Signals and Systems I

Lectures 10 & 11

Last Lecture

- Fourier Series & properties (two examples, time delay and time scaling)

Today

- Fourier Transform
- Fourier Transform Properties

Fourier Series vs Fourier Transform

Fourier series: Periodic signals	Fourier transform: More signals (including periodic ones)
$x_p(t) = \sum D_n e^{j\omega_0 n t}$	$x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
$D_n = \frac{1}{T_0} \int_{<T_0>} x_p(t) e^{-j\omega_0 n t} dt$	$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$
$\lvert D_n \rvert \& \angle D_n$ for each n at $\underbrace{\omega_0 n}_{\text{freq.}}$	$\lvert X(j\omega) \rvert \& \angle X(j\omega)$ for all ω

Similar to Fourier Series, Fourier Transform has periodic spirals in form of $e^{j\omega t}$ and they are being amplified with $|X(j\omega)|$ and rotated with $\angle(X(j\omega)$

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Fourier Series vs Fourier Transform

In Book $X(\omega)$, correct is $X(j\omega)$

Also $X(jf) = X(j2\pi f) = X(j\omega)$

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$x_p(t) = \sum D_n e^{j\omega_0 nt}$	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$
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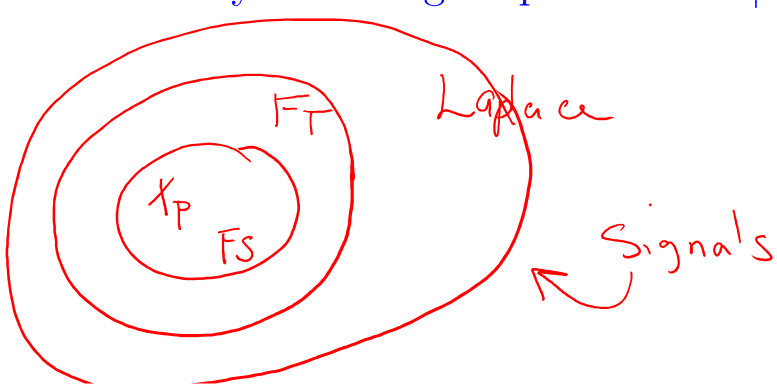

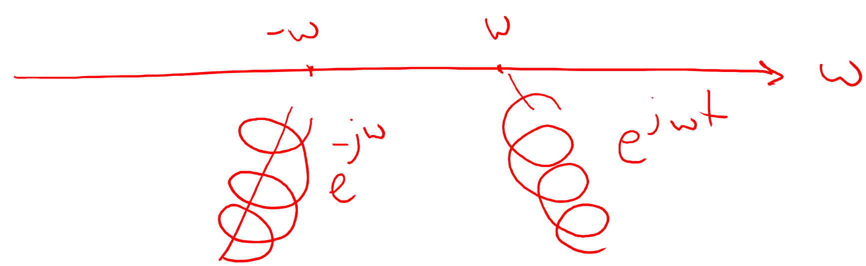

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Fourier Series vs Fourier Transform

Fourier series: Periodic signals	Fourier transform: More signals (including periodic ones)
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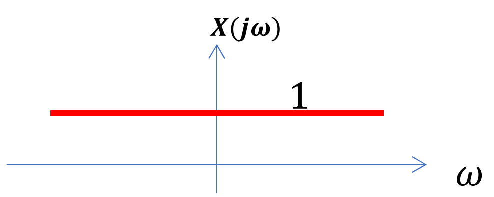
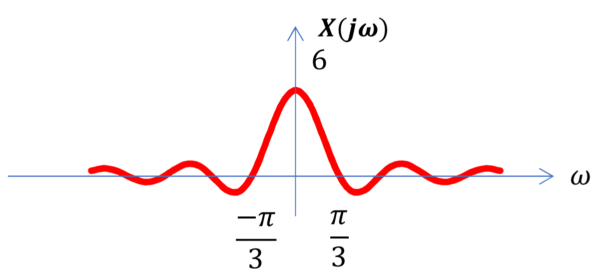
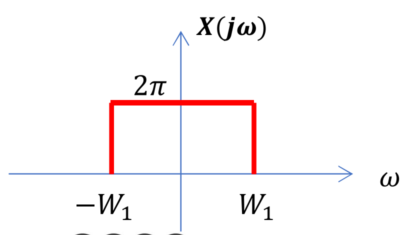
Similar to Fourier Series, Fourier Transform has periodic spirals in form of $e^{j\omega t}$ and they are being amplified with $|X(j\omega)|$ and rotated with $\angle(X(j\omega)$

What is the difference between the following FTs:







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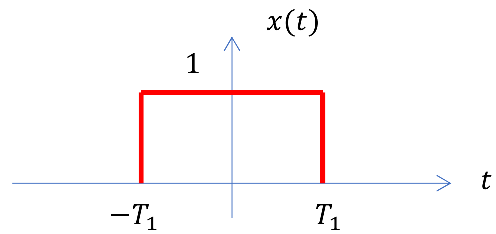
Fourier Transform

Example: Find and plot Fourier transform of the following signal:



$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

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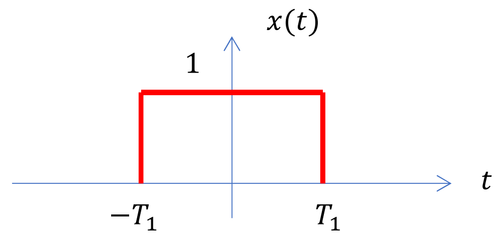
Fourier Transform

Example: Find and plot Fourier transform of the following signal:



Answer:

$$
\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-T_1}^{T_1} 1 \times e^{-j\omega t}dt \\
&= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega} \\
&= \frac{2sin(\omega T_1)}{\omega} \rightarrow Sinc\ Structure
\end{aligned}
$$

Pulse in time is always a sinc in frequnecy

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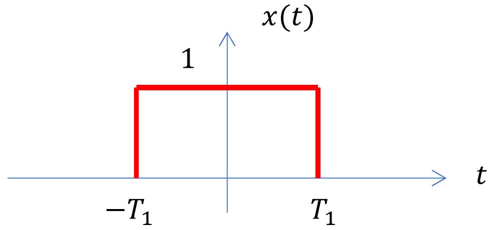
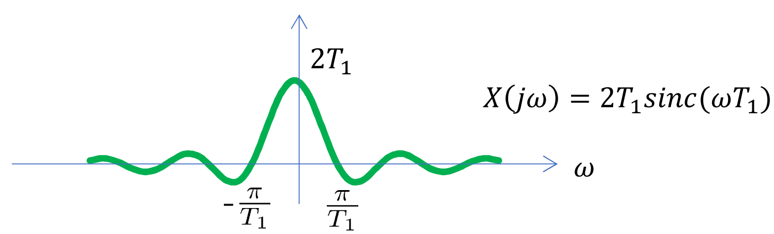
Fourier Transform

Find and plot Fourier transform for the following signal:



Answer:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$
$$= \int_{-T_1}^{T_1} 1.e^{-j\omega t}dt$$
$$= \frac{e^{j\omega T_1} - e^{-j\omega T_1}}{j\omega}$$
$$= \frac{2sin(\omega T_1)}{\omega} = \frac{2T_1 sin(\omega T_1)}{\omega T_1}$$



Pulse in time is always a sinc in frequnecy

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Fourier Transform

Example: Find and plot Fourier transform of the following signal:

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

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Fourier Transform

Example: Find and plot Fourier transform of the following signal:

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$\begin{aligned}
X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\
&= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\
&= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\
&= \frac{e^{-t(a+j\omega)}}{-(a+j\omega)}\Bigg|_0^{\infty} \\
&= \frac{e^{-\infty(a+j\omega)}}{-(a+j\omega)} - \frac{1}{-(a+j\omega)} \quad \text{since } a > 0, \ \lim_{t\to\infty} e^{-at} = 0 \text{ and } e^{j\omega t} \text{ doesn't have a limit, it rotates on a unit circle,} \\
&\qquad\qquad |e^{j\omega t}| = 1 \text{ for all } t \text{ even as } t \to -\infty \\
&= 0 - \frac{1}{-(a+j\omega)} \\
&= \frac{1}{a+j\omega}
\end{aligned}$$

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Fourier Transform

Example: Find and plot Fourier transform of the following signal:

$$x(t) = e^{-at}u(t), \quad a > 0$$

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \\ &= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt \\ &= \int_{0}^{\infty} e^{-at}e^{-j\omega t}dt \\ &= \frac{e^{-t(a+j\omega)}}{-(a+j\omega)}\Big|_{0}^{\infty} \\ &= 0 - \frac{1}{-(a+j\omega)} \\ &= \frac{1}{a+j\omega} \end{aligned}$$

$$|X(j\omega)| = \frac{1}{\sqrt{\omega^2 + a^2}} \longrightarrow |X(j\omega)| = |X(-j\omega)|$$

$$\angle X(j\omega) = \angle \frac{1}{j\omega + a} = -\angle(j\omega + a) = -tan^{-1}(\frac{\omega}{a})$$

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Fourier Transform

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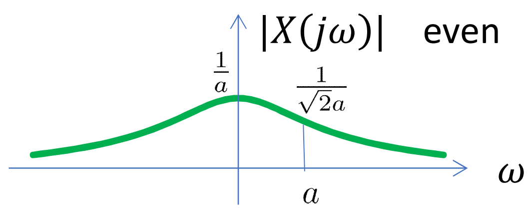

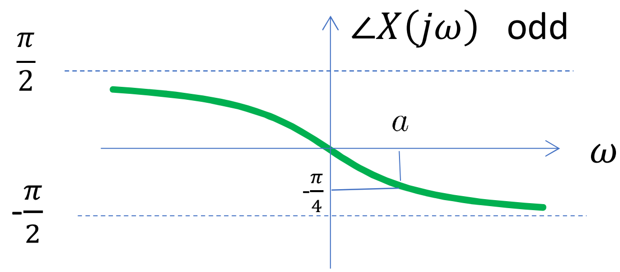

For a real signal in time, the absolute value of FT is even and the phase is odd.

How will this FT change as the value of a grows?

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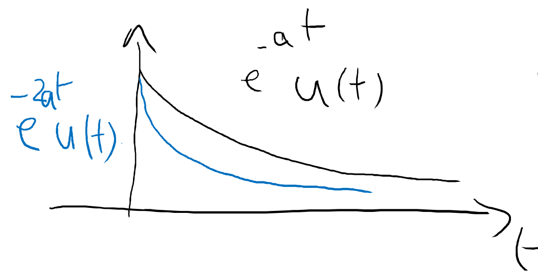
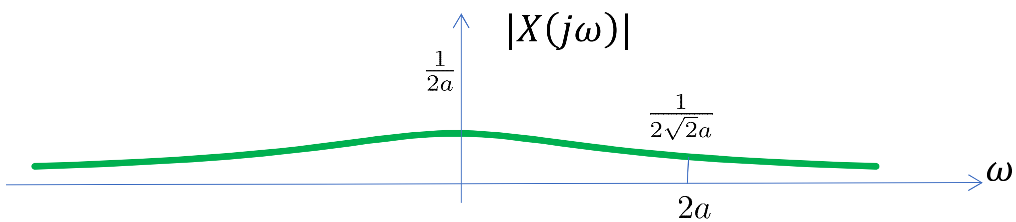
Fourier Transform

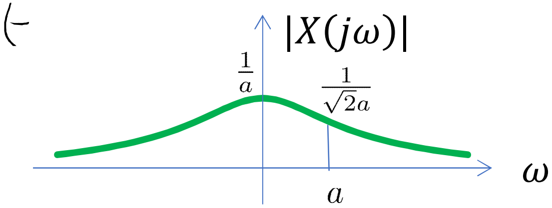
Example: Find and plot Fourier transform of the following signal:

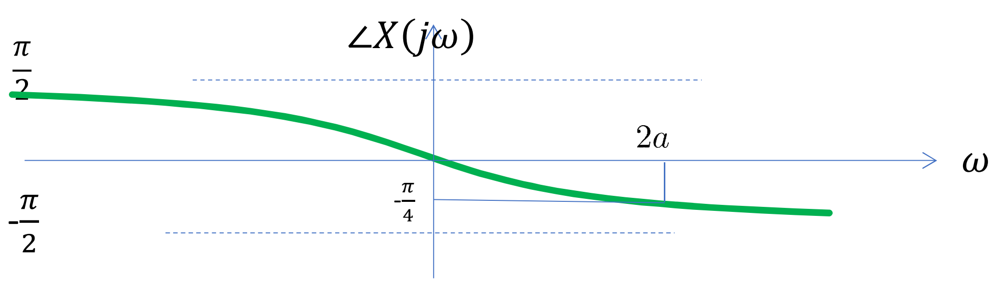
$$x(t) = e^{-2at}u(t), \quad a > 0$$

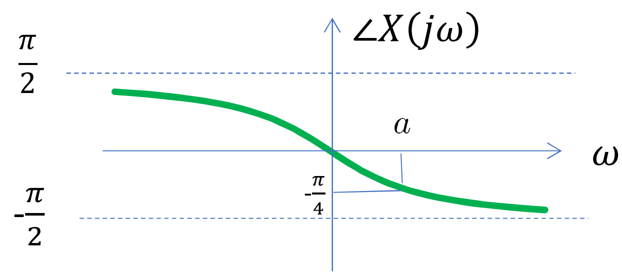


$$x(t) = e^{-at}u(t), \quad a > 0$$









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Connection between Fourier Series (in limit) and Fourier Transform

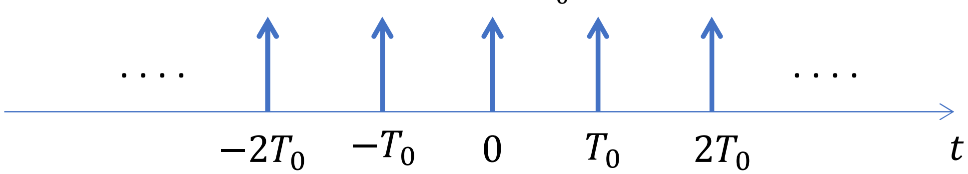
Remember the FS of impulse train:

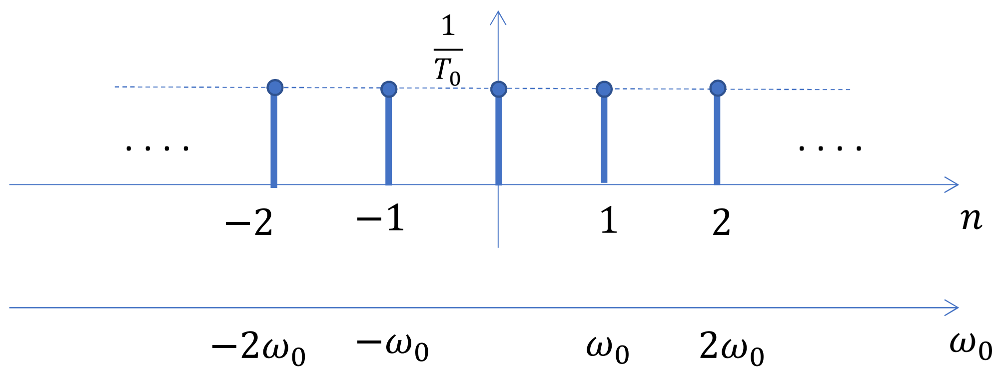
$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_0), \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$D_n = \frac{1}{T_0}\int_{T_0} \delta_{T_0}(t) e^{-jn\omega_0 t} dt = \int_{\frac{-T}{2}}^{\frac{-T}{2}} \delta(t) e^{-jn\omega_0 t} = \frac{1}{T_0}$$

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T_0} e^{jn\omega_0 t}$$

$x(t) = \delta_{T_0}(t)$





What will happen as T_0 grows to infinity?

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Connection between Fourier Series (in limit) and Fourier Transform

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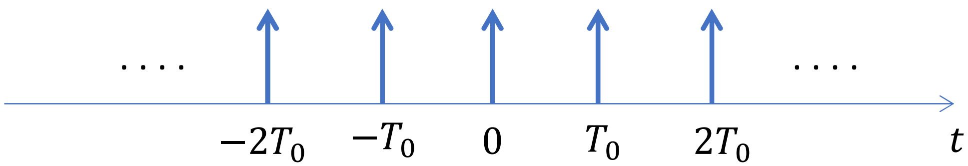
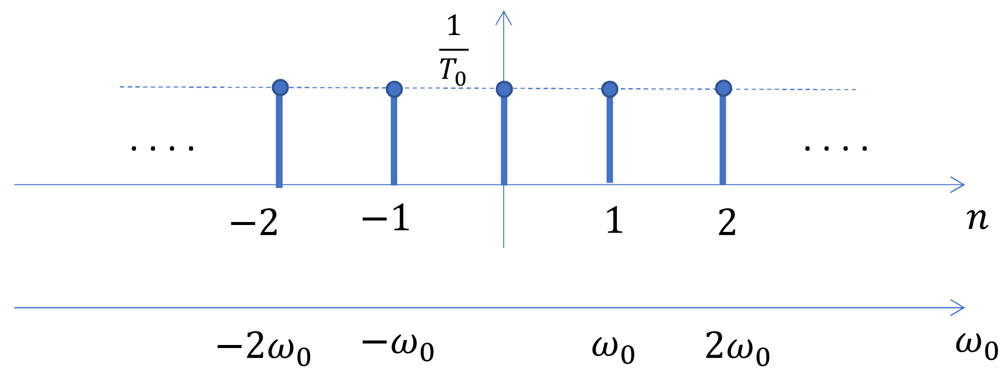
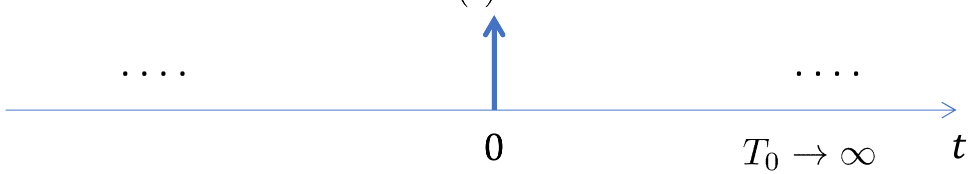




What will happen as T_0 grows to infinity?

$$\delta(t) = \lim_{T_0\to\infty} \delta_{T_0}(t)$$

$$\delta(t)$$



What is FT of $\delta(t)$?

$$\Delta(j\omega) = \int \delta(t)e^{-j\omega t}dt$$

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Connection between Fourier Series (in limit) and Fourier Transform

Remember the FS of impulse train:

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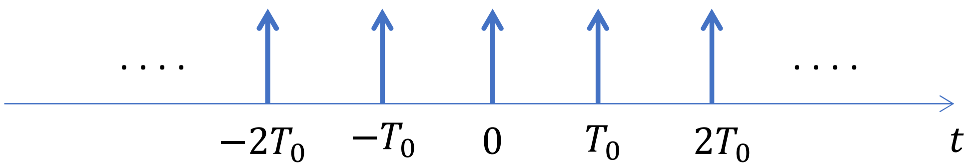
$$x(t) = \delta_{T_0}(t)$$



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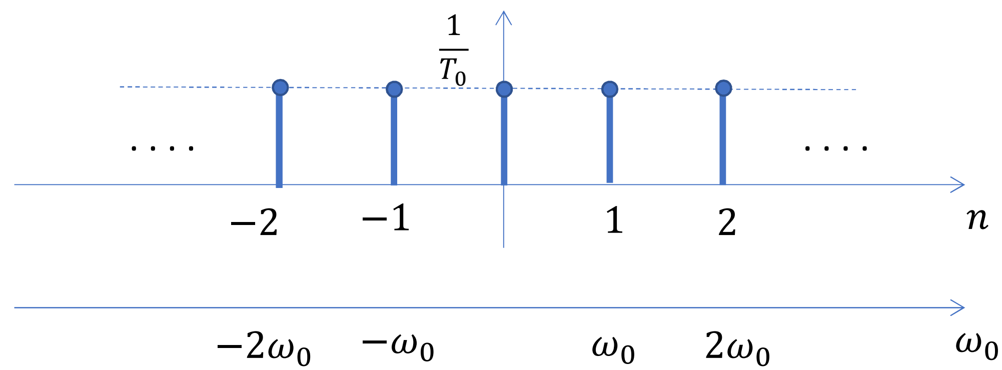
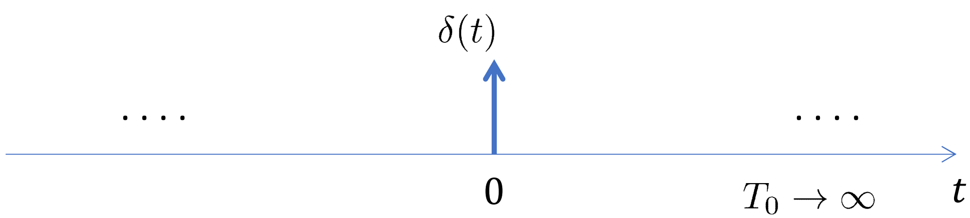
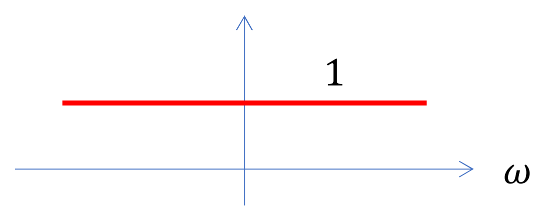


What will happen as T_0 grows to infinity?

$$\delta(t) = \lim_{T_0 \to \infty} \delta_{T_0}(t)$$



$$\Delta(j\omega) = \int \delta(t) e^{-j\omega t} dt = 1$$



$$\Delta(j\omega) = \lim_{T_0 \to \infty} T_0 D_n$$

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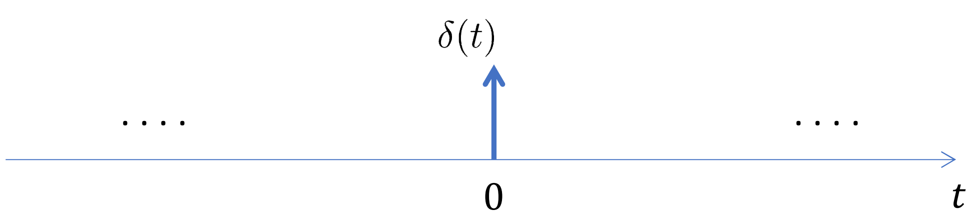
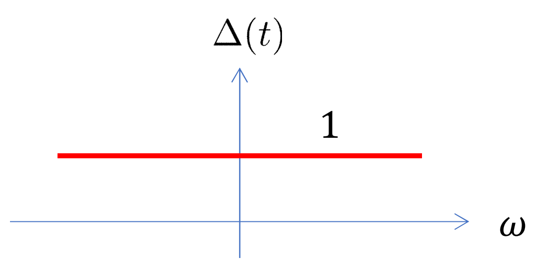
Fourier Transform of delta

$\delta(t)$ is built by adding periodic spirals of "all" frequencies!

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Delta(j\omega)e^{j\omega t}d\omega$$





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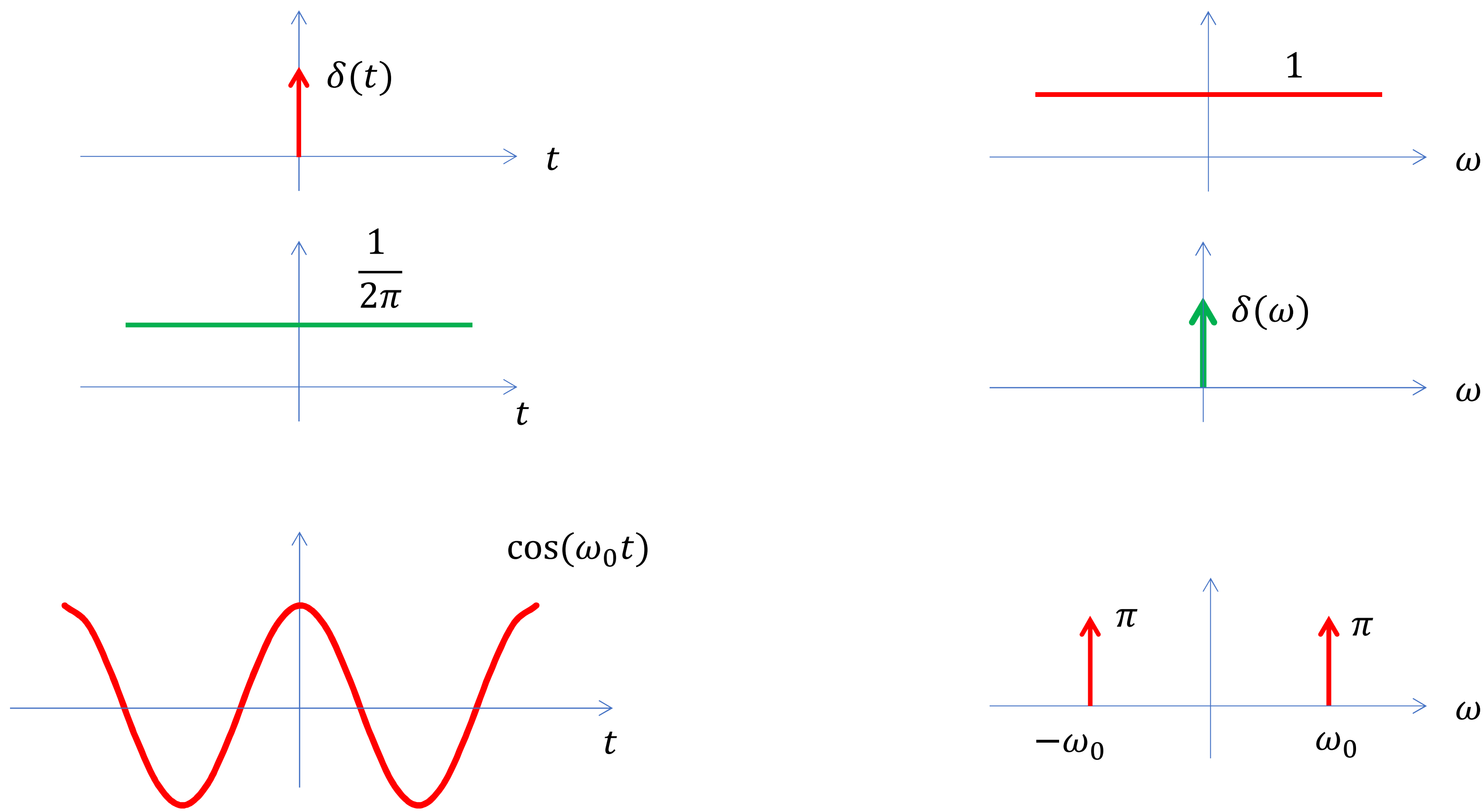
Important Signals Fourier Transforms

$\boldsymbol{x(t)}$	$\boldsymbol{X(j\omega)}$



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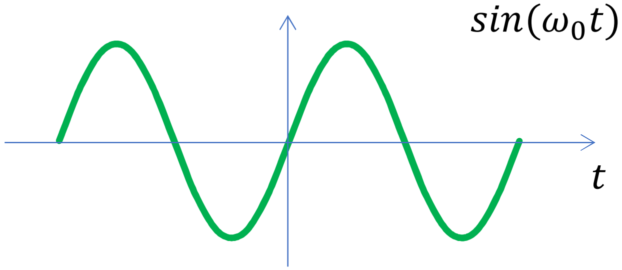
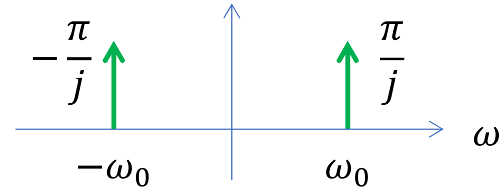
Important Signals Fourier Transforms

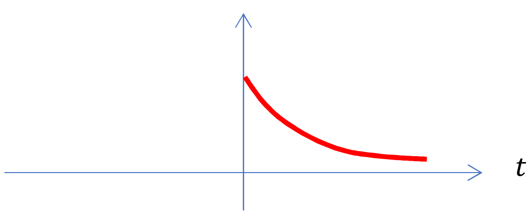
$x(t)$ | **$X(j\omega)$**

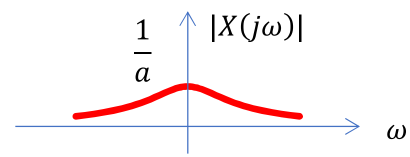
$sin(\omega_0 t)$

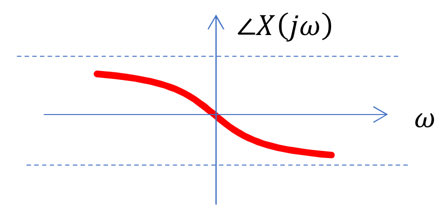




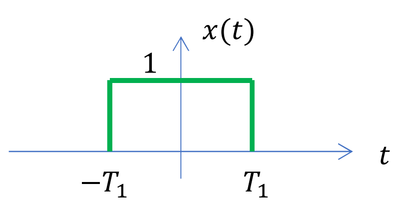
$e^{-at}u(t), \qquad a > 0$

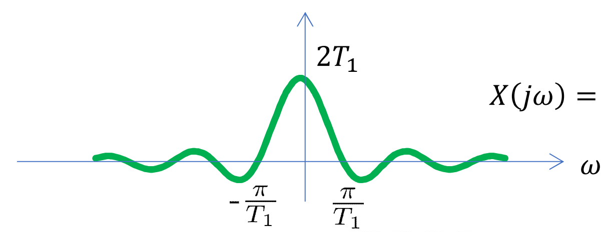






$$X(j\omega) = \frac{1}{j\omega + a}$$





$X(j\omega) = 2T_1 sinc(\omega T_1)$

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Fourier Transform Properties (Linearity)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

1. Linearity:

$$x_1(t) \xrightarrow{\text{FT}} X_1(j\omega)$$

$$x_2(t) \xrightarrow{\text{FT}} X_2(j\omega)$$

$$ax_1(t) + bx_2(t) \xrightarrow{\text{FT}} aX_1(j\omega) + bX_2(j\omega)$$

Example:

$$\begin{aligned}
x(t) &= \delta(t) + 2e^{-3t}u(t) \\
X(j\omega) &= FT\left(\delta(t)\right) + FT\left(2e^{-3t}u(t)\right) \\
&= 1 + 2FT\left(e^{-3t}u(t)\right) \\
&= 1 + 2\frac{1}{j\omega + 3}
\end{aligned}$$

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Fourier Transform Properties (Time Shift)

2. Time Shift

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) = x_1(t - t_0) \xrightarrow{FT} X_2(j\omega) = e^{-j\omega t_0} X_1(j\omega)$$

Delay in time by t_0 is a phase shift by ωt_0 in Fourier transform domain.

Proof:

$$\begin{aligned}
X_2(j\omega) &= \underbrace{\int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt}_{\text{definition of FT for } x_2(t)} \\
&= \int_{-\infty}^{\infty} x_1(\underbrace{t - t_0}_{\text{change of variable to } u}) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} x_1(u) e^{-j\omega(u+t_0)} du, \quad t - t_0 = u \to t = u + t_0 \to du = dt \\
&= \int_{-\infty}^{\infty} x_1(u) e^{-j\omega u} e^{-j\omega t_0} du \\
&= e^{-j\omega t_0} \underbrace{\int_{-\infty}^{\infty} x_1(u) e^{-j\omega u} du}_{\text{FT of } x_1(t)} \\
&= e^{-j\omega t_0} X_1(j\omega)
\end{aligned}$$

Note that boundaries of the integral also have to be adjusted. Here:
$t \to \infty$ then $u \to \infty$
similarly
$t \to -\infty$ then $u \to -\infty$
so the boundaries stay the same

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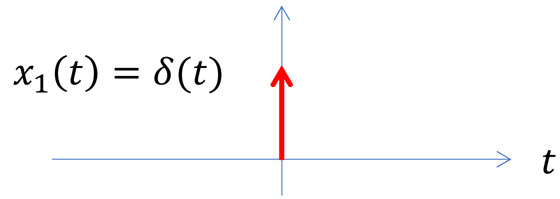
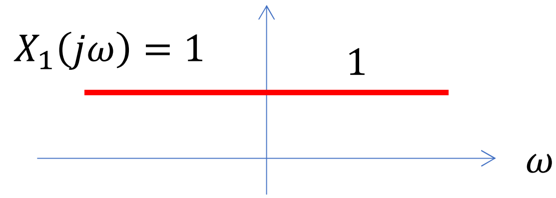
Fourier Transform Properties (Time Shift)

Example:



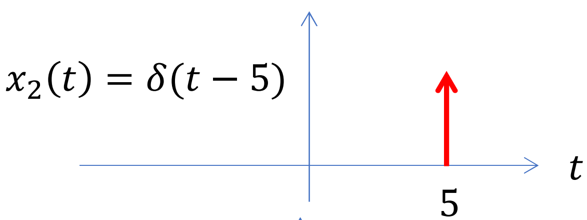


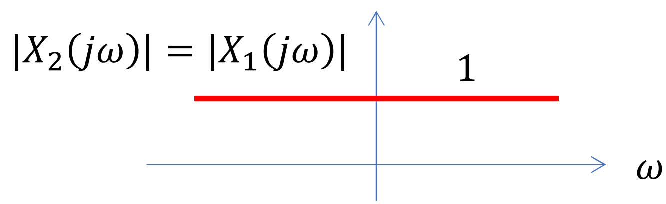
$$x_1(t) = \delta(t) \Rightarrow \quad X_1(j\omega) = 1$$

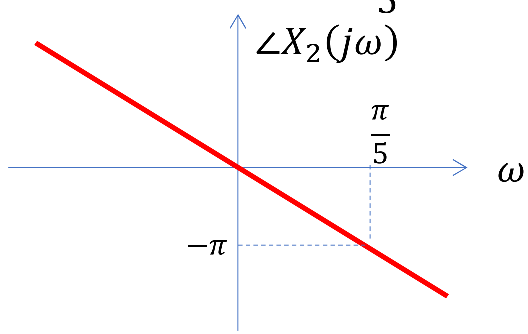
$$x_2(t) = \delta(t-5) = x_1(t-5) \Rightarrow \quad X_2(j\omega) = e^{-j5\omega}X_1(j\omega) = e^{-j5\omega}$$

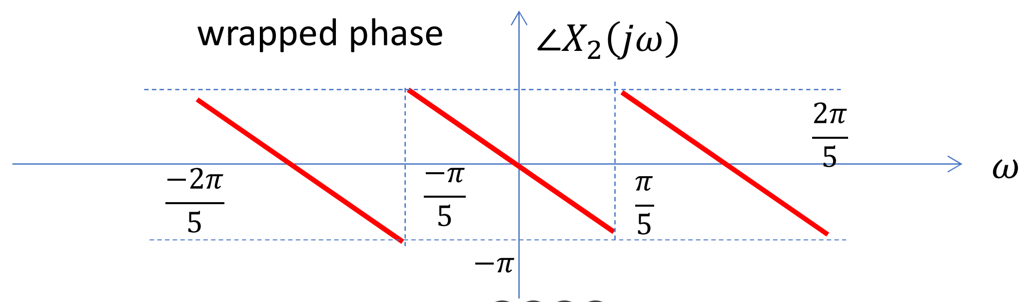
$$|X_2(j\omega)| = |e^{-j5\omega}| = 1$$

$$\angle X_2(j\omega) = \angle X_1(j\omega) - 5\omega = 0 - 5\omega$$









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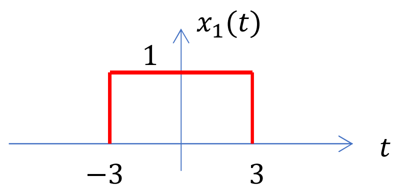
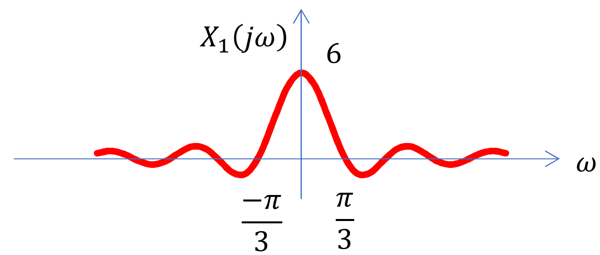
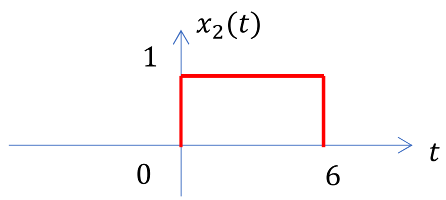
Fourier Transform Properties (Time Shift)

Examples: For the given signal $x(t)$ and its related Fourier transform $X_1(j\omega) = \frac{2sin(3\omega)}{\omega}$, what is FT of $x_2(t)$?







?

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_1(t-t_0) \xrightarrow{FT} e^{-j\omega t_0} X_1(j\omega)$$

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Fourier Transform Properties (Time Shift)

Examples: For the given signal $x(t)$ and its related Fourier transform $X_1(j\omega) = \frac{2sin(3\omega)}{\omega}$, what is FT of $x_2(t)$?

$$x_2(t) = x_1(t-3)$$

$$X_2(j\omega) = X_1(j\omega)e^{-j\omega 3} = e^{-j\omega 3}\left(\frac{2sin(3\omega)}{\omega}\right)$$

$$|X_2(j\omega) = |X_1(j\omega)|$$

$$\angle X_2(j\omega) = \angle X_1(j\omega) - 3\omega$$

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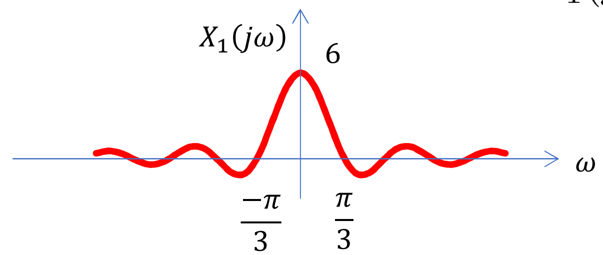
Fourier Transform Properties (Time Shift)

Examples: For the given signal $x(t)$ and its related Fourier transform $X_1(j\omega) = \frac{2sin(3\omega)}{\omega}$, what is FT of $x_2(t)$?



$$x_2(t) = x_1(t-3)$$

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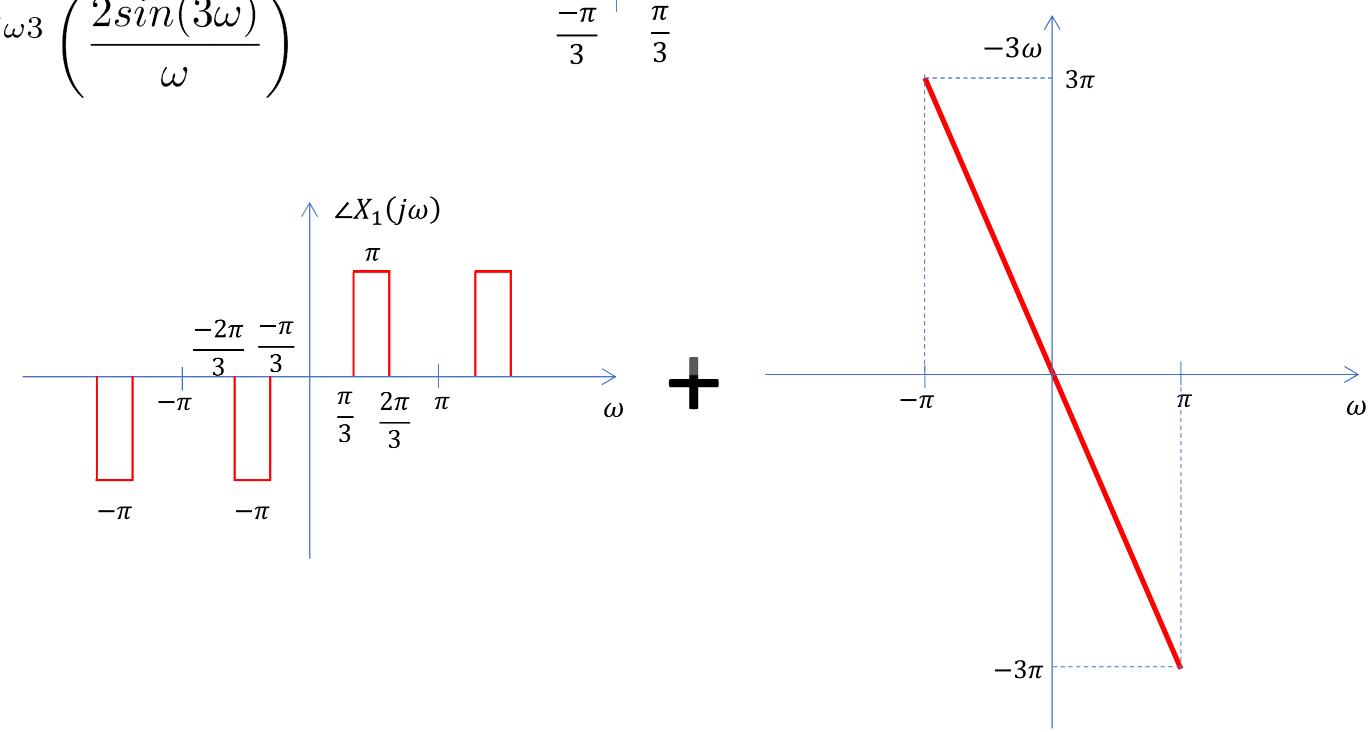

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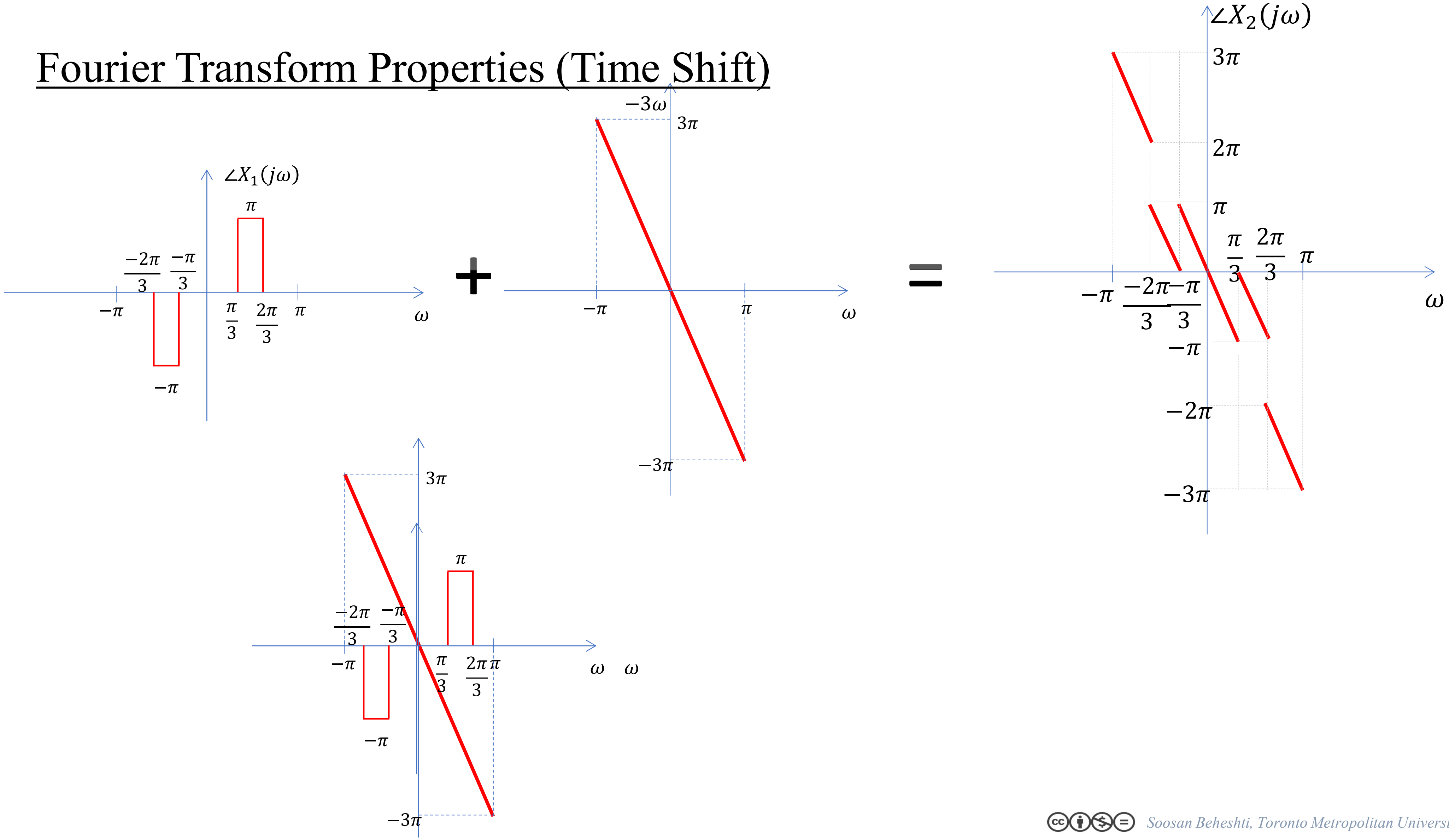
Fourier Transform Properties (Time Shift)



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$$x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform Properties (Frequency Shift)

3. Frequency Shift

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) =? \xrightarrow{FT} X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right)$$

Use change of variable similar to the time shift proof.

$$x_2(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(j\omega) e^{j\omega t} d\omega$$

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$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$
$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform Properties (Frequency Shift)

3. Frequency Shift

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$
$$x_2(t) =? \xrightarrow{FT} X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right)$$

$$\begin{aligned}
x_2(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_2(j\omega) e^{j\omega t} d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1\left(j(\omega - \omega_0)\right) e^{j\omega t} d\omega, \quad \omega - \omega_0 = V \rightarrow \omega = V + \omega_0 \rightarrow dV = d\omega \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(jV) e^{j(V+\omega_0)t} dV \\
&= e^{j\omega_0 t} \times \frac{1}{2\pi} \int_{-\infty}^{\infty} X_1(jV) e^{jVt} dV \\
&= e^{j\omega_0 t} x_1(t)
\end{aligned}$$

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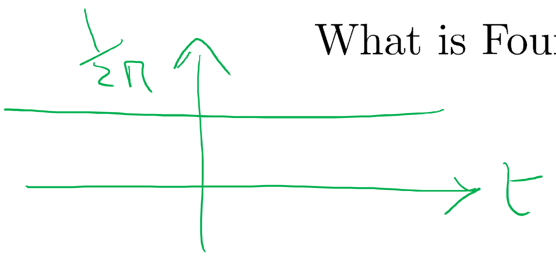
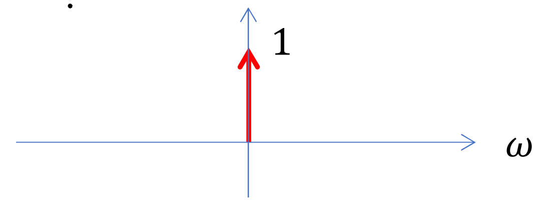
Fourier Transform Properties (Frequency Shift)

Important Example:



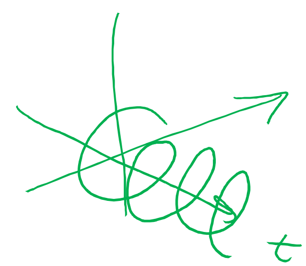
What is Fourier Transform of the periodic exponential $e^{j\omega_0 t}$?

$$x_1(t) = \frac{1}{2\pi} \xrightarrow{FT} X_1(j\omega) = \delta(\omega)$$

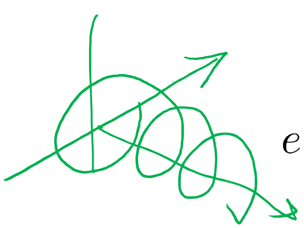


$$X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right) = \delta(\omega - \omega_0) \xrightarrow{IFT} x_2(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

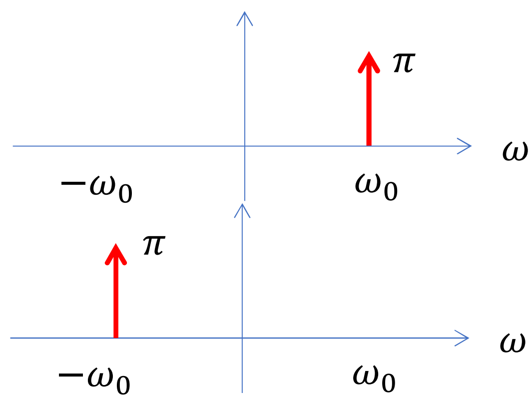
$$X_3(j\omega) = X_1\left(j(\omega + \omega_0)\right) = \delta(\omega + \omega_0) \xrightarrow{IFT} x_3(t) = \frac{1}{2\pi} e^{-j\omega_0 t}$$



$e^{j\omega_0 t} \xrightarrow{FT}$



$e^{-j\omega_0 t} \xrightarrow{FT}$



What were the Fourier Series of these signals?

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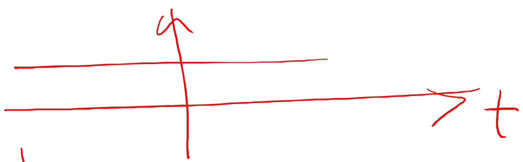
Fourier Transform Properties (Frequency Shift)

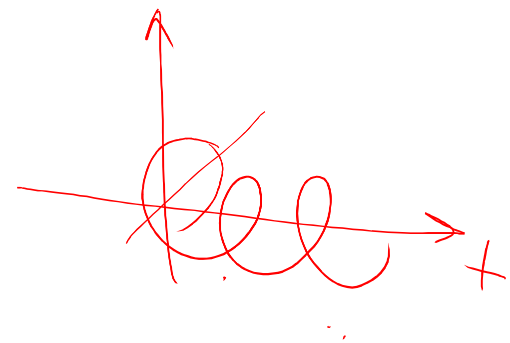
Example:



real signal

$$x_1(t) = \frac{1}{2\pi} \xrightarrow{FT} X_1(j\omega) = \delta(\omega)$$

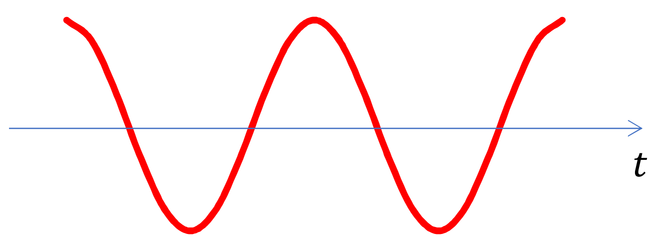
$$X_2(j\omega) = X_1\left(j(\omega - \omega_0)\right) = \delta(\omega - \omega_0) \xrightarrow{IFT} x_2(t) = \frac{1}{2\pi} e^{j\omega_0 t}$$

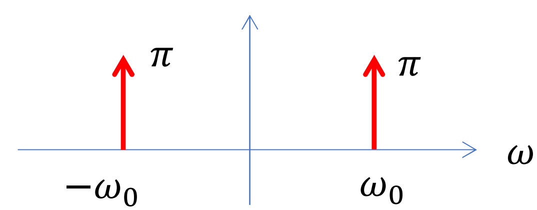


$$X_3(j\omega) = X_1\left(j(\omega + \omega_0)\right) = \delta(\omega + \omega_0) \xrightarrow{IFT} x_3(t) = \frac{1}{2\pi} e^{-j\omega_0 t}$$

$$X_2(j\omega) + X_3(j\omega) \xrightarrow{IFT} \pi\left(x_2(t) + x_3(t)\right) = \cos(\omega_0 t)$$

$\cos(\omega_0 t)$





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Fourier Transform Properties (Frequency Shift)

Example:

$$x_1(t) = e^{-3t}u(t) \longrightarrow \quad X_1(j\omega) = \frac{1}{j\omega + 3}$$

$$x_2(t) = e^{j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_2(j\omega) = X_1\left(j(\omega - \underbrace{\frac{\pi}{2}}_{\omega_0})\right) = \frac{1}{j(\omega - \frac{\pi}{2}) + 3}$$

$$x_3(t) = e^{-j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_3(j\omega) = X_1\left(j(\omega + \underbrace{\frac{\pi}{2}}_{\omega_0 = -\frac{\pi}{2}})\right) = \frac{1}{j(\omega + \frac{\pi}{2}) + 3}$$

$$x_4(t) = x_2(t) + x_3(t) = e^{-3t}u(t)\left(e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}\right) = \underbrace{e^{-3t}u(t) \times 2\cos(\frac{\pi}{2}t)}_{\text{Amplitude Modulation (AM) in time}}$$

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Fourier Transform Properties (Frequency Shift)

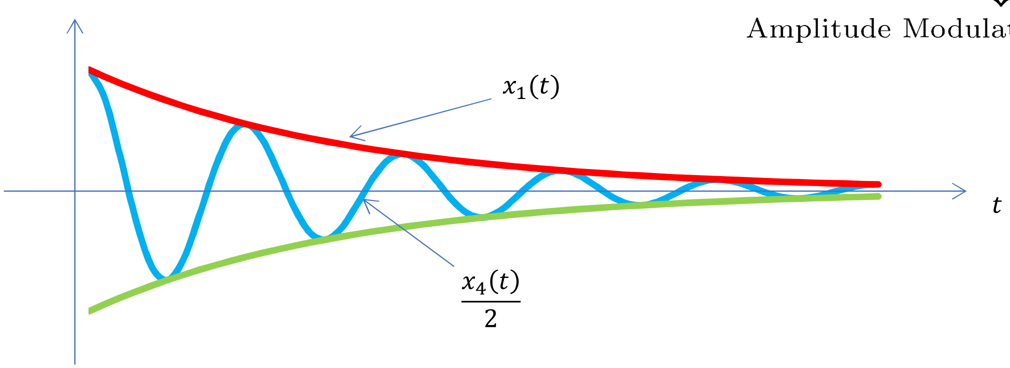
Example:

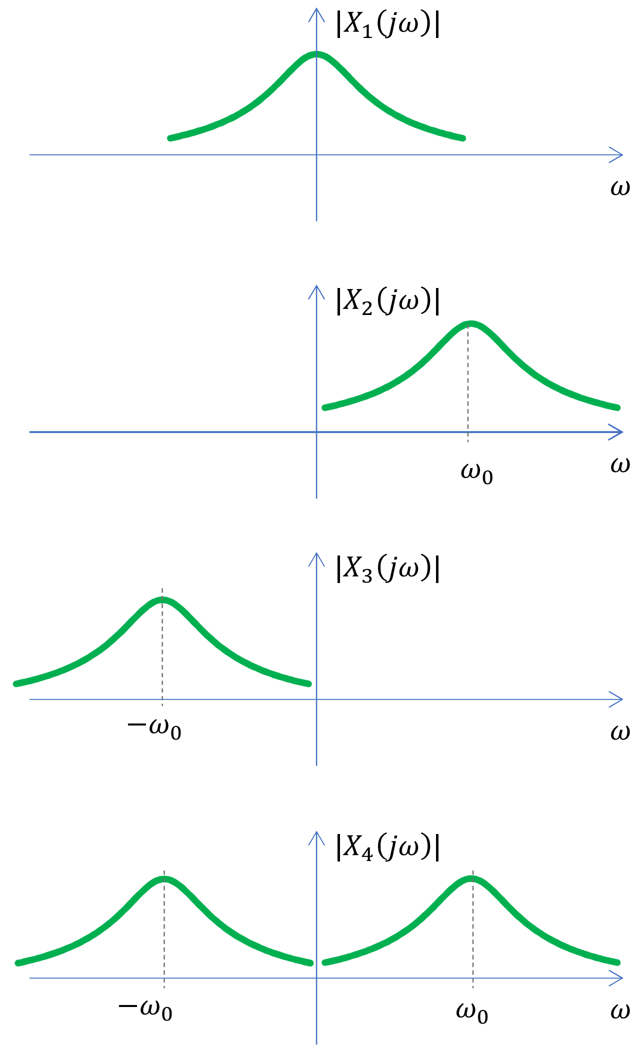
$$x_1(t) = e^{-3t}u(t) \longrightarrow \quad X_1(j\omega) = \frac{1}{j\omega + 3}$$

$$x_2(t) = e^{j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_2(j\omega) = X_1\left(j(\omega - \underbrace{\frac{\pi}{2}}_{\omega_0})\right) = \frac{1}{j(\omega - \frac{\pi}{2}) + 3}$$

$$x_3(t) = e^{-j\frac{\pi}{2}t}e^{-3t}u(t) \longleftarrow \quad X_3(j\omega) = X_1\left(j(\omega + \underbrace{\frac{\pi}{2}}_{\omega_0 = -\frac{\pi}{2}})\right) = \frac{1}{j(\omega + \frac{\pi}{2}) + 3}$$

$$x_4(t) = x_2(t) + x_3(t) = e^{-3t}u(t)\left(e^{j\frac{\pi}{2}t} + e^{-j\frac{\pi}{2}t}\right) = \underbrace{e^{-3t}u(t) \times 2\cos(\frac{\pi}{2}t)}_{\text{Amplitude Modulation (AM) in time}}$$





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Fourier Transform Properties (Frequency Shift)

Fourier Transform of causal part of real part of an exponential signal ($\sigma > 0$):

$$x(t) = e^{-\sigma t}\cos(\omega_0 t)u(t)$$

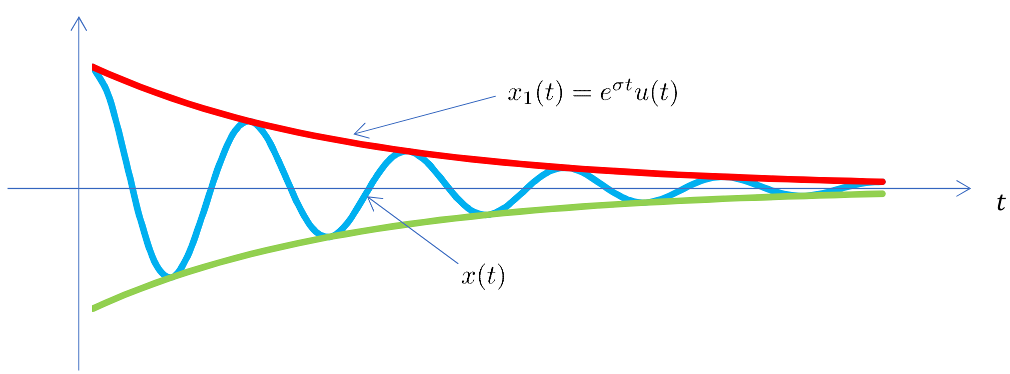
$$x_1(t) = e^{-\sigma t}u(t) \qquad FT(x_1(t)) = X_1(j\omega) = \frac{1}{j\omega + \sigma}$$

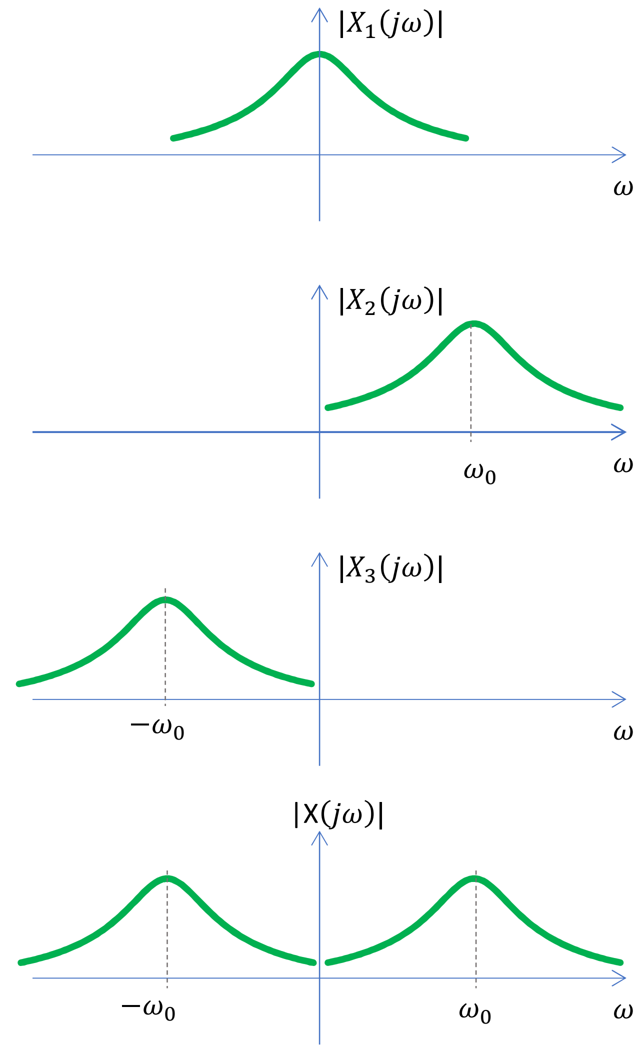
$$x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}x_1(t)$$

$$FT(x(t)) = \frac{1}{2}(X_1(j\omega - \omega_0) + X_1(j\omega + \omega_0))$$

$$= \frac{1}{2}(\frac{1}{j(\omega - \omega_0) + \sigma} + \frac{1}{j(\omega + \omega_0) + \sigma} = \frac{1}{2}(\frac{1}{j\omega + \sigma - j\omega_0} + \frac{1}{j\omega + \sigma + j\omega_0})$$

$$= \frac{1}{2}\frac{j\omega + \sigma - j\omega_0 + j\omega + \sigma + j\omega_0}{(j\omega + \sigma)^2 - (j\omega_0)^2} = \frac{j\omega + \sigma}{(j\omega + \sigma)^2 + \omega_0^2}$$





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$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Transform Properties (Time Scaling)

4. Scaling

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) = x_1(at) \xrightarrow{FT} X_2(j\omega) = ? \quad \text{as a function of} \quad X_1(j\omega)$$

Use change of variable similar to the previous properties

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt$$

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$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Fourier Transform Properties (Time Scaling)

4. Scaling

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) = x_1(at) \xrightarrow{FT} X_2(j\omega) =? \quad \text{as a function of} \quad X_1(j\omega)$$

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x_1(at)e^{-j\omega t}dt, \quad v = at \to t = \frac{v}{a} \to dv = adt$$

$$\text{If } a > 0 \quad = \int_{-\infty}^{\infty} x_1(v)e^{-j\omega\frac{v}{a}}\frac{dv}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x_1(v)e^{-j\frac{\omega}{a}v}dv = \frac{1}{a}X_1(j\frac{\omega}{a})$$

For $a < 0$ we have $dt = -\frac{dv}{|a|}$,
therefore
$t \to \infty$ then $u \to -\infty$
similarly
$t \to -\infty$ then $u \to \infty$

$$\text{If } a < 0 \quad = \frac{1}{|a|}\int_{-\infty}^{\infty} x_1(v)e^{-j\frac{\omega}{a}v}dv = \frac{1}{|a|}X_1(j\frac{\omega}{a})$$

So we have $\int_{\infty}^{-\infty} \cdots dv = -\int_{-\infty}^{\infty} \cdots dv$

In general for all a

$$X_2(j\omega) = \frac{1}{|a|}X_1(j\frac{\omega}{a})$$

If $a = -1$

$$x(-t) \longrightarrow X(-j\omega)$$

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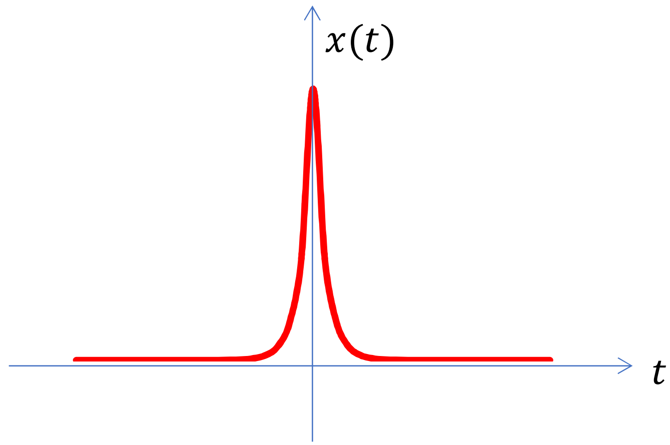
Fourier Transform Properties (Scaling)

Example:

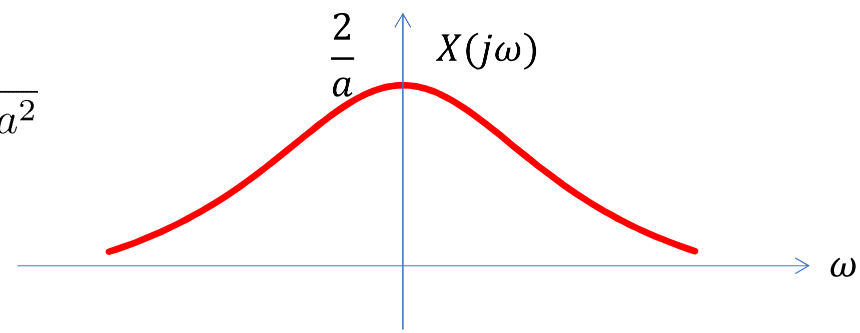
$$x(t) = e^{-2|t|}$$
$$= \underbrace{e^{-2t}u(t)}_{x_1(t)} + \underbrace{e^{2t}u(-t)}_{x_1(-t)}$$



$$X(j\omega) = X_1(j\omega) + X_1(-j\omega)$$
$$= \frac{1}{j\omega + 2} + \frac{1}{-j\omega + 2}$$
$$= \frac{2 \times 2}{\omega^2 + 4} = \frac{4}{\omega^2 + 4}$$

In general:

$$x(t) = e^{-|a|t} \longrightarrow X(j\omega) = \frac{2a}{\omega^2 + a^2}$$



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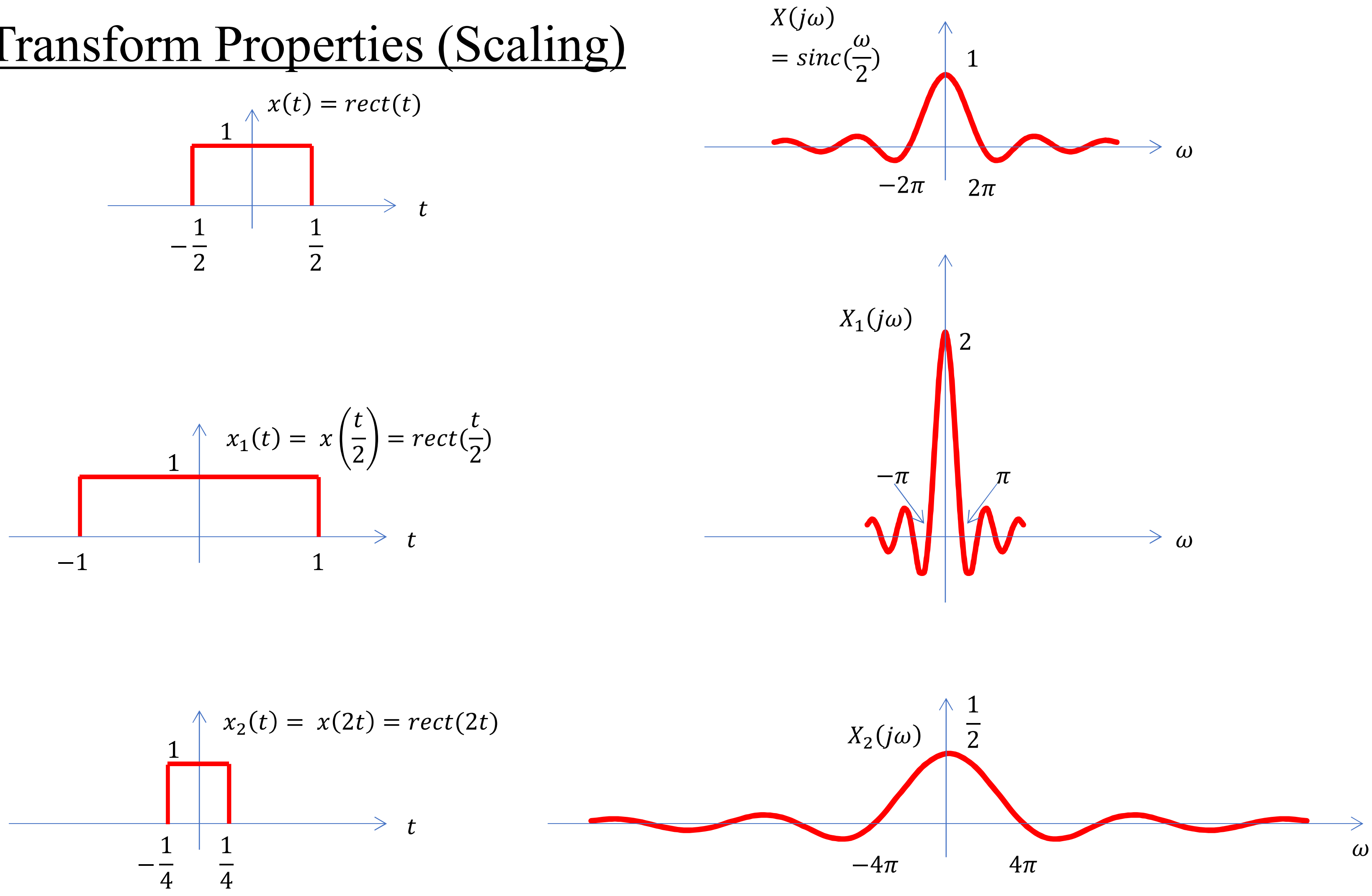
Fourier Transform Properties (Scaling)

Example:



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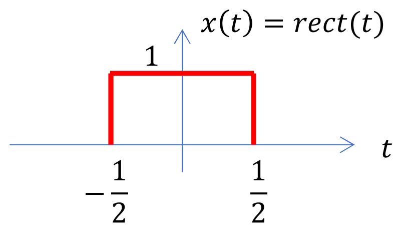
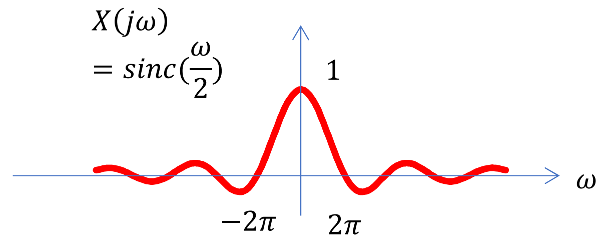
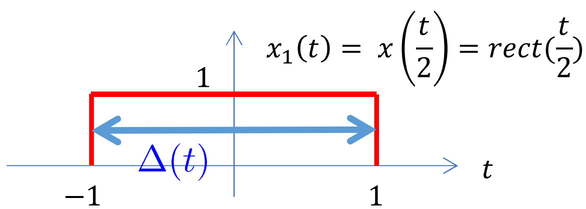
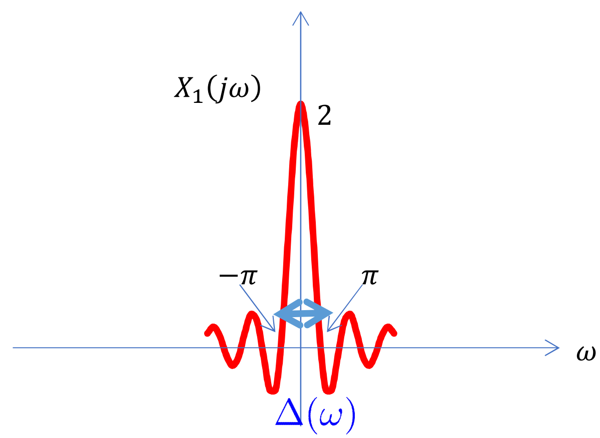
Fourier Transform Properties (Scaling)

Example:

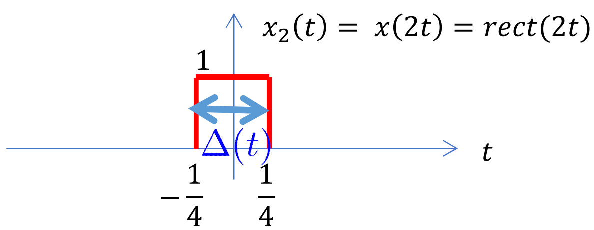


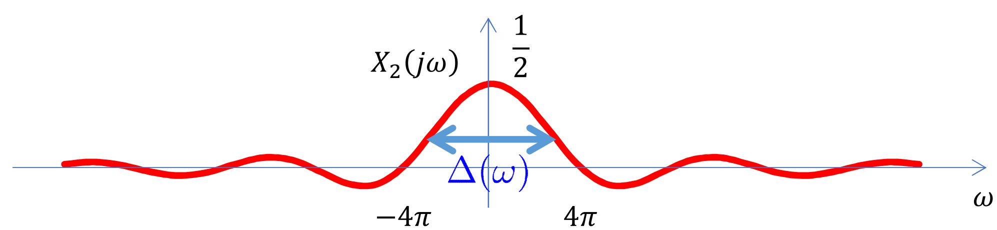






This is consistent with **Heisenberg's uncertainty principle:**
$\Delta t \times \Delta\omega > constant\ value$





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Fourier Transform Properties (Duality)

5. Duality

$$x_1(t) \longrightarrow X_1(j\omega)$$
$$x_2(t) = X_1(jt) \longrightarrow X_2(j\omega) = ?$$

$$X_2(j\omega) = \int_{-\infty}^{\infty} x_2(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} X_1(jt) e^{-j\omega t} dt$$

Replace ω with V

$$x_1(t) = \frac{1}{2\pi} \int X_1(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int X_1(jV) e^{jVt} dV$$

Replace t with ω

$$x_1(\omega) = \frac{1}{2\pi} \int X_1(jV) e^{jV\omega} dV$$

Replace V with t

$$2\pi x_1(-\omega) = \int X_1(jt) e^{-jt\omega} dt$$

$$2\pi x_1(-\omega) = X_2(j\omega)$$

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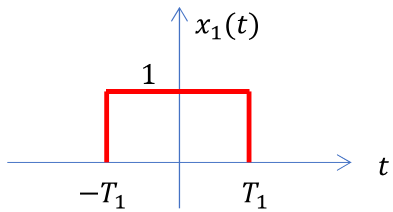
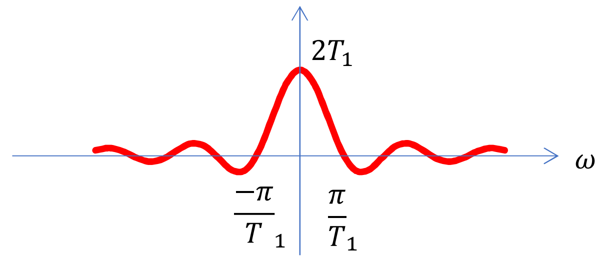
Fourier Transform Properties (Duality)

$$X_1(j\omega) = 2T_1 \frac{\sin(\omega T_1)}{\omega T_1}$$

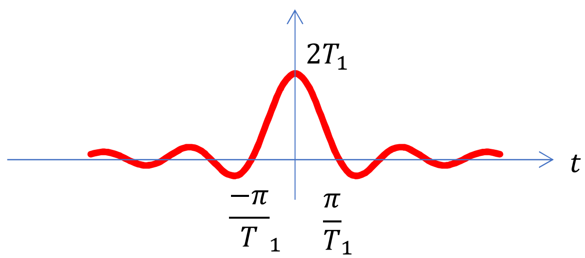
Example:

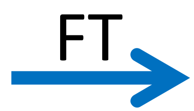


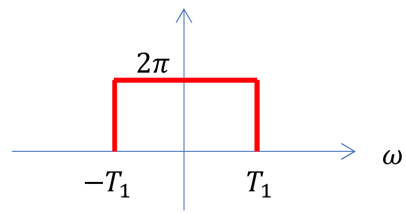


$$x_2(t) = X_1(jt) = 2T_1 \frac{\sin(tT_1)}{tT_1}$$

$$X_2(j\omega) = 2\pi x_1(-\omega)$$







Check this answer:

$$x_2(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_2(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-T_1}^{T_1} 2\pi e^{j\omega t}d\omega$$

$$= \frac{e^{jtT_1} - e^{-jtT_1}}{jt} = \frac{2sin(tT_1)}{t}$$

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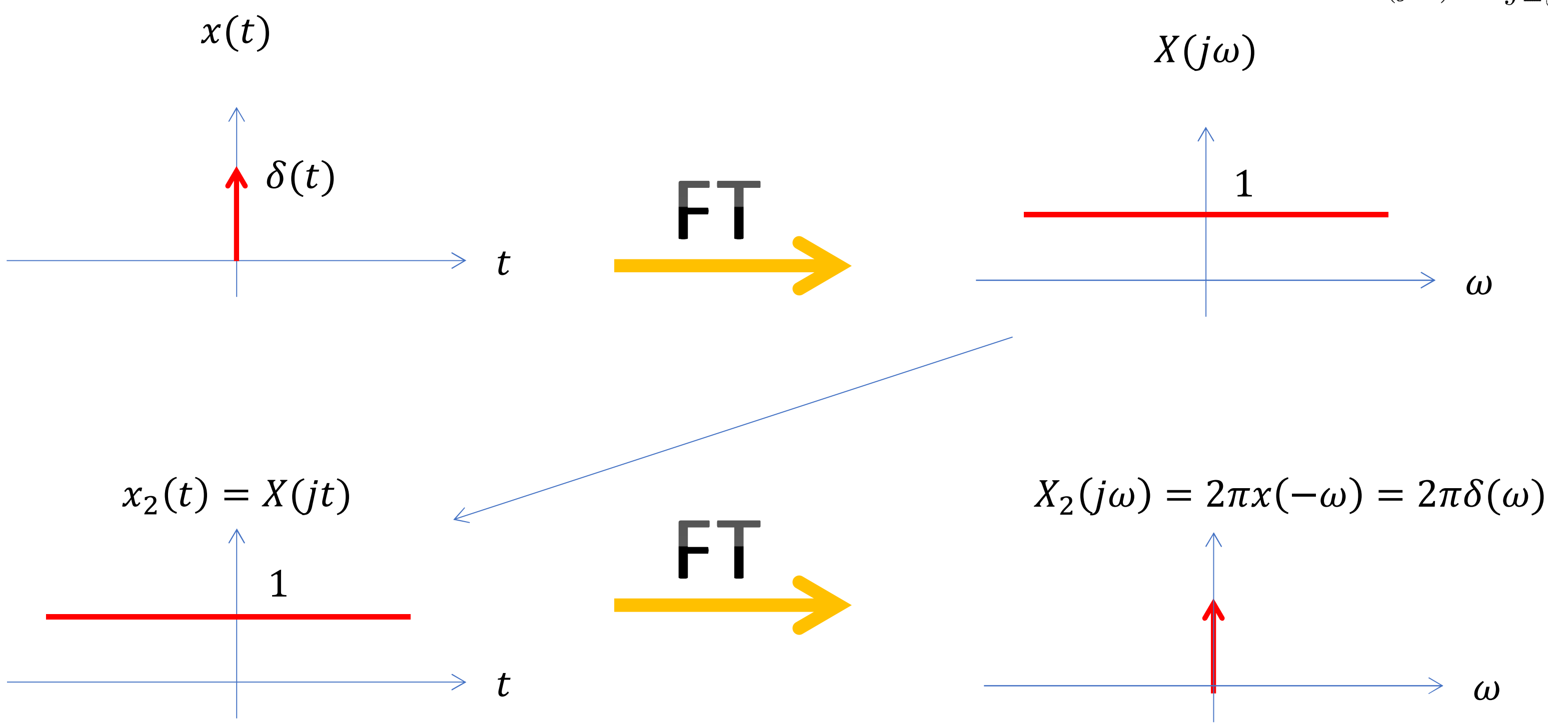
Fourier Transform Properties (Duality)

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$



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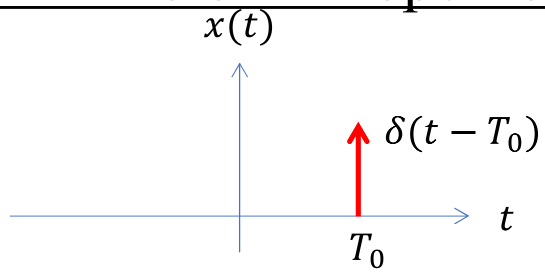
Fourier Transform Properties (Duality)

Example:



$\xrightarrow{\text{FT}}$

Periodic Spiral in Frequency domain with freq. T_0

$$X(j\omega) = e^{-j\omega T_0}$$

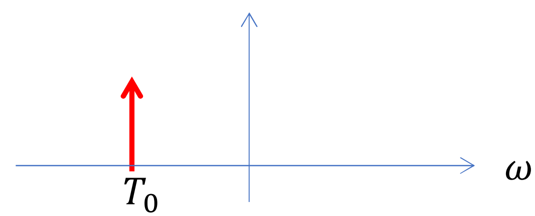
Periodic Spiral in Time with freq. T_0

$$x_2(t) = e^{-jT_0 t}$$

$\xrightarrow{\text{FT}}$

$$X_2(j\omega) = 2\pi \overbrace{\delta(-\omega - T_0)}^{x(-\omega)}$$

$$= 2\pi\delta(\omega + T_0)$$



Same for shift to other direction:

$$e^{jT_0 t} \xrightarrow{FT} \quad 2\pi\delta(\omega - T_0)$$

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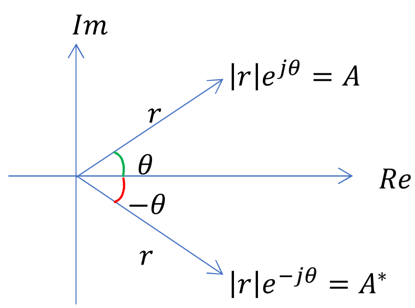
Fourier Transform Properties (Conjugate property)

6. Conjugate Property:

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$x^*(t) \xrightarrow{FT} X^*(-j\omega)$$

Reminder:



$A = a + jb$
$A^* = a - jb$

complex conjugate of a real number is itself!

Example:

$$x(t) = e^{j\omega_0 t} \longrightarrow \quad 2\pi\delta(\omega - \omega_0)$$

$$x^*(t) = (e^{j\omega_0 t})^* = e^{-j\omega_0 t} \longrightarrow X^*(-j\omega) = (2\pi\delta(-\omega - \omega_0))^* = 2\pi\delta(-\omega - \omega_0) = 2\pi\delta(\omega + \omega_0)$$

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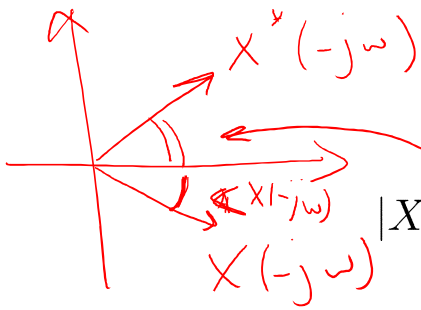
Conjugate symmetry property for real signals

If $x(t)$ is real: $x^*(t) = x(t)$ therefore:



$$FT(x^*(t)) = FT(x(t))$$
$$X^*(-j\omega) = X(j\omega)$$
$$|X(-j\omega)|e^{-j\angle X(-j\omega)} = |X(j\omega)|e^{j\angle X(j\omega)}$$

$$X(j\omega) = |X(j\omega)e^{j\angle X(j\omega)}$$
$$X(-j\omega) = |X(-j\omega)|e^{j\angle X(-j\omega)}$$
$$X^*(-j\omega) = |X(-j\omega)|e^{-j\angle X(-j\omega)}$$

For real signals:

1- $|X(-j\omega)| = |X(j\omega)|$ Absolute value is an even function
2- $-\angle X(-j\omega) = \angle X(j\omega)$ Phase is an odd function

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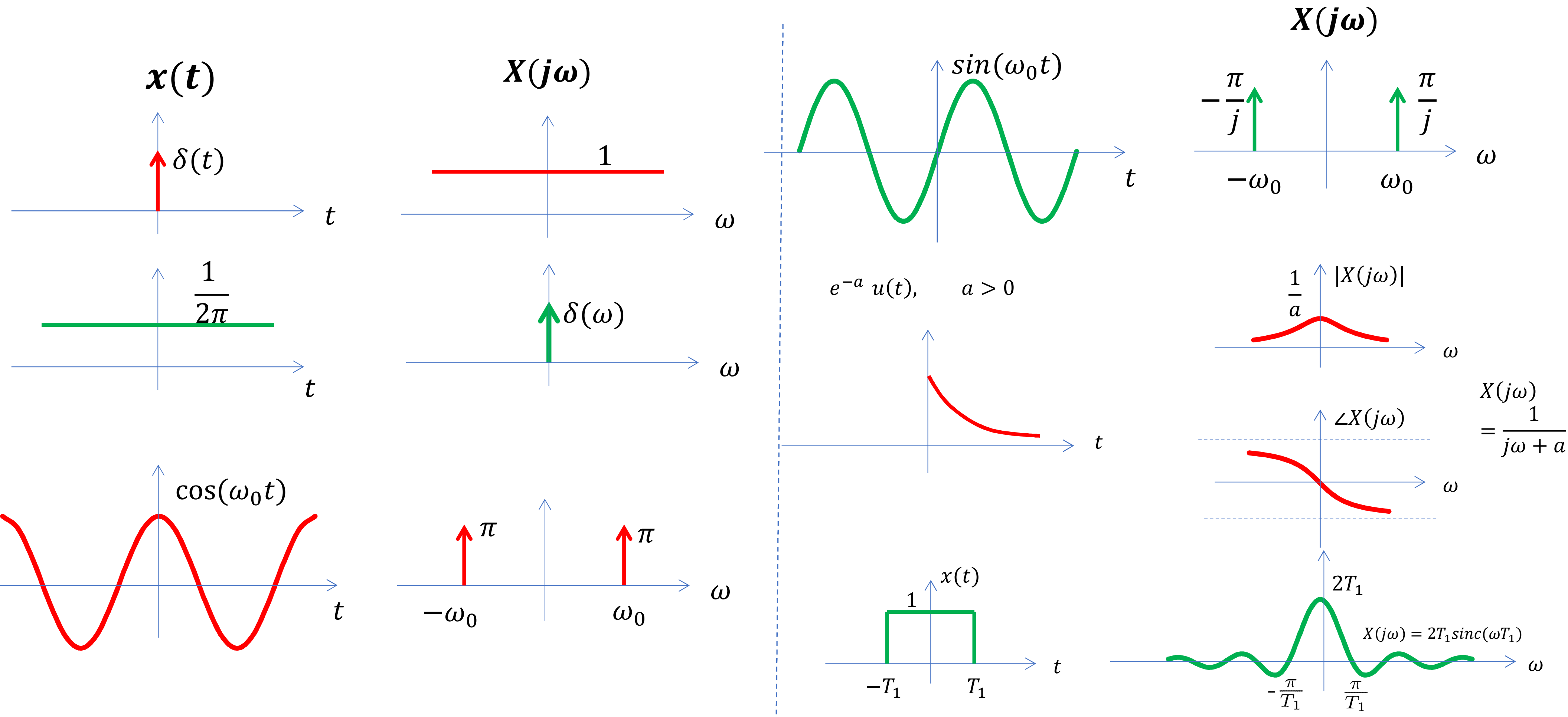
Complex Conjugate property for real signals



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Fourier Transform & Convolution

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

7. FT & Convolution

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$

$$x_2(t) \xrightarrow{FT} X_2(j\omega)$$

$$x_3(t) = x_1(t) * x_2(t) \xrightarrow{FT} X_3(j\omega) = X_1(j\omega) \times X_2(j\omega)$$

Convolution in time $\equiv$ Product in Frequency

Example:

$$x_1(t) = e^{-2t}u(t) \xrightarrow{FT} \frac{1}{j\omega + 2}$$

$$x_2(t) = \delta(t-3) \xrightarrow{FT} e^{-j\omega 3}$$

$$x_1(t) * x_2(t) = e^{-2t}u(t) * \delta(t-3) = e^{-2(t-3)}u(t-3) = e^6 e^{-2t} u(t-3)$$

FT transform using the product

$$X_1(j\omega) \times X_2(j\omega) = \frac{e^{-j\omega 3}}{j\omega + 2} = X_3(j\omega)$$

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Fourier Transform & Convolution

Example:

FT transform using the product

$$X_1(j\omega) \times X_2(j\omega) = \frac{e^{-j\omega 3}}{j\omega + 2} = X_3(j\omega)$$

FT transform using the definition

$$\begin{aligned}
X_3(j\omega) &= \int_{-\infty}^{\infty} x_3(t) e^{-j\omega t} dt \\
&= \int_{-\infty}^{\infty} e^6 e^{-2t} u(t-3) e^{-j\omega t} dt \\
&= e^6 \int_{3}^{\infty} e^{-2t} e^{-j\omega t} dt \\
&= e^6 \frac{e^{-t(2+j\omega)}}{-(2+j\omega)} \Bigg|_3^{\infty} \\
&= -e^6 \frac{e^{-3(2+j\omega)}}{-(2+j\omega)} \\
&= \frac{e^{-3j\omega}}{2+j\omega}
\end{aligned}$$

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$e^{j\omega_0 nt} \xrightarrow{FT} 2\pi\delta(\omega - \omega_0 n)$

Fourier Transform & Periodic Signals

8. FT & periodic signals

$$x_p(t) = \sum_n D_n e^{j\omega_0 nt} \xrightarrow{FT} X_p(j\omega) = FT\left(\sum D_n e^{j\omega_0 nt}\right)$$

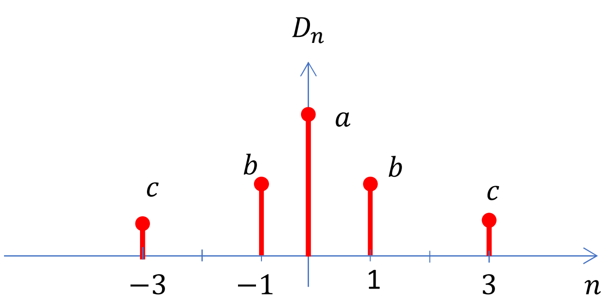
$$= \sum FT\left(D_n e^{j\omega_0 nt}\right)$$

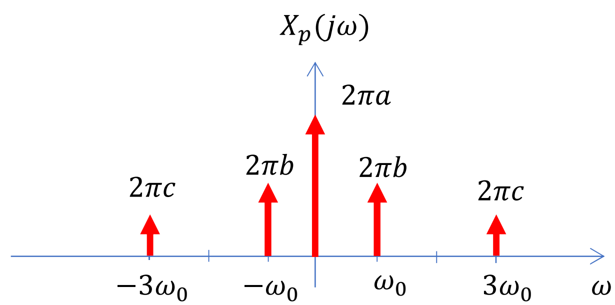
$$= \sum D_n FT\left(e^{j\omega_0 nt}\right)$$

$$= \sum D_n 2\pi\delta\left(\omega - \omega_0 n\right)$$

To calculate FT of a periodic signal first find its FS and then use this property.

$$X_p(j\omega) = \sum_n D_n 2\pi\delta(\omega - \omega_0 n)$$





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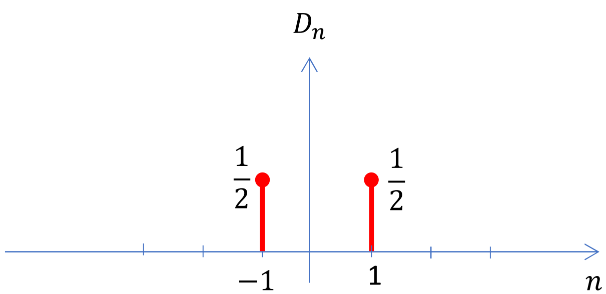
Fourier Transform & Periodic Signals

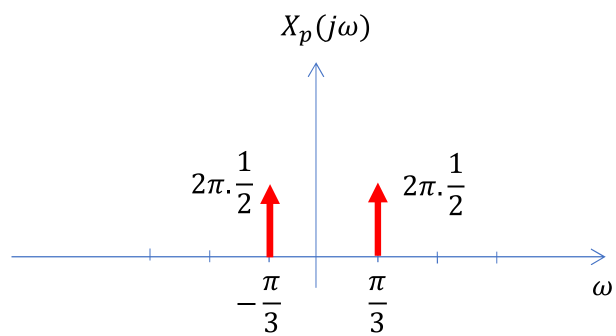
Example:

$$x_p(t) = cos(\underbrace{\frac{\pi}{3}}_{\omega_0} t) = \underbrace{\frac{1}{2}}_{D_1} e^{j\frac{\pi}{3}t} + \underbrace{\frac{1}{2}}_{D_{-1}} e^{-j\frac{\pi}{3}t}$$

$$X_p(j\omega) = \pi\delta(\omega - \underbrace{\omega_0}_{\omega_0 = \frac{\pi}{3}}) + \pi\delta(\omega + \omega_0)$$

There are two methods to find ft of sin and cos through shift and connection of periodic signals.





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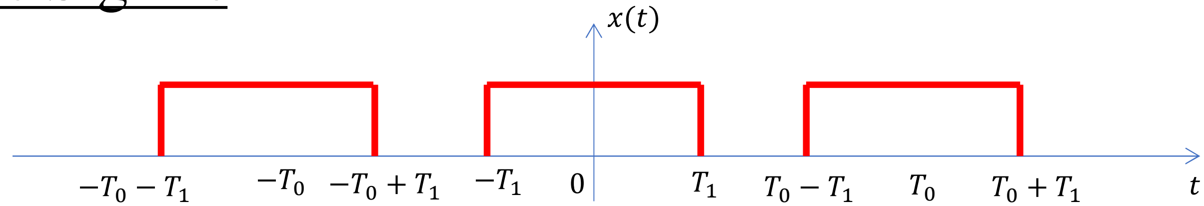
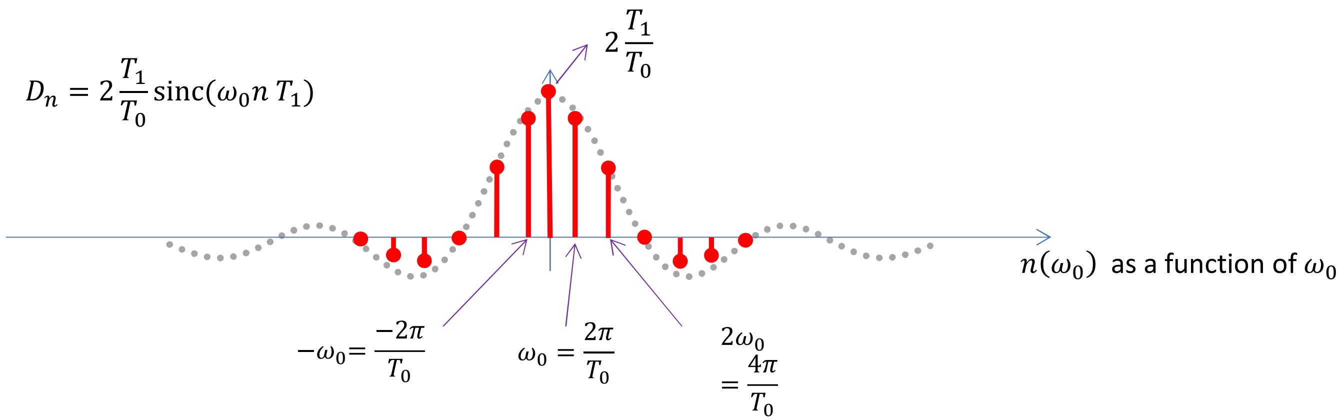
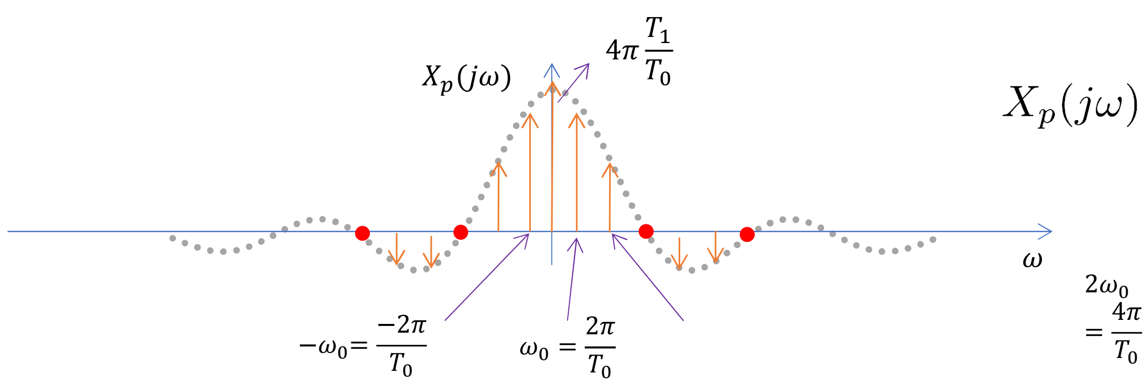
Fourier Transform & Periodic Signals







$$X_p(j\omega) = \sum 2\pi \left(\frac{2T_1}{T_0} sinc(\omega_0 n T_1)\right) \delta(\omega - n\omega_0)$$

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Fourier Transform & Periodic Signals

FT properties	**Signal**	**FT**
	$x(t)$	$X(j\omega)$
	$z(t)$	$Z(j\omega)$
Linearity	$ax(t) + bz(t)$	$aX(j\omega) + bZ(j\omega)$
Time shift	$x(t - T_0)$	$e^{-j\omega T_0} X(j\omega)$
Freq. shift	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X(j\frac{\omega}{a})$
Duality	$X(jt)$	$2\pi x(-\omega)$
Complex Conj.	$x^*(t)$	$X^*(-j\omega)$ (so for real signals $\lvert X(j\omega \rvert$ is even and $\angle(X(j\omega)$ is odd
Convolution	$x(t) * z(t)$	$X(j\omega) \times Z(j\omega)$
Periodic signals	$x_p(t)$ with D_n coeffs	$X_p(j\omega) = \sum_n D_n 2\pi\delta(\omega - \omega_0 n)$

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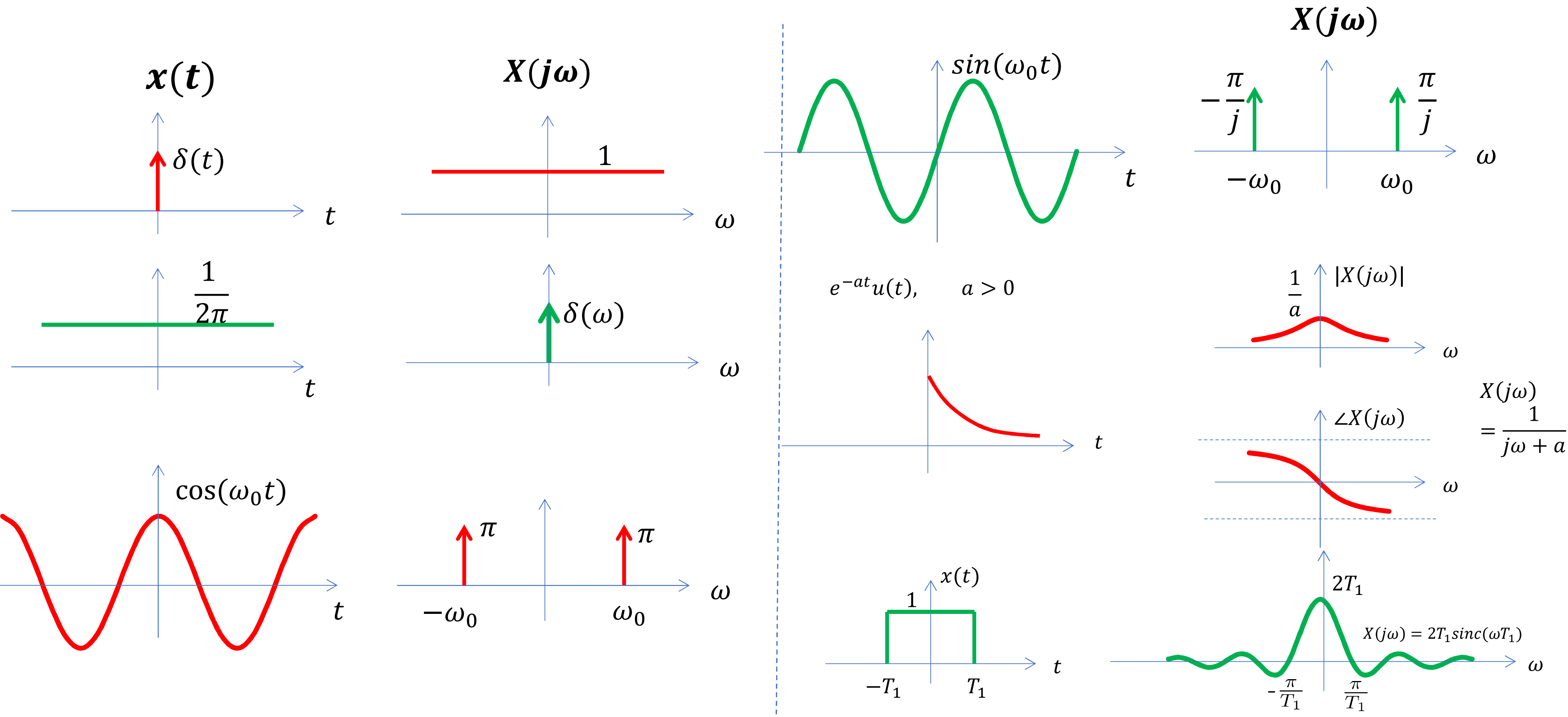
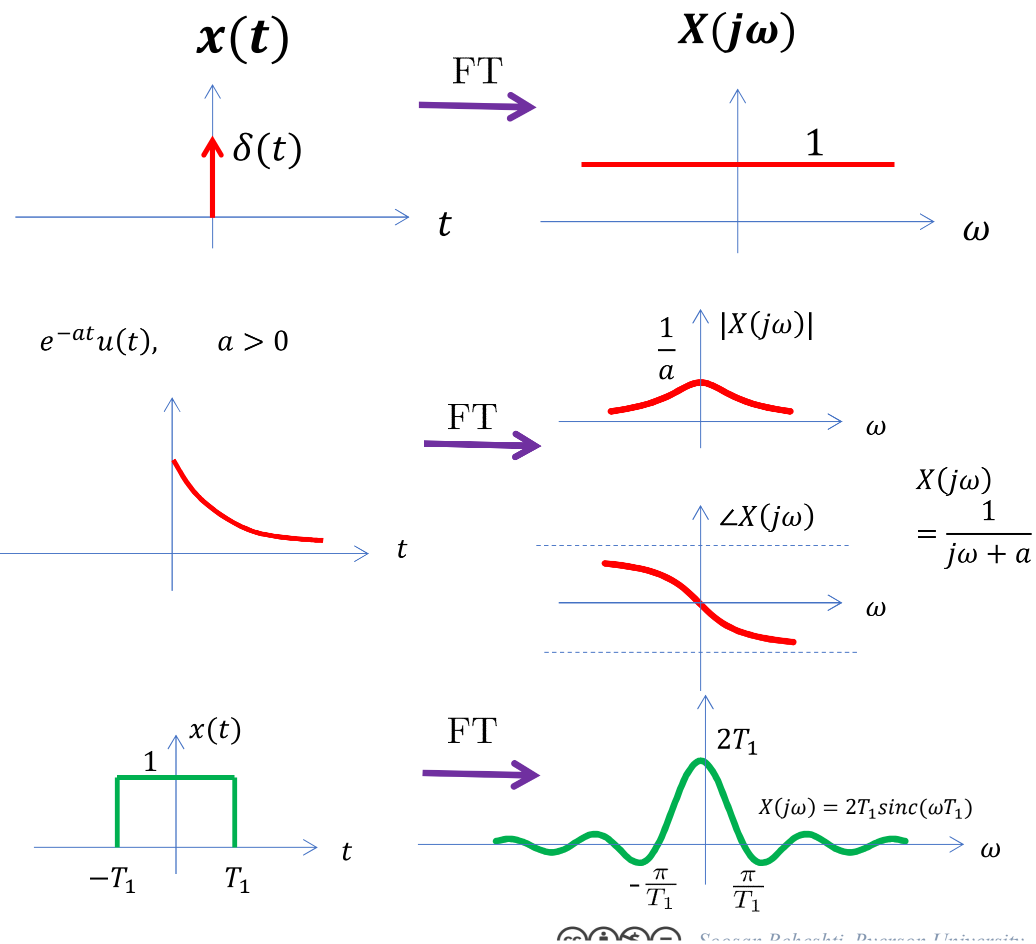
Fourier Transform of some Important Signals (review)



Fourier Transform of some Important Signals

Synthesis Equation: $x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

Analysis Equation: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$



The FTs can be calculated using the FT equation (The Analysis equation)

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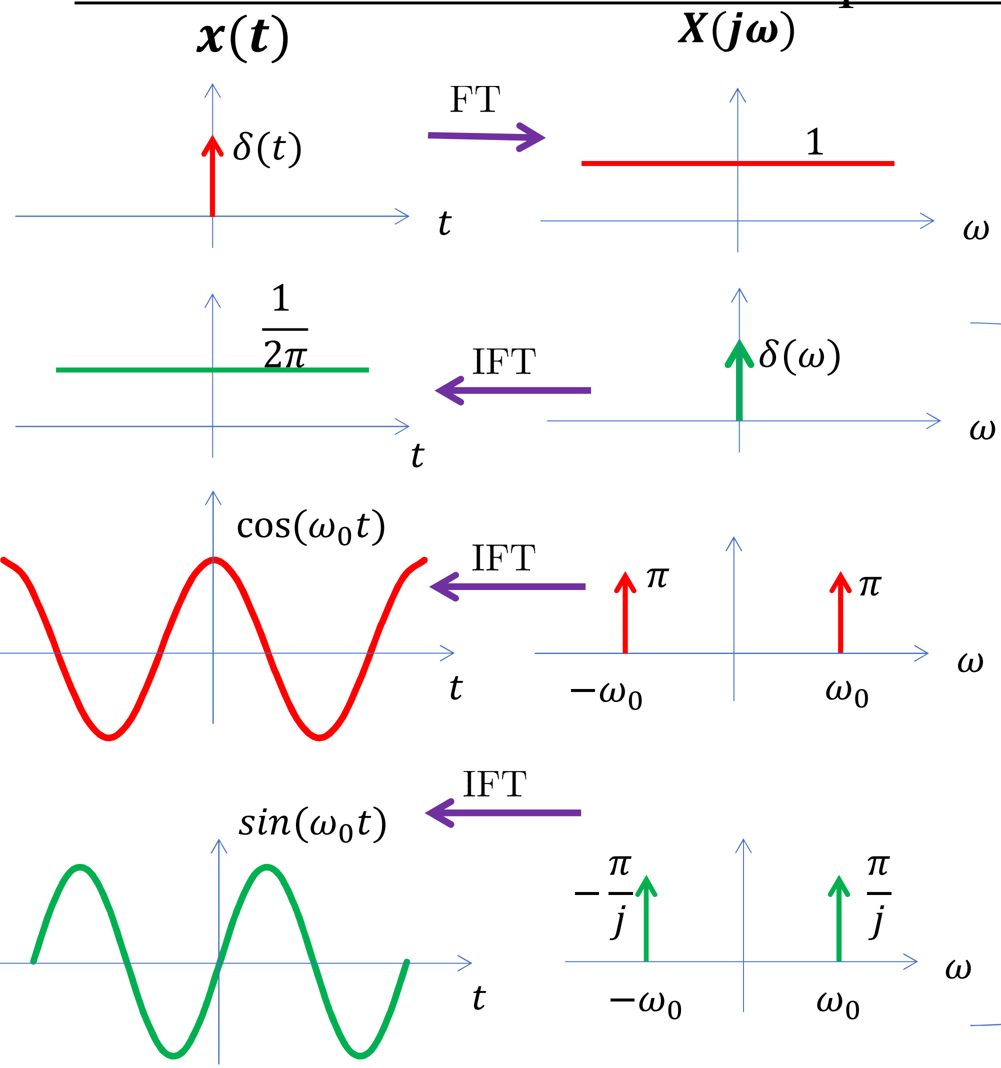
Fourier Transform of some Important Signals

Synthesis Equation: $x(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$

Analysis Equation: $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$



The FT can be easily calculated using the FT equation (The Analysis equation)

The inverse FT (IFT) can be easily calculated using:

1-The IFT equation (Synthesis equation)

2- Duality property

The FT can be calculated

3- using the periodic property

4- or directly by using the synthesis equation! (not as convenient as the above methods)

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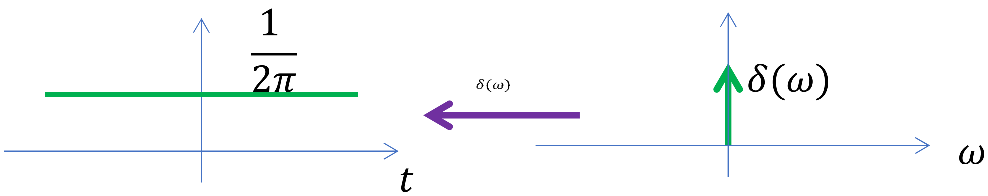
Fourier Transform of a constant signal



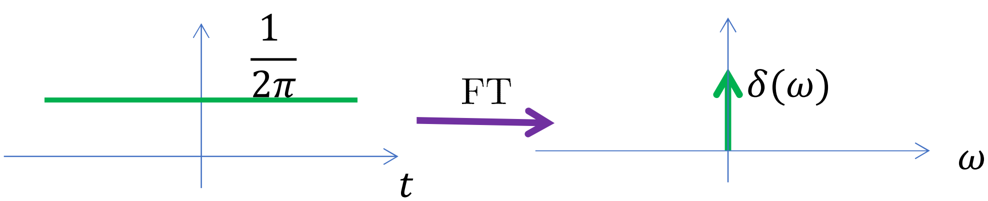
The inverse FT (IFT) can be easily calculated using:

1-The IFT equation (Synthesis equation)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)d\omega = \frac{1}{2\pi}$$

2- **Duality** property

$$\delta(t) \xrightarrow{\text{FT}} \Delta(j\omega) = 1 \xrightarrow{\text{FT}} 2\pi\delta(\omega)$$



The FT can be calculated

3- using the periodic property:

Constant signal is a pulse from $-T$ to T with period $2T$:

$D_0 = \frac{1}{2\pi}$ and the rest of the D_ns are zero. Therefore $X(j\omega) = 2\pi \times D_0\delta(\omega)$

4- or directly by using the synthesis equation! (not as convenient as the above methods):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \frac{1}{2\pi}e^{-j\omega t}dt = \begin{cases} \delta(\omega) & \text{if } \omega = 0 \\ 0 & \text{if } \omega \neq 0 \text{ (why?)} \end{cases}$$

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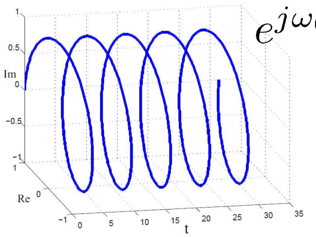
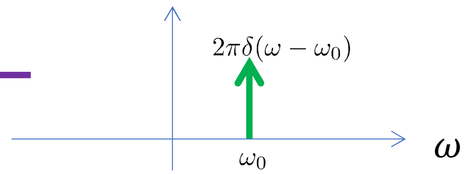
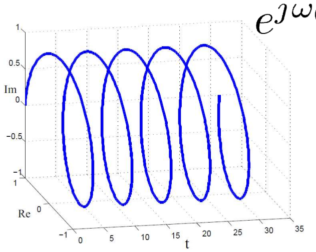
Fourier Transform of Periodic Spiral



$e^{j\omega_0 t}$

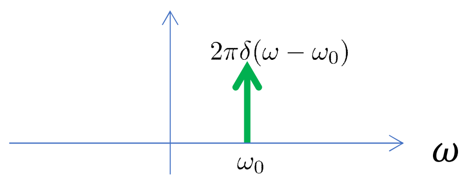
IFT $\longleftarrow$





$e^{j\omega_0 t}$

FT $\longrightarrow$



Orthogonality principle:

$$\int_{-\infty}^{\infty} e^{j\omega_0 t} e^{-j\omega t} dt = \begin{cases} 0 & \text{if} \quad \omega_0 \neq \omega \\ 2\pi\delta(\omega-\omega_0) & \text{if} \quad \omega_0 = \omega \end{cases}$$

Remember:

$$e^{j\omega_0 t} \xrightarrow{\text{Fourier Transform}} 2\pi\delta(\omega-\omega_0)$$

The inverse FT (IFT) can be easily calculated using:

1- **Direct** use of IFT definition:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega-\omega_0)e^{j\omega t}d\omega$$
$$= \int_{-\infty}^{\infty}\delta(\omega-\omega_0)e^{j\omega_0 t}d\omega = e^{j\omega_0 t}\underbrace{\int_{-\infty}^{\infty}\delta(\omega-\omega_0)d\omega}_{1} = e^{j\omega_0 t}$$

2- **Frequency Shift** property
$\delta(\omega) \xrightarrow{IFT} \frac{1}{2\pi}$ (constant signal)
Shifting the FT by ω is multiplication by $e^{j\omega_0 t}$ in time.

3- **Shift in time** property and **Duality** property

$x_1(t) = \delta(t+\omega_0) \xrightarrow{\text{FT}} X(j\omega) = e^{j\omega_0\omega}$
$x_2(t) = X(jt) = e^{j\omega_0 t} \xrightarrow{\text{FT}} 2\pi x_1(-\omega) = 2\pi\delta(\omega-\omega_0)$

The FT can be calculated

4- using the periodic property:

It is known that periodic spiral has only one nonzero D_n. If ω is positive $D_1 = 1$ and if it is negative $D_{-1} = 1$. Therefore $X(j\omega) = 2\pi \times D_1\delta(\omega-\omega_0)$ for positive ω_0 and $X(j\omega) = 2\pi \times D_{-1}\delta(\omega-\omega_0)$ for negative ω_0.

5- or directly by using the synthesis equation! (not as convenient as the above methods):

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{j\omega_0 t}e^{-j\omega t}dt = 2\pi\delta(\omega-\omega_0)$$

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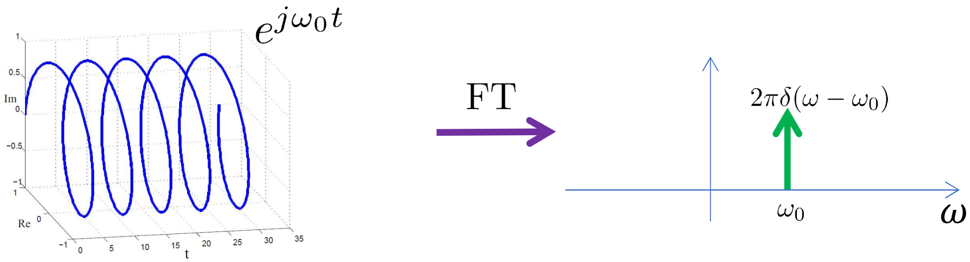
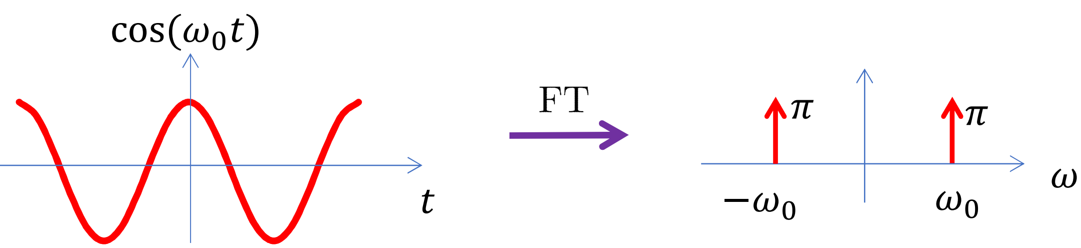
Fourier Transform of Cosine





Using either of five methods explained in the previous page, FT of cosine signal can be found through linear property of FT:

$$x(t) = \cos(t) = \tfrac{1}{2}e^{j\omega_0 t} + \tfrac{1}{2}e^{-j\omega_0 t} \xrightarrow{\text{FT}} X(j\omega) = \tfrac{1}{2}\text{FT}(e^{j\omega_0 t}) + \tfrac{1}{2}\text{FT}(e^{-j\omega_0 t})$$
$$= \tfrac{1}{2}2\pi\delta(\omega-\omega_0) + \tfrac{1}{2}2\pi\delta(\omega+\omega_0)$$

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Fourier Transform Properties (Product of Signals)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Remember convolution in time

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$
$$x_2(t) \xrightarrow{FT} X_2(j\omega)$$
$$x_3(t) = x_1(t) * x_2(t) \xrightarrow{FT} X_3(j\omega) = X_1(j\omega) \times X_2(j\omega)$$

Convolution in time $\equiv$ Product in Frequency

9. FT of product of signals

$$x_1(t) \xrightarrow{FT} X_1(j\omega)$$
$$x_2(t) \xrightarrow{FT} X_2(j\omega)$$
$$x_3(t) = x_1(t) \times x_2(t) \xrightarrow{FT} X_3(j\omega) = \frac{1}{2\pi} X_1(j\omega) * X_2(j\omega)$$

Product in time $\equiv$ Convolution in Frequency

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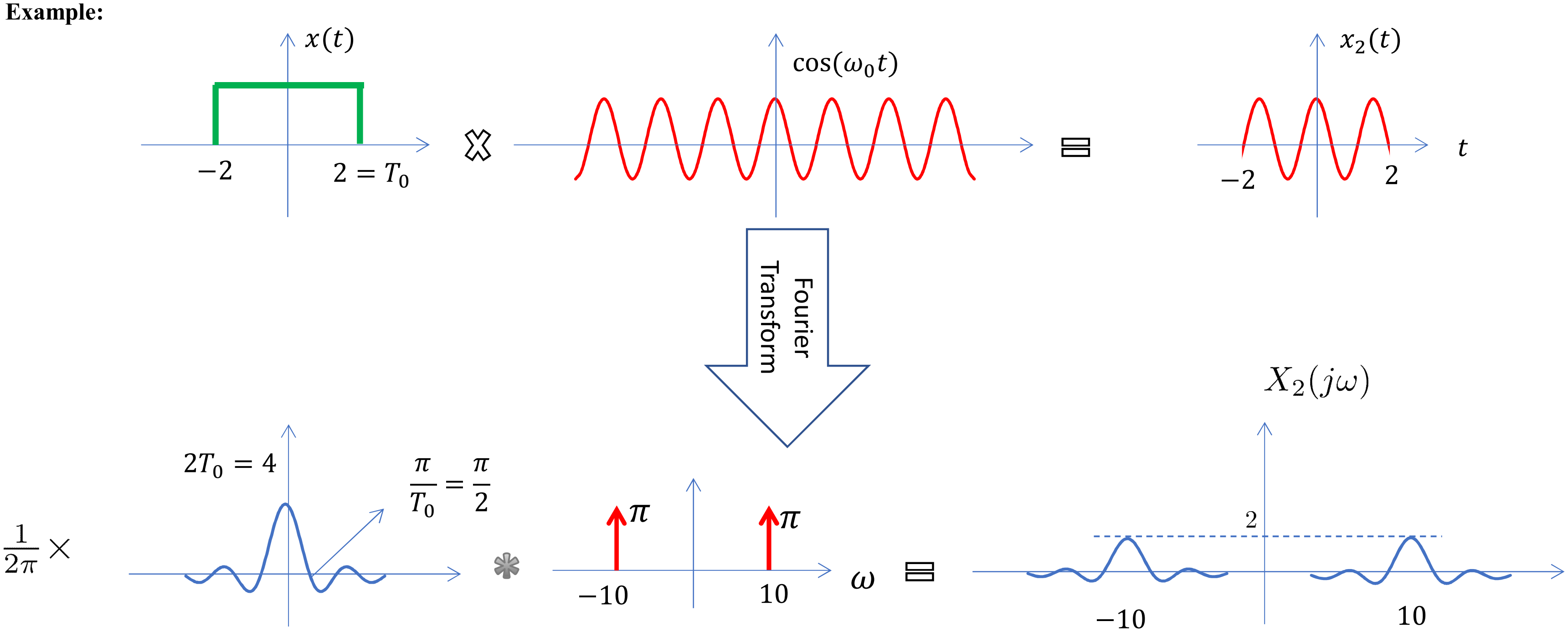
Fourier Transform Properties (Product of Signals)

Windowed version of cosine

Example:



$$x_2(t) = cos(\omega_0 t) \times x(t), \qquad \omega_0 = 10$$

$$x(t) \times \underbrace{cos(10t+\theta)}_{\frac{1}{2}e^{j\theta}e^{j\omega_0 t}+\frac{1}{2}e^{-j\theta}e^{-j\omega_0 t}} \xrightarrow{FT} \frac{1}{2\pi}X(j\omega) * \left(\pi e^{j\theta}\delta(j\omega - j\omega_0) + \pi e^{-j\theta}\delta(j\omega + j\omega_0)\right)$$

$$= \frac{1}{2}e^{j\theta}X\left(j(\omega-\omega_0)\right) + \frac{1}{2}e^{-j\theta}X\left(j(\omega+\omega_0)\right)$$

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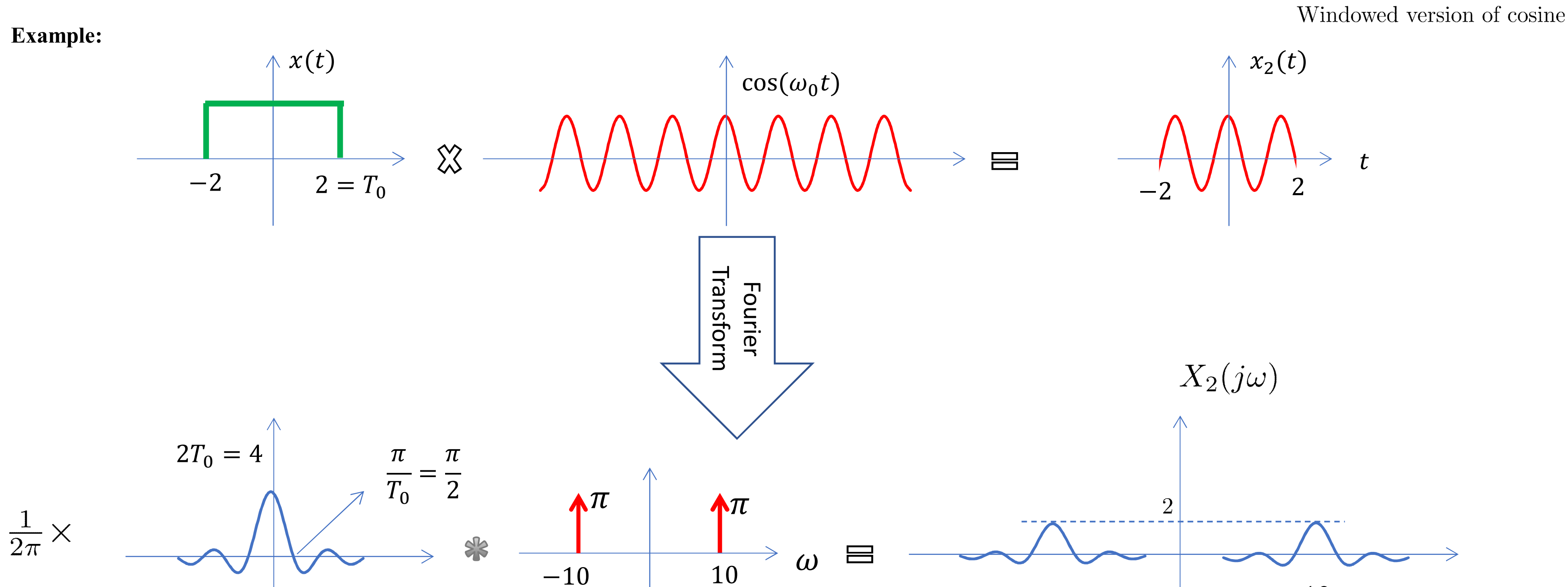
Fourier Transform Properties (Product of Signals)

Example:



Two interesting facts are shown by this simple example:

1- **Multiplication** of a signal (here a cosine) with a **pulse windows** the signal in time and equivalently **convolves** the FT of the signal with **sinc function**.

2- **Multiplication** of a signal (here a pulse) with **cosine Amplitude Modulates(AM)** the signal in time and equivalently the FT of the signal **shifts** to ω_o and $-\omega_0$

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Fourier Transform Properties (Derivative and Integral)

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

9. FT of derivative of a signal

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$v(t) = x'(t) \xrightarrow{FT} V(j\omega) = j\omega X(j\omega)$$

proof: take the derivative of both sides of the synthesis equation:

$$\frac{d}{dt}x(t) = \frac{d}{dt}\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}X(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d}{dt}e^{j\omega t}d\omega =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)j\omega e^{j\omega t}d\omega$$

10. FT of integral of a signal

$$x(t) \xrightarrow{FT} X(j\omega)$$

$$z(t) = \int_{-\infty}^{t} x(\tau)d\tau \xrightarrow{FT} Z(j\omega) = \frac{X(j\omega)}{j\omega}$$

Note that the integral property can be used only if $y(t) = \int_{-\infty}^{t} x(t)dt$ has a FT, i.e., the FT integral converges. For example it is known that $FT(\delta(t)) = 1$ and $u(t)$ is integral of $\delta(t)$ but $u(t)$ **doesn't have** a FT!

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Fourier Transform & Periodic Signals

FT properties	**Signal**	**FT**
	$x(t)$	$X(j\omega)$
	$z(t)$	$Z(j\omega)$
Linearity	$ax(t) + bz(t)$	$aX(j\omega) + bZ(j\omega)$
Time shift	$x(t - T_0)$	$e^{-j\omega T_0}X(j\omega)$
Freq. shift	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Scaling	$x(at)$	$\frac{1}{\|a\|}X(j\frac{\omega}{a})$
Duality	$X(jt)$	$2\pi x(-\omega)$
Complex Conj.	$x^*(t)$	$X^*(-j\omega)$ (so for real signals $\|X(j\omega\|$ is even and $\angle(X(j\omega)$ is odd
Convolution	$x(t) * z(t)$	$X(j\omega) \times Z(j\omega)$
Periodic signals	$x_p(t)$ with D_n coeffs	$X_p(j\omega) = \sum_n D_n 2\pi\delta(\omega - \omega_0 n)$
Product in time	$x(t) \times z(t)$	$\frac{1}{2\pi}X(j\omega) * Z(j\omega)$
Derivative	$x'(t)$	$j\omega X(j\omega)$
Integral	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{X(j\omega)}{j\omega}$ only if the integral of FT conveges

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