

# Signals and Systems I

## Lecture 3

## Last Lecture

- Combined Operation  $Ax(\alpha t - T)$
- Odd & Even Signals
- How to Calculate odd & even parts of the signal

## Today

- Build Signals with  $u(t)$  and  $\delta(t)$
- Closed form expression

## Operations on $u(t)$ & $\delta(t)$

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Operations on  $\delta(t)$

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Reminder:

$\delta(t)$  is infinity at 0 and its integral is 1.

$$*\delta(\alpha t) = \frac{1}{|\alpha|}\delta(t)$$

*proof:*

$$\int \delta(t) dt = 1 \rightarrow \int \delta(\alpha t) dt$$

$$\text{let } \alpha t = \omega \text{ then } dt = \frac{1}{\alpha} d\omega$$

$$\int \delta(\omega) \frac{1}{\alpha} d\omega$$

$$= \frac{1}{\alpha} \int \delta(\omega) d\omega = \frac{1}{\alpha} \cdot 1 = \frac{1}{\alpha}$$

$$\delta(\alpha(t - \beta)) = \frac{1}{|\alpha|}\delta(t - \beta)$$

Examples:

$$\delta(-t) = \delta(t) \rightarrow \text{Even Signal!}$$

$$\delta(2t) = \frac{1}{2}\delta(t)$$

$$\delta(-2t) = \frac{1}{2}\delta(t)$$

$$\delta\left(\frac{1}{3}t\right) = 3\delta(t)$$

$$\delta(5t - 3) = \delta\left(5\left(t - \frac{3}{5}\right)\right) = \frac{1}{5}\delta\left(t - \frac{3}{5}\right)$$

$$\text{Recall: } \delta(t)x(t) = \delta(t)x(0)$$

$$\delta(t - \tau_0)x(t - \tau_1) = \delta(t - \tau_0)x(\tau_0 - \tau_1)$$

## Operations on $u(t)$ & $\delta(t)$

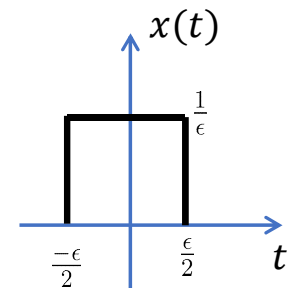
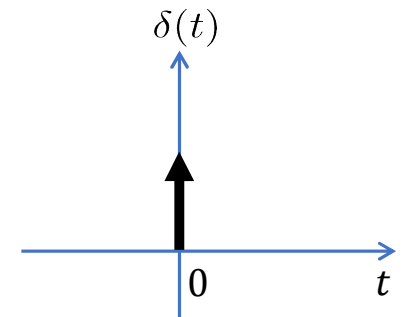
$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$

Remember that  $\delta(t)$  was limit of  $x(t)$  when  $\epsilon$  goes to zero.  
So  $\delta(\alpha t)$  is also limit of  $x(\alpha t)$

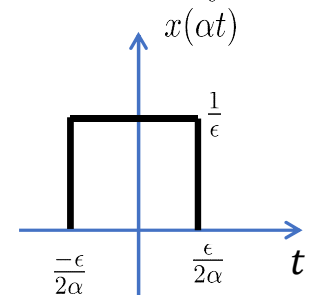
Also both  $\delta(t)$  and  $x(t)$  are even functions,  
 $\delta(t) = \delta(-t)$ ,  $x(t) = x(-t)$   
so  $\alpha$  and  $-\alpha$  operate the same.

Alternative method (for positive  $\alpha$ ): rename  $\alpha t$  as  $w$

$$\int \delta(\alpha t) dt = \int \delta(w) \frac{dw}{\alpha} = \frac{1}{\alpha} \int \delta(w) dw = \frac{1}{\alpha} \times 1$$
$$\alpha t = w, \alpha dt = dw$$



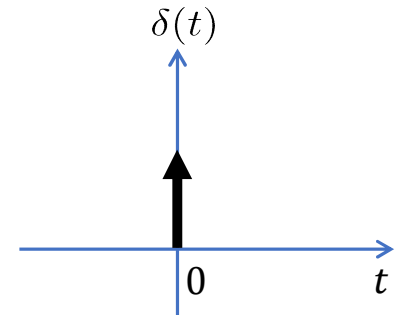
$$\int x(t) = \epsilon \times \frac{1}{\epsilon} = 1$$



$$\int x(\alpha t) = \frac{1}{\alpha} \epsilon \times \frac{1}{\epsilon} = \frac{1}{\alpha}$$

## Operations on $u(t)$ & $\delta(t)$

$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t)$$



In general:

$$\delta(\alpha(t - \beta)) = \frac{1}{|\alpha|} \delta(t - \beta)$$

←  $\delta(\alpha t - T)$ ,  $T = \alpha\beta$

Examples:  $\delta(-t) = \delta(t)$ , Even Signal

$$\delta(2t) = \frac{1}{2} \delta(t)$$

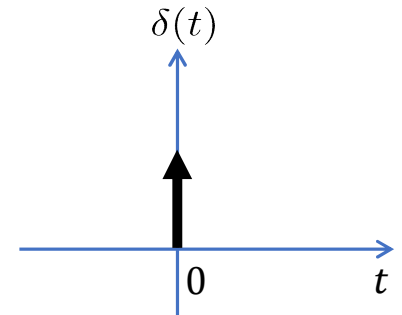
$$\delta(-2t) = \frac{1}{2} \delta(t)$$

$$\delta\left(\frac{1}{3}t\right) = 3\delta(t)$$

$$\delta(5t - 3) = \delta\left(5\left(t - \frac{3}{5}\right)\right) = \frac{1}{5} \delta\left(t - \frac{3}{5}\right)$$

## Reminder on $\delta(t)$

$$\delta(t - T_0)x(t - T_1) = \delta(t - T_0)x(T_0 - T_1)$$



Reminder: Multiplying  $\delta(t - T_0)$  by any function "Kills" the function for all values except at  $T_0$

$$\delta(t - T_0)y(t) = \delta(t - T_0)y(T_0)$$

Here  $y(t) = x(t - T_1)$ !

Example:

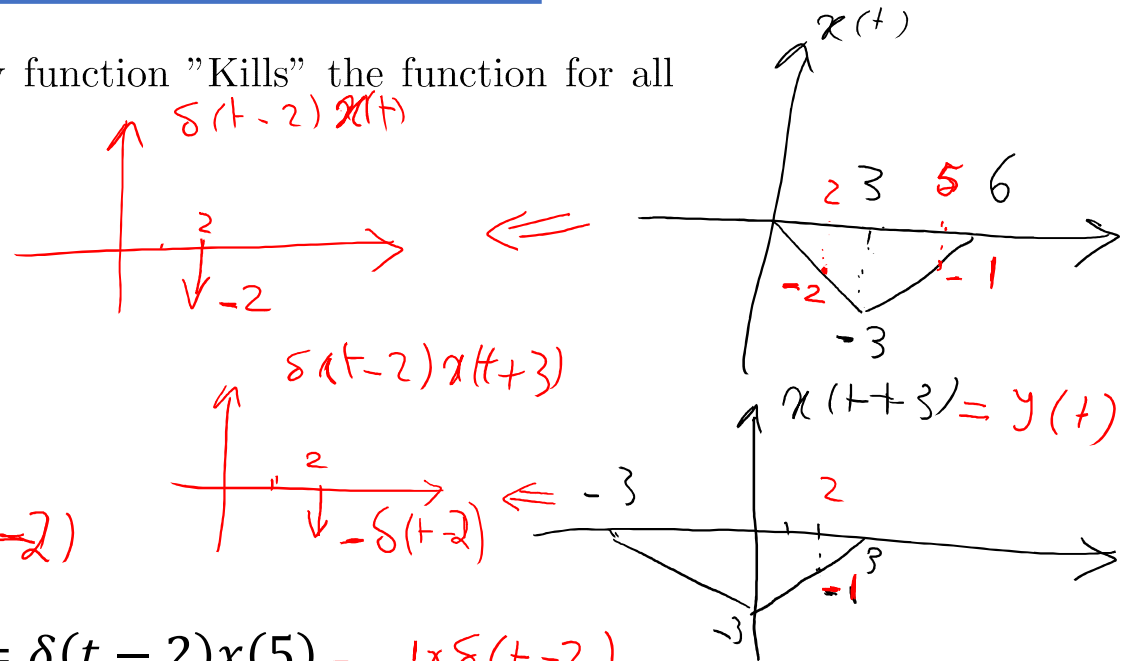
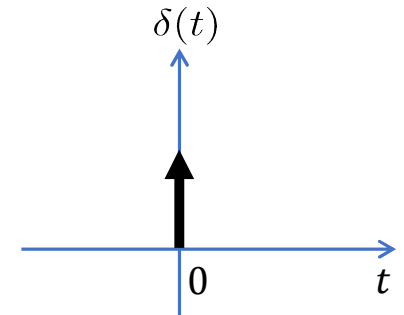
$$\delta(t - 2)x(t) = \delta(t - 2)x(2)$$

$$\delta(t - 2)x(t + 3) = \delta(t - 2)x(2 + 3) = \delta(t - 2)x(5)$$

## Reminder on $\delta(t)$

$$\delta(t - T_0)x(t - T_1) = \delta(t - T_0)x(T_0 - T_1)$$

Reminder: Multiplying  $\delta(t - T_0)$  by any function "Kills" the function for all values except at  $T_0$



Example:

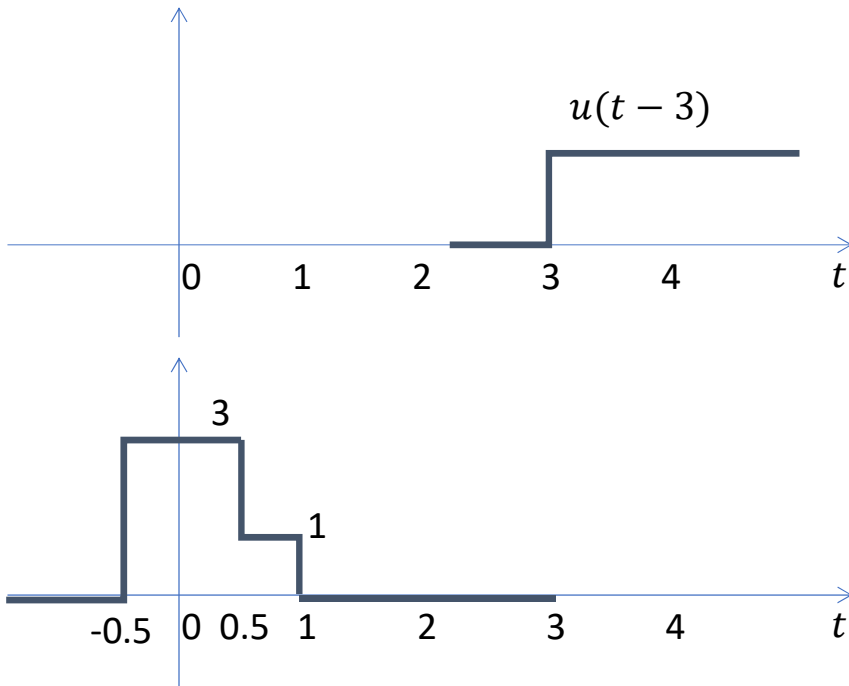
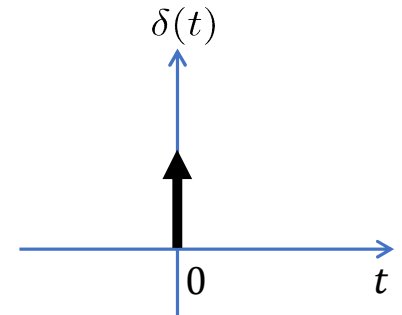
$$\delta(t - 2)x(t) = \delta(t - 2)x(2) = -2\delta(t - 2)$$

$$\delta(t - 2)x(t + 3) = \delta(t - 2)x(2 + 3) = \delta(t - 2)x(5) = -1 \times \delta(t - 2)$$

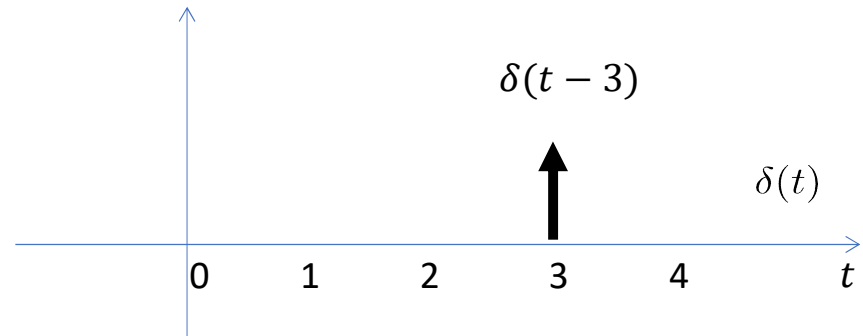
$\overline{y(2)}$

## Reminder on $\delta(t)$ and $u(t)$

$$\frac{d}{dt}u(t - T) = \delta(t - T)$$



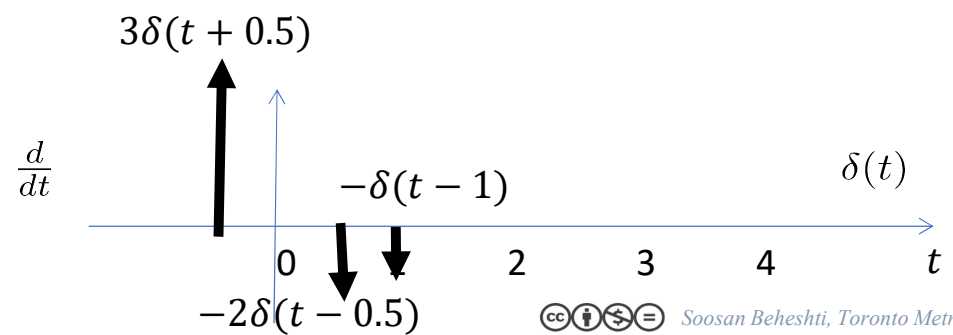
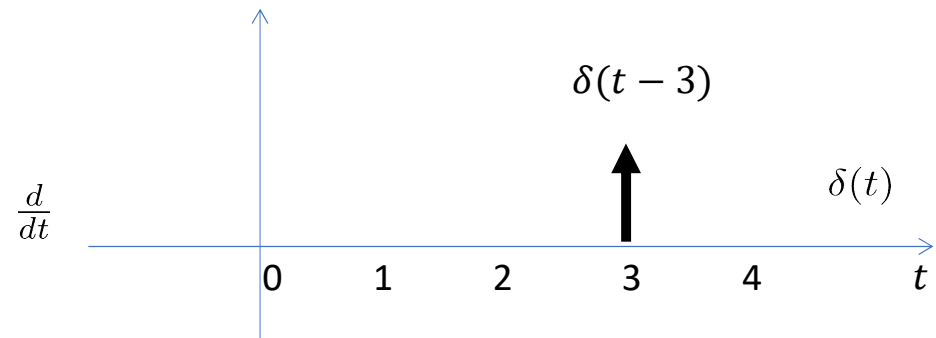
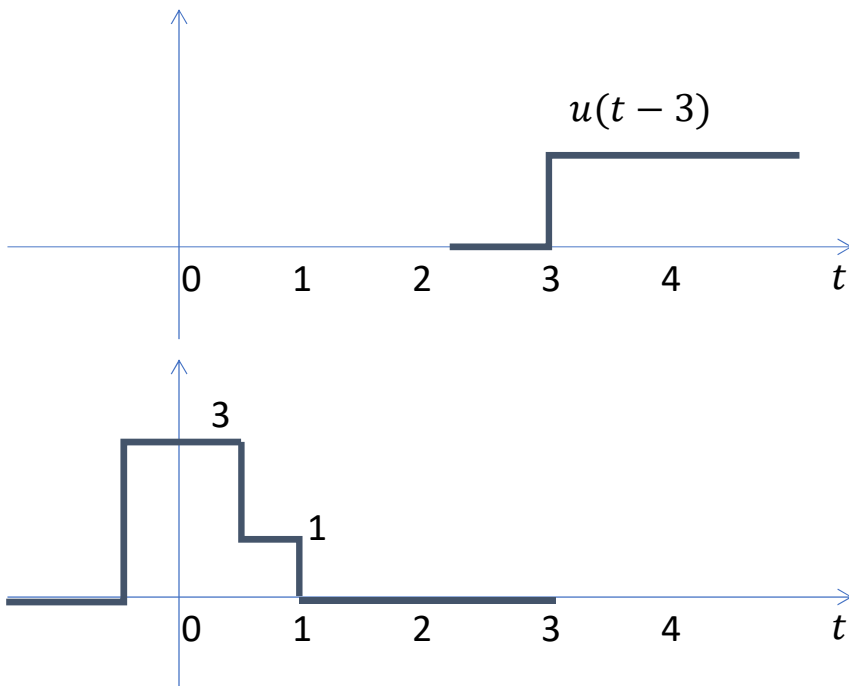
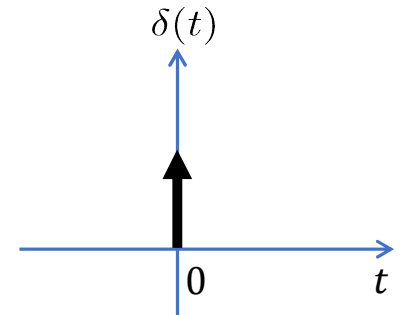
$\frac{d}{dt}$



$\frac{d}{dt}$

## Reminder on $\delta(t)$ and $u(t)$

$$\frac{d}{dt}u(t - T) = \delta(t - T)$$



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## Operations on $u(t)$

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$$u(\alpha t) = \begin{cases} u(t), & \text{if } \alpha > 0. \\ u(-t), & \text{if } \alpha < 0. \end{cases}$$

important

$$u(\alpha t - T) = \begin{cases} u\left(t - \frac{T}{\alpha}\right), & \text{if } \alpha > 0. \\ u\left(-t - \frac{T}{|\alpha|}\right), & \text{if } \alpha < 0. \end{cases}$$

Examples:

$$u(7t) = u(t), \quad u(-2.3t) = u(-t)$$

$$u(5t - 10) = u(5(t - 2)) = u(t - 2)$$

$$u(-5t - 10) = u\left(-t - \frac{10}{5}\right) = u(-t - 2)$$

## Operations on $u(t)$ & $\delta(t)$ (wrap up)

When dealing with  $u(t)$  and  $\delta(t)$ , consider the following two important properties:

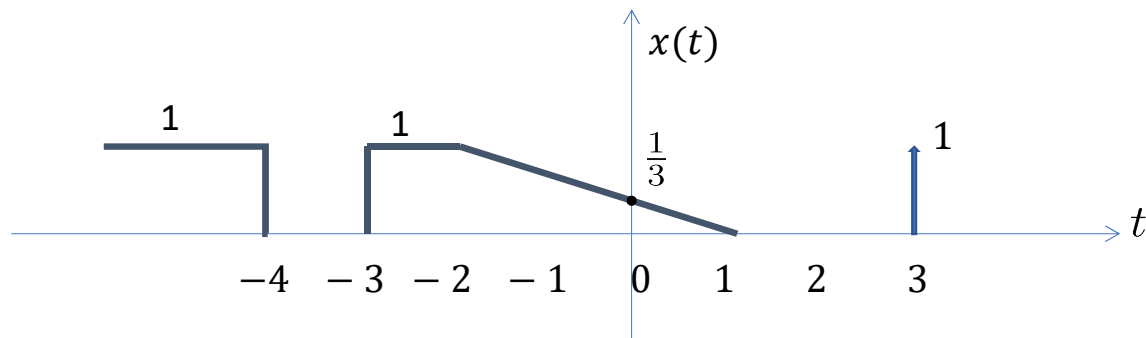
$$\delta(\alpha t) = \frac{1}{|\alpha|} \delta(t) \qquad u(\alpha t) = \begin{cases} u(t), & \text{if } \alpha > 0. \\ u(-t), & \text{if } \alpha < 0. \end{cases}$$

and recall that for any  $x(\alpha t - T)$ , it is **always** simpler to first take care of the shift by  $T$ . Once the shift is completed, use the above equations for  $\delta(t)$  and  $u(t)$ .

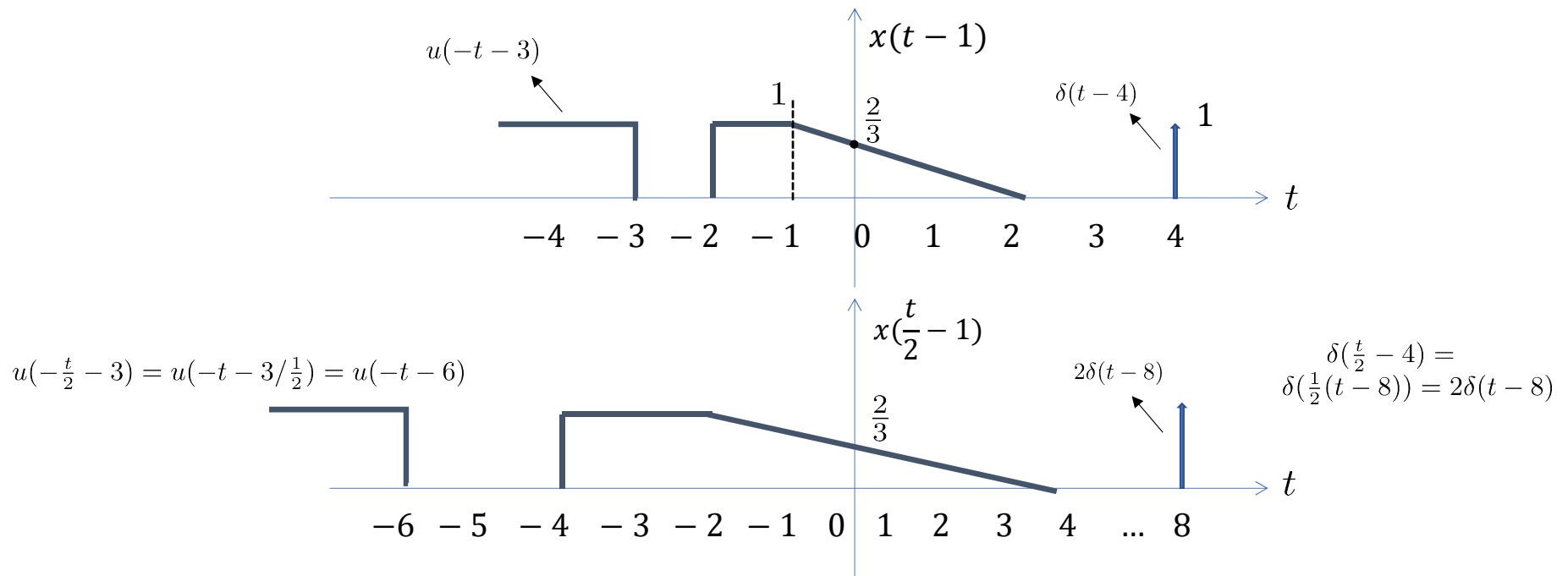
## Operations on $u(t)$ & $\delta(t)$

### **Example:**

Considering  $x(t)$  as the following signal, find and plot  $x(\frac{t}{2} - 1)$

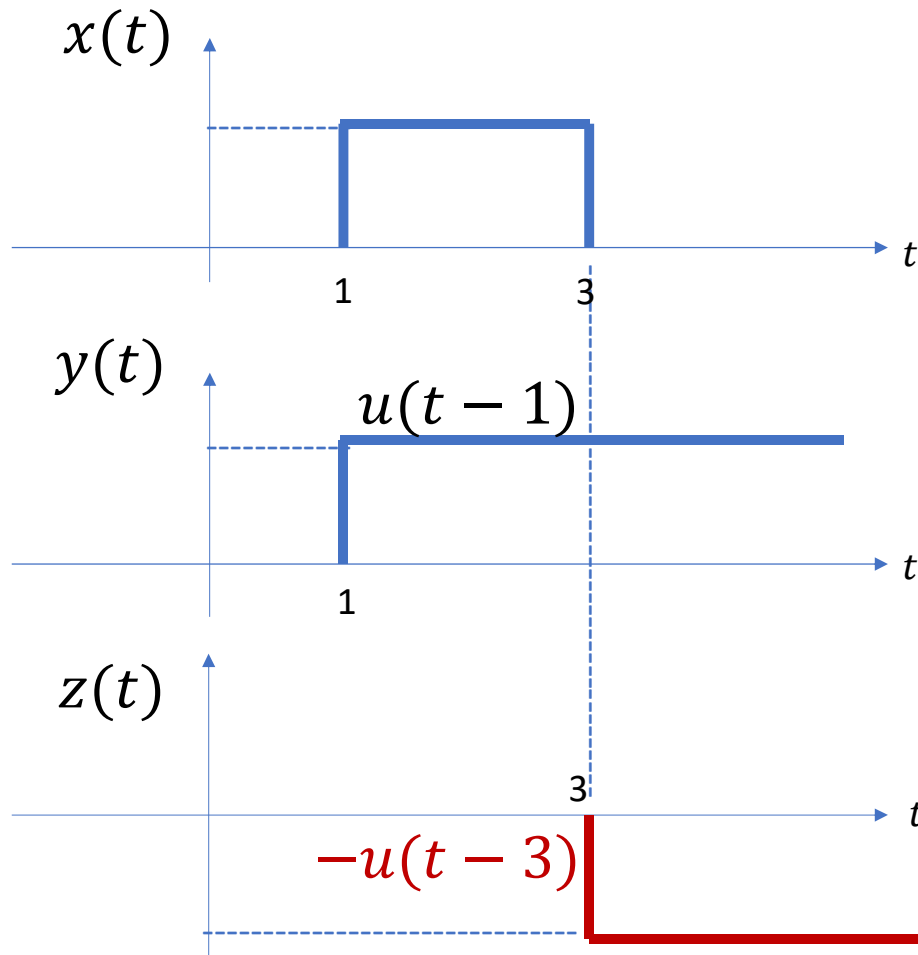


## Operations on $u(t)$ & $\delta(t)$



Having compression by  $\frac{1}{2}$  on the signal will effect BOTH location and amplitude of the  $\delta(t)$

## Using $u(t)$ to build segments

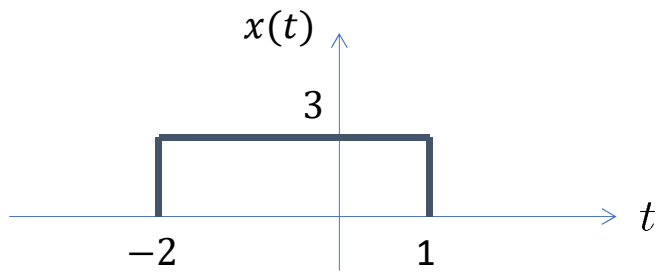


$$x(t) = y(t) + z(t)$$

A simple box can always be built by using  $u(t)$ .  
This ability of  $u(t)$  makes it very important specially in Continuous-Digital world, when we build functions by steps

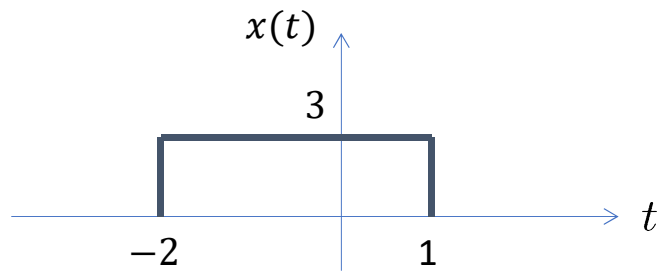
## Using $u(t)$ to build segments

- **Example:** Try to build the following signals:



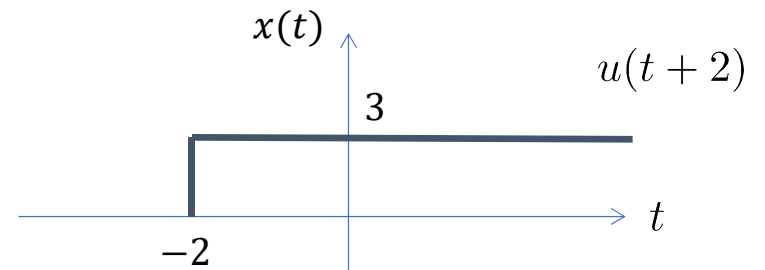
## Using $u(t)$ to build segments

- Example:** Try to build the following signals:

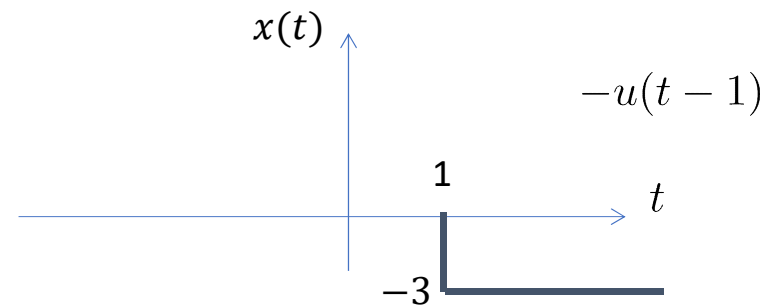


$$x(t) = 3(u(t + 2) - u(t - 1))$$

**Answer:**

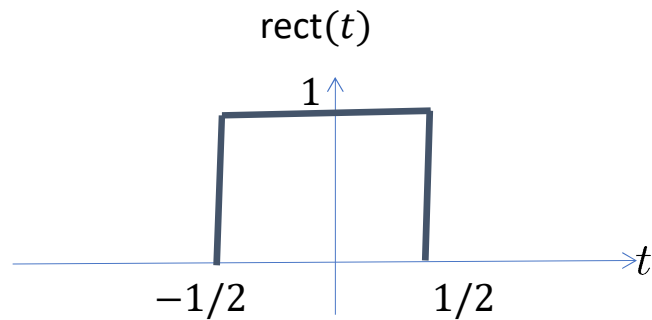


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## Using $u(t)$ to build segments

- **Example:** Try to build the following signals:

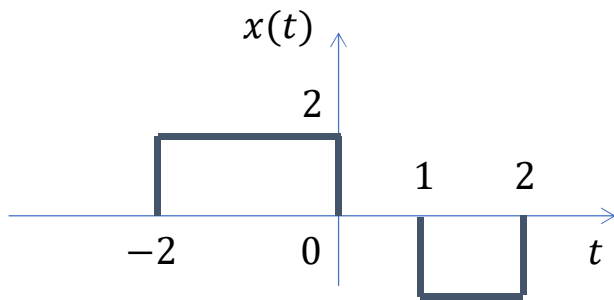


$$\text{rect}(t) = u(t + 1/2) - u(t - 1/2)$$

$$\text{rect}(t/T) = u(t + T/2) - u(t - T/2)$$

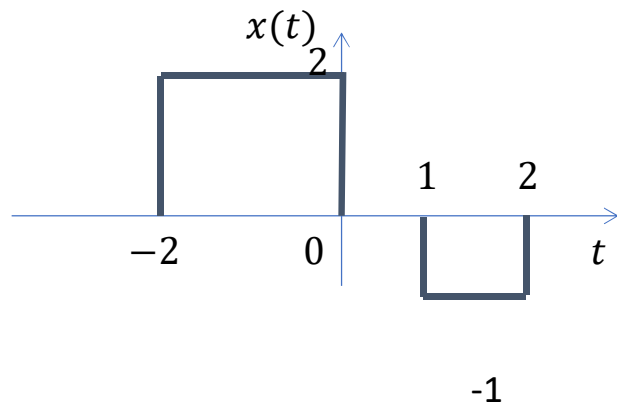
## Using $u(t)$ to build segments

- **Example:** Try to build the following signals

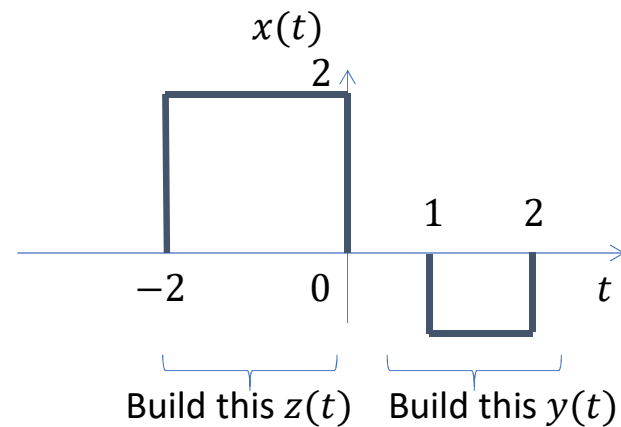


## Using $u(t)$ to build segments

- **Example:** Try to build the following signals



**Answer:**



$$x(t) = z(t) + y(t) .$$

$x(t)$  is super position of  $z(t)$  and  $y(t)$

$$x(t) = 2 u(t+2) - 2u(t) - u(t-1) + u(t-2)$$

## Using $u(t)$ to build segments

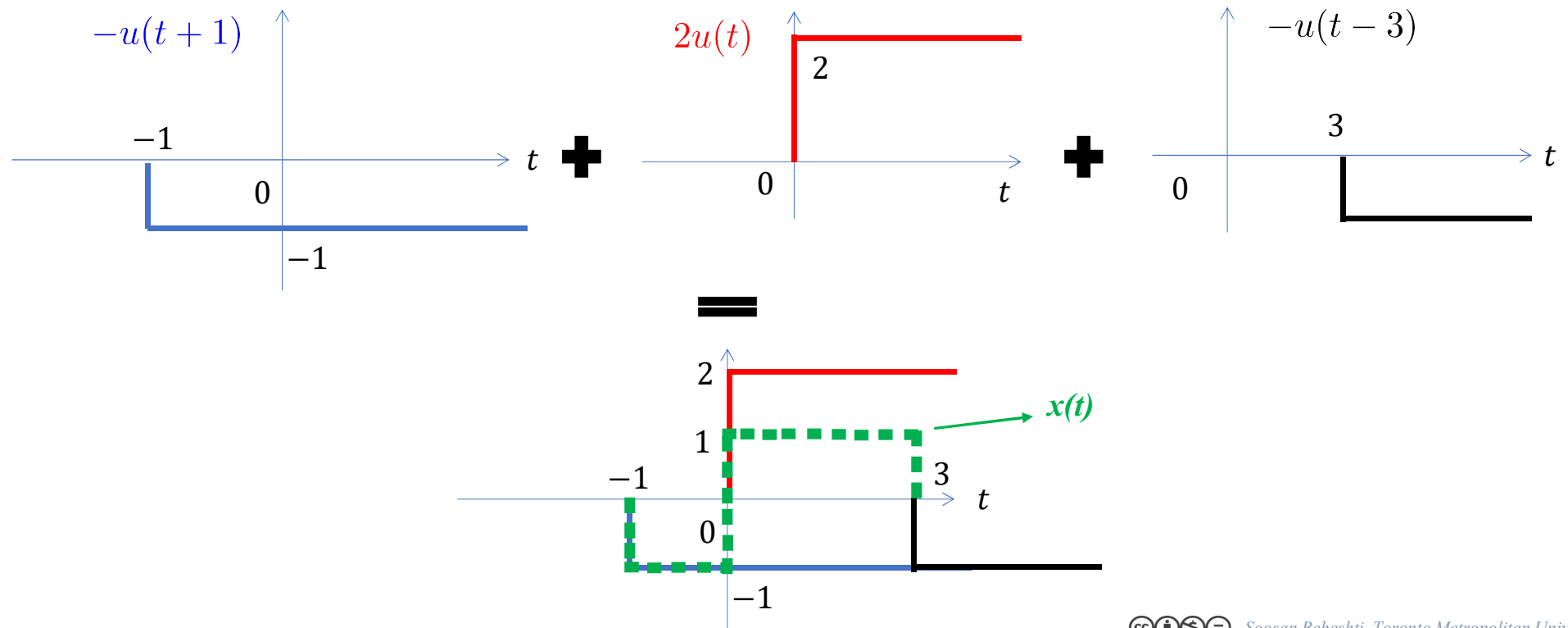
- **Example:** Plot the following signals:

$$1) \ x(t) = -u(t + 1) + 2u(t) - u(t - 3)$$

$$2) \ y(t) = 2u(t) - u(t - 1) - u(t - 2)$$

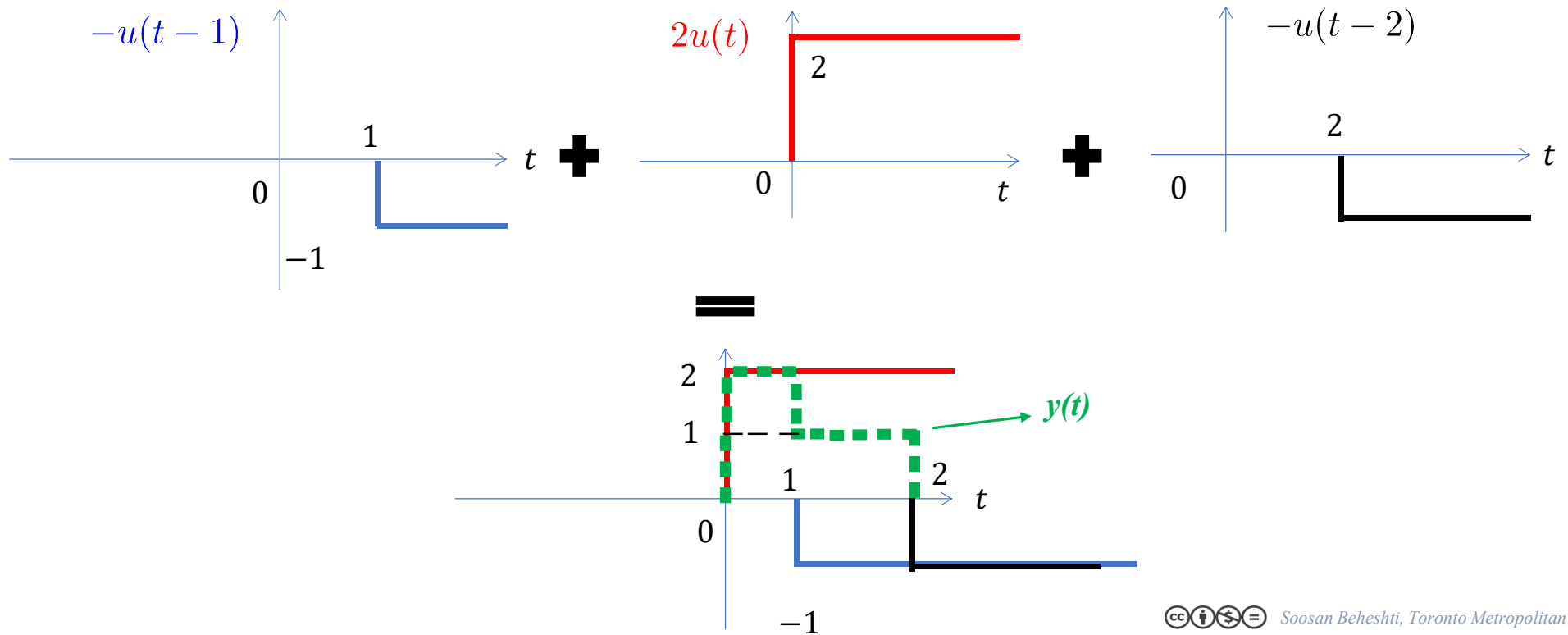
## Using $u(t)$ to build segments

$$x(t) = -u(t+1) + 2u(t) - u(t-3)$$



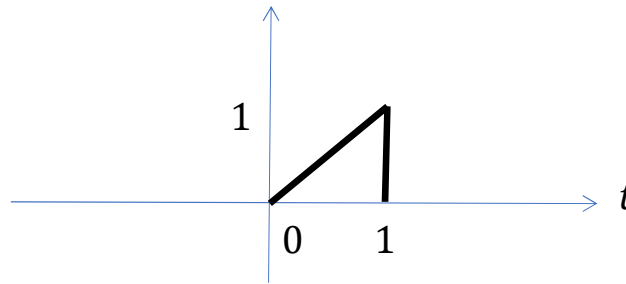
## Using $u(t)$ to build segments

$$y(t) = -u(t-1) + 2u(t) - u(t-2)$$



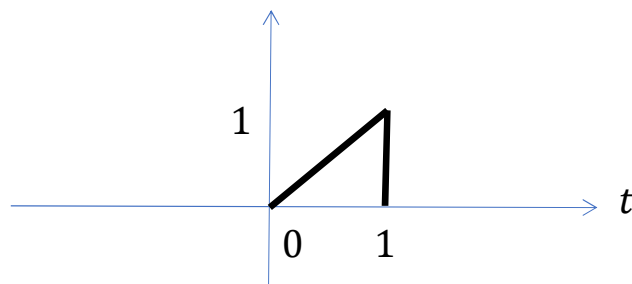
## Closed form expressions

What is the closed form expression of the following signal?



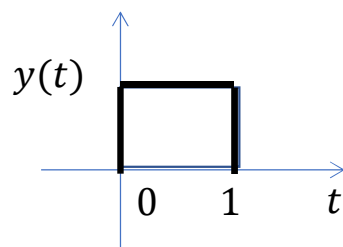
## Closed form expressions

What is the closed form expression of the following signal?



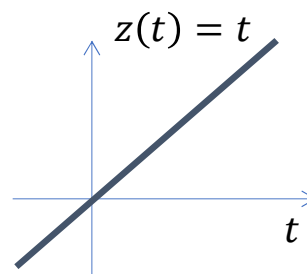
**Answer:**

=



$$u(t) - u(t - 1)$$

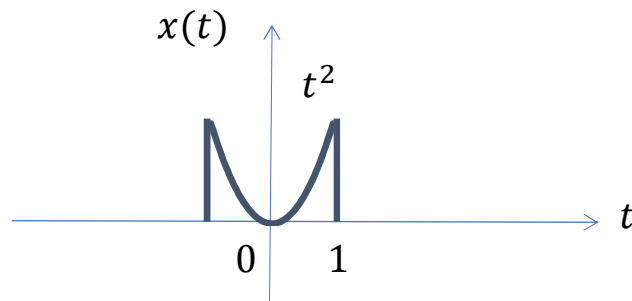
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$$\text{Therefore: } x(t) = (u(t) - u(t - 1))t$$

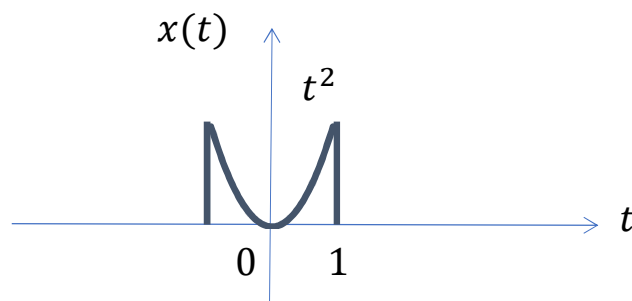
## Closed form expressions

**Example:**



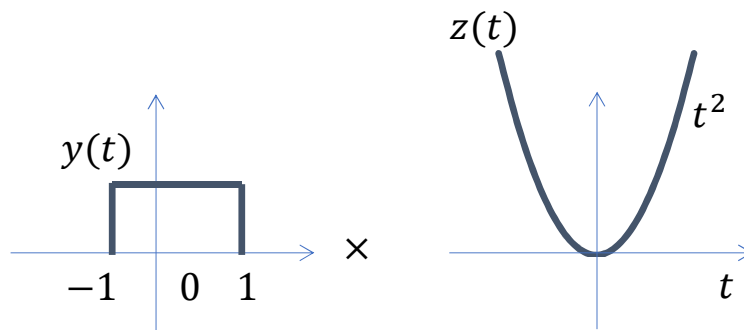
## Closed form expressions

**Example:**



**Answer:**

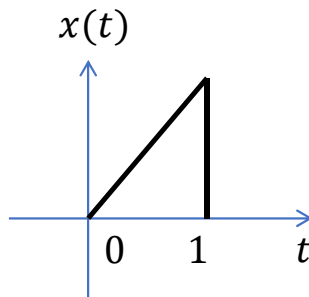
=



Therefore:  $x(t) = (u(t+1) - u(t-1))t^2$

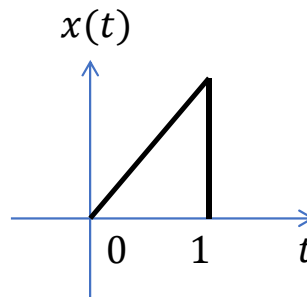
## Closed form expressions and combined operations

In the previous example we showed the closed form expression for the following signal:  
 $x(t) = (u(t) - u(t - 1)) t$ . Find the closed form expression for  $x(3t + 1)$  and plot it.

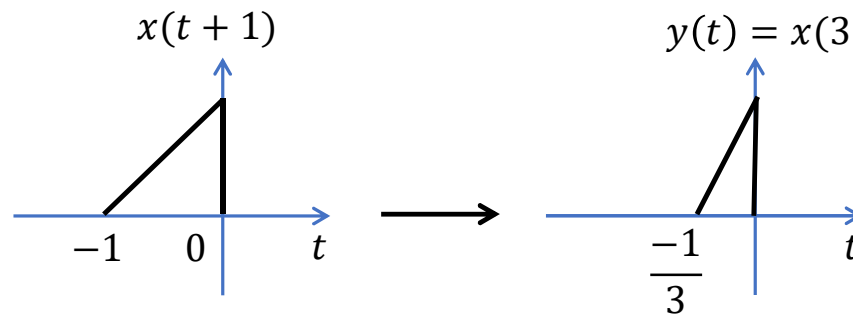


## Closed form expressions and combined operations

Consider the following signal  $x(t) = (u(t) - u(t - 1))t$ .  
Find the closed form expression for  $x(3t + 1)$  and plot it:



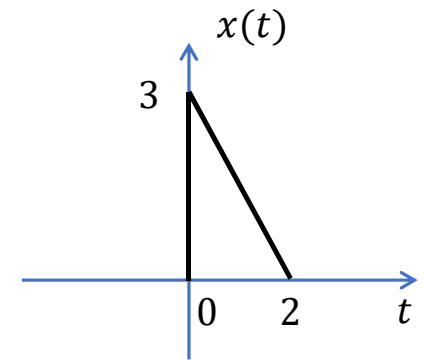
**Answer:**  $(3t + 1)(u(3t + 1) - u(3t + 1 - 1)) = (3t + 1)(u(t + \frac{1}{3}) - u(t))$



Here you can also verify your answer by direct use of the signal graph. But this is not an easy method for more complex signals

## Closed form expressions and combined operations

**Example:** Consider the following signal  $x(t)$  , write the closed form expression for  $y(t) = x(-\frac{t}{2})$  and plot it



## Closed form expressions and combined operations

**Example:** Consider the following signal  $x(t)$ , write the closed form expression for  $y(t) = x(-\frac{t}{2})$  and plot it

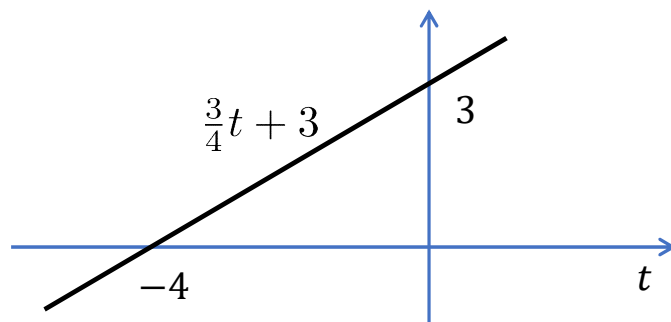
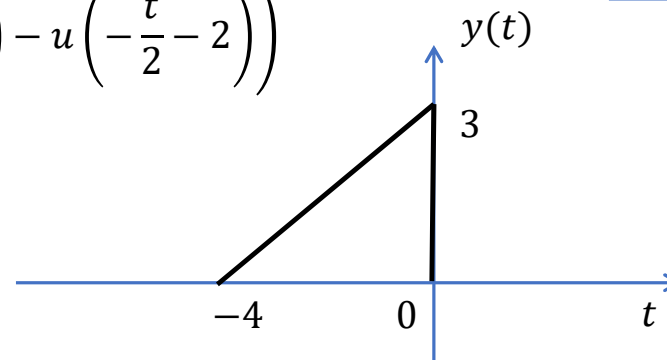
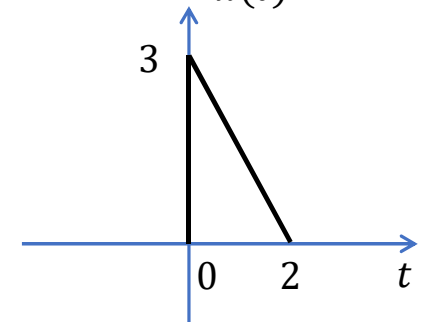
**Solution:**

$$x(t) = \left(-\frac{3}{2}t + 3\right)(u(t) - u(t - 2))$$

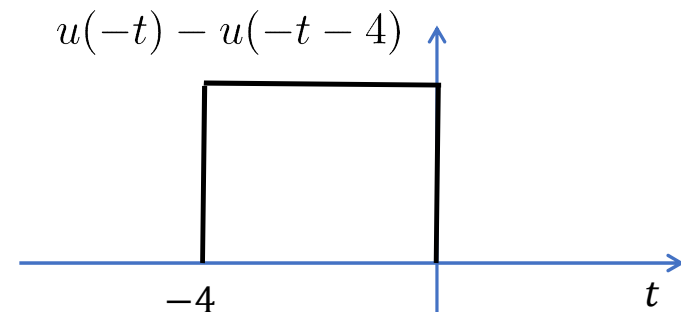
$$y(t) = x\left(-\frac{t}{2}\right) = \left(-\frac{3}{2}\left(-\frac{t}{2}\right) + 3\right)\left(u\left(-\frac{t}{2}\right) - u\left(-\frac{t}{2} - 2\right)\right)$$

$$= \left(\frac{3}{4}t + 3\right)(u(-t) - u(-t - 4))$$

$$u\left(-\frac{t}{2} - 2\right) = u(-t - 4)$$

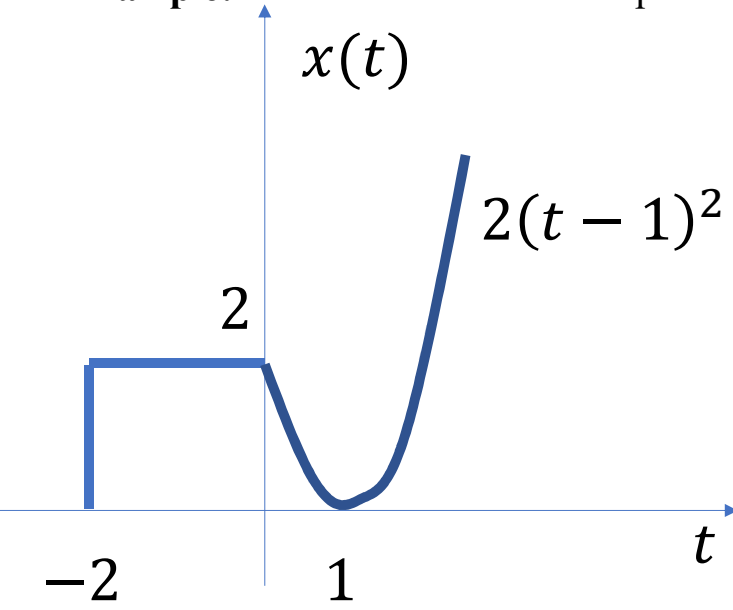


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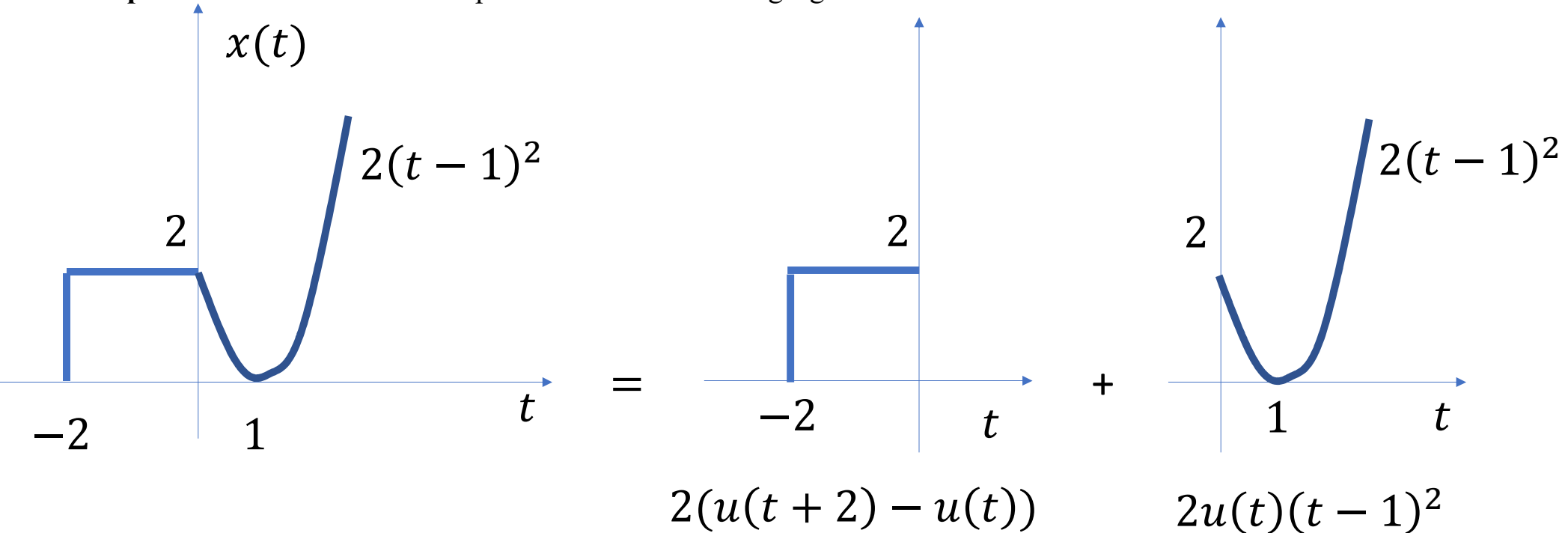
## Closed form expressions and combined operations

**Example:** Write the closed form expression for the following signal



## Closed form expressions and combined operations

**Example:** Write the closed form expression for the following signal

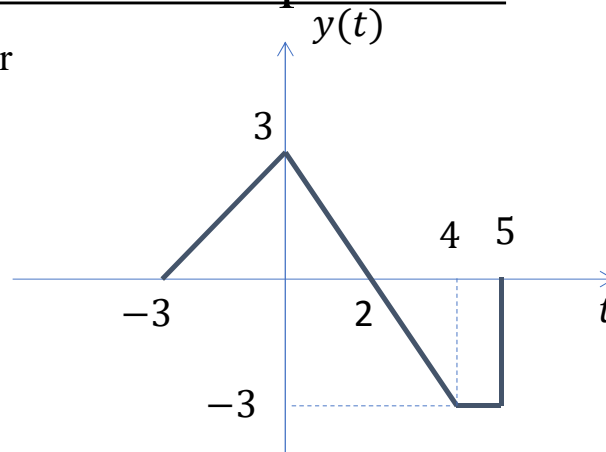


Through Super Position:

$$x(t) = 2(u(t + 2) - u(t)) + 2u(t)(t - 1)^2$$

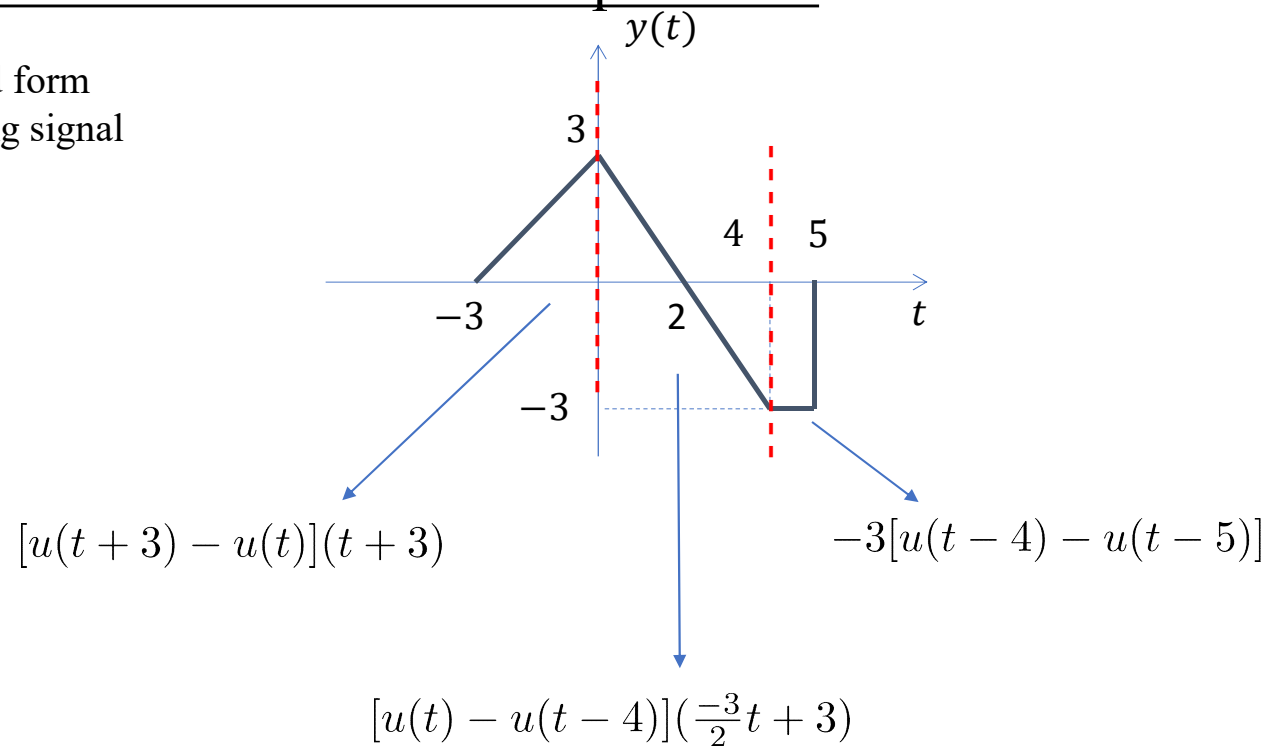
## Closed form expressions and combined operations

**Example:** Write the closed form expression for the following signal



## Closed form expressions and combined operations

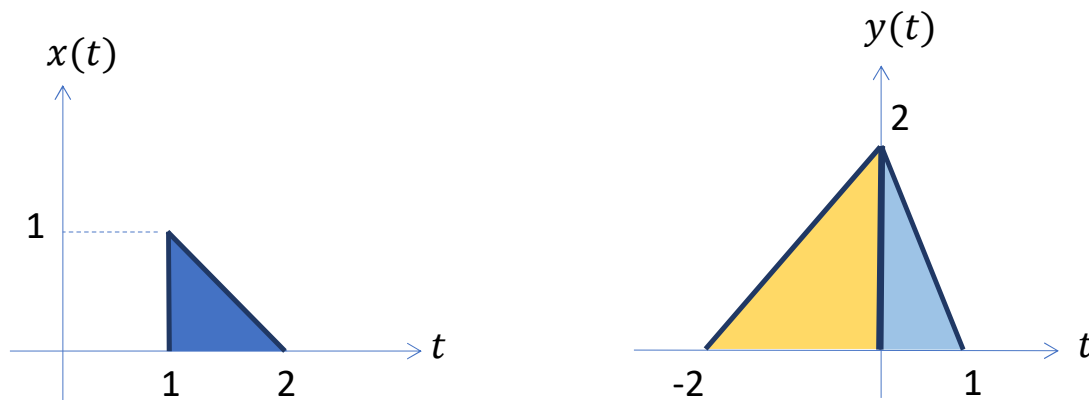
**Example:** Write the closed form expression for the following signal



$$y(t) = [-u(t) + u(t+3)](t+3) + [u(t) - u(t-4)](-\frac{3}{2}t + 3) - 3[u(t-4) - u(t-5)]$$

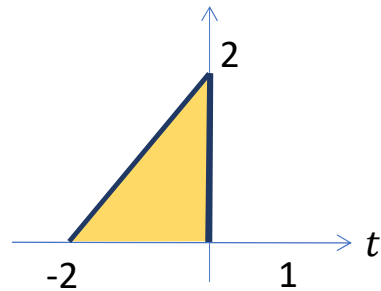
## Example of building signals from signals

- Write  $y(t)$  as a function of time shifted, time scaled, and amplitude scaled of  $x(t)$

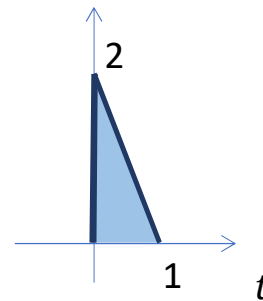


**Answer:**

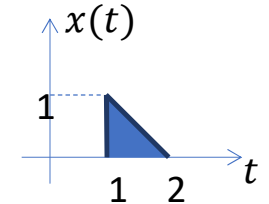
$$y(t) =$$



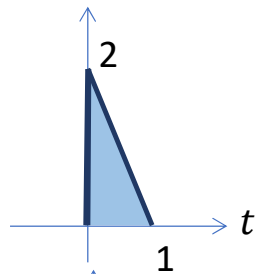
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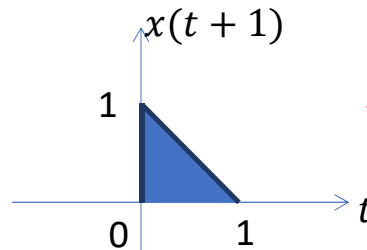
These terms are just shifted on x-axis and stretched on y-axis versions of  $x(t)$



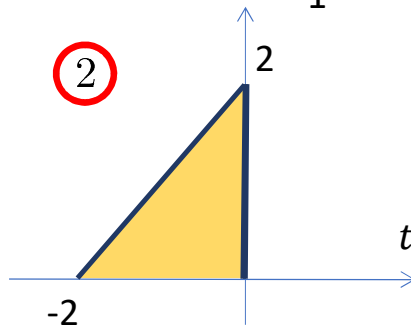
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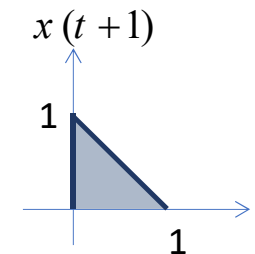
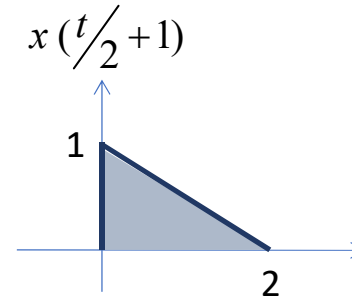
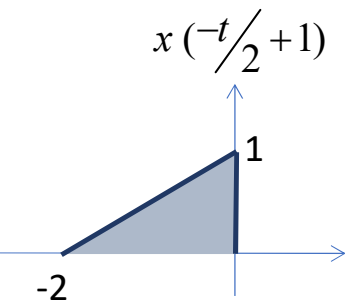
$$= 2 \times$$



②



$$= 2 \times$$



$$\therefore y(t) = 2x(t+1) + 2x\left(\frac{-t}{2} + 1\right)$$