Taylor Series

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1 Prerequisites:

1.1 Taylor Series Approximation:

The approximated expansion for the Taylor Series of the function $f(x)$ centered at $x = a$ is given by the following formula:

Taylor Series Approximate Expansion
$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a^3+\ldots)$

Where each constant is represented by an increasing derivative of the function $f(x)$ at some value a.

1.2 Linear Approximation:

Linear approximation, also known as linearization or tangent line approximation, is a method used in calculus to estimate the value of a function near a specific point by using the equation of the tangent line at that point. The linear approximation provides a good approximation of the function's behavior in the vicinity of the chosen point. The linear approximation is

based on the idea that a sufficiently smooth and well-behaved function can be approximated by a linear function (a straight line) near a particular point. The equation of the tangent line to the graph of the function at a given point $(a, f(a))$ is given by:

Linear Approximation
$L(x) = f(a) + f'(a)(x-a)$

Here:

- $L(x)$ is the linear approximation to the function,
- $f(a)$ is the value of the function at the point $x = a$,
- $f'(a)$ is the derivative of the function evaluated at $x = a$,
- x is the variable.

1.2.1 Example:

Find the value of $\sqrt{4.3}$ using the function $f(x) = \sqrt{x}$.

In this case the closest whole number that the function $f(x)$ outputs is $f(4) = \sqrt{4}$ which equals 2. So, our $a = 4$, $f(a) = 2$ and $f'(a) = \frac{1}{2\sqrt{4}}$ which is $\frac{1}{4}$.

Therefore, our equation of the tangent line is $L(x) = f(a) + f'(a)(x-a) = 2 + \frac{1}{4}(x-4)$. Plugging in $L(4.3)$ we get that $\sqrt{4.3} \approx 2.075$.

1.3 Absolute Value Equations:

Absolute Value Function
$\lvert x\rvert = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

The absolute value function $|x|$ returns the non-negative distance of x from zero on the number line.

Absolute Value Equation
$\lvert f(x)\rvert = g(x)$

Absolute value equations relate the absolute value of an expression $f(x)$ to another expression or constant $g(x)$. **Solving Procedure:**

1. Solve $f(x) = g(x)$
2. Solve $f(x) = -g(x)$

Solving each case may yield two sets of solutions, and validity is checked by substitution into the original equation.

1.3.1 Example:

Solve $|2x - 5| = 7$.

1. **Isolate the Absolute Value Expression:** $2x - 5 = 7$ or $2x - 5 = -7$
2. **Apply the Absolute Value Property:** $2x = 12$ or $2x = -2$
3. **Solve for the Variable:** Solutions: $x = 6$ and $x = -1$.
4. **Check for Extraneous Solutions:** Both solutions are valid.

2 Power Series:

A power series is an infinite series with the form:

Power Series
$\sum_{k=0}^{\infty} c_k x^k = c_o + c_1 x + c_2 x^2 + \ldots + c_n x^n + c_{n+1} x^{n+1} + \ldots$

This power series is centerd at 0. The more general formula that includes power series centered at a is:

Power Series
$\sum_{k=0}^{\infty} c_k (x-a)^k = c_o + c_1(x-a) + c_2(x-a)^2 + \ldots + c_n(x-a)^n + c_{n+1}(x-a)^{n+1}$

3 Taylor Polynomials:

Let f be a funcion with $f', f'', \ldots, f^{(n)}$ defined at a. The nth order Taylor Polynomial p' for f, centered at a, matches f in value, slope, and all derivatives up to the nth order at a.

Taylor Polynomials

4 Binomial Series:

Binomial Definition: The term ”binomial” is derived from the Latin words ”bi” which means two and ”nomen” which means name. Which shows that binomials involve expressions with two terms, for example: $(a + b)$.
Here are some examples of binomials:

$$(1 + x)^0 = 1$$

$$(1 + x)^1 = 1 + x$$

$$(1 + x)^2 = 1 + 2x + x^2$$

$$(1 + x)^3 = 1 + 3x + 3x^2 + x^3$$

$$(1 + x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

This pattern is derived from Pascals Triangle which lists out the coefficients for each term.

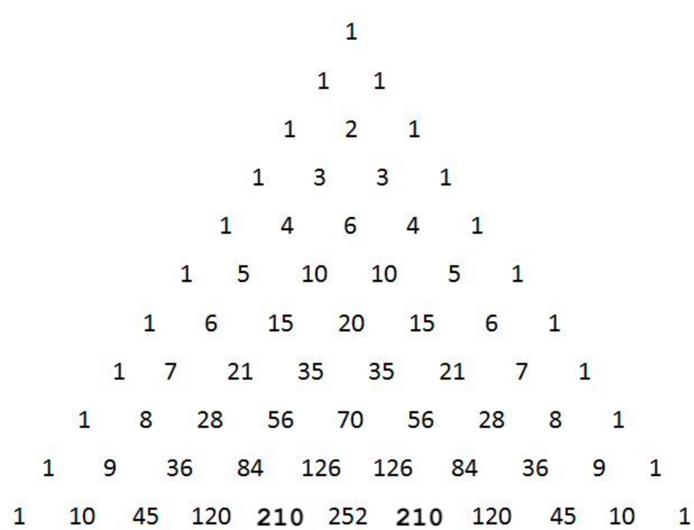

Figure 1: Pascals Triangle

However, if we wanted to calculate larger and larger powers of $(1+x)$ you can see that this would become difficult and cumbersome. For example if you wanted to calculate $(1+x)^n$ where n was a very large value like 8 it would not be wise to write out 8 rows of Pascals Triangle and then find the coefficients to write the answer.
We have a name for the terms that appear in the Pascals Triangle which are also the powers $(1+x)$. We call them *binomial coefficients.*

Binomial Coefficients
$\binom{p}{k} = \frac{p!}{k!(p-k)!}$

Note: The numbers $\binom{p}{0}, \binom{p}{1}, \binom{p}{2}, \ldots, \binom{p}{p}$ form the "pth" row on the Pascals Triangle.

More generally, if p is any real number and $k >= 1$ is an integer, then:

Binomial Coefficients
$\binom{p}{k} = \frac{p(p-1)(p-2)...(p-k+1)}{k!}$

The Special Symbol $\binom{p}{k}$ is pronounced "p choose k".

4.1 Examples:

4.1.1 Example 1:

If you had a class of 30 students and you had to pick a team of 3 students. This would be "30 choose 3" or $\binom{30}{3}$.

$$\binom{30}{3} = \frac{30!}{3!(27)!}$$

You can solve this equation to get the answer to the problem.

4.1.2 Example 2:

What if we wanted to find the 8th row of Pascals Triangle. This would be the same as doing:

$\binom{8}{0} = 1, \binom{8}{1} = \frac{8!}{1!(7)!}, \binom{8}{2} = \frac{8!}{2!(6)!}, \ldots, \binom{8}{n} = \frac{8!}{n!(8-n)!}$ Solving this you can calculate $(1+x)^8$

which equals $1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$.

4.1.3 Example 3:

Find $\binom{-3}{2}$ and $\binom{\frac{1}{2}}{3}$.

$$\binom{3}{-2} = \frac{-3(-3-1)}{2!}$$

Because $k = 2$ there are only 2 factors on the top $-3(-3-1)$.

$$\binom{3}{-2} = \frac{-3(-3-1)}{2!}$$

$$= \frac{(-3)(-4)}{2}$$

$$= 6$$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}$$

Because $k = 3$ there are only 3 factors on the top $\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)$

$$\binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}$$

$$= \frac{\frac{1}{2}(\frac{-1}{2})(\frac{-3}{2})}{6}$$

$$= \frac{\frac{3}{8}}{6}$$

$$= \frac{1}{16}$$

Binomial Series
$\sum_{k=0}^{\infty} \binom{p}{k} x^k = 1 + \sum_{k=1}^{\infty} \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!} x^k$