LS

$2x_1 + 3x_2 + 5x_3 = 1$

$x_1 - x_2 = 2$

$x_2 + x_3 = 5$

$A\,x = b$ → constant

coeff matrix — unknown vector

1- $\boxed{\text{Exact Method}}$

1- Gauss-Elim

2- Jordan

3- A^{-1} use

4- Cramer's

4.8 ITERATIVE METHODS

So far we have studied some direct methods which yield, after a certain amount of fixed computation, the solution to simultaneous linear equations. Now we shall discuss the *iterative* or *indirect methods*. In these methods we start from an approximation to the true solution and, if convergent, derive a sequence of closer approximations. We repeat the cycle of computations till the required accuracy is obtained.

But, the method of iteration is not applicable to all systems of equations. For this, each equation of the system must contain one large coefficient (much larger than the others in that equation) and the large coefficient must be attached to a different unknown in that equation.

In other words, the solution to a system of linear equations will exist by iterative procedure *if the absolute value of the largest coefficient is greater than the sum of the absolute values of all remaining coefficients in each equation* (condition for convergence).

Now let us see some of such methods in detail.

4.9 JACOBI METHOD OF ITERATION

row — columns

$\boxed{a_{11}} x_1 + \boxed{a_{12}} x_2 + a_{13} x_3 = b_1$

$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = b_2$

$\vdots$

This method is also known as Gauss–Jacobi method. Here, consider the following system of equations.

$$\boxed{a_1} x + b_1 y + c_1 z = d_1 \quad (4.19a)$$
$$a_2 x + \boxed{b_2} y + c_2 z = d_2 \quad (4.19b)$$
$$a_3 x + b_3 y + \boxed{c_3} z = d_3 \quad (4.19c)$$

→ 3-Eqs, 3-unknown

Let $|a_1| > |b_1| + |c_1|; \quad |b_2| > |a_2| + |c_2|; \quad |c_3| > |a_3| + |b_3|$

$|5| = 5$

$|-5| = 5$

That is, in each equation the coefficients of the diagonal terms are large. Hence the system (4.19) is ready for iteration. Solving for x, y and z respectively, we get

$a_1 x = d_1 - b_1 y - c_1 z$

$$\boxed{x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)} \quad (4.20a)$$

Initial

$x_0 = \quad y_0 = \quad z_0 =$

$$y_1 = \frac{1}{b_2}(d_2 - a_1 x_0 - c_2 z_0) \quad (4.20b)$$

$x_2 = \frac{1}{a_1}(d_1 - b y_1 - c_1 z_1)$

$y_2 = \frac{1}{b_2}(d_2 - a_1 x_1 - c_2 z_1)$

$z_2 =$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0) \quad (4.20c)$$

Let x_0, y_0 and z_0 be the initial approximations of the unknowns x, y and z. Substituting these on RHS of Eqns (4.20) the first approximations are given by

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Substituting the values x_1, y_1 and z_1 in the RHS of Eqn (4.20), the second approximations are given by

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_2 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_2 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

Proceeding in the same way, if x_r, y_r, z_r are the rth iterates then

$$x_{r+1} = \frac{1}{a_1}(d_1 - b_1 y_r - c_1 z_r)$$

$$y_{r+1} = \frac{1}{b_2}(d_2 - a_2 x_r - c_2 z_r)$$

$$z_{r+1} = \frac{1}{c_3}(d_3 - a_3 x_r - b_3 y_r)$$

$x_5 = 1.4142....$
$y_5 = 0.1237....$
$z_5 = 1.5357...$

$x_6 = 1.4142...$
$y_6 = 0.1237..$
$z_5 = 1.5357...$

$x_0 = 1,\ y_0 = 2,\ z_3 = 1$

The process is continued till convergency is secured.

Note: In the absence of any better estimates, the initial approximations are taken as $\boxed{x_0 = 0,\ y_0 = 0,\ z_0 = 0}$

Example 4.16 Solve by Jacobi iteration method the system $\boxed{8}x - 3y + 2z = 20$; $6x + 3y + \boxed{12}z = 35$; and $4x + 11y - z = 33$.

(Bangalore, B.E., 1994, M.U, B.E., 1981)

Solution Consider the given system as

$$\boxed{8}x - 3y + 2z = 20 \rightarrow x_{k+1} = \frac{1}{8}(20 + 3y_k - 2z_k)$$

$$4x + \boxed{11}y - z = 33 \qquad y_{k+1} = \frac{1}{11}(33 - 4x_k + z_k)$$

$$6x + 3y + \boxed{12}z = 35 \qquad z_{k+1} = \frac{1}{12}(35 - 6x_k - 3y_k)$$

$K=0$

$x_1 = \frac{1}{8}(20 + 3\cancel{y_0}^{0} - 2\cancel{z_0}^{0})$

$y_1 = \frac{1}{11}(33 - 4\cancel{x_0}^{0} + \cancel{z_0}^{0})$

$z_1 = \frac{1}{12}(35 - 6\cancel{x_0}^{0} - 3\cancel{y_0}^{0})$

$x_0 = 0$
$y_0 = 0$
$z_0 = 0$

$x_1 = 2.5$

$y_1 = 3$

$z_1 = 2.91$

so that the diagonal elements are dominant in the coefficient matrix. Now we write the equations in the form

$\underline{\underline{K=1}}$

$$x = \frac{1}{8}(20 + 3y - 2z) \tag{ia}$$

$x_2 = \frac{1}{8}(20 + 3y_1 - 2z_1)$

$$y = \frac{1}{11}(33 - 4x + z) \tag{ib}$$

$y_2 = \frac{1}{11}(33 - 4x_1 + z_1)$

$$z = \frac{1}{12}(35 - 6x - 3y) \tag{ic}$$

$z_2 = \frac{1}{12}(35 - 6x_1 - 3y_1)$

We start from an approximation $x_0 = y_0 = z_0 = 0$.

Substituting these on RHS of Eqns (i), we get

First approximation as

$$x_1 = \frac{1}{8}[20 + 3(0) - 2(0)] = \underline{2.5}$$

$$y_1 = \frac{1}{11}[33 - 4(0) + 0] = 3$$

$$z_1 = \frac{1}{12}[35 - 6(0) - 3(0)] = 2.9166667$$

Second approximation:

Substituting x_1, y_1, z_1 on RHS of Eqns (i), we get

$$x_2 = \frac{1}{8}[20 + 3(3) - 2(2.9166667)] = \boxed{2.895833}$$

$$y_2 = \frac{1}{11}[33 - 4(2.5) + 2.9166667] = \boxed{2.3560606}$$

$$z_2 = \frac{1}{12}[35 - 6(2.5) - 3(3)] = \boxed{0.9166666}$$

Third approximation:

Substituting x_2, y_2, z_2 on RHS of Eqns (i), we get

$$x_3 = \frac{1}{8}[20 + 3(2.3560606) - 2(0.9166666)] = 3.1543561$$

$$y_3 = \frac{1}{11}[33 - 4(2.8958333) + 0.9166666] = 2.030303$$

$$z_3 = \frac{1}{12}[35 - 6(2.8958333) - 3(2.3560606)] = 0.8797348$$

Fourth approximation:

Substituting x_3, y_3, z_3 on RHS of Eqns (i), we get

$$x_4 = \frac{1}{8}[20 + 3(2.030303) - 2(0.8797348)] = 3.0419299$$

$$y_4 = \frac{1}{11}[33 - 4(3.1543561) + 0.8797348] = 1.9329373$$

$$z_4 = \frac{1}{12}[35 - 6(3.1543561) - 3(2.030303)] = 0.8319128$$

Fifth approximation:

Substituting x_4, y_4, z_4 on RHS of Eqns (i), we get

$$x_5 = \frac{1}{8}[20 + 3(1.9329373) - 2(0.8319128)] = 3.0168733$$

$$y_5 = \frac{1}{11}[33 - 4(3.0414299) + 0.8319128] = 1.9696539$$

$$z_5 = \frac{1}{12}[35 - 6(3.0414299) - 3(1.9329373)] = 0.9127173$$

Sixth approximation:

Substituting x_5, y_5, z_5 on RHS of Eqns (i), we get

$$x_6 = \frac{1}{8}[20 + 3(1.9696539) - 2(0.9127173)] = \underline{3.01}04409$$

3.01 0.912

$$y_6 = \frac{1}{11}[33 - 4(3.0168733) + 0.9127173] = \underline{1.98}59295$$

$$z_6 = \frac{1}{12}[35 - 6(3.0168733) - 3(1.9696539)] = \underline{0.91}58165$$

Seventh approximation:

Substituting x_6, y_6, z_6 on RHS of Eqns (i), we get

2-digit

$$x_7 = \frac{1}{8}[20 + 3(1.9859295) - 2(0.9158165)] = \underline{3.01}57694$$

$$y_7 = \frac{1}{11}[33 - 4(3.0104409) + 0.9158165] = \underline{1.98}85503$$

$$z_7 = \frac{1}{12}[35 - 6(3.0104409) - 3(1.9859295)] = \underline{0.91}49638$$

Eighth approximation:

Substituting x_7, y_7, z_7 on RHS of Eqns (i), we get

$$x_8 = \frac{1}{8}[20 + 3(1.9885503) - 2(0.9149638)] = 3.0169654$$

$$y_8 = \frac{1}{11}[33 - 4(3.0157694) + 0.9149638] = 1.9865351$$

$$z_8 = \frac{1}{12}[35 - 6(3.0157694) - 3(1.9885503)] = 0.9116443$$

Ninth approximation:

Substituting x_8, y_8, z_8 on RHS of Eqns (i), we get

$$x_9 = \frac{1}{8}[20 + 3(1.9865351) - 2(0.9116443)] = 3.0170396$$

$$y_9 = \frac{1}{11}[33 - 4(3.0169654) + 0.9116443] = 1.9857984$$

$$z_9 = \frac{1}{12}[35 - 6(3.0169654) - 3(1.9865351)] = 0.9115501$$

Tenth approximation:

Substituting x_9, y_9, z_9 on RHS of Eqns (i), we get

$$x_{10} = \frac{1}{8}[20 + 3(1.9857984) - 2(0.9115501)] = 3.0167869$$

$$y_{10} = \frac{1}{11}[33 - 4(3.0170396) + 0.9115501] = 1.9857629$$

$$z_{10} = \frac{1}{12}[35 - 6(3.0170396) - 3(1.9857984)] = 0.9116972$$

Eleventh approximation:

Substituting x_{10}, y_{10}, z_{10} on RHS of Eqns (i), we get

$$x_{11} = \frac{1}{8}[20 + 3(1.9857629) - 2(0.9116972)] = 3.0167368$$

$$y_{11} = \frac{1}{11}[33 - 4(3.0167869) + 0.9116972] = 1.9858681$$

$$z_{11} = \frac{1}{12}[35 - 6(3.0167869) - 3(1.9857629)] = 0.9118326$$

Twelfth approximation:

Substituting x_{11}, y_{11}, z_{11} on RHS of Eqns (i), we get

$$x_{12} = \frac{1}{8}[20 + 3(1.9858681) - 2(0.91183226)] = 3.0167424$$

$$y_{12} = \frac{1}{11}[33 - 4(3.0167368) + 0.9118326] = 1.9858987$$

$$z_{12} = \frac{1}{12}[35 - 6(3.0167368) - 3(1.9858681)] = 0.9118312$$

$\therefore$ From the 11th and 12th approximations, the values of x, y, z are same correct to four decimal places. Stopping at this stage, we get $x = 3.0167$, $y = 1.9858$, $z = 0.9118$.

Example 4.17 Solve the following system of equations by Jacobi iteration method:

$3x + 4y + \boxed{15z} = 54.8$, $x + \boxed{12y} + 3z = 39.66$

and $\boxed{10x} + y - 2z = 7.74$ *(M.U, 1990)*

Solution The coefficient matrix of the given system is not diagonally dominant. Hence we rearrange the equations, as follows, such that the elements in the coefficient matrix are diagonally dominant

$$10x + y - 2z = 7.74$$
$$x + 12y + 3z = 39.66$$
$$3x + 4y + 15z = 54.8$$

Now we write the equations in the form

$$x = \frac{1}{10}(7.74 - y + 2z) \qquad \text{(ia)}$$

$$y = \frac{1}{12}(39.66 - x - 3z) \qquad \text{(ib)}$$

$$z = \frac{1}{15}(54.8 - 3x - 4y) \qquad \text{(ic)}$$

We start from an approximation $x_0 = y_0 = z_0 = 0$

Substituting these on RHS of (i), we get

First approximation as

$$x_1 = \frac{1}{10}[7.74 - 0 + 2(0)] = 0.774$$

$$y_1 = \frac{1}{12}[39.66 - 0 + 3(0)] = 1.1383333$$

$$z_1 = \frac{1}{15}[54.8 - 3(0) + 4(0)] = 3.6533333$$

Second approximation

$$x_2 = \frac{1}{10}[7.74 - 1.1383333 + 2(3.6533333)] = 1.3908333$$

$$y_2 = \frac{1}{12}[39.66 - 0.774 - 3(3.6533333)] = 2.3271667$$

$$z_2 = \frac{1}{15}[54.8 - 3(0.774) + 4(1.1383333)] = 3.1949778$$

Third approximation

$$x_3 = \frac{1}{10}[7.74 - 2.3271667 + 2(3.1949778)] = 1.1802789$$

$$y_3 = \frac{1}{12}[39.66 - 1.3908333 - 3(3.1949778)] = 2.3903528$$

$$z_3 = \frac{1}{15}[54.8 - 3(1.3908333) - 4(2.3271667)] = 2.7545889$$

Fourth approximation

$$x_4 = \frac{1}{10}[7.74 - 2.3903528 + 2(2.7545889)] = 1.0858825$$

$$y_4 = \frac{1}{12}[39.66 - 1.1802789 - 3(2.7545889)] = 2.5179962$$

$$z_4 = \frac{1}{15}[54.8 - 3(1.1802789) - 4(2.3903528)] = 2.7798501$$

Fifth approximation

$$x_5 = \frac{1}{10}[7.74 - 2.5179962 + 2(2.7798501)] = 1.0781704$$

$$y_5 = \frac{1}{12}[39.66 - 1.0858825 - 3(2.7798501)] = 2.5195473$$

$$z_5 = \frac{1}{15}[54.8 - 3(1.0858825) - 4(2.5179962)] = 2.7646912$$

Sixth approximation

$$x_6 = \frac{1}{10}[7.74 - 2.5195473 + 2(2.7646912)] = 1.0749835$$

$$y_6 = \frac{1}{12}[39.66 - 1.0781704 - 3(2.7646912)] = 2.5239797$$

$$z_6 = \frac{1}{15}[54.8 - 3(1.0781704) + 4(2.5195473)] = 2.76582$$

Seventh approximation

$$x_7 = \frac{1}{10}[7.74 - 2.5239797 + 2(2.76582)] = 1.074766$$

$$y_7 = \frac{1}{12}[39.66 - 1.0749835 - 3(2.76582)] = 2.523963$$

$$z_7 = \frac{1}{15}[54.8 - 3(1.0749835) + 4(2.5239797)] = 2.7652754$$

∴ From the sixth and seventh approximations
$x = 1.075,\ y = 2.524$ and $z = 2.765$ correct to three decimals.

4.10 $\boxed{\text{GAUSS–SEIDEL}}$ ITERATION METHOD

This is a modification of Gauss–Jacobi method. As before, the system of the linear equations

$|a| > |b_1| + |c_1|$

$$\boxed{a_1}x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

is written as

$$x = \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \tag{4.21a}$$

$$y = \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \tag{4.21b}$$

$$z = \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \tag{4.21c}$$

and we start with the initial approximation x_0, y_0, z_0. Substituting y_0 and z_0 in Eqn (4.21a), we get

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

Now substituting $x = x_1, z = z_0$ in Eqn (4.21b), we get

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_0)$$

Substituting $x = x_1, y = y_1$, in Eqn (4.21c), we get

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

This process is continued till the values of x, y, z are obtained to the desired degree of accuracy. The general algorithm is as follows:

If x_k, y_k, z_k are the kth iterates, then

$$x_{k+1} = \frac{1}{a_1}(d_1 - b_1 y_k - C_1 z_k)$$

$$x_{k+1} = \frac{1}{a_1}(d_1 - b_1 y_k - c_1 z_k)$$

$$y_{k+1} = \frac{1}{b_2}(d_2 - a_2 x_{k+1} - C_1 z_k)$$

$$y_{k+1} = \frac{1}{b_2}(d_2 - a_2 x_{k+1} - c_2 z_k)$$

$$z_{k+1} = \frac{1}{c_3}(d_3 - a_3 x_{k+1} - b_3 y_{k+1})$$

and

$$z_{k+1} = \frac{1}{c_3}(d_3 - a_3 x_{k+1} - b_3 y_{k+1})$$

Since the current values of the unknowns at each stage of iteration are used in proceeding to the next stage of iteration, this method is more rapid in convergence than Gauss–Jacobi method.

The rate of convergence of Gauss–Seidel method is roughly twice that of Gauss–Jacobi and the condition of convergence is same as we saw earlier in Section 4.8.

Note: Gauss–Seidel iteration method converges only for special systems of equations. In general, the round of errors will be small in iteration methods. Moreover, these are self-correcting methods; that is, any error made in computation will be corrected in the subsequent iteration.

Example 4.18 Solve the equations given in Example 4.9 by Gauss–Seidel iteration method. *(B.U, B.E., 1994)*

Gaus-Seidel

Gauss Jacobi

Solution From the given equations, we have

$x_1 = \frac{1}{8}(20 + 3y_0 - 2z_0)$

$$x = \frac{1}{8}(20 + 3y - 2z) \quad \text{(i)}$$

$x_1 = \frac{1}{8}(20 + 3y_0 - 2z_0)$

$y_1 = \frac{1}{11}(33 - 4x_1 + z_0)$

$$y = \frac{1}{11}(33 - 4x + z) \quad \text{(ii)}$$

$y_1 = \frac{1}{11}(33 - 4x_0 + z_0)$

$z_1 = \frac{1}{12}(35 - 6x_1 - 3y_1)$

$$z = \frac{1}{12}(35 - 6x - 3y) \quad \text{(iii)}$$

$z_1 = \frac{1}{12}(35 - 6x_0 - 3y_0)$

Putting $y = 0$, $z = 0$ in RHS of (i), we get $x_1 = \frac{20}{8} = 2.5$

Putting $x = 2.5$, $z = 0$ in RHS of (ii), we get

$$y_1 = \frac{1}{11}[33 - 4(2.5)] = 2.0909091$$

Putting $x = 2.5$, $y = 2.0909091$ in RHS of (iii), we get

$$z_1 = \frac{1}{12}[35 - 6(2.5) - 3(2.0909091)] = 1.1439394$$

For the *second approximation,*

$$x_2 = \frac{1}{8}[20 + 3y_1 - 2z_1]$$

$$= \frac{1}{8}[20 + 3(2.0909091) - 2(1.1439394)] = 2.9981061$$

$$y_2 = \frac{1}{11}[33 - 4x_2 + z_1]$$

$$= \frac{1}{11}[33 - 4(2.9981061) + 1.1439394] = 2.0137741$$

$$z_2 = \frac{1}{12}[35 - 6x_2 - 3y_2]$$

$$= \frac{1}{12}[35 - 6(2.9981061) - 3(2.0137741)] = \underline{0.9141701}$$

Third approximation:

$$x_3 = \frac{1}{8}[20 + 3(2.0137741) - 2(0.9141701)] = 3.0266228$$

$$y_3 = \frac{1}{11}[33 - 4(3.0266228) + 0.9141701)] = 1.9825163$$

$$z_3 = \frac{1}{12}[35 - 6(3.0266228) - 3(1.9825163)] = 0.9077262$$

Fourth approximation:

$$x_4 = \frac{1}{8}[20 + 3(1.9825163) - 2(0.9077262)] = 3.0165121$$

$$y_4 = \frac{1}{11}[33 - 4(3.0165121) + 0.9077262)] = 1.9856071$$

$$z_4 = \frac{1}{12}[35 - 6(3.0165121) - 3(1.9856071)] = 0.9120088$$

Fifth approximation:

$$x_5 = \frac{1}{8}[20 + 3(1.9856071) - 2(0.9120088)] = 3.0166005$$

$$y_5 = \frac{1}{11}[33 - 4(3.0166005) + 0.9120088] = 1.9859643$$

$$z_5 = \frac{1}{12}[35 - 6(3.0166005) - 3(1.9859643)] = 0.9118753$$

Sixth approximation:

Jocobi – 2 digit

Seidel – 3-digits

$$x_6 = \frac{1}{8}[20 + 3(1.9859643) - 2(0.9118753)] = 3.0167678$$

$$y_6 = \frac{1}{11}[33 - 4(3.0167678) + 0.9118753] = 1.9858913$$

approximate Soln → 4-digits

$$z_6 = \frac{1}{12}[35 - 6(3.0167678) - 3(1.9858913)] = 0.9118099$$

Seventh approximation:

$$x_7 = \frac{1}{8}[20 + 3(1.9858913) - 2(0.9118099)] = 3.0167568$$

$$y_7 = \frac{1}{11}[33 - 4(3.0167568) + 0.9118099] = 1.9858894$$

$$z_7 = \frac{1}{12}[35 - 6(3.0167568) - 3(1.9858894)] = 0.9118159$$

Since at the sixth and seventh approximations, the values of x, y, z are the same, correct to four decimal places, we can stop the iteration process.

$\therefore x = 3.0167, \quad y = 1.9858, \quad z = 0.9118$

We find that 12 iterations are necessary in Gauss–Jacobi method to get the same accuracy as achieved by 7 iterations in Gauss–Seidel method.

Example 4.19 Solve by Gauss–Seidel method, the following system of equations

$28x + 4y - z = 32, \; x + 3y + 10z = 24,$

and $2x + 17y + 4z = 35$ *(B.U. 1997, M.U. 1991)*

Solution The coefficient matrix of the given system is not diagonally dominant. Hence, we rearrange the equations as follows, such that the elements in the coefficient matrix are diagonally dominant.

$$28x + 4y - z = 32$$
$$2x + 17y + 4z = 35$$
$$x + 3y + 10z = 24$$

Hence we can apply Gauss–Seidal iteration method.

From the above equations

$$x = \frac{1}{28}[32 - 4y + z] \qquad \text{(i)}$$

$$y = \frac{1}{17}[35 - 2x - 4z] \qquad \text{(ii)}$$

$$z = \frac{1}{10}[24 - x - 3y] \qquad \text{(iii)}$$

First approximation

Putting $y = z = 0$ in (i), we get

$$x_1 = \frac{1}{28}(32) = 1.1428571$$

Putting $x = 1.1428571, \; z = 0$ in (ii), we get

$$y_1 = \frac{1}{17}[35 - 2(1.1428571)] = 1.9243697$$

Putting $x = 1.1428571, \; y = 1.9243697$ in (iii), we get

$$z_1 = \frac{1}{10}[24 - 1.1428571 - 3(1.9243697)] = 1.7084034$$

Second approximation

$$x_2 = \frac{1}{28}[32 - 4(1.9243697) + 1.7084034] = 0.9289615$$

$$y_2 = \frac{1}{17}[35 - 2(0.9289615) - 4(1.7084034)] = 1.5475567$$

$$z_2 = \frac{1}{10}[24 - 0.9289615 - 3(1.5475567)] = 1.8428368$$

Third approximation

$$x_3 = \frac{1}{28}[32 - 4(1.5475567) + 1.8428368] = 0.9875932$$

$$y_3 = \frac{1}{17}[35 - 2(0.9875932) - 4(1.8428368)] = 1.5090274$$

$$z_3 = \frac{1}{10}[24 - 0.9875932 - 3(1.5090274)] = 1.8485325$$

Fourth approximation

$$x_4 = \frac{1}{28}[32 - 4(1.5090274) + 1.8485325] = 0.9933008$$

$$y_4 = \frac{1}{17}[35 - 2(0.9933008)] = 4(1.8485325)] = 1.5070158$$

$$z_4 = \frac{1}{10}[24 - 0.9933008 - 3(1.5070158)] = 1.8485652$$

Fifth approximation

$$x_5 = \frac{1}{28}[32 - 4(1.5070158) + 1.8485652] = 0.9935893$$

$$y_5 = \frac{1}{17}[35 - 2(0.9935893) - 4(1.8485652)] = 1.5069741$$

$$z_5 = \frac{1}{10}[24 - 0.9935893 - 3(1.5069741)] = 1.8485488$$

Sixth approximation

$$x_6 = \frac{1}{28}[32 - 4(1.5069741) + 1.8485488] = 0.9935947$$

$$y_6 = \frac{1}{17}[35 - 2(0.9935947) - 4(1.8485488)] = 1.5069774$$

$$z_6 = \frac{1}{10}[24 - 0.9935947 - 3(1.5069774)] = 1.8485473$$

$\because$ the values of x, y, z in the fourth and fifth iteration are same upto four decimals, we stop the process here.

Hence $x = 0.9936,\ y = 1.5069,\ z = 1.8485$

Example 4.20 Using Gauss–Seidel iteration method, solve the system of equations

$$10x - 2y - z - w = 3; -2x + 10y - z - w = 15;$$
$$-x - y + 10z - 2w = 27; -x - y - 2z + 10w = -9$$

4 unknows → 4-equations

(*M.U, B.E., 1987*)

Solution The coefficient matrix of the given system is diagonally dominant. Hence we can apply Gauss–Seidel iteration method.

From the given equations, we can write

$$x = \frac{1}{10}[3 + 2y + z + w] \quad \text{(i)}$$

$$y = \frac{1}{10}[15 + 2x + z + w] \quad \text{(ii)}$$

$$z = \frac{1}{10}[27 + x + y + 2w] \quad \text{(iii)}$$

$$w = \frac{1}{10}[-9 + x + y + 2z] \quad \text{(iv)}$$

First approximation:

Putting $y = z = w = 0$ on RHS of (i), we get $x_1 = \frac{3}{10} = 0.3$

Putting $x = 0.3,\ z = w = 0$ on RHS of (ii),

we get $y_1 = \frac{1}{10}[15 + 2(0.3)] = 1.56$

Putting $x = 0.3,\ y = 1.56,\ w = 0$ on RHS of (iii),

we get $z_1 = \frac{1}{10}[27 + 0.3 + 1.56] = 2.886$

Putting $x = 0.3,\ y = 1.56,\ z = 2.886$ on RHS of (iv),

we get $w_1 = \frac{1}{10}[-9 + 0.3 + 1.56 + 2(2.886)] = -0.1368$

Second approximation:

$$x_2 = \frac{1}{10}[3 + 2(1.56) + 2.886 - 0.1368] = 0.88692$$

$$y_2 = \frac{1}{10}[15 + 2(0.88692) + 2.886 - 0.1368] = 1.952304$$

$$z_2 = \frac{1}{10}[27 + 0.88692 + 1.952304 + 2(-0.1368] = 2.9565624$$

$$w_2 = \frac{1}{10}[-9 + 0.88692 + 1.952304 + 2(2.9565624)] = -0.0247651$$

Third approximation:

$$x_3 = \frac{1}{10}[3 + 2(1.952304) + 2.9565624 - 0.0247651] = 0.9836405$$

$$y_3 = \frac{1}{10}[15 + 2(0.9836405) + 2.9565624 - 0.0247651] = 1.9899087$$

$$z_3 = \frac{1}{10}[27 + 0.9836405 + 1.9899087 + 2(-0.0247651)] = 2.9924019$$

$$w_3 = \frac{1}{10}[-9 + 0.983405 + 1.9899087 + 2(2.9924019)] = -0.0041647$$

Fourth approximation:

$$x_4 = \frac{1}{10}[3 + 2(1.9899087) + 2.9924019 - 0.0041647] = 0.9968054$$

$$y_4 = \frac{1}{10}[15 + 2(0.9968054) + 2.9924019 - 0.0041647] = 1.9981848$$

$$z_4 = \frac{1}{10}[27 + 0.9968054 + 1.9981848 + 2(-0.0041647)] = 2.9986661$$

$$w_4 = \frac{1}{10}[-9 + 0.9968054 + 1.9981848 + 2(2.9986661)] = -0.0007677$$

Fifth approximation:

$$x_5 = \frac{1}{10}[3 + 2(1.9981848) + 2.9986661 - 0.0007677] = 0.9994268$$

$$y_5 = \frac{1}{10}[15 + 2(0.9994268) + 2.998666 - 0.0007677] = 1.9996752$$

$$z_5 = \frac{1}{10}[27 + 0.9994268 + 1.9996752 + 2(-0.0007677)] = 2.9997567$$

$$w_5 = \frac{1}{10}[-9 + 0.9994268 + 1.9996752 + 2(2.9997567)] = -0.0001384$$

Sixth approximation:

$$x_6 = \frac{1}{10}[3 + 2(1.9996752) + 2.9997567 - 0.0001384] = 0.9998968$$

$$y_6 = \frac{1}{10}[15 + 2(0.9998968) + 2.9997567 - 0.0001384] = 1.9999412$$

$$z_6 = \frac{1}{10}[27 + 0.9998968 + 1.9999412 + 2(-0.0001384)] = 2.9999561$$

$$w_6 = \frac{1}{10}[-9 + 0.9998968 + 1.9999412 + 2(2.9999561)] = -0.0002498$$

Seventh approximation:

$$x_7 = \frac{1}{10}[3 + 2(1.9999412) + 2.9999561 - 0.0002498] = 0.9999588$$

$$y_7 = \frac{1}{10}[15 + 2(0.9999588) + 2.9999561 - 0.0002498] = 1.9999624$$

$$z_7 = \frac{1}{10}[27 + 0.9999588 + 1.9999264 + 2(-0.0002498)] = 2.9999422$$

$$w_7 = \frac{1}{10}[-9 + 0.9999588 + 1.9999624 + 2(2.9999422)] = -0.0001945$$

Now, from sixth and seventh approximations the values of x, y, z and w correct to four decimal places are

$$x = 0.9999 \qquad y = 1.9999 \qquad z = 2.9999 \qquad w = 0.0002$$

4.11 RELAXATION METHOD (SOR)

Consider the equations

$$a_1x + b_1y + c_1z = d_1 \rightarrow d_1 - a_1x - b_1y - c_1z = r_1$$

$$a_2x + b_2y + c_2z = d_2 \rightarrow d_2 - a_2x_2 - \quad = r_2$$

$$a_3x + b_3y + c_3z = d_3 \qquad = r_3$$

We define the residuals r_1, r_2, r_3 by the relations

$$r_1 = d_1 - a_1x - b_1y - c_1z \tag{4.22a}$$

$$r_2 = d_2 - a_2x - b_2y - c_2z \tag{4.22b}$$

$$r_3 = d_3 - a_3x - b_3y - c_3z \tag{4.22c}$$

To start with, we assume $x = y = z = 0$ and calculate the initial residuals. Then these residuals are reduced step by step giving increments to the variables. If we can find x, y, z such that residuals $r_1 = r_2 = r_3 = 0$ then those values of x, y, z are the exact values. Otherwise, we liquidate the residuals smaller and smaller and finally negligible to get better approximate values of x, y, z. For this purpose we construct an *operation table* as shown below.

x	y	z	r_1	r_2	r_3
1	0	0	$-a_1$	$-a_2$	$-a_3$
0	1	0	$-b_1$	$-b_2$	$-b_3$
0	0	1	$-c_1$	$-c_2$	$-c_3$

From Eqns (4.22), we can see that if x is increased by 1, keeping y and z constant, r_1, r_2 and r_3 decrease by a_1, a_2 and a_3, respectively.

This is shown in the above table along with the effects on the residuals when y and z are given unit increments. It can be noted that the operation table consists of the unit matrix I and transpose of the coefficient matrix.

At each step, the numerically largest residual is reduced almost to zero. To reduce a particular residual, the value of the corresponding variable is changed, i.e. to reduce, say, r_2 by α, y should be increased by $\frac{\alpha}{b_2}$. When all the residuals have been reduced to almost zero, then the increments in x, y and z are added separately to give the desired solution.

Note: After finding x, y and z, substitute them in Eqns (4.22), and check whether the residuals are negligible or not. If not, then there is some mistake and the entire process should be rechecked.

Convergency of Relaxation Method

This method can be applied successfully only if the diagonal elements of the coefficient matrix dominate the other coefficients in the corresponding row, i.e. if in Eqns (4.22)

$$|a_1| \geq |b_1| + |c_1|$$

$$|b_2| \geq |a_2| + |c_2|$$

and

$$|c_3| \geq |a_3| + |b_3|$$

with strict inequality for atleast one row.

Example 4.21 Solve by relaxation method, the equations $10x - 2y - 2z = 6$; $-x - 10y - 2z = 7$; $-x - y + 10z = 8$ (*M.U, B.E., 1991*)

Solution The residuals r_1, r_2 and r_3 are given by

$$r_1 = 6 - \boxed{10}x + 2y + 2z$$
$$r_2 = 7 + x - 10y + 2z$$
$$r_3 = 8 + x + y - 10z$$

The operation table is as follows.

x	y	z	r_1	r_2	r_3		
1	0	0	−10	1	1	←	L_1
0	1	0	2	−10	1	←	L_2
0	0	1	2	2	−10	←	L_3

The relaxation table is as follows

x	y	z	r_1	r_2	r_3		
0	0	0	6	7	8	←	L_4
0	0	1	8	9	−2	←	$L_5 = L_4 + L_3$
0	1	0	10	−1	−1	←	$L_6 = L_5 + L_2$
1	0	0	0	0	0	←	$L_7 = L_6 + L_1$

Explanation:

(i) In line L_4, the largest residual is 8. To reduce it, we give an increment of $\frac{8}{c_3} = \frac{8}{10} = 0.8 \simeq 1$. The resulting residuals are obtained by $L_4 + (1)\,L_3$, i.e. line L_5.

(ii) In line L_5, the largest residual is 9.

$$\therefore \text{ increment} = \frac{9}{b_2} = \frac{9}{10} = 0.9 \simeq 1$$

The resulting residuals $(= L_6) = L_5 + 1 . L_2$

(iii) In line L_6, the largest residual is 10.

$$\therefore \text{ increment} = \frac{10}{a_1} = \frac{10}{10} = 1$$

The resulting residuals ($= L_7$) $= L_6 + 1 . L_3$, which are all zeros.

$\Rightarrow$ Exact solution is arrived and it is $x = 1, y = 1, z = 1$

Example 4.22 Solve by relaxation method, the equations
$9x - y + 2z = 9,\ x + 10y - 2z = 15,\ 2x - 2y - 13z = -17$

(M.U. B.E.. 1996)

Solution The residuals r_1, r_2, r_3 are given by
$r_1 = 9 - 9x + y - 2z$; $r_2 = 15 - x - 10y + 2z$; $r_3 = -17 - 2x + 2y + 13z$ (i)

Operation table

x	y	z	r_1	r_2	r_3	
1	0	0	–9	–1	–2	$\leftarrow L_1$
0	1	0	1	–10	2	$\leftarrow L_2$
0	0	1	–2	2	13	$\leftarrow L_3$

Relaxation table

x	y	z	r_1	r_2	r_3	
0	0	0	9	15	–17	$\leftarrow L_4$
0	0	1	7	17	–4	$\leftarrow L_5 = L_4 + 1\ L_3$
0	1	0	8	7	–2	$\leftarrow L_6 = L_5 + 1. L_2$
0.89	0	0	–0.01	6.11	–3.78	$\leftarrow L_7 = L_6 + 0.89L_1$
0	0.61	0	0.6	0.01	–2.56	$\leftarrow L_8 = L_7 + 0.61L_2$
0	0	0.19	0.22	0.39	–0.09	$\leftarrow L_9 = L_8 + 0.1L_3$
0	0.039	0	0.259	0	–0.012	$\leftarrow L_{10} = L_9 + 0.039L_2$
0.028	0	0	0.007	–0.028	–0.068	$\leftarrow L_{11} = L_{10} + 0.028L_1$
0	0	0.00523	–0.00346	–1.01754	–0.00001	$\leftarrow L_{12} = L_{11} + 0.00523L_3$

Thus, $x = 0.89 + 0.028 = 0.918$; $y = 1 + 0.61 + 0.039 = 1.649$ and
$z = 1 + 0.19 + 0.00523 = 1.19523$.

Now substituting the values of x, y, z in (i), we get

$r_1 = 9 - 9(0.918) + 1.649 - 2(1.19523) = -0.00346$

$r_2 = 15 - 0.918 - 10(1.649) + 2(1.19523) = -0.1754$

$r_3 = -17 - 2(0.918) + 2(1.649) + 13(1.19523) = -0.00001$

which are in agreement with the final residuals in the table.

Example 4.23 Solve by relaxation method, the equations $10x - 2y + z = 12$, $x + 9y - z = 10$ and $2x - y + 11z = 20$ *(M.U, 1997)*

Solution The residuals r_1, r_2, r_3 are given by

$$r_1 = 12 - 10x + 2y - z$$
$$r_2 = 10 - x - 9y + z$$
$$r_3 = 20 - 2x + y - 11z$$

The operation table is

x	y	z	r_1	r_2	r_3	
1	0	0	−10	−1	−2	← L_1
0	1	0	2	−9	1	← L_2
0	0	1	−1	1	−11	← L_3

The relation table is

x	y	z	r_1	r_2	r_3	
0	0	0	12	10	20	← L_4
0	0	1.8	10.2	11.8	0.8	← $L_5 = L_4 + 1.8\, L_3$
0	1.31	0	12.82	0.01	1.51	← $L_6 = L_5 + 1.31\, L_2$
1.28	0	0	0.02	−1.27	−1.05	← $L_7 = L_6 + 1.28\, L_1$
0.02	0	0	0	−1.272	−1.054	← $L_8 = L_7 + 0.002\, L_1$
0	0	−0.1	0.1	−1.372	0.046	← $L_9 = L_8 - 0.1\, L_3$
0.01	0	0	0	1.382	0.026	← $L_{10} = L_9 + 0.01\, L_1$
0	0	0.0023	−0.0023	−1.3797	0.0007	← $L_{11} = L_{10} + 0.0023$

Thus $x = 1.28 + 0.02 + 0.01 = 1.31$; $y = 1.31$ and
$z = 1.8 - 0.1 + 0.0023 = 1.7023$.

EXERCISE 4.3

Solve the following system of linear equations by (i) Gauss and (ii) Gauss Seidel iteration method.

1. $2x + y + z = 4, x + 2y + z = 4; x + y + 2z = 4$
2. $8x + y + z = 18; 2x + 4y + z = 4; x + 3y + 5z = 5$
3. $5x + 2y + z = 12, x + 4y + 2z = 15, x + 2y + 5z = 20$

4. $9x + 2y + 4z = 20,\ x + 10y + 4z = 6,\ 2x - 4y + 10z = -15$
5. $54x + y + z = 110,\ 2x + 15y + 6z = 72,\ -x + 6y + 27z = 85$ (*M.U. B.E., 1993*)
6. $28x - 4y - z = 32,\ x + 3y + 10z = 24,\ 2x + 17y + 4z = 35$
7. $5x - y + z = 10,\ 2x + 4y = 12,\ x + y + 5z = -1$ (*Bangalore, B.E., 1990*)
8. $10x_1 - 5x_2 - 2x_3 = 3,\ 4x_1 - 10x_2 + 3x_3 = -3,\ x_1 + 6x_2 + 10x_3 = -3$
9. $10x + 2y + z = 9,\ 2x + 20y - 2z = -44,\ -2x + 3y + 10z = 22$
10. $10x_1 + 7x_2 + 8x_3 + 7x_4 = 32,\ 7x_1 + 5x_2 + 6x_3 + 5x_4 = 23;$
    $8x_1 + 6x_2 + 10x_3 + 9x_4 = 33,\ 7x_1 + 5x_2 + 9x_3 + 10x_4 = 31$

Solve by relaxation method the following equations:

11. $9x + 2y + z = 50,\ x + 5y - 3z = 18,\ -2x + 2y + 7z = 19$
12. $3x + 9y - 2z = 11,\ 4x + 2y + 13z = 24,\ 4x - 4y + 3z = -8$ (*M.U. B.E., 1993*)
13. $4.215x - 1.212y + 1.105z = 3.216$
    $-2.120x + 3.505y - 1.632z = 1.247$
    $1.122x - 1.313y + 3.986z = 2.112$
14. $10x - 2y - 3z = 305,\ -2x + 10y - 2z = 154,\ -2x - y + 10z = 120$
15. $8x_1 + x_2 + x_3 + x_4 = 14;\ 2x_1 + 10x_2 + 3x_3 + x_4 = -8$
    $x_1 - 2x_2 - 20x_3 + 3x_4 = 111,\ 3x_1 + 2x_2 + 2x_3 + 19x_4 = 53$

ANSWERS

1. $x = 1, y = 1, z = 1$
2. $x = 0.876, y = 0.919, z = 0.574$
3. $x = 0.996, y = 1.95, z = 3.16$
4. $x = 2.733, y = 0.986, z = -1.652$
5. $x = 1.926, y = 3.573, z = 2.425$
6. $x = 0.994, y = 1.507, z = 1.849$
7. $x = 2.556, y = 1.722, z = -1.055$
8. $x_1 = 0.342, x_2 = 0.285, x_3 = -0.505$
9. $x = 1.013, y = -1.996, z = 3.001$
10. $x_1 = 1, x_2 = 1, x_3 = 1, x_4 = 1$
11. $x = 6.13, y = 4.31, z = 3.23$
12. $x = 1.35, y = 2.103, z = 2.845$
13. $x = 0.943, y = 1.239, z = 0.673$
14. $x = 32, y = 26, z = 21$
15. $x_1 = 2, x_2 = 0, x_3 = -5, x_4 = 3$