# ON THE PROPERTIES OF SECOND ORDER LINEAR RECURRENCES WITH RATIONAL COEFFICIENTS

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Abstract. Linear recurrence sequences with integer parameters, particularly Lucas sequences, have been extensively studied in number theory. In this paper, we investigate the properties of second-order linear recurrences with non-integer rational parameters. We provide a characterization of when integer terms can appear, creating an algorithm that enables an arbitrary number of integers to appear in a sequence with given rational coefficients by altering the values of the first and second terms in the sequence. We establish conditions for particular sequences with predetermined parameters and values of the first and second terms, which outline a point for which integers fail to appear. Our results reveal intriguing divisibility properties of these sequences. Furthermore, we discuss how these number theoretic results can be applied to many fields, from pseudorandom number generation to data science.

## 1. Introduction

Second order linear recurrences are sequences of numbers defined by two things. First, by their two initial conditions; these initial conditions act as the first two terms of the sequence. For example, the most famous second order recurrence (the Fibonacci sequence) is defined by the starting conditions 0 and 1. Secondly, for each second order linear recurrence, there is a recurrence relation. This recurrence relation enables one to generate all of the rest of the terms in the recurrence. These recurrences are second order, meaning that they rely on two previous terms, and linear, meaning that they use constant multiples to generate the following term. Writing the $n$th term of the recurrence $S_n$, each recurrence can be written in the form:

$$S_0 = x, \quad S_1 = y, \quad S_{n+2} = aS_{n+1} + bS_n.$$

For almost a thousand years, second order linear recurrences have been studied in depth by mathematicians and researchers [5, 7]. The properties of these sequences enable them to be a very powerful tool for scientists. However, because they are so powerful, there is a large amount of pre-existing research on them, and their mathematical properties are very well known. Most of this research, though, is done when the constant multiples (values of $a$ and $b$) used in the second order recurrence are integers. We tried to figure out what would happen if these constant multiples were fractions. Thus, the purpose of this paper is to investigate the properties of terms in these new second order recurrences with fractional constant multiples.

*Date*: October 13, 2025.

## 2. Outline and Novelty

The sequences we mostly study in this paper are second order linear recurrences with rational (non-integer) coefficients:

$$S_{n+2} \;=\; \frac{a}{b}S_{n+1} + \frac{c}{d}S_n, \qquad b, d > 1.$$

**Example 2.1.** To lead into some of the main points of this paper, consider the following recurrence:

$$S_{n+2} \;=\; \frac{1}{2}S_{n+1} \;+\; \frac{1}{3}S_n, \qquad S_0 = 2, \quad S_1 = 7.$$

The first several terms, written in lowest terms as rational numbers, are:

$$\begin{aligned}
S_0 &= 2 = \frac{2}{1},\\
S_1 &= 7 = \frac{7}{1},\\
S_2 &= \frac{1}{2}\cdot 7 \;+\; \frac{1}{3}\cdot 2 \;=\; \frac{25}{6},\\
S_3 &= \frac{1}{2}\cdot\frac{25}{6} \;+\; \frac{1}{3}\cdot 7 \;=\; \frac{53}{12},\\
S_4 &= \frac{259}{72},\\
S_5 &= \frac{157}{48},\\
S_6 &= \frac{2449}{864},\\
S_7 &= \frac{4333}{1728},\\
S_8 &= \frac{22795}{10368},\\
S_9 &= \frac{40127}{20736},\\
S_{10} &= \frac{211{,}561}{124{,}416}.
\end{aligned}$$

After looking at this example, one notices:

- There are no integer terms past the first two initial conditions for $S_2$ through $S_{10}$.
- The denominators in each term grow explosively; thus enormous cancellation is necessary for integrality.

The reason the denominators grow so explosively is because one can write the recurrence in the equivalent form:

$$S_{n+2} \;=\; \frac{3S_{n+1} + 2S_n}{6}.$$

This shows that each sequential term continues to divide a factor of six out of some linear combination of the previous two terms. Thus, said linear combination must be very divisible by six to result in an integer term for large $n$.

After looking at many example sequences showing similar scarcity of integers, we developed two questions regarding these particular recurrences:

(1) **Question 1.** Can we choose initial values so that $S_n \in \mathbb{Z}$ at *specific indices we want*?
(2) **Question 2.** For a given recurrence with values for $S_0$ and $S_1$, does there exist an index $N$ after which $S_n \notin \mathbb{Z}$ *forever*?

In this paper, we attempt to answer these questions in Theorem 3.1 and Theorem 4.1.

**Remark 2.2.** In trying to answer these questions, we realize that some important conditions must be met for the integrality question to remain non-trivial.

$$\gcd(a, c) = \gcd(b, d) = 1 \quad \text{and} \quad \gcd(S_0, S_1) = 1.$$

Any recurrence of the form

$$S_{n+2} \;=\; \frac{a}{b} S_{n+1} \;+\; \frac{c}{d} S_n$$

can be rewritten as

$$S_{n+2} \;=\; \frac{ad\, S_{n+1} + bc\, S_n}{bd}.$$

If $\gcd(a, c) > 1$ or $\gcd(b, d) > 1$, then these common factors can be canceled (by dividing the numerator and denominator accordingly), giving an equivalent recurrence with a factor out front. Studying the reduced form with $\gcd(a, c) = \gcd(b, d) = 1$ just means we are working in lowest terms.

Similarly, if $\gcd(S_0, S_1) = g > 1$, write

$$S_n = g\, \widetilde{S}_n,$$

and define $\widetilde{S}_0 = S_0/g$, $\widetilde{S}_1 = S_1/g$. Then the same recurrence relation holds for $\widetilde{S}_n$:

$$\widetilde{S}_{n+2} \;=\; \frac{a}{b}\, \widetilde{S}_{n+1} \;+\; \frac{c}{d}\, \widetilde{S}_n,$$

with $\gcd(\widetilde{S}_0, \widetilde{S}_1) = 1$. This demonstrates a way to generate integers trivially. By making the gcd of the first two terms highly divisible by $bd$, this gcd enables guaranteed cancellation of the $bd$'s in the denominator of terms in the sequence and can generate many integers to start a sequence. However, this is both trivial and anticlimactic; it is a more interesting question to ask what happens when the first two terms are relatively prime. Thus, for the remainder of the paper, we assume that $\gcd(a, c) = \gcd(b, d) = 1$.

## 3. Integers at desired indices

It is known [3] that there is no recurrence

$$S_{n+2} = \frac{a}{b} S_{n+1} + \frac{c}{d} S_n, \qquad S_0 = x,\ S_1 = y,\ \gcd(x, y) = 1,\ \gcd(a, c) = \gcd(b, d) = 1,$$

that has the property that $S_n \in \mathbb{Z}$ holds at infinitely many indices $n$. This raises a question: for the same class of recurrences, can one choose coprime initial conditions $(x, y)$ so that $S_n$ is integral at an *arbitrarily long but still finite* list of indices? We give an answer in Theorem 3.1 below.

Throughout the section we assume $a, b, c, d \in \mathbb{Z}$ with $b, d \geq 2$, $\gcd(a, c) = \gcd(b, d) = \gcd(a, d) = 1$. Consider the recurrence:

$$S_{n+2} = \frac{a}{b} S_{n+1} + \frac{c}{d} S_n, \qquad S_0 = x,\ S_1 = y,\ \gcd(x, y) = 1. \tag{1}$$

Define $K := bd$ and $T_n := K^n S_n$, also let the $p$-adic valuation of a number $n$ be the exponent value in the prime factorization of $n$ for a prime $p$. We denote it $\nu_p(n)$ for the remainder of this paper.

**Theorem 3.1.** *Let $S_n$ be the recurrence described in* (1) *and let* $\mathrm{Set} = (1, 2, 3, \dots, n-1, n)$ *be a finite set of consecutive whole numbers starting at* 1*. Then there exist coprime integers $(x, y)$ and indices $m_1, m_2, m_3, \dots, m_n$ such that*

$$S_{m_j} \in \mathbb{Z} \quad \textit{for all } j = 1, \dots, k.$$

**Example 3.2.** Here, we see an example of what happens when a suitable pair of initial conditions are plugged into the recurrence:

$$S_{n+2} \;=\; \frac{1}{3} S_{n+1} \;+\; \frac{1}{2} S_n, \qquad S_0 = 1, \quad S_1 = 4683.$$

Using the recurrence to generate terms, we obtain:

$$\begin{aligned}
S_0 &= 1,\\
S_1 &= 4683,\\
S_2 &= \tfrac{1}{3}S_1 + \tfrac{1}{2}S_0 = \tfrac{4683}{3} + \tfrac{1}{2} = \frac{3123}{2},\\
S_3 &= \tfrac{1}{3}S_2 + \tfrac{1}{2}S_1 = \tfrac{3123}{6} + \tfrac{4683}{2} = 2862,\\
S_4 &= \tfrac{1}{3}S_3 + \tfrac{1}{2}S_2 = \tfrac{2862}{3} + \tfrac{3123}{4} = \frac{6939}{4},\\
S_5 &= \tfrac{1}{3}S_4 + \tfrac{1}{2}S_3 = \tfrac{6939}{12} + \tfrac{2862}{2} = \frac{8037}{4},\\
S_6 &= \tfrac{1}{3}S_5 + \tfrac{1}{2}S_4 = \tfrac{8037}{12} + \tfrac{6939}{8} = \frac{12297}{8},\\
S_7 &= \tfrac{1}{3}S_6 + \tfrac{1}{2}S_5 = \tfrac{12297}{24} + \tfrac{8037}{8} = 1517,\\
S_8 &= \tfrac{1}{3}S_7 + \tfrac{1}{2}S_6 = \tfrac{1517}{3} + \tfrac{12297}{16} = \frac{61163}{48},\\
S_9 &= \tfrac{1}{3}S_8 + \tfrac{1}{2}S_7 = \tfrac{61163}{144} + \tfrac{1517}{2} = \frac{170387}{144},\\
S_{10} &= \tfrac{1}{3}S_9 + \tfrac{1}{2}S_8 = \tfrac{170387}{432} + \tfrac{61163}{96} = \frac{891241}{864}.
\end{aligned}$$

In this example, the coprime initials $(S_0, S_1) = (1, 4683)$ produce

$$S_3, S_7 \in \mathbb{Z},$$

while all other displayed terms are non-integral.

To prove this theorem we took advantage of the following tools:
**(1) Scaling.** Starting from

$$S_{n+2} \;=\; \frac{a}{b} S_{n+1} + \frac{c}{d} S_n,$$

we used the scaled sequence $T_n := K^n S_n$, which (by a bit of algebraic manipulation) turns the $S_n \in \mathbb{Z}$ condition into a simple $p$-adic divisibility condition:

$$S_n \in \mathbb{Z} \quad \Longleftrightarrow \quad \nu_p(T_n) \ \geq \ n\,\nu_p(K) \text{ for every prime } p \mid K.$$

After this manipulation, the question of integrality turns into one of $p$-adic valuation.
**(2) Companion identity.** We then used a well known identity to express $T_n$ as a sum of terms in a Lucas sequence. Namely:

$$T_n \ = \ bdy\,U_n \ - \ b^2cd\,x\,U_{n-1},$$

where $Y = Ky$, $(U_n)$ is the Lucas sequence:

$$U_n = adU_{n-1} + b^2cdU_{n-2}.$$

This allowed us to turn the $p$-adic question about $T_n$ into one about the $p$-adic valuation of sums of Lucas Sequences.
**(3) Special vs Non-Special Primes.** Since the p-adic valuations of terms in Lucas Sequences differ when $p$ divides either $P$ and $Q$(called special primes) as opposed to dividing just one or none(called non-special primes). For the primes needing to divide terms in $T_n$ for $S_n$ to be integral, I broke up the prime factors of these terms, using a paper written by Ballot [1]. I was then able to outline conditions for $v_p(S_n)$ to be sufficiently large for each p dividing $bd$. **(4) Ballot use.** Because $T_n$ is a fixed linear combination of $U_n$ and $U_{n-1}$, we used the Chinese Remainder Theorem and combined it with the $p$-adic bounds for Lucas sequences given in [1] to find a method for producing an arbitrarily large set of indices $S = (m_1, m_2, \ldots, m_j)$ such that for all $m_i \in S$, $S_{m_i} \in \mathbb{Z}$.

**Lemma 3.3.** *Setting $K = bd$ and $T_n := K^n S_n$, the recurrence $(T_n)$ satisfies*

$$T_0 = x, \qquad T_1 = Ky, \qquad T_{n+2} = ad\,T_{n+1} + b^2cd\,T_n \quad (n \geq 0).$$

*Furthermore, $S_n \in \mathbb{Z}$ iff $K^n \mid T_n$.*

*Proof.* By definition $T_n = K^n S_n$. Thus

$$T_{n+2} = K^{n+2}S_{n+2} = \frac{a}{b}K^{n+2}S_{n+1} + \frac{c}{d}K^{n+2}S_n = a\frac{K}{b}(K^{n+1}S_{n+1}) + c\frac{K^2}{d}(K^nS_n).$$

Because $K/b = d$ and $K^2/d = b^2d$, we get

$$T_{n+2} = ad\,T_{n+1} + b^2cd\,T_n.$$

Also, $T_0 = K^0x = x$ and $T_1 = Ky$. Finally, $S_n = T_n/K^n \in \mathbb{Z}$ iff $K^n \mid T_n$. □

Convert the recurrence for $T_n$ into a simplified form by defining $P := ad, Q := -b^2cd$, so that

$$T_{n+2} = P\,T_{n+1} - Q\,T_n, \qquad T_0 = x, \quad T_1 = Ky. \tag{2}$$

Let $U_n$ be the Lucas sequence of the first kind for $(P, Q)$:

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+2} = P\,U_{n+1} - Q\,U_n. \tag{3}$$

**Lemma 3.4.** *$T_n$ satisfies the identity:*

$$T_n \ = \ Ky\,U_n \ - \ Q\,x\,U_{n-1}. \tag{4}$$

*Proof.* See, e.g., [7]. □

**Definition 3.5.** For a prime $p$, $\nu_p(\cdot)$ denotes the $p$-adic valuation. For any integers $X, Y$,

$$\nu_p(X+Y) \geq \min\{\nu_p(X), \nu_p(Y)\},$$

The two quantities are equal when $\nu_p(X) \neq \nu_p(Y)$. When the valuations are equal, we have to use the $\geq$ sign.

**Lemma 3.6.** *For $n \geq 0$, $S_n \in \mathbb{Z}$ if and only if*

$$\nu_p(T_n) \ \geq \ n\,\nu_p(K) \qquad \textit{for every prime } p \mid K.$$

*Proof.* Since $K = \prod_{p \mid K} p^{\nu_p(K)}$, we must have that $K^n$ divides $T_n$ iff $p^{n\nu_p(K)} \mid T_n$ for each $p \mid K$. This statement implies that $\nu_p(T_n)$ must be greater than or equal to $n\nu_p(K)$ for all such $p$. □

### 3.1. **Ensuring divisibility for special and non-special primes.**

**Definition 3.7.** A prime $p$ is called *special* for $(P, Q)$ if $p \mid P$ and $p \mid Q$. A prime $p$ is called *non-special* for $(P, Q)$ if $p$ divides either $P$ or $Q$ but not both.

**Remark 3.8. Primes dividing $d$.** These primes divide both $P = ad$ and $Q = -cb^2d$; clearly each prime dividing $d$ is special. Using [1], we find a method to generate coprime $x, y$ such that, for any number of indices coprime to $K$, for all $p \mid d$, $\nu_p(T_n) \geq n\nu_p(K)$.

**Primes dividing $b$.** These primes divide $Q = -cb^2d$ through the $b^2$ term. They do not divide $P = ad$ since $\gcd(b, d) = 1$, thus, they are non special. For those primes we find a method to generate coprime $x, y$ such that, for any desired set of indices, for all $p \mid b$, $\nu_p(T_n) \geq n\nu_p(K)$. This is a *stronger* condition than the "for any number of indices" method for special primes.

**Final method.** These two methods enable us to prove the theorem: we use the first method to generate coprime $(x_0, y_0)$ where for all $p \mid d$, $\nu_p(T_n) \geq n\nu_p(K)$ for any desired finite number of indices $n$. We call this set of indices $S = (2m_1, 2m_2, \ldots, 2m_j)$. Then we use the stronger condition for non-special primes and generate coprime $(x_1, y_1)$ such that, for all $p \mid b$, $\nu_p(T_n) \geq n\nu_p(K)$ for each index in $S$. Finally, we use the Chinese Remainder Theorem to find a combination of $(x_0, y_0)$ and $(x_1, y_1)$ that enables $\nu_p(T_n) \geq n\nu_p(K)$ for all $p \mid K$ at each index $n \in S$.

We record the $p$-adic size of the coefficients $P, Q$ at a special prime, which is exactly the hypothesis needed to invoke Ballot's bounds on $\nu_p(U_n)$.

**Lemma 3.9.** *Let $p$ be a special prime with $p \mid d$. If $a$ is defined to be the $p$-adic valuation of $P$, and $b$ is defined to be the $p$-adic valuation of $Q$, then $a = b$ and $2a > b$.*

*Proof.* Since $\gcd(b, d) = \gcd(c, d) = \gcd(a, d) = \gcd(a, c) = \gcd(a, b) = 1$

$$\nu_p(ad) = \nu_p(a) + \nu_p(d) = \nu_p(d) = \nu_p(c) + \nu_p(d) = \nu_p(-b^2) + \nu_p(c) + \nu_p(d) = \nu_p(-b^2cd).$$

Thus, $a = b$ and $2a > b$ follows. □

**Remark 3.10.** This allows us to use [1, Thm. 2.4] in our proofs.

3.1.1. *Non-special primes.*

**Proposition 3.11.** *Taking $S_n$ to be the second order recurrence defined above, for any finite increasing set of positive integers $S = \{n_1 < n_2 < \cdots < n_k\}$, there exists coprime integers $x, y$ with $T_0 = x$ and $T_1 = Ky$ such that for every prime $p \mid b$ and every $n \in S$,*

$$\nu_p\big(T_n\big) \ \geq \ n\,\nu_p(b),$$

*where $T_n := K^n S_n = (Ky)\,U_n - Q\,x\,U_{n-1}$. Equivalently, for all $n \in S$*

$$ybd\,U_n \ + \ cb^2d\,x\,U_{n-1} \ \equiv \ 0 \pmod{b^{\,n}}. \tag{5}$$

*Simplifying further:*

$$y\,U_n \ + \ cb\,x\,U_{n-1} \ \equiv \ 0 \pmod{b^{\,n-1}}. \tag{6}$$

*Proof.* First remember that for any prime $p \mid b$ we have $p \nmid P = ad$ (since $p \nmid a, d$) and $p \mid Q$. Using a bit of modular arithmetic and reducing the Lucas sequence in (6) mod $p$ gives $U_n \equiv P\,U_{n-1} \pmod p$, so for all values of $n \geq 1$: $U_n \equiv P^{n-1} \not\equiv 0 \pmod p$. Thus

$$\gcd(U_n, b) = 1 \quad \text{for all } n \geq 1, \tag{7}$$

so $U_n$ is always relatively prime to $b^t$. Now we construct $(x, y)$ satisfying (6) for all $n \in S$ by induction.

Set $x := 1$, now, find a $y_1$ solving (6) for $n = n_1$. Thanks to (7) the coefficient of $y$ has an inverse mod $b^{\,n_1-1}$, so we can find an initial value for $y_1$:

$$y_1 U_{n_1} + cbu_{n_1-1} \equiv 0 \pmod{b^{\,n_1-1}} \to y_1 \equiv -\,cb\,U_{n_1-1}\,U_{n_1}^{-1} \pmod{b^{\,n_1-1}}.$$

Suppose inductively that we have $y_i$ such that (6) holds for every $n \in \{n_1, \ldots, n_i\}$. We now force (6) to be solved at $n_{i+1}$. Define

$$y_{i+1} \ := \ y_i \ + \ t \cdot b^{\,n_i-1}, \qquad t \in \mathbb{Z}$$

with $t$ still yet to be determined. For any $n \leq n_i$ the left-hand side of (6) increases by $\ t\,b^{\,n_i-1}\,U_n$, which is divisible by $b^{\,n-1}$ because of $n \leq n_i$ and (7). So, the new sequence still satisfies (6) for $n \in \{n_1, \ldots, n_i\}$.

At $n = n_{i+1}$ we must solve

$$(y_i + tb^{\,n_i-1})U_{n_{i+1}} + cb\,x\,U_{n_{i+1}-1} \ \equiv \ 0 \pmod{b^{\,n_{i+1}-1}}.$$

We divide by $b^{\,n_i-1}$ and rearrange to get the following linear congruence for $t$ modulo $b^{n_{i+1}-n_i}$:

$$t \cdot \big(U_{n_{i+1}}\big) \ \equiv \ -\,\frac{y_i U_{n_{i+1}} + cb\,x\,U_{n_{i+1}-1}}{b^{\,n_i-1}} \pmod{b^{\,n_{i+1}-n_i}}.$$

By (7), $U_{n_{i+1}}$ is relatively prime to $b$, so this congruence is solvable for $t$. So, $y_{i+1} = y_i + tb^{n_i-1}$ creates a sequence such that (6) holds for all $n \in \{n_1, \ldots, n_i, n_{i+1}\}$, with $x = 1$. Finally, $\gcd(x, y_k) = \gcd(1, y_k) = 1$, so the initials are coprime. Since (6) is equivalent to $\nu_p(T_n) \geq n\,\nu_p(b)$ for each $p \mid b$, the proposition follows. $\square$

3.1.2. *Special primes: p-adic growth and linearization.* Here, we take care of ensuring terms of $T_n$ are divisible by sufficiently large powers of $p$. Write

$$P = p^a P', \qquad Q = p^b Q', \qquad p \nmid P'Q'.$$

By Lemma 3.9 we have that $2a > b$, so we can use the $p$-adic formulas for $U_n$ from [1, Thm. 2.4].

**Lemma 3.12.** *Let $a, b, c, d \in \mathbb{Z}$, $P, Q$ and $U_n$ all be as defined in* (1), (2) *and* (3) *respectively. Let $p$ be a special prime of $T_n$. Write $P = p^a P'$, $Q = p^b Q'$ with $p \nmid P'Q'$. Fix $m \geq 1$ with* $\gcd(m, d) = 1$, *so $p \nmid m$ for every special $p \mid d$. Then there exist integers $u_{2m}, u_{2m+1}$ with $p \nmid u_{2m}u_{2m+1}$ such that*

$$U_{2m} = p^{bm} u_{2m}, \qquad U_{2m+1} = p^{bm} u_{2m+1}.$$

*Define*

$$r_p(m) \equiv \frac{Q'bc}{P'}\, m \;+\; p^{2a-b} C_{p,t}(m) \pmod{p^t}. \tag{8}$$

*Furthermore, for all integers $h$,*

$$\nu_p\big(r_p(m+h) - r_p(m)\big) \;\geq\; \nu_p(h). \tag{9}$$

*Proof.* Since $\gcd(b, d) = \gcd(c, d) = \gcd(a, d) = 1$, we have $a = b > 0$, thus $2a > b$. Applying [1] to $U_{2m}$and $U_{2m+1}$ with $P = p^a P'$, $Q = p^b Q'$, $a = b$ and $\gcd(d, m) = 1$ returns

$$\nu_p(U_{2m}) = bm, \qquad \nu_p(U_{2m+1}) = bm.$$

So, one can factor

$$U_{2m} = p^{bm} u_{2m}, \qquad U_{2m+1} = p^{bm} u_{2m+1},$$

with $p \nmid u_{2m}u_{2m+1}$.

Now use the binomial expansion for $U_n$ [7]:

$$U_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-Q)^k \binom{n-k-1}{k} P^{n-1-2k}.$$

For $n = 2m$ and $p \nmid m$, the $k = m - 1$ term equals

$$(-Q)^{m-1} \binom{m}{m-1} P = p^{bm+(a-b)} (-Q')^{m-1} (P'm),$$

and every other term has at least one extra factor of $p$ (specifically $p^{2a-b}$ since $2a > b$). So, the explicit formula for $u_{2m}$ is:

$$u_{2m} = (-Q')^{m-1} \Big(P'm + p^{2a-b} A(m)\Big)$$

for some polynomial $A(m) \in \mathbb{Z}[m]$. Using the same trick for the $k = m$ term on $n = 2m + 1$ gets

$$u_{2m+1} = (-Q')^m \Big(1 + p^{2a-b} B(m)\Big)$$

for some polynomial $B(m) \in \mathbb{Z}[m]$. Thus, using the definition in (8) and canceling:

$$r_p(m) = -\frac{u_{2m+1}}{bcu_{2m}} = \frac{-Q'(1 + p^{2a-b} A(m))}{bc(P'm + p^{2a-b} B(m))} = -\frac{Q'}{bcP'}\, m^{-1} + p^{2a-b} C(m),$$

with $C(m) \in \mathbb{Z}_p[m]$, satisfying (**??**). If one reduces this mod $p^t$ one gets the desired congruence. To bound the $p$-adic difference of $r_p(m+h)$ and $r_p(m)$:

$$r_p(m+h) - r_p(m) = -\frac{Q'}{bcP'}\big(\,(m+h)^{-1} - \,(m)^{-1}\big) \;+\; p^{2a-b}\big(C(m+h) - C(m)\big).$$

Since $C$ has integer coefficients, $C(m+h) - C(m)$ is divisible by $h$ [6]. Taking the difference of the $m$ terms gets

$$-\frac{Q'}{bcP'}\big(\,(m+h)^{-1} - \,(m)^{-1}\big) = -\frac{Q'}{bcP'}\left(\frac{h}{m(m+h)}\right).$$

Since $\nu_p(Q') = \nu_p(bcP') = 0$ and $\gcd(m,d) = \gcd(m+h,d) = 1$ the ultrametric inequality then gives

$$\nu_p\big(r_p(m+h) - r_p(m)\big) \;\geq\; \min\big\{\nu_p(h),\; (2a-b) + \nu_p(h)\big\} = \nu_p(h),$$

because $2a - b > 0$. This proves (9). □

**Proposition 3.13.** *For a special prime $p \mid d$, and an $m, x, y$ coprime to $d$: $\nu_p(T_{2m+1}) \geq (2m+1)\,\nu_p(K)$ is true iff:*

$$\frac{x}{y} \;\equiv\; r_p(m) \pmod{p^{b(m-1)}}. \tag{10}$$

*Proof.* From Lemma 3.4 together with Lemma 3.12, and remembering that $a = b$

$$T_{2m+1} = Y\,U_{2m+1} - Q\,x\,U_{2m} = bd'p^{\,b(m+1)}u_{2m} + xb^2cd'p^{\,b(m+1)}\;u_{2m+1}.$$

For $T_{2m+1}$ to be an integer, $\nu_p(T_{2m+1}) > b(2m+1)$ for all $p$ dividing $d$, thus:

$$bd'p^{\,b(m+1)}u_{2m} + xb^2cd'p^{\,b(m+1)}\;u_{2m+1} \equiv 0 \pmod{p^{2bm}}.$$

Solving this for $\frac{x}{y}$ and reducing the modulo by $p^{b(m+1)}$ yields:

$$-xb^2cd'p^{\,b(m+1)}\;u_{2m+1} \equiv ybd'p^{\,b(m+1)}u_{2m} \pmod{p^{2bm}} \to \frac{x}{y} \equiv \frac{-u_{2m}}{bc\,u_{2m+1}} \pmod{p^{(b(m-1))}}.$$

This is exactly what is stated in the proposition. □

**Proposition 3.14.** *There exists a finite list of odd indices $2m_1+1 < \cdots < 2m_k+1$ with $\gcd(m_j, bd) = 1$ for all $j$ such that there exist coprime integers $(x,y)$, both of which do not share factors with d, with the property that for every special prime $p \mid d$ one has*

$$\nu_p\big(T_{2m_j+1}\big) \;\geq\; (2m_j+1)\,\nu_p(K) \qquad (1 \leq j \leq k).$$

*Proof.* Write $Y := Ky$ and use the companion identity, then for a special prime $p \mid d$, factor $Y = p^b bd'y$ with $p \nmid xy$. By Lemma 3.12, we have $u_{2m}, u_{2m+1}$ and

$$r_p(m) := \frac{-u_{2m}}{bc\,u_{2m+1}}$$

such that the following is true:

$$\nu_p(T_{2m+1}) \;\geq\; (2m+1)\,\nu_p(K) \iff \frac{x}{y} \;\equiv\; r_p(m) \pmod{p^{\,b(m-1)}}, \tag{11}$$

Furthermore, Lemma 3.12 gives the $p$-adic bound

$$\nu_p\big(r_p(m+h) - r_p(m)\big) \;\geq\; \nu_p(h) \qquad \text{for all } h \in \mathbb{Z} \text{ relatively prime to } d. \tag{12}$$

Fix the special prime $p$. Because the ratio $\frac{x}{y}$ is *fixed*, one must select a value of $m_{j+1}$ that also satisfies the right hand side of (11), in other words:

$$r_p(m_{i+1}) \equiv r_p(m_i) \pmod{p^{b(m_1-1)}}. \tag{13}$$

One can do this by choosing the indices recursively so that for each $j < k$,

$$m_{j+1} \equiv m_j \pmod{p^{b(m_1-1)}}. \tag{14}$$

Then for $h := m_{j+1} - m_j$, the following holds: $\nu_p(h) \geq b(m_1 - 1)$, and (12) yields

$$r_p(m_{j+1}) \equiv r_p(m_j) \pmod{p^{b(m_j-1)}}$$

which satisfies (13). Inductively, since for all $j+1 \leq k$ $m_{j+1} \equiv m_j \pmod{p^{b(m_1-1)}}$, each $m_j$ must be congruent with $m_k$ modulo $p^{b(m_j)}$. Because of this, solving all of the $k-1$ congruences ($m_1 \equiv m_2 \ldots m_{k-1} \equiv m_k$) is equivalent solving to the single strongest one at $m_k$:

$$\frac{x}{y} \equiv r_p(m_k) \pmod{p^{b(m_k-1)}}. \tag{15}$$

Solve the same congruence as (15) for each special prime $q \mid d$ simultaneously by imposing, for every $j < k$,

$$\frac{x}{y} \equiv r_p(m_k) \pmod{q^{\nu_q(d)(m_k-1)}}.$$

This keeps $\gcd(m_j, d) = 1$ and achieves the condition in the proposition at every special prime. Since the moduli $\{q^{(n-1)\nu_q(d)}\}_{q|d}$ are coprime, the Chinese Remainder Theorem produces a single ratio $x/y$ satisfying all these congruences at once. Thus, satisfying the property:

$$\nu_p(T_{2m_j+1}) \geq (2m_j + 1)\,\nu_p(K) \qquad \text{for every special } p \mid d \text{ and every } j,$$

and proving the proposition. $\square$

**Proposition 3.15.** *Let $S_n$ be the recurrence defined by*

$$S_{n+2} = \frac{a}{b}S_{n+1} + \frac{c}{d}S_n, \quad \gcd(a,c) = \gcd(b,d) = \gcd(a,d) = 1.$$

*For any finite subset of odd indices $2n_1{+}1, 2n_2{+}1, \ldots 2n_k{+}1$ with all $\gcd(n_i, bd) = 1$, one can select relatively prime $x, y$ such that $S_{2n_i+1} \in \mathbb{Z}$ for all $i$.*

*Proof.* Use Proposition 3.14 to generate relatively prime $x_0, y_0$ (these numbers are also coprime to $d$) such that:

$$\forall\, p \mid d \qquad \nu_p\big(T_{2n_j+1}\big) \geq (2n_j + 1)\,\nu_p(K) \qquad (1 \leq j \leq k).$$

Then, use Proposition 3.11 to generate relatively prime $x_1, y_1$ such that

$$\forall\, p \mid b \qquad \nu_p\big(T_{2n_j+1}\big) \geq (2n_j + 1)\,\nu_p(K) \qquad (1 \leq j \leq k).$$

Now, multiply these initial conditions to get the new initial conditions for $T_n$:

$$T_0 = x_0x_1, \qquad T_1 = Ky_0y_1.$$

These initial conditions satisfy both conditions, specifically:

$$\forall\, p \mid K \qquad \nu_p\big(T_{2n_j+1}\big) \geq (2n_j + 1)\,\nu_p(K) \qquad (1 \leq j \leq k).$$

Also, since $\gcd(x_0, d) = \gcd(y_0, d) = \gcd(x_1, b) = \gcd(y_1, b) = \gcd(x_0, y_0) = \gcd(x_1, y_1) = 1$, the only factors that could be shared by $(x_0x_1, y_0y_1)$ are ones relatively prime to

$K$. Because of this, factoring out this $\gcd(x_0x_1, y_0y_1)$ from the initial conditions for $T_n$ of $x_0x_1, Ky_0y_1$ will not change the $p$-adic valuation of terms in the sequence. Now, having generated a coprime $(x, y) = \left(\frac{x_0x_1}{\gcd(x_0x_1,y_0y_1)}, \frac{Ky_0y_1}{\gcd(x_0x_1,y_0y_1)}\right)$ such that:

$$\forall\, p \mid K \qquad \nu_p\big(T_{2n_j+1}\big) \;\geq\; (2n_j+1)\,\nu_p(K) \qquad (1 \leq j \leq k).$$

Using Lemma 3.3, one completes the proof. Furthermore, selecting $k$ to be as large as we desire, we satisfy the main theorem of the section. □

**Example 3.16** (Worked construction for Example 3.2)**.** We illustrate the method of this section by explaining how the coprime initial conditions $(S_0, S_1) = (1, 4683)$ in Example 3.2 were obtained, using the separate constructions for special and non-special primes and then gluing them together as in Proposition 3.16.

Consider the recurrence

$$S_{n+2} = \frac{1}{3}S_{n+1} + \frac{1}{2}S_n, \qquad S_0 = x,\ S_1 = y,$$

so that

$$a = c = 1, \quad b = 3, \quad d = 2.$$

Then $K = bd = 6$, and

$$P := ad = 2, \qquad Q := -b^2cd = -18.$$

The associated Lucas sequence $(U_n)$ for $(P, Q)$ satisfies

$$U_0 = 0, \quad U_1 = 1, \quad U_{n+2} = 2U_{n+1} + 18U_n,$$

and the companion identity gives

$$T_n := K^nS_n = Ky\,U_n - Qx\,U_{n-1} = 6y\,U_n + 18x\,U_{n-1}.$$

By Lemma 3.6 we have $S_n \in \mathbb{Z}$ if and only if

$$\nu_p(T_n) \;\geq\; n\,\nu_p(K) \qquad \text{for every prime } p \mid K.$$

We want to force integrality at the odd indices 3 and 7, i.e. at $2m_1 + 1 = 3$ and $2m_2 + 1 = 7$ with $m_1 = 1$ and $m_2 = 3$.

*Non-special prime $p = 3$: gluing across indices.* Here $3 \mid b$ and $3 \nmid P = ad$, so 3 is a non-special prime in the sense of Definition 3.8. Thus we are in the setting of Proposition 3.12 with

$$b = 3, \quad c = 1, \quad x = 1,$$

and with our desired index set

$$S = \{n_1, n_2\} = \{3, 7\}.$$

Proposition 3.12 says that we can construct coprime integers $(x_1, y_1)$ with $x_1 = 1$ such that for every $n \in S$ and every $p \mid b$ (here just $p = 3$) we have

$$\nu_3(T_n) \;\geq\; n\,\nu_3(K) = n,$$

and, equivalently, that for each $n \in S$

$$y\,U_n + cbx\,U_{n-1} \equiv 0 \pmod{b^{n-1}} \qquad (\text{with } c = 1,\ b = 3,\ x = 1). \tag{16}$$

We now carry out the inductive "gluing" construction from the proof of Proposition 3.12 in this concrete case.

First, we work at the smallest index $n_1 = 3$. Using

$$U_2 = 2, \quad U_3 = 22,$$

the condition (16) for $n = 3$ becomes

$$y_1 U_3 + 3U_2 \equiv 0 \pmod{3^{3-1}} \quad \Longleftrightarrow \quad 22y_1 + 6 \equiv 0 \pmod{9}.$$

Since $\gcd(U_3, 3) = 1$, this linear congruence has a unique solution modulo $3^{3-1} = 9$; explicitly one finds

$$y_1 \equiv 3 \pmod{9}.$$

This gives an initial choice of $y_1$ which makes $\nu_3(T_3) \geq 3$.

Next, we "glue in" the second index $n_2 = 7$ without losing the condition at $n_1 = 3$. Following the proof of Proposition 3.12, we look for a new value

$$y_2 := y_1 + t \cdot b^{n_1 - 1} = y_1 + t \cdot 3^2$$

for some integer $t$. For every $n \leq n_1$, the extra term $t \cdot 3^2$ is divisible by $3^{n-1}$, so (16) continues to hold at $n = 3$.

We now impose (16) at $n_2 = 7$ using $y_2$. With

$$U_6 = 2552, \quad U_7 = 15112,$$

this condition is

$$(y_1 + 9t)\, U_7 + 3U_6 \equiv 0 \pmod{3^{7-1}}.$$

Dividing by $3^{n_1-1} = 3^2$ produces a linear congruence for $t$ modulo $3^{n_2 - n_1} = 3^4$; since $\gcd(U_7, 3) = 1$ (cf. (7) in the proof of Proposition 3.12), such a congruence is always solvable. In this example the resulting congruence for $t$ modulo $3^4 = 81$ has the solution

$$t \equiv 34 \pmod{81},$$

and hence

$$y_2 = y_1 + 9t \equiv 3 + 9 \cdot 34 \equiv 309 \pmod{3^6}.$$

Thus, after gluing in the second index, we have a value $y_2$ such that

$$y_2 U_n + 3U_{n-1} \equiv 0 \pmod{3^{n-1}} \quad \text{for } n = 3, 7.$$

Equivalently, for $p = 3$ we have

$$\nu_3(T_3) \geq 3, \qquad \nu_3(T_7) \geq 7.$$

Any integer $y$ with

$$y \equiv 309 \pmod{3^6}$$

behaves like this $y_2$ in the 3-adic sense, and will therefore give the desired divisibility at the indices 3 and 7 for the non-special prime 3.

*Special prime $p = 2$.* Since $2 \mid d$, the prime 2 is special. In this case we directly use the companion identity to compute

$$T_3 = 6y\, U_3 + 18x\, U_2 = 6y \cdot 22 + 18 \cdot 2 = 132y + 36 = 4(33y + 9),$$

$$T_7 = 6y\, U_7 + 18x\, U_6 = 6y \cdot 15112 + 18 \cdot 2552 = 90672y + 45936 = 16(5667y + 2871),$$

again with $x = 1$. Thus

$$\nu_2(T_3) = 2 + \nu_2(33y + 9), \qquad \nu_2(T_7) = 4 + \nu_2(5667y + 2871).$$

The conditions $\nu_2(T_3) \geq 3$ and $\nu_2(T_7) \geq 7$ are therefore equivalent to

$$33y + 9 \equiv 0 \pmod 2 \quad \text{and} \quad 5667y + 2871 \equiv 0 \pmod 8.$$

Reducing the second congruence modulo 8, using $5667 \equiv 3$ and $2871 \equiv 7$, we obtain

$$3y + 7 \equiv 0 \pmod 8 \qquad \Longleftrightarrow \qquad y \equiv 3 \pmod 8.$$

This is exactly the condition supplied by Proposition 3.15 in this concrete example.

*Chinese remainder theorem and gluing across primes.* We now glue the 2-adic and 3-adic information together as in the "final method" and Proposition 3.16. From the special prime $p = 2$ we must have

$$y \equiv 3 \pmod 8,$$

and from the non-special prime $p = 3$ (after gluing the indices 3 and 7) we must have

$$y \equiv 309 \pmod{3^6 = 729}.$$

Since $\gcd(8, 729) = 1$, the Chinese remainder theorem yields a unique solution modulo $8 \cdot 729 = 5832$, namely

$$y \equiv 4683 \pmod{5832}.$$

Taking the smallest positive representative, we choose $y = 4683$, and with $x = 1$ we obtain the coprime initial conditions

$$(S_0, S_1) = (1, 4683)$$

used in Example 3.2.

With these initials, the first few terms of the recurrence are

$$S_2 = \frac{3123}{2}, \quad S_3 = 2862, \quad S_4 = \frac{6939}{4}, \quad S_5 = \frac{8037}{4}, \quad S_6 = \frac{12297}{8}, \quad S_7 = 1517,$$

so indeed $S_3, S_7 \in \mathbb{Z}$ while the other displayed terms are non-integral. Thus the method of Propositions 3.12, 3.15, and 3.16 realises Theorem 3.1 for the pair of odd indices $\{3, 7\}$ in this concrete example.

## 4. Eventual non-integrality and an explicit cutoff

In this section we answer Question 2 from the introduction: given a fixed rational-multiple second-order recurrence and fixed coprime initial values, does there exist a finite index $k$ past which no further terms are integers? In Theorem 4.1, for all but a very small subset of sequences, we provide both an affirmative answer and a sequence-specific formula for the index $k$. Throughout this section we keep the hypotheses and notation of Section 3:

$$S_{n+2} = \frac{a}{b} S_{n+1} + \frac{c}{d} S_n, \qquad S_0 = x,\ S_1 = y, \qquad \gcd(a, c) = \gcd(b, d) = \gcd(x, y) = 1,$$

with $b, d \geq 2$. Write $K := bd$ (so $K = \operatorname{lcm}(b, d)$ because $\gcd(b, d) = 1$). To answer the question, we prove the following theorem:

**Theorem 4.1.** *For each sequence $S_n$ defined by unique values of $a, b, c, d, x, y$ such that*

$$S_{n+2} = \frac{a}{b} S_{n+1} + \frac{c}{d} S_n, \qquad S_0 = x,\ S_1 = y, \qquad \gcd(a,c) = \gcd(b,d) = \gcd(x,y) = 1,$$

*and $\nu_p(x) \neq \frac{\nu_p(d)}{2}$, there exists a $k$ such that for all $n \geq k$, $S_n \notin \mathbb{Z}$.*

To prove this theorem, we use three main tools. We start off by using the fact that each term in $S_n$ can be written as follows:

$$S_n = \alpha r_1^n + \beta r_2^n$$

for explicit values of $\alpha, \beta, r_1, r_2$. Similarly using a bit of algebra and intuition, we turn the fact that $S_n = \alpha r_1^n + \beta r_2^n$ into a lower bound for $p$-adic valuation of the term $\left(\frac{r_1}{r_2}\right)^n + \frac{\beta}{\alpha}$ that is dependent on the linear growth of $n$. Finally we use [2, Theorem 2.1] to prove that $\nu_p\Big(\left(\frac{r_1}{r_2}\right)^n + \frac{\beta}{\alpha}\Big)$ has an upper bound which is dependent on a factor of $\log(n)$. Thus, comparing our lower bound to the upper bound in [2] yields a value of $k$ such that for all $n > k$: $S_n \notin \mathbb{Z}$.

4.1. **Setting up to use Chim's Bounds.**

**Definition 4.2.** Let $P, Q \in \mathbb{Q}$ and let $r_1, r_2 \in \mathbb{Q}(r_1, r_2)$ be the roots of $R^2 - \frac{a}{b}R - \frac{c}{d} = 0$. $S_{n+2} = \frac{a}{b} S_{n+1} + \frac{c}{d} S_n$ can also be written as:

$$S_n = \alpha\, r_1^n + \beta\, r_2^n$$

With $\alpha, \beta$ coming from the values of $S_0, S_1$. Specifically: for initial conditions $S_0 = x$, $S_1 = y$,

$$\alpha = \frac{y - r_2 x}{r_1 - r_2}, \qquad \beta = \frac{r_1 x - y}{r_1 - r_2},$$

This comes from [4].

**Lemma 4.3.** *For a given second order linear recurrence, we have:*

$$\nu_p(S_n) = \nu_p\big(\Lambda^n + \Gamma^{-1}\big) \ + n\nu_p(r_2) - \nu_p(\beta) - \nu_p(\Gamma).$$

*Proof.* Using Definition 4.2, let $r_1, r_2$ be the (distinct) roots of $R^2 - \frac{a}{b}R - \frac{c}{d} = 0$. Then every term of $S_n$ can be written as:

$$S_n = \alpha\, r_1^n + \beta\, r_2^n.$$

Factoring out $r_2^n \beta \Gamma$ gives

$$S_n = \Gamma\beta\, r_2^n\Big(\Lambda^n + \Gamma^{-1}\Big), \qquad \Gamma := \frac{\alpha}{\beta}, \quad \Lambda := \frac{r_1}{r_2}, \tag{17}$$

hence

$$\nu_p(S_n) = \nu_p(\beta) + n\,\nu_p(r_2) + \nu_p(\Gamma) + \nu_p\big(\Lambda^n + \Gamma^{-1}\big). \tag{18}$$

$\square$

We know that $\nu_p(a) = \nu_p(b) = \nu_p(c) = 0$ for all $p$ dividing $d$, so consider the characteristic equation for $S_n$, namely:

$$R^2 - \frac{a}{b}R - \frac{c}{d} = 0.$$

Write
$$u := \frac{a}{b}, \qquad v := \frac{c}{d}, \qquad \Delta := u^2 + 4v,$$
and let $r_1, r_2$ be the roots
$$r_{1,2} = \frac{u \pm \sqrt{\Delta}}{2}.$$
Fixing a prime $p \mid d$ and setting $\delta := \nu_p(d) \geq 1$ results in $\gcd(a,d) = \gcd(b,d) = \gcd(c,d) = 1$, then we have
$$\nu_p(u) = 0, \qquad \nu_p(v) = -\delta.$$

**Lemma 4.4.** *If $p \geq 3$ divides $d$ (so $\delta = \nu_p(d) \geq 1$), then*
$$\nu_p(r_1) = \nu_p(r_2) = -\frac{\delta}{2}.$$

*Proof.* Using the ultrametric inequality: $\nu_p(r_{1,2}) \geq \min(\nu_p(u), \frac{1}{2}\nu_p(\Delta)) - \nu_p(2)$. We know that
$$\nu_p(\Delta) \geq \min(2\nu_p(u), 2\nu_p(2) + \nu_p(v))$$
(again from the ultrametric inequality). We also know that $\nu_p(2) = \nu_p(u) = 0$ and $\nu_p(v) = -\delta$. Since $\nu_p(u) \neq 2\nu_p(2) + \nu_p(v)$:
$$\nu_p(\Delta) = -\delta.$$
Plugging in for $r_{1,2}$:
$$\nu_p(r_{1,2}) \geq \min(0, -\tfrac{\delta}{2}).$$
Finally, because $0 \neq \frac{\delta}{2}$:
$$\nu_p(r_{1,2}) = -\frac{\delta}{2}.$$
□

**Lemma 4.5.** *Let $p = 2$ divide $d$ and put $\delta = \nu_2(d) \geq 1$. Then we still have that:*
$$\nu_2(r_1) = \nu_2(r_2) = -\frac{\delta}{2}.$$

*Proof.* Write $s := u + \sqrt{\Delta}$ and $t := u - \sqrt{\Delta}$. Note that
$$s + t = 2u, \qquad s - t = 2\sqrt{\Delta}, \qquad st = u^2 - \Delta = \left(\frac{a}{b}\right)^2 - \left(\frac{a}{b}\right)^2 + 4\frac{c}{d} = 4v.$$
Since $\nu_2(u) = 0$ and $\nu_2(v) = -\delta$, we have that
$$\nu_2(st) = \nu_2(-4v) = 2 - \delta.$$

*Case 1:* $\delta \geq 3$. Here we have $\nu_2(v) \leq -3$. Thus
$$\nu_2(\Delta) \geq \min(2\nu_2(u), 2\nu_2(2) + \nu_2(v)) = \min(0, 2 - \delta) = 2 - \delta.$$
Since $0 \neq 2-\delta$ and $2-\delta < 0$: $\nu_p(\Delta) = 2-\delta$. Thus: $\nu_2(r_{1,2}) \geq \min(\nu_p(u), 1-\frac{1}{2}\delta) - 1$, again since $\nu_p(u) \neq 1 - \frac{1}{2}\delta$:
$$\nu_2(r_{1,2}) = 1 - \frac{1}{2}\delta - 1 = -\frac{1}{2}\delta.$$

*Case 2:* $\delta = 2$. In this case $\nu_2(st) = \nu_p(4v) = 2-\delta = 0$ and $\nu_2(s+t) = \nu_p(2u) = 1$. If $\nu_2(s) \neq \nu_2(t)$, then:
$$\nu_2(s+t) = \min\{\nu_2(s), \nu_2(t)\} \leq 0,$$

this is a contradiction. Thus $\nu_2(s) = \nu_2(t)$. Call $\nu_p(s) = \nu_p(t) = \lambda$, because: $2\lambda = \nu_2(st) = 0$, we must have that $\lambda = 0$. Thus

$$\nu_2(r_{1,2}) = \lambda - \nu_2(2) = 0 - 1 = -1 = -\delta/2.$$

*Case 3:* $\delta = 1$. Now $\nu_2(st) = \nu_p(4v) = 2 - \delta = 1$, $\nu_2(s+t) = \nu_p(2u) = 1$, and $\nu_2(s-t) = \nu_2(2\sqrt{\Delta}) = 1 + \nu_2(\sqrt{\Delta})$. If $\nu_2(s) \neq \nu_2(t)$, then

$$\nu_2(s+t) = \min\{\nu_2(s), \nu_2(t)\} = 1$$

forces $\nu_2(s) + \nu_2(t) \geq 2$, contradicting $\nu_2(st) = 1$. Thus $\nu_2(s) = \nu_2(t) = \lambda$, so $2\lambda = \nu_2(st) = 1$ and $\lambda = \frac{1}{2}$. Hence

$$\nu_2(r_1) = \nu_2(r_2) = \lambda - \nu_2(2) = \tfrac{1}{2} - 1 = -\tfrac{1}{2} = -\delta/2.$$

$\square$

**Corollary 4.6.** *For every special prime $p \mid d$ with $\delta = \nu_p(d) \geq 1$ one has*

$$\nu_p(\Lambda) = \nu_p(r_1) - \nu_p(r_2) = 0.$$

*Thus, $\Lambda$ is $p$-free.*

**Lemma 4.7.** *Assume $p \mid d$ and $\frac{\delta}{2} \neq \nu_p(x)$. Then $\nu_p(\Gamma^{-1}) = 0$.*

*Proof.* Since $\alpha = \frac{y - r_2x}{r_1 - r_2}$, and $\beta = \frac{r_1x - y}{r_1 - r_2}$, we have

$$\Gamma^{-1} = \frac{r_1x - y}{y - r_2x}.$$

Analyzing the $p$-adic valuations of the numerators and denominators of $\Gamma^{-1}$ results in:

$$\nu_p(r_1x - y) \geq \min\left(-\frac{\delta}{2} + \nu_p(x), \nu_p(-y)\right) \qquad \nu_p(y - r_2x) \geq \min\left(\nu_p(y), \nu_p(-x) - \frac{\delta}{2}\right).$$

Now, because $\gcd(x, y) = 1$,

$$\nu_p(r_1) = \nu_p(r_2) = -\frac{\delta}{2} \quad \text{and} \quad \nu_p(x) \neq \frac{\delta}{2},$$

if $\nu_p(x) = 0$, then set $\nu_p(y) = \kappa \geq 0$ (the $\kappa \geq 0$ condition holds because $x$ is an integer):

$$\nu_p(r_1x - y) \geq \min\left(-\frac{\delta}{2}, \kappa\right) \qquad \nu_p(y - r_2x) \geq \min\left(\kappa, -\frac{\delta}{2}\right).$$

Because $\kappa$ is positive $\nu_p(r_1x - y) = \nu_p(y - r_2x) = -\frac{\delta}{2}$. To complete the proof, if $\nu_p(x) = \kappa > 0$ and $\nu_p(y) = 0$, we have that:

$$\nu_p(r_1x - y) \geq \min\left(-\frac{\delta}{2} + \kappa, 0\right) \qquad \nu_p(y - r_2x) \geq \min\left(0, -\frac{\delta}{2} + \kappa\right).$$

Finally, since $\nu_p(x) = \kappa \neq \frac{\delta}{2}$, the $\geq$ signs turn into $=$ signs, and $\nu_p(r_1x - y) = \nu_p(y - r_2x)$. Thus, $\nu_p(\Gamma) = \nu_p(r_1x - y) - \nu_p(y - r_2x) = 0$, completing the proof. $\square$

4.2. **Using Chim's Bounds.** In the paper [2], Chim provides an upper bound for the $p$-adic valuation of $\alpha_1^{b_1} + \alpha_2^{b_2}$ when the $\alpha_i$ are algebraic numbers (roots of polynomials with integer coefficients) with $p$-adic valuation of 0, and the $b_i$ are positive integers.

From Corollary 4.6 and Lemma 4.7, we have that $\Lambda = \alpha_1$, $\Gamma^{-1} = \alpha_2$, $n = b_1$ and $1 = b_2$ are suitable choices for the $\alpha_i$ and $b_i$. This is because, $\Lambda$ and $\Gamma^{-1}$ are $p$-free, and $n, 1$ are positive integers.

**Lemma 4.8.** *For some special prime $p \mid d$ with $\delta = \nu_p(d)$, in order for $\nu_p(S_n) \geq 0$:*

$$\nu_p(\Lambda^n + \Gamma^{-1}) \geq \frac{\delta}{2}n - \nu_p(\beta) - \nu_p(\Gamma).$$

*Proof.* After re-arranging Lemma 4.3, and setting $\nu_p(S_n) \geq 0$, we have that:

$$\nu_p(\Lambda^n + \Gamma^{-1}) \geq \frac{\delta}{2}n - \nu_p(\beta) - \nu_p(\Gamma) \tag{19}$$

exactly as stated. □

4.2.1. *Proof of Theorem 4.1.*

*Proof.* For the remainder of this proof, let $C_n$ be computable constants depending only on the values of $a, b, c, d, x, y, p$ for a given recurrence and a given prime. From Theorem 2.1 in [2], denote $b'$ to be the value $\frac{b_1}{C_1} + \frac{b_2}{C_2}$. In our case with $b_1 = n$ and $b_2 = 1$, this returns

$$\frac{n}{C_1} + \frac{1}{C_2}. \tag{20}$$

Furthermore, write $H = \max(\log(b') + C_3, C_4, C_5)$, thus, $H$ never exceeds

$$|\log(b') + C_3| + |C_4| + |C_5|.$$

Plugging in for $b'$ using (20) yields:

$$\left|\log\left(\frac{n}{C_1} + \frac{1}{C_2}\right) + C_3\right| + |C_4| + |C_5|.$$

Since $\log\left(\frac{n}{C_1} + \frac{1}{C_2}\right) < \log\left(\frac{2n}{C_1}\right)$, and setting $|C_4| + |C_5| = C_6$ we have:

$$H \leq \left|\log\left(\frac{2n}{C_1}\right) + C_3\right| + C_6.$$

The rest of the bound in [2] does not involve $n$ and never changes when looking at the same sequence. Thus, looking at the full bound, we see that:

$$\nu_p(\Lambda^n + \Gamma^{-1}) < C_7\left(\left|\log\left(\frac{2n}{C_1}\right) + C_3\right| + C_6\right). \tag{21}$$

Finally, combining (21) with (19), we get the final bound:

$$\frac{\delta}{2}n - \nu_p(\beta) - \nu_p(\Gamma) < C_7\left(\left|\log\left(\frac{2n}{C_1}\right) + C_3\right| + C_6\right). \tag{22}$$

Since the left hand side is linear in $n$ while the right hand side is logarithmic, after explicitly solving for $C_1, C_3, C_6$ and $C_7$ using the values described in [2], and rearranging this inequality to solve for $n$, one receives an index $k$, such that for all $n$ greater than $k$, (22) stops holding. Thus $S_n \notin \mathbb{Z}$. □

## References


1. C. Ballot, *Density of prime divisors of linear recurrences*, Mem. Amer. Math. Soc. **261** (2019), no. 1260.
2. K. C. Chim, *A p-adic analogue of a formula of Ramanujan*, J. Number Theory **248** (2025), 295–329.
3. P. Corvaja and U. Zannier, *Finiteness of integral values for the ratio of two linear recurrences*, Invent. Math. **149** (2002), no. 2, 431–451.
4. G. Everest, A. van der Poorten, I. Shparlinski, and T. Ward, *Recurrence sequences*, Mathematical Surveys and Monographs, vol. 104, American Mathematical Society, Providence, RI, 2003.
5. V. E. Hoggatt, Jr., *Fibonacci and Lucas numbers*, Houghton Mifflin Co., Boston, Mass., 1969.
6. N. Koblitz, *p-adic numbers, p-adic analysis, and zeta-functions*, second ed., Graduate Texts in Mathematics, vol. 58, Springer-Verlag, New York, 1984.
7. S. Vajda, *Fibonacci and Lucas numbers, and the golden section: Theory and applications*, Dover Publications, Inc., Mineola, NY, 2008, Reprint of the 1989 edition.



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