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EXAMPLE 1:-

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

Checking Symmetry about Y-axis :-

$$y = \frac{2(-x)^2 - 8}{(-x)^2 - 16}$$

$$= \frac{2x^2 - 8}{x^2 - 16}$$

* Since final answer is equal, function is symmetric along the Y-axis

Checking Symmetry about Origin :-

$$-y = \frac{2(-x)^2 - 8}{(-x^2) - 16}$$

$$-y = \frac{2x^2 - 8}{x^2 - 16}$$

* Since final answer is not equal to orignal function hence function is not symmetric along Origin.

'X' and 'Y' Intercept :-

Y-intercept :-

$x = 0 \quad , y = ?$

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

$$y = \frac{2(0)^2 - 8}{(0)^2 - 16}$$

$$y = \frac{\cancel{-}8}{\cancel{-}16}$$

$$y = \frac{1}{2}$$

$$\boxed{Y \text{ intercept } (0, \tfrac{1}{2})}$$

X-intercept :-

$x = ? \quad , y = 0$

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

$$\frac{2x^2 - 8}{x^2 - 16} = 0$$

$$2x^2 - 8 = 0$$

$$2x^2 = 8$$

$$x^2 = \frac{8}{4}$$

$$\sqrt{x^2} = \sqrt{\pm 2}$$

$$x = \pm 2$$

$$\boxed{X \text{ intercept } (-2, 0), (2, 0)}$$

Vertical Asymptotes :-

$$\frac{2x^2 - 8}{x^2 - 16}$$

$x^2 - 16 \longrightarrow (x)^2 - (4)^2$

$(x + 4)(x - 4)$

$$\frac{2(x^2 - 4)}{(x+4)(x-4)}$$

* No cutting is possible

$x + 4 = 0 \qquad\qquad x - 4 = 0$

$x = -4 \qquad\qquad x = +4$

∴ Vertical Asymtotes at $x = +4$, $x = -4$

Sign of 'Y' :- (Found after using 1st Derivative)

Interval	Test Point	Value of y	Comments
$(-\infty, -4)$	-5	$14/3$	+ increasing
$(-4, -2)$	-3	$-10/7$	− decreasing
$(-2, -2)$	0	$1/2$	+ increasing
$(2, 4)$	3	$-10/7$	− decreasing
$(4, +\infty)$	5	$14/3$	+ increasing

End Behaviour :-

$$\lim_{x \to +\infty} \frac{2x^2-8}{x^2-16}$$

$$= \frac{x^2(2 - 8/x^2)}{x^2(1 - 16/x^2)}$$

$$= \frac{2 - 8/{+\infty}}{1 - 16/{+\infty}}$$

$$= \frac{2-0}{1-0}$$

$$\lim_{x \to +\infty} f(x) = 2$$

$$\lim_{x \to -\infty} \frac{2x^2-8}{x^2-16}$$

$$= \frac{x^2(2 - 8/x^2)}{x^2(1 - 16/x^2)}$$

$$= \frac{2 - 8/{-\infty}}{1 - 16/{-\infty}}$$

$$= \frac{2-0}{1-0}$$

$$\lim_{x \to -\infty} f(x) = 2$$

Horizontal asymptote at $x = 2$

1^{st} Derivative :-

$$y = \frac{(2x^2 - 8)}{x^2 - 16}$$

$$y' = \frac{(x^2 - 16)(4x) - (2x^2 - 8)(2x)}{(x^2 - 16)^2}$$

$$y' = \frac{4x^3 - 64x - (4x^3 - 16x)}{(x - 16)^2}$$

$$y' = \frac{4x^3 - 64x - 4x^3 + 16x}{(x^2 - 16)^2} = \boxed{\frac{-48x}{(x^2 - 16)^2}}$$

Finding Critical point :-

$$\frac{-48x}{(x^2 - 16)^2} = 0$$

$$-48x = 0$$

$$x = 0$$

$\Rightarrow$ Putting X- coordinate in function to find Y value

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

$$y = \frac{2(0)^2 - 8}{(0)^2 - 16}$$

$$y = \frac{-8}{-16}$$

$$y = \frac{+1}{+2}$$

HENCE CRITICAL POINT :-

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2^{nd} Derivative :-

$$y' = \frac{-48x}{(x^2-16)^2}$$

$$y'' = \frac{(x^2-16)^2(-48) - (-48x)(2)(x^2-16)(2x)}{((x^2-16)^2)^2}$$

$$y'' = \frac{(-48)(x^2-16)^2 + 48x(4x(x^2-16))}{(x^2-16)^4}$$

$$y'' = \frac{-48(x^2-16)^2 + 192x^2(x^2-16)}{(x^2-16)^4}$$

$$y'' = \frac{\cancel{(x^2-16)}(-48(x^2-16) + 192x^2)}{(x^2-16)^{4-1}}$$

$$y'' = \frac{-48(x^2-16) + 192x^2}{(x^2-16)^3}$$

$$y'' = \frac{-48x^2 + 768 + 192x^2}{(x^2-16)^3}$$

$$y'' = \frac{144x^2 + 768}{(x^2-16)^3}$$

$$y'' = \frac{48(3x^2+16)}{(x^2-16)^3}$$

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Finding inflation points:-

$$\frac{48(3x^2+16)}{(x^2-16)^3} = 0$$

$$48(3x^2+16) = 0$$

$$3x^2+16=0$$

$$3x^2=-16$$

$$x^2=\frac{-16}{3}$$

$$\sqrt{x^2}=\sqrt{\frac{-16}{3}}$$

$$x = \text{Math Error}$$

No inflation point as -ve sign appeared in under root.

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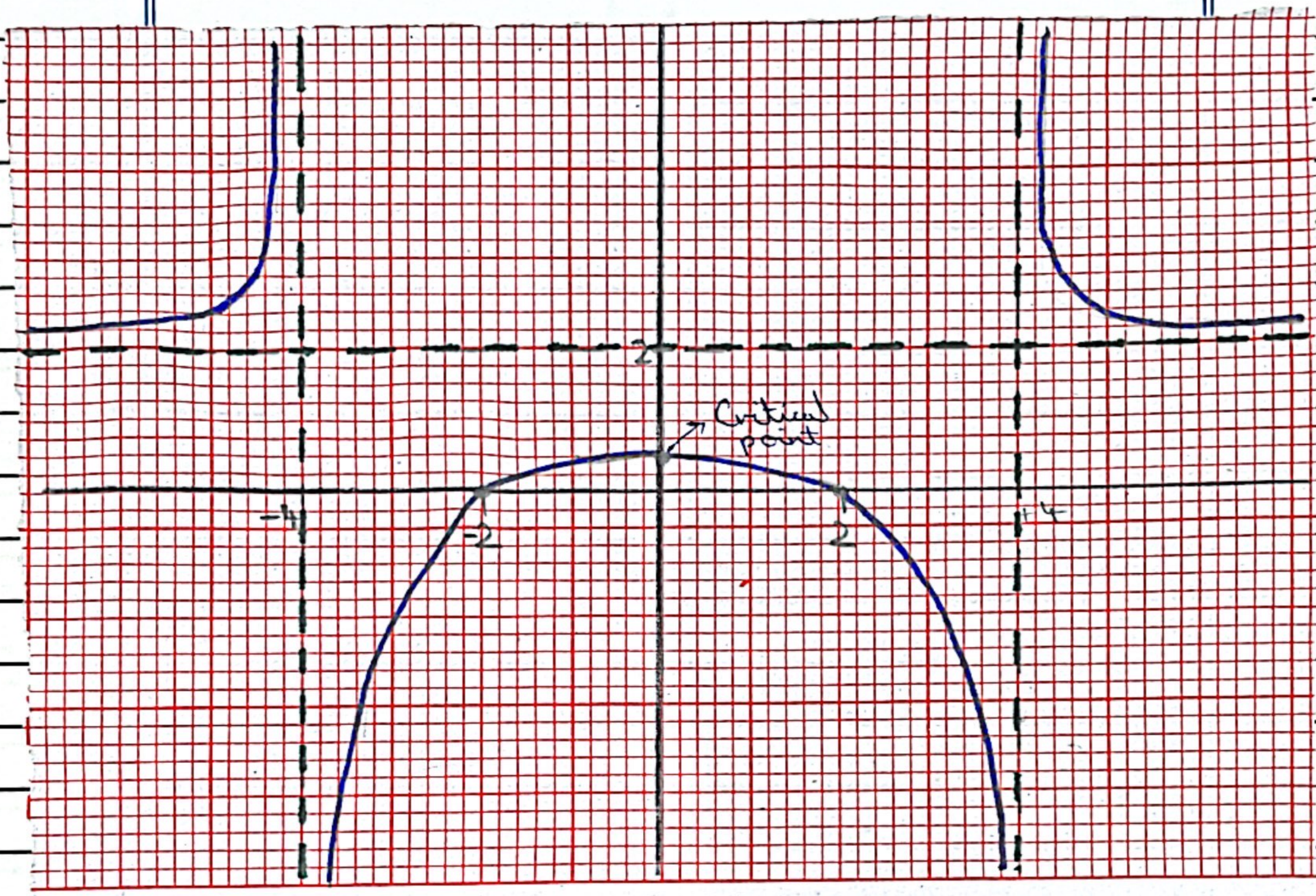
GRAPH AND CONCLUSION



Graph was drawn using data collected in previous steps.

Key Points :-

→ Horizontal asymptotes are shown at $x = 2$
→ Critical point at $(0, 0.5)$ → Considered Relative Maxima
→ Vertical asymptotes at $x = 4$, $x = -4$
→ Graph after $x = 4$, $x = -4$ is concave up whereas before that it is concave down
→ Graph is symmetric along Y axis

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EXAMPLE 2:-

$$y = \frac{x^2 - 1}{x^3}$$

Checking Y-axis symmetry :-

$$y = \frac{(-x)^2 - 1}{(-x)^3}$$

$$y = \frac{x^2 - 1}{-x^3}$$

Function is not symmetric about y-axis.

Checking Origin Symmetry:-

$$-y = \frac{(-x)^2 - 1}{(-x)^3}$$

$$+y = +\frac{x^2 - 1}{x^3}$$

$$y = \frac{x^2 - 1}{x^3}$$

Function is symmetric about the origin.

'X' AND 'Y' - Intercept :-

Y-INTERCEPT	X-INTERCEPT
$x = 0, y = ?$	$y = 0, x = ?$
$y = \frac{x^2 - 1}{x^3}$	$0 = \frac{x^2 - 1}{x^3}$
$y = \frac{(0)^2 - 1}{(0)^3}$	$x^2 - 1 = 0$
	$x^2 = 1$
$y = \infty$	$\sqrt{x^2} = \sqrt{1}$
	$x = \pm 1$
$(0, \infty)$	$(1, 0); (-1, 0)$

Vertical Asymptotes :-

$$y = \frac{x^2 - 1}{x^3}$$

Asymptote at $x = 0$, since function will approach infinity.

$$y = \frac{(0)^2 - 1}{(0)^3}$$

$$y = \frac{-1}{0}$$

$$y = -\infty$$

Sign of 'y''

Interval	Test Point	Value of 'y''	Comments
$(-\infty, -1.73)$	-2	$-(-0.375)$	Decreasing
$(-1.73, +1.73)$	1.5	$+(0.370)$	Increasing
$(1.73, +\infty)$	2	$+(0.375)$	Increasing

USING SECOND DERIVATIVE :-

Interval	Test point	Value of 'y''	Comment
$(-\infty, -2.45)$	-3	$-(-0.296)$	Concave down
$(-2.45, 2.45)$	2	$+(0.375)$	Concave up
$(2.45, +\infty)$	3	$+(+0.296)$	Concave up

End Behaviour :-

$\lim\limits_{x \to +\infty} \frac{x^2-1}{x^3}$	$\lim\limits_{x \to -\infty} \frac{x^2-1}{x^3}$
$\frac{(+\infty)^2 - 1}{(+\infty)^3}$	$\frac{(-\infty)^2 - 1}{(-\infty)^3}$
$\frac{1}{\infty} = 0$	$\frac{-1}{\infty} = 0$

Horizontal Asymptotes $y = 0$

1^{st} Derivative :-

$$y = \frac{x^2 - 1}{x^3}$$

$$y' = \frac{x^3 (2x) - (x^2 - 1)(3x^2)}{(x^3)^2}$$

$$y' = \frac{2x^4 - (3x^4 - 3x^2)}{x^6}$$

$$y' = \frac{2x^4 - 3x^4 + 3x^2}{x^6}$$

$$y' = \frac{-x^4 + 3x^2}{x^6}$$

OR

$$y' = \frac{-x^2(x^2 + 3)}{x^{6-2}}$$

$$y' = -\frac{(x^2 + 3)}{x^4}$$

Finding Critical Point :-

$$\frac{-x^2+3}{x^4} = 0$$

$$-x^2+3=0$$
$$+x^2=+3$$
$$\sqrt{x^2}=\sqrt{3}$$
$$x=\pm\sqrt{3} \quad \text{or} \quad x=\pm 1.73$$

Finding y-value of critical point :-

$\frac{x^2-1}{x^3}$	$\frac{x^2-1}{x^3}$
$\frac{(1.73)^2-1}{(1.73)^3}$	$\frac{(-1.73)^2-1}{(-1.73)^3}$
$y=0.384$	$y=-0.384$
$(1.73, 0.384)$	$(-1.73, -0.384$

2nd Derivative :-

$$y' = \frac{-x^2+3}{x^4}$$

$$y'' = \frac{(x^4)(-2x) - (-x^2+3)(4x^3)}{(x^4)^2}$$

$$y'' = \frac{-2x^5 - (-4x^5+12x^3)}{x^8}$$

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$$y'' = \frac{-2x^5 + 4x^5 - 12x^3}{x^8}$$

$$y'' = \frac{2x^5 - 12x^3}{x^8}$$

$$y'' = \frac{2x^3(x^2-6)}{x^{8-3}}$$

$$y'' = \frac{2(x^2-6)}{x^5}$$

Finding inflection point :-

$$\frac{2(x^2-6)}{x^5} = 0$$

$$2(x^2-6)=0$$

$$x^2-6=0$$

$$x^2=6$$

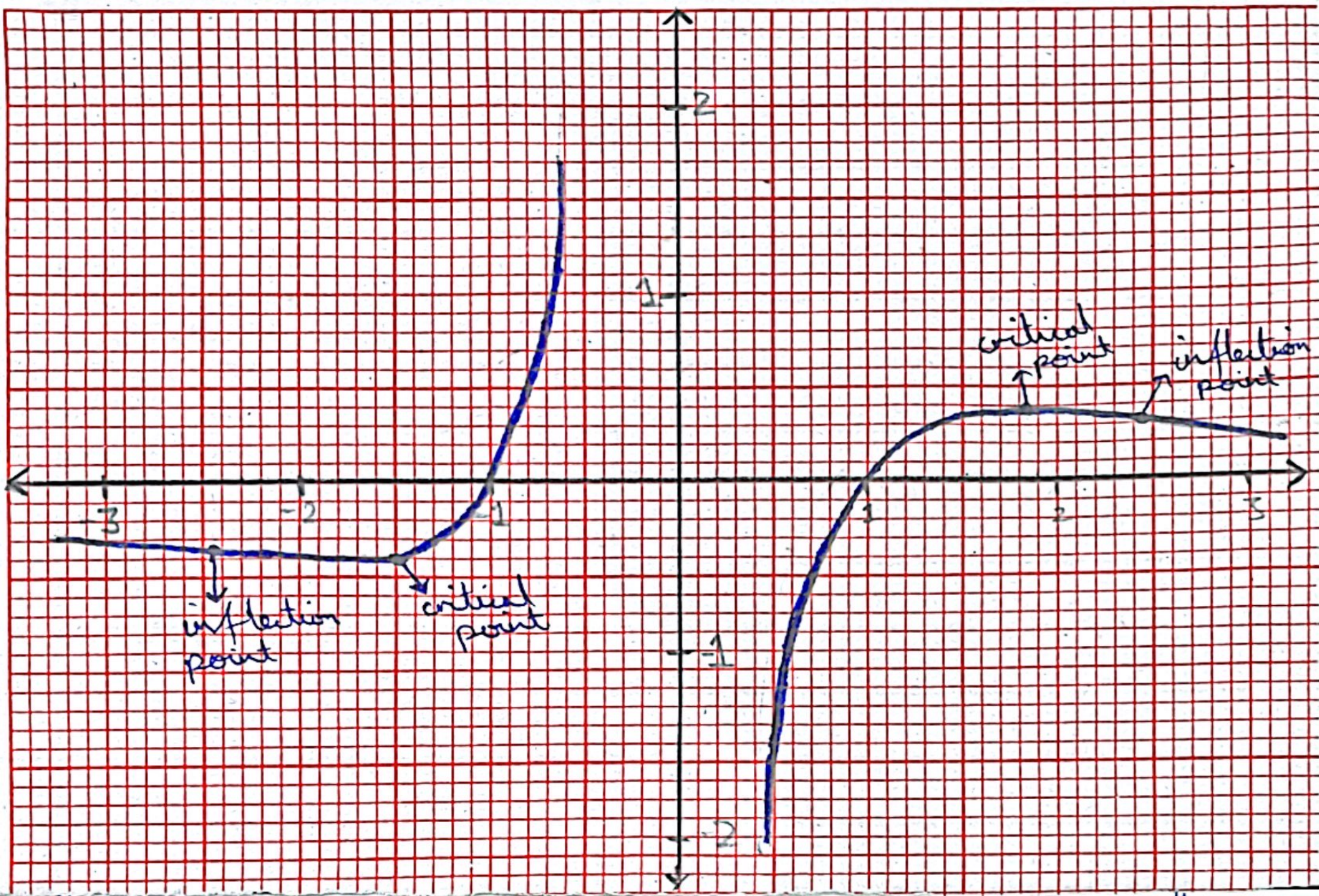
$$\sqrt{x^2}=\sqrt{6}$$

$$x=\sqrt{6}$$

$$x=\pm\sqrt{6} = \pm 2.45$$

Finding y-value of inflection point

$\frac{x^2-1}{x^3}$	$\frac{x^2-1}{x^3}$
$\frac{(+2.45)^2-1}{(2.45)^3}$	$\frac{(-2.45)^2-1}{(-2.45)^3}$
$y=+0.340$	$y=-0.340$
$(2.45, 0.340)$	$(-2.45, -0.340)$



Key Points :-

- X - interepts $(-1, 0)$ and $(1, 0)$ are shown on the graph
- Graph is symmetri about the origin
- Concavity changes at inflection point $(\pm 2.45, 0.340)$
- $x = 0$ is Vertical asymptote as function is discontinuos at $x = 0$.
- Relative minima is $(-1.73, 0.384)$
- Relative maxima is $(1.73, 0.384)$

EXAMPLE 3:-

$$y = (x-4)^{2/3}$$

Checking Symmetry about the Y-axis

$$y = \sqrt[3]{((-x) - 4)^2}$$

$$y = \sqrt[3]{(-x)^2 + 2(-x)(-4) + (-4)^2}$$

$$y = \sqrt[3]{x^2 - 8x + 16}$$

⟹ y is not equal to orignal function hence function is not symmetric along y-axis

Checking Symmetry about Origin:-

$$-y = \sqrt[3]{((-x) - 4)^2}$$

$$-y = \sqrt[3]{(-x)^2 - 2(-x)(-4) + (-4)^2}$$

$$-y = \sqrt[3]{x^2 - 8x + 16}$$

Function is not symmetric about origin

'X' and 'Y' Intercept :-

Y intercept	X intercept
$x=0 \,,\; y=?$	$y=0 \,,\; x=?$
$y=\sqrt[3]{(0-4)^2}$	$\sqrt[3]{(x-4)^2}=0$
	$\left[\sqrt[3]{(x-4)^2}\right]^3=(0)^3$
$y=\sqrt[3]{16}$	
$y=2.5$	$(x-4)^2=0$
$(0,2.5)$	$\sqrt{(x-4)^2}=\sqrt{0}$
	$x-4=0$
	$x=4$
	$(4,0)$

Vertical Asymptotes:-

$$y=\sqrt[3]{(x-4)^2}$$

- Since function is not rational hence no asymptotes form
- Function is continuos everywhere

Sign of 'y'

Interval	Test Point	Sign	Comments
$(-\infty, 0)$	-1	$+(2.98)$	Increasing
$(0, 4)$	$+1$	$+(2.08)$	Increasing
$(4, \infty)$	$+5$	$+(1)$	Increasing

End Behaviour :-

$y = \sqrt[3]{(x-4)^2}$

$\lim_{x \to +\infty} y = \sqrt[3]{(\infty + 4)^2}$

$= \sqrt[3]{\infty}$

$= +\infty$

$y = \sqrt[3]{(x-4)^2}$

$\lim_{x \to -\infty} y = \sqrt{(-\infty - 4)^2}$

$= \sqrt{(-\infty)^2}$

$= +\infty$

1st Derivative

$$y = \sqrt[3]{(x-4)^2}$$

$$y = (x-4)^{2/3}$$

$$y = \frac{2}{3}(x-4)^{\frac{2}{3}-1} \cdot (1)$$

$$y = \frac{2}{3}(x-4)^{-\frac{1}{3}}$$

$$y = \frac{2}{3(x-4)^{\frac{1}{3}}}$$

Finding critical points:-

$$\frac{2}{3(x-4)^{\frac{1}{3}}} = 0$$

If we insert $x = 4$ the function will be equal to infinity hence discontinuity at $x = 4$.

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2^{nd} Derivative :-

$$y' = \frac{2}{3(x-4)^{\frac{1}{3}}}$$

$$y'' = \frac{2}{3}(x-4)^{\frac{-1}{3}}$$

$$y'' = \frac{2}{3}\left(\frac{-1}{3}\right)(x-4)^{\frac{-1}{3}-1}(1)$$

$$y'' = \frac{-2}{9}(x-4)^{\frac{-4}{3}}$$

$$y'' = \frac{-2}{9(x-4)^{\frac{4}{3}}}$$

Inflection Point :-

$$\frac{-2}{9(x-4)^{4/3}} = 0$$

There is no solution to this equation hence no inflection points.

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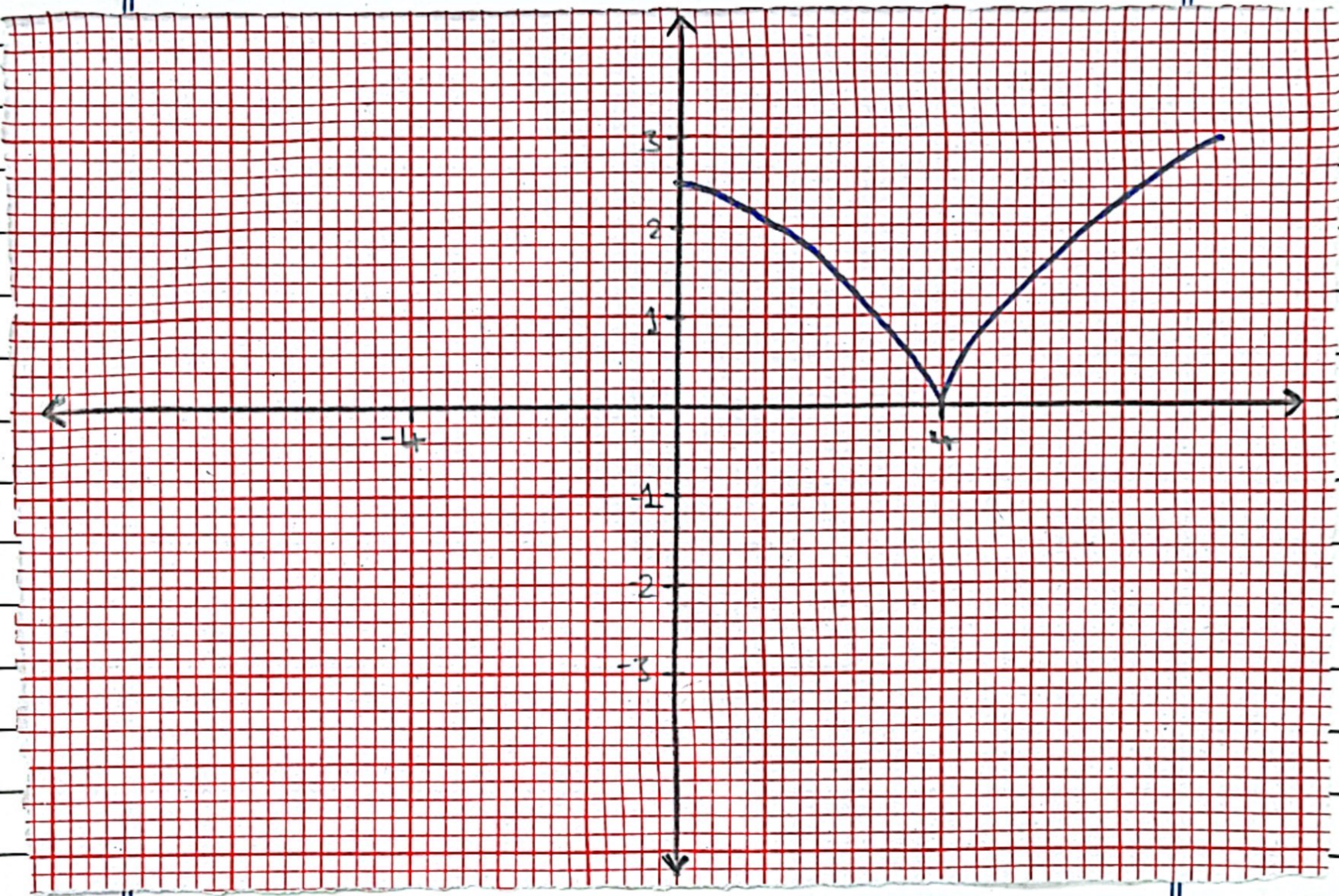
GRAPH AND CONCLUSION :-



Key Points :-

→ Y intercept shown on graph (0, 2.5)

→ Cusp is formed at $x = 4$ which shows discontinuity at $x = 4$

→ No changes in concavity

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EXAMPLE 4 :-

$$y = 6x^{\frac{1}{3}} + 3x^{4/3}$$

Symmetry about Y-axis

$$y = 6(-x)^{\frac{1}{3}} + 3(-x)^{\frac{4}{3}}$$

$$y = 6\left(\sqrt[3]{-x}\right) + 3\left(\sqrt[3]{(-x)^4}\right)$$

$$y = -6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$$

Not symmetric about Y axis since function is not equal to orignal function.

Origin symmetry :-

$$y = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$$

$$-y = 6\left(\sqrt[3]{-(x)}\right) + 3\sqrt[3]{(-x)^4}$$

$$-y = -6x^{\frac{1}{3}} + 3\sqrt[3]{x^4}$$

$$-y = -6x^{\frac{1}{3}} + 3\sqrt[3]{x^4}$$

No origin Symmetry because function is not equal to original function

'X' and 'Y' Intercept :-

Y intercept	X intercept
$x = 0$, $y = ?$	$y = 0$, $x = ?$
$y = 6(0)^{\frac{1}{3}} + 3(0)^{\frac{4}{3}}$	$0 = 6x^{\frac{1}{3}} + 3x^{4/3}$
	$0 = 3x^{\frac{1}{3}}(2+x)$
$y = 0$	$2 + x = 0$
	$x = -2$
$(0, 0)$	$(-2, 0)$

Vertical Asymptotes

$$y = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$$

Since function is not rational and hence no vertical asymptotes will be formed

Sign of 'y'' :-

Interval	Test Point	Sign	Comment
$(-\infty, -0.5)$	-1	$-(-3)$	Decreasing
$(-0.5, +\infty)$	1	$+9$	Increasing
$(-\infty, 1)$	-2		Concave up
$(1, +\infty$	2		Concave down

End Behaviour :-

$y = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$	$y = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$
$y = \lim_{x \to +\infty} 6(\infty)^{\frac{1}{3}} + 3(\infty)^{\frac{4}{3}}$	$y = \lim_{x \to -\infty} 6(-\infty)^{\frac{1}{3}} + 3(-\infty)^{\frac{4}{3}}$
$y = +\infty$	$y = +\infty$

- No Vertical and Horizontal Asymptotes

1st Derivative

$$y = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$$

$$y' = 6\left(\frac{1}{3}\right)x^{\frac{-2}{3}}(1) + 3\left(\frac{4}{3}\right)x^{\frac{1}{3}}(1)$$

$$y' = 2x^{\frac{-2}{3}} + 4x^{\frac{1}{3}}$$

Finding Critical Point :-

$$y' = 0$$

$$2x^{\frac{-2}{3}} + 4x^{\frac{1}{3}} = 0$$

$$2x^{\frac{-2}{3}}(1 + 2x) = 0$$

$$1 + 2x = 0$$

$$2x = -1$$

$$x = \frac{-1}{2}$$

If we rearrange the y' we can get

$$\frac{2(1+2x)}{x^{\frac{2}{3}}}$$

- If we take $x = 0$ function will be discontinuous
- Therefore $(0, 0)$ is also an inflection point

__2nd Derivative :-__

$$y' = 2x^{\frac{-2}{3}} + 4x^{\frac{1}{3}}$$

$$y'' = 2\left(\frac{-2}{3}\right)x^{\frac{-5}{3}} + 4\left(\frac{1}{3}\right)(x)^{\frac{-2}{3}}$$

$$y'' = -\frac{4}{3}x^{\frac{-5}{3}} + \frac{4}{3}x^{\frac{-2}{3}}$$

Finding inflection points other than $(0, 0)$

$$-\frac{4}{3}x^{\frac{-5}{3}}(1-x) = 0$$

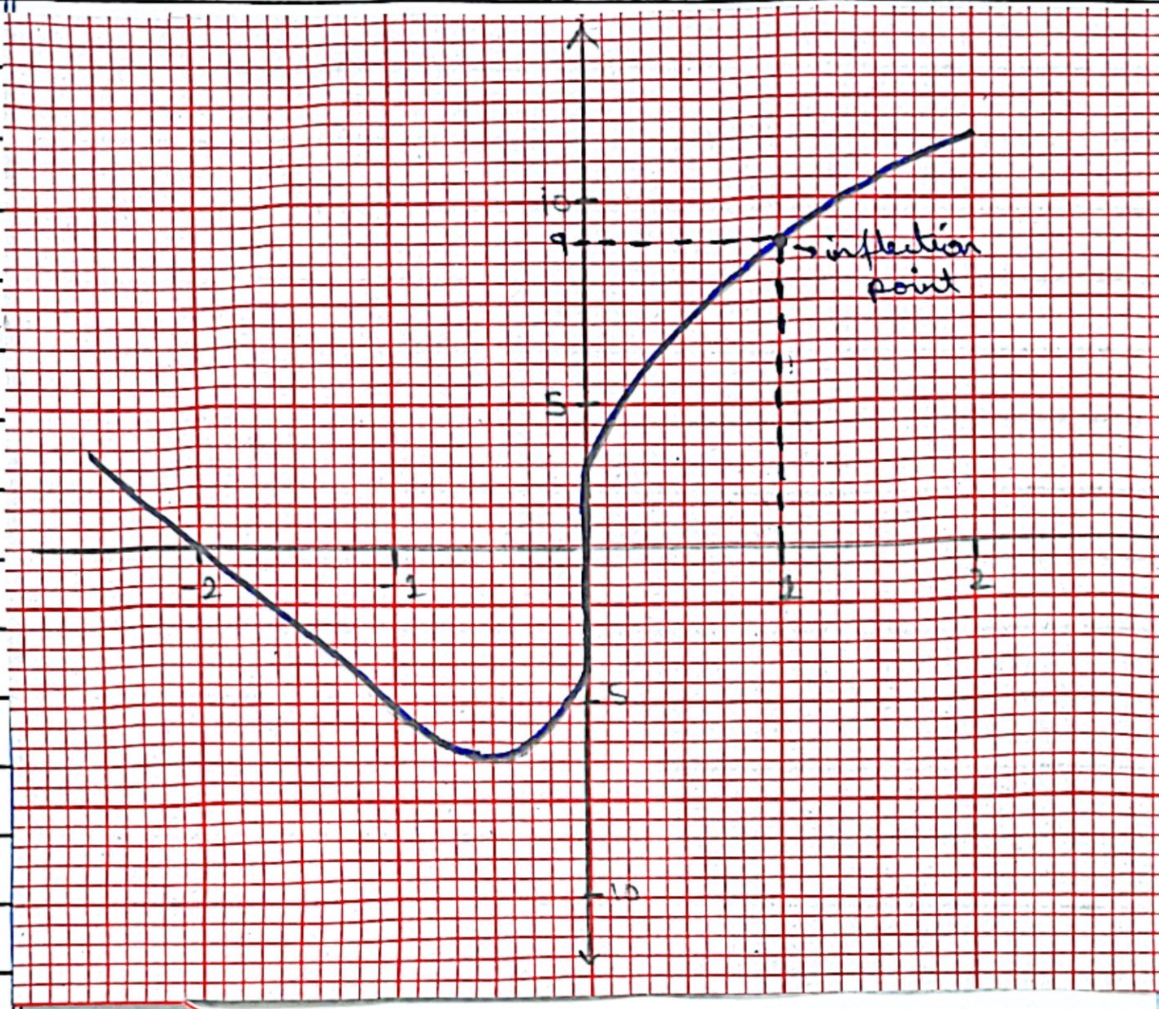
$$1 - x = 0$$
$$x = 1$$

Finding y-value of inflection point :-

$$y = 6x^{\frac{1}{3}} + 3x^{\frac{4}{3}}$$
$$y = 6(1)^{\frac{1}{3}} + 3(1)^{\frac{4}{3}}$$
$$y = 9$$

Inflection point $(1, 9)$

GRAPH AND CONCLUSION



Key Points :-

* No symmetries in the graph
* $(1, 9)$ inflection point where concavity changes from concave up to concave down
* At the graph if $x=0$ is inserted $y=\infty$ hence graph is shown in that way.
* X intercept is also shown on ~~co~~. $(-2,0)$.