

Signals and Systems I

Lecture 3

Last Lecture

- Combined Operation $Ax(\alpha t - T)$
- Odd & Even Signals
- How to Calculate odd & even parts of the signal

Today

- Build Signals with $u(t)$ and $\delta(t)$
- Closed form expression

Operations on $u(t)$ & $\delta(t)$

Operations on $\delta(t)$

Examples:

Reminder:

$\delta(t)$ is infinity at 0 and its integral is 1.

$$^*\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$

proof:

$$\int \delta(t) dt = 1 \rightarrow \int \delta(\alpha t) dt$$

$$\text{let } \alpha t = \omega \text{ then } dt = \frac{1}{\omega} d\omega$$

$$\begin{aligned} \int \delta(\omega) \frac{1}{\omega} d\omega \\ = \frac{1}{\omega} \int \delta(\omega) d\omega = \frac{1}{\omega} \cdot 1 = \frac{1}{\omega} \end{aligned}$$

$$\delta(\alpha(t - \beta)) = \frac{1}{|\alpha|} \delta(t - \beta)$$

$$\delta(-t) = \delta(t) \rightarrow \text{Even Signal!}$$

$$\delta(2t) = \frac{1}{2} \delta(t)$$

$$\delta(-2t) = \frac{1}{2} \delta(t)$$

$$\delta\left(\frac{1}{3}t\right) = 3\delta(t)$$

$$\delta(5t - 3) = \delta\left(5(t - \frac{3}{5})\right) = \frac{1}{5} \delta(t - \frac{3}{5})$$

$$\text{Recall: } \delta(t)x(t) = \delta(t)x(0)$$

$$\delta(t - \tau_0)x(t - \tau_1) = \delta(t - \tau_0)x(\tau_0 - \tau_1)$$

Operations on $u(t)$ & $\delta(t)$

$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$

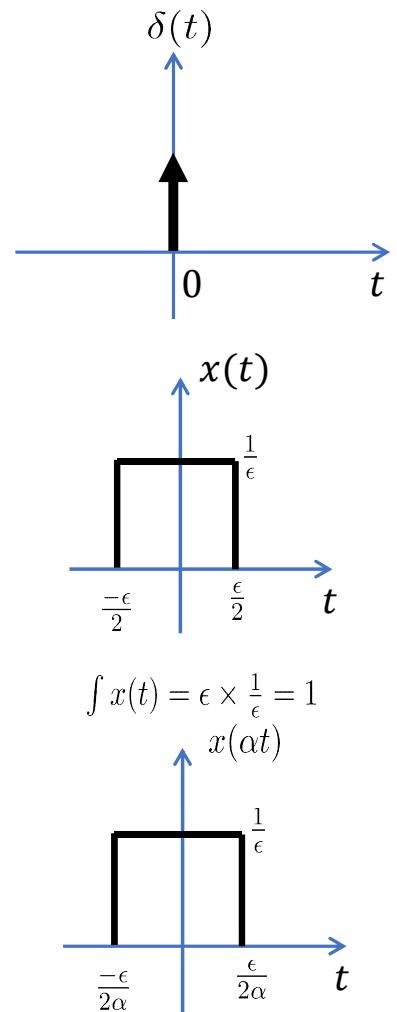
Remember that $\delta(t)$ was limit of $x(t)$ when ϵ goes to zero.
So $\delta(\alpha t)$ is also limit of $x(\alpha t)$

Also both $\delta(t)$ and $x(t)$ are even functions,
 $\delta(t) = \delta(-t)$, $x(t) = x(-t)$
so α and $-\alpha$ operate the same.

Alternative method (for positive α): rename αt as w

$$\int \delta(\alpha t) dt = \int \delta(w) \frac{dw}{\alpha} = \frac{1}{\alpha} \int \delta(w) dw = \frac{1}{\alpha} \times 1$$

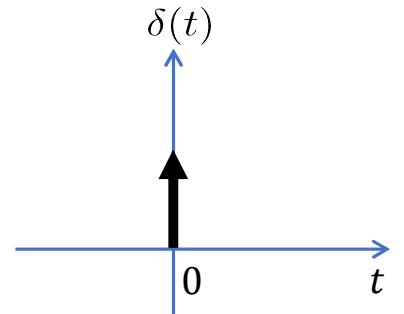
$$\alpha t = w, \alpha dt = dw$$



$$\int x(\alpha t) dt = \frac{1}{\alpha} \int x(t) dt = \frac{1}{\alpha} \times 1 = \frac{1}{\alpha}$$

Operations on $u(t)$ & $\delta(t)$

$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$



In general:

$$\delta(\alpha(t - \beta)) = \frac{1}{|\alpha|} \delta(t - \beta)$$

$\leftarrow \delta(\alpha t - \beta), T = \alpha \beta$

Examples: $\delta(-t) = \delta(t)$, Even Signal

$$\delta(2t) = \frac{1}{2} \delta(t)$$

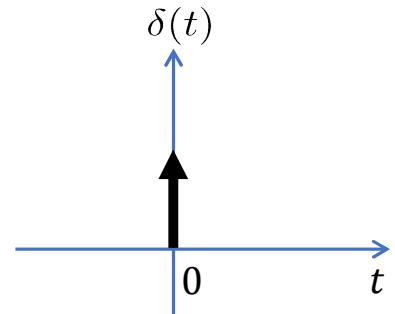
$$\delta(-2t) = \frac{1}{2} \delta(t)$$

$$\delta\left(\frac{1}{3}t\right) = 3\delta(t)$$

$$\delta(5t - 3) = \delta\left(5(t - \frac{3}{5})\right) = \frac{1}{5} \delta(t - \frac{3}{5})$$

Reminder on $\delta(t)$

$$\delta(t - T_0)x(t - T_1) = \delta(t - T_0)x(T_0 - T_1)$$



Reminder: Multiplying $\delta(t - T_0)$ by any function "Kills" the function for all values except at T_0

$$\delta(t - T_0)y(t) = \delta(t - T_0)y(T_0)$$

Here $y(t) = x(t - T_1)$!

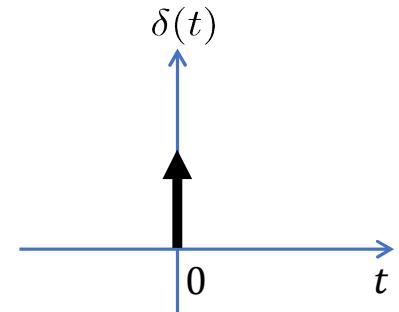
Example:

$$\delta(t - 2)x(t) = \delta(t - 2)x(2)$$

$$\delta(t - 2)x(t + 3) = \delta(t - 2)x(2 + 3) = \delta(t - 2)x(5)$$

Reminder on $\delta(t)$

$$\delta(t - T_0)x(t - T_1) = \delta(t - T_0)x(T_0 - T_1)$$

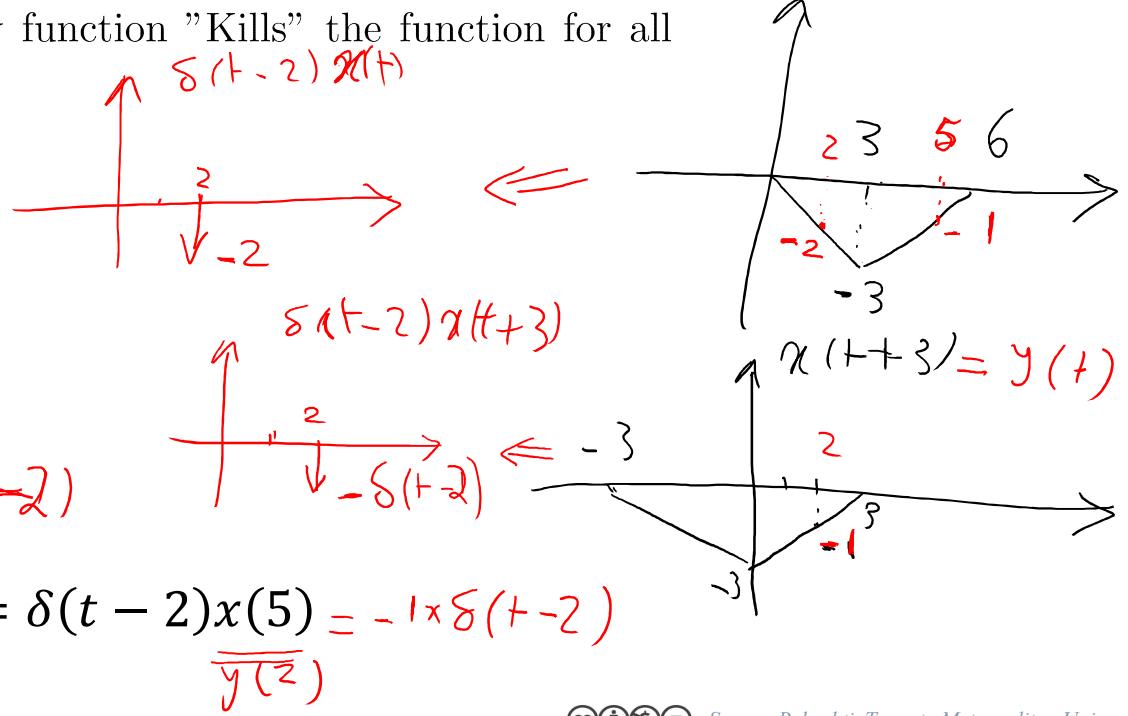


Reminder: Multiplying $\delta(t - T_0)$ by any function "Kills" the function for all values except at T_0

Example:

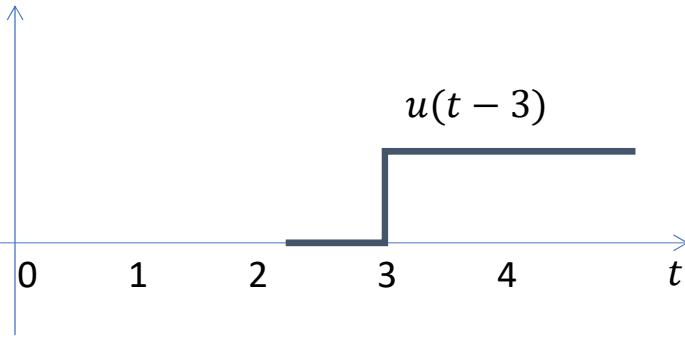
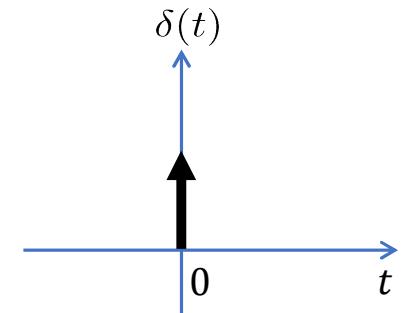
$$\delta(t - 2)x(t) = \delta(t - 2)x(2) = -2\delta(t - 2)$$

$$\delta(t - 2)x(t + 3) = \delta(t - 2)x(2 + 3) = \delta(t - 2)x(5) = -1 \times \delta(t - 2) \overline{y(z)}$$

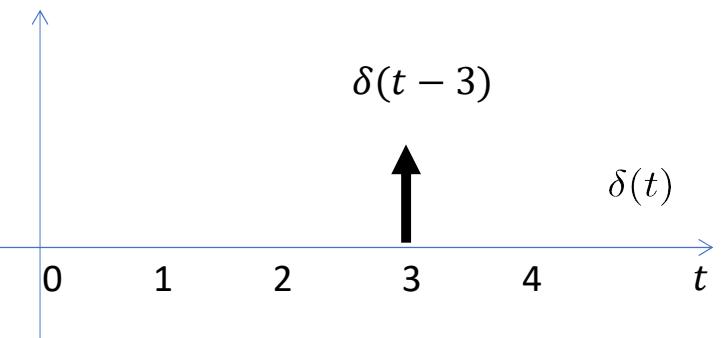


Reminder on $\delta(t)$ and $u(t)$

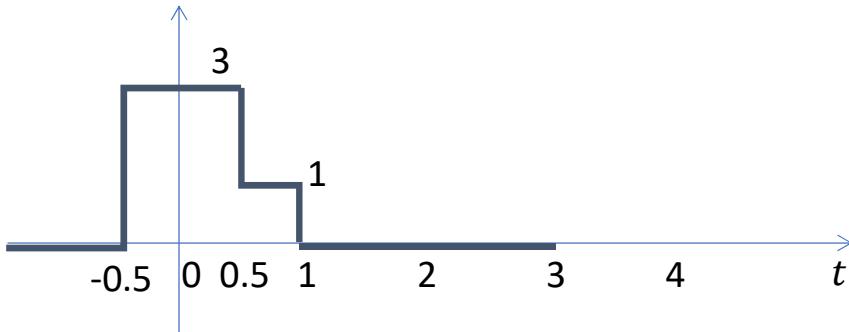
$$\frac{d}{dt}u(t - T) = \delta(t - T)$$



$$\frac{d}{dt}$$

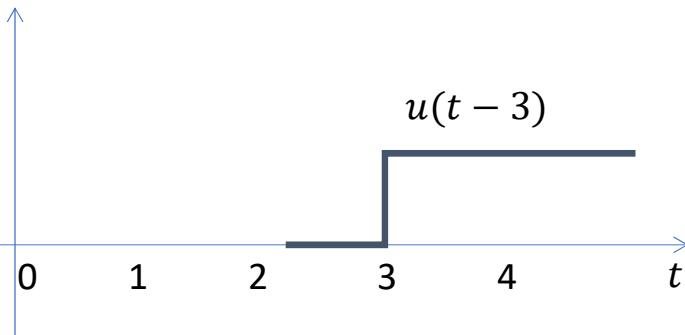
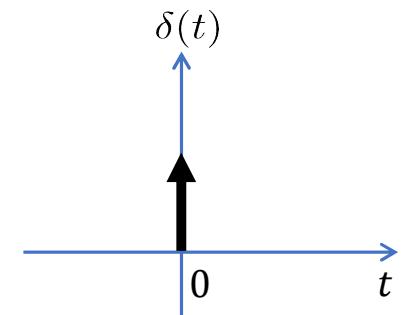


$$\frac{d}{dt}$$

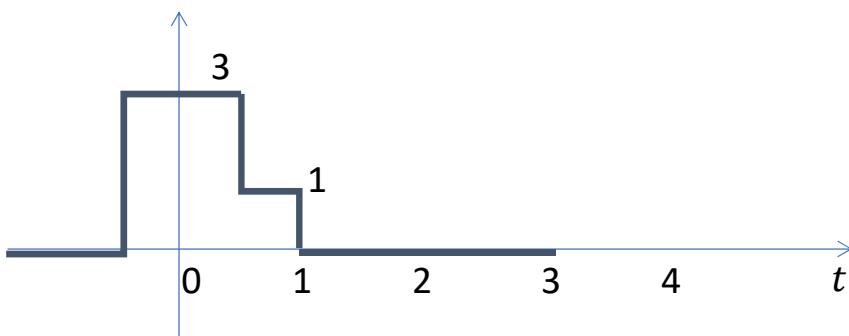
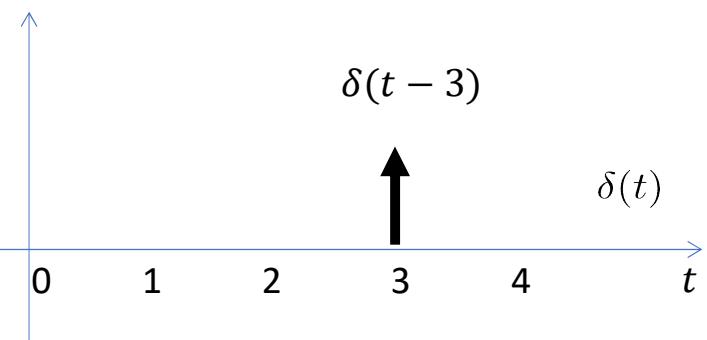


Reminder on $\delta(t)$ and $u(t)$

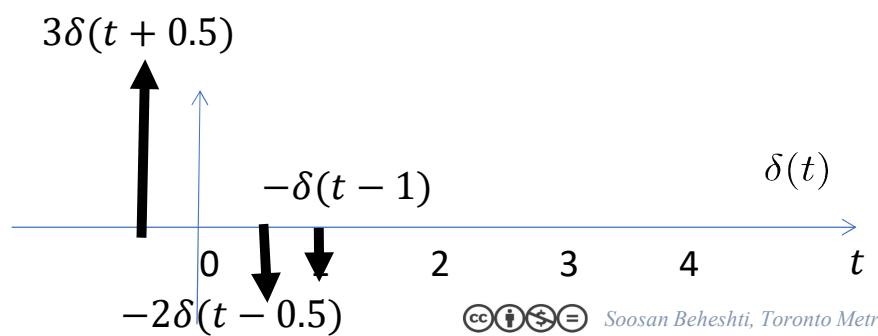
$$\frac{d}{dt}u(t - T) = \delta(t - T)$$



$$\frac{d}{dt}$$



$$\frac{d}{dt}$$



Operations on $u(t)$

$$u(\alpha t) = \begin{cases} u(t), & \text{if } \alpha > 0. \\ u(-t), & \text{if } \alpha < 0. \end{cases}$$

important

$$u(\alpha t - T) = \begin{cases} u\left(t - \frac{T}{\alpha}\right), & \text{if } \alpha > 0. \\ u\left(-t - \frac{T}{|\alpha|}\right), & \text{if } \alpha < 0. \end{cases}$$

Examples:

$$u(7t) = u(t), \quad u(-2.3t) = u(-t)$$

$$u(5t - 10) = u(5(t - 2)) = u(t - 2)$$

$$u(-5t - 10) = u\left(-t - \frac{10}{5}\right) = u(-t - 2)$$

Operations on $u(t)$ & $\delta(t)$ (*wrap up*)

When dealing with $u(t)$ and $\delta(t)$, consider the following two important properties:

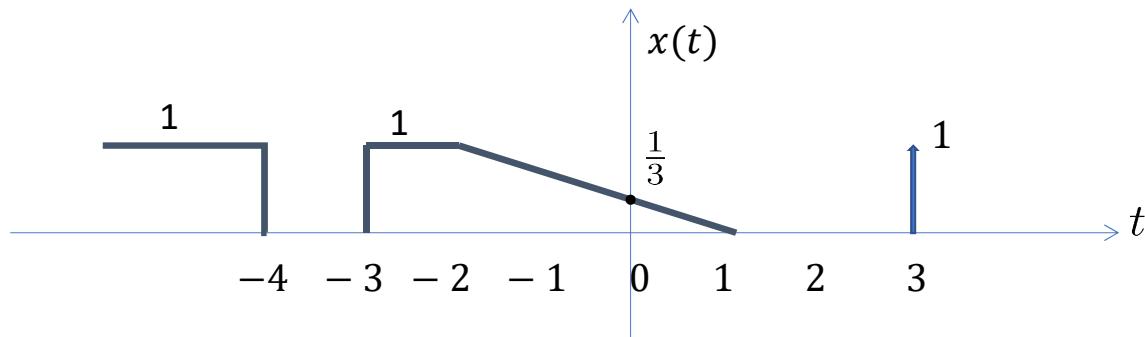
$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t) \quad u(\alpha t) = \begin{cases} u(t), & \text{if } \alpha > 0. \\ u(-t), & \text{if } \alpha < 0. \end{cases}$$

and recall that for any $x(\alpha t - T)$, it is **always** simpler to first take care of the shift by T . Once the shift is completed, use the above equations for $\delta(t)$ and $u(t)$.

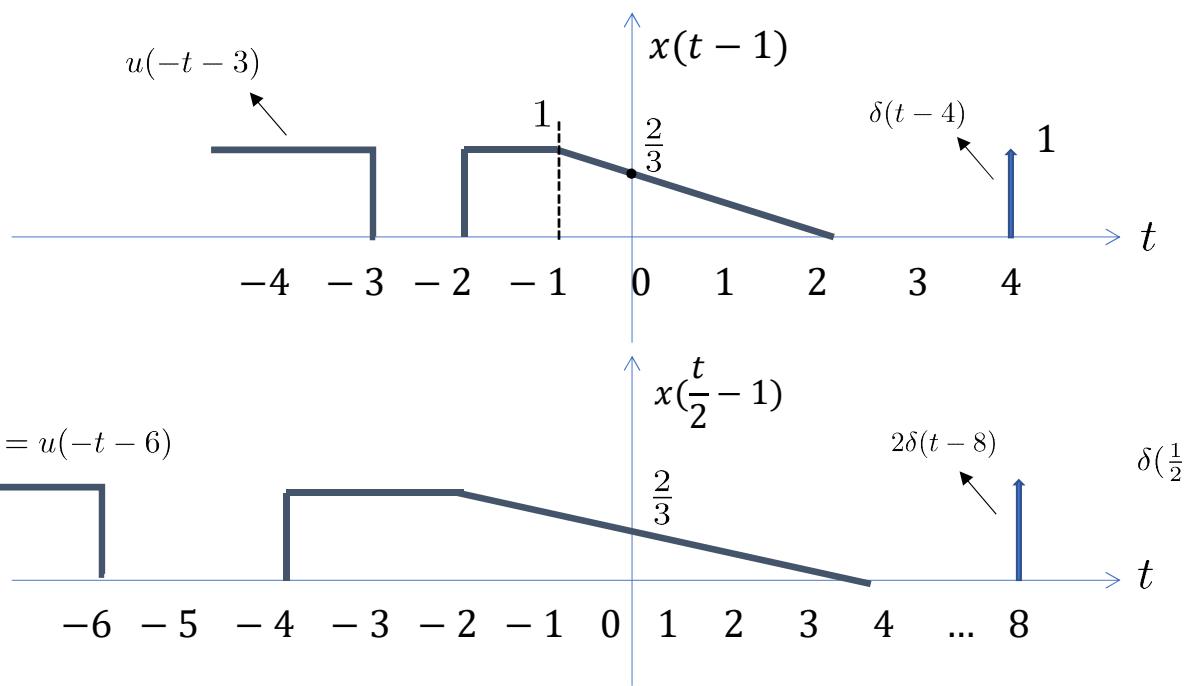
Operations on $u(t)$ & $\delta(t)$

Example:

Considering $x(t)$ as the following signal, find and plot $x(\frac{t}{2} - 1)$

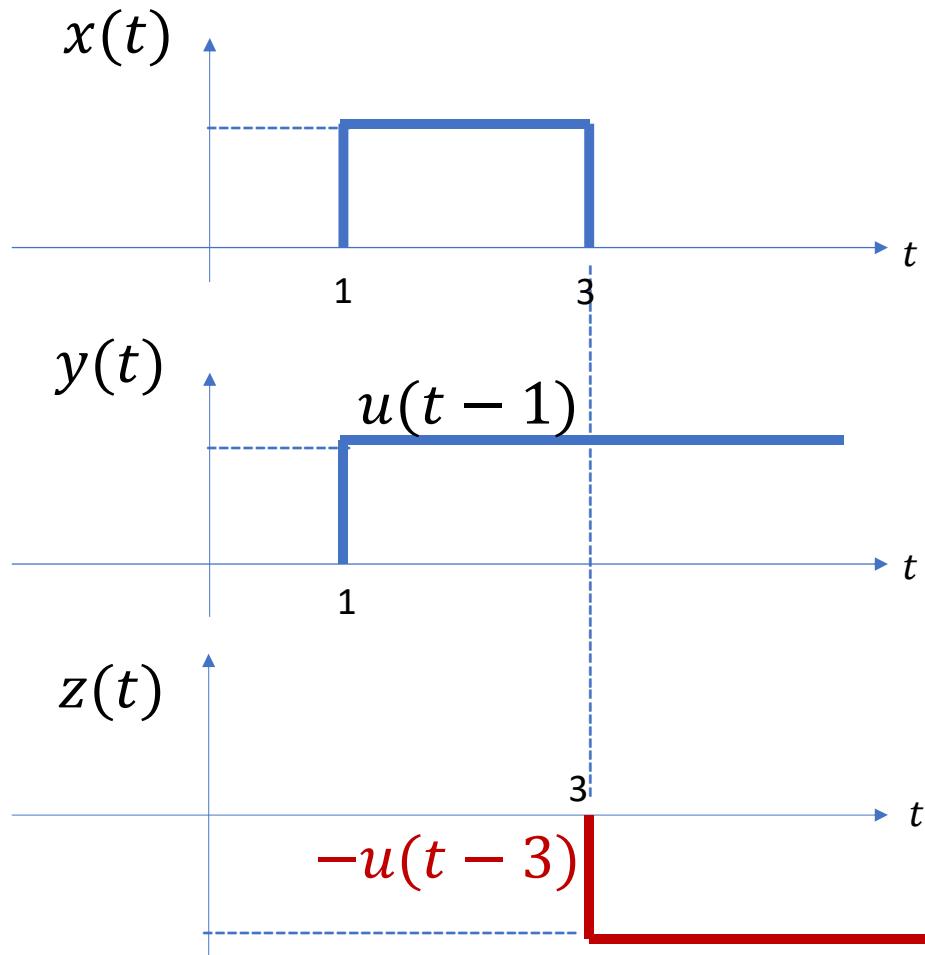


Operations on $u(t)$ & $\delta(t)$



Having compression by $\frac{1}{2}$ on the signal will effect BOTH location and amplitude of the $\delta(t)$

Using $u(t)$ to build segments

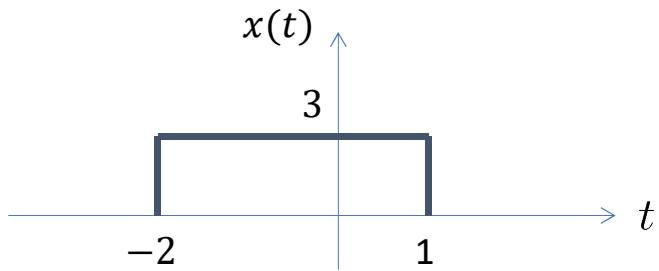


$$x(t) = y(t) + z(t)$$

A simple box can always be built by using $u(t)$.
This ability of $u(t)$ makes it very important specially
in Continues-Digital world, when we build functions
by steps

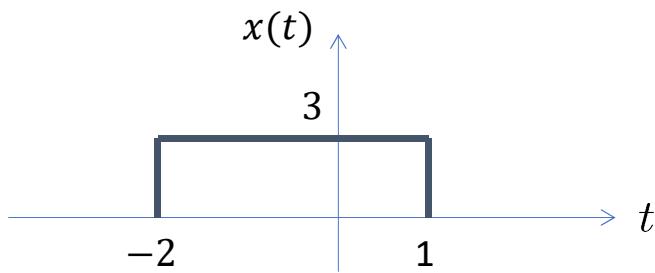
Using $u(t)$ to build segments

- **Example:** Try to build the following signals:

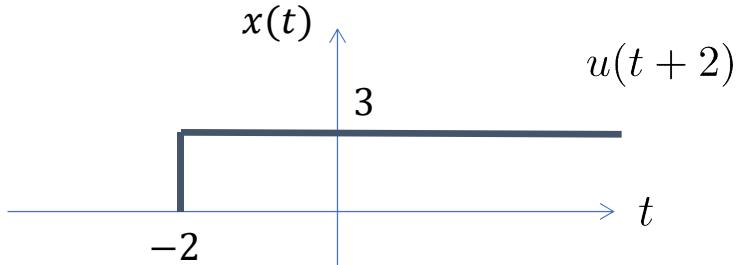


Using $u(t)$ to build segments

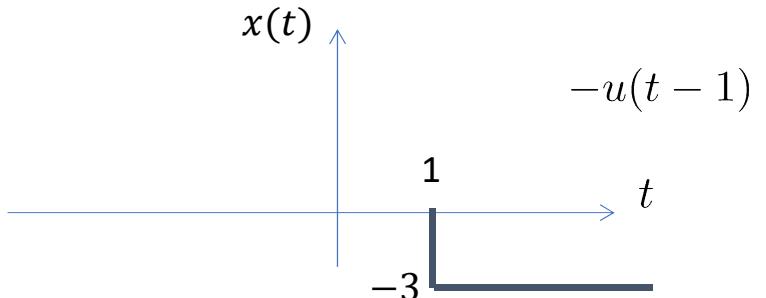
- **Example:** Try to build the following signals:



Answer:

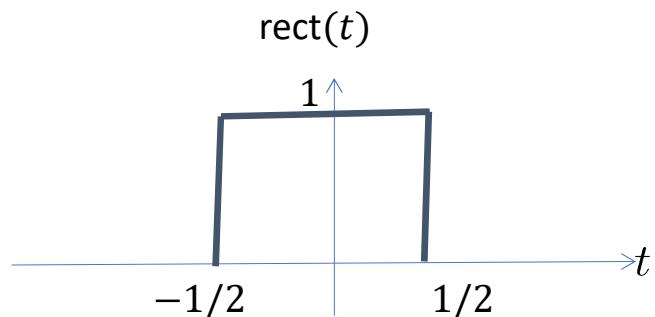


$$x(t) = 3(u(t + 2) - u(t - 1))$$



Using $u(t)$ to build segments

- **Example:** Try to build the following signals:

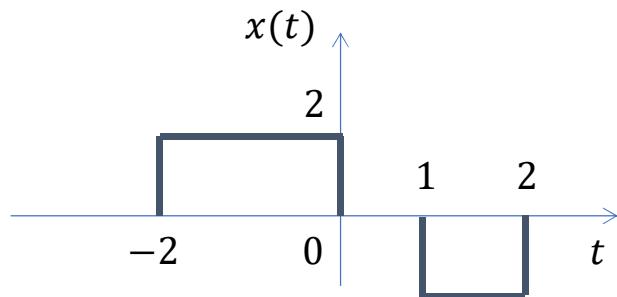


$$\text{rect}(t) = u(t + 1/2) - u(t - 1/2)$$

$$\text{rect}(t/T) = u(t + T/2) - u(t - T/2)$$

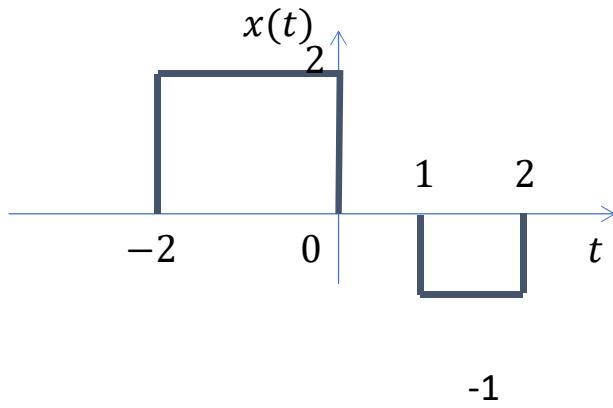
Using $u(t)$ to build segments

- **Example:** Try to build the following signals

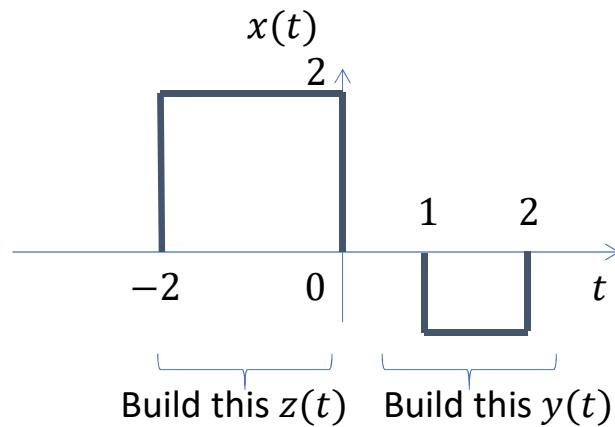


Using $u(t)$ to build segments

- **Example:** Try to build the following signals



Answer:



$$x(t) = z(t) + y(t) .$$

$x(t)$ is upper position of $z(t)$ and $y(t)$

$$x(t) = 2u(t+2) - 2u(t) - u(t-1) + u(t-2)$$

Using $u(t)$ to build segments

- **Example:** Plot the following signals:

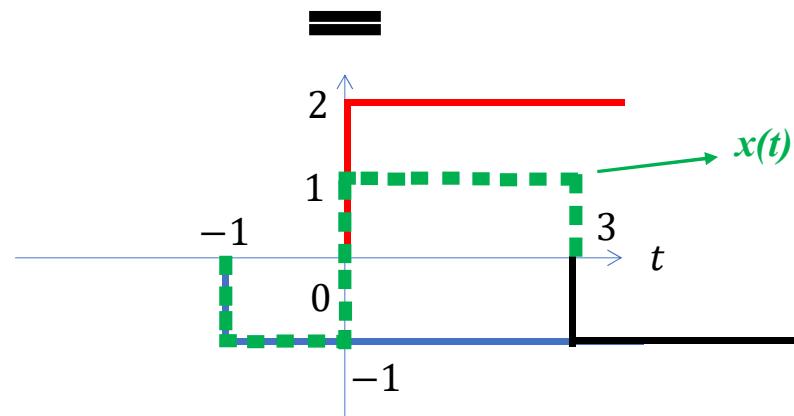
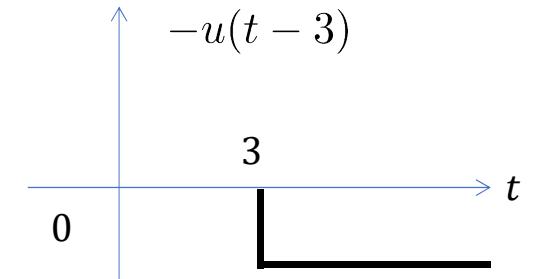
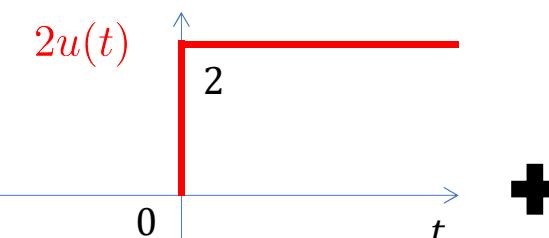
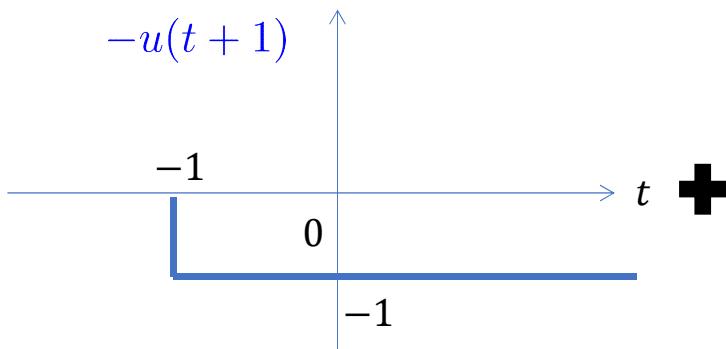
$$1) \ x(t) = -u(t + 1) + 2u(t) - u(t - 3)$$

$$2) \ y(t) = 2u(t) - u(t - 1) - u(t - 2)$$



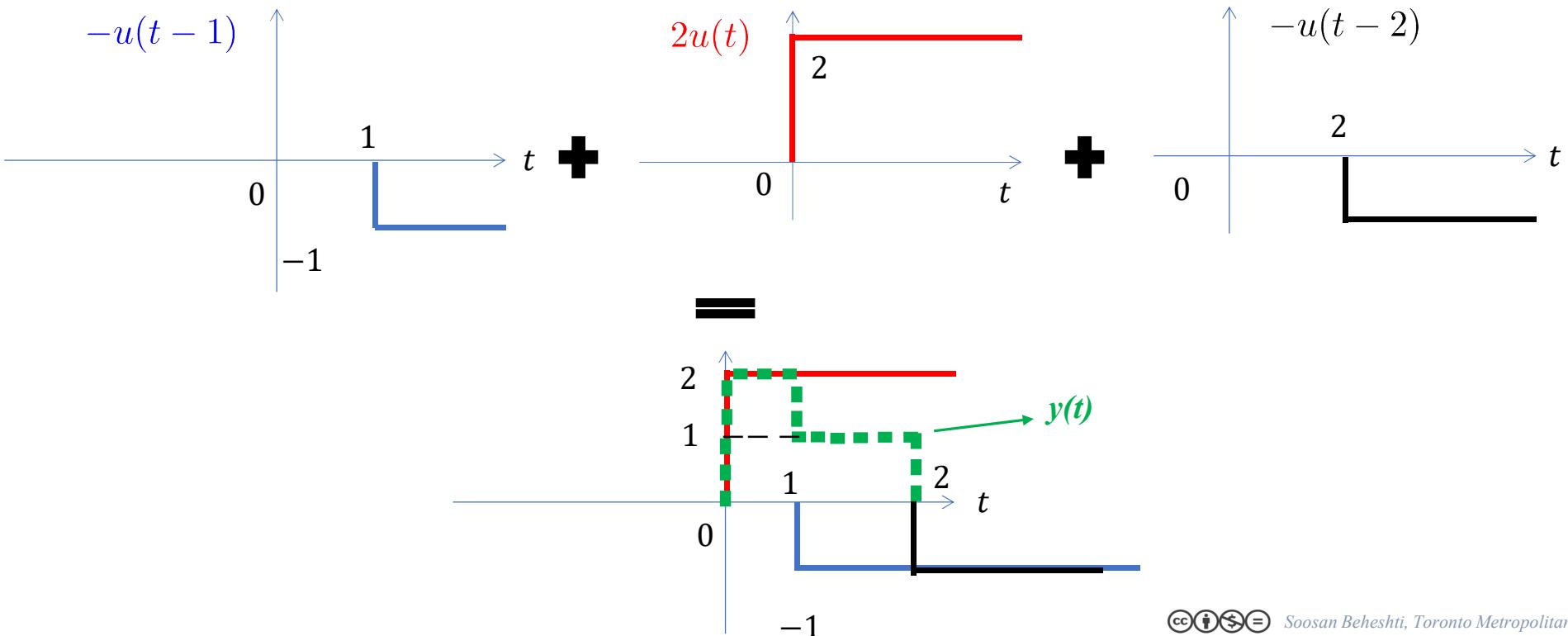
Using $u(t)$ to build segments

$$x(t) = -u(t+1) + 2u(t) - u(t-3)$$



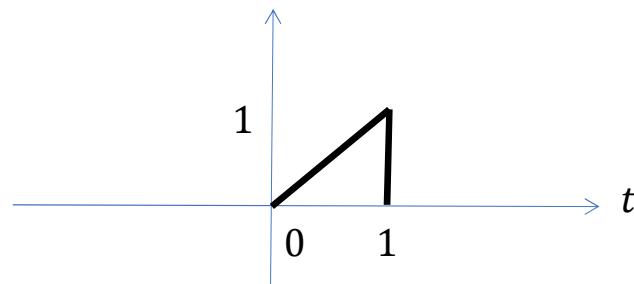
Using $u(t)$ to build segments

$$y(t) = -u(t-1) + 2u(t) - u(t-2)$$



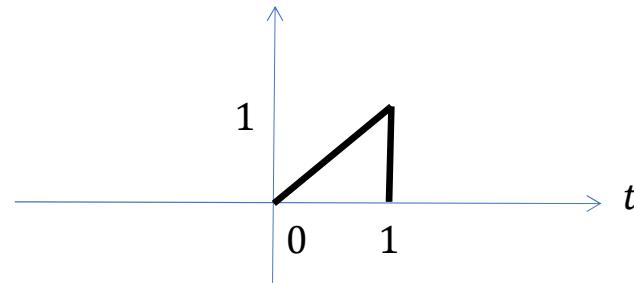
Closed form expressions

What is the closed form expression of the following signal?

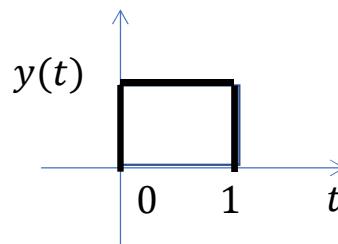


Closed form expressions

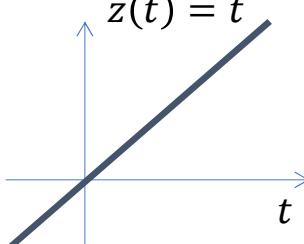
What is the closed form expression of the following signal?



Answer:

$$=$$


A graph of a unit step function $y(t)$ versus t . The vertical axis is labeled $y(t)$. The horizontal axis is labeled t , with marks at 0 and 1. The function is 0 for $t < 0$, jumps to 1 at $t = 0$, remains at 1 until $t = 1$, and then drops back to 0.

$$\times$$


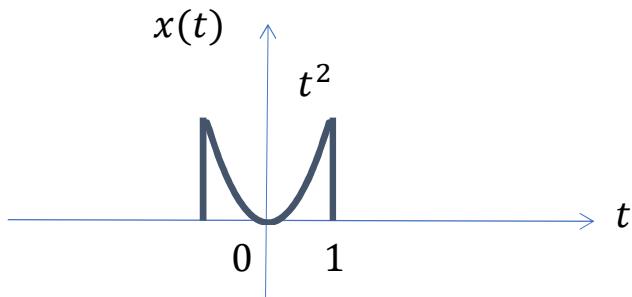
A graph of a linear function $z(t) = t$ versus t . The vertical axis has an upward-pointing arrow. The horizontal axis is labeled t . A straight line passes through the origin (0,0) and extends upwards and to the right.

$$u(t) - u(t - 1)$$

Therefore: $x(t) = (u(t) - u(t - 1))t$

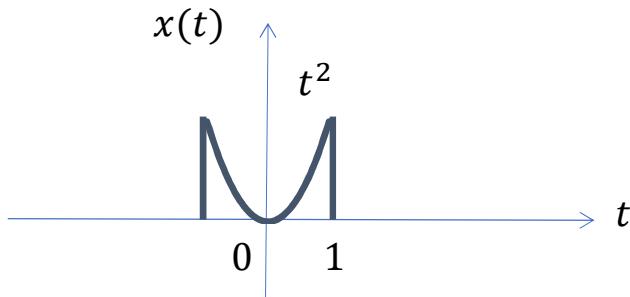
Closed form expressions

Example:

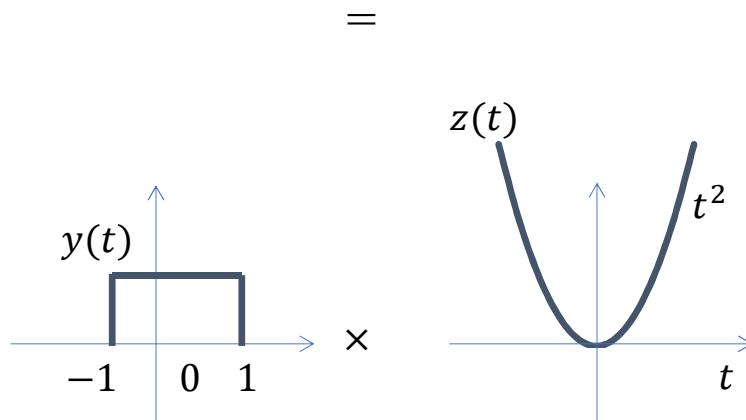


Closed form expressions

Example:



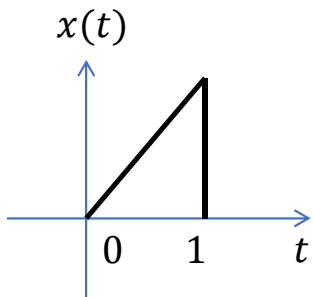
Answer:



$$\text{Therefore: } x(t) = (u(t+1) - u(t-1))t^2$$

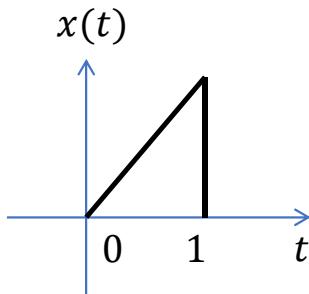
Closed form expressions and combined operations

In the previous example we showed the closed form expression for the following signal:
 $x(t) = (u(t) - u(t - 1)) t$. Find the closed form expression for $x(3t + 1)$ and plot it.

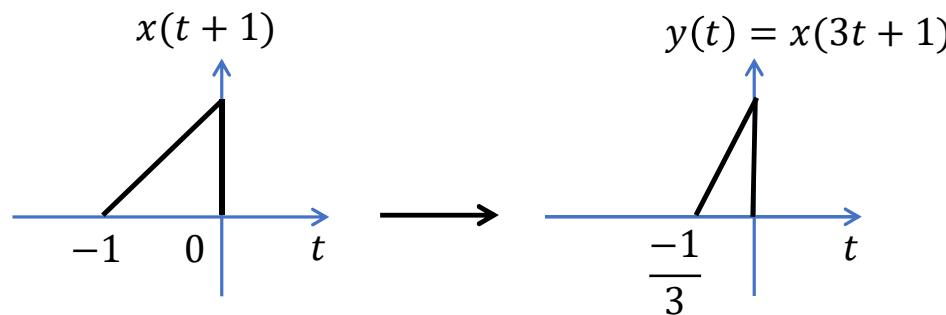


Closed form expressions and combined operations

Consider the following signal $x(t) = (u(t) - u(t - 1))t$.
Find the closed form expression for $x(3t + 1)$ and plot it:



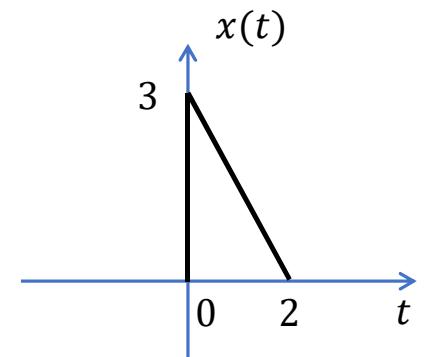
Answer: $(3t + 1)(u(3t + 1) - u(3t + 1 - 1)) = (3t + 1)(u\left(t + \frac{1}{3}\right) - u(t))$



Here you can also verify your answer by direct use of the signal graph. But this is not an easy method for more complex signals

Closed form expressions and combined operations

Example: Consider the following signal $x(t)$, write the closed from expression for $y(t) = x(-\frac{t}{2})$ and plot it



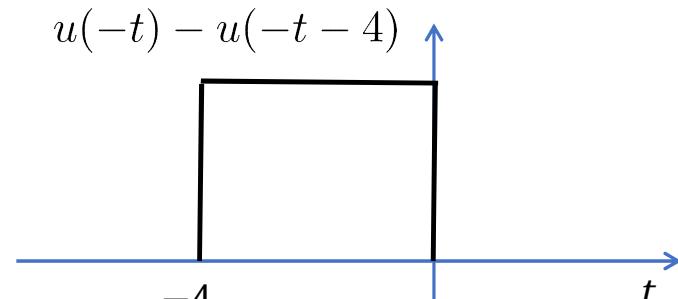
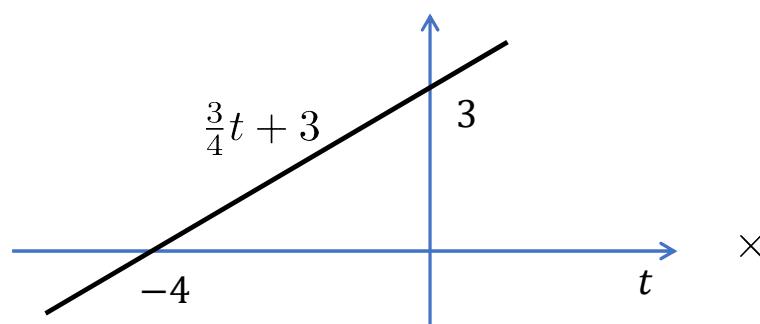
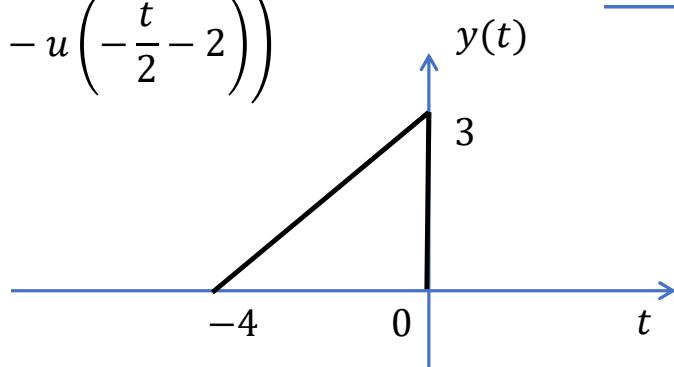
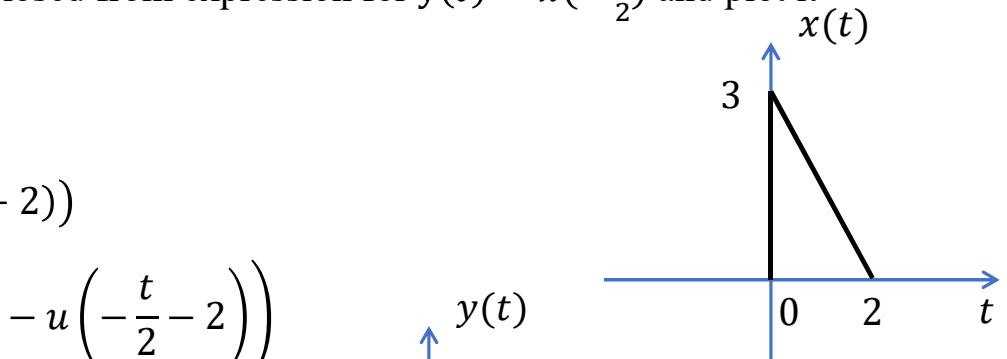
Closed form expressions and combined operations

Example: Consider the following signal $x(t)$, write the closed from expression for $y(t) = x(-\frac{t}{2})$ and plot it

Solution:

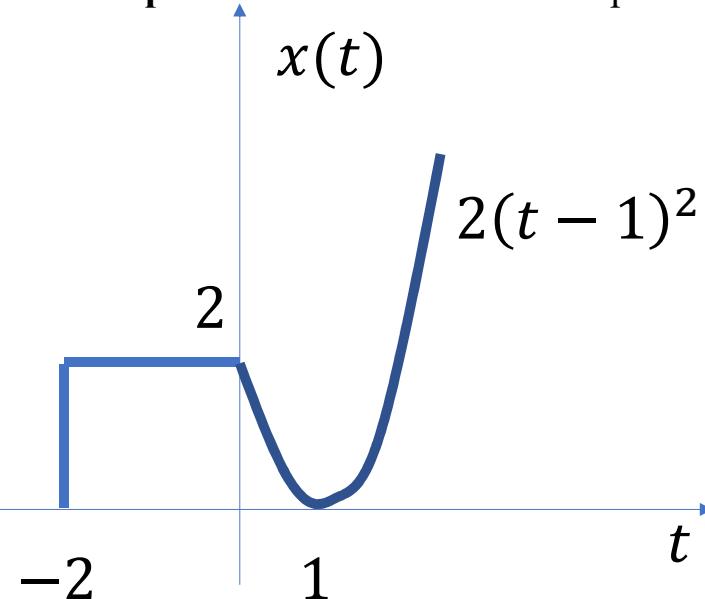
$$\begin{aligned}x(t) &= \left(-\frac{3}{2}t + 3\right)(u(t) - u(t-2)) \\y(t) &= x\left(-\frac{t}{2}\right) = \left(-\frac{3}{2}\left(-\frac{t}{2}\right) + 3\right)\left(u\left(-\frac{t}{2}\right) - u\left(-\frac{t}{2}-2\right)\right) \\&= \left(\frac{3}{4}t + 3\right)(u(-t) - u(-t-4))\end{aligned}$$

$$u\left(-\frac{t}{2}-2\right) = u(-t-2/|1/2|)$$



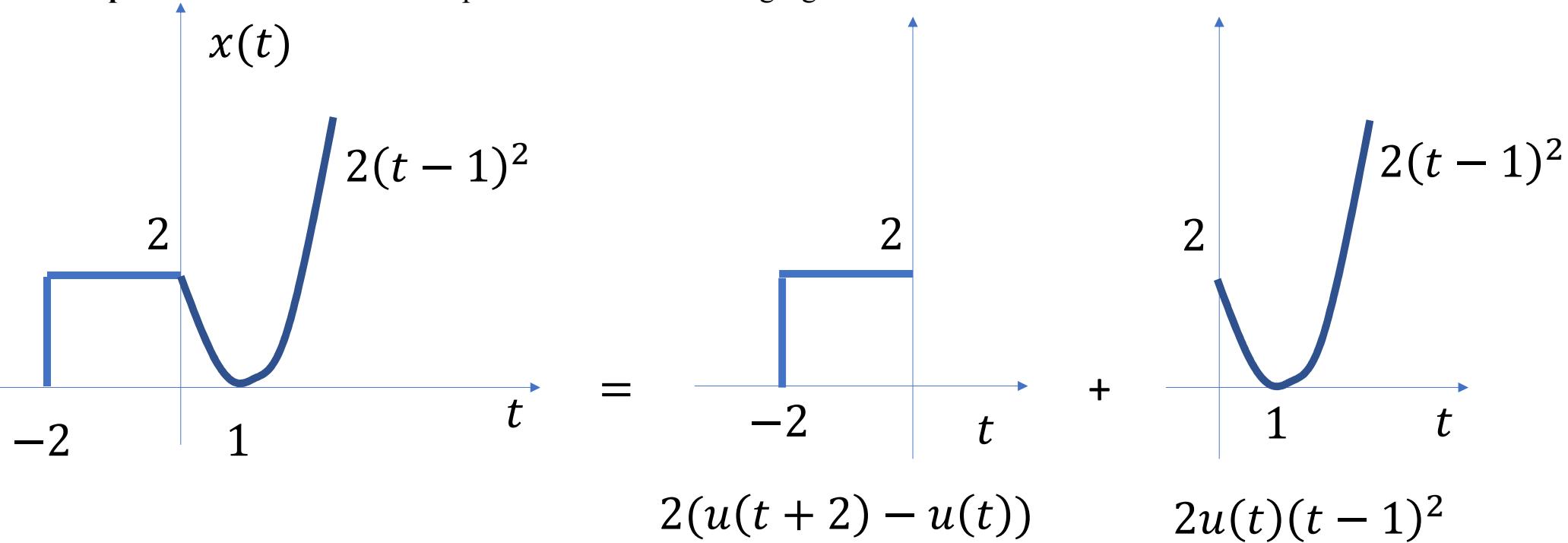
Closed form expressions and combined operations

Example: Write the closed form expression for the following signal



Closed form expressions and combined operations

Example: Write the closed form expression for the following signal

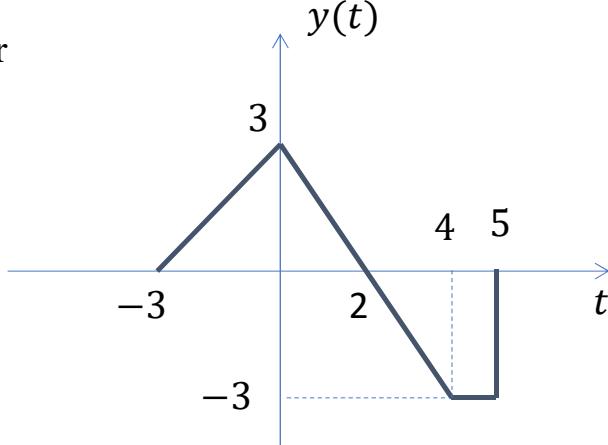


Through Super Position:

$$x(t) = 2(u(t + 2) - u(t)) + 2u(t)(t - 1)^2$$

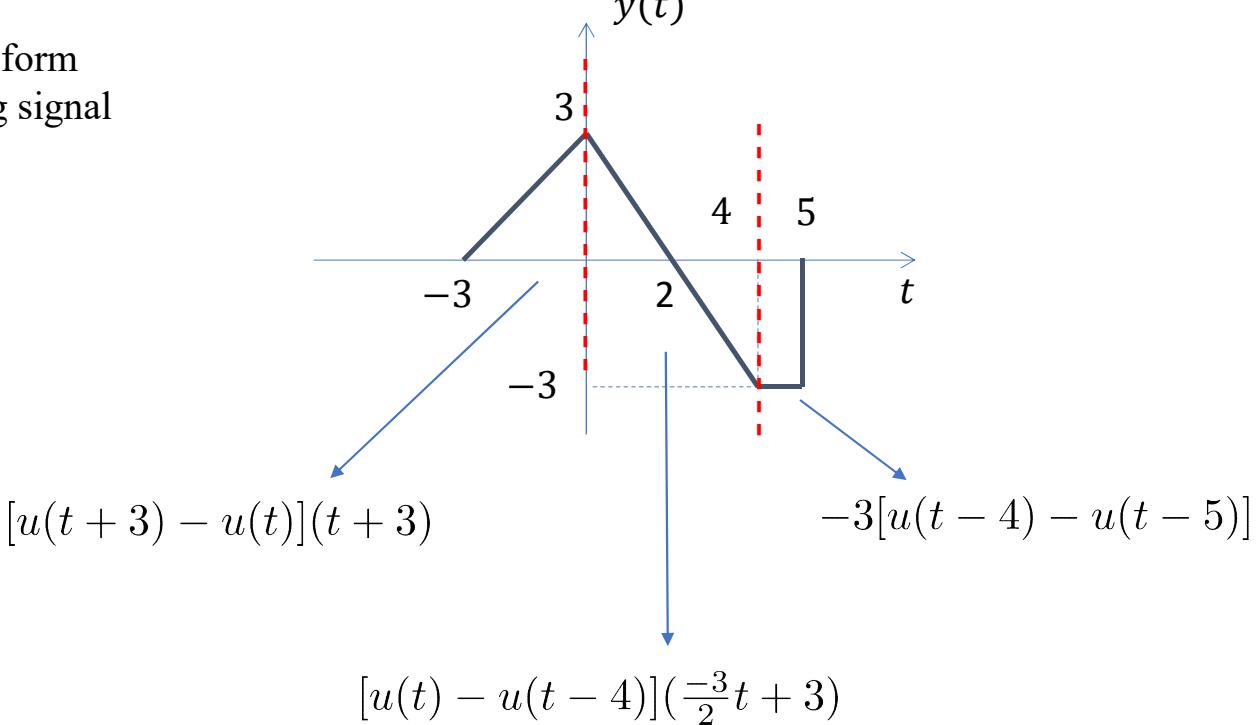
Closed form expressions and combined operations

Example: Write the closed form expression for the following signal



Closed form expressions and combined operations

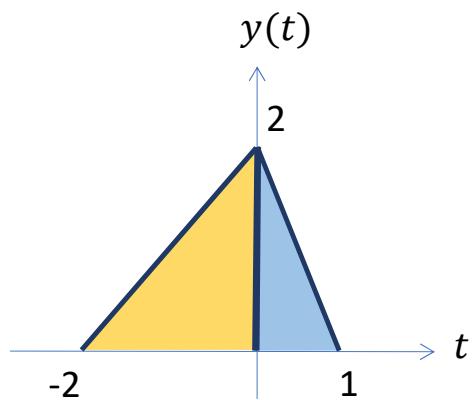
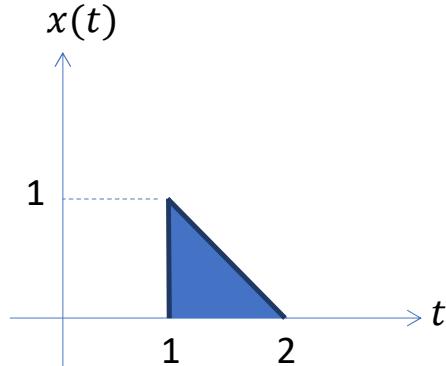
Example: Write the closed form expression for the following signal



$$y(t) = [-u(t) + u(t+3)](t+3) + [u(t) - u(t-4)]\left(\frac{-3}{2}t + 3\right) - 3[u(t-4) - u(t-5)]$$

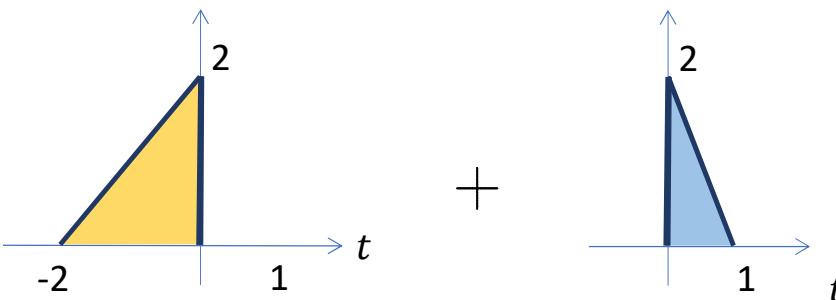
Example of building signals from signals

- Write $y(t)$ as a function of time shifted, time scaled, and amplitude scaled of $x(t)$

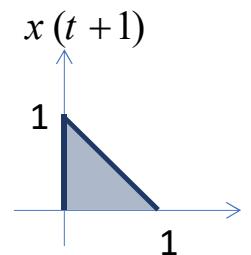
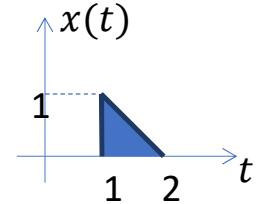
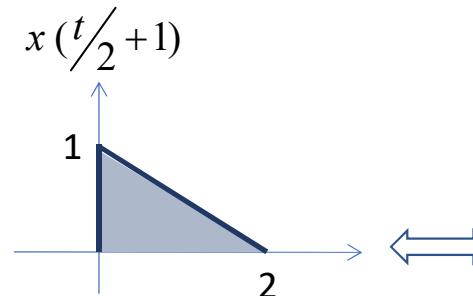
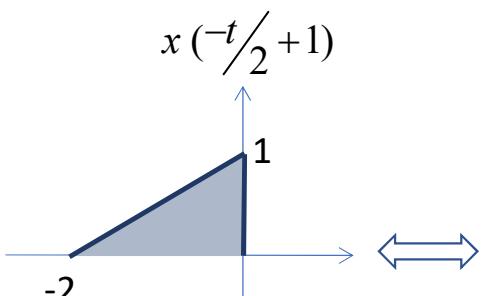
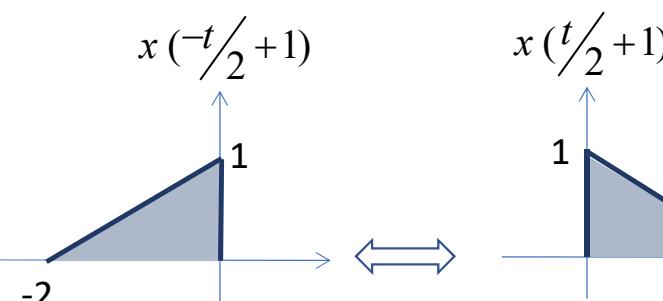
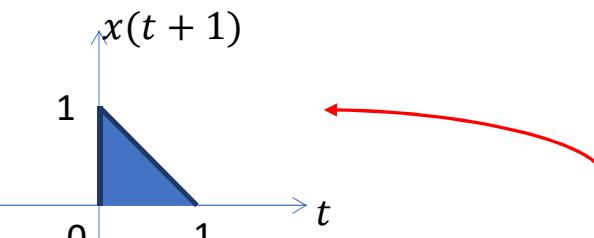
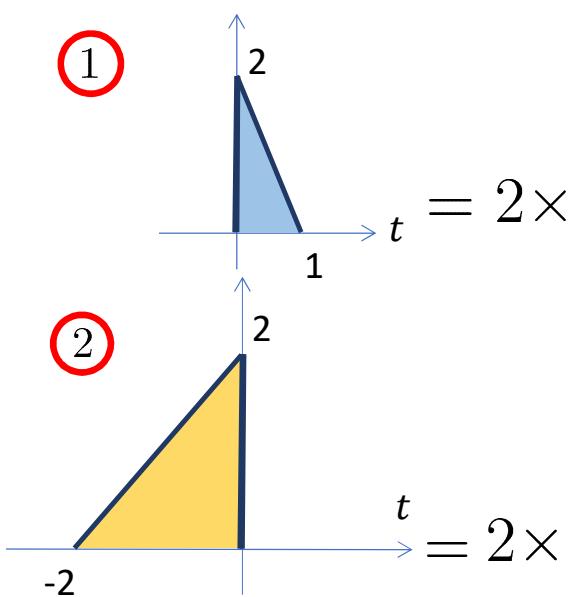


Answer:

$$y(t) =$$



These terms are just shifted on x-axis and stretched on y-axis versions of $x(t)$



$$\therefore y(t) = 2x(t+1) + 2x\left(\frac{-t}{2} + 1\right)$$