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A04

EXAMPLE 1:- Find the area of the region bounded above by $y = x+6$, bounded below by $y = x^2$ and bounded on the sides by the lines 0 and 2.

Step 1:- Finding intervals for upper and lower curve

$$x^2 = x+6$$
$$x^2 - x - 6 = 0$$
$$x(x-3)+2(x-3) = 0$$
$$(x+2)(x-3) = 0$$
$$x+2=0 \qquad x-3=0$$
$$x=-2 \qquad x=3$$

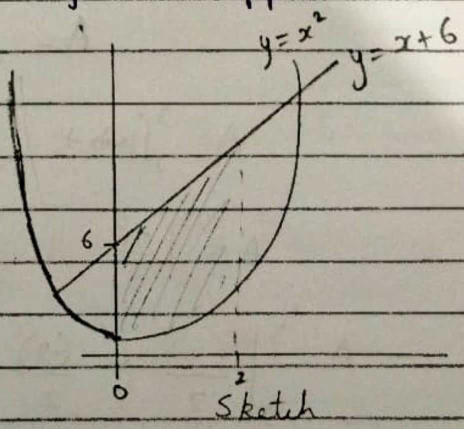
But we need interval from $x=0$ to $x=2$ hence we will find upper and lower for it.

$$y = x^2 \quad , y = x+6$$

Step 2:- Choosing point between $x=0, x=2$

$x = 1$	$x = 1$
$y = x^2$	$y = x+6$
$y = (1)^2$	$y = 1+6$
$y = 1$ Lower	$y = 7$ Upper

Step 3:- Finding Area



$$A = \int_0^b f(x) - g(x)\, dx$$

$$A = \int_0^2 x+6 - x^2\, dx$$

$$A = \int_0^2 x\,dx + \int_0^2 6\,dx - \int_0^2 x^2\,dx$$

$$A = \left.\frac{x^2}{2}\right|_0^2 + \left.6x\right|_0^2 - \left.\frac{x^3}{3}\right|_0^2$$

$$A = \frac{(2)^2}{2} - \frac{(0)^2}{2} + 6(2) - 6(0) - \frac{(2)^3}{3} - \frac{(0)^3}{3}$$

$$A = 2 + 12 - \frac{8}{3}$$

$$\boxed{A = \frac{34}{3}}$$

EXAMPLE 2 :- Find the area of the region bounded above by $y = x+6$ and enclosed between $y = x^2$.

From previous example we have the total region between the two curves.

$[-2, 3]$ calculated in step 1 of previous question.

Splitting the two into more regions

$[-2, 0]$, $[0, 3]$

$y = x^2$	$y = x+6$	$y = x^2$	$y = x+6$
Taking $x = -1$	Taking $x = -1$	Taking $x = 1$	$x = 1$
$y = (-1)^2$	$y = -1+6$	$y = (1)^2$	$y = 1+6$
$y = 1$	$y = 5$	$y = 1$	$y = 7$
Lower	Upper	Lower	Upper

* Since in Both curves $x+6$ is the Upper curve we are computing the limits together:

$$A = \int_{-2}^{3} x+6 - x^2 \, dx$$

$$A = \int_{-2}^{3} x\,dx + \int_{-2}^{3} 6\,dx - \int_{-2}^{3} x^2\,dx$$

$$A = \left.\frac{x^2}{2}\right|_{-2}^{3} + \left. 6x \right|_{-2}^{3} - \left.\frac{x^3}{3}\right|_{-2}^{3}$$

$$A = \left.\frac{(3)^2}{2} - \frac{(-2)^2}{2}\right|_{-2}^{3} + \left. 6(3) - 6(-2) \right|_{-2}^{3} - \left.\frac{(3)^3}{3} - \frac{(-2)^3}{3}\right|_{-2}^{3}$$

$$A = \left[\frac{9-4}{2}\right] + [18+12] - \left[\frac{27-(-8)}{3}\right]$$

$$A = \frac{5}{2} + 30 - \frac{35}{3}$$

$$\boxed{A = \frac{125}{6}}$$

EXAMPLE #4 :- Find the area of the region enclosed by $x = y^2$ and $y = x - 2$

$x = y^2$ $\qquad\qquad$ $x = y + 2$

$y = \sqrt{x}$ $\qquad\qquad$ $y = x - 2$

Step 1 :- Finding Interval $\qquad$ $\sqrt{x} = x - 2$

Taking square on both sides

$(\sqrt{x})^2 = (x-2)^2$

$x = x^2 - 4x + 4$

$x^2 - 4x - x + 4 = 0$

$x(x-4) - 1(x-4) = 0$

$(x-1)(x-4) = 0$

$x - 1 = 0$ $\qquad\qquad$ $x - 4 = 0$

$x = 1$ $\qquad\qquad$ $x = 4$

Step 2 :- Finding intervals

1 $\qquad$ 4

$y = \sqrt{x}$ $\qquad\qquad$ $y = x - 2$

$x = 3$ $\qquad\qquad$ $x = 3$

$y = \sqrt{3}$ $\qquad\qquad$ $y = 3 - 2$

$y = 1.73$ $\qquad\qquad$ $y = 1$

Upper $\qquad\qquad$ Lower

$$A = \int_1^4 \sqrt{x} - x - 2 \, dx$$

$$A = \int_1^4 \sqrt{x} \, dx - \int x \, dx - \int 2 \, dx$$

$$A = \left| \frac{x^{3/2}}{\frac{3}{2}} \right|_1^4 - \left| \frac{x^2}{2} \right|_1^4 - \left| 2x \right|_1^4$$

$$A = \left| \frac{2(4)^{3/2}}{3} - \frac{2(1)^{3/2}}{3} \right|_1^4 - \left| \frac{(4)^2}{2} - \frac{(1)^2}{2} \right|_1^4 - \left| 2(4) - 2(1) \right|_1^4$$

$$A = \left[\frac{2}{3}(7)\right] - \left[\frac{15}{2}\right] + [6]$$

$$\boxed{A = \frac{19}{6}}$$

EXAMPLE 5 :- Find the area of region enclosed by $x = y^2$ and $y = x-2$ integrating with respect to y.

$$x = y^2 \quad , \quad y = x-2$$

Step 1 :- Equating x :-

$$x = y + 2$$

$$y^2 = y + 2$$
$$y^2 - y - 2 = 0$$
$$y^2 - 2y + y - 2 = 0$$
$$y(y-2) + 1(y-2) = 0$$
$$(y+1)(y-2) = 0$$

$y + 1 = 0 \qquad y - 2 = 0$

$\boxed{y = -1} \qquad \boxed{y = 2}$

Step 2 :- Sign Test for finding intervals.

-1 to 0

$x = y^2$	$x = y + 2$
Taking -0.5	Taking -0.5
$x = (-0.5)^2$	$x = -0.5 + 2$
$x = 0.25$	$x = 1.5$
Lower	Upper

0 to 2

$x = y^2$	$x = y + 2$
Taking $x = 1$	Taking $x = 1$
$x = (1)^2$	$x = 1 + 2$
$x = 1$	$x = 3$
Lower	Upper

Since both intervals the upper curve is $y+2$ the intervals are being taken together

$$A = \int_{-1}^{2} y + 2 - y^2 \, dy$$

$$A = \int_{-1}^{2} y\,dy + \int_{-1}^{2} 2\,dy - \int_{-1}^{2} y^2\,dy$$

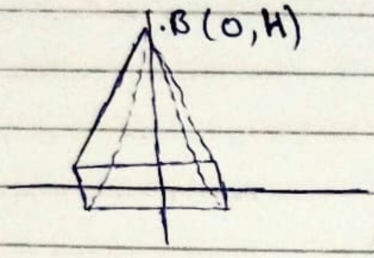
$$A = \Big|_{-1}^{2} \frac{y^2}{2} + \Big|_{-1}^{2} 2y - \Big|_{-1}^{2} \frac{y^3}{3}$$

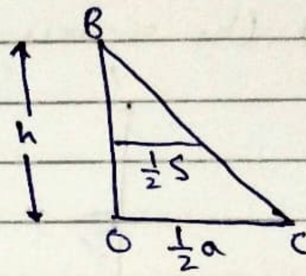
$$A = \Big|_{-1}^{2} \frac{(2)^2}{2} - \frac{(1)^2}{2} + \Big|_{-1}^{2} 2(2) - 2(-1) - \Big|_{-1}^{2} \frac{(2)^3}{3} - \frac{(-1)^3}{3}$$

$$A = \left[\frac{4-1}{2}\right] + (2(3)) - \left(\frac{8+1}{3}\right)$$

$$\boxed{A = 9/2}$$

EXAMPLE 1 :- Derive the formula for the volume of a right Pyramid whose attitude h and whose base is a square with slides of length a.





From the pyramid we remove a triangle that can be used to calculate the area. We have to only find right pyramid so we will take $\frac{1}{2}a$.

$$\frac{\frac{1}{2}s}{\frac{1}{2}a} = \frac{h-y}{h}$$

$$\frac{2}{2}\left(\frac{s}{a}\right) = \frac{h-y}{h}$$

$$s = \frac{a}{h}(h-y)$$

Since it has a square

$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2$$

$$V = \int_0^h A(y)\, dy$$

$$V = \int_0^h \frac{a^2}{h^2}(h-y)^2\, dy$$

$$V = \frac{a^2}{h^2}\int_0^h (h-y)^2\, dy$$

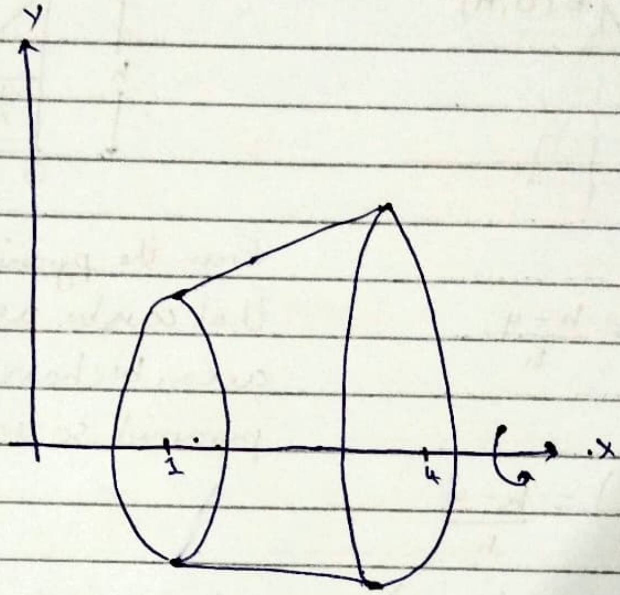
$$V = \frac{a^2}{h^2}\left[-\frac{1}{3}(h-y)^3\right]_0^h$$

$$V = \frac{a^2}{h^2}\left[0 + \frac{1}{3}h^3\right]$$

$$V = \frac{1}{3}a^2h$$

The volume is $\frac{1}{3}$ of the area of the base times the altitude

EXAMPLE 2:- Find the volume of solid that is obtained when the region under the curve $y=\sqrt{x}$ over the interval $[1,4]$ is resolved about the x-axis



$$V = \int_a^b \pi [f(x)]^2 \, dx$$

$$V = \int_1^4 \pi (\sqrt{x})^2 \, dx$$

$$V = \int_1^4 \pi \, x \cdot dx$$

$$V = \pi \int_1^4 x \, dx$$

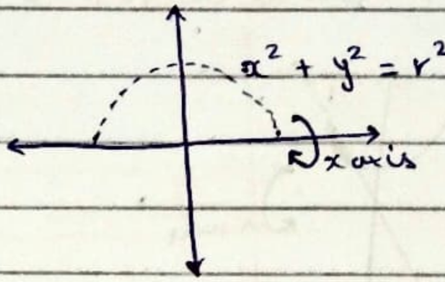
$$V = \pi \left| \frac{x^2}{2} \right|_1^4$$

$$V = \pi \left| \frac{(4)^2}{2} \right| - \left| \frac{(1)^2}{2} \right|$$

$$V = \pi \left| \frac{16-1}{2} \right|$$

$$V = \frac{15\pi}{2}$$

EXAMPLE 3:- Derive the formula for the volume of a sphere of radius r.



If we rotate a semi circle around x-axis it will form a sphere.

$$x^2+y^2=r^2$$

So,

$$y=f(x).$$

$$x^2+y^2=r^2$$

$$x^2=r^2-y^2$$

$$x=\sqrt{r^2-y^2}$$

Along x-axis

$$y=\sqrt{r^2-x^2}$$

Volume:-

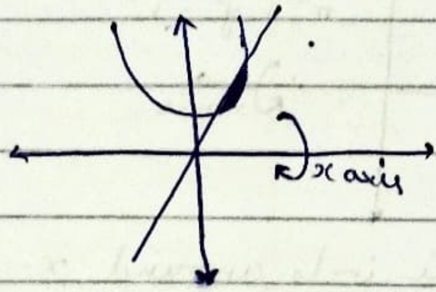
$$V=\int_a^b \pi\,[f(x)]^2\,dx$$

$$V=\pi\int_{-r}^{r} r^2-x^2\,dx$$

$$V=\pi\left|r^2x-\frac{x^3}{3}\right|_{-r}^{r}$$

$$V=\frac{4}{3}\pi r^3$$

EXAMPLE 4:- Find the volume of the solid generated when the region between the graphs of the equation $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis



$$V = \int_a^b \pi [f(x)]^2 - (g[x])^2 \, dx$$

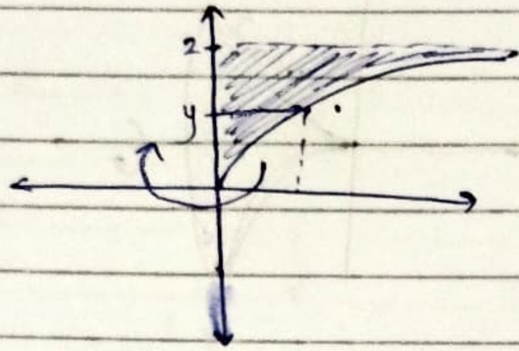
$$V = \int_0^2 \pi \left(\frac{1}{2} + x^2\right)^2 - (x^2) \, dx$$

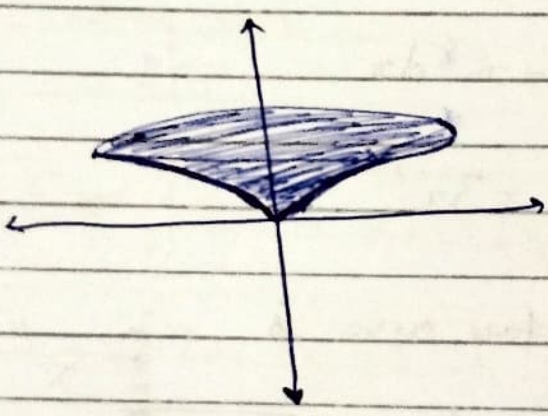
$$V = \int_0^2 \pi \cdot \left(\frac{1}{4} + x^4\right) . dx$$

$$V = \pi \left[\frac{x}{4} + \frac{x^5}{5}\right]_0^2$$

$$V = \frac{69\pi}{16}$$

EXAMPLE 5 : Find the volume of the solid generated when the region enclosed by $y=\sqrt{x}$, $y=2$ and $x=0$ and $x=1$ revolved about the y-axis :





$$V = \int_c^d \pi [u(y)]^2 dy$$

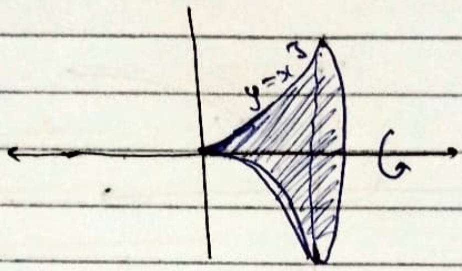
$$V = \int_0^2 \pi y^4 dy$$

$$V = \left. \frac{\pi y^5}{5} \right|_0^2$$

$$V = \frac{(2)^5 \pi}{5} - \frac{(0)^5 \pi}{5}$$

$$\boxed{V = \frac{32\pi}{5}}$$

EXAMPLE 1 :- Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis



$$y = x^3$$

$$\frac{dy}{dx} = x^3 \, dx$$

$$\frac{dy}{dx} = 3x^2$$

Now the Surface area is

$$S = \int_0^1 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

$$S = \int_0^1 2\pi x^3 \sqrt{1 + (3x^2)^2} \, dx$$

$$S = 2\pi \int_0^1 x^3 (1 + 9x^4) \, dx$$

$$S = \frac{2\pi}{36} \int_0^{10} u^{\frac{1}{2}} \, du \qquad u = 1 + 9x^4$$

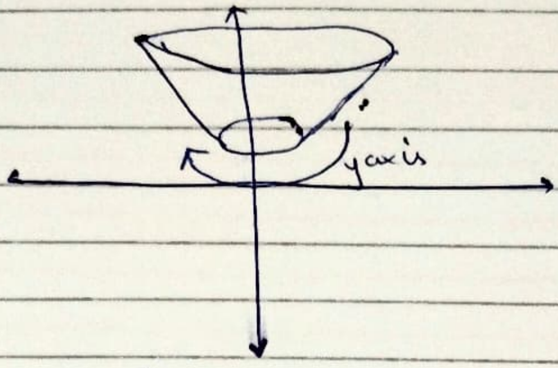
$$du = 36x^3 \, dx$$

$$S = \frac{2\pi}{36} \times \frac{2}{3} u^{3/2} \Big|_{u=1}^{10}$$

$$S = \frac{\pi}{27} \left(10^{3/2} - 1\right)$$

$$S \approx 3.56$$

EXAMPLE 2 :- Find the area of surface that is generated by revolving the Portion of curve $y = x^2$ b/w $x = 1$ and $x = 2$ about y-axis



$y = x^2$

Taking sqrt

$x = \sqrt{y}$

$$S = \int_1^4 2\pi \; x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$$

$$S = \int_1^4 2\pi \sqrt{y} \sqrt{1 + \frac{1}{2\sqrt{y}}} \, dy$$

$$S = \pi \int_1^4 \sqrt{4y+1} \, dy$$

Using u substitution :-

$u = 4y + 1$

$du = 4\,dy$

$$S = \pi \int_1^4 u \, du$$

$$S = \pi \int_1^4 \frac{u^2}{2} \, du$$

$$S = \pi \int_1^4 \frac{\sqrt{4y+1}}{2}$$

$$S = \frac{\pi}{4} \Big|_5^{17} \frac{2\pi^{3/2}}{3} \Big|$$

$$S = \frac{\pi}{6} \left[17^{3/2} - 5^{3/2} \right]$$

$$\boxed{S \approx 30.85}$$