INTEGER OCCURRENCES IN RATIONAL LINEAR RECURRENCES

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Abstract. Linear recurrence sequences with integer coefficients, particularly Lucas sequences, have been extensively studied in number theory. In this paper, we investigate the properties of second-order linear recurrences with non-integer rational coefficients. We provide a characterization of when integer terms can appear, proving that under a mild coprimality condition, no two consecutive terms beyond S_2 can both be integers. We establish conditions involving special primes under which isolated integer terms can occur arbitrarily far into the sequence, providing a constructive method using p-adic valuations and results from Ballot. We also demonstrate that any finite number of consecutive integers can appear at the beginning of a suitably chosen sequence. Finally, we prove that for fixed initial conditions, only finitely many integers appear, providing an explicit cutoff using p-adic linear forms in logarithms.

1. Introduction

While one would guess that inputting an integer into the polynomial $x^2 + 2x + 7$ would result in an integer output, it is surprising to learn that inputting an integer into the polynomial $\frac{1}{2}x^2 + \frac{1}{2}x$ would also output an integer. In this paper we study a similar phenomenon relating to Second Order Recurrences. In a similar vain, one would expect that the terms in the sequence $S_0 = 1, S_1 = 7, S_{n+2} = 3S_{n+1} + 2S_n$ would be integers, however in this paper, much like the unexpected integer producing polynomial $\frac{1}{2}x^2 + \frac{1}{2}x$ we investigate integer occurrences in second order recurrences and prove and discover many interesting results.

We investigate second-order linear recurrence sequences where the coefficients are rational numbers, not necessarily integers. We explore the occurrences of integers within these sequences.

Definition 1.1. A sequence $(S_n)_{n\geq 0}$ is generated by a *rational second-order linear recurrence* if, given initial terms $S_0, S_1 \in \mathbb{Z}$, subsequent terms are defined by

$$S_{n+2} = c_1 S_{n+1} + c_0 S_n \quad \text{for } n \geq 0, \tag{1}$$

where the coefficients $c_1 = \frac{c}{d}$ and $c_0 = \frac{a}{b}$ are irreducible fractions with $a, c, b, d \in \mathbb{Z}$. To focus on the non-integer case, we generally assume $b > 1$ or $d > 1$.

Our investigation focuses on understanding when and how integer terms can appear in these sequences. Specifically, we address questions regarding the length of initial runs of integers, the possibility of isolated integers ("integer islands"), restrictions on consecutive integers, and the eventual finiteness of integer terms.

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2. Characterization and Tools

This section introduces the fundamental tools for analyzing integer occurrences, including an auxiliary integer sequence and p-adic criteria.

We study the recurrence $S_{n+2} = c_1 S_{n+1} + c_0 S_n$ with $c_1 = \frac{c}{d}$ and $c_0 = \frac{a}{b}$ irreducible. Let $Q = \operatorname{lcm}(b, d)$. We use $\nu_p(\cdot)$ to denote the p-adic valuation.

2.1. Denominator bounds.

Lemma 2.1. *Let $(S_n)_{n\geq 0}$ satisfy the recurrence (1) with $S_0, S_1 \in \mathbb{Z}$. Let $S_n = \frac{A_n}{B_n}$ be the reduced fraction representation with $B_n > 0$. Then for every $n \geq 0$, B_n divides Q^n.*

Proof. We use strong induction. The base cases $n = 0, 1$ hold as $B_0 = B_1 = 1$, and $1 \mid Q^0, 1 \mid Q^1$. Assume $B_k \mid Q^k$ for all $k \leq n+1$. Let $L_d = Q/d$ and $L_b = Q/b$. These are integers since Q is the LCM. We rewrite the recurrence:

$$S_{n+2} = \frac{c}{d} S_{n+1} + \frac{a}{b} S_n = \frac{c L_d S_{n+1} + a L_b S_n}{Q}.$$

Substituting the fractional forms:

$$S_{n+2} = \frac{c L_d \frac{A_{n+1}}{B_{n+1}} + a L_b \frac{A_n}{B_n}}{Q}.$$

Let $X = \operatorname{lcm}(B_{n+1}, B_n)$. By the inductive hypothesis, $B_{n+1} \mid Q^{n+1}$ and $B_n \mid Q^n$. Therefore, $X \mid Q^{n+1}$. We combine the fractions in the numerator:

$$S_{n+2} = \frac{c L_d A_{n+1}(X/B_{n+1}) + a L_b A_n (X/B_n)}{QX}.$$

The denominator QX divides $Q \cdot Q^{n+1} = Q^{n+2}$. Since S_{n+2} is defined by this fraction, its reduced denominator B_{n+2} must divide Q^{n+2}. □

2.2. The Auxiliary Sequence and p-adic Criteria.

To analyze integrality using divisibility, we introduce an auxiliary sequence of integers.

Definition 2.1. Define the *auxiliary integral sequence* $(T_n)_{n\geq 0}$ by $T_n := Q^n S_n$. Let $L_d = Q/d$ and $L_b = Q/b$.

Proposition 2.2. *The sequence (T_n) consists of integers and satisfies the integer-coefficient linear recurrence*

$$T_{n+2} = P\,T_{n+1} + R\,T_n \quad (n \geq 0), \tag{2}$$

where $P := cL_d$ and $R := aQL_b$. The initial conditions are $T_0 = S_0$ and $T_1 = QS_1$. Furthermore, $S_n \in \mathbb{Z}$ if and only if $Q^n \mid T_n$.

Proof. By definition $T_n = Q^n S_n$. We analyze T_{n+2}:

$$\begin{aligned} T_{n+2} = Q^{n+2} S_{n+2} &= Q^{n+2}\left(\frac{c}{d} S_{n+1} + \frac{a}{b} S_n\right) \\ &= c\frac{Q}{d}(Q^{n+1} S_{n+1}) + a\frac{Q^2}{b}(Q^n S_n) \\ &= cL_d(T_{n+1}) + aQL_b(T_n) = PT_{n+1} + RT_n. \end{aligned}$$

(Note that $Q^2/b = QL_b$). Since $P, R \in \mathbb{Z}$ and $T_0 = S_0, T_1 = QS_1$ are integers, all terms (T_n) are integers. Finally, $S_n = T_n/Q^n \in \mathbb{Z}$ if and only if $Q^n \mid T_n$. □

The *ultrametric inequality* states $\nu_p(x+y) \geq \min(\nu_p(x), \nu_p(y))$, with equality if $\nu_p(x) \neq \nu_p(y)$.

Proposition 2.3. *Fix $n \geq 0$. Then $S_n \in \mathbb{Z}$ if and only if*

$$\nu_p(T_n) \ \geq \ n\,\nu_p(Q) \qquad \textit{for every prime } p \mid Q.$$

Proof. Q^n divides T_n if and only if $p^{n\nu_p(Q)}$ divides T_n for every prime p dividing Q. This is equivalent to $\nu_p(T_n) \geq n\nu_p(Q)$ for all such p. □

2.3. **Connection to Lucas Sequences.** We relate T_n to its associated Lucas sequence.

Definition 2.2. Let (U_n) be the Lucas sequence of the first kind associated with the parameters (P, R):

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+2} = P\,U_{n+1} + R\,U_n.$$

Proposition 2.4 (Companion identity)**.** *Let (T_n) be the auxiliary sequence defined by (2) with $T_0 = S_0$ and $T_1 = QS_1$. Then*

$$T_n \ = \ T_1 U_n \ + R\,T_0\,U_{n-1} \qquad (n \geq 1). \tag{3}$$

Proof. Let $V_n := T_1U_n + RT_0U_{n-1}$. We verify the initial conditions for $n = 1, 2$. $V_1 = T_1U_1 + RT_0U_0 = T_1(1) + 0 = T_1$. $V_2 = T_1U_2 + RT_0U_1 = T_1(P) + RT_0(1) = PT_1 + RT_0$. By the definition of the auxiliary recurrence, this is T_2. Since (V_n) satisfies the same linear recurrence $V_{n+2} = PV_{n+1} + RV_n$ (as it is a linear combination of solutions (U_n) and (U_{n-1})) and matches (T_n) for the first two terms, we have $V_n = T_n$ for all $n \geq 1$. □

3. Constructing Integer Terms

This section explores the construction of sequences exhibiting specific patterns of integers, utilizing the p-adic criteria and the connection to Lucas sequences.

3.1. **Arbitrarily Long Initial Runs.** We provide a constructive proof that arbitrarily long initial runs of integers can be achieved by choosing appropriate (potentially non-coprime) initial conditions.

Theorem 3.1 (Constructive Initial Integrality)**.** *Let $m \geq 1$. For any coefficients $c_1 = \frac{c}{d}, c_0 = \frac{a}{b}$, there exist integers S_0, S_1 such that $S_n \in \mathbb{Z}$ for all $0 \leq n \leq m$.*

Proof. We utilize the auxiliary sequence T_n and the companion identity (Proposition 2.4). Let $X = S_0$ and $Y = QS_1$. We require $\nu_p(T_n) \geq n\nu_p(Q)$ for $0 \leq n \leq m$ and all $p \mid Q$.

Let $\kappa_p = \nu_p(Q)$. We analyze $T_n = YU_n + RXU_{n-1}$ (for $n \geq 1$). By the ultrametric inequality, it is sufficient to ensure both terms satisfy the valuation requirement:

$$\nu_p(YU_n) = \nu_p(Y) + \nu_p(U_n) \geq n\kappa_p,$$
$$\nu_p(RXU_{n-1}) = \nu_p(R) + \nu_p(X) + \nu_p(U_{n-1}) \geq n\kappa_p.$$

We define the necessary thresholds for the valuations of Y and X:

$$A_{1,p}(m) := \max_{1\leq n\leq m} \big(n\kappa_p - \nu_p(U_n)\big),$$
$$A_{0,p}(m) := \max_{1\leq n\leq m} \big(n\kappa_p - \nu_p(R) - \nu_p(U_{n-1})\big).$$

If we choose X, Y such that $\nu_p(X) \geq A_{0,p}(m)$ and $\nu_p(Y) \geq A_{1,p}(m)$ for all $p \mid Q$, the conditions are met.

An explicit choice is:

$$S_0 = X := \prod_{p|Q} p^{A_{0,p}(m)}, \qquad Y := \prod_{p|Q} p^{A_{1,p}(m)}.$$

We must ensure $S_1 = Y/Q$ is an integer. This requires $\nu_p(Y) \geq \nu_p(Q) = \kappa_p$ for all $p \mid Q$. We check the definition of $A_{1,p}(m)$. Since $m \geq 1$, we consider $n = 1$. $U_1 = 1$, so $\nu_p(U_1) = 0$. The threshold requires $A_{1,p}(m) \geq (1\kappa_p - 0) = \kappa_p$. Thus $Q \mid Y$, and S_1 is an integer. This construction guarantees $S_n \in \mathbb{Z}$ for $0 \leq n \leq m$. □

3.2. **Extending Integer Runs (The Gluing Method).** We can iteratively extend the length of an initial run of integers by combining existing sequences, while maintaining control over the coprimality of the initial conditions relative to Q.

Proposition 3.2 (Gluing Method)**.** *Suppose we have two sequences $(S_n^{(0)})$ and $(S_n^{(1)})$ satisfying the recurrence, with corresponding auxiliary sequences $(T_n^{(0)})$ and $(T_n^{(1)})$. Suppose $(S_n^{(i)}) \in \mathbb{Z}$ for $0 \leq n \leq m_i$. Let $m := \max\{m_0, m_1\}$. Assume that for every $p \mid Q$,*

$$\textit{at least one of } S_0^{(0)},\ S_0^{(1)} \textit{ is not divisible by } p. \tag{$*$}$$

Then there exists a new sequence (S_n), formed as a specific integer linear combination of scaled versions of $(S_n^{(0)})$ and $(S_n^{(1)})$, such that $S_n \in \mathbb{Z}$ for $0 \leq n \leq m+1$, and its initial term S_0 satisfies $\gcd(S_0, Q) = 1$.

Proof. We work with the auxiliary sequences $(T_n^{(i)})$. Let $\kappa_p = \nu_p(Q)$. The hypothesis implies $\nu_p(T_n^{(i)}) \geq n\kappa_p$ for $n \leq m_i$.

Step 1: Equalize valuations at $n = m$. For each $p \mid Q$, define scaling exponents:

$$s_0(p) := \max\{0,\ \nu_p(T_m^{(1)}) - \nu_p(T_m^{(0)})\},$$
$$s_1(p) := \max\{0,\ \nu_p(T_m^{(0)}) - \nu_p(T_m^{(1)})\}.$$

Define the scaling factors $K_0 := \prod_{p|Q} p^{s_0(p)}$ and $K_1^{(0)} := \prod_{p|Q} p^{s_1(p)}$. Let $\widehat{T}_n^{(0)} := K_0 T_n^{(0)}$ and $\widehat{T}_n^{(1,0)} := K_1^{(0)} T_n^{(1)}$. By construction, $\nu_p(\widehat{T}_m^{(0)}) = \nu_p(\widehat{T}_m^{(1,0)})$ for every $p \mid Q$. The scaled sequences still satisfy the integrality condition up to index m.

Step 2: Force integrality at $n = m+1$. We seek integers u, v and an integer R' with $\gcd(R', Q) = 1$ such that the sequence $T_n := u\widehat{T}_n^{(0)} + vR'\widehat{T}_n^{(1,0)}$ satisfies $Q^{m+1} \mid T_{m+1}$.

For each $p \mid Q$, we require:

$$u\widehat{T}_{m+1}^{(0)} + vR'\widehat{T}_{m+1}^{(1,0)} \equiv 0 \pmod{p^{(m+1)\kappa_p}}.$$

We can choose u, v via the Chinese Remainder Theorem (CRT) to satisfy these congruences simultaneously.

Step 3: Ensure $\gcd(S_0, Q) = 1$. The new initial term is $S_0 = T_0 = uK_0S_0^{(0)} + vR'K_1^{(0)}S_0^{(1)}$. We analyze S_0 modulo p for $p \mid Q$.

$$S_0 \equiv \underbrace{(vK_1^{(0)}S_0^{(1)})}_{\alpha_p} R' + \underbrace{(uK_0S_0^{(0)})}_{\beta_p} \pmod{p}.$$

By the definition of $K_0, K_1^{(0)}$, p divides at most one of them (since $s_0(p)s_1(p) = 0$). Combined with the hypothesis $(*)$, it is impossible for both α_p and β_p to be 0 (mod p) (assuming u, v are chosen appropriately).

If $\alpha_p \not\equiv 0 \pmod p$, we can choose $R' \pmod p$ such that $\alpha_p R' + \beta_p \not\equiv 0 \pmod p$. If $\alpha_p \equiv 0 \pmod p$, then $\beta_p \not\equiv 0 \pmod p$, and any choice of R' (coprime to p) works. By CRT, we can choose R' such that $S_0 \not\equiv 0 \pmod p$ for all $p \mid Q$. ☐

3.3. **Integer Islands and Special Primes.** We investigate when integers can appear far out in the sequence. This depends on the p-adic properties of the auxiliary parameters P and R.

Definition 3.1. A prime p is called *special* for the auxiliary recurrence (P, R) if $p \mid P$ and $p \mid R$.

We analyze the valuations of P and R.

Lemma 3.3 (Valuation identity). *For every prime p, let $\kappa_p = \nu_p(Q)$. Then*

$$\nu_p(P) = \nu_p(c) + (\kappa_p - \nu_p(d)),$$
$$\nu_p(R) = \nu_p(a) + 2\kappa_p - \nu_p(b).$$

Consequently,

$$2\nu_p(P) - \nu_p(R) = 2\nu_p(c) - \nu_p(a) + \nu_p(b) - 2\nu_p(d).$$

Proof. Since $P = cL_d$ and $L_d = Q/d$, $\nu_p(P) = \nu_p(c) + \nu_p(Q) - \nu_p(d)$. Since $R = aQL_b$ and $L_b = Q/b$, $\nu_p(R) = \nu_p(a) + \nu_p(Q) + \nu_p(Q) - \nu_p(b)$. The identity for the difference follows by direct calculation. ☐

We now adapt results from Ballot [1] concerning the valuation of Lucas sequences for special primes. Ballot uses the subtraction convention $X_{n+2} = P'X_{n+1} - Q'X_n$. Our convention is $U_{n+2} = PU_{n+1} + RU_n$. The results are adapted by setting $P' = P$ and $Q' = -R$.

Lemma 3.4 (Valuations of U_n for special primes, adapted from [1, Thm. 1.2]). *Suppose p is a special prime for (P, R). Let $a = \nu_p(P)$ and $b = \nu_p(R)$. Let $P = p^a P', R = p^b R'$ with $p \nmid P'R'$.*

(1) If $b > 2a$, then $\nu_p(U_n) = (n-1)a$ for all $n \geq 1$.
(2) If $b = 2a$, then $\nu_p(U_n) = (n-1)a + \nu_p(U'_n)$, where $U'_n = U(P', R')$.
(3) If $b < 2a$, then $\nu_p(U_{2n+1}) = bn$ for $n \geq 0$, and

$$\nu_p(U_{2n}) = bn + (a - b) + \nu_p(n) + \lambda_n,$$

where $\lambda_n = 0$ unless $p \in \{2, 3\}$, $2a = b+1$, and $p|n$. In this exceptional case, $\lambda_n = \nu_p((P')^2 + R')$.

Proof. The proofs follow directly from the analysis in [1]. The adaptation from the subtraction convention (P', Q') to the addition convention (P, R) involves replacing Q' with $-R$. This sign change only affects the analysis in the exceptional case for $p \in \{2, 3\}$, where the valuation depends on $\nu_p((P')^2 - Q')$. Replacing Q' with $-R'$ yields the stated result $\nu_p((P')^2 + R')$. ☐

The fastest growth occurs in Case 3 ($b < 2a$). We use this structure to force integrality at specific indices.

Definition 3.2. Let p be a special prime with $b < 2a$. Define the unit parts of the Lucas sequence terms based on Lemma 3.4. Let u_{2m} and u_{2m-1} be the integers coprime to p such that:

$$U_{2m} = p^{\,b(m-1)+(a-b)+\nu_p(m)+\lambda_m} u_{2m},$$
$$U_{2m-1} = p^{\,b(m-1)} u_{2m-1}.$$

Define the ratio $r_p(m) := u_{2m}/(R' u_{2m-1})$. It is known that this ratio exhibits stability modulo powers of p as m varies p-adically (see [1]).

Proposition 3.5. *Fix a special prime $p \mid Q$ with $b < 2a$. Let $S_0 = X = p^{a_0} X_0$ and $QS_1 = Y = p^{a_1} Y_0$, with $p \nmid X_0 Y_0$. The condition $\nu_p(T_{2m}) \geq 2m\nu_p(Q)$ is equivalent to a congruence condition on X_0/Y_0:*

$$\frac{X_0}{Y_0} \equiv -p^{\,E(m)}\, r_p(m) \pmod{p^{\,L_p(m)}}, \tag{4}$$

where $E(m) = a_1 + (a-b) + \nu_p(m) + \lambda_m - a_0$, and $L_p(m)$ is determined by the required valuation minus the valuation already present.

Proof. By the companion identity, $T_{2m} = YU_{2m} + RXU_{2m-1}$. We substitute the factored forms from Definition 3.2:

$$\begin{aligned} T_{2m} &= (p^{a_1}Y_0)(p^{b(m-1)+(a-b)+\nu_p(m)+\lambda_m} u_{2m}) + (p^b R')(p^{a_0}X_0)(p^{b(m-1)} u_{2m-1}) \\ &= p^{b(m-1)} \left(Y_0 p^{a_1+(a-b)+\nu_p(m)+\lambda_m} u_{2m} + X_0 p^{a_0+b} R' u_{2m-1} \right). \end{aligned}$$

We require $\nu_p(T_{2m}) \geq 2m\kappa_p$. This means the bracketed term must be divisible by $p^{2m\kappa_p - b(m-1)}$. The congruence condition (4) arises from requiring the two terms inside the bracket to cancel modulo the required power of p, after dividing by their greatest common power of p. □

Theorem 3.6 (Synthesis of Prescribed Integers)**.** *Assume that for* every *prime $p \mid Q$, p is special for (P, R) and satisfies $2\nu_p(P) > \nu_p(R)$. Let $\{2m_1 < \cdots < 2m_k\}$ be any finite set of even indices. Then there exist coprime integers S_0, S_1 such that*

$$S_{2m_j} \in \mathbb{Z} \quad \textit{for } j = 1, \ldots, k.$$

Proof. For each $p \mid Q$, Proposition 3.5 gives a congruence condition on the ratio of the p-coprime parts of S_0 and QS_1 for each index m_j:

$$\frac{X_0}{Y_0} \equiv -p^{\,E(m_j)}\, r_p(m_j) \pmod{p^{\,L_p(m_j)}}.$$

To satisfy these simultaneously for $j = 1, \ldots, k$, we rely on the stability of the ratios $r_p(m)$. We impose spacing conditions on the indices:

$$m_{j+1} \equiv m_j \pmod{p^{\,L_p(m_j)}} \qquad \text{for each } p \mid Q.$$

A specific choice is $m_{j+1} := m_j + \prod_{q \mid Q} q^{L_q(m_j)}$. Under this spacing, the stability of $r_p(m)$ ensures that the k congruences at prime p are compatible and reduce to the single strongest one (at m_k).

We then combine these single congruences across all $p \mid Q$ using the Chinese Remainder Theorem with modulus $\mathfrak{N} := \prod_{p \mid Q} p^{\,L_p(m_k)}$. This yields a residue class $R_{\mathfrak{N}}$. We choose coprime integers X_0, Y_0 such that $X_0 \equiv R_{\mathfrak{N}} Y_0 \pmod{\mathfrak{N}}$. Choosing S_0, S_1 based on these X_0, Y_0 (and appropriate powers of p for a_0, a_1 to ensure $S_1 \in \mathbb{Z}$) satisfies all congruences, ensuring $S_{2m_j} \in \mathbb{Z}$. □

4. Restrictions on consecutive integers

We now investigate limitations on the occurrence of consecutive integers.

4.1. **Case 1: Coprime denominators.**

When the denominators b and d are coprime, and the initial conditions are coprime, consecutive integers are highly restricted.

Theorem 4.1. *Let the sequence $(S_n)_{n\geq 0}$ be generated by $S_{n+2} = \frac{c}{d}S_{n+1} + \frac{a}{b}S_n$, where the coefficients are irreducible and satisfy:*

(1) $b > 1$ *and* $d > 1$.
(2) $\gcd(b, d) = 1$.

If the initial conditions S_0, S_1 are coprime integers, then S_n and S_{n+1} cannot both be integers for any $n \geq 2$.

Proof. We first establish the implications of two consecutive terms being integers. Let $k \geq 1$. Assume $S_k, S_{k+1} \in \mathbb{Z}$. Since $\gcd(b, d) = 1$, $Q = bd$. We rewrite the recurrence:

$$bdS_{k+1} = cbS_k + adS_{k-1}.$$

Analyze divisibility by b. We have $b \mid bdS_{k+1}$ and $b \mid cbS_k$. Thus $b \mid adS_{k-1}$. Since $\gcd(a, b) = 1$ (as a/b is irreducible) and $\gcd(d, b) = 1$, we have $\gcd(b, ad) = 1$. Thus $b \mid S_{k-1}$. This also implies S_{k-1} must be an integer (as adS_{k-1} is the difference of integers, and b divides S_{k-1}).

Analyze divisibility by d. We have $d \mid bdS_{k+1}$ and $d \mid adS_{k-1}$ (since $S_{k-1} \in \mathbb{Z}$). Thus $d \mid cbS_k$. Since $\gcd(c, d) = 1$ (as c/d is irreducible) and $\gcd(b, d) = 1$, we have $\gcd(d, cb) = 1$. By Euclid's Lemma, $d \mid S_k$.

Summary: If $S_k, S_{k+1} \in \mathbb{Z}$ $(k \geq 1)$, then $S_{k-1} \in \mathbb{Z}$, $d \mid S_k$, and $b \mid S_{k-1}$.

Now we prove the main statement by contradiction. Assume $S_n, S_{n+1} \in \mathbb{Z}$ for some $n \geq 2$. We apply the implications iteratively downward:

- $(k = n)$: $S_n, S_{n+1} \in \mathbb{Z} \implies S_{n-1} \in \mathbb{Z}$.
- ...
- $(k = 2)$: $S_2, S_3 \in \mathbb{Z} \implies S_1 \in \mathbb{Z}, d|S_2, b|S_1$.
- $(k = 1)$: $S_1, S_2 \in \mathbb{Z} \implies S_0 \in \mathbb{Z}, d|S_1, b|S_0$.

Since $n \geq 2$, the implications for $k = 1$ and $k = 2$ must hold. From $k = 1$, we have $b|S_0$. From $k = 2$, we have $b|S_1$.

Since $b > 1$, b is a common divisor of S_0 and S_1. This contradicts the assumption $\gcd(S_0, S_1) = 1$. (The edge cases $S_0 = 0$ or $S_1 = 0$ also lead to contradictions as they imply the other term is ± 1, but they must be divisible by $b > 1$). $\square$

Remark 4.2. The restriction starts at $n = 2$. It is possible for (S_0, S_1, S_2) to be integers with $\gcd(S_0, S_1) = 1$. E.g., $S_{n+2} = \frac{1}{3}S_{n+1} + \frac{1}{2}S_n$ with $S_0 = 2, S_1 = 3$. $S_2 = 2$. $S_3 = 13/6$.

4.2. **Case 2: Relaxed Conditions.**

If $\gcd(b, d) > 1$, or if one of the coefficients is an integer (e.g., $d = 1$), the restriction does not hold, as shown by examples such as $S_{n+2} = \frac{9}{2}S_{n+1} + 4S_n$ with $S_0 = 7, S_1 = 8$, which starts with ten consecutive integers despite coprime initial conditions.

5. Eventual absence of integer terms

For a fixed sequence, under the assumption of non-degeneracy, there are only finitely many integer terms. We establish an explicit bound beyond which no integer terms can appear, using results on p-adic linear forms in logarithms.

Theorem 5.1 (Eventual nonintegrality with an explicit cutoff). *Consider the recurrence $S_{n+2} = c_1 S_{n+1} + c_0 S_n$ with fixed initial conditions S_0, S_1. Assume the characteristic equation $X^2 - c_1 X - c_0 = 0$ has roots r_1, r_2 such that r_1/r_2 is not a root of unity (non-degenerate).*

Then there exists an explicitly computable constant N, depending on the coefficients and initial conditions, such that for all $n > N$, $S_n \notin \mathbb{Z}$.

Proof. We use the Binet formula. Since the sequence is non-degenerate, $S_n = \alpha r_1^n + \beta r_2^n$. We assume $\beta \neq 0$. Let $\Gamma := \alpha/\beta$ and $\Lambda := r_1/r_2$.

$$S_n = \beta r_2^n (\Gamma \Lambda^n + 1).$$

Step 1: Establishing a linear lower bound. Fix a prime $p \mid Q$. We analyze the valuations in the number field $K = \mathbb{Q}(r_1, r_2)$. Since the coefficients c_0, c_1 are not both integers, there must be a prime valuation ν_p such that $\nu_p(c_1) < 0$ or $\nu_p(c_0) < 0$.

Consider the product of the roots $r_1 r_2 = -c_0 = -a/b$. If $p \mid b$, then $\nu_p(r_1) + \nu_p(r_2) = \nu_p(-a/b) \leq -\nu_p(b) \leq -1$ (assuming $\gcd(a, b) = 1$). A similar argument applies if $p \mid d$. This implies at least one root must have a negative valuation. Let us label the roots such that $\nu_p(r_2) < 0$. Define $\rho := -\nu_p(r_2) > 0$.

For $S_n \in \mathbb{Z}$, we must have $\nu_p(S_n) \geq 0$.

$$\nu_p(\beta) + n\nu_p(r_2) + \nu_p(\Gamma\Lambda^n + 1) \geq 0.$$

This leads to the necessary condition:

$$\nu_p(\Gamma\Lambda^n + 1) \ \geq \ n\rho - \nu_p(\beta). \tag{5}$$

The right-hand side grows linearly with n.

Step 2: Establishing a logarithmic upper bound. We seek an upper bound on $\nu_p(\Gamma\Lambda^n + 1)$.

Case 2a: $\nu_p(\Lambda) \neq 0$. By the ultrametric inequality, $\nu_p(\Gamma\Lambda^n + 1) = \min(\nu_p(\Gamma\Lambda^n), 0)$. As $n \to \infty$, this value is bounded (it is eventually 0 or $\nu_p(\Gamma\Lambda^n)$ depending on the signs, but the linear growth required by (5) will eventually exceed it).

Case 2b: $\nu_p(\Lambda) = 0$. If $\nu_p(\Gamma) \neq 0$, then $\nu_p(\Gamma\Lambda^n) = \nu_p(\Gamma) \neq 0$. By the ultrametric inequality, $\nu_p(\Gamma\Lambda^n + 1) = \min(\nu_p(\Gamma), 0)$. This is constant, again contradicting (5) for large n.

Case 2c: $\nu_p(\Lambda) = 0$ and $\nu_p(\Gamma) = 0$. Here Λ and $-\Gamma^{-1}$ are p-adic units. We are analyzing $\nu_p(\Lambda^n - (-\Gamma^{-1}))$. Since Λ is not a root of unity, these elements are multiplicatively independent. We apply explicit bounds for p-adic linear forms in two logarithms, such as those provided by Chim [2] (Theorem 2.1).

These results provide an upper bound of the form:

$$\nu_p(\Gamma\Lambda^n + 1) \ \leq \ \mathcal{K}_p\big(\log n + C_H\big), \tag{6}$$

where $\mathcal{K}_p$ and C_H are explicitly computable constants depending on p, the algebraic properties (heights) of Γ and Λ, and the degree of the number field K.

Step 3: Determining the cutoff. Comparing the linear lower bound and the logarithmic upper bound:

$$n\rho - \nu_p(\beta) \leq \mathcal{K}_p\big(\log n + C_H\big).$$

Define the function $F_p(n) = n\rho - \mathcal{K}_p \log n - (\nu_p(\beta) + \mathcal{K}_p C_H)$. Since $\rho > 0$, the linear term $n\rho$ dominates the logarithmic term $\mathcal{K}_p \log n$ for large n. Therefore, $F_p(n) \to \infty$ as $n \to \infty$.

There exists a computable cutoff N_p such that $F_p(n) > 0$ for all $n > N_p$. For $n > N_p$, the integrality condition (5) cannot be satisfied.

Let $N = \max_{p|Q} N_p$. Then for all $n > N$, $S_n \notin \mathbb{Z}$. □

References

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