dx = \frac{1}{6\sqrt{\log 2}}\, t^{-1/2}\,dt$$

$$= \frac{1}{6\sqrt{\log 2}} \Gamma(1/2)$$

$$= \frac{1}{6\sqrt{\log 2}} \sqrt{\pi}$$

$$\int_0^\infty 4^{-x^2} dx = \int_0^\infty e^{-x^2 \log 4} dx$$

put $x^2 \log 4 = t$

$$x^2 = \frac{t}{\log 4}$$

$$x = \frac{1}{\sqrt{\log 4}} t^{1/2} \Rightarrow dx = \frac{1}{2\sqrt{\log 4}} t^{-1/2} dt$$

$$= \int_0^\infty e^{-t} \frac{1}{2\sqrt{\log 4}} t^{-1/2} dt$$

$$= \frac{1}{2\sqrt{\log 4}} \int_0^\infty e^{-t} t^{-1/2}$$

$$= \frac{1}{2\sqrt{\log 4}} \Gamma 1/2 = \frac{1}{2\sqrt{\log 4}} \sqrt{\pi}$$

$$\int_0^{\pi/2} \sqrt{\tan x}\, dx$$

when the integrals of trignometric and different functions are given then we will use Another Special function Beta function.

<u>Beta function</u>:-

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} dx \quad , (m>0, n>0)$$

is. called Beta function..

$\beta(m,n) = \beta(n,m) \Longrightarrow$ symmetry

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

$$= \int_0^{\pi/2} x^{m-1}(1-x)^{n-1}\,dx$$

put $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\,d\theta$

when $x=0,\ \theta=0$

$x=1,\ \theta=\pi/2$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\,2\sin\theta\cos\theta\,d\theta$$

$$\boxed{\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta}$$

Relation b/w $\beta(m,n)$ and Γm, Γn :-

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = 1/2\ \beta(m,n)$$

$$= 1/2\ \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$\int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\,d\theta$$

$2m-1 = 1/2, \quad 2n = -1/2$

$m = 3/4, \quad n = 1/4$

$$\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \frac{1/2\ \Gamma 3/4\ \Gamma 1/4}{\Gamma(3/4+1/4)} = 1/2\ \Gamma 3/4\ \Gamma 1/4$$

Let $I_1 = \int_1^{\infty} \frac{dx}{x^2(1+e^x)}$ and $I_2 = \int_1^{\infty} \frac{x+1}{x\sqrt{x}}\, dx$

which of the following statement is true

a) I_1 convergens, I_2 diverges

) I_2 " , I_1 "

) Both converges

) Both diverges.

$$I_1 = \int_1^{\infty} \frac{dx}{x^2(1+ex)}$$

$$I_2 = \int_1^{\infty} \frac{1}{\sqrt{x}}\, dx + \int_1^{\infty} \frac{1}{x^{3/2}}\, dx$$

$$= 2\left|\sqrt{x}\right|_1^{\infty} - 2\left|\sqrt{x}\right|_1^{\infty}$$

$I_2 = \infty$ (divergens)

$$I_1 = \int_1^{\infty} \frac{dx}{x^2(1+ex)}$$

$$f(x) = \frac{1}{x^2(1+ex)}$$

$$g(x) = \frac{1}{x^2}$$

$$\int_1^{\infty} g(x)\, dx = \int_1^{\infty} (1/x^2)\, dx = -\left|1/x\right|_1^{\infty} = 1$$

(converges)

By comparision .

$\int_1^\infty f(x)dx$ also converges

$\int_1^\infty \frac{dx}{x^2(1+cx)}$ converges

what is the Nature of the Series.

$$\sum_{n=1}^{\infty} \frac{n^2+1}{n^5+1} = \frac{n^2(1+\frac{1}{n^2})}{n^5(1+\frac{1}{n^5})} = \frac{1(1+1/n^2)}{n^3(1+1/n^5)}$$

$$V_n = \frac{1}{n^3}.$$

$$\sum v_n = \frac{1}{n^3}.$$

$$\boxed{\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + ---- \quad \begin{matrix} p>1 \text{ converges.} \\ p\leq 1 \text{ diverges} \end{matrix}}$$

$$\sum \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + --$$

Here $p>1 = (3>1)$ Hence the given Series is converges.

<u>Applications</u>:- (definite integrals)

<u>Areas</u>:- If a curve is symmetrical about x-axis

then $\boxed{\text{area} = \int_a^b y\,dx}$

b) If a curve is symmetrical about y-axis.

$$\boxed{\text{Area} = \int_a^b x\,dy}$$

c) Area Enclosed b/w two curves.

$y_1 = f(x)$, $y_2 = g(x)$ is

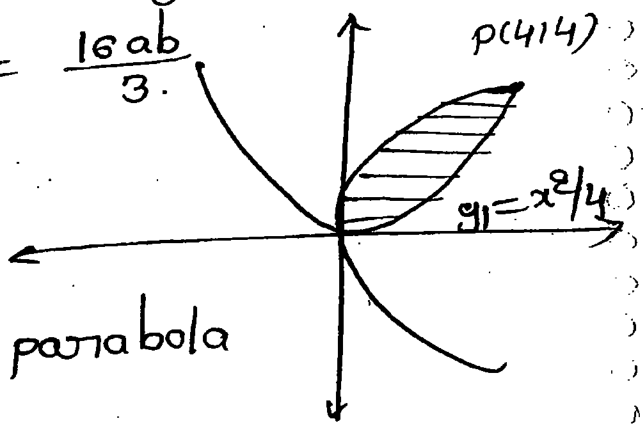
$$A = \int_a^b (y_1 - y_2)\,dx \quad \text{if } y_1 > y_2$$

$$A^a = \int_a^b (y_2 - y_1)\,dx \quad \text{if } y_2 > y_1$$

If the area bounded by the parabolas.

(i) $y^2 = 4x$ and $x^2 = 4y$ is

a) $\frac{10}{3}$ b) $\frac{12}{5}$ c) $\cancel{\frac{16}{3}}$ d) $\frac{20}{3}$.



Area b/w the parabolas $= \frac{16ab}{3}$.

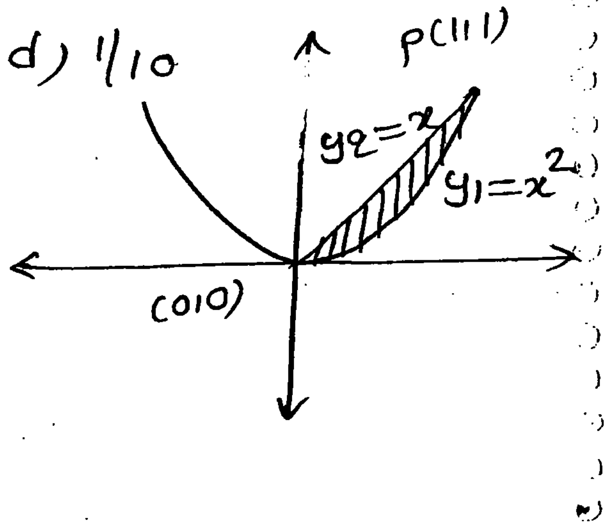
$a = 1$, $b = 1$

$A = 16/3$.

EC, CIVIL

The area bounded by the parabola $y = x^2$ and $y = x$ is.

a) $1/3$ b) $1/6$ c) $1/8$ d) $1/10$



$y_1 = x^2$, $y = x$

$x^2 = x \Longrightarrow x^2 - x = 0$

$x(x-1) = 0$

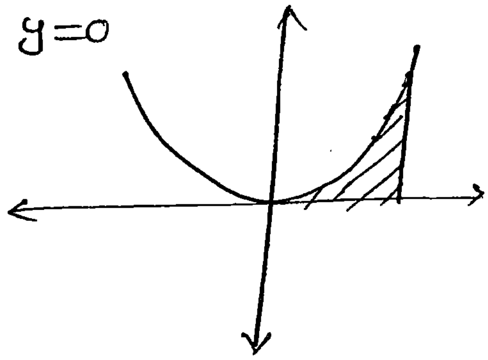
$$\boxed{x = 0,\ x = 1}$$

$$= \int_0^1 (x - x^2)\,dx = \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 = \left(1/2 - 1/3\right) = 1/6$$

3) $y = x(x-1)$, $x=0$, $x=1$ (ਗ)

$$\int_0^1 x(x-1)\,dx = \left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^1 = 1/6.$$

) $y = x^2$; $x = 4$, $y = 0$



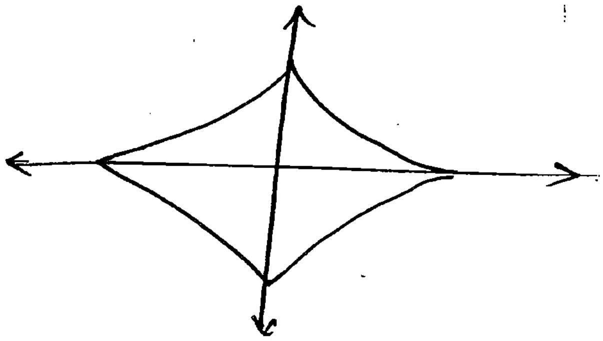
$$\text{Area} = \int_0^4 y\,dx$$

$$= \int_0^4 x^2\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^4 = 1/3\,(64) = \frac{64}{3}$$

* The area bounded by the curve $x = \cos^3\theta$, $y = \sin^3\theta$, $\theta = 0$, $\theta = \pi/2$ is $x^{2/3} + y^{2/3} = 1$

Sol



$$\text{Area} = \int_0^{\pi/2} y\,dx$$

$$= \int_0^{\pi/2} \sin^3\theta\,(-3\cos^2\theta\,\sin\theta)\,d\theta$$

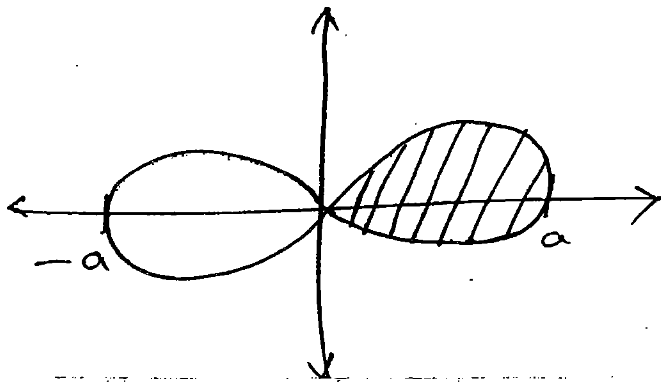
$$= -3\int_0^{\pi/2} \sin^4\theta\,\cos^2\theta\,d\theta = \frac{-3\cdot\cancel{3}\cdot 1\cdot 1\cdot \pi/2}{2\,\cancel{6}\cdot 4\cdot 2\cdot 2} = \frac{-3\pi}{32}$$

$$= 3\pi/32$$

* The area of a Loop of a curve $y^2 = x^2(a^2 - x^2)$ is

Sol

$$y = \pm x\sqrt{a^2 - x^2}$$



when $x = 0$, $y = 0$

$x = a$, $y = 0$

$x = -a$, $y = 0$

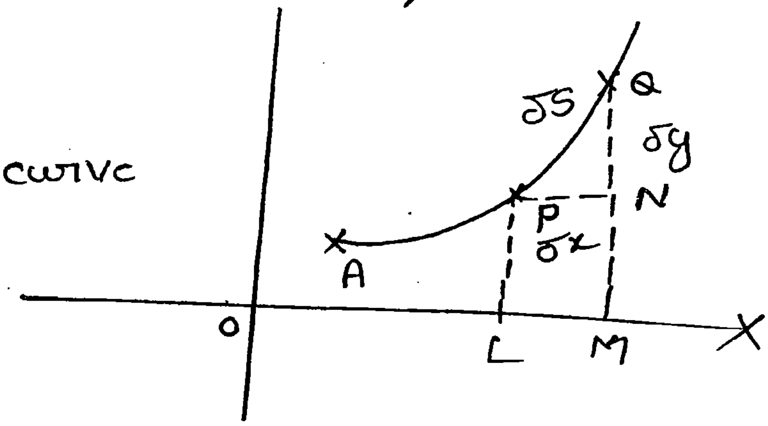
$$= 2\int_0^a y\,dx = 2\int_0^a x\sqrt{a^2-x^2}\,dx$$

put $x = a\sin\theta$ when $x=0,\ \theta=0$

$dx = a\cos\theta\, d\theta$ $\quad x=a,\ \theta=\pi/2$

$$= 2\int_0^{\pi/2} a^3 \sin\theta\cos^2\theta\, d\theta$$

$$= 2a^3\left|\frac{-\cos 3\theta}{3}\right|_0^{\pi/2} = \frac{2a^3}{3}.$$



ngth of the curve :-

ngth of the arc of the curve

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

From the triangle PQN

$$(PQ)^2 = (PN)^2 + (NQ)^2$$

$$(\delta s)^2 = (\delta x)^2 + (\delta y)^2$$

$$\left(\frac{\delta s}{\delta x}\right)^2 = 1+\left(\frac{\delta y}{\delta x}\right)^2$$

as $Q \to P,\ \delta s \to 0$

$$\boxed{\frac{ds}{dx} = \sqrt{1+(dy/dx)^2}}$$

$$\frac{ds}{dx} = \sqrt{1+(dy/dx)^2}$$

$$\int \frac{ds}{dx} = \int\sqrt{1+(dy/dx)^2}\,dx$$

$$\boxed{S = \int_{\alpha}^{\beta} \sqrt{1+(dy/dx)^2}\, dx}$$

1) The length of the curve $y = 2/3\, x^{3/2}$ b/w $x=0$ to $x=1$ is

a) 1.66 ~~b)~~ 1.22 c) 1.34 d) 1.86.

Sol

$$y = 2/3\, x^{3/2}$$

$$\frac{dy}{dx} = 2/3 \cdot 3/2\, x^{1/2} = \sqrt{x}$$

$$S = \int_0^1 \sqrt{1+x}\, dx = 2/3 \left| (1+x)^{3/2} \right|_0^1$$

$$= 2/3 \left(2^{3/2} - 1\right) = 1.22$$

2) $y = \log \sec x$, $x=0$, $x=\pi/4$ then

$S =$

$$y = \log \sec x$$

$$\frac{dy}{dx} = \frac{1}{\cancel{\sec} x} (\cancel{\sec} x \tan x) = \tan x$$

$$dy/dx = \tan x$$

$$S = \int_0^{\pi/4} \sqrt{1+\tan^2 x}\, dx = \int_0^{\pi/4} (\sec x)\, dx$$

$$= \left[\log (\sec x + \tan x)\right]_0^{\pi/4}$$

$$\boxed{S = \log(\sqrt{2}+1)}$$

The length of the parabola $y=x^2$, $x=0$, $x=1$ is.

$$y=x^2$$

$$\frac{dy}{dx}=2x \qquad S=\int_0^1 \sqrt{1+4x^2}\,dx$$

$$S=\int_0^1 (1+4x^2)^{1/2}\,dx = \cancel{\frac{2}{3}\left[(1+4x^2)^{3/2}\right]_0^1}$$

$$=\left| \tfrac{1}{2}\sin^{-1}(2x) + \frac{2x}{2}\sqrt{1+4x^2}\right|_0^1 \qquad \cancel{= 2/3\left[(1+4)^{3/2}-1\right] = 2/3\left[5^{3/2}-1\right] = 6.}$$

$$= \tfrac{1}{2}\sinh^{-1}(2)+\sqrt{5}$$

$$= \underline{\underline{1.4}}$$

Length of the parabola $y=x^3$. $x=0, x=1$

$$dy/dx = 3x^2$$

$$S=\int_0^1 \sqrt{1+9x^4}\,dx = 1.54$$

The Length of the curve $x=\cos^3\theta$, $y=\sin^3\theta$, $\theta=0$, $\theta=\pi/2$ is

a) $3/2$ b) $9/2$ c) $11/2$ d) $12/5$

$$y=\sin^3\theta$$

$$\frac{dy}{d\theta} = -3\sin^2\theta\cos\theta$$

If $x=f(\theta)$, $y=g(\theta)$ then

$$S=\int_\alpha^\beta \sqrt{f'(\theta)^2+g'(\theta)^2}\,d\theta$$

$$s = \int_0^{\pi/2} \sqrt{9\cos^4\theta \sin^2\theta + 9\sin^4\theta \cos^2\theta}$$

$$s = 3\int_0^{\pi/2} \sqrt{\cos^2\theta \sin^2\theta (\cos^2\theta + \sin^2\theta)}\, d\theta$$

$$= 3\int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = 3/2 \left| \sin^2\theta \right|_0^{\pi/2}$$

$$= 3/2(1) = 1.5$$

(অ)

$$s = \int_\alpha^\beta \sqrt{1 + (dy/dx)^2}\; \frac{dx}{d\theta} . d\theta$$

7) Length of the curves are given by $x = a(\theta + \sin\theta)$ $y = a(1 - \cos\theta)$, $\theta = 0, \pi/2$ then $s =$ ______

$$s = \int_0^{\pi/2} \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}\; d\theta$$

$$= \int_0^{\pi/2} a\sqrt{1 + \cos^2\theta + \cos\theta + \sin^2\theta}\; d\theta$$

$$= a\int_0^{\pi/2} \sqrt{2 + 2\cos\theta}\; d\theta$$

$$= \sqrt{2}a \int_0^{\pi/2} \sqrt{1 + \cos\theta}\; d\theta$$

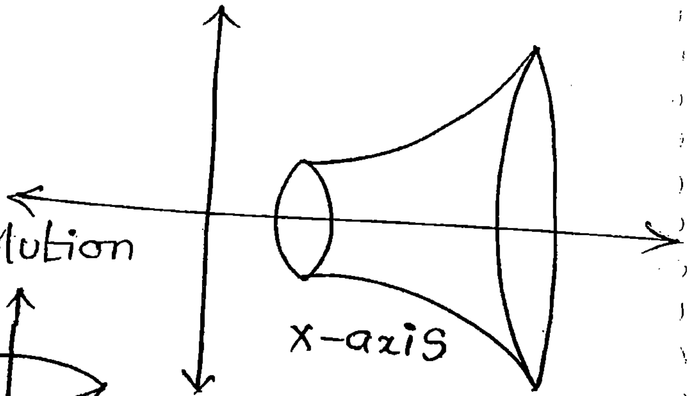
$$= 2a \int_0^{\pi/2} \cos\theta/2 \; d\theta$$

$$= 2a \times 2 \left| \sin\theta/2 \right|_0^{\pi/2} = 4a . \frac{1}{\sqrt{2}} = 2\sqrt{2}a$$

lumes:-

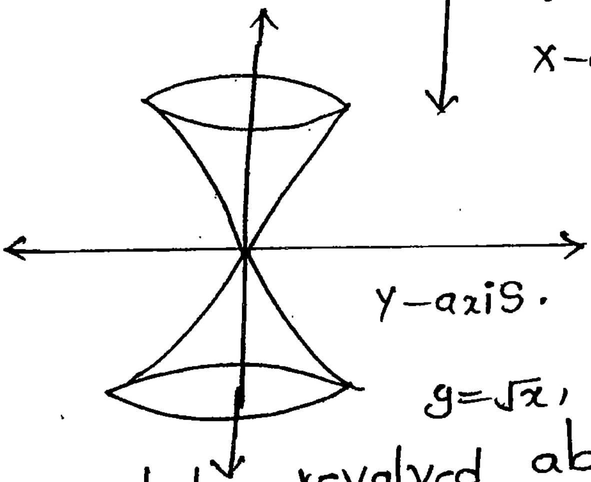
The volume of Solid of Revolution about x-axis is

$$V = \int_a^b \pi y^2 dx$$



The volume of Solid of Revolution about y-axis is

$$V = \int_a^b \pi x^2 dy$$



2M

One arc of the parabola $y=\sqrt{x}$, $x=1$, $x=2$ revolved about x-axis

Volume of Solid so generated by the curve is

$$V = \int_1^2 \pi x\, dx = \pi \left| \frac{x^2}{2} \right|_1^2 = \pi/2\,(4-1)$$

$$= 3\pi/2$$

$y=\cos x$, $x=0$, $x=\pi/2$ Revolved around the x-axis

the volume of the solid is.

$$V = \int_0^{\pi/2} \pi (\cos^2 x)\, dx$$

$$= \pi \int_0^{\pi/2} \cos^2 x\, dx = \pi \cdot 1/2 \cdot \pi/2 = \pi^2/4$$

The area of the parabola $y^2 = 8x$, $x=2$ revolved

around y-axis then the volume of Solid of

Revolution is

$$V=\int_a^b \pi x^2\, dy$$

$x=y^2/8$ $\quad y^2=16$

$y=\pm 4$

$y^2=8x$

$2y\,dy=8\,dx$

$dy=\left(\frac{8}{2}\right)\frac{dx}{y}$

$dy=4\,\frac{dx}{2\sqrt{2}\sqrt{x}}$

$$V=\pi\int_{-4}^{4}\frac{y^4}{64}\,dy = \frac{\pi}{64}\,2\int_0^4 y^4\,dy$$

$$=\frac{\pi}{32}\left[\frac{y^5}{5}\right]_0^4 = \frac{\pi}{32\times 5}\,[4]^5$$

$$=32\pi/5$$

partial differentiation :-

Function of two variables :-

Let $z=f(x,y)$ be given function z is a function of x,y $\quad V=f(r,h)$.

$z=f(x,y)$ where x and y are two independent variables the value of 'z' depending upon x,y and this type of functions are called as implicity functions.

Eg: $\quad z=x^3+y^3+3xy$

$u=\log(x^3+y^3+z^3)$

$u=f(x,y,z,r,s,t \text{ ------ })$ function of several variables.

Limit of function of two variables :-

If $f(x,y)\to l$ as $x\to a,\ y\to b$ then

$|f(x,y) - l| \leq \varepsilon$, $\varepsilon > 0$ ε = epsilon

$= .00001$

however $|x-a| < \delta_1$, $\delta_1 > 0$

$|y-b| < \delta_2$, $\delta_2 > 0$

Such that $\underset{\substack{x\to a \\ y\to b}}{Lt} f(x,y) = l$

If $\boxed{\underset{\substack{x\to a \\ y\to b}}{Lt} f(x,y) = \underset{\substack{y\to b \\ x\to a}}{Lt} f(x,y)}$ then only limit Exist otherwise limit does not Exist.

) $\underset{\substack{x\to 1 \\ y\to 2}}{Lt} \dfrac{x \cdot y}{x+y}$

$\underset{x\to 1}{Lt}\left[\underset{y\to 2}{Lt} \dfrac{xy}{x+y}\right] = \underset{x\to 1}{Lt}\left(\dfrac{2x}{2+x}\right) = 2/3.$

$\underset{y\to 2}{Lt}\left[\underset{x\to 1}{Lt} \dfrac{xy}{x+y}\right] = \underset{y\to 2}{Lt}\left(\dfrac{y}{1+y}\right) = 2/3.$

Hence both are Equal. Hence limit Exist.

) $\underset{\substack{x\to 0 \\ y\to 0}}{Lt} \dfrac{x-y}{x+y} = 0/0$ (Limit does not Exist)

$\underset{x\to 0}{Lt}\left|\underset{y\to 0}{Lt} \dfrac{x-y}{x+y}\right| = \underset{x\to 0}{Lt} \dfrac{x-0}{x+0} = x/x = 1$

$\underset{y\to 0}{Lt}\left|\underset{x\to 0}{Lt} \dfrac{x-y}{x+y}\right| = \underset{y\to 0}{Lt} -y/y = -1$

$\underset{\substack{x\to 0\\ y\to 0}}{Lt} f(x,y) \neq \underset{\substack{y\to 0\\ x\to 0}}{Lt} f(x,y)$ then Limit does not Exist.

For solving the above type of Equations

Substitute $y = mx$

$y \to 0 \quad mx \to 0$

$= \underset{\substack{x\to 0\\ mx\to 0}}{Lt} \dfrac{1-m}{1+m}$. The limit depend upon 'm'

For different values to 'm' we get different limits it is Not unique. Therefore limit does not Exist.

<u>continuous:-</u>

$\underset{\substack{x\to a\\ y\to b}}{Lt} f(x,y) = f(a,b)$ then the function is said to be continuous.

① $f(x,y) = \dfrac{2xy}{x^2+y^2}$ when $(x,y) \neq (1,2)$

$= 4/5$ when $(x,y) = (1,2)$

if $f(x,y)$ is continuous at $(1,2)$

$\underset{\substack{x\to 1\\ y\to 2}}{Lt} f(x,y) = \underset{\substack{y\to 2\\ x\to 1}}{Lt} f(x,y) = 4/5 = f(1,2)$

It is continuous funct

finition of partial differentiation:-

Let $z=f(x,y)$ be a given function where x,y are two independent variables. Keeping one variable kept constant. Let 'y' be kept constant then the 'z' reduces to the function 'x' only.

$$z+\delta z = f(x+\delta x, y)$$

$$\boxed{\delta z = f(x+\delta x, y) - f(x,y)}$$

$$\underset{\delta x\to 0}{Lt} \frac{\delta z}{\delta x} = \underset{\delta x\to 0}{Lt} \frac{f(x+\delta x, y)-f(x,y)}{\delta x}$$ is called

$\frac{\partial z}{\partial x}$ (or) f_x i.e partial differential co-efficient of 'z' w.r.to 'x'

$$\underset{\delta y\to 0}{Lt} \frac{\delta z}{\delta y} = \underset{\delta y\to 0}{Lt} \frac{f(x, y+\delta y)-f(x,y)}{\delta y}$$

is called partial differential co-efficient of 'z' w.r.to 'y' and it is denoted as

$$\frac{\partial z}{\partial y} \text{ (or) } f_y$$

partial differentiation is Nothing but ordinary differentiation keeping one of the variable is constant.

physically $\frac{dz}{dx}$, $\frac{dz}{dy}$ gives the Rate of change of z w.r.to x and y.

→ Geometrically partial differentiation represents the Equation of surfaces.

$$\int \frac{dz}{dx} = \int xy$$

$$\boxed{z = \frac{x^2}{2}y + \phi(y)}$$ ← gntegral curves

$\phi(x, y, z) = 0$ Equations of the surface.

$\phi(x, y, z) = c$ Equations of the level surface.

→ Therefore partial differentiation is the ordinary differentiation with one variable constant.

① $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

$\frac{d}{dy}(u \pm v) = \frac{du}{dy} \pm \frac{dv}{dy}$

2) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{d}{dy}(uv) = u\frac{dv}{dy} + v\frac{du}{dy}$.

3) $\frac{d}{dx}(u/v) = \frac{v\frac{du}{dx} - u\, dv/dx}{v^2}$

• If $u = x^y$ then

$$\boxed{\frac{\partial u}{\partial y} = x^y \log x}$$

If $u = x\log(xy)$ then

$$\frac{\partial u}{\partial x} = \cancel{x}\,\frac{1}{\cancel{x}y}\cdot y + \log(xy)$$

$$\boxed{\frac{\partial u}{\partial x} = 1+\log(xy)}$$

If $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^3-3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z} = \frac{3(\cancel{x^2+y^2+z^2-xy-yz-zx})}{(x+y+z)(\cancel{x^2+y^2+z^2-xy-yz-zx})}$$

$$= \frac{3}{(x+y+z)}$$

Q. If $x = r\cos\theta$, $y = r\sin\theta$ then

$\frac{\partial\theta}{\partial x}$ a) $\frac{\sin\theta}{r}$ b) $\frac{\cos\theta}{r}$ c) $\frac{-\sin\theta}{r}$ d) $-\cos\theta/r$

$\tan\theta = y/x$ $\quad x^2+y^2=r^2$

$\theta = \tan^{-1}(y/x)$

$$\frac{d\theta}{dx} = \frac{1}{1+y^2/x^2}(-y/x^2) = \frac{x^2}{x^2+y^2}(-y/x^2)$$

$$= \frac{-r\sin\theta}{r^2} = \frac{-\sin\theta}{r}$$

$$\boxed{d\theta/dx = \frac{-\sin\theta}{r}}$$

$$\frac{dr}{dx} = \frac{x}{r} \qquad x^2+y^2=r^2$$

$$2x+0 = 2r\frac{dr}{dx}$$

$$\boxed{\frac{dr}{dx} = \cos\theta}$$

$$\boxed{\frac{dr}{dy} = \sin\theta}$$

If $v=r^n$ and $r^2 = x^2+y^2+z^2$ then $\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2}$ is

~~a)~~ $n(n+1)r^{n-2}$ b) $n(n-1)r^{n-2}$ c) $n(n+2)r^{n-2}$

d) $n(n-2)r^n$

Sol

$V = r^n$

$$\frac{dv}{dx} = nr^{n-1}\frac{dr}{dx} = nr^{n-1}\frac{x}{r} = nr^{n-2}\cdot x$$

$$\frac{d^2v}{dx^2} = n(n-2)r^{n-3}\frac{dr}{dx}\cdot x + nr^{n-2}$$

$$= n(n-2)r^{n-4}x^2 + nr^{n-2}$$

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = n(n-2)r^{n-4}(x^2+y^2+z^2) + 3nr^{n-2}$$

$= n(n-2)\, r^{n-2} + 3n r^{n-2}$

$= [n(n-2) + 3n]\, r^{n-2}$

$= n(n+1)\, r^{n-2}$

If $u = (x^2+y^2+z^2)^{-1/2}$ then find

$u_{xx} + u_{yy} + u_{zz}$

a) -1 $\quad$ b) 0 $\quad$ c) $1/2$ $\quad$ d) 1

$$\frac{\partial u}{\partial x} = -1/2\,(x^2+y^2+z^2)^{-3/2}(2x)$$

$$= -(x^2+y^2+z^2)^{-3/2}(x)$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x(-3/\cancel{2})(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2}\right]$$

$$= +\left[3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2}\right]$$

$$u_{xx} + u_{yy} + u_{zz} = 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2}$$
$$-3(x^2+y^2+z^2)^{-3/2} = 0.$$

If $u = f(r)$ where $r^2 = x^2+y^2$ then

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$

a) $f''(r) - \frac{1}{r} f'(r)$

b) $f''(r) - 2/r\, f'(r)$

c) $f''(r) + 1/r\, f'(r)$

d) $f''(r) + \frac{2}{r} f'(r)$

$u = f(r)$

$$\frac{du}{dx} = f'(r)\frac{dr}{dx} = f'(r)\frac{x}{r} = \frac{xf'(r)}{r}$$

$$\frac{d^2u}{dx^2} = \cancel{\frac{f''(r)}{r}} \quad \frac{r\left[xf''(r)\frac{dr}{dx} + f'(r)\cdot 1\right] - xf'(r)\frac{dr}{dx}}{r^2}$$

$$\frac{d^2u}{dx^2} = \frac{x^2f''(r) + rf'(r) - \frac{x^2}{r}f'(r)}{r^2} \qquad \frac{dr}{dx} = x/r$$

$$\frac{\partial^2u}{\partial x^2} + \frac{d^2u}{dy^2} = \frac{(x^2+y^2)f''(r) + 2rf'(r) - \left(\frac{x^2+y^2}{r}\right)f'(r)}{r^2}$$

$$= \frac{r^2f''(r) + 2rf'(r) - \frac{r^2}{r}f'(r)}{r^2}$$

$$= \frac{r^2f''(r) + 2rf'(r) - rf'(r)}{r^2}$$

$$\boxed{\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = f''(r) \cdot \frac{1}{r}f'(r)}$$

Homogeneous function:-

definition:- Let $z = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \cdots + a_ny^n$ be a polynomial of nth order in x,y

$$z = x^n\left[a_0 + a_1(y/x) + a_2(y/x)^2 + \cdots a_n(y/x)^n\right]$$

$z = x^n f(y/x)$ is called homogeneous function of nth order

where $f(y/x) = a_0 + a_1(y/x) + a_2(y/x)^2 + \cdots$

: ① $z = x^2+y^2+2xy$

$= x^2\left[1+(y/x)^2+2(y/x)\right]$ order $n=2$

$u = 1/x+1/y = \frac{x+y}{xy} = \frac{\cancel{x}(1+y/x)}{\cancel{x^2}(y/x)}$

$= x^{-1}f(y/x)$

$\boxed{\text{order } n=-1}$

If the given function is homogenous function then order is.

$u = \frac{x^3+y^3}{xy} = \frac{x^3\left(1+(y/x)^3\right)}{x^2(y/x)} = x\,f(y/x)$

order $=1$

$u = \frac{x^{1/3}-y^{1/3}}{x^{1/2}+y^{1/2}}$ = then order $= 1/3-1/2 = -1/6$

$u = \frac{xy(x^3+y^3)^2}{(x+y)} = \frac{x^2(y/x)\,(x^3)^2\left[1+(y/x)^3\right]}{x(1+y/x)}$

$= \frac{x^8}{x}\,f(y/x) = x^7 f(y/x)$

$\boxed{\text{order } n=7}$

$u = x^3+y^3+3xy$

It is Not Expressed in the above form then it will be not a homogenous function.

Euler's theorem:-

→ It is applicable to homogenous function

If $u \to f(x,y)$ be a homogenous function of the nth order then $\boxed{x\frac{du}{dx} + y\frac{du}{dy} = nu.}$

1) If $u = \frac{x^3+y^3}{xy}$ then $xu_x + yu_y = u$ $\frac{x^{\cancel{3}}(1+(y/x)^3)}{x^{\cancel{2}}(y/x)}$

order = 1

2) $u = x^2+y^2+xy$ then $xu_x + yu_y = \underline{2u}$ order = 2

3) $u = \frac{xy}{x+y}$ then $x\cdot u_x + y u_y = 1\cdot u = \underline{u}.$ order = 1

) If $u = \frac{x^3+y^3}{xy}$ then $xu_x + yu_y = 1\cdot u = \underline{u}.$

) If $u = \frac{x^{1/3}-y^{1/3}}{x^{1/2}+y^{1/2}}$ then $xu_x + yu_y = \underline{-1/6\,u}$

5) If $u = \log\left\{\frac{x^3+y^3}{x+y}\right\}$ then $xu_x + yu_y$

a) 2 b) −1 c) 3 d) 4

Sol $f(u) = e^u$

$n = 2$

→ whenever $\log(\ )$ function given then $2\frac{e^u}{e^u}$

$xu_x + yu_y$ is the order of the function $= 2$

$\log\left[\frac{x^{\cancel{3}}(1+(y/x)^3)}{\cancel{x}(1+y/x)}\right]$ order = 2

7) $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then $xu_x + yu_y$

$$\boxed{\text{F} \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)}}$$

Hence from the above formula

$$xu_x + yu_y = ①\left[\frac{\sin u}{\cos u}\right] = \underline{\tan u}$$

$$f(u) = \sin u.$$

$$u = \tan^{-1}\left[\frac{x^3+y^3}{x+y}\right] \text{ then } xu_x + yu_y$$

$$= n\frac{f(u)}{f'(u)} \qquad f(u) = \tan u$$

$$f'(u) = \sec^2 u$$

$$= 2\frac{\tan u}{\sec^2 u} = 2\left(\frac{\sin u}{\cancel{\cos u}}\right)(\cos^{\cancel{2}} u)$$

$$\boxed{xu_x + yu_y = \sin 2u}$$

cond order theorem:-

atement:-

If $u \to f(x,y)$ be a homogeneous function of n^{th} order then $\boxed{x^2 u_{xx} + 2xy\, u_{xy} + y^2 u_{yy} = n(n-1)u.}$

1) $u = \frac{x+y}{xy}$ then $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy}$

$$= n(n-1)u$$

$$\frac{x(1+y/x)}{x^2(y/x)}$$

$$= -1(-1-1)u$$

$$= -1(-2)u = \underline{2u}$$

(2) $u = x^2+y^2/xy$ then $x^2u_{xx} + 2xy\,u_{xy} + y^2u_{yy}$

$$\frac{\cancel{x^2}\,(1+(y/x)^2)}{\cancel{x^2}\,(y/x)} \qquad n=0 \qquad = n(n-1)u$$

$$= 0(0-1)u$$

$$= -u$$

) $u = \log\left\{\frac{x^3+y^3}{x+y}\right\}$ then $x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial x} + 2xy\frac{\partial^2 u}{\partial x\partial y} = \_\_\_\_$

a) -1 ~~b) -2~~ ~~(c)~~ 2 d) 1

→ If the log, cos, sin different forms are given

then (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{n f(u)}{f'(u)} = \underline{F(u)}$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \underline{F(u)\left[F'(u)-1\right]}$

Sol $f(u) = e^u$ $\qquad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2 = F(u)$

$= 2(\cancel{2-1}) = \cancel{2}$

1) $x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = 2(0-1) = \underline{\underline{-2}}$

→ $u = \cos^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then $x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy}$

$x\,u_x + y\,u_y = -\cot u = F(u)$

$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy} = -\cot u\left[\operatorname{cosec}^2 u - 1\right]$

$= \underline{-\cot^3 u}$

5) $u = \sin^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then

$x^2u_{xx} + y^2u_{yy} + 2xy\,u_{xy}$ is

$x u_x + y u_y = \tan u = f(u)$

$^2 u_{xx} + y^2 u_{yy} + 2xy\, u_{xy} = \tan u\,(\sec^2 u - 1) = \tan^3 u$

<u>rivative of composite function :-</u>

Let $z = f(x, y)$ be a given function
nd $x = \phi(t)$, $y = \psi(t)$ then z is said to be
mposite function of (t)

then

$$\boxed{\frac{dz}{dt} = \frac{\partial t}{\partial x}\frac{dx}{dt} + \frac{\partial t}{\partial y}\frac{dy}{dt}}$$ is called derivative
f <u>composite function</u> (या) <u>total derivative</u>
<u>function</u>.

) $z = x^2 + y^2$, $x = t^2 + 1$, $y = 2t + 1$

$$\frac{dz}{dt} = \frac{\partial t}{\partial x}\frac{dx}{dt} + \frac{\partial t}{\partial y}\frac{dy}{dt}.$$

$$= (2t)(2x) + (2)(2y)$$

$$= 4tx + 4y$$

$$\boxed{\frac{dz}{dt} = 4t(t^2+1) + 4(2t+1)}$$

$z = x^2 - y^2$, $x = t^2 + 1$, $y = t^2 - 1$

$$\frac{dz}{dt}\,(t=1) = 2x(2t) + 2t(-2y)$$

$$= 4xt - 4ty$$

$= 4t(t^2+1) - 4t(t^2-1)$

$= 8t$

$\boxed{\frac{dz}{dt} = 8}$

Formula

I. If $u = f(x, y)$ then $du = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ is called total derivative of x^2y is

$u = x^2y$

$\boxed{du = 2xy\,dx + x^2 dy}$

2) If $u = xy$ then

$du = x\,dy + y\,dx$

) $u = x\log(xy)$ then

$du = \left[\cancel{x}\frac{1}{\cancel{x}y}(y) + \log(xy)\right]dx + \left[\cancel{x}\frac{1}{\cancel{x}y}(x)\,dy\right]$

$\boxed{du = [1+\log(xy)] + (x/y)\,dy}$

) $u = x\sin xy$

$du = [x\cos(xy)(y) + \sin xy]\,dx + [x\cos(xy)(x)]\,dy$

$= (xy\cos(xy) + \sin xy)\,dx + (x^2\cos(xy))\,dy$

III If $u = f(x, y)$ then

$\frac{du}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\,dy/dx$ is called the total derivative of (u) w.r.to (x)

The total derivative of x^2y when x & y are [re]lated by the $x^2+y^2+xy=1$ is.

$u = x^2 y$.

$$x^2+y^2+xy=1$$

$$\frac{du}{dx} = 2xy + x^2\left(\frac{dy}{dx}\right)$$

$$2x + 2y\,dy/dx + x\frac{dy}{dx} + y = 1$$

$$\frac{du}{dx} = 2xy + \frac{x^2(-2x+y)}{2y+x}$$

$$\cancel{dy/dx\,(2y+x) = 1}$$

$$\boxed{\frac{du}{dx} = 2xy - x^2(2x+y)/(x+2y)}$$

$$\boxed{\cancel{dy/dx = \frac{1}{2y+x}}}$$

$$dy/dx\,[2y+x] = -1\,[2x+y]$$

$$dy/dx = \frac{-(2x+y)}{(2y+x}$$

If $u = x\log(xy)$, $x^3+y^3=3xy$ then

$$\frac{du}{dx} = \left(\cancel{x}\frac{1}{\cancel{x}y}(y) + \log xy\right] + \left(\cancel{x}\frac{1}{\cancel{x}y}x(x)\right)\frac{dy}{dx}$$

$$\frac{du}{dx} = [1+\log xy] + (x/y)\,dy/dx$$

$$x^3+y^3 = 3xy$$

$$3x^2+3y^2\,dy/dx = 3x\,dy/dx + 3y$$

$$[3x^2-3y] = dy/dx\,[3x-3y^2]$$

$$dy/dx = \frac{3x^2-3y}{3x-3y^2} = \left(\frac{x^2-y}{x-y^2}\right)$$

$$\boxed{\frac{du}{dx} = [1+\log xy] + \frac{x}{y}\left[\frac{x^2-y}{x-y^2}\right]}$$

3) If $u = x^2+y^2$, $xy = 1$ then $\frac{du}{dx}$

$$\frac{du}{dx} = 2x + 2y\,(dy/dx)$$

$$\left[\begin{aligned} xy &= 1 \\ x\,dy/dx + x &= 1 \\ x\,dy/dx &= 1-x \\ dy/dx &= \frac{1-x}{x} \end{aligned}\right]$$

$y = 1/x$

$y' = -1/x^2$

$$\frac{du}{dx} = 2x + 2y\,\cancel{\left(\frac{1-x}{x}\right)}\,(-1/x^2)$$

$$\boxed{\frac{du}{dx} = 2x + 2y\,(-1/x^2)}$$

IV If $u = f(x,y) = c$ then $\boxed{du = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0}$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y}\right)dy = 0$$

$$\boxed{dy/dx = -\left[\partial f/\partial x \,/\, \partial f/\partial y\right]}$$

is called derivative of implicity function

1) If $u = \log(x+y) + \sin(x+y) = 0$ then the value of dy/dx is

$$\frac{dy}{dx} = -\left[\frac{\frac{1}{(x+y)} + \cos(x+y)}{\frac{1}{(x+y)} + \cos(x+y)}\right] = 1$$

$$\boxed{dy/dx = 1}$$

If $u = x\sin(xy) = 0$ then

$$dy/dx = \left[\frac{-x\cos(xy)y + \sin xy}{+x\cos(xy)(x)}\right]$$

$$dy/dx = \frac{-xy\cos(xy) + \sin xy}{x^2\cos(xy)}$$

$$\boxed{dy/dx = \frac{-y}{x} + \frac{1}{x^2}\tan(xy)}$$

1 If $u = f(x-y, y-z, z-x)$ then the value of

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \_\_\_\_$$

> The Method is

Let $r = x-y$, $s = y-z$, $t = z-x$ then the unction $u = f(r,s,t) \rightarrow$ is called the function of function of implicity function then

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial x}$$

$$\frac{\partial r}{\partial x} = 1 \qquad \frac{\partial s}{\partial x} = 0 \qquad \frac{\partial t}{\partial x} = -1$$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} - \frac{\partial f}{\partial t} \longrightarrow (1)$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\cdot\frac{\partial r}{\partial y} + \frac{\partial f}{\partial s}\cdot\frac{\partial s}{\partial y} + \frac{\partial f}{\partial t}\cdot\frac{\partial t}{\partial y}$$

$$\frac{\partial r}{\partial y} = -1 \qquad \frac{\partial s}{\partial y} = 1$$

$$\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial r} + \frac{\partial f}{\partial s} \longrightarrow ②$$

$$\frac{\partial u}{\partial z} = -\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \longrightarrow ③$$

$$u_x + u_y + u_z = \cancel{\frac{\partial f}{\partial r}} - \cancel{\frac{\partial f}{\partial t}} - \cancel{\frac{\partial f}{\partial r}} + \cancel{\frac{\partial f}{\partial s}} - \cancel{\frac{\partial f}{\partial s}} + \cancel{\frac{\partial f}{\partial t}}$$

$$\boxed{u_x + u_y + u_z = 0.}$$

7. If $u = f(x/y, y/x)$ then $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$.

$r = x/y \quad s = y/x$

$$\frac{\partial u}{\partial x} = \frac{\partial f}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial f}{\partial s} \cdot \frac{\partial s}{\partial x}$$

$$\frac{\partial r}{\partial x} = 1/y \quad \frac{\partial s}{\partial x} = -y/x^2$$

$$\frac{\partial u}{\partial x} = 1/y \left(\frac{\partial f}{\partial r}\right) - y/x^2 \left(\partial f/\partial s\right) \longrightarrow ①$$

$$\frac{\partial u}{\partial y} = \frac{\partial f}{\partial r}\left(-x/y^2\right) + 1/x \frac{\partial f}{\partial s}$$

$$\boxed{u_x + u_y = \frac{\partial f}{\partial r}\left[1/y - x/y^2\right] + \frac{\partial f}{\partial s}\left[1/x - y/x^2\right]}$$

Applications of partial derivatives:-

Maximum value and Minimum value for the function of single variable:-

The Necessary condition for the function $f(x)$ is said to have maxima & minima is $\boxed{f'(x) = 0}$

Solve for 'x' the value of 'x' are called the

ationary values

fficient condition :-

find the $f''(x)$ at the point

f $f''(x) > 0$ at the point then $f(x)$ is said to be

~~Maxima~~ at the point (Local minima)

ninima

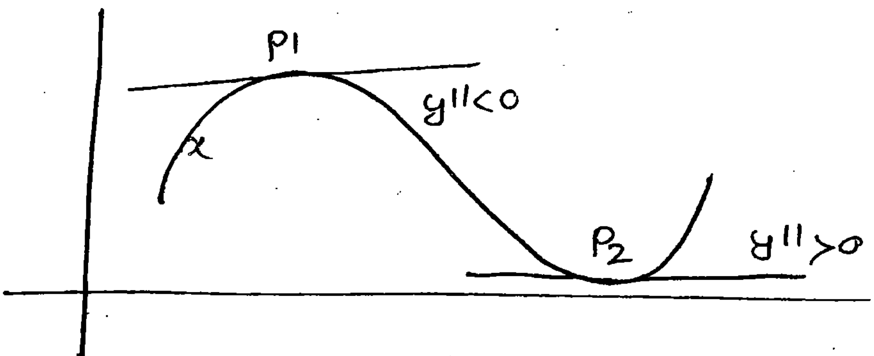
- If $f''(x) < 0$ at the point then $f(x)$ is said to

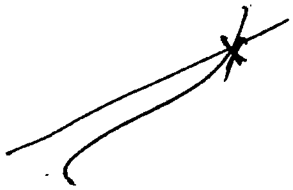
be Maxima at the point (Local Maxima)

If $f''(x) = 0$ at the point then $f(x)$ is not an

Extreme. (No maxima, No minima). This point is

called point of inflection.





- For the function $f(x) = (x^2-4)^2$ has

i) only one minima b) two minima c) two maxim

d) Three Maxi

$$f(x) = (x^2-4)^2$$

$$f'(x) = 2(x^2-4)(2x)$$

$$f''(x) = 4[(x^2-4) + x(2x)]$$

$$f''(4) = 4[x^2-4+2x^2]$$

$$f''(x) = 4[3x^2-4]$$

$$f'(x) = 0$$

$$4x(x^2-4) = 0$$

$$x=0, \quad x^2-4=0$$

$$x=0, \quad x=\pm 2$$

$$f''(x) = 4[3x^2-4]$$

$$f''(x) = -16 < 0 \quad \text{Maxima}$$

$$\left.\begin{array}{l} f''(-2) > 0 \\ f''(2) > 0 \end{array}\right\} \text{Minima.}$$

Two Minima

* The minimum value of $f(x) = x\log x$ is.

$$f'(x) = \frac{x}{x} + \log x$$

$$f'(x) = 1 + \log x = 0$$

$$\log x = -1$$

$$x = e^{-1}$$

$$x = 1/e$$

$$f''(x) = 1/x = e$$

Minim value is

$f(1/e) = 1/e \log(1/e)$

$= 1/e\ [\log(1) - \log e]$

$f(1/e) = -1/e$

The function $f(x) = x^2 e^{-x}$, the Maximum occurs when x is equal to

a) 2 b) 1 c) 0 d) -1

a) $f(x) = x^2 e^{-x}$

$f'(x) = 2x e^{-x} - x^2 e^{-x}$

$e^{-x}(2x - x^2) = 0$

$x = 0, \quad x = 2$

$f''(x) = e^{-x}(2 - 2x) - e^{-x}(2x - x^2)$

ON substituting $x = 0$

$f''(0) = e^{-0}(2) - {}^{-0}(0)$

$f''(0) = 2 > 0$ Minima

$f''(2) = e^{-2}(2-4) - e^{-2}(0) < 0$

Maximum at $x = 2$

The Maxima and minima of the function

$f(x) = 2x^3 - 15x^2 + 36x + 10$ occurs at

1) $x = 3$ (or) $x = 2$ b) $x = 2$ (&) $x = 3$ c) $x = 3$ & $x = 1$

d) $x = 4$ & $x = 1$

$f'(x) = 6x^2 - 30x + 36.$

$f'(x) = x^2 - 5x + 6$

$f'(x) = (x-2)(x-3)$

$x = 2, 3$

$f''(x) = 12x - 30$

$f''(2) = 12(2) - 30 < 0$ Maxima

$f''(3) = 12(3) - 30 > 0$ Minima

- The function $f(x) = x^2 - x - 2$, the Maximum value of $f(x)$ in the closed interval $[-4, 4]$ is.

a) 18 b) 10 c) -2.25 d) None

$f'(x) = 2x - 1 = 0$ $x = 1/2$

$f''(x) = 2$

$f(-4) = 16 + 4 - 2 = 18$

$f(4) = 16 - 4 - 2 = 10$

$f(-4) > f(4)$ then the Maximum value is 18 //

Maxima & Minima function of two variables

Let $f(x, y) = 0$ be given function the necessary condition for $f(x,y)$ is said to be $\frac{\partial f}{\partial x} = 0$ ① , $\frac{\partial f}{\partial y} = 0$ ②

→ Solving Eq ①, ② for x and y

Sufficient condition:-

find $r = \partial^2 f/\partial x^2$, $t = \partial^2 f/\partial y^2$, $s = \frac{\partial^2 f}{\partial x \partial y}$ at the point

If $rt-s^2>0$, $r>0$ at the point then $f(x,y)$ is

id to be minima at that point.

If $rt-s^2>0$, $r<0$ at the point then $f(x,y)$ is

id to be Maxima at that point.

If $rt-s^2<0$ at the point $f(x,y)$ is not an

xtreme (No maxima, No minima) this point is

alled <u>saddle point</u>

If $rt-s^2=0$ at the point then further

consideration [i-e choose other point]

$$\boxed{f_{xy}^2 - f_{xx} f_{yy} > 0 \quad \text{saddle point}}$$

At (0,0) the function $f(x,y) = x^2-y^2$ has

) Minima b) Maxima c) Saddle point d) None

Sol

$$\frac{df}{dx} = 2x \qquad \frac{d^2f}{dx^2} = 2 = r$$

$$\frac{df}{dy} = -2y \qquad \frac{d^2f}{dy^2} = -2 = t$$

$$\frac{df}{dydx} = 0 = s \qquad rt-s^2 = -4<0$$

$= \text{Saddle point}$

The minimum Value of $f(x,y) = xy + \frac{9}{x} + \frac{3}{y}$ is

a) 3 b) 6 c) 9 d) 12

$$\frac{df}{dx} = y - \frac{9}{x^2} = 0 \qquad x^2y = 9 \Rightarrow y = \frac{9}{x^2}$$

$$\frac{df}{dy} = x - \frac{3}{y^2} = 0 \qquad xy^2 = 3$$

$$x = 3/y^2$$

$$\frac{9}{y^4}(y) = 9 \qquad x = \frac{3}{81/x^4}$$

$$\frac{\cancel{9}}{y^3} = \cancel{9} \qquad x^3 = 27$$

$$x = 3.$$

$$y^3 = 1$$

$$y = 1$$

$$(x,y) = (3,1)$$

$$f(3,1) = 3+3+3 = 9$$

* The minimum value of $f(x,y) = x^3 + y^3 - 3xy$ is.

$$\frac{df}{dx} = 3x^2 - 3y \qquad x^2 - y = 0 = r$$

$$\frac{df}{dy} = 3y^2 - 3x \qquad y^2 - x = 0 \qquad \frac{df}{dxdy}$$

$$t$$

$$x^2 = y$$

$$x^4 - x = 0$$

$$x(x^3 - 1) = 0$$

$$x = 0 \quad x = 1 \Rightarrow y = 0, \quad y = 1.$$

Stationary values $(0,0)$ $(1,1)$

$$r = 6x$$

$$t = 6y$$

$$s = 3.$$

at $(1,1)$ $r=6,\ t=6,\ s=-3$

$rt-s^2=36-9>0,\ r>0.$

min at $(1,1)$

$f(1,1)=1+1-3=-1$

10

$f(x,y)=4x^2+6y^2-8x-4y+8$ the optimal value of

(x,y) is

a) Minimum value to $10/3$.

b) Maximum value to $10/3$.

c) min $8/3$ d) Max $8/3$.

$$\frac{df}{dx}=8x-8=0 \Rightarrow x=1$$

$$\frac{df}{dy}=12y-4=0 \Rightarrow y=1/3$$

$r=8$ $rt-s^2=(8)(12)=96>0,\ r>0.$

$t=12$ Minimum at $(1,1/3)$.

$s=0$

$$f(1,1/3)=4+6/9-8/3-4/3+8$$

$$=10/3.$$

09 2M

The distance b/w the origin and the point nearest to it on the surface $z^2=1+xy$ is

a) 1 b) $\sqrt{3}/2$ c) $\sqrt{3}$ d) 2

$$(OP)^2=f(x,y,z)=(x-x_1)^2+(y-y_1)^2+(z-z_1)^2$$

$(OP)^2 = (x^2) + y^2 + z^2$

put $z^2 = 1+xy$

$f(x,y) = x^2+y^2+1+xy$

$\frac{\partial f}{\partial x} = 2x+y = 0 \longrightarrow (1)$

$\frac{\partial f}{\partial y} = 2y+x = 0 \longrightarrow (2)$

from (1), (2)
stationary points.
$(0,0)$

$\frac{\partial^2 f}{\partial x^2} = 2 = r \quad \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] = 1 = s$

$\frac{\partial^2 f}{\partial y^2} = 2 = t \quad r=2, \; t=2 \quad s=1$

$rt - s^2 = 4-1 > 0, \quad r>0.$

Minimum at $(0,0)$

$f(x,y) = 1+0+0+0 = 1$

10/

Numerical Methods (2-M).

Solution for Linear Equations / Non linear Equations / Transidental Equation.

Solution for N differential Equation.

Solution for N. Eq / L. Eq / T. Eq :-

definition of transdental Equation

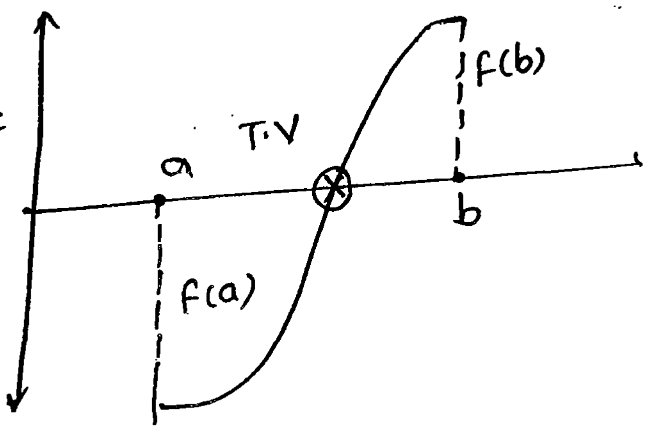
An Equation which involves, Exponsial, trignometric and logarthemic terms then the Equation is known as transdental Equation

$$f(x) = xe^{x} - \cos x$$

$$f(x) = \cos x - \log x + x^2$$

Intermediate value property

If $f(x)$ is continous $[a, b]$. $f(a), f(b)$ are different signs $(f(a) \cdot f(b) < 0)$ then there exist **atleast one** Root in the interval (a, b)



In general we can find the initial approximations for the Solution of transdental Equation using intermediate value property

Rate of convergence:-

If $\frac{e_{i+1}}{e_i}$ is almost constant. the Rate of convergence is known as slow convergence and the ... nvergence is ... der of convergence is linear (or) first order.

If $\frac{e_{i+1}}{e^p}$ is Nearer to the constant the Rate ... convergence is known as faster. and the order of convergence is p^{th} order $(p>1)$

Methods:-

- Bisection Method
- Regular false Method
- Secant Method
- Newton Raphson's Method.

Bisection Method:- (Halving)

Iterative formula $= \frac{(+ve)+(-ve)}{2}$

Procedure:- If $f(x)$ is continous in $[a,b]$, $f(a)$ is Negative, $f(b)$ +ve

$f(x) \rightarrow [a,b]$

$f(a) = -ve$

$f(b) +ve$

$f_A = b+a/2 = x_1 \qquad f(x_1) = -ve$

$SA = \frac{b+x_1}{2} = x_2$; $f(x_2). = +ve$

$SA = \frac{x_1+x_2}{2} = x_3$ (till accuracy)

This method is guarantee to convert but very slow ince we are reaching to the true value on two sides of the polynomial.

→ overall rate of convergence is slow convergence and order of convergence is linear

→ In this method we are reducing 1/2 factor of Error on step by step. Therefore length of the interval at the N^{th} step.

$$\boxed{\frac{|b-a|}{2^n} \leq \epsilon}$$

where (ϵ) is a small Error quantity. for

Example $[0,1]$, $\epsilon = 10^{-2}$. $n = ?$

$$\frac{1-0}{2^n} \leq 10^{-2}$$

$$\frac{1}{2^n} \leq \frac{1}{100} \Longrightarrow n = 6.67$$

$$n \simeq 7$$

→ By using this method we cannot locate the complex Roots of the Equation

Find 3rd approximation for the function

$f(x) = x^3 - 4x - 9 \longrightarrow [2, 3]$

$f(2) = (2)^3 - 8 - 9 = -9$

$f(3) = 27 - 12 - 9 = 6.$

$F = \frac{2+3}{2} = 2.5$; $f(2.5) = -ve$

$A = \frac{2.5+3}{2} = 2.75$; $f(2.75) = +ve$

$TA = \frac{2.5+2.75}{2} = \underline{\underline{2.625}}$

$f(x) = xe^x - 1$ lies b/w $[0, 1]$ then find the 3rd approximation.

$f(0) = 0 \cdot e^0 - 1 = -1$

$f(1) = 1 \cdot e^1 - 1 = 1.718$

$FA = \frac{0+1}{2} = 0.5$; $f(0.5) = +ve$

$SA = \frac{1+0.5}{2} = 0.75$; $f(0.75) = +ve$

$TA = \frac{0.5+0.75}{2} = 0.625$

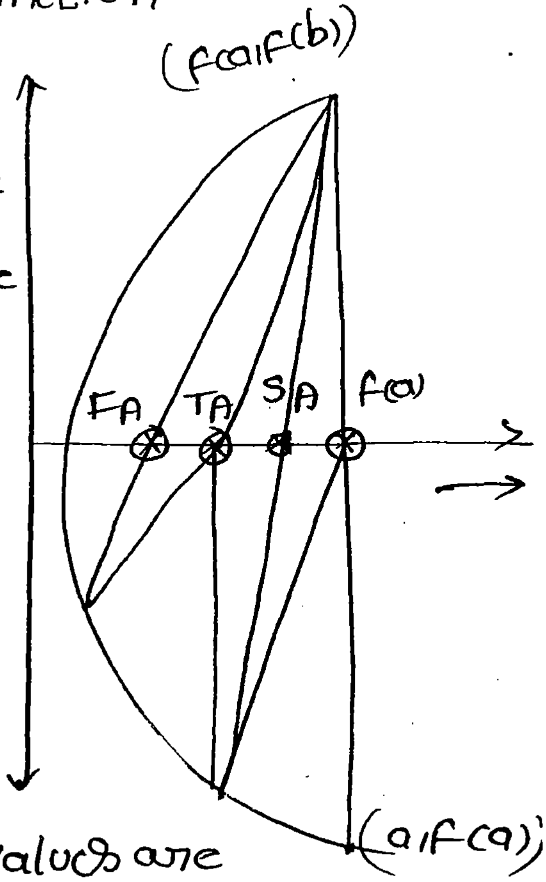
<u>Regular</u> <u>false</u> <u>Method</u> (false position)

$$\boxed{x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \cdot f(x_0)}$$

x_0 = Negative Root of the function

x_1 = positive " "

→ This method is also guarantee to convert and also faster than the Bisection method.



→ Since we reaching to the true value on one side.

→ overall rate of convergence is slow and order of converge is linear

⇒ If the first approximation values are -ve till the end of problem approximations are -ve (Root is approaches to positive otherwise opposite direction)

By using this method also we cannot find out the complex Roots of the Equation.

* find 3rd approximation for the function

$f(x) = xe^x - \cos x \rightarrow [0, 1]$

$f(0) = -1$

$f(1) = 2.179$. (Radian)

$$f_A = 0 - \frac{1-0}{2.179+1}(-1)$$

$$= 1/3.179 = 0.3146$$

$$\boxed{f(0.3146) = -0.51987}$$

$$S_A = 0.3146 - \frac{1-0.3146}{2.179+0.519}(1-0.51987)$$

$$\boxed{S_A = 0.4462}$$

$$f(0.4462) = -0.2049$$

$$T_A = 0.4462 + \frac{1-0.4462}{2.179+0.2049}(+0.2049)$$

$$\boxed{T_A = 0.49402}$$

$f(x) = x\log_{10} x - 1.2$ which lies b/w [2,3]

It is a transcidental function

$$f(2) = -0.5979$$

$$f(3) = 0.2316.$$

$$f_A = 2 - \frac{3-2}{0.2316+0.5979}(-0.5979)$$

$$f_A = 2.72102$$

$$f(2.72102) = -0.01709$$

$$SA = 2.72102 + \frac{3-2.72102}{0.2316+0.01709} \cdot (0.01709)$$

$SA = 2.7401 \simeq 2.740$

$f(2.740) = -0.00038$

$$TA = 2.7401 - \frac{3-2.7401}{0.2316+0.00038} (-0.00038)$$

$= 2.7404 \simeq 2.740.$

$\boxed{\text{True value} = 2.7404}$

Secant method:- (chord method)

The iterative formula for the secant method.

$$\boxed{x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})} \cdot f(x_n)} \quad \text{for } n \geq 1$$

→ This method is No guarantee to convert. Since for finding the successive approximation last two current combinations (both +ve, both -ve, +ve, -ve)

→ overall rate of convergence is faster convergence and the order of converges is 1.6C

→ If at all converges it is faster than the regular false method 1.62 times.

By using this method we can locate the mplex Roots of the Equation.

find 3rd app of function $f(x) = xe^{x} - \cos x$ [0, 1]

$\boxed{\text{Mode: Rad}}$

$f(0) = -1$

$f(1) = 2.179$

$$fA = 1 - \frac{1-0}{2.179+1}(2.179)$$

$FA = 0.3146$

$f[0.3146] = -0.51987.$

$$SA = 0.3146 - \frac{0.3146-1}{-0.51987 - 2.179}(-0.51987)$$

$SA = 0.4462$

$\boxed{f(0.4462) = -0.2049.}$

$$TA = 0.4462 - \frac{0.4462 - 0.3146}{-0.2049 + 0.51987}(-0.2049)$$

$\boxed{TA = 0.5314}$

<u>Newton</u> <u>Rapson's</u> <u>Method</u>:- (Tangent Method)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$\boxed{\because f'(x_n) \neq 0}$

This method is also guarantee to convert as
If the initial approximation is Nearer to the
true value)

- overall rate of convergence is faster convergence
and order of convergence is second order
(quadratic)

→ For finding the successive approximations
use positive root of functional value.

→ If the derivative of functional value is
Larger then method moves directly otherwise it
is very slow sometimes diverges.

→ If this method fails $[f'(x_n) = 0]$ apply the
Regular false method.

→ This method can also be known as tangent
method (Geometrically)

→ This is the best method for Locating the complex
Roots of Equation.

→ By using this method we can find $\sqrt{n}$, $n^{1/3}$,
N, $\frac{1}{N}$, $\frac{1}{\sqrt{N}}$ ---

Square Root $= \sqrt{N}$

$x_{n+1} = 1/2\ (x_n + \frac{N}{x_n})$

cube Root $- (N)^{1/3}$.

$\cancel{x_{n+1} = 1/3\ (2x_n + \frac{N}{x_n^2})}$

pth Root $(N)^{1/P}$.

$x_{n+1} = \frac{1}{p}\left((p-1)x_n + \frac{N}{x_n^{p-1}}\right)$

Inverse pth Root : $\frac{1}{\sqrt[P]{N}}$

$x_{n+1} = \frac{x_n}{p}\left[(p+1) - N x_n^P\right]$.

Inverse

$x_{n+1} = x_n(2 - N x_n)$

Inverse square Root

$\frac{1}{\sqrt{N}}$

$x_{n+1} = \frac{x_n}{2}\ (3 - N x_n^2)$

Inverse cube Root $\frac{1}{\sqrt[3]{N}}$

$x_{n+1} = \frac{x_n}{3}\ (4 - N x_n^3)$

Find $\sqrt{12}$ by using Newton's Method.

$$\boxed{x_{n+1} = 1/2\ (x_n + N/x_n)}$$

$x_0 = \frac{3+4}{2} =$ 3.5

First approximation $= 1/2\ (3.5 + 12/3.5) = 3.4642$

SA $= 1/2\ (3.4642 + 12/3.4642) = 3.4641$

TA $= 1/2\ (3.4641 + 12/3.4641) = 3.4641$

Find $\sqrt[3]{10}$ by using Netwons Method.

$(10)^{1/3}$ $\quad (8)^{1/3}$ $\quad (27)^{1/3}$.

$x_0 = \frac{2+3}{2} = 2.5$

$x_{n+1} = 1/3\ (2x_n + N/x_n^2)$

$$FA = 1/3\left[2(2.5) + \frac{10}{(2.5)^2}\right] = 2.2$$

$$SA = 1/3\left[2(2.2) + \frac{10}{(2.2)^2}\right] = 2.1553$$

$$TA = 1/3\left[2(2.1553) + \frac{10}{(2.1553)^2}\right] = 2.1543$$

$$fA = 1/3\left[2(2.1543) + \frac{10}{(2.1543)^2}\right] = 2.1543$$

* find the second approximation for the function

$f(x) = xe^x - 1 \qquad x_0 = 1$

Sol $f(x) = xe^x - 1$

$f'(x) = xe^x + e^x \implies f(1) = 1.e^1 - 1 = 1.718$

$f'(1) = e^1 + e^1 = 5.436$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 1 - \frac{1.718}{5.436} = 0.6839$$

$= 1$

$f(0.6839) = 0.3552$

$f'(0.6839) = 3.3362$

$$SA = 0.6839 - \frac{0.3552}{3.3362} = 0.5774$$

Solution for the Numerical differential Equation:-

definition of single step Method:-

If the y value can be increment by only one step at a time then the methods are known as single step method. The General rule is

New value = old value + (slop) (step size)

ultistep method

If 'y' values are incremented by more than ne step at a time then the methods are known as Multistep method. These methods are also nown as predictors correctors method

Single step
- wer series
  - picards
  - Taylor's
- step by step
  - Eulers
  - R.K method

Multistep
- Milne's
- Adams

predictors & correctors.

icards method :-

$dy/dx = f(x,y) \quad ; \quad y(x_0) = y_0$

$dy/dx = f(x,y)$

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx$$

$$y - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

$$\boxed{y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx}$$

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

$$\vdots$$

$$\boxed{y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx}$$

* Solve the differential Equation $dy/dx = x+y$. Such that $y(0)=1$ upto 3rd approximation.

Sol $f(x,y) = x+y$; $x_0 = 0$; $y_0 = 1$

$$y_1 = 1 + \int_0^x (x+1)\, dx = 1 + x + x^2/2$$

$$y_2 = 1 + \int_0^x (f(x, y_1)\, dx = 1 + \int_0^x x + 1 + x + x^2/2$$

$$= 1 + \int_0^x 2x + 1 + x^2/2$$

$$y_2 = 1 + x^2 + x + \frac{x^3}{6}$$

$$y_3 = 1 + \int_0^x f(x, y_2)\, dx = 1 + \int_0^x x + 1 + x^2 + x + x^3/6$$

$$= 1 + x + 2(x^2/2) + \frac{x^3}{3} + \frac{x^4}{24}$$

$$\boxed{y_3 = 1 + x + x^2 + x^3/3 + x^4/24}$$

iylor's Method :- It is the application for the
ifferentiation

$\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

$$y = y_0 + \frac{(x-x_0)^1}{1!}(y')_0 + \frac{(x-x_0)^2}{2!}(y'')_0 + \cdots \frac{(x-x_0)^n}{n!}(y^n)_0$$

$$(y')_0 = (dy/dx)_{(x_0, y_0)}$$

$$(y'')_0 = (d^2y/dx^2)_{(x_0, y_0)}$$

In this Method the successive approximations are representing with successive order of derivative f.A (y'), S.A (y'') and ——

Solve the differential Equation

$dy/dx = 2y + xe^x + e^{2x}$ such that $y(0)$ find 3rd approximation.

$y' = 2y + xe^x + e^{2x}$; $x_0 = 0$, $y = 1$

$y' = 2y + xe^x + e^{2x}$; $(y')_{(0,1)} = 2(1) + e^0$
$= 3$

$y'' = 2y' + xe^x + e^x + 2e^{2x}$; $(y'')_{(0,1)} = 2(3) + 0 + 1 + 2$
$= 9$

$y''' = 2y'' + xe^x + e^x + e^x + 4e^{2x}$

$y'''_{(0,1)} = 2(9) + 1 + 1 + 4 = 24$

$$y_{incr} = 1 + \frac{(x-0)^1}{1!}(3) + \frac{(x-0)^2}{2!}(9) + \frac{(x-0)^3}{3!}(24)$$

$$y_{Inc} = 1 + 3x + \frac{9x^2}{2} + 4x^3 + \text{-----}$$

<u>Eulers</u> <u>Method</u>:- It is the application of the <u>integration</u> and Extension of <u>Taylor's</u> Method.

$$dy/dx = f(x,y) \; ; \; y(x_0) = y_0.$$

$$y_1 = y_0 + hf(x_0, y_0)$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$\vdots$$

$$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$$

→ The degree of the polynomial for the <u>Eulers</u> <u>integration</u> Method is <u>1st</u> degree.

→ The order of truncation Error is <u>$O(h^2)$</u>

→ This Method can also be known as <u>predictors</u> & <u>correctors</u> Method (Single Step)

→ It is a application for the <u>trapezoidal</u> <u>rule</u>

* Solve the differential Equation $dy/dx = x+y$, Such that $y(0)=1$. find $y(0.1)$ in steps of 0.02

<u>Sol</u>

$f(x,y) = x+y$, $x_0 = 0$, $y=1$, $h = 0.02$

x	y	$f(x,y) = x+y$	New Value.
$=1$	$y_0 = 1$	$f(x_0, y_0) = 1$	$y_1 = 1 + 0.02(1) = 1.02$
0.02	$y_1 = 1.02$	$f(x_1, y_1) = 1.04$	$y_2 = 1.02 + 0.02(1.04) = 1.04$
0.04	$y_2 = 1.04$	$f(x_2, y_2) = 1.08$	$y_3 = 1.04 + 0.02(1.08) = 1.06$
$=0.06$	$y_3 = 1.06$	$f(x_3, y_3) = 1.12$	$y_4 = 1.06 + 0.02(1.12) = 1.08$
0.08	$y_4 = 1.08$	$f(x, y_4) = 1.16$	$y_5 = 1.08 + 0.02(1.16) = 1.1$
$=0.1$	$y_5 = 1.1$		

$$\boxed{y(0.1) = 1.1}$$

Runge Kutta's Method :- By default it is a 4th order

$$\frac{dy}{dx} = f(x, y) \qquad y(x_0) = y_0.$$

$$\boxed{y_1 = y_0 + k}$$

$$\boxed{k = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]}$$

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$$

$$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

* Apply R-K Method and Solve the differential Equation $dy/dx = x+y$, $y(0)=1$ find $y(0.2)$

Sol $f(x,y) = x+y$; $x_0=0$, $y_0=1$

$x_0+h=0.2$

$0+h=0.2$

$h=0.2$

$K_1 = 0.2\,f(0,1) = 0.2\,[0+1] = 0.2$

$K_2 = 0.2\,f(0.1,\ 1.1) = 0.2\,[0.1+1.1] = 0.24$

$K_3 = 0.2\,f(0.1,\ 1.12) = 0.2\,(0.1+1.12) = 0.244$

$K_4 = 0.2\,f(0.2,\ 1.244) = 0.2\,(1.444) = 0.288.$

$$K = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$K = 1/6\left[0.2 + 2(0.24) + 2(0.244) + 0.288\right]$$

$$\boxed{K = 0.2428}$$

$$y(0.2) = y_0 + K$$

$$= 1 + 0.2428$$

$$\boxed{y(0.2) = 1.2428}$$

R.K (3rd order):

$$y_1 = y_0 + K$$

$$K = \frac{1}{6}\left[K_1 + 4K_2 + K_3\right]$$

$$K_1 = hf(x_0, y_0)$$

$$K_2 = h\,f(x_0+h/2,\ y_0+K/2)$$

$$K_3 = h\,f\ (x_0+h/2,\ y_0+K')$$

$k' = hf(x_0+h, y_0+k_1)$

(Rong's Method)

K (2nd):

$$y_1 = y_0 + k$$

$$k = 1/2\,[k_1 + k_2]$$

$$k_1 = h\,f(x_0, y_0)$$

$$k_2 = hf(x_0+h, y_0+k_1)$$

Modified Euler's Method.

R.K (1st order):-

$$y_1 = y_0 + hf(x_0, y_0).$$

Euler's (Simple).

Multistep Methods:-

Milne's Method:-

$$dy/dx = f(x, y) \qquad y(x_0) = y_0$$

$$y_{4,P} = y_0 + \frac{4h}{3}\left[2f_1 - f_2 + 2f_3\right]$$

$$y_{4,c} = y_2 + \frac{h}{3}\left[f_2 + 4f_3 + f_{4P}\right]$$

$$y_{4,c(1)} = y_2 + h/3\left[f_2 + 4f_3 + f_{4,c}\right] \quad \text{(till accuracy)}$$

> The origin of this Method is Newton's forward interpolation formula (Simpson's rule)

→ In this method 1st approximation is 4th iteration for Evaluating this we need starting 3-iterations (y_1, y_2, y_3) which can be generate using any one of the Single step methods.

→ continue the corrector iterations untill the accuracy

→ R-K Method is meant for smaller values of t and milnes Method for the Larger values of t. This method gives more stable solution than any one of the single step Method.

Adamas Bash Forth Method :-

$$dy/dx = f(x, y) \qquad y(x_0) = y_0$$

$$y_{1,P} = y_0 + \frac{h}{24}\left[55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}\right]$$

$$y_{1,C} = y_0 + \frac{h}{24}\left[9f_1 + 19f_0 - 5f_{-1} + f_{-2}\right]$$

$$y_{1,C}^{(1)} = y_0 + \frac{h}{24}\left[9f_{1,C} + 19f_0 - 5f_{-1} + f_{-2}\right]$$

Initial	x_{-3} y_{-3}	f_{-3}
1st	$x_{-3}+h$ x_{-2} y_{-2}	f_{-2}
2nd	$x_{-3}+2h$ x_{-1} y_{-1}	f_{-1}
3rd	$x_{-3}+3h$ x_0 y_0	f_0

: origin of this Method is Newtons backward
terpolation

In this Method also first approximation is
ith iteration.

For Evaluating this we need starting values
f_{-3}, f_{-2}, f_{-1} which can be generate by using
any one of single step Method

continue the corrector iteration until the accuracy

This is the best Method for finding the stable
Solution of Numerical differential Equations

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Inverse Z - Transform (continuation)

→ $Z^{-1}\left(\frac{Z}{(Z-3)(Z+5)}\right)$ for $3<|Z|<5$

$$Z^{-1}\left[\frac{Z\cdot Z^{n-1}=Z^{n}}{(Z-3)(Z+5)}\right]$$

$$=\left(\frac{3^n}{8}+\frac{(-5)^n}{-8}\right)u(n)$$

$$=\left[\frac{3^n}{8}-\frac{(-5)^n}{8}\right]u(n) \quad \text{for } |Z|>3 \text{ \& } |Z|>5.$$

$$=\frac{3^n u(n)}{8}+\frac{(-5)^n}{8}u(-n-1)$$

$$=\begin{cases} 3^n/8\,; & n\geq 0 \\ (-5)^n/8\,; & n<0 \end{cases} \qquad \begin{cases} 3^n/8 & : n\geq 0 \\ (-5)^n/8 & : n\leq 0. \end{cases}$$

2 $Z^{-1}\left[\frac{1}{(Z-2)(Z-3)}\right]$ for $2<|Z|<3$.

$$=Z^{-1}\left[\frac{Z^{n-1}}{(Z-2)(Z-3)}\right]$$

$$=\left[\frac{(2)^{n-1}}{-1}+\frac{(3)^{n-1}}{1}\right]u(n)$$

$$=\left(-2^{n-1}+(3)^{n-1}\right)u(n)$$

$$=\left[\underset{u(n)}{-2^{n-1}}-(3)^{n-1}\,u(-n-1)\right]$$

$$=\left[-2^{n-1}u(n-1)-3^{n-1}u(-n)\right]$$

$$a^n * a^n = \sum_{m=0}^{n} a^m \cdot a^{n-m}$$

a) na^n
b) $2a^n$
c) a^{2n}
~~d)~~ None

$$= \sum_{m=0}^{n} (a^n) = a^n \sum_{m=0}^{n} (1)$$

$$\boxed{a^n * a^n = a^n(n+1)}$$

$$a^n * b^n = \sum_{m=0}^{n} a^m \cdot b^{n-m}$$

$$\boxed{S_n = \frac{a(r^n - 1)}{(r-1)}}$$

$$= b^n \sum_{m=0}^{n} (a/b)^m$$

$$= b^n \left[1 + a/b + (a/b)^2 + \cdots \right]$$

$$= b^n \left[\frac{(a/b)^{n+1} - 1}{-1 + a/b} \right] = \cancel{b^n \left[\frac{b}{b-a} \right]}$$

$$= \cancel{b^n} \left(\frac{a^{n+1} - b^{n+1}}{(a-b)/\cancel{b}} \cancel{b^{n+1}} \right) \cancel{= \frac{b^{n+1}}{(b-a)}}$$

$$Z^{-1}\left(\frac{Z}{(Z-a)} \cdot \frac{Z}{(Z-b)} \right)$$

$$\boxed{Z^{-1}\left[\bar{f}(z) \cdot \bar{g}(z) \right] = f(n) * g(n)}$$

$$Z^{-1}\left[\frac{Z}{(Z-a)} \frac{Z}{(Z-b)} Z^{n-1} \right]$$

$$= Z^{-1}\left[\frac{Z^{n+1}}{(Z-a)(Z-b)} \right]$$

$$= \left[\frac{a^{n+1}}{(a-b)} + \frac{b^{n+1}}{(b-a)} \right] = \frac{a^{n+1} - b^{n+1}}{(a-b)}$$

$$Z^{-1}\left[\frac{Z^2}{(Z-a)^2} Z^{n-1} = Z^{n+1} \right]$$

$$= Z^{-1}\left(\frac{Z^{n+1}}{(Z-a)^2} \right) = \underset{Z \to a}{Lt} (n+1) Z^n$$

$$= (n+1) a^n$$

* If $Z[f(n)] = \frac{z}{(z-1/2)(z-3)(z+2)}$ in $|z| = 3/2$ then $f(1) = ?$

$$f(n) = Z^{-1}\left[\frac{z}{(z-1/2)(z-3)(z+2)}\right]$$

$$f(n) = Z^{-1}\left[\frac{z \cdot z^{n-1} = z^n}{(z-1/2)(z-3)(z+2)}\right]$$

$$= r_1 \,|\, z=1/2 = \frac{(1/2)^n}{(1/2-3)(1/2+2)}$$

$$= -4/25\,(1/2)^n.$$

$$= r_2 \,|\, z=3$$

$$= \frac{(3)^n}{(3-1/2)(3+2)} = \frac{(3)^n}{(5/2)(5)} = \frac{2(3^n)}{25}.$$

Since $|z| = 3/2$ is the circle condition and hence the $z = 1/2$ is only lies inside of the circle and remaining two are lies outside of this.

Hence $f(n) = -4/25\,(1/2)^n$

$$f(1) = -4/25\,(1/2) = -2/25.$$

__Application of Z-transform to differential Equations__

→ Equation consisting of derivaties. is called differential Equation

Equation consisting of discrete functions is known as difference Equation.

Solve $u_{n+1} + u_n = 0$; $u_0 = 1$

$$z[u_{n+1} + u_n] = z[0]$$

$$z[\bar{u}(z) - u_0] + \bar{u}(z) = 0$$

$$(z+1)\bar{u}(z) = z u_0 = z$$

$$\boxed{\bar{u}(z) = \frac{z}{z+1}} \qquad u_n = z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$$

$$u_{n+2} - 3u_{n+1} + 2u_n = u(n) \quad ; \quad \begin{matrix} u_0 = 0 \\ u_1 = 1 \end{matrix}$$

$$z[u_{n+2} - 3u_{n+1} + 2u_n] = z[u(n)]$$

$$z^2[\bar{u}(z) - \cancel{z}u_0 - u_1/z] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-1}$$

$$z^2\bar{u}(z) - z - 3z\,\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-1}$$

$$\bar{u}(z)[z^2 - 3z + 2] = \frac{z}{z-1} + z$$

$$\bar{u}(z)[(z-1)(z-2)] = \frac{z + z(z-1)}{(z-1)} = \frac{\cancel{z} + z^2 - \cancel{z}}{(z-1)}$$

$$\bar{u}(z)[(z-1)(z-2)] = \frac{z^2}{(z-1)}$$

$$\bar{u}(z) = \left(\frac{z^2}{(z-1)^2(z-2)}\right)$$

$$z^2 z^{n-1} = z^{n+1}$$

$$\bar{u} = z^{-1}\left(\frac{z^2 z^{n-1}}{(z-1)^2(z-2)}\right)$$

$$= r_1\Big|_{z=1} + r_2\Big|_{z=2}$$

$$= \underset{z\to 1}{Lt}\ \frac{d}{dz}\left(\frac{z^{n+1}}{z-2}\right) + 2^{n+1}$$

$$= \underset{z\to 1}{Lt}\left[\frac{(z-2)(n+1)z^{n} - z^{n+1}(1)}{(z-2)^2}\right] + 2^{n+1}$$

$$= \left(\frac{(-1)(n+1)-1}{1}\right) + 2^{n+1}$$

$$= \frac{-n-1-1}{1} + 2^{n+1} = -n-2+2^{n+1}$$

Fourier Transformation

If f(x) is a periodic function and satisfy the conditions of dirichlet's then it can be Expressed as fourier ies. By Expanding this concept for Non periodic unction it can be Expressed as fourier integral

ourier integral :-

If f(x) is defined in (−c, c) and satisfy the conditions of dirchlet as $c \to \infty$

$$\boxed{f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\int_{0}^{\infty} f(t)\cos\lambda(t-x)\,dt\,d\lambda}$$

↳ fourier integral

λ is a parameter

If f(x) is an odd function

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\sin\lambda x\int_{0}^{\infty} f(t)\sin\lambda t\,dt\,d\lambda$$

↳ sine integral

If f(x) is an Even function

$$f(x) = \frac{2}{\pi}\int_{0}^{\infty}\cos\lambda x\int_{0}^{\infty} f(t)\cos\lambda t\,dt\,d\lambda$$

↳ cosine integral

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\int_{-\infty}^{\infty} f(t)\,e^{i\lambda t}\,dt\,d\lambda$$

↳ gen complex formation integral

Fourier transformation in different forms:-

1) $F(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{ist} dt = \bar{F}(s)$

↳ fourier transformation

$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{F}(s) e^{-ist} ds.$

↳ inverse fourier transfo

2) $F(f(t)) = \int_{-\infty}^{\infty} f(t) e^{ist} dt = \bar{F}(s)$

$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \bar{F}(s) e^{-ist} ds.$

3) $F(f(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{ist} dt = \bar{F}(s)$

$f(t) = \int_{-\infty}^{\infty} \bar{F}(s) e^{-ist} ds.$

Fourier Sine/cosine Transform :-

1) $F_s(f(t)) = \bar{F}_s(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin st \, dt$

↳ F.S.T

$f(t) = F_s^{-1}(\bar{F}_s(s)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{F}_s(s) \sin st \, ds$

↳ I.F.S.T

2) $F_c(f(t)) = \bar{F}_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \cos st \, dt$

$t) = F_c^{-1}(F_c(S)) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} F_c(S) \cos St \, dS$

↳ I.F.C.T

<u>nite Fourier Sine/cosine Transform:-</u>

If $f(x)$ is defined in $(0, l)$

then (i) $F_S(f(x)) = \int_0^l f(x) \sin \frac{s\pi x}{c} \, dx$

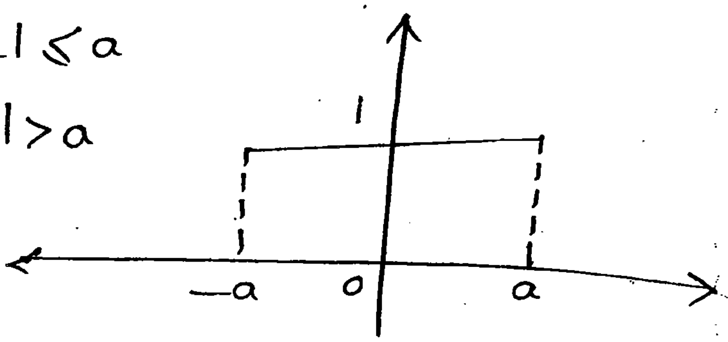
(Here 's' is an integer)

$F_c(f(x)) = \int_0^l f(x) \cos \frac{s\pi x}{l} \, dx$

<u>ote</u>:- Fourier transformation is applicable for table functions. Not applicable for astable functions

The no of discontinuous points are finite

F.T of $f(x) = \begin{cases} 1 & ; |x| \leq a \\ 0 & : |x| > a \end{cases}$



$s) = \int_{-\infty}^{\infty} f(x) e^{isx} \, dx$

$= \int_{-a}^{a} (1) e^{isx} \, dx$

$$= \left(\frac{e^{isx}}{is}\right)_{-a}^{a} = \frac{e^{isa} - e^{-isa}}{is} = \left(\frac{2\sin as}{s}\right)$$

(i) $\frac{2\sin as}{s}$ (ii) $\frac{1}{\sqrt{2\pi}} \frac{2\sin as}{s}$ (iii) $\frac{1}{2\pi} \frac{2\sin as}{s}$

Find Fourier transform

$$f(x) = \begin{cases} 1-x^2 ; & |x| \leq 1 \\ 0 ; & |x| > 1 \end{cases}$$

$$F(s) = \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

<u>Leibnitz's Rule</u>

$$\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$v_1, v_2 \to$ integrals.

$u', u'' \to$ Derivaties.

$$F(s) = \int_{-1}^{1} (1-x^2)\, e^{isx}\, dx$$

$$= \left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\left[\frac{e^{isx}}{(is)^2}\right] + (-2)\left[\frac{e^{isx}}{(is)^3}\right] \right]_{-1}^{1}$$

$$= \left[\frac{2e^{is}}{s^2} - \frac{2e^{is}}{-is^3} - 2(-1)\frac{e^{-is}}{-s^2} + \frac{2e^{-is}}{-is^3} \right]$$

$$= -2/s^2 \,(e^{is} + e^{-is}) + \frac{2}{is^3}(e^{is} - e^{-is})$$

$= -\frac{2}{s^2}(2\cos s) + \frac{2}{is^3}(2i\sin s)$

$= \frac{4}{s^3}(\sin s - s\cos s)$

$$\boxed{\begin{aligned} &F[\delta(t)] = 1 \\ &F[\delta(t-a)] = e^{isa} \text{ (or) } e^{-isa} \\ &F_s[\delta(t-a)] = \sin as \end{aligned}}$$

Fourier transform of $\sin x$ does not Exist because it is periodic signal.

S.T of e^{-ax} :-

$$f_s(F(t)) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(xs)\, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin(sx)\, dx$$

$$F_s(e^{-ax}) = \sqrt{2/\pi}\left(\frac{s}{s^2+a^2}\right)$$

$$F_c(e^{-ax}) = \sqrt{2/\pi} \int_0^\infty e^{-ax} \cos(sx)\, dx$$

$$= \sqrt{2/\pi}\left(\frac{a}{s^2+a^2}\right)$$

$F_S(e^{ax}) \longrightarrow$ unstable system.

$F_S(\sin x) \longrightarrow$ doesnot Exist

- find F.S.T of $\frac{e^{-ax}}{x}$

$$\bar{f}_S(s) = \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

$$= \sqrt{2/\pi} \left[\cot^{-1}(a/s) \right]$$

(or)

$$\bar{f}_S(s) = \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx$$

differentiate w.r.to s.

$$\frac{d}{ds}\left[\bar{f}_S(s)\right] = \sqrt{2/\pi} \int_0^\infty \frac{e^{-ax}}{\cancel{x}} \cancel{x} \cos sx \, dx$$

$$\frac{d}{ds}\left[\bar{f}_S(s)\right] = \sqrt{2/\pi} \frac{a}{s^2+a^2}$$

integrate w.r.to s.

$$\bar{f}_S(s) = \sqrt{2/\pi} \tan^{-1}(s/a) \text{ (or) } \sqrt{2/\pi} \cot^{-1}(a/s)$$

* find F.S.T of xe^{-ax}

$$\bar{f}_S(s) = \sqrt{2/\pi} \int_0^\infty xe^{-ax} \sin sx \, dx$$

$$\bar{f_s}(s) = \sqrt{2/\pi} \int_0^\infty \cancel{x} e^{-ax} \frac{(-\cos sx)}{\cancel{x}} dx$$ ~~differentiate~~ integrate

$$= \sqrt{2/\pi} \int_0^\infty (e^{-ax}) (-\cos sx)\, dx$$

$$= \sqrt{2/\pi} \int_0^\infty \frac{a}{s^2+a^2}\, ds.$$

differentiate w.r.t s.

$$\boxed{\bar{f_s}(s) = \frac{2as}{s^2+a^2}}$$

find F.S.T of $(1/x)$

$$\bar{f_s}(s) = \sqrt{2/\pi} \int_0^\infty (1/x)(\sin sx)\, dx$$

By Laplace transform

$$= \sqrt{2/\pi}\ (\pi/2)$$

$$\boxed{\bar{f_s}(s) = \sqrt{\pi/2}}$$

find F.S.T of $\frac{1}{\sqrt{x}}$

$$\bar{f_s}(s) = \sqrt{2/\pi} \int_0^\infty \frac{1}{\sqrt{x}} \sin sx\, dx$$

$$\Gamma n = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$$

$$\bar{F}_S(S) = -\sqrt{2/\pi}\int_0^\infty \frac{1}{\sqrt{x}}\, e^{-isx}\, dx$$

Imaginary part of

$isx = t$

$x = \frac{t}{is}$

$\boxed{dx = dt/is}$

$$= -\text{Img part of } \frac{\sqrt{2}}{\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{t/is}}\, e^{-t}\, \frac{dt}{is}.$$

$$= -\text{Img part of } \frac{\sqrt{2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{is}}\int_0^\infty t^{-1/2}e^{-t}\, dt$$

$$= -\text{Img part of } \frac{\sqrt{2}}{\sqrt{\pi}}\,\frac{1}{\sqrt{is}}\int_0^\infty t^{1/2-1}e^{-t}\, dt$$

$$= -\text{Img part of } \frac{\sqrt{2}}{\cancel{\sqrt{\pi}}}\,\frac{1}{\sqrt{i}\sqrt{s}}\,\cancel{\sqrt{\pi}}$$

$$= \frac{\sqrt{2}}{\sqrt{s}}\left(\frac{1}{\sqrt{i}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{s}}\left(\text{Img of } \frac{1}{\sqrt{i}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{s}}\left(\text{Img of } \frac{1}{\sqrt{e^{i\pi/2}}}\right)$$

$$= \frac{\sqrt{2}}{\sqrt{s}}\left(-\text{Img of } e^{-i\pi/4}\right]$$

$$= \sqrt{2}/\sqrt{s}\left[-\cos\pi/4 + i\sin\pi/4\right]$$

$$= \sqrt{2}/\sqrt{s}\left(+\frac{1}{\sqrt{2}}\right] = +1/\sqrt{s}.$$

F.C.T of $\left(\frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{s}}$.

If the transformation of a function is itself then it is called as <u>Self reciprocal</u> <u>functions</u>

<u>Eg:</u> $\frac{1}{\sqrt{x}}$ is a Self reciprocal functions.

$e^{-x^2/2}$ is also Self reciprocal under fourier transformation.

$$F\left[e^{-x^2/2}\right] = e^{-s^2/2}$$

$$= \cancel{\frac{1}{\sqrt{2\pi}}}\,\cancel{\sqrt{2\pi}}\; e^{-s^2/2}$$

$$\int_0^\infty f(x)\sin tx\,dx = \begin{cases} 1\;; & 0\le t\le 1 \\ 2\;; & 1\le t<2 \\ 0\;; & t\geq 2 \end{cases}$$

$$F_S(f(x)) = \int_0^\infty f(x)\sin tx\,dx = \overline{f_S}(t)$$

$$f(x) = F_S^{-1}\left(\overline{f_S}(t)\right) = \frac{2}{\pi}\int_0^\infty \overline{f_S}(t)\;\sin tx\,dt$$

$$= \frac{2}{\pi}\left[\int_0^1 (1)\sin tx\,dt + \int_1^2 2\sin tx\,dt + \int_2^\infty (0)\,dt\right]$$

$$= \frac{2}{\pi}\left[\left(\frac{-\cos tx}{x}\right)_0^1 + \left(\frac{-2\cos tx}{x}\right)_1^2\right]$$

$$= \frac{2}{\pi}\left[\frac{-\cos x}{x} + 1/x - \frac{2\cos 2x}{x} + \frac{2\cos x}{x}\right].$$

7 F.F.S.T of $f(x)=1$; $[0,\pi]$

Sol $$FS(f(x)) = \int_0^l f(x) \sin \frac{s\pi x}{l}\, dx$$

$$= \int_0^{\pi} (1) \sin \frac{s\cancel{\pi} x}{\cancel{\pi}}\, dx$$

$$= \int_0^{\pi} (1) \sin(sx)\, dx$$

$$= \left(\frac{-\cos sx}{s}\right)_0^{\pi} = -1/s\,[\cos s\pi - 1]$$

$$= \frac{+1}{s}[1-\cos s\pi]$$

$$\boxed{FS(f(x)) = \frac{-(-1)^s+1}{s}.}$$

$\int_{-1}^{1} \frac{1}{x^2}\, dx =$ does not Exist. because at $x=0$ it is discontinous.

Proporties:-

Linearity:-

$$I\{a f(t) \pm b g(t)\} = a I\{f(t)\} \pm b I(g(t))$$

Scaling:-

$$F\{f(at)\} = \frac{1}{a}\bar{F}(s/a)$$

$$F[f(t/a)] = a\bar{F}(as)$$

fting property:-

If $F(f(t)) = \bar{f}(s)$. e^{ist} ↓ e^{-ist} ↓

then $F\{f(t)\, e^{iat}\} = \bar{f}(s+a)$ $\bar{f}(s-a)$

$F[f(t)\, e^{-iat}] = \bar{f}(s-a)$ $\bar{f}(s+a)$

$F[f(t-a)] = e^{isa}\, \bar{f}(s)$. $e^{-isa}\, \bar{f}(s)$

dolation property:-

If $F(f(t)) = \bar{F}(s)$

$F(f(t)\cos at) = 1/2\, [\bar{f}(s+a) + \bar{f}(s-a)]$

$F_s\,(f(t)\sin at) = \frac{1}{2}[\bar{f_c}(s-a) - \bar{f_c}(s+a)]$

$F_s\,(f(t)\cos at) = \frac{1}{2}[\bar{F_s}(s+a) + \bar{F_s}(s-a)]$

$F_c(f(t)\sin at) = 1/2\, [\bar{F_s}(s+a) - \bar{F_s}(s-a)]$

$F_c\,(f(t)\cos at) = 1/2\, [\bar{F_s}(s+a) + \bar{F_c}(s-a)]$

lation b/w Laplace transformation & fourier transformation

If $f(t) = \begin{cases} e^{-xt}\, g(t) & ; t \geq 0 \\ 0 & ; t < 0 \end{cases}$

then $\boxed{F(f(t)) = L\{g(t)\}}$

convolution theorem:-

$$F^{-1}\{\bar{f}(s)\cdot\bar{g}(s)\} = f(t) * g(t)$$

$$= \int_{-\infty}^{\infty} f(x)\, g(t-x)\, dx$$

parseval's Identities:-

$F(S) \xrightarrow{F.T of} f(t)$

$G(S) \xrightarrow{F.T of} g(t)$

$\overline{G}(S) \xrightarrow{conjugate} G(S)$

$\overline{g(t)} \xrightarrow{conjugate} g(t)$

1) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(S)\,\overline{G(S)}\, ds = \int_{-\infty}^{\infty} f(t)\,\overline{g(t)}\, dt.$

2) $\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(S)|^2\, ds = \int_{-\infty}^{\infty} |f(t)|^2\, dt.$

3) $\frac{2}{\pi}\int_{0}^{\infty} F_S(f(x))\; F_S(g(x))\, ds = \int_0^{\infty} f(x)\, g(x)\, dx$

4) $\frac{2}{\pi}\int_{0}^{\infty} F_c(f(x))\cdot F_c(g(x))\, ds = \int_0^{\infty} f(x)\cdot g(x)\, dx$

$$\int_0^{\infty} \frac{x^2}{(x^2+a^2)^2}\, dx$$

$$\int_0^{\infty} \left[\frac{x}{x^2+a^2}\right]^2 dx$$

put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$\left[\frac{(x^2+a^2)^2 (2x) - x^2 (2)(x^2+a^2)(2x)}{(x^2+a^2)^4}\right]$$

$$= \int_0^{\pi/2} \frac{a^2\tan^2\theta}{(a^2+a^2\tan^2\theta)^2}\, a\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{\cancel{a^2}\tan^2\theta}{\cancel{a}^4\sec^4\theta}\, \cancel{a}\sec^2\theta\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= \frac{1}{a}\int_0^{\pi/2} \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2a}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2a}\left[\pi/2 - 1/2\sin\pi\right] = 1/2a\,(\pi/2) = \pi/4a$$

$$\int_0^\infty \frac{s^2}{(s^2+a^2)^2}\, ds.$$

$$= \int_0^\infty \left(\frac{s}{s^2+a^2}\right)\cdot\left(\frac{s}{s^2+a^2}\right) ds.$$

$$= \pi/2 \int_0^\infty e^{-ax} e^{-ax}\, dx$$

$$= \pi/2 \left[\frac{e^{-2ax}}{-2a}\right]_0^\infty = \pi/2\left[0+1/2a\right] = \pi/4a.$$ 110

$\rightarrow \int_0^\infty \frac{1}{(x^2+a^2)^2} dx.$

put $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$$= \int_0^{\pi/2} \frac{1}{(a^2\tan^2\theta + a^2)^2} a\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta}$$

$$= \frac{1}{a^3}\int_0^{\pi/2}\left(\frac{1}{\sec^2\theta}\right)d\theta$$

$$= \frac{1}{a^3}\int^{\pi/2}\cos^2\theta\, d\theta$$

$$= \frac{1}{a^3}\left[\frac{1+\cos 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{1}{2a^3}\left[\theta + 1/2\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{1}{2a^3}\left[\pi/2\right] = \pi/4a^3.$$

(or)

$$\int_0^\infty \frac{1}{(s^2+a^2)^2} ds.$$

$$= \frac{1}{a^2}\int_0^\infty \frac{a}{s^2+a^2}\cdot\frac{a}{s^2+a^2} ds$$

$$= \frac{\pi}{2a^2} \int_0^\infty e^{-ax} \cdot e^{-ax}\, ds.$$

$$= \pi/2a^2 \left[\frac{e^{-2ax}}{-2a} \right]_0^\infty = \pi/2a^2 (0 + 1/2a)$$

$$= \pi/4a^3.$$

$$\int_0^\infty \frac{1}{(x^2+a^2)(x^2+b^2)}\, dx$$

$$= \frac{1}{ab} \int_0^\infty \frac{a}{(x^2+a^2)} \cdot \frac{b}{(x^2+b^2)}.$$

$$= \pi/2ab \int_0^\infty e^{-ax} \cdot e^{-bx}\, dx$$

$$= \frac{\pi}{2ab} \left[\frac{e^{-ax}}{-a} \cdot \frac{e^{-bx}}{-b} \right]_0^\infty$$

$$= \frac{\pi}{2ab} \left(\frac{1}{a+b} \right).$$

If $F(f(x)) = \bar{f}(s)$ then fin $F(f(-x))$

$$F(f(-x)) = \int_{-\infty}^{\infty} f(-x) e^{isx}\, dx \qquad -x = t$$

$$x = -t$$

$$\int_{-\infty}^{\infty} f(t) e^{i(-s)t}\, dt = \int_{\infty}^{-\infty} f(t) e^{is(-t)}\, dt. \qquad dx = -dt.$$

$$\boxed{F(f(-x)) = \bar{f}(-s)}$$

* $F\left[x^n f(x)\right] = (-i)^n \dfrac{d^n}{ds^n}\left[\bar{f}(s)\right]$.

Sol

$$\bar{f}(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$\frac{d}{ds}\left[\bar{f}(s)\right] = \int_{-\infty}^{\infty} f(x)\,(ix)\, e^{isx}\, dx$$

$$\frac{1}{i}\frac{d}{ds}\left[\bar{f}(s)\right] = \int_{-\infty}^{\infty} f(x)\, x\, e^{isx}\, dx$$

Similarlly this process continues.

$$\boxed{F\left[x^n f(x)\right] = (-i)^n \frac{d^n}{ds^n}\left[\bar{f}(s)\right]}$$

* $F_s(x f(x)) = -\dfrac{d}{ds}\bar{f_c}(s)$

* $F_c(x f(x)) = \dfrac{d}{ds}\bar{f_s}(s)$

$$\bar{f_c}(s) = \int_0^{\infty} f(x)\cos sx\, dx$$

$$\frac{d}{ds}\bar{f_c}(s) = -\int_0^{\infty} f(x)\, x \sin sx\, dx$$

$$\boxed{-\frac{d}{ds}\bar{f_c}(s) = \int_0^{\infty} f(x)\, x \sin sx\, dx}$$

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Z-Transformation

Z- Transformation has also similar properties as Laplace transformation but the only difference is that it operated in the sequence but not on the continous function. It is called discrete Analog of continous function

definition:- If $f(n)$ is defined for $\forall n \geq 0$ $(n \in Z)$ and $f(n)=0$ $\forall n<0$.

$$\text{then } z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n} = \bar{F}(z)$$

↳ Right sided Expansion.

Here 'z' is a complex variable.

2) If $f(n)$ is defined $\forall n<0$ $(n \in z)$ and $f(n)=0$, $\forall n \geq 0$ then

$$z[f(n)] = \sum_{n=-\infty}^{-1} f(n) z^{-n} = \bar{F}(z)$$

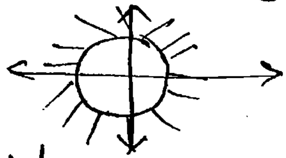
↳ Left sided Transform

$$z[f(n)] = \sum_{n=-\infty}^{\infty} f(n) z^{-n} = \bar{F}(z)$$

↳ Two sided Z- Transform

Note:- The transformation Exist if the series is convergent series.

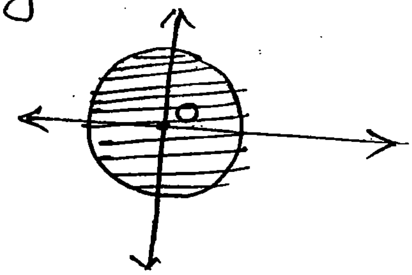
Region of convergence:- of Right sided transformation

Roc of Right sided transformation in the form 

$|z| > |a|$ outside of a circle.

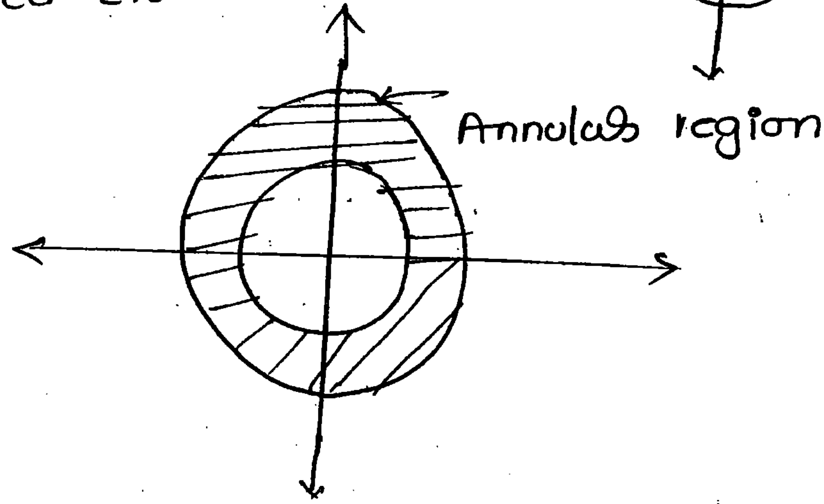
c of Left-sided transformation in the form

$|z| < |a|$. where 'a' is Any point may be real (or) complex



Roc of two sided transformation

$|a| < |z| < |b|$



Properties:-

Linearity:-

$Z\{af(n) \pm bg(n)\} = aZ[f(n)\} \pm b\, Z\{g(n)\}$

Z-transform of Some Standard functions:-

$$Z\{1\} = \frac{z}{z-1} \quad : |z|>1$$

$$\sum_{n=0}^{\infty} (1)(z^{-n}) = 1 + 1/z + 1/z^2 + \text{——}$$

$$= \frac{1}{1-1/z} = \frac{z}{z-1} \quad \boxed{|z|>1}$$

) $Z[a^n] = \dfrac{z}{z-a} \quad : |z|>|a|$

) $Z\{a^{-n}\} = \dfrac{z}{z-1/a} = \dfrac{az}{az-1} \quad ; |z|>1/|a|$

4) $Z\{(-a)^n\} = \dfrac{z}{z+a}$: $|z| > |a|$

5) $Z\{K\} = K\left(\dfrac{z}{z-1}\right)$: $|z| > 1$

6) $Z\{u(n)\} = \dfrac{z}{z-1}$: $|z| > 1$

where $u(n)$ is the unit step function.

<u>unit step sequence:-</u>

$$u(n) = \begin{cases} 1 : n \geq 0 \\ 0 : n < 0 \end{cases}$$

<u>unit impulse sequence:-</u>

$$\delta(n-K) = \begin{cases} 1 & n = K \\ 0 & n \neq K \end{cases}$$

7) $Z\{\delta(n-K)\} = \dfrac{1}{z^K}$; $|z| > 0$

8) $Z[\delta(n)] = 1; \forall z$

9) $Z\left(\dfrac{1}{n!}\right) =$

$$\sum_{n=0}^{\infty} \left(\frac{1}{n!}\right) z^{-n} = 1 + \frac{1}{z} + \frac{1}{2!\,z^{2}} + \frac{1}{3!\,z^3} + \cdots$$

$$= 1 + 1/z + \frac{(1/z^2)}{2!} + \frac{(1/z)^3}{3!} + \cdots$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = e^{1/z} ; \quad |z| > 0}$$

0) $Z\left[\dfrac{1}{(n+1)!}\right]$

$$\sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n} = 1 + \frac{1}{2!} z^{-1} + \frac{1}{3!} z^{-2} + \cdots$$

$= z \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-(n-1)} \qquad = z\left[e^{1/z} - 1\right]$.

$\frac{1}{(n-1)!}\} = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} z^{-n} = z^{-1} \cdot e^{1/z}$.

$= 1/z\left(1 + 1/z + \frac{1}{2! z^2} + \frac{1}{3! z^3} + \cdots\right)$

$\frac{1}{n}\} = \sum_{n=0}^{\infty} (1/n)(z^{-n})$

$= \sum_{n=1}^{\infty} (1/n)(z^{-n})$

$= \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3z^3} + \cdots$

$\boxed{\log(1-x) = -x - x^2/2 - x^3/3 - x^4/4}$

$= -\log(1 - 1/z)$ for $|z| < 1$

$\boxed{z\{\frac{1}{n}\} = \log\left(\frac{z}{z-1}\right) \text{ for } |z| > 1}$

$z\{\frac{1}{n-1}\} = \sum_{n=2}^{\infty} \frac{1}{(n-1)} e^{-z/(z-1)}$

$\boxed{z\left[\frac{1}{n-1}\right] = z^{-1} \log\left(\frac{z}{z-1}\right)}$

$z\left(\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \frac{1}{(n+1)} z^{-n}$

$= z \log\left(\frac{z}{z-1}\right) + \frac{1}{2!} z^{-1} +$

$1/3! \, z^{-2}$

$$= z\log\left(\frac{z}{z-1}\right)$$

** $Z\{n\} = \sum_{n=0}^{\infty} n z^{-n}$

$$= 1/z + 2/z^2 + 3/z^3 + \text{---}$$

$$= \frac{1}{z}\left[1 + 2/z + 3/z^2 + \text{-----}\right]$$

$$= \frac{1}{z}\left[(1-1/z)^{-2}\right] = \frac{z}{(z-1)^2} \quad : \quad |z|>1$$

$$\boxed{Z[n] = \frac{z}{(z-1)^2}}$$

<u>Recurrence</u> <u>Relation</u>:-

$$Z\{n^p\} = -z\frac{d}{dz}\left[Z(n^{p-1})\right] \quad ; \quad p \in Z^+$$

<u>Ex:</u> $Z[n^2] = -z\frac{d}{dz}\left[Z(n)\right]$

$$= -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left(\frac{(z-1)^2(1) - 2(z-1)z}{(z-1)^4}\right)$$

$$= -z\left[\frac{z^2+1-2z-2(z-1)z}{(z-1)^4}\right]$$

$$= -z\left[\frac{-z^2-2z+1}{(z-1)^4}\right]$$

$$= z(z^2+2z-1)/(z-1)^4$$

$$= \frac{z^2+z}{(z-1)^3}$$

$$z\left[\cos n\theta + i\sin n\theta\right]$$

$$= z\left[e^{in\theta}\right] = \frac{z}{z-e^{i\theta}}$$

$$z\left[e^{in\theta}\right] = \frac{z(z-e^{-i\theta})}{(z-e^{i\theta})(z-e^{-i\theta})}$$

$$= \frac{z^2 - z(\cos\theta - i\sin\theta)}{z^2 - z(e^{i\theta}+e^{-i\theta}) + \cancel{e^{i\theta}}\cdot\cancel{e^{-i\theta}}}$$

$$z\left[e^{in\theta}\right] = \frac{(z^2 - z\cos\theta) + iz\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\boxed{\begin{aligned} z\left[\sin n\theta\right] &= \frac{z\sin\theta}{z^2-2z\cos\theta+1} \\ z\left[\cos n\theta\right] &= \frac{(z^2-z\cos\theta)}{z^2-2z\cos\theta+1} \end{aligned}}$$

$$z(\cosh n\theta + i\sinh n\theta)$$

$$= \frac{e^{i\theta}+e^{-i\theta}}{\quad}$$

$$\boxed{\begin{aligned} z\left[\sinh n\theta\right] &= \frac{z\sinh\theta}{z^2-2z\cosh\theta+1} \\ z\left[\cosh n\theta\right] &= \frac{(z^2-z\cosh\theta)}{z^2-2z\cosh\theta+1} \end{aligned}}$$

Properties:-

Damping Rule:- If $Z(u(n)) = \bar{U}(Z)$

① $Z\{a^n u_n\} = \bar{U}(Z/a)$

2) $Z[a^{-n} u(n)] = \bar{U}(aZ)$

First Shifting (or) Right Shifting theorem:-

If $Z\{u(n)\} = \bar{U}(Z)$.

$\Rightarrow \quad Z[u_{n-k}] = Z^{-k}\,\bar{U}(Z)\ ; \quad n \geq k$

Left shifting properties:-

$$Z\{u_{n+k}\} = Z^{k}\left[\bar{U}(Z) - u_0 - \frac{u_1}{2} - \frac{u_2}{Z^2} - \cdots - \frac{u_{k-1}}{Z^{k-1}}\right]$$

Multiplication by n^p ($p \in Z^+$)

If $Z[u_n] = \bar{U}(Z)$

① $Z[n u_n] = -Z \dfrac{d}{dZ}(\bar{U}Z)$

2) $Z[n^p u_n] = (-Z)^p \dfrac{d^p}{dZ^p}[\bar{U}(Z)]$

$$= \left(-Z\frac{d}{dZ}\right)^p \bar{U}(Z)$$

$$= (-Z)^p \frac{d^p}{dZ^p}\,\bar{U}(Z)$$

vision by n:-

$$\left[\frac{u(n)}{n}\right] = -\int_0^z z^{-1} u(z)\, dz$$

discontinous cannot be differentiable.

nvolution property:-

If u_n & v_n are two descrete functions then

$$u_n * v_n = \sum_{m=0}^{n} u_m \cdot v_{n-m}$$

$$z[u_n * v_n] = \overline{u(z)} \cdot \overline{v(z)}$$

nitial value theorem:-

If $z\{u_n\} = \overline{u(z)}$ then

$$u_0 = \lim_{z\to\infty} \overline{u}(z)$$

$$u_k = \lim_{z\to\infty} z\{u_{n+k}\}$$

Eg:

$$u_1 = \lim_{z\to\infty} \left[z(\overline{u(z)} - u_0)\right]$$

$$u_2 = \lim_{z\to\infty} z^2 \left[\overline{u(z)} - u_0 - u_1/z\right]$$

nal value theorem:-

$$\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)\, \overline{u(z)}$$

$Z[2n^2 + 5\sin n\pi/2 + a^4]$

$= 2\dfrac{(z^2+z)}{(z-1)^3} + \dfrac{5z}{z^2+1} + a^4 \cdot \dfrac{z}{(z-1)}$.

* $Z[2^{2n+3}] = Z[2^{2n} \cdot 2^3] = 8Z[4^n] = 8\dfrac{z}{z-4}$

$|z| > 4$

$Z[\cos n\pi/2] = \dfrac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$

$= \dfrac{z^2 - z\cos\pi/2}{z^2 - 2z\cos\pi/2 + 1} = \dfrac{z^2}{z^2+1}$

$Z[\sin(n+1)\theta] = Z[\sin n\theta\cos\theta + \cos n\theta\sin\theta]$

$= \dfrac{z^2\sin\theta}{z^2 - 2z\cos\theta + 1}$

* $Z[a^n + \frac{1}{n}]$

$= \sum_{n=0}^{\infty} (a^n + \quad) z^{-n}$

$= \sum_{n=1}^{\infty} a^n z^{-n} + \sum_{n=1}^{\infty} (1/n)(z^{-n})$

$= \dfrac{z}{z-a} - 1 \quad + \log\left(\dfrac{z}{z-1}\right)$

$= \dfrac{z - z + a}{z-a} + \log\left(\dfrac{z}{z-1}\right)$

$= \dfrac{a}{(z-a)} + \log\left(\dfrac{z}{z-1}\right)$

$$z\left[\frac{1}{n(n-2)}\right] = \frac{1}{2} z\left[\frac{1}{n-2} - \frac{1}{n}\right]$$

$$= \frac{1}{2} z\left(\frac{1}{(n-2)} - \frac{1}{n}\right)$$

$$= \frac{1}{2} \sum_{n=3}^{\infty} \left[\frac{1}{(n-2)} - \frac{1}{n}\right] z^{-n}$$

$$= 1/2 \left[\sum_{n=3}^{\infty} \frac{1}{n-2} z^{-n} - \sum_{n=3}^{\infty} \frac{1}{n} z^{-n}\right]$$

$$z\left[\frac{1}{n(n-2)}\right] = 1/2\left(z^{-2}\log\left(\frac{z}{z-1}\right) - \log\left(\frac{z}{z-1}\right) + 1/z + 1/2z^2\right]$$

Domain		Transformation	ROC
$\{a^n\}$	$n \geq 0$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$\{a^n\}$	$n > 0$	$\frac{a}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
$z(a^n)$	$n \leq 0$	$\frac{a}{a-z}$	$\lvert z\rvert < \lvert a\rvert$
$z(a^n)$	$n <$	$\frac{z}{a-z}$	$\lvert z\rvert < \lvert a\rvert$

$z[2^n + 3^n]$ $\quad n \geq 0$

$$= \frac{z}{z-2} + \frac{z}{z-3} \qquad \boxed{\text{Roc } \lvert z\rvert > 3}$$

$$z[2^n + 3^n]\, u(-n) = \frac{2}{2-z} + \frac{3}{3-z} \qquad \boxed{\text{Roc } \lvert z\rvert < 2}$$

$$u_n = \begin{cases} 2^n; & n < 0 \\ 3^n; & n \geq 0 \end{cases}$$

$$z[u(n)] = z[2^n] = \frac{z}{2-z} \qquad \lvert z\rvert < 2$$

$z[3^n] = \frac{z}{z-3}$; $|z|>3$ ✗

$= \frac{z}{2-z} +$ does Not Exist because there is No common point do the ROC. Hence it will Not Exist

* $x_n = \begin{cases} 2^n ; & n>0 \\ 3^n ; & n\leq 0 \end{cases}$

$z[x_n] = ?$

$= \frac{2}{(z-2)} + \frac{3}{(3-z)} = \frac{2(3-z)+3(z-2)}{(z-2)(3-z)}$

$= \frac{\cancel{6}-2z+3z-\cancel{6}}{(z-2)(3-z)}$ ROC

$= \frac{z}{(z-2)(3-z)}$ $2<|z|<3$.

* If $x_6 = \{ \overset{x_0}{1}, \overset{x_1}{3}, \overset{x_2}{5}, \overset{x_3}{7}, \overset{x_4}{0}, \overset{x_5}{9} \}$

$z\{x_6\} = ?$

$z\{x_6\} = \sum_{n=0}^{5} x_n z^{-n} = 1+3/z+5/z^2+7/z^3+9/z^5$

ROC: $|z|>0$. (or) $z \neq 0$

* If $x_6 = \{ \overset{x_{-3}}{1}, \overset{x_{-2}}{3}, \overset{x_{-1}}{5}, \underset{\uparrow}{\overset{x_0}{7}}, \overset{x_1}{0}, \overset{x_2}{9} \}$

$z\{x_6\} = z^3+3z^2+5z+7+9/z^2$

$0<|z|<\infty$.

$Z\left\{\frac{1}{4^K}\right\}$; $-2 \leq K \leq 2$

$$= \sum_{K=-2}^{2} \left(\frac{1}{4^K}\right) z^{-K}$$

$$= 16z^2 + 4z + 1 + \frac{1}{4z} + \frac{1}{16z^2} \qquad 0 < |z| < \infty.$$

$\{{}^nc_K\}$ n is a scalar

n is a constant and 'k' is variable.

$$= \sum_{K=0}^{\infty} {}^nc_K\, z^{-K}$$

$$= {}^nc_0 + {}^nc_1 z^{-1} + {}^nc_2 z^{-2} + {}^nc_3 z^{-3} + {}^nc_4 z^{-4} + \text{---} \; {}^nc_n z^{-n}$$

$$= (1 + 1/z)^n \quad [\text{Binomial Expansion}]$$

$\boxed{\text{ROC: } |z| > 0.}$

$Z\{{}^nc_K\}$ K is a scalar and n is a variable

$$= \sum_{n=K}^{\infty} {}^nc_K\, z^{-n} \qquad \boxed{{}^nc_r = {}^nc_{n-r}}$$

$$= \sum_{n=K}^{\infty} {}^nc_{n-K}\, z^{-n}$$

$$= {}^kc_0 z^{-K} + {}^{(K+1)}c_1 z^{-(K+1)} + {}^{(K+2)}c_2 z^{-(K+2)} + \text{------}$$

$$= z^{-K}\left[{}^kc_0 + {}^{(K+1)}c_1\, 1/z + {}^{(K+2)}c_2\, 1/z^2 + \text{-----}\right]$$

$$= z^{-k}\left[(1-1/z)^{-(k+1)}\right].$$

* $$\frac{1}{n!} * \frac{1}{n!} = \sum_{m=0}^{n}\left(\frac{1}{m!}\right)\frac{1}{(n-m)!}$$

$$= \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{2!\,(n-2)!} + \frac{1}{3!\,(n-3)!} + \cdots \frac{1}{n!}$$

$$= \frac{1}{n!}\left[1 + n + \frac{n(n-1)}{2!} + \cdots\cdots\right]$$

$$= \frac{1}{n!}\left[{}^{n}c_0 + {}^{n}c_1 + {}^{n}c_2 + \cdots\cdots + {}^{n}c_n\right] = \frac{1}{n!}2^n$$

$$\boxed{\frac{1}{n!} \times \frac{1}{n!} = \frac{1}{n!}2^n}$$

* $$\boxed{\frac{1}{n!} * \frac{1}{n!} * \frac{1}{n!} = \frac{3^n}{n!}}$$

* $$\cancel{1*1} = \cancel{\frac{(2)^1}{(1!)}} \neq \cancel{\frac{2}{1}} = \cancel{2}$$

* $$\boxed{(1*1) = \sum_{m=0}^{n} (1)(1) = (r-1)}$$

* $$\boxed{u(n) * u(n) = n+1}$$

* $$\boxed{u(t) * u(t) = t}$$

$L(u(t))\ L(u(t))$

$1/s \cdot 1/s = 1/s^2$ $\qquad L^{-1}[1/s^2] = t\,u(t).$

* $\cos 2t * \sin t$

$$L^{-1}\left[\frac{s}{(s^2+4)}\frac{1}{(s^2+1)}\right] = \frac{1}{3}L^{-1}\left[\frac{s}{s^2+4} - \frac{s}{(s^2+1)}\right]$$

$= \frac{-1}{3}\left[\cos 2t - \cos t\right]$

$Z\left[a^n\, n u(n)\right]$

$Z\left[a^n\, u(n)\right] = \frac{z}{z-a}$

$\downarrow$ multiplication

$n a^n u(n)] = -z\frac{d}{dz}\left[\frac{z}{z-a}\right] = \frac{az}{(z-a)^2}$

$Z\left[n u(n)\right] = \frac{z}{(z-1)^2}$

$Z\left[a^n . n u(n)\right] = u(z/a) = \frac{z/a}{(z/a-1)^2} = \frac{az}{(z-a)^2}$

$Z\left[e^{-a}\frac{a^n}{n!}\right] = e^{-a}\, Z\left(\frac{a^n}{n!}\right)$

$= e^{-a} . e^{1/(z/a)}$ $\qquad Z(1/n!) = e^{1/z}$

$= e^{-a}. e^{a/z} = e^{-(a-a/z)}$

$Z\left\{(2+3i)^n \frac{1}{n!}\right\} \xrightarrow{DR} e^{1/z\,(2+3i)} = e^{\frac{(2+3i)}{z}}$

$Z\left[e^{-an}\sin n\theta\right] =$ ~~(z+a) sin~~

$= Z\left\{(e^a)^{-n}\sin n\theta\right\}$

$= \frac{z e^a \sin\theta}{(ze^a)^2 - 2(ze^a)\cos\theta + 1}$

$L[2^t] = L\left[e^{\log 2^t}\right] = L\left[e^{t\log 2}\right] = \frac{1}{s-\log 2}$

$$Z\left[n\sin n\theta\right] = -z\frac{d}{dz}\left[\frac{z\sin\theta}{z^2-2z\cos\theta+1}\right]$$

$$= -z\left[\frac{(z^2-2z\cos\theta+1)(+\cancel{z\cos\theta}\ \sin\theta) - z\sin\theta\,(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{z^2\sin\theta - 2z\cos\theta\sin\theta + \sin\theta - 2z^2\sin\theta - 2z\sin\theta\cos\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{-z^2\sin\theta - 4z\sin\theta\cos\theta + \sin\theta}{(z^2-2z\cos\theta+1)^2}\right]$$

$=$

* $Z[(n-1)^2] =$

(a) $\frac{z+1}{(z-1)^2}$ b) $\frac{z^2}{(z-1)^3}$ c) $\frac{z+1}{(z-1)^3}$ d) $\frac{z^3-3z^2+4z}{(z-1)^3}$ F

e) None

Sol

$$Z\left[n^2-2n+1\right]$$

$$= \frac{z^2+z}{(z-1)^3} - \frac{2z}{(z-1)^2} + \frac{z}{(z-1)}$$

$$= \frac{z^2+z-2z(z-1)+z(z-1)^2}{(z-1)^3}$$

$$= \frac{z^2+z-2z^2+2z+z(z^2-2z+1)}{(z-1)^3} = \frac{z^3-3z^2+4z}{(z-1)^3}$$

If $u_n = n^2$ then $Z[u_{n-1}] = ?$

from the first shifting theorem.

$$Z\{u_{n-1}\} = Z^{-1} \cdot Z(u_n)$$

$$= Z^{-1} \cdot Z[n^2] = Z^{-1}\left(\frac{Z^2+Z}{(Z-1)^3}\right)$$

$$\boxed{Z[u_{n-1}] = \frac{Z+1}{(Z-1)^3}}$$

$$Z[\sin(n-1)\theta] = Z[\sin n\theta \cos\theta - \cos n\theta \sin\theta]$$

$$= \frac{Z\sin\theta\cos\theta - (Z^2 - 2\cos\theta)\sin\theta}{(Z^2 - 2Z\cos\theta + 1)^2}$$

$$Z\{u_n\} = \frac{Z}{Z-1} + \frac{Z}{Z^2+1} \text{ then } Z[u_{n+2}] = ?$$

$$Z[u_{n+2}] = Z^2\left[\bar{u}(Z) - u_0 - u_1/Z\right] \longrightarrow (1)$$

$$u_0 = \lim_{Z\to\infty} \bar{u}(Z) = \lim_{Z\to\infty} \frac{Z}{(Z-1)} + \frac{Z}{(Z^2+1)} = 1+0 = 1$$

$$u_1 = \lim_{Z\to\infty} Z[\bar{u}(Z) - u_0] = \lim_{Z\to\infty} Z\left(\frac{Z}{Z-1} + \frac{Z}{Z^2+1} - 1\right)$$

$$= \lim_{Z\to\infty} Z\left[\frac{Z-(Z-1)}{(Z-1)} + \frac{Z}{Z^2+1}\right]$$

$$= \lim_{Z\to\infty} Z\left[\frac{1}{(Z-1)} + \frac{Z}{(Z^2+1)}\right]$$

$$= 1+1 = \underline{2}$$

II Method.

Apply transform $u_n = 1+\sin n\pi/2$

$u_{n+2} = 1+\sin (n+2)\pi/2$

$u_{n+2} = 1-\sin n\pi/2$

$$\boxed{z[u_{n+2}] = \frac{z}{z-1} - \frac{z}{z^2+1}}$$

$z\{u_n\} = \dfrac{z^2-3z+4}{(z-3)^3}$ for $|z|<3$ then $u_3 = ?$

$$= \frac{z^2-3z+4}{z^3(1-3/z)^3} = \frac{z^2-3z+4}{z^3}\left[1-3/z\right]^{-3}$$

$$z\{u_n\} = \frac{z^2-3z+4}{z^3}\left[1+3\cdot 3/z+6(3/z)^2+10(3/z)^3+\cdots\right]$$

$$=(z^2-3z+4)\left[1/z^3+9/z^4+54/z^5+\cdots\right]$$

$u_3 =$ co-efficient of $1/z^3$

$= (1\times 54)+(9\times -3)+(4\times 1)$

$= 54-27+4 = 58-27 = 31.$

* Inverse Z-transform:-

If $z(f(n)) = \bar{f}(z)$ then $f(n) = z^{-1}(\bar{f}(z))$

Linearity:-

$$z^{-1}(a\bar{f}(z)\pm b\bar{g}(z)) = az^{-1}(\bar{f}(z))\pm bz^{-1}(\bar{g}(z))$$

Methods:- ① Standard formula (formula based Ans) power series / division method.

In this method Expand $F(z)$ as infinite series in powers of 'z' as per ROC. By using eclarinus, taylors, Laurents and etc. then it can be Expressed as $\sum_n f(n) z^{-n}$.

partial fraction

Here Split the partial fractions for $\frac{F(z)}{z}$ not or $F(z)$ directly.

convolotion property

$$z^{-1}\left[f(z)\cdot g(z)\right] = f(n) * g(n)$$
$$= \sum_{m=0}^{n} f(m)\, g(n-m)$$

Con / integration:- tc a

$$\oint_c f(z)\, dz$$

$$z^{-1}(F(z)) = \frac{1}{2\pi i}\oint_c z^{n-1} F(z)\, dz.$$

cauchys Residue theorem.

$$= \frac{1}{2\pi i} \times 2\pi i \text{ (Sum of Residues at each pole of } z^{n-1} F(z) \text{ which are inside c)}$$

$$= \sum_i r_i$$

1) If $z=a$ is a pole of order one

$$\text{Res}\left[f(z) : z=a\right] = \underset{z\to a}{Lt} (z-a)f(z)$$

2) If $z=a$ is a pole of order 'm'

$$\text{Res} = \frac{1}{(m-1)!} \underset{z\to a}{Lt} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]$$

→ If 'c' is not given then consider all the poles are inside

→ $z^{-1}(1) = \delta(n)$

→ $z^{-1}\left(\frac{z}{z-1}\right) = u(n)$

→ $z^{-1}\left(\frac{z}{z+1}\right) = (-1)^n u(n)$

→ $z^{-1}\left(\frac{z}{z-a}\right) = a^n u(n)$

→ $z^{-1}\left(\frac{z}{z+a}\right) = (-a)^n u(n)$

→ $z^{-1}(z^2) = \delta(n+2)$

→ $z^{-1}(1/z^2) = \delta(n-2)$

$= u(n-2) - u(n-3)$

→ $z^{-1}(e^{2/z}) = z^{-1}\left[1 + 2/z + \frac{1}{2!}(2/z)^2 + \frac{1}{3!}(2/z)^3 + \text{------}\right]$

$$= z^{-1}\left[\sum_{n=0}^{\infty} \frac{2^n}{n!} z^{-n}\right] = \frac{2^n}{n!} u(n)$$

→ $z^{-1}\left(\log\left(\frac{z}{z+1}\right)\right) = -z^{-1}\left(\log\left(\frac{z+1}{z}\right)\right)$

$$= -z^{-1}\left(\left(\log(1+1/z)\right)\right)$$

$$= -z^{-1}\left[1/z - 1/2z^2 + \frac{1}{3z^3} - \frac{1}{4z^4} + \text{----}\right]$$

$$= -z^{-1}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^{-n}\right)$$

$$= -\frac{(-1)^{n-1}}{n} u(n-1)$$

$$= \frac{(-1)^n}{n} u(n-1)$$

$Z^{-1}\left(\frac{1}{Z-2}\right)$ for $|Z|>2$

a) $2^{n-1}u(n)$ b) $2^n u(n-1)$ ~~c)~~ $2^{n-1}u(n-1)$ d) $2^n u(n)$

$$|2/Z| < 1$$

$$= \frac{1}{Z(1-2/Z)} = 1/Z\left[1-2/Z\right]^{-1}$$

$$= 1/Z\left[1+2/Z+(2/Z)^2+(2/Z)^3+ \cdots\right]$$

$$= 1/Z + 2/Z^2 + 4/Z^3 + 8/Z^4 + \cdots$$

$$= \sum_{n=1}^{\infty} 2^{n-1} Z^{-n} = 2^{n-1}u(n-1)$$

(37)

By using Residue theorem

$$= r_1 \big|_{Z=2}$$

$$= \lim_{Z\to 2} \cancel{(Z-2)} \frac{Z^n}{\cancel{(Z-2)}} = 2^n u(n)$$

$$Z^{-1}\left[\frac{Z}{(Z-3)(Z+5)}\right]$$

$$Z^{-1}\left(\frac{Z \cdot Z^{n-1}}{(Z-3)(Z+5)} = Z^n\right)$$

$$= r_1\big|_{Z=3} + r_2\big|_{Z=-5}$$

$$= \left[\frac{3^n}{8} + \frac{(-5)^n}{-8}\right] u(n)$$

$$* \quad Z^{-1}\left[\frac{1}{(Z-1)(Z+8)}\right] \quad (Z^{n-1})$$

$$= \left[1/9 + \frac{(-8)^{n-1}}{-9}\right] u(n-1)$$

$$* \quad Z^{-1}\left[\frac{8Z^2}{(4Z-1)(2Z+1)}\right]$$

$$= Z^{-1}\left[\frac{8Z^2 Z^{n-1} = Z^{n+1}}{4(Z-1/4)\ 2(Z+1/2)}\right]$$

$$= r_1 \Big|_{Z=1/4} + r_2 \Big|_{Z=-1/2}$$

$$= \left[\frac{(1/4)^{n+1}}{1/4+1/2} + \frac{(-1/2)^{n+1}}{-1/2-1/4}\right]$$

$$= 4/3\left[(1/4)^{n+1} - (-1/2)^{n+1}\right] u(n)$$

$$* \quad Z^{-1}\left[\frac{Z}{Z^2+2Z+2}\right]$$

$$Z_1 = -1+i$$

$$Z_2 = -1-i$$

$$= Z^{-1}\left[\frac{Z \cdot Z^{n-1} = Z^n}{(Z-Z_1)(Z-Z_2)}\right]$$

$$= r_1 \Big|_{Z=Z_1} + r_2 \Big|_{Z=Z_2}$$

$$= \left[\frac{(Z_1)^n}{Z_1-Z_2} + \frac{(Z_2)^n}{Z_2-Z_1}\right]$$

$$= \left[\frac{(-1+i)^n}{2i} + \frac{(-1-i)^n}{2i}\right]$$

$$= (\sqrt{2})^n \left[\frac{(-1/\sqrt{2}+i/\sqrt{2})^n - (-1/\sqrt{2}-i/\sqrt{2})^n}{2i} \right]$$

$$= (\sqrt{2})^n \left[\frac{(e^{i3\pi/4})^n - (e^{-i3\pi/4})^n}{2i} \right] u(n)$$

$$= (\sqrt{2})^n \sin 3n\pi/4$$

$$z^{-1}\left[\frac{z^2}{(z-3)^5}\right]$$

$$= z^{-1}\left[\frac{z^2 \cdot z^{n-1} = z^{n+1}}{(z-3)^5}\right]$$

$$= r_1 \Big|_{z=3}$$

$$\frac{1}{(5-1)!} \lim_{z\to 3} \frac{d^4}{dz^4} \left(\cancel{(z-3)^5} \frac{z^{n+1}}{\cancel{(z-3)^5}} \right)$$

$$= \frac{1}{4!} \left[(n+1)(n)(n-1)(n-2)\,(3)^{n-3} \right] u(n+1)$$

$$z^{-1}\left(\frac{z}{(z-2)(z+3)}\right) \text{ for } |z|<2$$

$$z^{-1}\left(\frac{z \cdot z^{n-1}}{(z-2)(z+3)}\right)$$

$$= r_1 \Big|_{z=2} + r_2 \Big|_{z=-3}$$

$$|z|>3, \quad |z|>2$$

$$= \left(\frac{2^n}{5} + \frac{(-3)^n}{-5} \right) u(n)$$

$$= -\left(\frac{2^n}{5} + \frac{(-3)^n}{-5} \right) u(-n-1)$$

Vector calculus

① vector differentiation (2) vector integration

vector: $\bar{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\bar{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

dot product: $\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|\cos\theta$.

$$\boxed{\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}||\bar{b}|}}$$

$|\bar{a}| = \sqrt{\bar{a}\cdot\bar{a}}$

$= \sqrt{a_1^2 + a_2^2 + a_3^2}$

$i\cdot i = j\cdot j = k\cdot k = 1$

$i\cdot j = j\cdot k = k\cdot i = 0$.

$\bar{a}\cdot\bar{b} = 0$ ⊥r

$\bar{a}\cdot\bar{b} = |\bar{a}||\bar{b}|$ parallel

cross product:- $\boxed{\bar{a}\times\bar{b} = |\bar{a}||\bar{b}|\sin\theta\,\hat{n}}$

unit vector $\bar{c} = \frac{\bar{a}}{|\bar{a}|}$

$$\bar{a}\times\bar{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$i\times i = j\times j = k\times k = 0$

$\bar{i}\times\bar{j} = \bar{k}$; $\bar{j}\times\bar{k} = \bar{i}$, $\bar{k}\times\bar{i} = \bar{j}$

$= i\,(a_2b_3 - b_2a_3) - j\,(a_1b_3 - b_1a_3) + k\,(a_1b_2 - a_2b_1)$

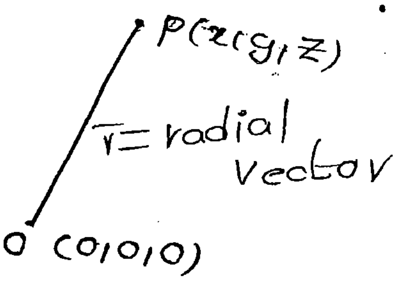
Triple product:-

→ $\bar{a}\cdot(\bar{b}\times\bar{c}) = (\bar{a}\times\bar{b})\cdot\bar{c} = [abc] = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

→ ~~If~~ $\bar{a}\cdot(\bar{b}\times\bar{c}) = 0$ then the vectors

$\bar{a}, \bar{b}, \bar{c}$ are called coplanar vectors.

sition vector:-



$\overline{OP} = \bar{r} = (x-0)\,\bar{i} + (y-0)\,\bar{j} + (z-0)\,\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ is called position vector

$|\bar{r}| = \sqrt{x^2+y^2+z^2} \implies r^2 = x^2+y^2+z^2$

$= x(t)\,i + y(t)\,j + z(t)\,k$ is called parametric form of position vector, where 't' is parameter.

$\bar{r} = \cos t\,\bar{i} + \sin t\,\bar{j} + t\,k = f(t)$

rivative of a vector

If $\bar{r} = f(t)$ be a given position vector

$\bar{r} + \delta\bar{r} = f(t+\delta t)$

$\delta\bar{r} = f(t+\delta t) - f(t)$.

$\lim_{\to 0} \frac{\delta\bar{r}}{\delta t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t}$ is called $\frac{d\bar{r}}{dt}$

i.e. derivative of $\bar{r}$ w.r.to 't')

→ If 't' is a time variable then $\frac{d\bar{r}}{dt}$ is called velocity vector

$\left|\frac{d\bar{r}}{dt}\right| = \sqrt{\frac{d\bar{r}}{dt} \cdot \frac{d\bar{r}}{dt}}$ magnitude of velocity vector

$\bar{r} = t\,i + t^2 j + t^3 k$

$d\bar{r}/dt = i + 2t\,j + 3t^2 k$

$= i + 2j + 3k$

$$\left|\frac{dr}{dt}\right| = \sqrt{1+4+9} = \sqrt{14}$$

Properties

1) $\frac{d}{dt}(\bar{A} \pm \bar{B}) = \frac{d\bar{A}}{dt} \pm \frac{d\bar{B}}{dt}$ where $\bar{A} = A(t)$

$\bar{B} = B(t)$

2) $\frac{d}{dt}(\bar{A} \cdot \bar{B}) = \bar{A}\frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt} \cdot \bar{B}$

3) $\frac{d}{dt}(\bar{A} \times \bar{B}) = \bar{A} \times \frac{d\bar{B}}{dt} + \frac{d\bar{A}}{dt} \times B.$

Vector operator ∇ (DEL):-

definition:-

$\boxed{\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}}$ is called vector operator

$|\nabla|$ Not possible. It is Not vector, Not Scalar It i. only operator.

Gradient:- If $\phi(x, y, z)$ be a given Scalar function then $\nabla\phi$ is called gradient.

$$\nabla\phi = i\frac{d\phi}{dx} + j\frac{d\phi}{dy} + k\frac{d\phi}{dz}$$

$\boxed{|\nabla\phi| = \sqrt{\nabla\phi \cdot \nabla\phi}}$ magnitude of gradient of Scalar function

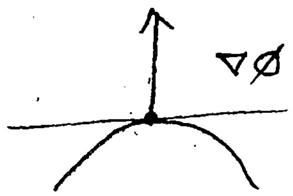
physically gradient gives rate of change of ϕ w.r.to x, y, z

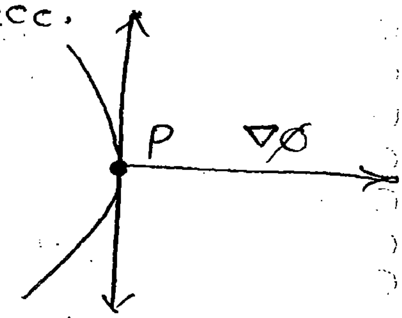
metrically

$\phi(x,y,z) = c$ be a given level Surface.

which is always Normal to the

...el Surface.





$$\boxed{\phi(x,y,z) = c}$$

$$\phi = xyz$$

$$\nabla\phi = i\frac{d}{dx}(xyz) + j\frac{d}{dy}(xyz) + k\frac{d}{dz}(xyz)$$

$$\boxed{\nabla\phi = yzi + xzi + xyk}$$

If $\phi = xy^2z$ and the point $p(1,2,3)$

$$\nabla\phi = y^2zi + 2xyzj + xy^2k$$

$$\boxed{\nabla\phi = 12i + 12j + 4k}$$

If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ then

$$\nabla r = i\frac{d}{dx}(r) + j\frac{d}{dy}(r) + k\frac{d}{dz}(r)$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{dr}{dx} = x/r, \quad \frac{dr}{dy} = y/r, \quad \frac{dr}{dz} = z/r$$

$$\nabla r = i(x/r) + j(y/r) + k(z/r)$$

$$= \frac{i(x) + j(y) + k(z)}{r} = \frac{\bar{r}}{r}$$

$$\boxed{\nabla r = \frac{\bar{r}}{r}}$$

) · Gradion (r^n) (∇r^n) ·

$$\nabla r^n = i\frac{d}{dx}(r^n) + j\frac{d}{dy}(r^n) + k\frac{d}{dz}(r^n)$$

$$= nr^{n-1}\left[i\frac{dr}{dx} + j\frac{dr}{dy} + k\frac{dr}{dz}\right]$$

$$= nr^{n-1}\left[i(x/r) + j(y/r) + k(z/r)\right]$$

$$= nr^{n-2}\left[x\bar{i} + y\bar{j} + z\bar{k}\right]$$

$$\boxed{\nabla r^n = nr^{n-2}(\bar{r})}$$

5) $$\nabla(\log r) = i\frac{d}{dx}(\log r) + j\frac{d}{dy}(\log r) + k\frac{d}{dz}(\log r)$$

$$= \frac{1}{r}\left[i\frac{dr}{dx} + j\frac{dr}{dy} + k\frac{dr}{dz}\right]$$

$$= \frac{1}{r}\left[i(x/r) + j(y/r) + k(z/r)\right]$$

$$\boxed{\nabla(\log r) = \frac{1}{r^2}(\bar{r})}$$

6) $$\nabla(\sin r) = i\frac{d}{dx}(\sin r) + j\frac{d}{dy}(\sin r) + k\frac{d}{dz}(\sin r)$$

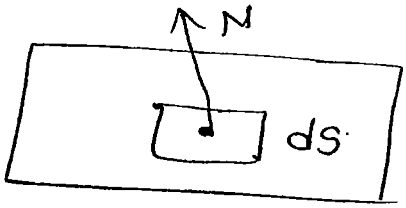
$$= \cos r \cdot \frac{\bar{r}}{r}$$

7) $$\nabla(e^{x+y+z}) = e^{x+y+z}(i+j+k)$$

<u>Unique Normal vector to the surface:-</u>

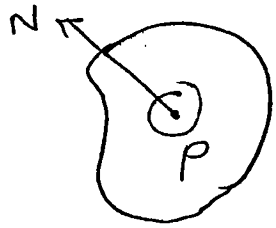
The unique Normal vector to the surface $\phi(x,y,z)$ is defined as

$$\overline{N} = \frac{\nabla \phi}{|\nabla \phi|}$$



$\phi = xyz$, $P(1,2,3)$

then find $\overline{N}$



$$\nabla \phi = yz\overline{i} + xz\overline{j} + xy\,\overline{k}$$

$$= 6i + 3j + 2k$$

$$|\nabla \phi| = \sqrt{36+9+4} = 7$$

$$N = \frac{\nabla \phi}{|\nabla \phi|} = \frac{6i+3j+2k}{7}$$

$\phi = x^2 - y^2 + z - 2$ at point $(1,-1,2)$ then $\overline{N}$

$$\nabla \phi = 2x\overline{i} - 2y\overline{j} + k$$

$$= 2\overline{i} + 2\overline{j} + \overline{k}$$

$$|\nabla \phi| = \sqrt{4+4+1} = 3.$$

$$\boxed{\overline{N} = \frac{2\overline{i}+2\overline{j}+\overline{k}}{3}}$$

Angle b/w two Surfaces:-

Let $\phi_1(x,y,z) = c_1$, $\phi_2(x,y,z) = c_2$ be given level surfaces then

$$\boxed{\cos\theta = \frac{\nabla \phi_1 \, \nabla \phi_2}{|\nabla \phi_1|\,|\nabla \phi_2|}}$$

gives angle between them.

The Angle b/w two surfaces is Nothing but angle b/w their Normals. If $\boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$ then the surfaces are said to be <u>orthogonal</u> <u>surfaces</u>

- The Angle b/w the surfaces $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$ is

<u>Sol</u>

$$\phi_1 = x^2+y^2+z^2-9 \qquad \nabla\phi_1 = 2x\bar{i}+2y\bar{j}+2z\bar{j}$$
$$= 4i-2j+4k$$

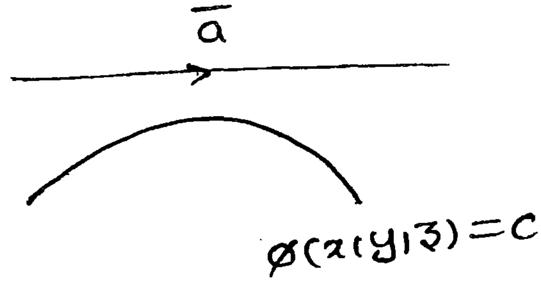
$$\phi_2 = x^2+y^2-z-3 \qquad \nabla\phi_2 = 2x\bar{i}+2y\bar{j}-\bar{k}$$
$$= 4i-2j-k$$

$$|\nabla\phi_1| = \sqrt{4+4+4} = \sqrt{12} \quad 6$$

$$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21}$$

$$\cos\theta = \frac{(4i-2j+4k)(4i-2j-k)}{6\sqrt{21}}$$

<u>directional</u> <u>derivative</u>:-



<u>definition</u>:- The directional derivative of a scalar function in the direction of the vector is defined as $\nabla\phi\cdot\bar{e}$

$\nabla\phi\cdot\bar{e}$ gives directional derivative of $\nabla\phi$.

$$\boxed{\text{where } \bar{e} = \frac{\bar{a}}{|\bar{a}|}}$$

fo $\nabla\phi\cdot\bar{e}$ is Negative then it is opposite
ection

$\phi = x+y$ and the point $p(1,1)$ $\bar{a} = \bar{i}+\bar{j}$ then the
ectional derivative is

$$\nabla\phi = \overline{i+j} \qquad \nabla\phi\cdot\bar{e} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\bar{e} = \frac{\overline{i+j}}{\sqrt{2}}$$

The directional derivative of the scalar function
$= x^2+2y^2+z$ at the point $(1,1,2)$ in the direction
$3i-4j$ is

a) -4 b) -2 c) -1 d) 1

$$\nabla f = 2x\bar{i}+4y\bar{j}+z\bar{k}$$

$$= 2\bar{i}+4\bar{j}+\bar{k}$$

$$\bar{e} = \frac{3i-4j}{5} \qquad \nabla f\cdot\bar{e} = \frac{6-16}{5} = \frac{-10}{5} = -2$$

$f = x^2+3y^2+2z^2$ at the point $p(1,2,-1)$
$\bar{a} = \bar{i}-\bar{j}+2\bar{k}$ then find the directional
derivative is

a) $3\sqrt{6}$ b) $-3\sqrt{6}$ c) 18 d)

$$\nabla f = 2x\bar{i}+6y\bar{j}+4z\bar{k}$$

$$= 2\bar{i}+12\bar{j}-4\bar{k}$$

$$\bar{e} = \frac{\bar{a}}{|\bar{a}|} = \frac{i-j+2k}{\sqrt{6}}.$$

$$\nabla f \cdot \bar{e} = \frac{2-12-8}{\sqrt{6}} = \frac{-18}{\sqrt{6}} = -3\sqrt{6}$$

divergence of a vector function:-

definition:- $\nabla \cdot F$ is called divergence.

where $\bar{F} = F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k}$

$$\nabla \cdot F = \left(i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz}\right) \cdot (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})$$

$$\boxed{\nabla \cdot F = \frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz}}$$

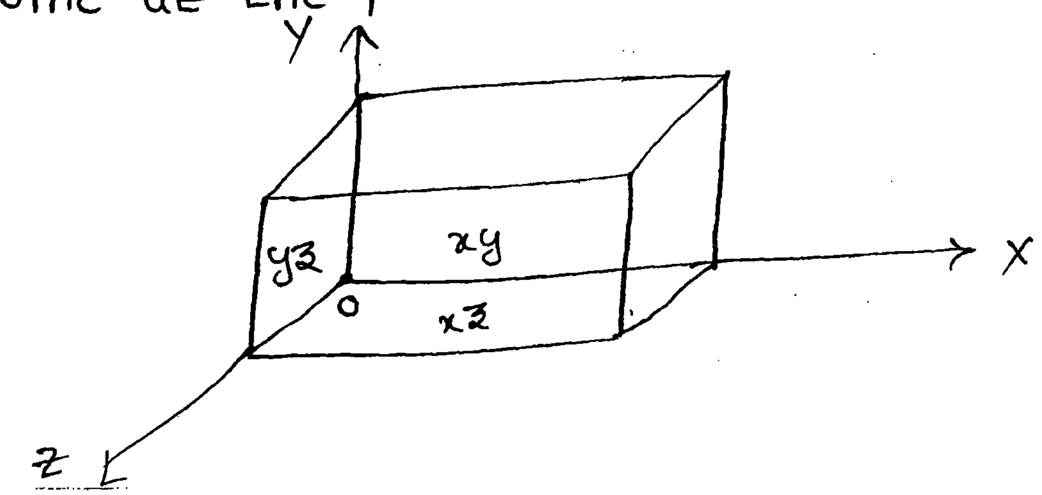
physically divergence measures outflow - inflow.

If $\boxed{\nabla \cdot \bar{F} = 0}$ is called divergence free vector.

$\boxed{\text{outflow} = \text{inflow} = \text{constant}}$

Geometrically

divergence gives the rate at which fluid entering in the Rectangular parallelepiped for unit volume at the point is called divergency



If $\overline{F} = (x-y)\overline{i} + (y-z)\overline{j} + (x+y+z)\overline{K}$ then

$$\cdot\overline{F} = \frac{d}{dx}(x-y) + \frac{d}{dy}(y-z) + \frac{d}{dz}(x+y+z)$$

$$= 1+1+1 = 3$$

$$\boxed{\nabla\cdot\overline{F} = 3}$$

$$\overline{F} = (y-z)\overline{i} + (x-z)\overline{j} + (x+y)\overline{K}$$

$$\boxed{\nabla\cdot\overline{F} = 0}$$

$$\overline{r} = x\overline{i} + y\overline{j} + z\overline{K}$$

$$\nabla\cdot\overline{r} = \frac{d}{dx}(x) + \frac{d}{dy}(y) + \frac{d}{dz}(z) = 1+1+1 = 3.$$

$\nabla(r^n.\overline{r})$ where $\overline{r} = x\overline{i} + y\overline{j} + z\overline{K}$

$$= \frac{d}{dx}(r^n.\overline{r}) + \frac{d}{dy}(r^n.\overline{r}) + \frac{d}{dz}(r^n.\overline{r})$$

$$= \left[i\frac{d}{dx} + j\frac{d}{dy} + K\frac{d}{dz}\right]\left[r^n(x\overline{i} + y\overline{j} + z\overline{K})\right]$$

$$= \frac{d}{dx}(xr^n) + \frac{d}{dy}(yr^n) + \frac{d}{dz}(zr^n)$$

$$= x\cdot nr^{n-1}\frac{dr}{dx} + r^n + ynr^{n-1}\frac{dr}{dy} + r^n + znr^{n-1}\frac{dr}{dz}$$

$$= 3r^n + nr^{n-1}\left(x\frac{dr}{dx} + y\frac{dr}{dy} + z\frac{dr}{dz}\right)$$

$$= 3r^n + nr^{n-1}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right)$$

$$= 3r^n + nr^{n-1}(r^2/r)$$

$$= 3r^n + nr^n = \boxed{(n+3)r^n = \nabla(r^n\overline{r})}$$

* ~~If the divergence of~~ If $r^n \cdot \bar{r}$ is a divergence free vector then the value of (n) is $\underline{-3}$

i) $\nabla(r^2 \cdot \bar{r}) = \underline{5r^2}$

ii) $\nabla\ (\bar{r}/r^3) = (n+3)r^n = (-3+3)r^3 = 0$

7) If $\phi(x,y,z) = ax^2y - y^3$ for an ideal fluid $\nabla^2\phi = 0$ in that case the value of (a) is ——

$$\nabla^2\phi = \nabla(\nabla\phi)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$$

$$\frac{\partial\phi}{\partial x} = 2axy \Rightarrow \frac{\partial^2\phi}{\partial x^2} = 2ay$$

$$\frac{\partial\phi}{\partial y} = ax^2 - 3y^2 \Longrightarrow \frac{\partial^2\phi}{\partial y^2} = -6y.$$

$$\nabla^2\phi = 0$$

$$2a\ \ 6y = 0 \qquad y$$

$$\boxed{a=3}$$

8) If $\phi = xyz$ then

$$\nabla\cdot(\nabla\phi) = \nabla(yz\bar{i} + xz\bar{j} + xy\bar{k})$$

$$= 0$$

9) If $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\nabla\cdot(\bar{a}\times\bar{r}) = 0 \qquad \bar{a}\times\bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$$

$$= i(a_2z - a_3y) - j(a_1z - a_3x) + k(a_1y - a_2x)$$

$$= 0$$

7 A vector function is given

$= -\{x\cos(xy)+y\}\, i + [\bar{y}\cos(xy)]\,\bar{j} + [\sin(z^2) + x^2 y^2]\, K$

then $\nabla\cdot\bar{V}$

$\nabla\cdot\bar{V} = i\frac{d}{dx} + [\cancel{-}x\cos(xy)+y] + \frac{d}{dy}[\bar{y}\cos(xy)] + \frac{d}{dz}[\sin(z^2) + x^2 + y^2]$

$= x\sin(x\cancel{y})y - \cancel{\cos}(xy) + \cancel{\cos}(xy) - xy\cancel{\sin}(xy) + \cos(z^2)\,2z$

$$\boxed{\nabla\cdot\bar{V} = 2z\cos(z^2)}$$

<u>curl of a vector point function</u>

<u>definition</u>:- It is defined as. $\nabla\times\bar{F}$ is called curl

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_1 & F_2 & F_3 \end{vmatrix} = i\left(\frac{dF_3}{dy} - \frac{dF_2}{dz}\right), j\left(\frac{dF_3}{dx} - \frac{dF_1}{dz}\right) + k\left(\frac{dF_2}{dx} - \frac{dF_1}{dy}\right)$$

If $\boxed{\nabla\times\bar{F} = 0}$ is called <u>irrotational vector</u>

$$\nabla\times\bar{V} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & y & z \end{vmatrix} = i(0) - j(0) + k(0)$$

$$= \bar{0}$$

) If $\bar{F} = yz\bar{i} + xz\bar{j} + xy\bar{k}$ then

$$\nabla \times \bar{F} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ yz & xz & xy \end{vmatrix} = i(x-x) + j(y-y) + k(z-z)$$
$$= \bar{0}$$

) $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\nabla \times (\bar{a} \times \bar{r}) =$

a) $\bar{a}$ b) $2\bar{a}$ c) a d) 0

$$\bar{a} \times \bar{r} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix} = i(a_2 z - a_3 y) - j(a_1 z - a_3 x) + k(a_1 y - a_2 x)$$

$$\nabla \times (\bar{a} \times \bar{r}) = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ (a_2 z - a_3 y) & (a_1 z - a_3 x) & (a_1 y - a_2 x) \end{vmatrix}$$

$$= i\left[\frac{d}{dy}(a_1 y - a_2 x) - \frac{d}{dz}(a_1 z - a_3 x)\right]$$
$$- j\left[\frac{d}{dx}(a_1 y - a_2 x) - \frac{d}{dz}(a_2 z - a_3 y)\right]$$
$$+ k\left[\frac{d}{dx}(a_1 z - a_3 x) - \frac{d}{dy}(a_2 z - a_3 y)\right]$$

$$= i(a_1 + a_1) - j(-a_2) + j(a_2) + k(+a_3) + a_3 k$$

$$= 2(a_1)\bar{i} + 2a_2\bar{j} + 2a_3\bar{k} = 2\bar{a}$$

ysically

et $\overline{v} = \overline{\omega} \times \overline{r}$ (linear velocity)

$\overline{\omega} = \omega_1 \overline{i} + \omega_2 \overline{j} + \omega_3 \overline{k}$ (Angular velocity)

$\overline{r} = x\overline{i} + y\overline{j} + z\overline{k}$

$\nabla \times \overline{v} = \nabla \times (\overline{\omega} \times \overline{r}) = 2\overline{\omega}$

$$\boxed{\overline{\omega} = 1/2\,(\nabla \times \overline{v})}$$

divergence of curl $\overline{F}$ is always zero

$$\boxed{\begin{array}{l} \nabla \cdot (\nabla \times \overline{F}) = 0 \\ \nabla \times (\nabla \times \overline{F}) = \nabla(\nabla \cdot \overline{F}) - \nabla^2 \overline{F} \end{array}}$$

vector identity

If $\overline{F} = xi - yj$

a) divergence free, Not irrotational

b) irrotational but not divergence free

c) divergence free and irrotational

d) not divergence free, but irrotational

$\nabla \cdot \overline{F} = 1$

$$\nabla \times \overline{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & 0 \end{vmatrix} = \overline{0}$$

$$= \frac{6}{4} - \frac{16}{6} = -7/6$$

2010

The line integral of $\bar{F} = (x^2+xy)\,i + (y^2+xy)\,j$ and 'c' is a curve line joining the points $(0,2)$ to $(2,0)$ is —

a) -8 b) 4 c) 0 d) 8

Sol

$$\int_c \bar{F}\cdot d\bar{r} = \int_c (x^2+xy)\,dx + (y^2+xy)\,dy$$

put $y = 2-x \Rightarrow dy = -dx$

$x = 0$ to $x = 2$

$y-y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$

$y-2 = \frac{0-2}{2-0}(x)$

$y-2 = -x$

$\boxed{y = 2-x}$

$$\int \bar{F}\cdot d\bar{r} = \int_0^2 \Big[x^2+x(2-x)\,dx + (2-x)^2+x(2-x)\,(-dx)\Big]$$

$$= \int_0^2 \Big[(\cancel{x^2}-\cancel{x^2}+2x)\,dx - dx\,\big(4+\cancel{x^2}-4x+2x-\cancel{x^2}\big)\Big]$$

$$= \int_0^2 \big[2x\,dx - dx(-2x+4)\big]$$

$$= \Big[2(x^2/2)+2(x^2/2)-4x\Big]_0^2$$

$$= 4+4-8 = 0.$$

$\int_c \bar{F}\cdot d\bar{r} = 0$. Hence it is conservative

* The value of $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = (3x+2y)\,i + (4x-2y)\,j$ where 'c' is a curve $x^2+y^2=4$

Vector integration:-

Let $\frac{d}{dt}[f(t)] = F(t)$

$$\int \frac{d}{dt}[F(t)]\,dt = \int F(t)\,dt$$

$\boxed{f(t) = \int F(t)\,dt}$ is called vector integration

$$\int_a^b F(t)\,dt = \bar{F}(b) - \bar{F}(a)$$

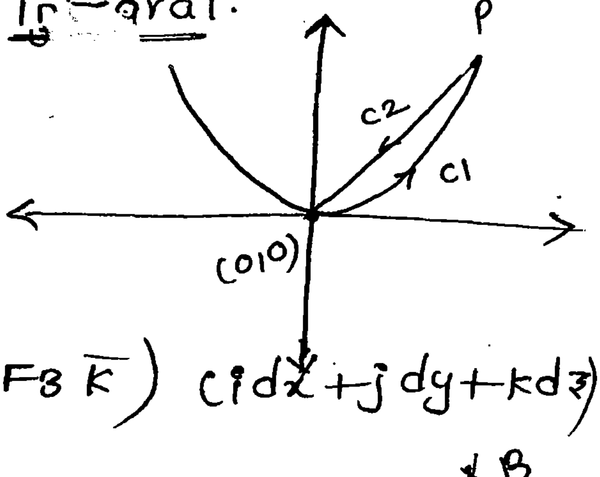
① $\int_0^{\pi/2} [\cos t\, i + \sin t\, j + t k]\,dt$

$$= i\,|\sin t|_0^{\pi/2} + j\,[-\cos t]_0^{\pi/2} + k\,[t^2/2]_0^{\pi/2}$$

$$= i + j + \frac{\pi^2}{8} k$$

Line integrals:-

definition:- An integral which is to be evaluated along a curve is called line integral.

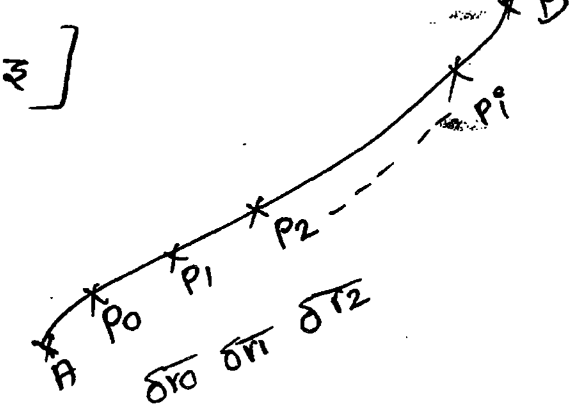


$y = x^2$, $y = x$

Mathematically it is defined as

$$\int_c \bar{F}\cdot \bar{dr} = \int_c (F_1 \bar{i} + F_2 \bar{j} + F_3 \bar{k})\,(i\,dx + j\,dy + k\,dz)$$

$$= \int_c [F_1\,dx + F_2\,dy + F_3\,dz]$$



$\Sigma F\cdot d\bar{r_i}$ as Lt $n \to \infty$

$\delta r_i \to 0$

If 'c' is a closed curve then $\boxed{\oint_C \vec{F} \cdot \vec{dr}}$ is called
colation

put $x = r\cos\theta$, $y = r\sin\theta$

$\theta = 0 \Rightarrow \theta = 2\pi$

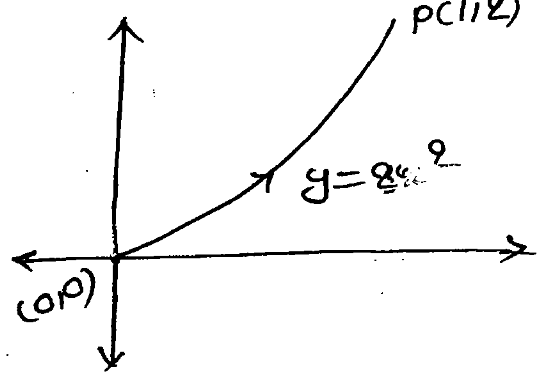
If F is a force the total workdone by a force is
$\int \overline{F} \cdot \overline{dr}$

If $\boxed{\int_C \overline{F} \cdot \overline{dr} = 0}$ is called conservative forces
(or) Loss of Energy

$\int_C \overline{F} \cdot \overline{dr} \neq 0$ Non conservative forces.

The $\int \overline{F} = 3xyi - y^2j$ and 'c' is a curve $y = 2x^2$
from $(0,0)$ to $(1,2)$ is

a) $-6/7$ b) $-7/6$ c) $5/3$ d) $2/3$.



$\int \overline{F} \cdot \overline{dr} = \int_C [3xy\,dx - y^2 dy]$

put $y = 2x^2$

$dy = 4x\,dx$

$x = 0$ to 1

$$\int_C \overline{F} \cdot \overline{dr} = \int_0^1 [3x(2x^2)dx - (2x^2)^2 4x\,dx]$$

$$= \int_0^1 [6x^3 dx - 16x^5 dx]$$

$$= \left(\frac{6(x^4)}{4}\right)_0^1 - \left(\frac{16(x^6)}{6}\right)_0^1 = \frac{6}{4} - \frac{4}{6} = \frac{36-16}{24}$$

$$= \frac{6}{4} - \frac{16}{6} = -7/6$$

<u>2010</u>

The line integral of $F = (x^2+xy)i + (y^2+xy)j$ and 'c' is a curve line joining the points (0,2) to (2,0) is —

a) −8 b) 4 c) 0 d) 8

<u>Sol</u>

$$\int_c F\cdot dr = \int_c (x^2+xy)dx + (y^2+xy)dy$$

put $y = 2-x \Rightarrow dy = -dx$

$x = 0$ to $x = 2$

$$y - y_1 = \frac{y_2-y_1}{x_2-x_1}(x-x_1)$$

$$y - 2 = \frac{0-2}{2-0}(x)$$

$$y - 2 = -x$$

$$\boxed{y = 2-x}$$

$$\int_c F\cdot dr = \int_0^2 \Big[x^2 + x(2-x)\,dx + (2-x)^2 + x(2-x)(-dx)\Big]$$

$$= \int_0^2 \Big[\left(x^2 - x^2 + 2x\right)dx - dx\left[4 + \cancel{x^2} - 4x + 2x - \cancel{x^2}\right]\Big]$$

$$= \int_0^2 \Big[2x\,dx - dx(-2x+4)\Big]$$

$$= \Big[2(x^2/2) + 2(x^2/2) - 4x\Big]_0^2$$

$$= 4 + 4 - 8 = 0.$$

$\int_c F\cdot dr = 0$. Hence it is conservative

* The value of $\int_c \vec{F}\cdot\vec{dr}$ where $\vec{F} = (3x+2y)i + (4x-2y)j$. where 'c' is a curve $x^2+y^2=4$

$$\int_C (3x+2y)\,dx + (4x-2y)\,dy \qquad x^2+y^2=4 \text{ (Circle Equation)}$$

put $x = 2\cos\theta$, $y = 2\sin\theta$

$dx = -2\sin\theta\, d\theta$ $\quad dy = 2\cos\theta\, d\theta$

$\theta = 0,\ \theta = 2\pi$

$$\int_0^{2\pi} (6\cos\theta + 4\sin\theta)(-2\sin\theta\,d\theta) + (4(2\cos\theta) - 2(2\sin\theta))(2\cos\theta\, d\theta)$$

$$\int_0^{2\pi} \left[\left(-12\sin\theta\cos\theta - 8\sin^2\theta + 16\cos^2\theta - 8\sin\theta\cos\theta\right) d\theta\right]$$

$$\int_0^{2\pi} \left[-20\sin\theta\cos\theta - 8\sin^2\theta + 16\cos^2\theta\right]$$

$$-20\left|\frac{\sin^2\theta}{2}\right|_0^{2\pi} - \frac{8}{2}\left|\theta - \frac{\sin 2\theta}{2}\right|_0^{2\pi} + \frac{16}{2}\left(\theta + \frac{\sin 2\theta}{2}\right)_0^{2\pi}$$

$= 0 - 4(2\pi) + 8(2\pi)$

$= -8\pi + 16\pi = 8\pi$

The line integral of vector function

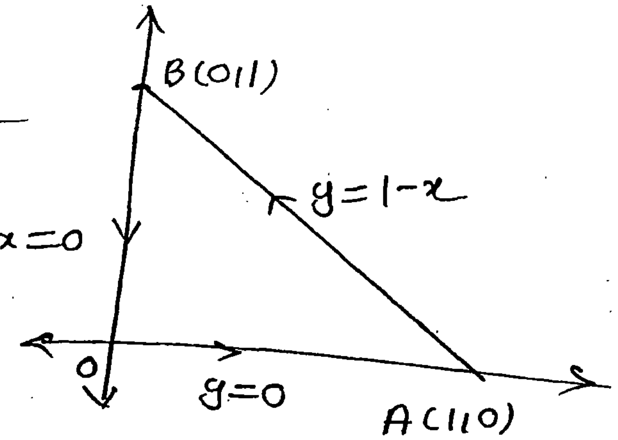
$\bar{F} = (3x^2 - 8y^2)i + (4y - 6xy)j$ and C is $x=0,\ y=0$

and $x+y=1$

a) $2/3$ b) $5/3$ c) $1/2$ d) $3/2$

Sol $\int_C \vec{F}\cdot d\vec{r} = \int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$

$\int_C \vec{F}\cdot d\vec{r} = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BO} \vec{F}\cdot d\vec{r}$



Along OA:- $y=0$

$dy=0 \quad x=0,\ x=1$

$\int_0^1 3x^2\,dx = 3\left(x^3/3\right)_0^1 = 1$

Along AB:- $y=1-x \Rightarrow dy=-dx$

$x=1,\ y=0$

$= \int_1^0 \left[(3x^2) - 8(1-x)^2 - \{4(1-x) - 6x(1-x)\right]dx$

$= \int_0^1 \left[3x^2 + 8x^2 - 8 - 16x - [4 \ldots x - 6x + 24x^2\right]$

$= \int_0^1 \left[-11x^2 + 26x - 12\right]$

$= -11(x^3/3) + 26(x^2/2) - 12(x)$

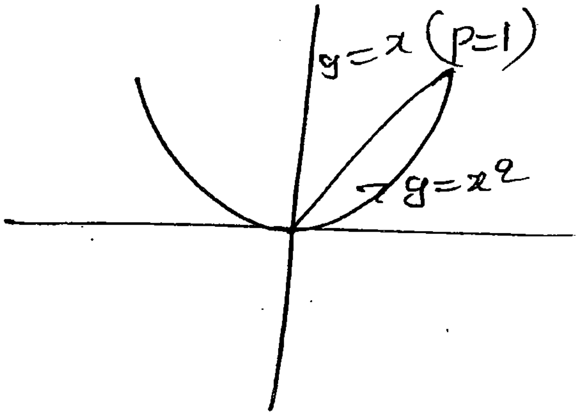
$= 11/3 - 13 + 12 = 8/3 \longrightarrow (2)$

Along BO:- $x=0,\ dx=0,\ y=1,\ y=0$

$\int_1^0 4y\,dy = 4\left(y^2/2\right)_1^0 = -4/2 = -2$

$\int_c F \cdot dr = \cdot 1 + 8/3 - 2 = \frac{3+8-6}{3} = 5/3$

$\int F = (3x^2 - 8y^2) i + (4y - 6xy) j$ and i is $y = x$, $y = x^2$



$F \cdot dr = \int_c (3x^2 - 8y^2)\, dx +$

$(4y - 6xy)\, dy$

Along $c_1 : x^2$

$dy = 2y\, d\cancel{y}$

$x = 0, x = 1$

$\int \left[(3x^2 - 8x^2) + (4x^2 - 6x^3)(2x) \right] dx$

$= \int_0^1 \cancel{(-5x^2 + 8x^4 - 12x^4)}\, dx$

$= \int_0^1 \cancel{(-4x^4} \quad dy \quad = \int_0^1 \left[3x^2 - 8x^4 + 8x^3 - 12x^4 \right] dx$

$= \int_0^1 (3x^2 - 20x^4 + 8x^3)\, dx$

$= 1 - 4 + 2 = -1$

Along c_2: $\quad y = x$

$dy = dx$

$x = 1, x = 0$

$$= \int_1^0 (-5x^2 - 6x^2 + 4x) \, dx$$

$$= \left| \frac{-11x^3}{3} + \frac{4x^2}{2} \right| = 11/3 - 2 = 5/3 \rightarrow ①$$

① + ②

$$= -1 + 5/3 = 2/3.$$

Green's theorem:-

statement:- If $m(x,y)$, $N(x,y)$ having continuous first order partial derivative line bounded by a curve 'c' in x-y plane then

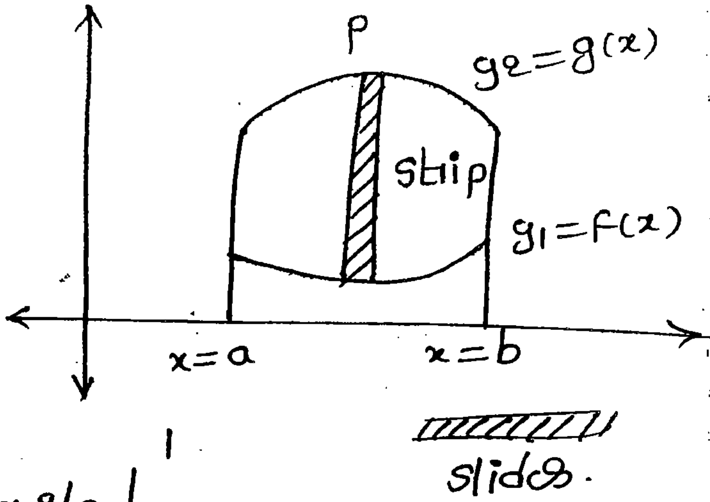
$$\int_C (M dx + N dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial m}{\partial y} \right) dx\,dy$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \left(\frac{\partial N}{\partial x} \cdot \frac{\partial M}{\partial y} \right) dy\,dx.$$

Method to Evaluate the double integrator:-

If y_1, y_2 are functions of x only and (x_1, x_2) are constants the order of integration is first integrate w.r.to 'y' and treating 'x' as a constant, the remaining Expression is integrated i.r. to x

$$\iint_R f(x,y)\,dy\,dx = \int_{x_1=a}^{x_2=b}\int_{y_1=f(x)}^{y_2=g(x)} f(x,y)\,dy\,dx$$



= value of $\int_0^1\int_0^x e^{y/x}\,dy\,dx$

$$\int_0^1 \left|e^{y/x}\right|_0^x (x)\,dx$$

$$= \int_0^1 x(e-1)\,dx = (e-1)\left|x^2/2\right|_0^1$$

$$= \frac{(e-1)}{2}$$

Alternative put $y/x = t$

$y = xt$

$dy = x\,dt$

$y=0,\ t=0,\quad y=x,\ t=1$

$$= \int_0^1\int_0^1 x e^t\,dt\,dx = \int_0^1 x\left|e^t\right|_0^1 dx = (e-1)\left|x^2/2\right|_0^1$$

$$= \frac{(e-1)}{2}$$

$$\int_0^1\int_{x^2}^{x} xy\,dy\,dx$$

$$\int_0^1 x\left[y^2/2\right]_{x^2}^{x} dx = \frac{1}{2}\int_0^1 x(x^2-x^4)\,dx$$

$$\frac{1}{2}\int_0^1 (x^3 - x^5)\,dx = 1/2\left[x^4/4 - x^6/6\right]_0^1$$

$$= 1/2\left[1/4 - 1/6\right] = 1/\cancel{2}\,\frac{(\cancel{2})}{(24)}$$

$$= 1/24$$

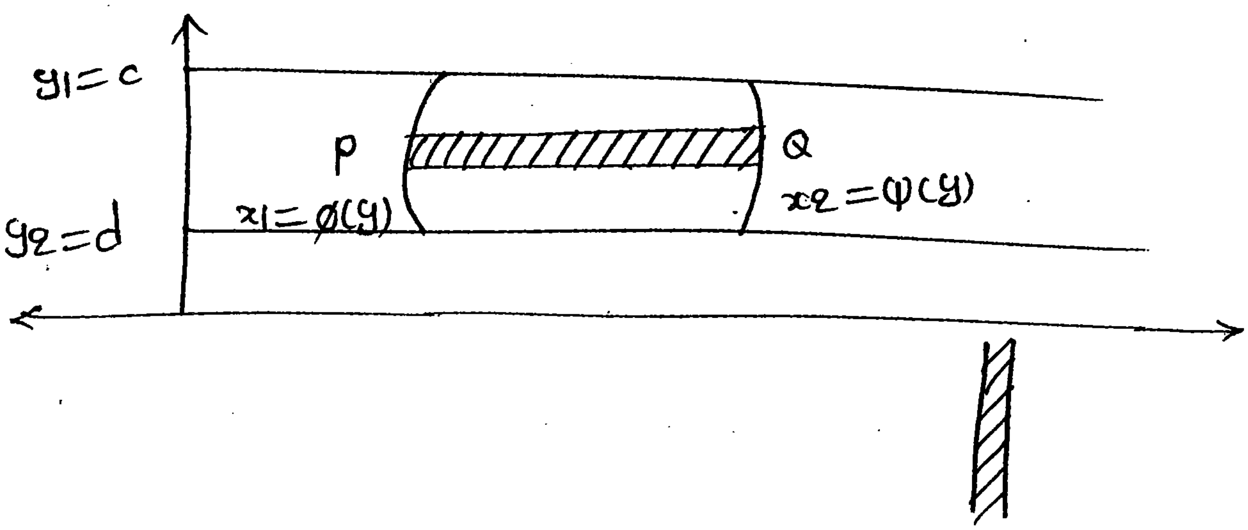
** $$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{1+x^2+y^2}$$

$$\int_0^1\int_0^{\sqrt{1+x^2}} \frac{dy\,dx}{\left(\sqrt{1+x^2}\right)^2+\left(y^2\right)} = \int_0^1 \frac{1}{\sqrt{1+x^2}}\left|\tan^{-1}\frac{y}{\sqrt{1+x^2}}\right|_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}}\left(\tan^{-1}(1) - \tan^{-1}(0)\right)dx$$

$$= \pi/4\left[\log\left(x+\sqrt{1+x^2}\right)\right|_0^1 = \pi/4\,\log\left(1+\sqrt{2}\right)$$

$$\boxed{\iint_R f(x,y)\,dx\,dy = \int_{y_1=c}^{y_2=d}\int_{x_1=\phi(y)}^{x_2=\psi(y)} f(x,y)\,dx}$$



$$\int_0^1 \int^{y^2} e^{x/y}\,dx\,dy$$

-1 b) 0   ~~c) 1/2~~   d) 1

$$= \int_0^1 y\left|e^{x/y}\right|_0^{y^2} dy = \int_0^1 y(e^y - 1)\,dy$$

$$= \left| ye^y - e^y - y^2/2 \right|_0^1$$

$$= (\cancel{e} - \cancel{e} - 1/2) - (0 - 1 - 0)$$

$$= -1/2 + 1 = 1/2$$

$$\int_0^1 \int_y^{y^2} (x+y)\,dx\,dy$$

$$= \int_0^1 \left[\left| x^2/2 \right|_y^{y^2} + y \cdot |x|_y^{y^2} \right] dy$$

$$= \int_0^1 \left[1/2\,(y^4 - y^2) + y\,(y^2 - y) \right] dy$$

$$= 1/2\left[y^5/5 - y^3/3 \right] + \frac{y^4}{4} \; \frac{-y^3}{3} \Big]_0^1$$

$$= 1/2\,(1/5 - 1/3) + (1/3 - 1/2)$$

$$= \cancel{-1/15} \quad -3/20$$

~~If~~ the limits of the first integral and second integral are constant and Equal. The order of integration is not required.

$$\iint_R f(x,y)\,dx\,dy = \int_{x_1=a}^{x_2=b}\int_{y_1=c}^{y_2=d} f(x,y)\,dx\,dy$$

) $\int_0^1\int_0^1 e^{x+y}\,dx\,dy$

Sol
$$= \int_0^1 e^x \left|e^y\right|_0^1 dx$$
$$= \int_0^1 e^x(e-1)\,dx = (e-1)\left|e^x\right|_0^1 = (e-1)^2$$

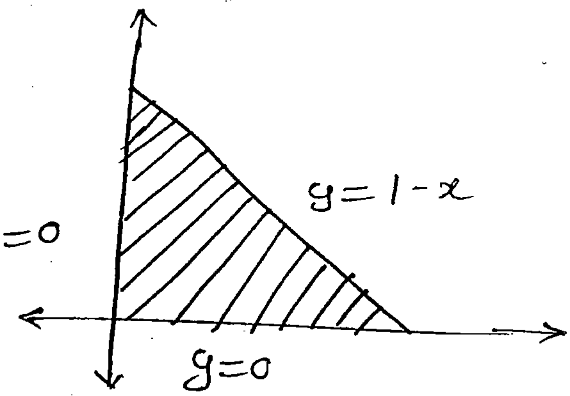
2) $\int_0^{\pi/2}\int_0^{\pi/2} \sin(x+y)\,dy\,dx$

a) 1 b) 0 ~~c)~~ 2 d) 1/2 e) 3.

Sol
$$= \int_0^{\pi/2}\left[-\cos(x+y)\right]_0^{\pi/2} dx$$
$$= \int_0^{\pi/2} -\cos(x+\pi/2) + \int_0 \cos(x+0)\,dx$$
$$= \int_0^{\pi/2}(\sin x+\cos x)\,dx = -\left|\cos x\right|_0^{\pi/2} + \left|\sin x\right|_0^{\pi/2}$$
$$= 1+1 = 2$$

3) $\int_0^1\int_1^2 xy\,dx\,dy = \int_0^1 y\left|x^2/2\right|_1^2 dy$
$$= 1/2\,(4-1)\int_0^1 y\,dy = 3/2\left|y^2/2\right|_0^1$$
$$= 3/4\,(1)$$

$$ca = \iint_R dA$$



$$= \int_{x_1}^{x_2}\int_{y_1}^{y_2} dy\,dx$$

$$= \int_0^1\int_0^{1-x} dy\,dx$$

$$= \int_0^1 (1-x)\,dx = \left| x - x^2/2 \right|_0^1 = 1 - 1/2 = 1/2$$

$\iint_R xy\,dx\,dy$ where (R) is the region $x^2+y^2 \leq 1$

$$y^2 \leq 1-x^2$$

$$y = \pm\sqrt{1-x^2}$$

put $y=0$, $x = \pm 1$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} xy\,dy\,dx$$ 14

$$= \int_{-1}^{1} x \left| y^2/2 \right|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx$$

$$= 1/2 \int_{-1}^{1} x\left[(1-x^2) - (1-x^2)\right] dx$$

$$= 0$$

problems on Green's theorem :-

① By Greeni's theorem the value of $\int_c [xy\,dy - y^2 dx]$ where 'c' is a square cut from the quadrant bounded by the line $x=1, y=1$ is —

Sol $\quad M = -y^2, \quad N = xy$

$$\frac{dm}{dy} = -2y \qquad \frac{dN}{dx} = y$$

$$= \int_0^1 \int_0^1 3y\,dy\,dx = 3 \int_0^1 \left| y^2/2 \right|_0^1 dx = 3/2 \left| x \right|_0^1$$

$$= 3/2$$

2) $\int_c (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \qquad x=0, y=0, x+y=1$

(use Greeni's theorem).

Sol $\quad M = 3x^2 - 8y^2 \qquad N = 4y - 6xy. \qquad x=0 \quad x=1$

$\qquad y=0 \quad y=1-x$

$$\frac{dm}{dy} = -16y \qquad \frac{dN}{dx} = -6y$$

$$= \int_0^1 \int_0^{1-x} 10y\,dy\,dx = 5 \int_0^1 (1-x)^2 dx$$

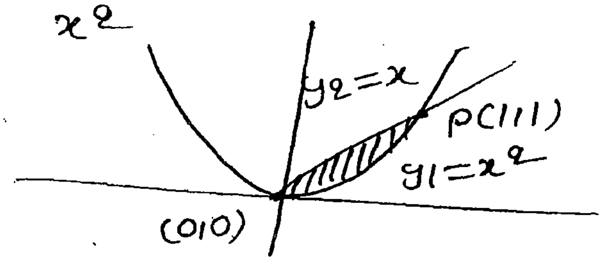
$$= \frac{-5}{3} \left| (1-x)^3 \right|_0^1 = 5/3$$

3) By Greeni's theorem the valueof

$\int_c [xy\,dy - y^2 dx], \quad y=x, \quad y=x^2$

$M = -y^2 \qquad N = xy \qquad = \int_0^1 \int_{x^2}^{x} 3y \, dy \, dx$

$\frac{\partial M}{\partial y} = -2y \qquad \frac{\partial N}{\partial x} = y$



$= 3/2 \int_0^1 (x^2 - x^4) \, dx$

$= 3/2 \left(1/3 - 1/5 \right) \quad = \cancel{3/2 (1/6)} = \cancel{3/12} = \cancel{1/4}$

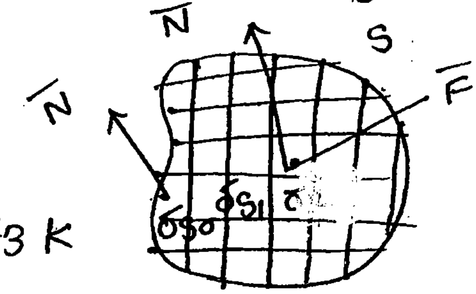
$= 3/2 \left[\frac{2}{15} \right] = 3/15 = 1/5 .$

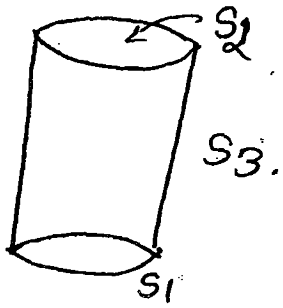
<u>rface Integrals:-</u>

It is defined as an integral which is to be valuated over a Surface is called Surface integral.

Mathematically it is defined as

$\iint_S \bar{F} \cdot \bar{N} \, ds$





$\equiv \bar{F} = F_1 i + F_2 j + F_3 k$

$\Sigma \bar{F} \cdot \bar{N} \, \delta S_i \quad$ as $Lt \; n \to \infty , \; \delta S_i \to 0$

$\boxed{\bar{F} \cdot \bar{N} \, \delta S_i = \iint_S \bar{F} \cdot \bar{N}_i \, ds}$ is called Surface integral

$\bar{F} = F_1 i + F_2 j + F_3 k$

$\bar{N} =$ outward unit Normal vector

$ds =$ projection of the Surface onto the plane

Method to evaluate Surface integral:-

1) If the surface 'S' on x-y plane ($z=0$)

$$\iint_S \bar{F}\cdot\bar{N}\,ds = \int_{x_1}^{x_2}\int_{y_1}^{y_2} \bar{F}\cdot\bar{N}\,\frac{dy\,dx}{|\bar{N}\cdot\bar{K}|}$$

where $\bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$ i-e $\phi(x,y,z) = c$ is given

otherwise $\bar{N} = \bar{K}$

2) If the surface 'S' is on yz plane ($x=0$)

$$\iint_S \bar{F}\cdot\bar{N}\,ds = \int_{y_1}^{y_2}\int_{z_1}^{z_2} \frac{\bar{F}\cdot\bar{N}\,dz\,dy}{|\bar{N}\cdot i|}$$ where

$\bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$ i-e $\phi(x,y,z) = c$ given otherwise $\bar{N} = i$

3) If the surface 'S' is , xz-plane ($y=0$)

$$\iint_S \bar{F}\cdot\bar{N}\,ds = \int_{x_1}^{x_2}\int_{z_1}^{z_2} \frac{\bar{F}\cdot\bar{N}\,dz\,dx}{|\bar{N}\cdot j|}$$

where $\bar{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$ i-e $\phi(x,y,z) = c$ given

otherwise $\bar{N} = j$

Stokes theorem:- (line integral into Surface integral)

→ **Statement:-** Let 'S' be two sided open Surface bounded by a closed curve c, Let $\bar{F}$ be a differentiable vector function then.

$$\int_C \bar{F}\cdot d\bar{r} = \iint_{\nabla} (\nabla\times\bar{F})\cdot\bar{N}\,ds$$

By Stoke's theorem the value of $\int_C \bar{r}\cdot d\bar{r}$ where

is $x^2+y^2=4$, $z=0$

$$= \iint_S (\nabla\times\bar{r})\cdot\bar{N}\,ds$$

$$\nabla\times\bar{r} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ x & y & z \end{vmatrix}$$

$$= i(0) - j(0) + k(0) = 0$$

$$= \iint_S \bar{0}\cdot\bar{N}\,ds$$

$$= 0$$

By Stoke's theorem, the value of

$$\int_C [yz\,dx + xz\,dy + xy\,dz]$$

where c is $x=0$, $y=0$

$x+y=1$, $z=0$

$$\bar{F} = yzi + xzj + xyk$$

$$\nabla\times\bar{F} = \begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & d/dz \\ yz & xz & xy \end{vmatrix} = i(0) - j(0) + k(0)$$

$$= 0$$

$$= \iint_S (\nabla\times\bar{F})\cdot\bar{N}\,ds = \iint_S 0\cdot\bar{N}\,ds = 0$$

$\int_C \bar{F}\cdot d\bar{r}$ where $\bar{F} = (y-z+2)i + (yz+4)j - xz\bar{k}$

and 'c' is boundary of the surface of the cube

$x=0,\ y=0,\ z=0,\ x=2,\ y=2$ above the xy-plane.

~~a)~~ -4 b) -2 c) 4 d) 2

sol

$$\nabla\times\bar{F}=\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y-z+2 & yz+4 & -xz \end{vmatrix}$$

$$= i(0-y)-j(-z+1)+k(-1)$$

$$= -iy-j(1-z)-k.$$

$\boxed{\bar{N}=\bar{K}}$ since $z=0$

$$ds=\frac{dy.dx}{|\bar{K}\cdot\bar{K}|}=dydx.\ (xy\text{-plane})$$

$$\int \bar{F}\cdot d\bar{r}=\int_{x_1}^{x_2}\int_{y_1}^{y_2}(\nabla\times\bar{F})\cdot\bar{N}\frac{dydx}{\bar{K}|}$$ above the xy-plane.

$$(\nabla\times\bar{F})\bar{N}=\left[-iy-j(1-z)-\bar{K}\right]\cdot\bar{K}=-1.$$

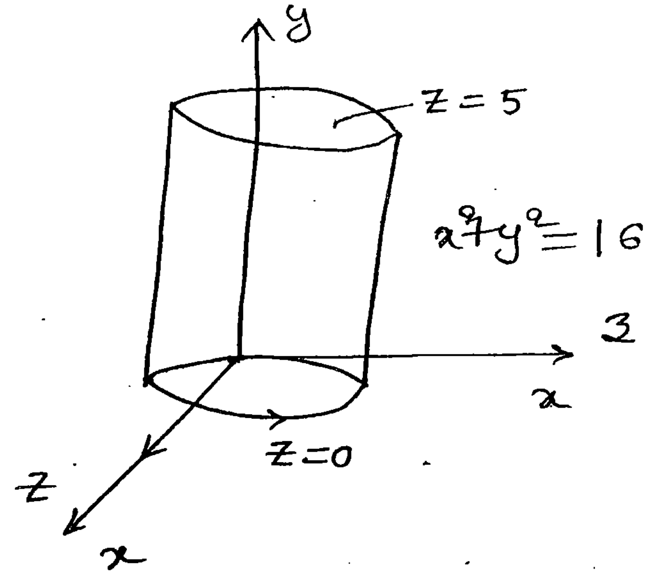
$$\int_c \bar{F}\cdot d\bar{r}=\int_0^2\int_0^2(-1)\,dydx=\int_0^2 -2\,dx=\underline{\underline{-4}}$$

~~Gauss th~~

* The value of $\iint_S \bar{F}\cdot\bar{N}\,ds$ where

$F=zi+xj+3y^2zk$ and 'S' is the surface of the cylinder $x^2+y^2=16$ included in the first octant between $z=0$, $z=5$ is

80 b) 90 c) 110 d) 115.



$F = zi + xj - 3y^2zk$

$\phi = x^2+y^2 = 16$

$\nabla\phi = 2xi + 2yj$

$|\nabla\phi| = \sqrt{4(x^2+y^2)}$

$= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{\cancel{2}(xi+yj)}{\cancel{2}\sqrt{16}} = \dfrac{xi+yj}{4}$

$\bar{F}\cdot\bar{N} = (zi+xj-3y^2zk)\dfrac{(xi+yj)}{4}$

$= x/4\,(z+y)$

Let 's' be on yz-plane $(x=0)$

$ds = \dfrac{dz\,dy}{|\bar{N}\cdot i|}$

$\iint_S \bar{F}\cdot\bar{N}\,ds = \int_{y_1}^{y_2}\int_{z_1}^{z_2} \bar{F}\cdot\bar{N}\dfrac{dz\,dy}{|\bar{N}\cdot i|}$

$= \int_{y_1}^{y_2}\int_{z_1}^{z_2} \dfrac{\cancel{x}}{\cancel{4}}(z+y)\dfrac{\cancel{4}}{\cancel{x}}\,dz\,dy$

$= \int_0^4\int_0^5 (z+y)\,dz\,dy = \dfrac{25}{2}|y|_0^4 + \dfrac{5}{2}|y^2|_0^4$

$= 50 + 40 = 90.$

Gauss divergence theorem:-

Let (v) be the volume and (s) is a closed surface and $\bar{F}$ be a differentiable vector function then

$$\boxed{\iint_S \bar{F} \cdot \bar{N}\, ds = \iiint_V (\nabla \cdot \bar{F})\, dv}$$

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} \left(\frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz} \right) dz\, dy\, dx$$

1) $$\int_0^1 \int_0^x \int_0^y xyz\, dz\, dy\, dx = \int_0^1 \int_0^x xy \left| z^2/2 \right|_0^y dy\, dx$$

$$= \frac{1}{2} \int_0^1 \int_0^x xy^3\, dy\, dx$$

$$= \frac{1}{8} \int_0^1 (x^5)\, dx = \frac{1}{(8)(6)} (x^6)_0^1 = 1/48.$$

2) $$\int_0^1 \int_0^1 \int_0^1 e^{x+y+z}\, dz\, dy\, dx =$$

$$= \int_0^1 \int_0^1 (e^x)(e^y)(e^z)_0^1\, dy\, dx$$

$$= \int_0^1 \int_0^1 (e^x)(e^y)(e-1)\, dy\, dx$$

$$= (e-1) \int_0^1 \int_0^1 (e^x)(e^y)\, dy\, dx$$

$$= (e-1) \int_0^1 (e^x)(e^y)_0^1\, dx = (e-1)^3,$$

The volume of an object is Expressed in spherical co-ordinates is given by

$$V = \int_0^{2\pi} \int_0^{\pi/3} \int_0^1 r^2 \sin\phi \; \underset{①}{dr} \, \underset{②}{d\phi} \, \underset{③}{d\theta}$$ the value of the integral is.

a) ~~$\pi/3$~~ b) $\pi/6$ c) $2\pi/3$ d) $\pi/4$

$\underline{\underline{I}}$

$$V = \int_0^{2\pi} \int_0^{\pi/3} \left(\frac{r^3}{3}\right)_0^1 d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} \int_0^{\pi/3} \sin\phi (1) \, d\phi \, d\theta$$

$$= \frac{1}{3} \int_0^{2\pi} (-\cos\phi)_0^{\pi/3} \, d\theta$$

$$= 1/3 \int_0^{2\pi} (-1/2 + 1) \, d\theta$$

$$= \frac{1}{6} |\theta|_0^{2\pi} = \frac{1}{6}(2\pi) = \pi/3.$$

<u>Volume</u> <u>integral</u>:-

$\iiint_V F \cdot dv$ is called volume integral.

$$= \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} [F_1 \bar{i} + F_2 \bar{j} + F_3 k] \, dz \, dy \, dx$$

$$= i \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_1 \, dz \, dy \, dx + j \int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} F_2 \, dz \, dy \, dx$$

$$+K \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} F_3\, dz\, dy\, dx$$

Prob. Gauss divergence theorem:-

1) By Gauss divergence theorem $\iint_S \bar{F}\cdot\bar{N}\, ds$

where $\bar{F} = 3xi - 4yj + 8zK$

and S is $x=0 \quad y=0 \quad z=0$

$x=2 \quad y=3 \quad z=1$

Sol

$$\iiint_V (\nabla\cdot\bar{F})\, dv$$

$$= \int_0^2\int_0^3\int_0^1 7\, dz\, dy\, dx = 7\times 1\times 3\times 2 = 42$$

) By Gauss divergence theorem the value of $\iint_S \bar{F}\cdot\bar{N}\, ds$

where $\bar{F} = xi + yj + zK$ and S is $x^2+y^2+z^2 = 9$

$$= \iiint_V (3)(dv) = \cancel{3}\, \frac{4\pi}{\cancel{3}}(3)^3 = 27\times 4\pi$$

3) By Gauss divergence theorem

$$\iint_S [\, yz\, dz\, dy + xz\, dz\, dx + xy\, dy\, dx\,] \quad 0\le x\le 1,$$

$0\le y\le 1, \quad 0\le z\le 1$

Sol $\quad F = yz\bar{i} + xz\bar{j} + xy\bar{k}$

$$\nabla\cdot\bar{F} = \frac{d}{dx}(yz) + \frac{d}{dy}(xz) + \frac{d}{dz}(xy)$$

$$= 0$$

$$\iiint_V (\nabla\cdot\bar{F})\, dv = \iiint_V 0\, dv = 0$$

By Gauss divergence theorem

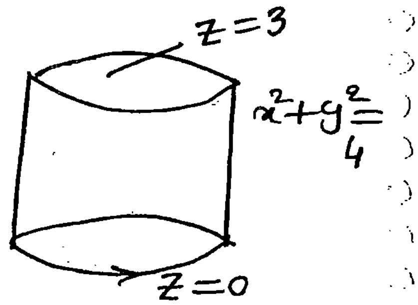
$\iint_S F \cdot \overline{N} ds$ where $\overline{F} = 4xi - 2y^2j + z^2k$,

$x^2+y^2=4$, $z=0$, $z=3$.

$$\nabla \cdot \overline{F} = \frac{d}{dx}(4x) - \frac{d}{dy}(2y^2) + \frac{d}{dz}(z^2)$$

$$\boxed{\nabla \cdot \overline{F} = (4-4y+2z)}$$

$$\int F \cdot \overline{N} dS = \iiint_V (\nabla \cdot \overline{F}) dv = \iiint_V (4-4y+2z)\, dz\, dy\, dx$$



$x^2+y^2=4$

put $y=0 \Rightarrow x=\pm 2$

$$\boxed{y = \pm\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \int_{0}^{3} (4-4y+2z)\, dz\, dy\, dx$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} (12-12y+9)\, dy\, dx$$

$$= \int_{-2}^{2} 21 \left| y \right|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx - 12 \int_{-2}^{2} \left| y^2/2 \right|_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= 42 \int_{-2}^{2} \sqrt{4-x^2}\, dx - 12(0)$$

$$= 84 \int_{0}^{2} \sqrt{4-x^2}\, dx - 0$$

$$= 84\left[\frac{4}{2}\sin^{-1}(x/2) + x/2\sqrt{4-x^2}\right]_0^2$$

$$= 84 \cdot \cancel{2} \cdot \pi/\cancel{2} + 0 - 0 = 84\pi$$

Fourier Series

periodic function:- A function which repeats itself with in a period is called periodic function.

$\sin x$, $\cos x$ are periodic function with period 2π

$$f(t) = f(t+T) = f(t+2T) = f(t+3T) = \cdots f(t+nT)$$

where T is a period.

Given No's are arranged according to the same law is called series.

Eg: $1+2+3+\cdots\cdots n$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

Definition:- The fourier series of the function $f(x)$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \longrightarrow (1)$$

$$f(x) = \frac{a_0}{2} + [a_1\cos x + a_2\cos 2x + \cdots\cdots] + [b_1\sin x + b_2\sin 2x + \cdots]$$

where a_0, a_n, b_n are called Fourier co-efficients.

deduction formulae:-

$$\int_\alpha^{\alpha+2\pi} \sin nx\, dx = 0$$

$$\int_\alpha^{\alpha+2\pi} \cos nx\, dx = 0$$

$$\int_\alpha^{\alpha+2\pi} \sin^2 nx\, dx = \pi$$

$$\int_\alpha^{\alpha+2\pi} \sin mx \cos nx\, dx = 0 \quad (m \neq n)$$

④ $$\int_\alpha^{\alpha+2\pi} \cos^2 nx\, dx = \pi$$

⑤ $$\int_\alpha^{\alpha+2\pi} \cos mx \cos nx\, dx = 0 \quad (m \neq n)$$

⑥ $$\int_\alpha^{\alpha+2\pi} (\sin mx \sin nx)\, dx = 0 \quad (m \neq n)$$

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$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\, dx + \sum_{n=1}^{\infty} b_n \sin nx\, dx$$

$+2\pi$

$$f(x) = \frac{a_0}{2}\int_\alpha^{\alpha+2\pi} (1)\, dx + \sum_{n=1}^{\infty} a_n \int_\alpha^{\alpha+2\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_\alpha^{\alpha+2\pi} \sin nx\, dx$$

$$= \frac{a_0}{2} \left| x \right|_\alpha^{\alpha+2\pi} + 0 + 0$$

$$= \frac{a_0}{2} [\cancel{\alpha} + 2\pi - \cancel{\alpha}] = \frac{a_0}{\cancel{2}} (\cancel{2}\pi) = a_0 \pi$$

$$\boxed{a_0 = \frac{1}{\pi}\int_\alpha^{\alpha+2\pi} f(x)\, dx}$$

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_{\alpha}^{\alpha+2\pi} \cos nx\,dx + \sum_{n=1}^{\infty} a_n \int_{\alpha}^{\alpha+2\pi} \cos^2 nx\,dx + \sum_{n=1}^{\infty} b_n \int_{\alpha}^{\alpha+2\pi} \cos nx \sin nx\,dx$$

$$\boxed{a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx}$$

Similarlly $\boxed{b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx}$

II) $0 \leq x \leq 2\pi$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx \qquad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

III) $0 \leq x \leq \pi$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx \qquad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

IV) $-\pi \leq x \leq \pi$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

If $f(x) = x^2 \quad -\pi \leq x \leq \pi$ is

$\pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \sin nx$

✓ $\pi^2/3 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx$

$\pi^2/3 - 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ d) 0

$$\cos n\pi = (-1)^n$$

$$\sin n\pi = 0.$$

$\int uv = uv_1 - u'v_2 + u''v_3$

$f(x) = x^2, \quad -\pi \leq x \leq \pi$ is an Even function $\boxed{b_n = 0}$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} + \frac{2\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi\cos n\pi}{n^2} + 0 - 0\right] = \frac{4(-1)^n}{n^2}$$

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx\,.dx$$

$$= \frac{\cancel{2}\pi^2}{3\times\cancel{2}} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx$$

$$x^2 = \pi^2/3 + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \cdots\right]$$

at $x=\pi$

$$\pi^2 = \pi^2/3 + 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{3\pi^2 - \pi^2}{3} = 4\left[1/1^2 + 1/2^2 + 1/3^2 + \cdots\right]$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = (\pi^2/6)$$

2) If $f(x) = 0, \; -\pi < x < 0$

$= \sin x, \quad 0 < x < \pi$. the term independent of the a_0 in the fourier series.

a) $2/\pi$ b) $1/\pi$ c) d) $1/2$

Sol

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(0)\,dx + \int_{0}^{\pi}(\sin x)\,dx\right]$$

$$= \frac{1}{\pi}\left[-\cos x\right]_0^{\pi} = 2/\pi$$

The term independent of 'x' in the fourier series

$$a_0/2 = 1/\pi$$

The fourier series $f(x) = x\sin x$, $0 \le x \le 2\pi$

then $a_1 =$

$1/2$ $\quad$ ~~b)~~ $-1/2$ $\quad$ c) 0 $\quad$ d) 2

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx \, dx$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x\cos x \, dx$$

$$a_1 = \frac{1}{2\pi}\int_0^{2\pi} x(2\sin x\cos x)\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\sin 2x \, dx = \frac{1}{2\pi}\left(-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right)_0^{2\pi}$$

$$a_1 = \frac{1}{2\pi}\left[-2\pi\cos 4\pi/2 + 0 - 0\right]$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi\cos 4\pi}{2}\right] = -1/2$$

The co-efficient of $\sin x$ in the Expansion of

$f(x) = x^2$ in $(-\pi, \pi)$ is

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$ $\quad$ b) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ $\quad$ c) $\frac{\pi^2}{6}$ $\quad$ ~~d) 0~~

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\sin x \, dx$$

$$b_n = \frac{1}{\pi}\left[x^2(-\cos x) + 2x\sin x + 2(-\cos x)\right]_{-\pi}^{\pi}$$

$$b_n = \frac{1}{\pi}\left[\left(-\pi^2\cos\pi + 0 - 2\cos\pi\right) - \left(-\pi^2\cos\pi - 2\cos\pi\right)\right]$$

$$= \frac{1}{\pi}\left[-\cancel{2\cos\pi} + \cancel{2\cos\pi} - \cancel{\pi^2\cos\pi} + \cancel{\pi^2\cos\pi}\right]$$

$$\boxed{b_n = 0}$$

(change of interval problems)

→ If the interval is $c, c+2l$.

The Fourier Expansion of the function in the $[c, c+2l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int^{c+2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$[-l, l] = [-1, 1]$ $[-2, 2]$ $[0, 3] = [0, 2(3/2)]$

* If $f(x) = 0 \quad -2 < x < 0$

$= 1 \quad 0 < x < 2$ then the term independent of 'x' in the Fourier Series is.

Sol

$$a_0 = 1/2\int_{-2}^{2} f(x)\,dx = \frac{1}{2}\left[\int_{-2}^{0} f(x)\,dx + \int_0^2 f(x)\,dx\right]$$

$a_0 = 1/2 \left[x\right]_0^2 = 1/2\,(2) = 1$

$\boxed{\frac{a_0}{2} = 1/2}$

In the fourier series of the function

$f(x) = 0 \qquad -2 < x < -1$

$\quad = k \qquad -1 < x < 1$

$\quad = 0 \qquad 1 < x < 2$ then the constant term in the

series is.

a) k ~~b) $k/2$~~ c) $2k$ d) $4k$

$[-2, 2]$

$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)dx = 1/2 \left[\int_{-2}^{-1} \cancel{f(x)}\,(0) + \int_{-1}^{1} k\,dx + \int_{1}^{2}(0)dx\right]$

$= 1/2 \left[(k)\right]_{-1}^{1} = \frac{2k}{2} = k$

$a_0 = k \Rightarrow \boxed{\frac{a_0}{2} = k/2}$

Q10 The fourier series of $\sin^2 x$ is

a) $0.5 - \cos 2x$ b) $0.5 - \sin 2x$

~~c)~~ $0.5 - 0.5\cos 2x$ d) $0.5 + 0.5\cos 2x$

! $\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx$

Any function $f(x)$ can be Expanded as a fourier series if it satisfy the following conditions 1995

① f(x) is period, single value, ~~period~~ finite.

2) f(x) has a finite no of discontinous in any one period.

3) f(x) has atmost finite maxima & Minima.

The above conditions are called dirchellt's condition.

→ If $f(x) = 1-x^2$, $-\pi \leq x \leq \pi$ contains

a) only sine terms b) ~~only~~ cosine terms c) both d) Nor

Sol

$f(x) = 1-x^2$ is an Even function.

Hence $b_n = 0$. Hence it contains only cosine terms

The fourier Expansion of the function

Half range Sine series:- If the interval $(0, c)$ which is half of $[-c, c]$ the half range sine series is defined as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{c}$$

$$b_n = \frac{2}{c} \int_0^c f(x) \sin \frac{n\pi x}{c} \, dx$$

The graph is reflected about origin a X-axis

Half range cosine series:-

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{c}$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx \quad , \quad a_n = \frac{2}{c}\int^c f(x) \cos \frac{n\pi x}{c}\,dx$$

The graph is reflected about y-axis

If $f(x) = x$ is Expressed as a half range sine series in $(0,2)$ then the co-efficient of $\sin \pi x$ in the series is.

a) $2/\pi$ b) $-2/\pi$ c) $1/\pi$ d) $-1/\pi$

$c = 2$

$$x = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2}$$

$$= b_1 \sin(\pi x/2) + b_2 \sin \pi + - -$$

$$b_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c}\,dx$$

$$b_n = \frac{2}{2}\int_0^2 x \sin \frac{\pi x}{\cancel{2}}\,dx$$

$$= \left[-\frac{x \cos \pi x}{x} + \frac{\sin \pi x}{\pi^2}\right]_0^2$$

$$b_n = -2/\pi + 0 = -2/\pi$$

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:) If $f(x) = x$ is Expressed as half range cosine series in $(0,2)$

Sol

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2} \qquad a_0 = \frac{2}{2}\int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

$$a_n = 2/2 \int_0^2 x \cos\frac{n\pi x}{2}\,dx$$

$$= \left[x \sin\frac{n\pi x}{2} \cdot \frac{2}{n\pi} + \cos\frac{n\pi x}{2} \cdot \frac{4}{n^2\pi^2} \right]_0^2$$

$$= \left[0 + \cos n\pi \frac{4}{n^2\pi^2} - \left(0 + \frac{4}{n^2\pi^2}\right)\right]$$

$$= \frac{4}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Mathematics