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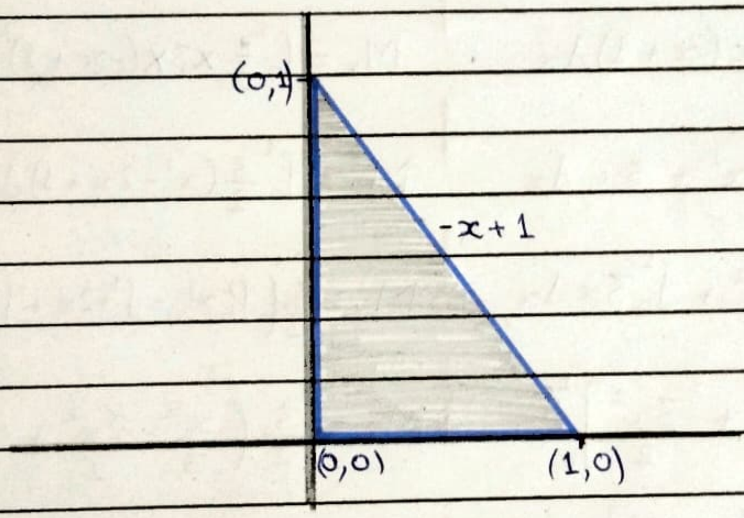
PBL 02

EXAMPLE 2:- Find the center of gravity of the triangular lamina with vertices (0,0), (0,1) and (1,0) and density $\delta = 3$



Using the formula :-

$$M = \delta A$$

$$M = \delta \times \frac{1}{2} \times b \times h$$

$$M = 3 \times \frac{1}{2} \times 1 \times 1$$

$$M = 3 \times \frac{1}{2}$$

$$M = \frac{3}{2}$$

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Finding the Centre of Gravity :-

X-COORDINATE	Y-COORDINATE
$\bar{x} = \frac{M_y}{M}$	$\bar{y} = \frac{M_x}{M}$
Finding M_y	Finding M_x
$M_y = \int_0^1 \delta x f(x)\,dx$	$M_x = \int_0^1 \frac{1}{2} \delta (f(x))^2\,dx$
$M_y = \int_0^1 3x(-x+1)\,dx$	$M_x = \int_0^1 \frac{1}{2} \times 3 \times (-x+1)^2\,dx$
$M_y = \int_0^1 -3x^2 + 3x\,dx$	$M_x = \int_0^1 \frac{3}{2}(x^2-2x+1)\,dx$
$M_y = \int_0^1 -3x^2 + \int_0^1 3x\,dx$	$M_x = \frac{3}{2}\left(\int_0^1 x^2 - \int_0^1 2x + \int_0^1 1\right)$
$M_y = \left[\frac{-3x^3}{3} + \frac{3x^2}{2}\right]_0^1$	$M_x = \frac{3}{2}\left(\frac{x^3}{3} - \frac{\cancel{2}x^2}{\cancel{2}} + x\right)_0^1$
$M_y = \left[\frac{-3(1)^3}{3} + \frac{3(1)^2}{2}\right] - 0$	$M_x = \frac{3}{2}\left[\frac{(1)^3}{3} - (1)^2 + 1\right] - 0$
$M_y = \left[-1 + \frac{3}{2}\right]$	$M_x = \frac{3}{2}\left[\frac{1}{3}\right]$
$M_y = \frac{1}{2}$	$M_x = \frac{1}{2}$
$\bar{x} = \frac{M_y}{M}$	$\bar{y} = \frac{M_x}{M}$
$\bar{x} = \frac{1/2}{3/2} = \frac{1}{3}$	$\bar{y} = \frac{1/2}{3/2} = \frac{1}{3}$

CENTER OF GRAVITY :- $\left(\frac{1}{3}, \frac{1}{3}\right)$

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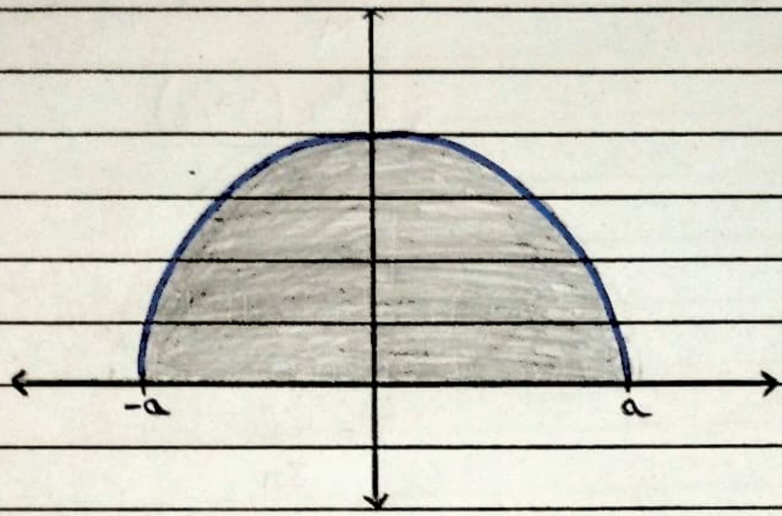
EXAMPLE 3:- Find the centroid of the semicircular region.



$f(x) = \sqrt{a^2 - x^2}$

$$M = A$$

- Area of Circle $= \pi r^2$
- Area of Semi-Circle $= \frac{1}{2}\pi r^2$ since $r = a$

$$= \frac{1}{2}\pi a^2$$

Finding Centroid:-

X-COORDINATE

By symmetry the center will always lie on x-axis

$$\bar{x} = 0$$

Y-COORDINATE

$$\bar{y} = \frac{M_x}{M}$$

$$M_x = \int_{-a}^{a} \frac{1}{2}\left(\sqrt{a^2 - x^2}\right)^2 dx$$

$$M_x = \int_{-a}^{a} \frac{1}{2}(a^2 - x^2)\, dx$$

$$M_x = \int_{-a}^{a} \frac{1}{2} a^2 - \int_{-a}^{a} x^2$$

$$M_x = \frac{1}{2}\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a}$$

$$M_x = \frac{1}{2}\left[\left(a^3 - \frac{1}{3}a^3\right) - \left(-a^3 + \frac{1}{3}a^3\right)\right]$$

$$M_x = \frac{1}{2}\left(\frac{4a^3}{3}\right)$$

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$\bar{x} = 0$ $\qquad \bar{y} = \frac{M_x}{M}$

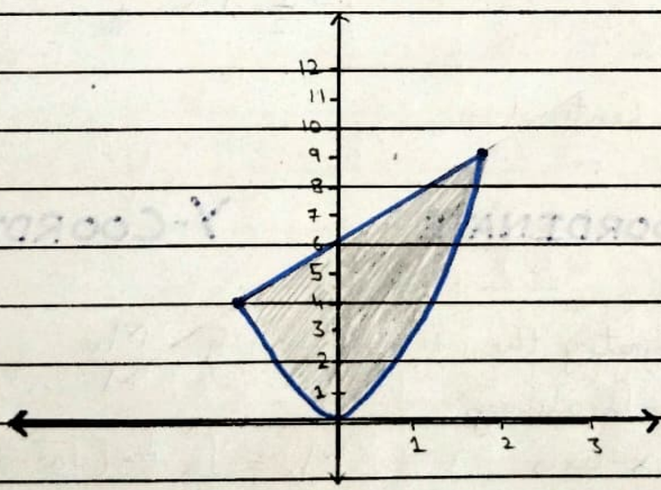
$$\bar{y} = \frac{1/2\,(4a^3/3)}{1/2\,\pi a^2}$$

$$\bar{y} = \frac{4a^{3-2}}{3} \times \frac{1}{\pi \cancel{a^2}}$$

$$\bar{y} = \frac{4a^3}{3\pi}$$

CENTROID OF THE SEMICIRCLE $\left(0, \frac{4a^3}{3\pi}\right)$

EXAMPLE 4:- Find the centroid of the region R enclosed between the curves $y = x^2$ and $y = x + 6$



→ Finding Area between the two curves for Mass.

$y = x^2 \qquad , y = x + 6$

$$x^2 = x + 6$$

$$x^2 - x - 6 = 0$$

$$x^2 - 3x + 2x - 6 = 0$$

$$x(x-3) + 2(x-3) = 0$$

$$(x+2)(x-3) = 0$$

$$x = -2 \qquad x = 3$$

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-2 ——— 3

LOWER	UPPER
$y = x^2$	$y = x + 6$
$y = (1)^2$	$y = 1 + 6$
$y = 1$	$y = 7$

$M = A$

$$M = \int_{-2}^{3} [(x+6) - x^2]\, dx$$

$$M = \int_{-2}^{3} x\,dx + \int_{-2}^{3} 6\,dx - \int_{-2}^{3} x^2\,dx$$

$$M = \frac{x^2}{2} + 6x - \frac{x^3}{3} \Big|_{-2}^{3}$$

$$M = \left(\frac{(3)^2}{2} - 6(3) - \frac{(3)^3}{3}\right) - \left(\frac{(-2)^2}{2} + 6(-2) - \frac{(-2)^3}{3}\right)$$

$$M = \frac{125}{6}$$

$\bar{x} = \frac{M_y}{M}$	$\bar{y} = \frac{M_x}{M}$
Finding M_y using $\int_a^b x[f(x) - g(x)]dx$	Finding M_x using $\int_a^b \frac{1}{2}(f(x))^2 dx$
$M_y = \int_{-2}^{3} x[(x+6) - x^2]\,dx$	$M_x = \int_{-2}^{3} \frac{1}{2}[(x+6)^2 - (x)^2]\,dx$
$M_y = \int_{-2}^{3} x^2 + 6x - x^3\,dx$	$M_x = \int_{-2}^{3} \frac{1}{2}(x^2 + 12x + 36 - x^4)\,dx$
$M_y = \int_{-2}^{3} x^2 dx + \int_{-2}^{3} 6x\,dx - \int_{-2}^{3} x^3 dx$	$M_x = \frac{1}{2}\left(\frac{1}{3}x^3 + \frac{12x^2}{2} + 36x - \frac{x^5}{5}\right)\Big|_{-2}^{3}$
$M_y = \frac{x^3}{3} + \frac{6x^2}{2} - \frac{x^4}{4}\Big|_{-2}^{3}$	$M_x = \frac{3}{250}$

$$M_y = \left(\frac{(3)^3}{3} + \frac{6(3)^2}{2} - \frac{(3)^4}{4}\right) - \left(\frac{(-2)^3}{3} + \frac{6(-2)^2}{3} - \frac{(-2)^4}{4}\right)$$

$$M_y = \frac{63}{4} - \frac{16}{3}$$

$$M_y = \frac{125}{12}$$

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$$\bar{x} = \frac{M_y}{M}$$

$$\bar{x} = \frac{125}{12} \div \frac{125}{6}$$

$$\bar{x} = \frac{125}{12} \times \frac{6}{125}$$

$$\bar{x} = \frac{1}{2}$$

$$\bar{y} = \frac{M_x}{M}$$

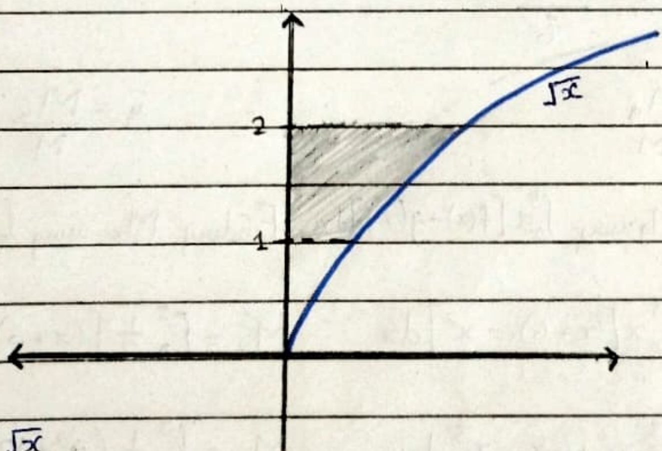
$$\bar{y} = \frac{3/250}{125/6}$$

$$\bar{y} = \frac{3}{250} \times \frac{6}{125}$$

$$\bar{y} = 4$$

CENTROID POINT :- $\left(\frac{1}{2}, 4\right)$

EXAMPLE 5:- Find the centroid of the region R enclosed between the curves $y = \sqrt{x}$, $y = 1$, $y = 2$ and the y-axis



$y = \sqrt{x}$

$x = y^2$ [Since Area is about Y-axis]

$$M = A$$

$$M = \int_1^2 y^2 \, dy$$

$$M = \left.\frac{y^3}{3}\right|_1^2$$

$$M = \frac{(2)^3}{3} - \frac{(1)^3}{3}$$

$$M = \frac{7}{3}$$

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$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

* Since Area is about Y-axis we will use Y axis formulas

$$M_y = \int_1^2 \frac{1}{2}(y^2)^2\,dy$$

$$M_y = \int_1^2 \frac{1}{2} y^4\,dy$$

$$M_y = \frac{1}{2}\left(\frac{y^5}{5}\right)\Bigg|_1^2$$

$$M_y = \frac{y^5}{10}\Bigg|_1^2$$

$$M_y = \frac{(2)^5}{10} - \frac{(1)^5}{10}$$

$$M_y = \frac{31}{10}$$

$$\bar{x} = \frac{M_y}{M}$$

$$\bar{x} = \frac{31/10}{7/3}$$

$$\bar{x} = \frac{31}{10} \times \frac{3}{7}$$

$$\bar{x} = \frac{93}{70}$$

$$M_x = \int_1^2 y\,w(y)\,dy$$

$$M_x = \int_1^2 y(y^2)\,dy$$

$$M_x = \int_1^2 y^3\,dy$$

$$M_x = \frac{y^4}{4}\Bigg|_1^2$$

$$M_x = \frac{(2)^4}{4} - \frac{(1)^4}{1}$$

$$M_x = \frac{15}{4}$$

$$\bar{y} = \frac{M_x}{M}$$

$$\bar{y} = \frac{15/4}{7/3}$$

$$\bar{y} = \frac{15}{4} \times \frac{3}{7}$$

$$\bar{y} = \frac{45}{28}$$

CENTROID POINT :- $\left(\frac{93}{70}, \frac{45}{28}\right)$

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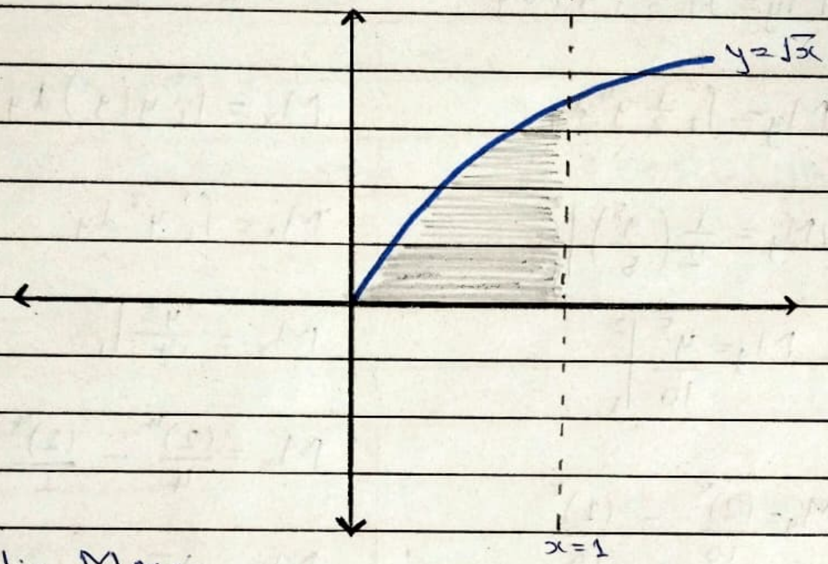
Q Find the mass and center of gravity of the lamina with density δ.

23. A lamina bounded by the x-axis, the line $x=1$ and the curve $y=\sqrt{x}$, $\delta=2$.



Finding MASS:-

Finding Area for formula $M=\delta A$

$$A=\int_0^1 \sqrt{x}\,dx$$

$$A=\int_0^1 x^{\wedge\frac{1}{2}}\,dx$$

$$A=\int_0^1 x^{\frac{1}{2}}\,dx$$

$$A=\left.\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right|_0^1$$

$$A=\frac{x^{3/2}}{3/2}=\left.\frac{2x^{3/2}}{3}\right|_0^1$$

$$A=\frac{2(1)^{3/2}}{3}-0$$

$$\boxed{A=\frac{2}{3}}$$

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$M = \rho A$

$\rho = 2, A = \frac{2}{3}$

$M = 2 \times \frac{2}{3} = \frac{4}{3}$

$$\boxed{M = \frac{4}{3}}$$

Finding Centre of Gravity :-

X - COORDINATE

$\bar{x} = \frac{1}{\text{area of } R} \int_a^b x f(x)\, dx$

Area of $R = \frac{2}{3}$

$\bar{x} = \frac{1}{\frac{2}{3}} \int_0^1 x(\sqrt{x})\, dx$

$\bar{x} = \frac{3}{2} \int_0^1 x^{3/2}\, dx$

$\bar{x} = \frac{3}{2} \left(\frac{x^{3/2+1}}{3/2+1} \right)_0^1$

$\bar{x} = \frac{3}{2} \left(\frac{x^{5/2}}{5/2} \right)_0^1$

$\bar{x} = \frac{3}{2} \left(\frac{2x^{5/2}}{5} \right)_0^1$

$\bar{x} = \frac{3}{2} \left(\frac{2(1)}{5} \right) - 0$

$\bar{x} = \frac{6}{10} = \frac{3}{5}$

Y - COORDINATE

$\bar{y} = \frac{1}{\text{Area of } R} \int_a^b \frac{1}{2} (f(x))^2 dx$

Area of $R = \frac{2}{3}$

$\bar{y} = \frac{1}{2/3} \int_0^1 \frac{1}{2} f(x)^2 dx$

$\bar{y} = \frac{3}{2} \int_0^1 \frac{1}{2} (\sqrt{x})^2 dx$

$\bar{y} = \frac{3}{2} \int_0^1 \frac{1}{2} x\, dx$

$\bar{y} = \frac{3}{2} \times \frac{1}{2} \int_0^1 x\, dx$

$\bar{y} = \frac{3}{4} \left(\frac{x^2}{2} \right)_0^1$

$\bar{y} = \frac{3}{4} \left(\frac{(1)^2}{2} \right) - \frac{3}{4} \left(\frac{(0)^2}{2} \right)$

$\bar{y} = \frac{3}{8}$

CENTRE OF GRAVITY :- $\left(\frac{3}{5}, \frac{3}{8} \right)$

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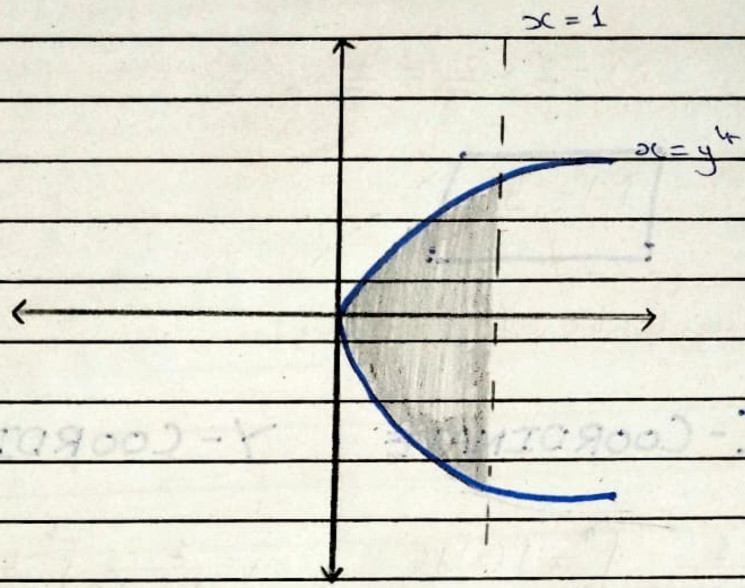
24. A lamina bounded by the graph of $x = y^4$ and the line $x = 1$, $\delta = 15$



Finding Mass using $M = \delta A$

$\delta = 15, A = ?$

Finding Area:-

$y^4 = 1$

$y^4 - 1 = 0$

$(y^2+1)(y+1)(y-1) = 0$

$y^2 + 1 = 0$

$a = 1$; $b = 0$, $c = 1$

Using Quadratic formula

$$y = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{0 \pm \sqrt{0 - 4(1)(1)}}{2(1)}$$

$$y = \frac{\pm\sqrt{-4}}{2}$$

No Real Roots

$y + 1 = 0$

$\boxed{y = -1}$

$y - 1 = 0$

$\boxed{y = 1}$

Interval :-

-1 $\quad\quad\quad\quad$ 1

$x = y^4$

Taking $y = 0.75$

$x = (0.75)^4$

$x = 0.31$

$\therefore$ Lower Curve

$x = 1$

$x = 1$

$\therefore$ Upper Curve

$$A = \int_{-1}^{1} 1 - y^4 \, dy$$

$$A = \int_{-1}^{1} 1 \, dy - \int_{-1}^{1} y^4 \, dy$$

$$A = y \Big|_{-1}^{1} - \frac{y^5}{5} \Big|_{-1}^{1}$$

$$A = \left[y - \frac{y^5}{5} \right]_{-1}^{1}$$

$$A = \left[1 - \frac{(1)^5}{5} \right] - \left[-1 - \frac{(-1)^5}{5} \right]$$

$$A = \left[\frac{4}{5} \right] - \left[-\frac{4}{5} \right]$$

$$A = \frac{8}{5}$$

$$M = \delta A$$

$$\delta = 15 \, , \, A = \frac{8}{5}$$

$$M = 15 \times \frac{8}{5}$$

$$M = 24$$

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Now finding center of Gravity:-

X-COORDINATE

* Since Area is bounded by Y-axis region; we ~~ge~~ use this formula:-

$$\bar{x} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2} f(y)^2 \, dy$$

$$\text{Area of } R = \frac{8}{5}$$

$$\bar{x} = \frac{1}{8/5} \int_{-1}^{1} \frac{1}{2} (1-y^4)^2 \, dy$$

$$\bar{x} = \frac{5}{8} \times \frac{1}{2} \int_{-1}^{1} (1-y^4)^2 \, dy$$

$$\bar{x} = \frac{5}{16} \int_{-1}^{1} 1 - y^8 \, dy$$

$$\bar{x} = \frac{5}{16} \int_{-1}^{1} 1 \, dy - \frac{5}{16} \int_{-1}^{1} y^8 \, dy$$

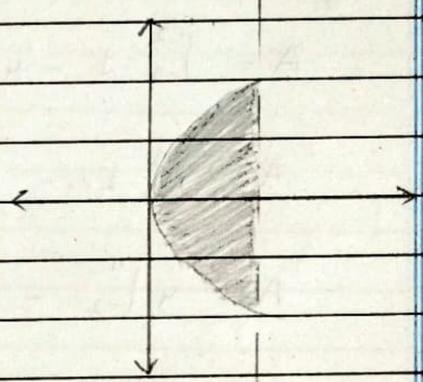
$$\bar{x} = \frac{5}{16} \left[y - \frac{y^9}{9} \right]_{-1}^{1}$$

$$\bar{x} = \frac{5}{16} \left[1 - \frac{(1)^9}{9} \right] - \frac{5}{16} \left[1 - \frac{(-1)^9}{9} \right]$$

$$\bar{x} = \frac{5}{9}$$

Y-COORDINATE

By Symmetry the shape is semicircle divided by x-axis hence



centre point will always lie on x-axis

so:-

$$\bar{y} = 0$$

CENTRE OF GRAVITY :- $\left(\frac{5}{9}, 0\right)$

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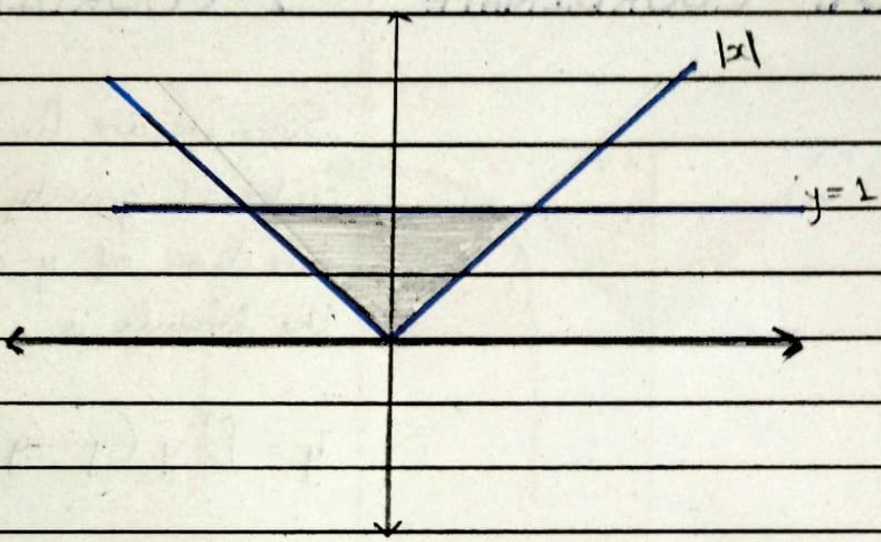
25 A lamina bounded by the graph $y = |x|$ and the line $y = 1$, $\rho = 3$



Finding Mass using $M = \rho A$

$\rho = 3$ $A \to ?$

$$A = \frac{1}{2} \times b \times h$$

$$A = 2\left(\frac{1}{2} \times 1 \times 1\right)$$

$$A = 1$$

$$M = \rho A$$

$$M = 3 \times 1$$

$$\boxed{M = 3}$$

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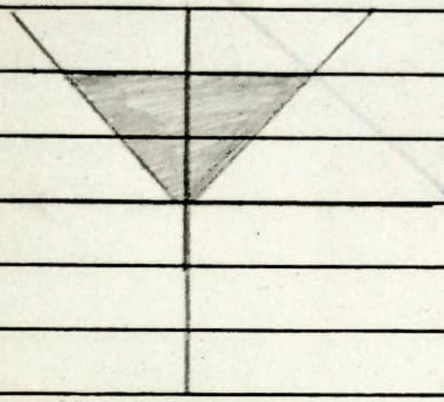
Finding Center of Gravity :-

X - COORDINATE



Since shape is symmetrical the centre will be somewhere along the y-axis

$\therefore \quad \bar{x} = 0$

Y - COORDINATE

Since we are finding centre of gravity in the direction of y-axis the formula is

$$\bar{y} = \int_0^1 y\,(y-(-y))\,dy$$

$$\bar{y} = \int_0^1 y(2y)\,dy$$

$$\bar{y} = \int_0^1 2y^2\,dy$$

$$\bar{y} = \frac{2y^3}{3}\Big|_0^1$$

$$\bar{y} = \frac{2(1)^3}{3} - \frac{2(0)^3}{3}$$

$$\bar{y} = \frac{2}{3}$$

CENTER OF GRAVITY :- $(0, 2/3)$

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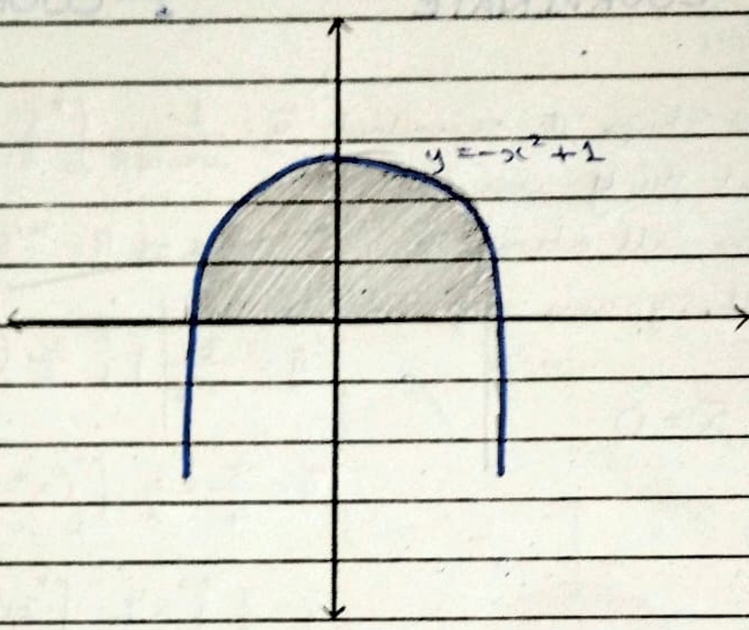
26. A lamina bounded by the x-axis and the graph of the equation $y = 1 - x^2$, $S = 3$.



Finding mass using $M = SA$

$S = 3$, $A = ?$

$$A = \int_{-1}^{1} -x^2 + 1 \, dx$$

$$A = \left[-\frac{x^3}{3} + x \right]_{-1}^{1}$$

$$A = \left[\frac{-(+1)^3}{3} + 1 \right] - \left[\frac{-(-1)^3}{3} + (-1) \right]$$

$$A = \left[-\frac{1}{3} + 1 \right] - \left[\frac{1}{3} - 1 \right]$$

$$A = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$$

$M = SA$

$M = 3 \times \frac{4}{3}$

$\boxed{M = 4.}$

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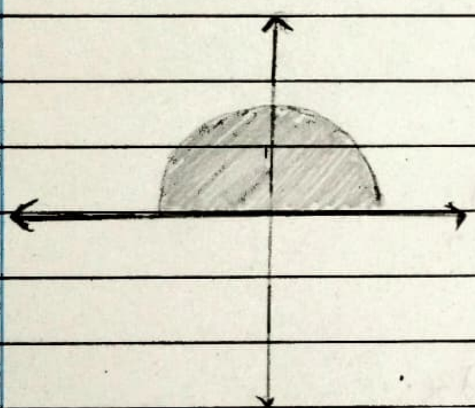
Now we find the center of Gravity:-

X-COORDINATE

Since Shape is symmetrical about the y-axis the centre will always be at the y-axis so,

$\bar{x} = 0$



Y-COORDINATE

$$\bar{y} = \frac{1}{\text{area of } R} \int_a^b \frac{1}{2} f(x)^2 \, dx$$

area of $R = 4/3$

$$\bar{y} = \frac{1}{4/3} \int_{-1}^{1} \frac{1}{2} (-x^2+1)^2 \, dx$$

$$\bar{y} = \frac{3}{4} \times \frac{1}{2} \int_{-1}^{1} (x^4 - 2x^2 + 1) \, dx$$

$$\bar{y} = \frac{3}{8} \int_{-1}^{1} x^4 dx - \int_{-1}^{1} 2x^2 dx + \int_{-1}^{1} 1 \, dx$$

$$\bar{y} = \frac{3}{8} \left[\frac{x^5}{5} - \frac{2x^3}{3} + x \right]_{-1}^{1}$$

$$\bar{y} = \frac{3}{8} \left[\frac{(1)^5}{5} - \frac{2(1)^3}{3} + 1 \right] - \frac{3}{8} \left[\frac{(-1)^5}{5} - \frac{2(-1)^3}{3} + (-1) \right]$$

$$\bar{y} = \left[\frac{1}{5} \right] - \left[-\frac{1}{5} \right]$$

$$\bar{y} = \frac{2}{5}$$

CENTER OF GRAVITY :- $\left(0, \frac{2}{5}\right)$