Signals and Systems I

Lecture 9

Last Lecture: Fourier Series

Fourier Series

- $e^{j\omega t}$
- $\sin(\omega t + \theta)$
- $\cos(\omega t + \theta)$
- Periodic pulse in time → Fourier Series in the form of Sinc

Today

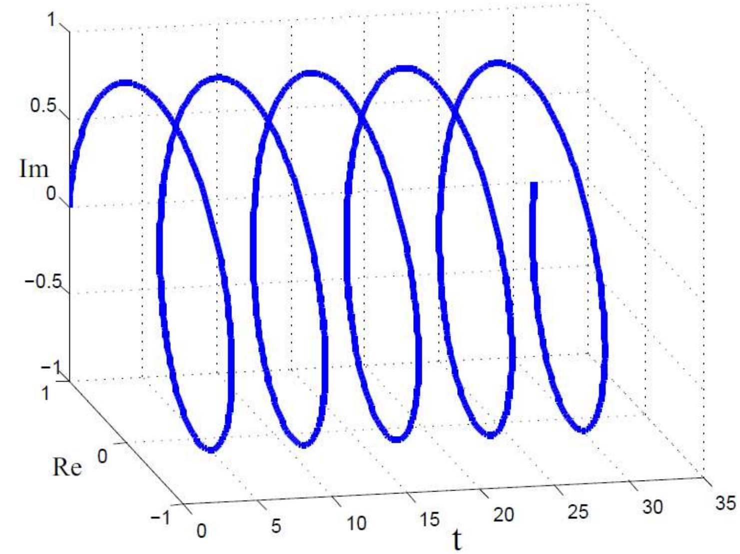
- Trigonometric Fourier Series & Exponential Fourier Series
- LTI Systems, Eigen Function of LTI System (e^{st})
- LTI Systems & Fourier Series
- More Examples
- Some properties of FS

Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \qquad \text{(Fourier series)} \Longleftarrow \textit{Synthesis}$$

$$D_n = \frac{1}{T_0}\int_{T_0} x(t)e^{-j\omega_0 nt}dt \qquad \text{(Finds } D_n \text{ from } x(t)) \Longleftarrow \textit{Analysis}$$

$$D_n e^{jn\omega_0 t}$$



Componencts of Fourier Series are periodic spirals in form of $e^{jn\omega_0 t}$ which is a periodic spiral with frequency $n\omega_0$.

Each spiral is then rotated by angle of D_n and amplified by $|D_n|$

$x_1(t) = je^{-j5t}$

$\omega_0 = -5, \;\; |D_{-1}| = 1, \; \angle(D_{-1}) = \frac{\pi}{2}$

$x_2(t) = 2e^{j\frac{\pi}{3}}e^{j2\pi t}$

$\omega_0 = 2\pi, \;\; |D_1| = 2, \; \angle(D_1) = \frac{\pi}{3}$

$x_3(t) = -2e^{j\frac{5}{6}\pi t}$

$\omega_0 = \frac{5}{6}\pi, \;\; |D_1| = 2, \; \angle(D_1) = \pi$

Fourier Series

Euler Formulas:

$$\cos(\alpha t+\beta) = \frac{e^{j(\alpha t+\beta)} + e^{-j(\alpha t+\beta)}}{2}$$

$$\sin(\alpha t+\beta) = \frac{e^{j(\alpha t+\beta)} - e^{-j(\alpha t+\beta)}}{2j}$$

Associated FS coefficients: $\omega_0 = \alpha$

$$\cos(\alpha t+\beta) = \underbrace{\frac{1}{2}e^{j\beta}}_{D_1} e^{j(\alpha t)} + \underbrace{\frac{1}{2}e^{-j\beta}}_{D_{-1}} e^{j(-\alpha t)}$$

$$\sin(\alpha t+\beta) = \underbrace{\frac{1}{j2}e^{j\beta}}_{D_1} e^{j(\alpha t)} + \underbrace{\frac{-1}{2j}e^{-j\beta}}_{D_{-1}} e^{j(-\alpha t)}$$

$\rightarrow$ One Spiral = One Coefficient

$\rightarrow$ One Sine/Cosine = Two Coefficients (one positive and one negative: $D_1 \& D_{-1}$

Trigonometric Fourier Series and Exponential Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 nt} \qquad \text{(Exponential Fourier series)}$$

$$x(t) = D_0 + \underbrace{D_1 e^{j\omega_0 t} + D_{-1} e^{-j\omega_0 t}} + \underbrace{D_2 e^{j2\omega_0 t} + D_{-2} e^{-j2\omega_0 t}} + \underbrace{D_3 e^{j3\omega_0 t} + D_{-3} e^{-j3\omega_0 t}} + \cdots$$

$$x(t) = a_0 + \overbrace{a_1 \cos(\omega_0 t) + b_1 \sin(\omega_0 t)} + \overbrace{a_2 \cos(2\omega_0 t) + b_2 \sin(2\omega_0 t)} + \overbrace{a_3 \cos(3\omega_0 t) + b_3 \sin(3\omega_0 t)}$$

Euler formula

$$\overbrace{a_1\left(\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right) + b_1\left(\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right)}$$

$$\overbrace{\left(\frac{a_1}{2} + \frac{b_1}{2j}\right)e^{j\omega_0 t} + \left(\frac{a_1}{2} - \frac{b_1}{2j}\right)e^{-j\omega_0 t}}$$

$$\text{for} \quad n > 0 \quad \begin{cases} D_0 = a_0 \\ D_n = \frac{a_n}{2} + \frac{b_n}{2j} \\ D_{-n} = \frac{a_n}{2} - \frac{b_n}{2j} \end{cases} \overset{\text{or}}{\Longrightarrow} \begin{cases} D_n + D_{-n} = a_n \\ D_n - D_{-n} = \frac{b_n}{j} \end{cases}$$

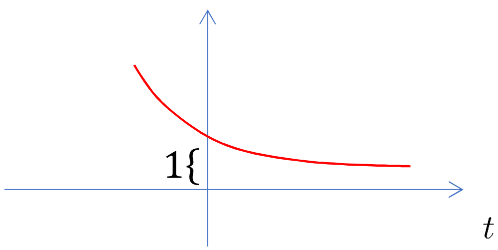
Simple Example:

$$x(t) = cos(\omega_0 t)$$

Trigonometric FS: $a_0 = 0, a_1 = 1, a_2 = 0, ..., b_n = 0, \quad \forall n$

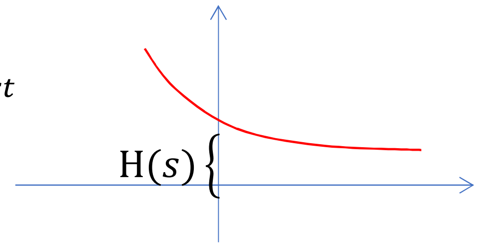
Exponential FS: $D_0 = 0, D_1 = \dfrac{1}{2}, D_{-1} = \dfrac{1}{2}$

Exponentials e^{st} and LTI Systems



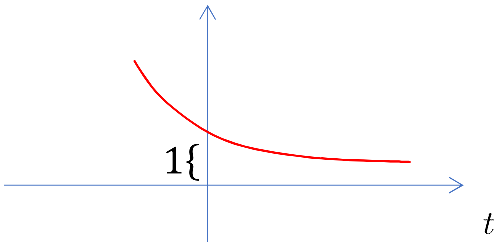
$s = \alpha < 0$

$$x(t) = e^{st} \longrightarrow \boxed{\begin{matrix}\text{LTI system} \\ h(t)\end{matrix}} \longrightarrow \underbrace{y(t) = H(s)}_{number}\, e^{st}$$



$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

Exponentials e^{st} and LTI Systems



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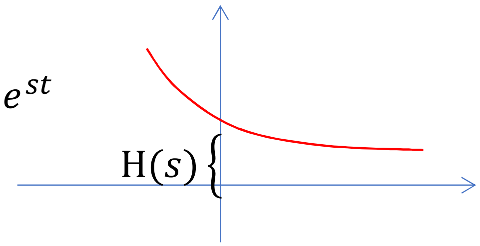


$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

matrix eigenvalue & eigenvector:

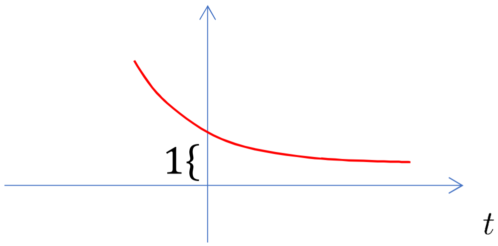
$$Av = \lambda v$$

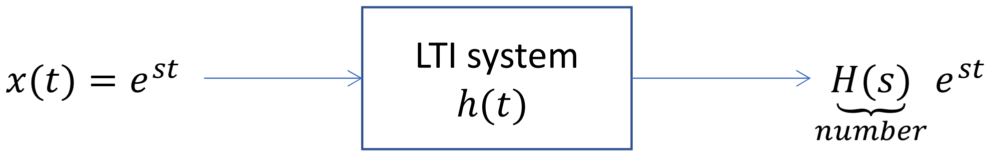
matrix A

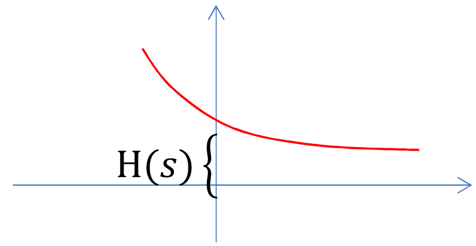
eigenvector

eigenvalue

Exponentials e^{st} and LTI Systems







$s = \alpha < 0$

$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$

Why?

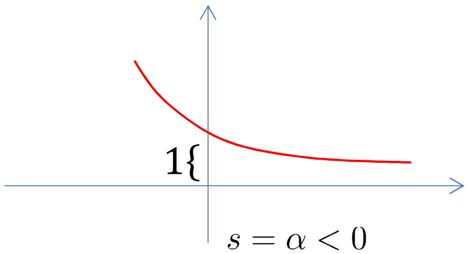
$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= e^{st} * h(t) \\ &= \int_{-\infty}^{\infty} x(t-\tau) h(\tau) d\tau \\ &= \int_{-\infty}^{\infty} e^{s(t-\tau)} h(\tau) d\tau \\ &= e^{st} \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau \\ &= e^{st} H(s) \end{aligned}$$

e^{st} is eigenfunction of LTI systems.

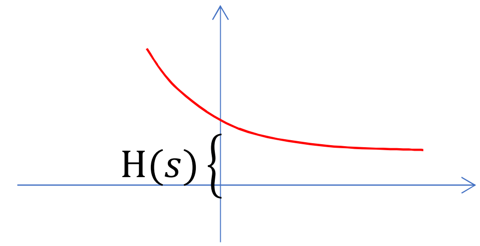
and $H(s)$ is its eigenvalue.

Reminder of eigenvalue and eigenvector for Matrix A transformation: $y = Ax$ where input is vector x and output is vector y. Eigenvalue is the a v with eigenvalue λ such that $\lambda v = Av$

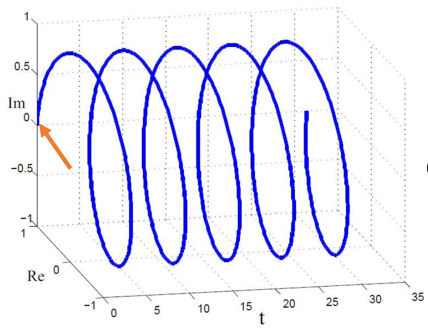
Exponentials e^{st} and LTI Systems



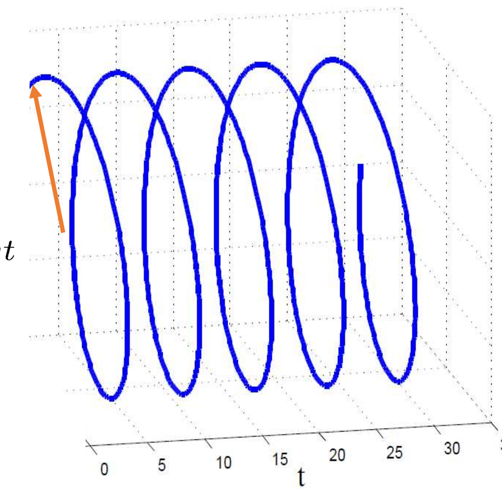
$$x(t) = e^{st} \longrightarrow \boxed{\begin{matrix}\text{LTI system}\\ h(t)\end{matrix}} \longrightarrow \underbrace{H(s)}_{number}\ e^{st}$$

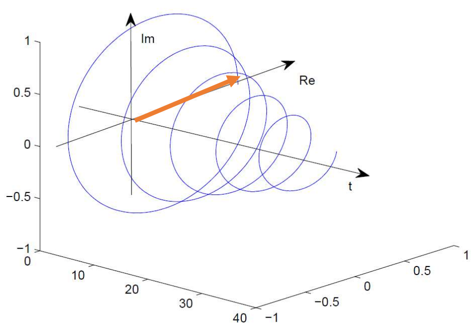


$$H(s) = \int_{-\infty}^{\infty} e^{-s\tau} h(\tau) d\tau$$



$e^{j\omega t}$

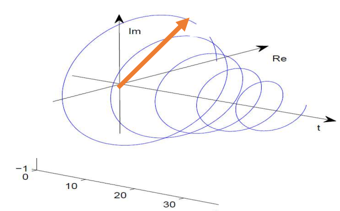




e^{st}

$s = -2 + j5$

$H(s)e^{st}$



Periodic Signals and LTI Systems

Consider the case for $s = j\omega$ (Pure Imaginary)

$$x(t) = e^{j\omega t} \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ h(t)\end{array}} \longrightarrow \underbrace{H(j\omega)}_{number}\, e^{j\omega t}$$

$$H(j\omega) = \int e^{-j\omega\tau} h(\tau) d\tau$$

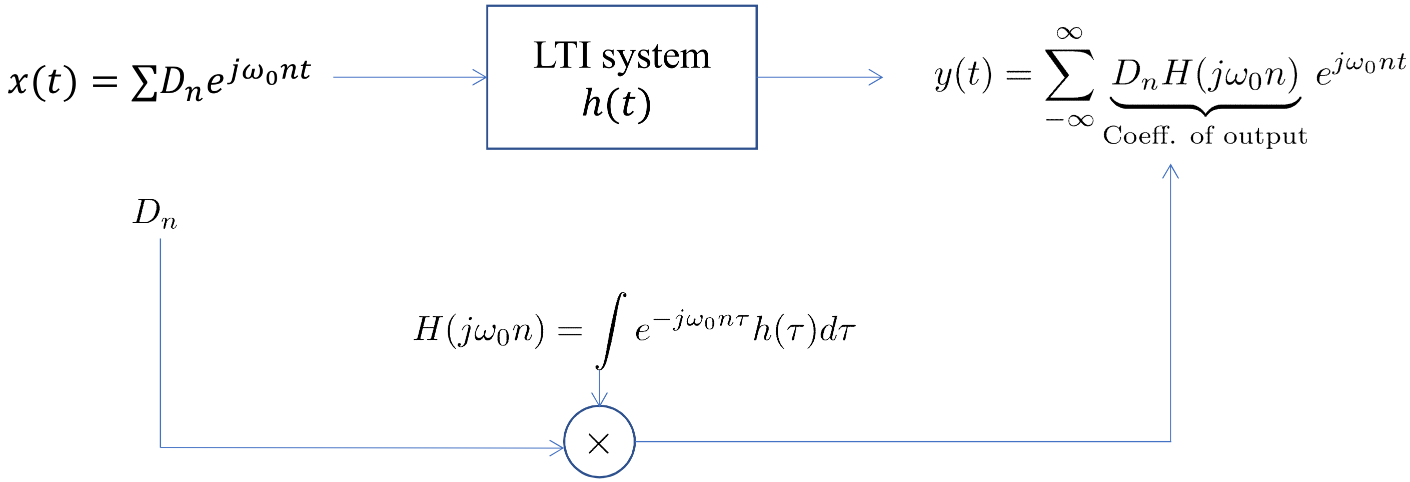
This property can be used to find the output of LTI systems to periodic inputs.
If $x(t)$ is periodic: $x(t) = \sum_{-\infty}^{\infty} D_n e^{j\omega_0 n t}$

$$x(t) = \sum D_n e^{j\omega_0 n t} \longrightarrow \boxed{\begin{array}{c}\text{LTI system}\\ h(t)\end{array}} \longrightarrow y(t) = ?$$

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= \left(\sum_{-\infty}^{\infty} D_n e^{j\omega_0 n t}\right) * h(t) \\ &= \sum_{-\infty}^{\infty} D_n (e^{j\omega_0 n t} * h(t)) \\ &= \sum_{-\infty}^{\infty} \underbrace{D_n H(j\omega_0 n)}_{\text{Coeff. of output}}\, e^{j\omega_0 n t} \end{aligned}$$

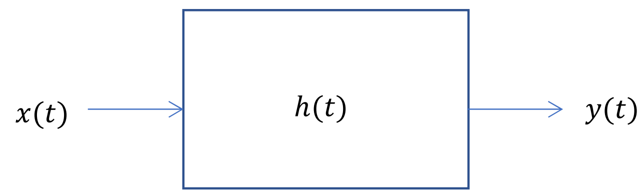
Convolution is transformed to
"product of eigenvalues and Fourier Series Coefficients!"

Periodic Signals and LTI Systems



It is faster to find the Convolution results using $H(j\omega_0 n)$s and Fourier Series Coefficients

Periodic Signals and LTI Systems



Output is also periodic with coefficients $D_n H(j\omega_0 n)$.

To Find output of an LTI system to a periodic input $x(t)$:

1- Find D_n coefficients of Fourier series of $x(t)$.

$$x(t) = \sum D_n e^{j\omega_0 nt}$$

2- For ω_0, $2\omega_0$, $3\omega_0, \ldots,$ $n\omega_0$ (where ω_0 is the fundamental freq. of $x(t)$) find $H(j\omega_0 n)$:

$$H(j\omega_0 n) = \int_{-\infty}^{\infty} e^{-j\omega_0 n\tau} h(\tau) d\tau$$

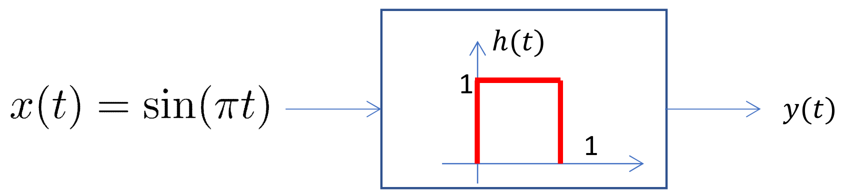
3- Coefficients of $y(t)$ (output of the system to $x(t)$) are $D_n H(j\omega_0 n)$ and the output is as follows:

$$y(t) = \sum D_n H(j\omega_0 n) e^{j\omega_0 nt}$$

Output of LTI system to periodic signal has the <u>same</u> period

Periodic Signals and LTI Systems

Example: Find the output of the following LTI system to $x(t) = \sin(\pi t)$?



Periodic Signals and LTI Systems

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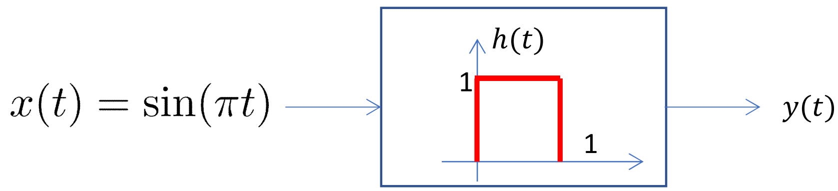

To Find output of an LTI system to a periodic input $x(t)$:

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$$x(t) = \sum D_n e^{j\omega_0 n t}$$

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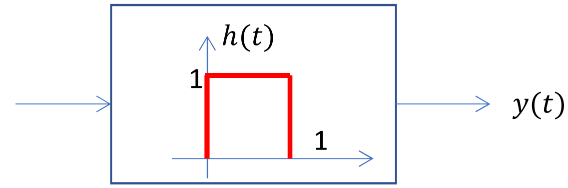
Periodic Signals and LTI Systems

$$x(t) = \sin(\pi t)$$



1- $x(t)$ is periodic $(\omega_0 = \pi)$

$$x(t) = sin(\pi t) = \underbrace{\frac{1}{2j}}_{D_1^x} e^{j\pi t} - \underbrace{\frac{1}{2j}}_{D_{-1}^x} e^{-j\pi t}$$

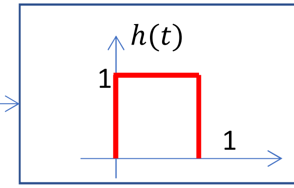
2- Since only D_1^x & D_{-1}^x are non-zero, we have to find $H(j\omega_0)$ & $H(-j\omega_0)$:

$$\begin{aligned} H(j\omega_0) = H(j\pi) &= \int_{-\infty}^{\infty} e^{-j\pi\tau} h(\tau) d\tau \\ &= \int_0^1 e^{-j\pi\tau} 1 d\tau \\ &= \left.\frac{e^{-j\pi\tau}}{-j\pi}\right|_0^1 = \frac{1 - e^{-j\pi}}{j\pi} = \frac{2}{j\pi} \end{aligned}$$

$$\begin{aligned} H(-j\pi) &= \int_0^1 e^{j\pi\tau} 1 d\tau \\ &= \frac{e^{j\pi} - 1}{j\pi} = \frac{-2}{j\pi} \end{aligned}$$

Periodic Signals and LTI Systems

$x(t) = \sin(\pi t)$



$y(t)$

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3- Coefficients of $y(t)$ are:

$$D_1^y = D_1^x.H(j\pi) = \frac{1}{2j}.\frac{2}{j\pi} = \frac{-1}{\pi}$$

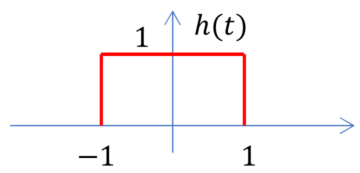
$$D_{-1}^y = D_{-1}^x.H(-j\pi) = \frac{-1}{2j}.\frac{-2}{j\pi} = \frac{-1}{\pi}$$

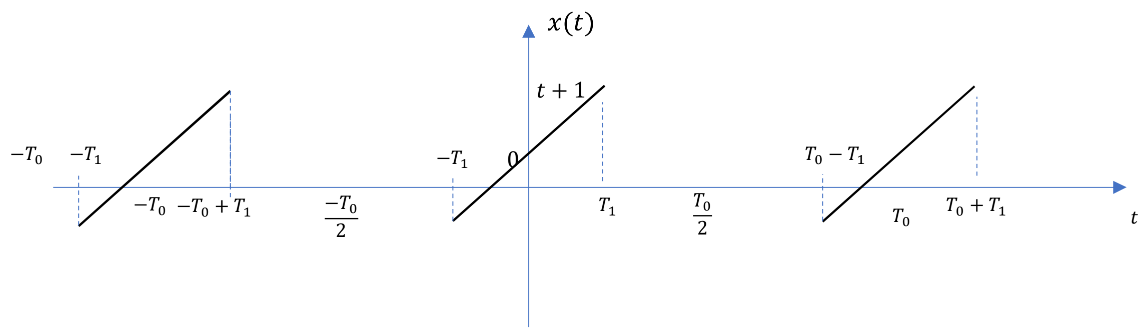
$$\begin{aligned} y(t) &= \sum D_n^y e^{j\omega_0 nt}, \quad (\omega_0 = \pi) \\ &= D_1^y e^{j\pi t} + D_{-1}^y e^{-j\pi t} \\ &= \frac{-1}{\pi}\left(e^{j\pi t} + e^{-j\pi t}\right) \\ &= \frac{-2}{\pi} cos(\pi t) \end{aligned}$$

Output of LTI system to periodic signal has the same period

Periodic Signals and LTI Systems

Example: Find Fourier series of $x(t)$ and use it to find $y(t) = x(t) * h(t)$ where $h(t) = u(t+1) - u(t-1)$. Solve for when $T_1 = 2$ and $T_0 = 4$.





Periodic Signals and LTI Systems

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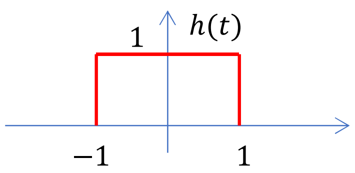
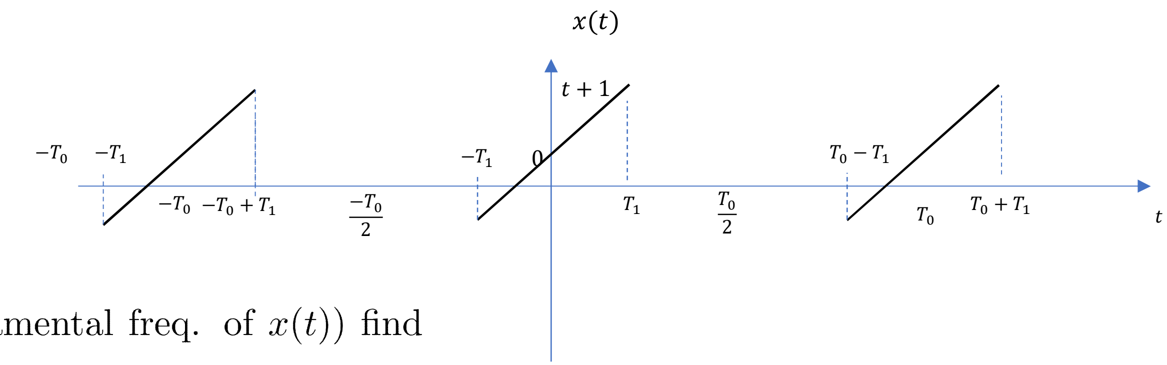




1- Find D_n coefficients of Fourier series of $x(t)$.

$$x(t) = \sum D_n e^{j\omega_0 n t}$$

2- For $\omega_0,\ 2\omega_0,\ 3\omega_0, \ldots,\ n\omega_0$ (where ω_0 is the fundamental freq. of $x(t)$) find $H(j\omega_0 n)$:

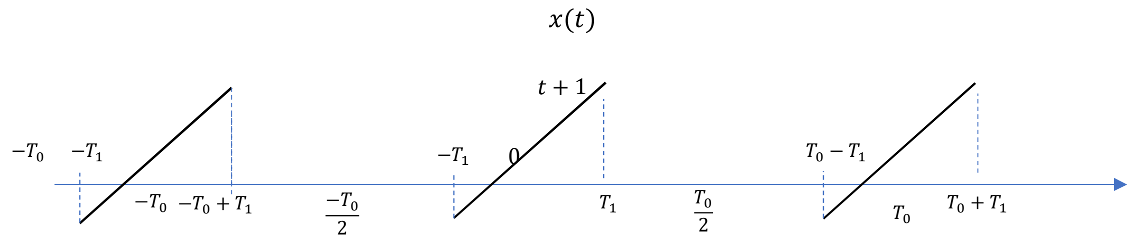
$$H(j\omega_0 n) = \int_{-\infty}^{\infty} e^{-j\omega_0 n\tau} h(\tau) d\tau$$

3- Coefficients of $y(t)$ (output of the system to $x(t)$) are $D_n H(j\omega_0 n)$ and the output is as follows:

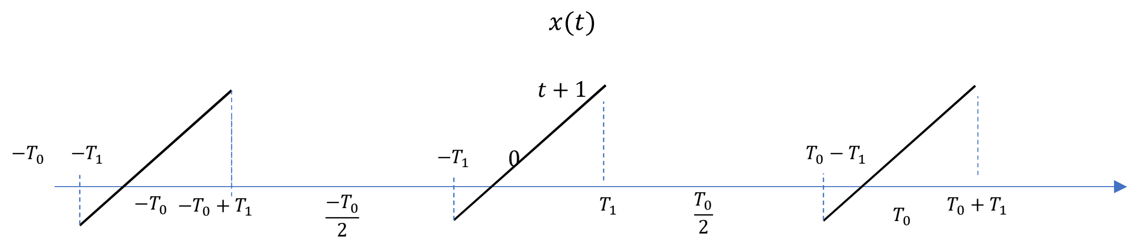
$$y(t) = \sum D_n H(j\omega_0 n) e^{j\omega_0 n t}$$

Periodic Signals and LTI Systems

$$D_n = \frac{1}{T_0} \int_{<T_0>} x(t) e^{-j\omega_0 n t} dt$$



Periodic Signals and LTI Systems



First find D_n s for $x(t)$:

$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt,\ \omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0}\int_{-T_1}^{T_1}(t+1)e^{-j\omega_0 nt}dt = \underbrace{\frac{1}{T_0}\int_{-T_1}^{T_1} te^{-j\omega_0 nt}dt}_{(1)} + \underbrace{\frac{1}{T_0}\int_{-T_1}^{T_1} e^{-j\omega_0 nt}dt}_{(2)}$$

(2): Calculation of integral in (2) for $n \neq 0$:

$$= \frac{1}{T_0}\frac{1}{-j\omega_0 n}e^{-j\omega_0 nt}\Big|_{-T_1}^{T_1} = \frac{1}{j\omega_0 nT_0}\left(e^{j\omega_0 nT_1} - e^{-j\omega_0 nT_1}\right) = \frac{2}{\omega_0 nT_0}sin(\omega_0 nT_1) = \alpha_n$$

Calculation of integral in (2) for $n = 0$

$$= \frac{1}{T_0}\int_{-T_1}^{T_1} 1dt = \frac{2T_1}{T_0} = \alpha_0$$

(1): In this step we use integration by part

$$\int_a^b \frac{d(uv)}{dt}dt = \int_a^b u\frac{dv}{dt}dt + \int_a^b v\frac{du}{dt}dt \Rightarrow UV\big|_a^b = \int_a^b u\frac{dv}{dt}dt + \int_a^b v\frac{du}{dt}$$

Consider the following substitutions:

$$\begin{cases} u = t \rightarrow \frac{du}{dt} = 1 \\ \frac{dv}{dt} = e^{-j\omega_0 nt} \rightarrow v = \frac{e^{-j\omega_0 nt}}{-j\omega_0 n} \end{cases}$$

Now we can write:

$$\Rightarrow \frac{1}{T_0}\left(\int_{-T_1}^{T_1} te^{-j\omega_0 nt}dt = t.\frac{-e^{-j\omega_0 nt}}{j\omega_0 n}\Big|_{-T_1}^{T_1} - \int_{-T_1}^{T_1}\frac{-e^{-j\omega_0 nt}}{j\omega_0 n}.\underbrace{1}_{\frac{du}{dt}}dt\right)$$

$$= \frac{1}{T_0}\left(\frac{T_1e^{-j\omega_0 nT_1} + T_1e^{j\omega_0 nT_1}}{-j\omega_0 n} - \frac{e^{-j\omega_0 nt}}{(-j\omega_0 n)^2}\Big|_{-T_1}^{T_1}\right)$$

$$= \frac{1}{T_0}\left(\frac{j2T_1}{\omega_0 n}cos(\omega_0 nT_1) - j\frac{2sin(\omega_0 nT_1)}{\omega_0^2 n^2}\right)$$

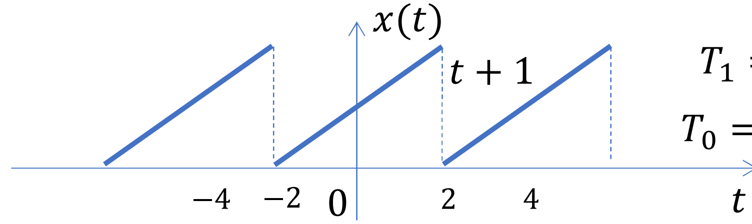
$$= j\left(\frac{2T_1}{2\pi n}cos(\omega_0 nT_1) - \frac{2sin(\omega_0 nT_1)}{T_0\omega_0^2 n^2}\right), \qquad T_0\omega_0 = 2\pi$$

$$= j\beta_n$$

$$D_n = \alpha_n + j\beta_n$$

Periodic Signals and LTI Systems

For $T_1 = 2$ and $T_0 = 4$



$T_1 = 2$

$T_0 = 4,\quad \omega_0 = \dfrac{2\pi}{T_0} = \dfrac{\pi}{2}$

$$\begin{aligned}\alpha_n &= \frac{2}{\omega_0 n T_0} sin(\omega_0 n T_1)\\ &= \frac{2}{2\pi n} sin\left(\frac{\pi}{2}.2.n\right)\\ &= \frac{1}{n\pi} sin(n\pi)\\ &= \begin{cases} 1 & n = 0\\ 0 & otherwise\end{cases}\end{aligned}$$

For $n \neq 0$

$$\beta_n = \left(\frac{2T_1}{2\pi n} cos(\omega_0 n T_1) - \frac{2 sin(\omega_0 n T_1)}{T_0 \omega_0^2 n^2}\right), \qquad T_0\omega_0 = 2\pi$$

$$= \frac{1}{4}\left(\frac{2.2}{\frac{\pi}{2} n} cos(\pi n) - \frac{2}{\omega_0^2 n^2} sin(\pi n)\right) = 2\frac{(-1)^n}{\pi n}$$

For $n = 0$

$$\beta_0 = \frac{1}{4}\int_{-2}^{2} t dt = 0$$

$\sin(\pi n) = 0$

$\cos(\pi n) = (-1)^n$

$$D_n = \begin{cases} 1 & n = 0\\ \frac{j2(-1)^n}{\pi n} & n \neq 0\end{cases}$$

Periodic Signals and LTI Systems

Reminder:

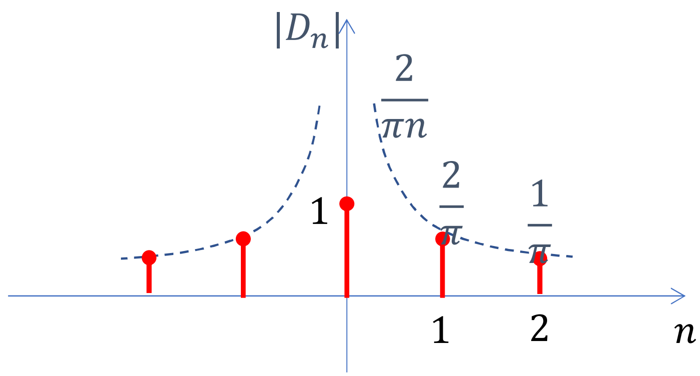
If $D_n = \alpha_n + j\beta_n$ then:

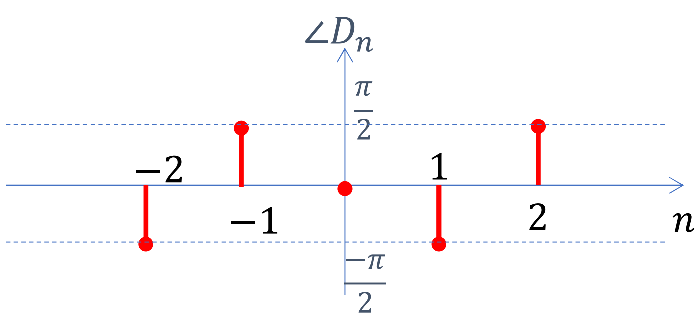
$$|D_n| = \sqrt{\alpha_n^2 + \beta_n^2}$$

$$\angle D_n = tan^{-1}\frac{\beta_n}{\alpha_n}$$

$$D_n = \begin{cases} 1 & n = 0 \\ \frac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$

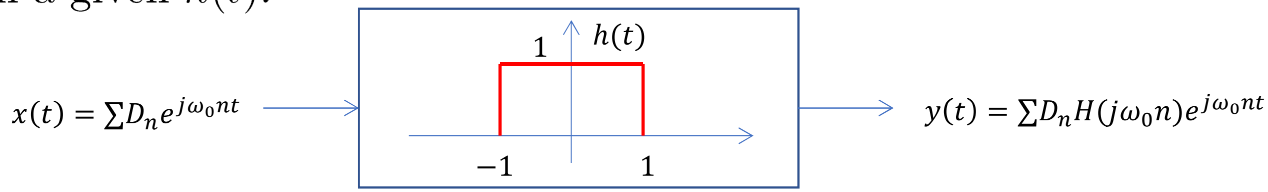
$$|D_n| = \begin{cases} 1 & n = 0 \\ \frac{2}{\pi|n|} & n \neq 0 \end{cases} \quad \Rightarrow \quad \angle D_n = \begin{cases} 0 & n = 0 \\ \frac{\pi}{2} & (n > 0,\ n = \text{even}) \text{ and } (n < 0,\ n = \text{odd}) \\ \frac{-\pi}{2} & (n > 0,\ n = \text{odd}) \text{ and } (n < 0,\ n = \text{even}) \end{cases}$$





Periodic Signals and LTI Systems

Passing $x(t)$ through LTI system with a given $h(t)$.



To Find output of an LTI system to a periodic input $x(t)$:

1- Find D_n coefficients of Fourier series of $x(t)$.

$$x(t) = \sum D_n e^{j\omega_0 nt}$$

$$D_n = \begin{cases} 1 & n = 0 \\ \frac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$

2- For ω_0, $2\omega_0$, $3\omega_0, \ldots,$ $n\omega_0$ (where ω_0 is the fundamental freq. of $x(t)$) find $H(j\omega_0 n)$:

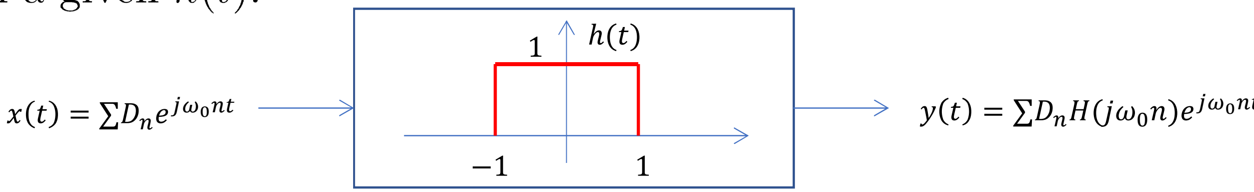
$$H(j\omega_0 n) = \int_{-\infty}^{\infty} e^{-j\omega_0 n\tau} h(\tau) d\tau$$

3- Coefficients of $y(t)$ (output of the system to $x(t)$) are $D_n H(j\omega_0 n)$ and the output is as follows:

$$y(t) = \sum D_n H(j\omega_0 n) e^{j\omega_0 nt}$$

Periodic Signals and LTI Systems

Passing $x(t)$ through LTI system with a given $h(t)$.



For the given signal $\omega_0 = \frac{\pi}{2}$

$$H(j\frac{\pi}{2}n) = \int_{-\infty}^{\infty} h(t)e^{-j\frac{\pi}{2}nt}dt = \int_{-1}^{1} 1e^{-j\frac{\pi}{2}nt}dt$$

$$= \frac{e^{-j\frac{\pi}{2}nt}}{-j\frac{\pi}{2}n}\Bigg|_{-1}^{1} = \frac{e^{-j\frac{\pi}{2}n} - e^{j\frac{\pi}{2}n}}{-j\frac{\pi}{2}n}$$

$$= \frac{2sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n} \leftarrow Sinc\ Structure$$

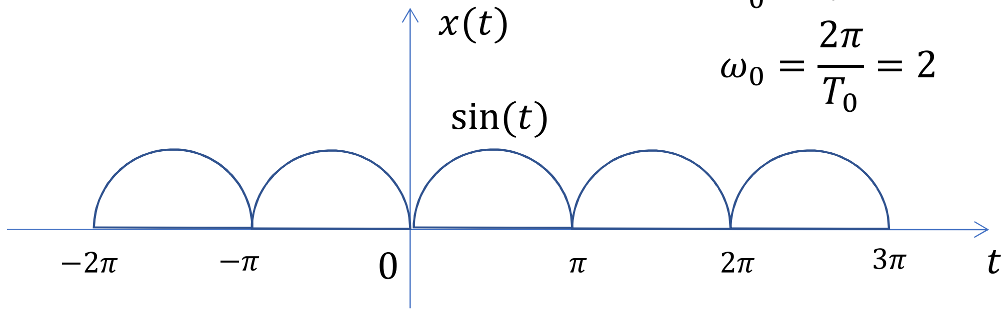
$$D_n = \begin{cases} 1 & n = 0 \\ \frac{j2(-1)^n}{\pi n} & n \neq 0 \end{cases}$$

$$y(t) = \sum \frac{2sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n} D_n e^{j\omega_0 nt}$$

$$= 2e^{j\omega_0 .0.t} + \sum_{n=-\infty,\ n\neq 0}^{\infty} 2\frac{sin(\frac{\pi}{2}n)}{\frac{\pi}{2}n}\frac{2j(-1)^n}{\pi n}e^{j\omega_0 nt}$$

Periodic Signals

Example: Find Fourier series of the following signal:



$T_0 = \pi$

$$\omega_0 = \frac{2\pi}{T_0} = 2$$

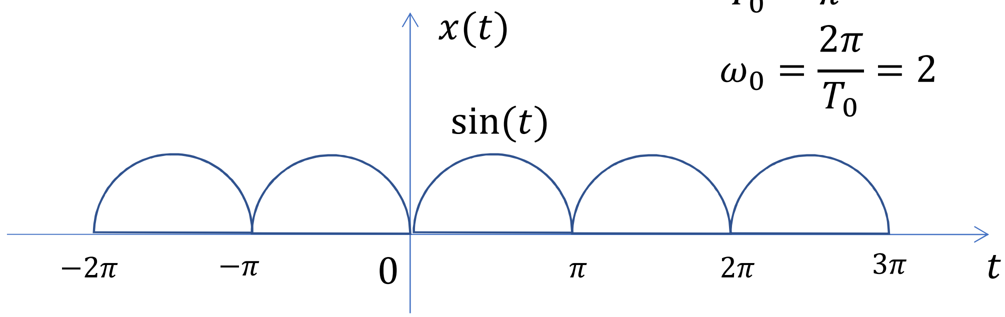
$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 n t}dt$$

Periodic Signals

Example: Find Fourier series of the following signal:

$T_0 = \pi$

$\omega_0 = \frac{2\pi}{T_0} = 2$



$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{\pi}\int_0^{\pi} \sin(t)e^{-j2nt}dt$$

$$= \frac{1}{\pi}\int_0^{\pi} (\frac{1}{2j}e^{jt} - \frac{1}{2j}e^{-jt})e^{-j2nt}dt$$

$$= \frac{1}{\pi}\int_0^{\pi} \frac{1}{2j}e^{j(1-2n)t}dt - \frac{1}{\pi}\int_0^{\pi} \frac{1}{2j}e^{-j(1+2n)t}dt$$

$$= \frac{1}{\pi}\frac{1}{2j}\frac{e^{j(1-2n)t}}{j(1-2n)}\Big|_0^{\pi} - \frac{1}{\pi}\frac{1}{2j}\frac{e^{-j(1+2n)t}}{-j(1+2n)}\Big|_0^{\pi} = \frac{1}{\pi}\left[\frac{e^{j\pi(1-2n)}}{2(-1)(1-2n)} - \frac{1}{2(-1)(1-2n)}\right] - \frac{1}{\pi}\left[\frac{e^{-j\pi(1+2n)}}{2(1+2n)} - \frac{1}{2(1+2n)}\right]$$

$$= \frac{2}{\pi}\frac{1}{1-2n} + \frac{2}{\pi}\frac{1}{1+2n}$$

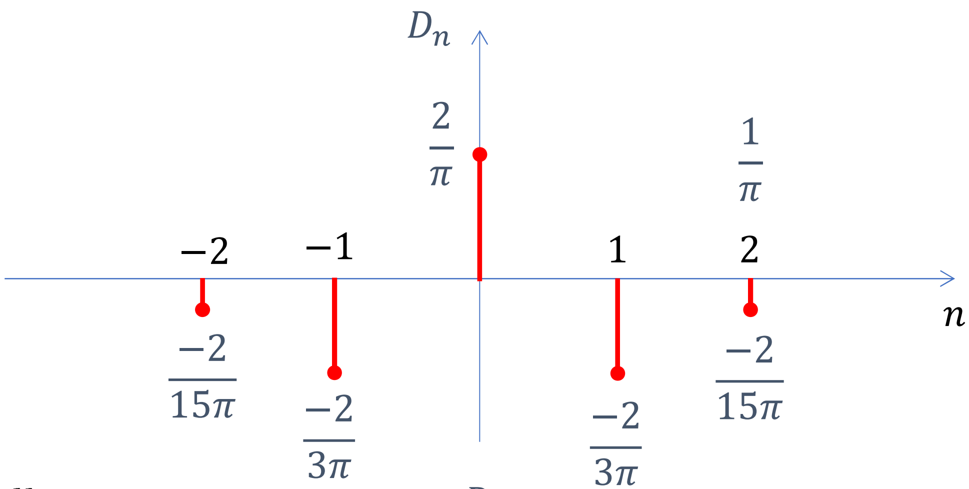
$$= \frac{2(1+2n)+2(1-2n)}{2\pi(1-4n^2)} = \frac{2}{\pi(1-4n^2)}$$

Reminder:
$e^{j\pi} = -1$
$e^{j\pi(1-2n)} = e^{j\pi}e^{-j2n\pi} = -1 \times 1$

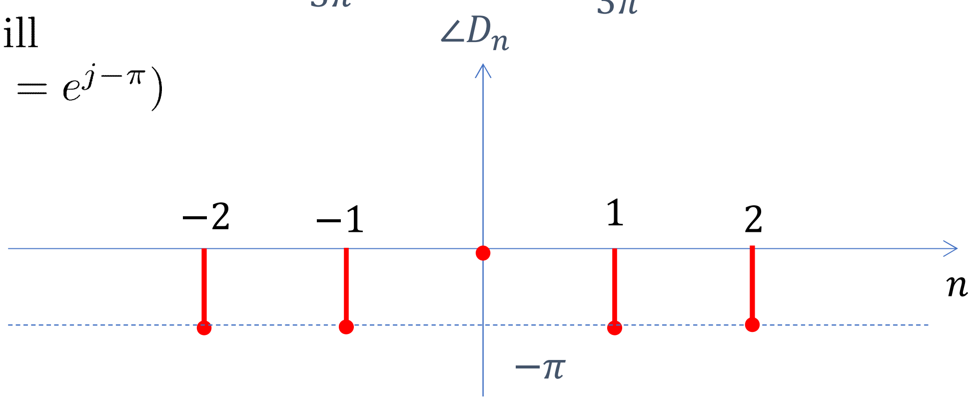
Plot $|D_n|$ and $\angle(D_n)$

Periodic Signals

$$D_n = \frac{2}{\pi(1-4n^2)}$$

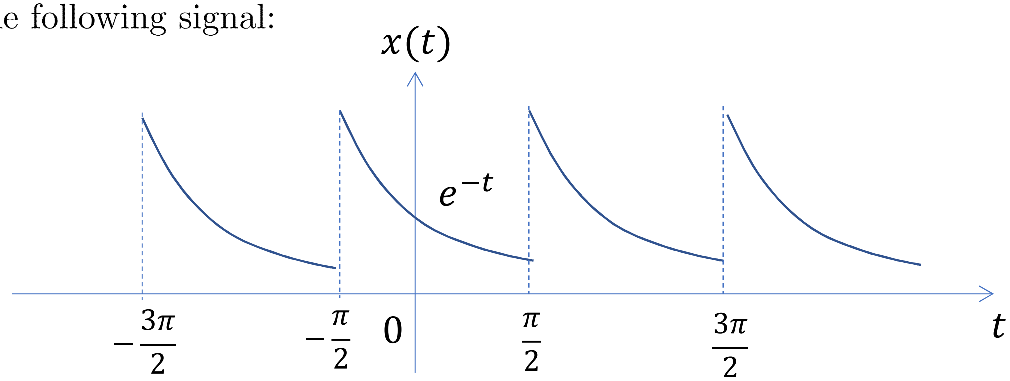


Note that this phase is still an odd function of n $(e^{j\pi} = e^{j-\pi})$



Periodic Signals

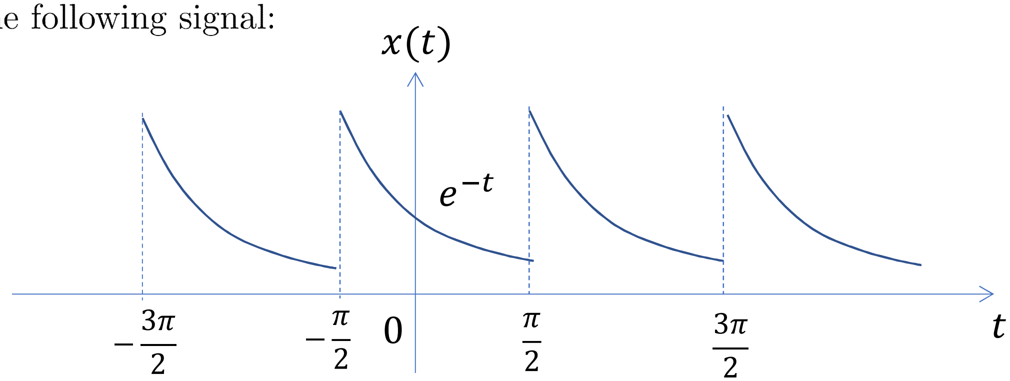
Example: Find the Fourier series of the following signal:



$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt$$

Periodic Signals

Example: Find the Fourier series of the following signal:



$$\omega_0 = \frac{2\pi}{\pi} = 2$$

$$T_0 = \pi$$

$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-t}e^{-j2nt}dt$$

$$= \frac{1}{\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} e^{-t(j2n+1)}dt = \frac{1}{-\pi(2nj+1)}\left(e^{-(j2n+1)\frac{\pi}{2}} - e^{(j2n+1)\frac{\pi}{2}}\right)$$

$$= \frac{1}{-\pi(2nj+1)}\left[e^{-jn\pi}e^{-\frac{\pi}{2}} - e^{jn\pi}e^{\frac{\pi}{2}}\right]$$

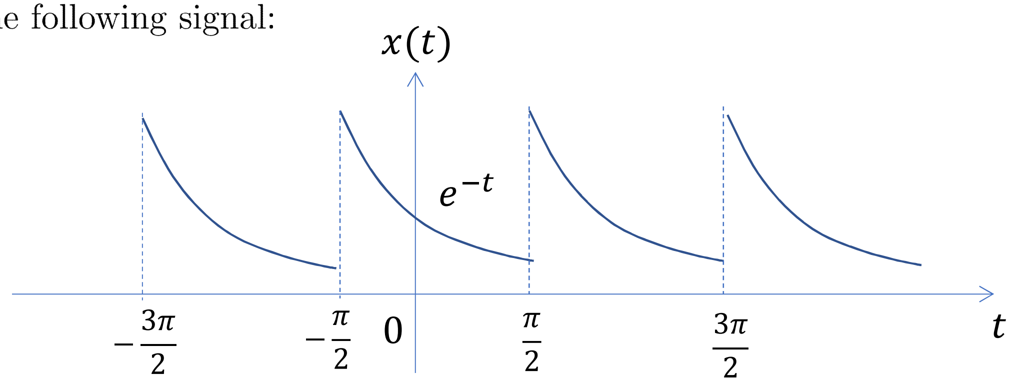
$$= \frac{1}{-\pi(2nj+1)}\left((-1)^n e^{-\frac{\pi}{2}} - (-1)^n e^{\frac{\pi}{2}}\right)$$

$$= \frac{1}{\pi(2nj+1)}(-1)^n\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)$$

What are $|D_n|$s and $\angle(D_n)$?

Periodic Signals

Example: Find the Fourier series of the following signal:



$$\omega_0 = \frac{2\pi}{\pi} = 2$$

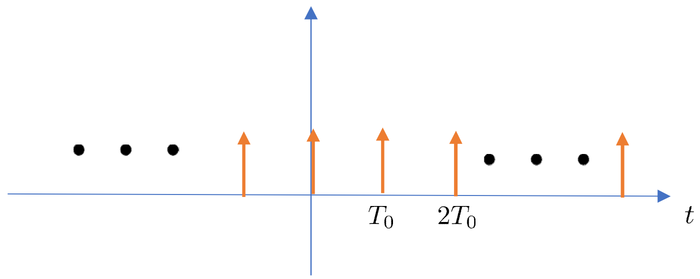
$$T_0 = \pi$$

$$\begin{aligned} |D_n| &= \left|\frac{1}{\pi(2nj+1)}(-1)^n \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)\right| \\ &= \left|\frac{1}{\pi(2nj+1)}\right| |(-1)^n| \left|\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)\right| \\ &= \frac{1}{\sqrt{(2n\pi)^2 + \pi^2}} \times 1 \times \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) \end{aligned}$$

$$\begin{aligned} \angle(D_n) &= \angle\left(\frac{1}{\pi(2nj+1)}(-1)^n \left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)\right) \\ &= \angle\left(\frac{1}{\pi(2nj+1)}\right) + \angle((-1)^n) + \angle\left(\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right)\right) \\ &= \angle 1 - \angle(\pi(2nj+1)) + \angle((-1)^n) + 0 \\ &= \begin{cases} 0 - \tan^{-1}(2n) + 0 & n \ \ even \\ 0 - \tan^{-1}(2n) + \pi & n \ \ odd \end{cases} \end{aligned}$$

Periodic Signals

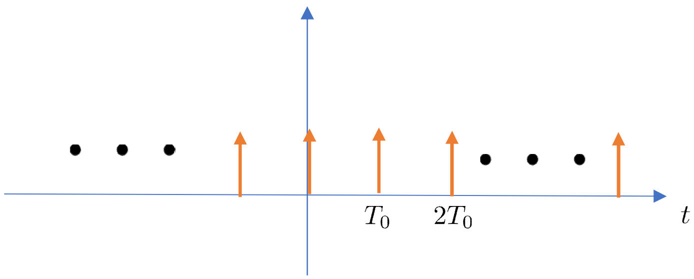
Example: Find FS of impulse train



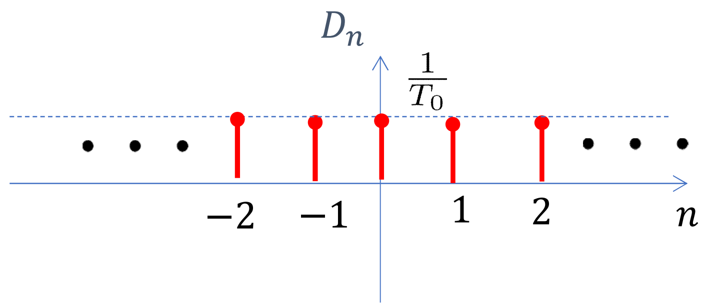
$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt$$

Periodic Signals

Example: Find FS of impulse train



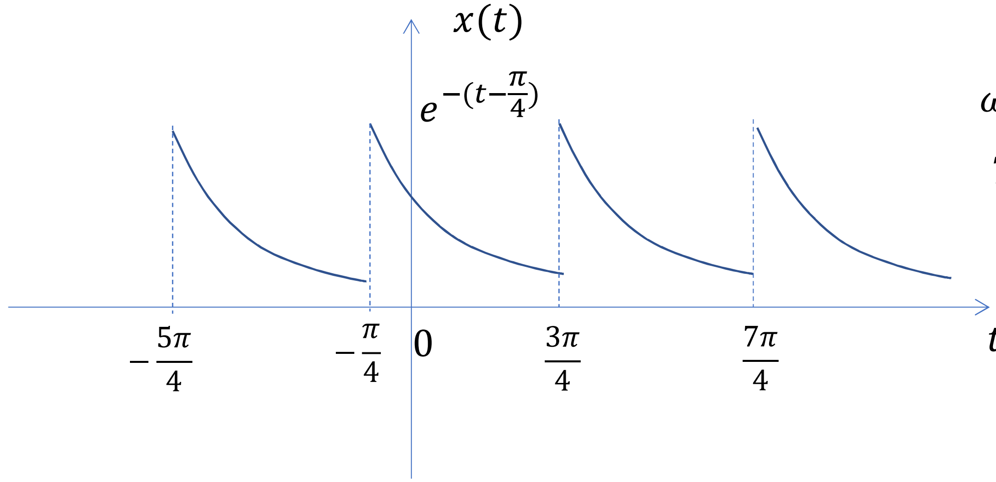
$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-j\omega_0 n0}dt = \frac{1}{T_0}$$



Important signal in sampling

Periodic Input Delay, and LTI Systems

Find the fourier series of the following signal which is shifted version of the previous example by $\frac{\pi}{4}$

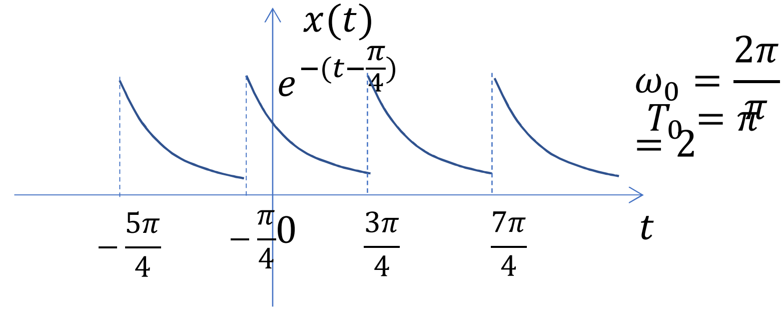


$$\omega_0 = \frac{2\pi}{\pi} = 2$$

$$T_0 = \pi$$

$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt$$

Periodic Input Delay, and LTI Systems



$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{\pi}\int_{\frac{-\pi}{4}}^{\frac{3\pi}{4}} e^{-(t-\frac{\pi}{4})}e^{-j2nt}dt$$

$$= \frac{e^{\frac{\pi}{4}}}{\pi}\int_{\frac{-\pi}{4}}^{\frac{3\pi}{4}} e^{-t(j2n+1)}dt = \frac{1}{-\pi(j2n+1)}e^{-(j2n+1)t}\Big|_{-\frac{\pi}{4}}^{\frac{3\pi}{4}}$$

$$= \frac{e^{\frac{\pi}{4}}}{-\pi(j2n+1)}\left(e^{-j2n.\frac{3\pi}{4}}e^{\frac{-3\pi}{4}} - e^{j2n.\frac{\pi}{4}}e^{\frac{\pi}{4}}\right)$$

$$= \frac{1}{-\pi(j2n+1)}\left(e^{-j2n.\frac{3\pi}{4}}e^{\frac{-\pi}{2}} - e^{j2n.\frac{\pi}{4}}e^{\frac{\pi}{2}}\right)$$

$$= \frac{1}{-\pi(j2n+1)}\frac{e^{j\frac{\pi}{2}n}}{e^{j\frac{\pi}{2}n}}\left(e^{-j2n.\frac{3\pi}{4}}e^{\frac{-\pi}{2}} - e^{j2n.\frac{\pi}{4}}e^{\frac{\pi}{2}}\right)$$

$$= \frac{1}{-\pi(j2n+1)}\frac{1}{e^{j\frac{\pi}{4}2n}}\left(e^{-j\pi n}e^{\frac{-\pi}{2}} - e^{j\pi n}e^{\frac{\pi}{2}}\right)$$

$$= e^{-j\frac{\pi}{4}2n}\underbrace{\frac{1}{-\pi(j2n+1)}\left(e^{-j\pi n}e^{\frac{-\pi}{2}} - e^{j\pi n}e^{\frac{\pi}{2}}\right)}_{\text{FS. of previous example}}$$

In case of delay the following is usually useful:

$$e^{ja} + e^{jb} = e^{j\frac{a+b}{2}}(e^{j\frac{a-b}{2}} + e^{j\frac{-(a-b)}{2}})$$
$$= e^{j\frac{a+b}{2}} \times 2\cos(\frac{a-b}{2})$$

$$\boxed{\begin{array}{ll} \text{If} \quad x(t) \xrightarrow{\text{Fourier series}} & D_n \\ \text{then} \quad x(t-t_0) \xrightarrow{\text{Fourier series}} & D_n.e^{-j\omega_0 t_0 n} \end{array}}$$

Some Properties of FS

Time Shift

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x(t-t_0) \xrightarrow{\text{Fourier series}} D_n.e^{-j\omega_0 t_0 n}$$

Linearity: If $x(t)$ and $y(t)$ have same fundamental Freqs.

$$x(t) \xrightarrow{\text{Fourier series}} D_n^x$$

$$y(t) \xrightarrow{\text{Fourier series}} D_n^y$$

$$Ax(t)+By(t) \xrightarrow{\text{Fourier series}} AD_n^x + BD_n^y$$

Product in time (convolution in Freq)

If $x(t)$ and $y(t)$ have same fundamental Freqs.

$$x(t) \xrightarrow{\text{Fourier series}} D_n^x$$

$$y(t) \xrightarrow{\text{Fourier series}} D_n^y$$

$$x(t)y(t) \xrightarrow{\text{Fourier series}} D_n^x * D_n^y$$

Scalar Multiplication

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$Ax(t) \xrightarrow{\text{Fourier series}} AD_n$$

Time Reversal

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x(-t) \xrightarrow{\text{Fourier series}} D_{-n}$$

Conjugation in time

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x^*(t) \xrightarrow{\text{Fourier series}} D^*_{-n}$$

Since $(e^{j\theta})^* = e^{-j\theta}$

Some Properties of FS

Time Scaling $(\alpha > 0)$

$$x(t) \xrightarrow{\text{Fourier series}} D_n$$

$$x(\alpha t) \xrightarrow{\text{Fourier series}} D_n$$

Only fundamental freq. is changed from ω_0 to $\alpha\omega_0$

$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t}$$

Fundamental freq is ω_0

$$x(\alpha t) = \sum_{n=-\infty}^{\infty} D_n e^{j(\alpha\omega_0) n t}$$

Fundamental freq is $\alpha\omega_0$ for $\alpha > 0$)

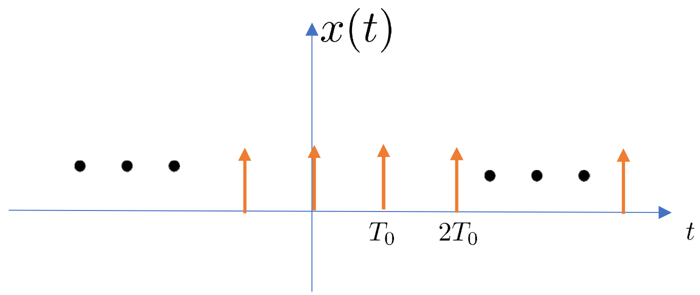
For $\alpha < 0$ the fundamental frequency is $|\alpha|\omega_0$ and the D_ns are therefore D_{-n}s

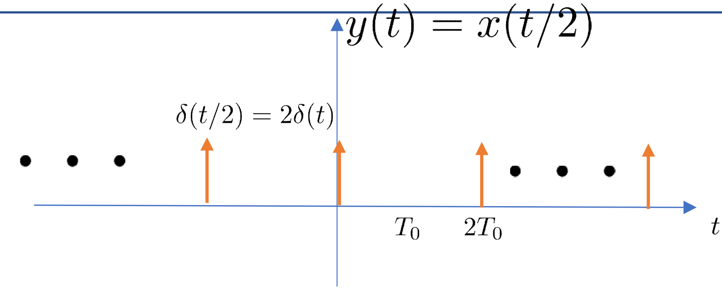
Time Scaling and FS

Time Scaling

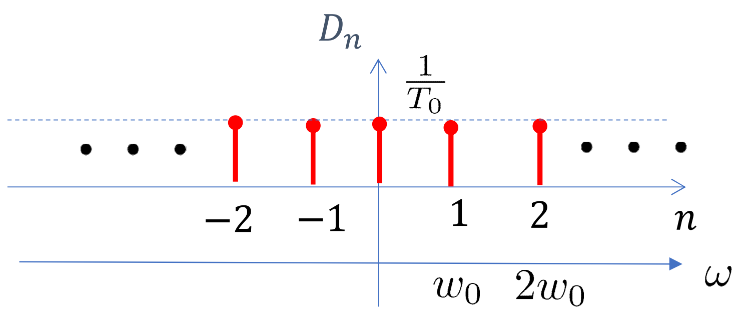
$$x(t) \xrightarrow{\text{Fourier series}} D_n \quad \text{Fundamental freq is } \omega_0$$

$$x(\alpha t) \xrightarrow{\text{Fourier series}} D_n \quad \text{Fundamental freq is } \alpha\omega_0$$





$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-j\omega_0 n0}dt = \frac{1}{T_0}$$

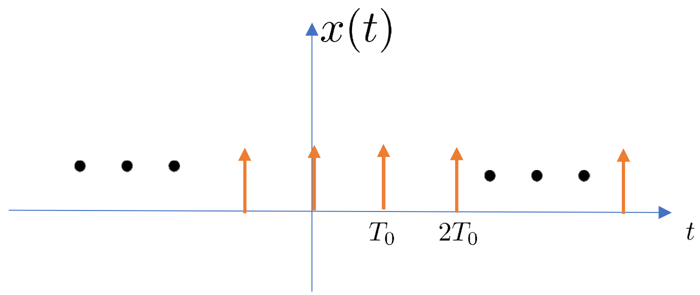


Time Scaling and FS

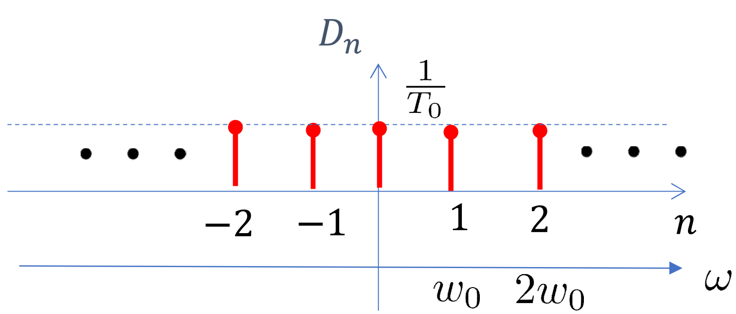
Time Scaling

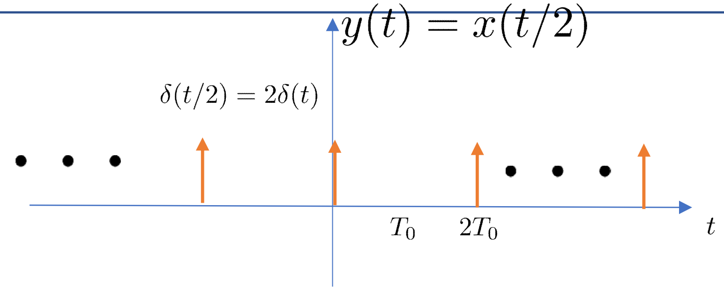
$$x(t) \xrightarrow{\text{Fourier series}} D_n \quad \text{Fundamental freq is } \omega_0$$

$$x(\alpha t) \xrightarrow{\text{Fourier series}} D_n \quad \text{Fundamental freq is } \alpha\omega_0$$

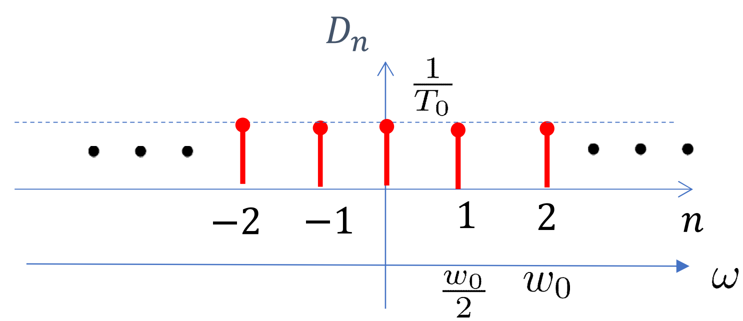


$$D_n = \frac{1}{T_0}\int_{<T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t)e^{-j\omega_0 n0}dt = \frac{1}{T_0}$$

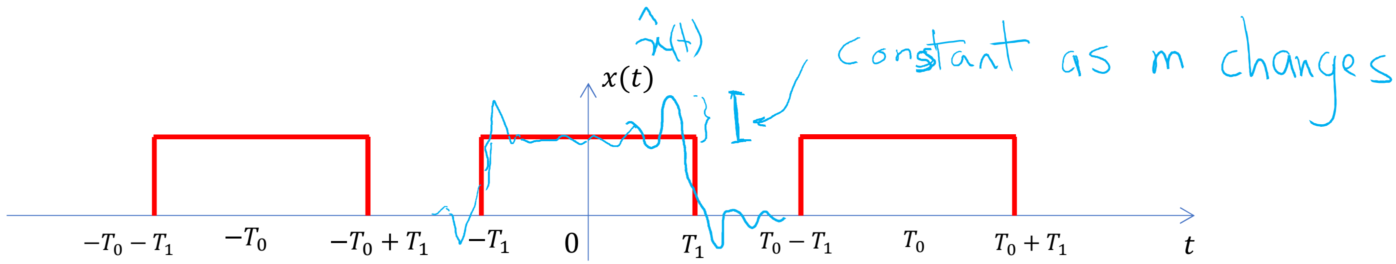




$$D_n = \frac{1}{2T_0}\int_{<2T_0>} x(t)e^{-j\omega_0 nt}dt = \frac{1}{2T_0}\int_{-T_0}^{T_0} 2\delta(t)e^{-j\omega_0 n0}dt = \frac{1}{T_0}$$



Using Finite Number of Coefficients (Gibbs Phenomenon)



$$x(t) = \sum_{n=-\infty}^{\infty} D_n e^{j\omega_0 n t}$$

$$\hat{x}(t) = \sum_{n=-m}^{m} D_n e^{j\omega_0 n t}$$

Windowing D_ns in frequency domain is equivalent to convolving with a sinc function in the time domain that results in the Gibbs Phenomenon.