Scientific Computing

$ax^2 + bx + c = 0$

$\cancel{x = -b \pm \sqrt{4}}$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow \text{Analytical}$$

$x^2 - x - 1 = 0$

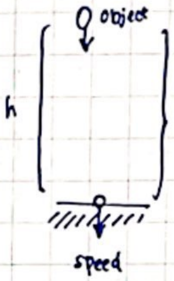
$$x = \frac{1 \pm \sqrt{1+4}}{2} \rightarrow \text{exact value}$$

Errors
- Rounding
- chopping
- Modelling

$\pi = 3.1415\overset{926}{\cancel{296}}54 \ldots$

Rounding 4dp $\rightarrow$ 3.1416

Chopping 4dp $\rightarrow$ 3.1415



$v = v_0 + at$

$s = v_0 t + \frac{1}{2} a t^2$

Binary and Decimal

convert 1110.1011 to decimal

$(1110.1011)_2 = (\ldots\ldots)_{10}$

$2^3\ 2^2\ 2^1\ 2^0 \quad 2^{-1}\ 2^{-2}\ 2^{-3}\ 2^{-4}$

$(1 \times 2^3) + (1 \times 2^2) + (1 \times 2^1) + (0 \times 2^0) + (1 \times 2^{-1})$

$+ (0 \times 2^{-2}) + (1 \times 2^{-3}) + (1 \times 2^{-4})$

$= 8 + 4 + 2 + 0 + 0.5 + 0 + 0.125 + 0.0625$

$= (14.6875)_{10}$

Types of errors
- absolute $= e_a = |x_t - x_a|$
- Relative $= e_r = \left|\frac{e_a}{x_t}\right| \times 100\%$

$x_t = x_{true}$

$x_a = x_{approximate}$

Decimal to Binary

$(14.6375)_{10}$

$14/2 = 7$ remainder 0

$7/2 = 3$ remainder 1

$3/2 = 1$ remainder 1

$1/2 = 0$ remainder 1

$0.6375 \times 2 = 1.275$	integer 1	$0.8 \times 2 = 1.6$	integer 1
$0.275 \times 2 = 0.55$	integer 0	$0.6 \times 2 = 1.2$	integer 1
$0.55 \times 2 = 1.1$	integer 1	$0.2 \times 2 = 0.4$	integer 0
$0.1 \times 2 = 0.2$	integer 0	$0.4 \times 2 = 0.8$	integer 0
$0.2 \times 2 = 0.4$	integer 0		
$0.4 \times 2 = 0.8$	integer 0		

$\therefore$ Binary $\Rightarrow$ 1110.101100110011

$$\begin{cases} x + y + z = 4 \\ x - y - z = -2 \\ 2x + 8y + z = 19 \end{cases}$$

Gauss Elimination

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & -1 & -1 & -2 \\ 2 & 8 & 1 & 19 \end{array}\right] \begin{array}{l} \\ r_2 - r_1 \\ r_3 - 2r_1 \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 6 & -1 & 11 \end{array}\right] \begin{array}{l} \\ \\ r_3 + 3r_2 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 0 & -7 & -7 \end{array}\right] \quad \Delta \text{upper} \quad \text{Gauss stops}$$

$-7z = -7 \rightarrow z = 1$

$-2y - 2(1) = -6 \rightarrow y = 2$

$x + y + z = 4$

$x + 2 + 1 = 4 \rightarrow x = 1$

$\{(1, 2, 1)\}$

Gauss Jordan

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & -2 & -2 & -6 \\ 0 & 0 & -7 & -7 \end{array}\right] \begin{array}{l} \\ r_2 \times -\frac{1}{2} \\ r_3 \times -\frac{1}{7} \end{array} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 - r_3 \\ r_2 - r_3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \begin{array}{l} r_1 - r_2 \\ \\ \\ \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} x = 1 \\ y = 2 \\ z = 1 \end{array}$$

A = LU

L = lower Δ matrix

U = upper Δ matrix

$$A = LU = \begin{array}{cc} L & U \\ \begin{bmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ +2 & -3 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -7 \end{bmatrix} \end{array}$$

Solving using Doolittle Decomposition method

$$\begin{cases} x + y + z = 4 \\ x - y - z = -2 \\ 2x + 8y + z = 19 \end{cases}$$

AX = B is changed into LUX = B

Find y_i from LY = B and then solve X_i from UX = y

$$\underset{A}{\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 2 & 8 & 1 \end{bmatrix}} \underset{X}{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}} = \underset{B}{\begin{bmatrix} 4 \\ -2 \\ 19 \end{bmatrix}}$$

↓

LU

$$\underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix}} \underbrace{\underset{U}{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -7 \end{bmatrix}} \underset{X}{\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}}}_{Y} = \underset{B}{\begin{bmatrix} 4 \\ -2 \\ 19 \end{bmatrix}}$$

L . Y . = B

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 19 \end{bmatrix}$$

$y_1 = 4$

$y_1 + y_2 = -2 \rightarrow 4 + y_2 = -2$

$y_2 = -6$

$2y_1 - 3y_2 + y_3 = 19$

$2(4) - 3(-6) + 3y_3 = 19$

$y_3 = -7$

UX = Y

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -7 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ -7 \end{bmatrix}$$

$-7X_3 = -7$

$X_3 = 1$

$-2X_2 - 2X_3 = -6$

$-2X_2 - 2(1) = -6$

$X_2 = 2$

$X_1 + X_2 + X_3 = 4$ ∴ $\{(1,2,1)\}$

$X_1 + 2 + 1 = 4$

$X_1 = 1$

PIVOTING

$$\begin{cases} -2x + 4y - z = 1 \\ x - y + 3z = 3 \\ -4x + 2y - z = -3 \end{cases}$$

Using diagonally dominant (absolute value of each diagonal element) should be dominant.

$$\left[\begin{array}{ccc|c} -2 & 4 & 1 & 1 \\ 1 & -1 & 3 & 3 \\ -4 & 2 & -1 & -3 \end{array}\right] \quad \begin{array}{l} |-2| < |4| + |-1| \\ |-1| < |1| + |3| \\ |-1| < |-4| + |2| \end{array}$$

} not diagonally dominant.

$$r_{13} \left[\begin{array}{ccc|c} \boxed{-4} & 2 & -1 & -3 \\ 1 & -1 & 3 & 3 \\ -2 & 4 & -1 & 1 \end{array}\right] \quad |-4| > |2| + |-1|$$

$$r_{23} \left[\begin{array}{ccc|c} \boxed{-4} & 2 & -1 & -3 \\ -2 & \boxed{4} & -1 & 1 \\ 1 & -1 & \boxed{3} & 3 \end{array}\right] \quad \begin{array}{l} |-4| > |2| + |-1| \\ |4| > |-2| + |-1| \\ |3| > |1| + |-1| \end{array}$$

→ diagonally dominant

EXAMPLE :

$$\begin{cases} 2x + y + z = 4 \\ x + 3y + z = \cancel{4}\,5 \\ 2x + y + 8z = 11 \end{cases}$$

Notice that this system is alredy diagonally dominant solve with Gauss-Seidel method. Acceptance Error = 12%. Initial values (0.0.0) use. Adp.

Gauss Seidel

→ $\{(1,1,1)\}$ → the solution

$x = \frac{4 - y - z}{2}$

$y = \frac{5 - x - z}{3}$

$z = \frac{11 - 2x - y}{8}$

x = 0 y = 0 z = 0

1st iteration $x = \frac{4-0-0}{2} = 2$

$y = \frac{5-2-0}{3} = 1$

$z = \frac{11-2(2)-1}{8} = \cancel{2}\ 0.75$

→ error $= \frac{2-0}{2} \times 100\% = 100\%$

error $= \frac{1-0}{1} \times 100\% = 100\%$

error $= \frac{0.75-0}{0.75} \times 100\% = 100\%$

2nd iteration $x = \frac{4-1-0.75}{2} = \frac{2.25}{2} = 1.125$

$y = \frac{5-1.125-0.75}{3} = \ldots\ldots$

$z = \ldots\ldots$

Task 1

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 9x_2 + 4x_3 = 9 \\ 8x_1 + x_2 + 2x_3 = 15 \end{cases}$$

a). Gaussian Elimination

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 9 & 4 & 9 \\ 8 & 1 & 2 & 15 \end{array}\right] \xrightarrow{\substack{r_2 - 2r_1 \\ r_3 - 8r_1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 5 & -2 & 7 \\ 0 & -15 & -22 & 7 \end{array}\right]$$

$$\xrightarrow{r_2/5} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -2/5 & 7/5 \\ 0 & -15 & -22 & 7 \end{array}\right] \xrightarrow{\substack{r_1 - 2r_2 \\ r_3 + 15r_2}} \left[\begin{array}{ccc|c} 1 & 0 & 3.8 & -1.8 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & 0 & -28 & 28 \end{array}\right]$$

$$\xrightarrow{r_3/-28} \left[\begin{array}{ccc|c} 1 & 0 & 3.8 & -1.8 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{r_1 - 3.8r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\xrightarrow{r_2 + 0.4r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$\therefore \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$$

b). LU Decomposition

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 9 & 4 \\ 8 & 1 & 2 \end{bmatrix}$$

$$r_2 = r_2 - 2r_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 8 & 1 & 2 \end{bmatrix} \quad r_3 = r_3 - 8r_1 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & -15 & -22 \end{bmatrix}$$

$$r_3 = r_3 + 3r_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & -28 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 15 \end{bmatrix}$$

$y_1 = 1 \ldots$ ①
$2y_1 + y_2 = 9 \ldots$ ②
$8y_1 + (-3)y_2 + y_3 = 15 \ldots$ ③

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & -28 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 28 \end{bmatrix}$$

$x_1 + 2x_2 + 3x_3 = 1 \ldots$ ③'
$5x_2 - 2x_3 = 7 \ldots$ ②'
$-28x_3 = 28 \ldots$ ①'

① $y_1 = 1$

② $2(1) + y_2 = 9$
$2 + y_2 = 9$
$y_2 = 7$

③ $8(1) + (-3)(7) + y_3 = 15$
$8 - 21 + y_3 = 15$
$y_3 = 15 + 21 - 8$
$y_3 = 28$

①' $-28x_3 = 28$
$x_3 = -1$

②' $5x_2 - 2(-1) = 7$
$5x_2 + 2 = 7$
$5x_2 = 5$
$x_2 = 1$

③' $x_1 + 2(1) + 3(-1) = 1$
$x_1 + 2 - 3 = 1$
$x_1 - 1 = 1$
$x_1 = 2$

$\therefore \{(2, 1, -1)\}$

c). NO.
The coefficient of the system is not diagonally dominant. In this case, the diagonal elements are 1, 9, 2. However, the sum of the absolute value of the other elements in each row is greater than the absolute value of the diagonal element in the 2nd and 3rd row

d). Multiply each element by 2/9 in 2nd row and each element by 4 in the 3rd row

$$\begin{bmatrix} 1 & 2 & 3 \\ 4/9 & 5/9 & -8/9 \\ 32/9 & 4/9 & 8/9 \end{bmatrix}$$

∴ However the result will be different with the LU Decomposition results, due to the changes in the coefficients.

Task 2

1. $\begin{cases} 9x + 2y + 3z = 23 \\ 2x + 5y + z = 10 \\ 2x + 3y + 9z = 16 \end{cases}$

a). ① equation → 9 > sum of 2+3 = Dominant

② equation → 5 is not > sum of 2+1 = NOT Dominant

③ equation → 9 is > sum of 2+3 = Diagonally Dominant

∴ system is not Diagonally Dominant as a whole.

b). Gauss-seidel

Iteration 1

: $x = (23-2y-3z)/9 = 2.556$

$y = (10-2x-z)/5 = 1.4112$

$z = (16-2x-3y)/9 = 1.6131$

Iteration 2

: $x = (23-2y-3z)/9 = 2.7986$

$y = (10-2x-z)/5 = 1.3745$

$z = (16-2x-3y)/9 = 1.6486$

Iteration 3

: $x = (23-2y-3z)/9 = 2.8848$

$y = (10-2x-z)/5 = 1.3610$

$z = (16-2x-3y)/9 = 1.6697$

∴ $x = 2.8848$

$y = 1.3610$

$z = 1.6697$

Check ERROR %

$/|x(i)|$

↳ $\text{error} = \max(|x(i)-x(i-1)| / |y(i)-y(i-1)| / |y(i)| / |z(i)-z(i-1)| / |z(i)|) \times 100\%$

$\text{error} = \max(|2.8848-2.8672|/|2.8848|, |1.3610-1.3639|/|1.3610|, |1.6697-1.6651|/|1.6697|) \times 100\%$

$= \max(0.0061, 0.0021, 0.0027) \times 100\%$

$= 0.61\% < 1\%$

2. $\begin{cases} 0.1x + 2y + 3z = 5.2 \\ 2x + 0.5y + z = 5.5 \\ 2x + 3y + 0.2z = 7.2 \end{cases}$

a). Iteration 1:

$X = (5.2 - 2y - 3z) / 0.1 = -52y - 30z + 52$

$y = (5.5 - 2x - z) / 0.5 = -4x - 2z + 11$

$Z = (7.2 - 2x - 3y) / 0.2 = -10x/3 - 15y/2 + 36$

Iteration 2:

$x = (5.2 - 2y - 3z) / 0.1 = -52y - 30z + 52$

$Y = (5.5 - 2x - z) / 0.5 = -4x - 2z + 11$

$z = (7.2 - 2x - 3y) / 0.2 = -10x/3 - 15y/2 + 36$

b). conclusion, the values of x, y, and z does not change after the first iteration. Therefore, the system does not have a unique solution and the Gauss-Seidel method is considered to be not converge.

$\begin{cases} -2x + 4y - z = 1 \\ x - y + 3z = 3 \\ -4x + 2 - y = -3 \end{cases}$

$\left[\begin{array}{ccc|c} -2 & 4 & -1 & 1 \\ 1 & -1 & 3 & 3 \\ -4 & 2 & -1 & -3 \end{array}\right]$ $\quad |-2| \not\geq |4| + |-1|$, $\quad |-1| \not\geq |1| + |3|$, $\quad |-1| \not\geq |-4| + |2|$ $\quad \therefore$ not diagonally dominated

$r_{1 \leftrightarrow 3} \left[\begin{array}{ccc|c} -4 & 2 & -1 & -3 \\ 1 & -1 & 3 & 3 \\ -2 & 4 & -1 & 1 \end{array}\right]$

$r_{2 \leftrightarrow 3} \left[\begin{array}{ccc|c} -4 & 2 & -1 & -3 \\ -2 & 4 & -1 & 1 \\ 1 & -1 & 3 & 3 \end{array}\right]$ $\quad |-4| \geq |2| + |-1|$, $\quad |4| \geq |-2| + |-1|$, $\quad |3| \geq |1| + |-1|$

$\rightarrow$ diagonally dominated

$-4x + 2y - z = -3 \rightarrow x = \dfrac{-3 - 2y + z}{-4}$

$-2x + 4y - z = 1 \rightarrow y = \dfrac{1 + 2x + z}{4}$

$x - y + 3z = 3 \rightarrow z = \dfrac{3 - x + y}{3}$

$$x = \frac{(-3-2y+z)}{-4}, \quad y = \frac{(1+2x+z)}{4}, \quad z = \frac{(3-x+y)}{3}$$

1^{st} iteration,

$$x: \frac{-3-2(0)+0}{-4} = 0.75 \quad , e = \left|\frac{0.75-0}{0.75}\right| \times 100\% = 100\%.$$

$$y = \frac{1+2(0.75)+0}{4} = 0.625 \quad , e = \left|\frac{0.625-0}{0.625}\right| \times 100\% = 100\%.$$

$$z = \frac{3-0.75+0.625}{3} = 0.9583 \quad , e = \left|\frac{0.9583-0}{0.9583}\right| \times 100\% = 100\%.$$

2^{nd} iteration,

$$x = \cancel{-3(-2)(0.6} \quad \frac{-3-2(0.625)+0.9583}{-4} = \cancel{0.88}\ 0.8229 \quad , e = \left|\frac{0.8229-0.75}{0.8229}\right| \times 100\% = 8.86\%.$$

$$y = \frac{1+2(0.8229)+0.9583}{4} = 0.9010 \quad , e = \left|\frac{0.9010-0.625}{0.9010}\right| \times 100\% = 30.63\%.$$

$$z = \frac{3-0.8229+0.9010}{3} = 1.0260 \quad , e = \left|\frac{1.0260-0.9583}{1.0260}\right| \times 100\% = 6.60\%.$$

$$\begin{cases} -0.46x + y = 1.1 \\ -0.5x + y = 1 \end{cases}$$

* system has exact answer $\{(2.5, 2.5)\}$
* solve x and Y with Gauss-Elimination (2dp) without diagonally dominant (2dp)

$$\left[\begin{array}{cc|c} -0.46 & 1 & 1.1 \\ -0.5 & 1 & 1 \end{array}\right] \begin{array}{l} \rightarrow -0.46(2.5) + 1(2.25) = 1.1 \\ \rightarrow -0.5(2.5) + 1(2.25) = 1 \end{array}$$

$$r_2 - \frac{(-0.5)}{(-0.46)} r_1 \rightarrow \left[\begin{array}{cc|c} -0.46 & 1 & 1.1 \\ 0 & -0.09 & -0.20 \end{array}\right]$$

$-0.09y = -0.20 \rightarrow y = 2.22$

$-0.46x + 1(2.22) = 1.1$

$\hookrightarrow x = 2.43$

* Interchanging rows cannot make diagonally dominant matrix. We have to use <u>sealed row</u> ~~prio~~ <u>pivoting</u>

$$\left[\begin{array}{cc|c} -0.46 & 1 & 1.1 \\ -0.5 & 1 & 1 \end{array}\right] \begin{array}{l} \rightarrow \max\{|-0.46|, |1|\} = 1 \Rightarrow s = 1 \\ \rightarrow \max\{|-0.5|, |1|\} = 1 \rightarrow s = 1 \end{array}$$

$$\max\left\{\left|\frac{-0.46}{1}\right|, \left|\frac{-0.5}{1}\right|\right\} = 0.5 \rightarrow 2^{nd} \text{ row}$$

$$r_{1 \leftrightarrow 2} \left[\begin{array}{cc|c} -0.5 & 1 & 1 \\ -0.46 & 1 & 1.1 \end{array}\right] \begin{array}{l} s = 1 \\ s = 1 \end{array}$$

$$r_2 - \frac{(-0.46)}{(-0.5)} r_1 \sim \left[\begin{array}{c} \cancel{-0.5} \\ \cancel{-0.46} \end{array}\right] \left[\begin{array}{cc|c} -0.5 & 1 & 1 \\ 0 & 0.08 & 0.18 \end{array}\right]$$

$0.08y = 0.18 \Rightarrow y = 2.25$

$-0.5x + 1(2.25) = 1 \Rightarrow x = 2.5$

Example

$$\begin{cases} 2x - 2y + 6z = 16 \\ -2x + 4y + 3z = 0 \\ x - 8y - 4z = 1 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 6 & 16 \\ -2 & 4 & 3 & 0 \\ 1 & -8 & -4 & 1 \end{array}\right]$$ → cannot make diagonally dominant matrix. Use scaled row pivoting.

$\max\left[|2|, |-2|, |6|\right] = 6$

$\max\left[|-2|, |4|, |3|\right] = 4$

$\max\left[|1|, |-8|, |-4|\right] = 8$

$$\left[\begin{array}{ccc|c} 2 & -2 & 6 & 16 \\ -2 & 4 & 3 & 0 \\ 1 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} s=6 \\ s=4 \\ s=8 \end{array}$$

$\max : \left\{ \left|\frac{2}{6}\right|, \left|\frac{-2}{4}\right|, \left|\frac{1}{8}\right| \right\} = \frac{1}{2}$

2nd row

$$r_{1 \leftrightarrow 2} \left[\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ 2 & -2 & 6 & 16 \\ 1 & -8 & -4 & 1 \end{array}\right] \begin{array}{l} s=4 \\ s=6 \\ s=8 \end{array}$$

$$\begin{array}{l} r_2 + r_1 \\ r_3 + \frac{1}{2}r_1 \end{array} \left[\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ 0 & 2 & 9 & 16 \\ 0 & -6 & -2\frac{1}{2} & 1 \end{array}\right] \begin{array}{l} s=4 \\ s=6 \\ s=8 \end{array}$$

$\max \left\{ \left|\frac{2}{6}\right|, \left|\frac{-6}{8}\right| \right\} = \frac{6}{8}$

↑ 3rd row

$$r_{2 \leftrightarrow 3} \left[\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ 0 & -6 & -2\frac{1}{2} & 1 \\ 0 & 2 & 9 & 16 \end{array}\right] \begin{array}{l} s=4 \\ s=6 \\ s=8 \end{array}$$

$$r_3 + \frac{1}{3}r_2 \left[\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ 0 & -6 & -2\frac{1}{2} & 1 \\ 0 & 0 & 8\frac{1}{6} & 16\frac{1}{3} \end{array}\right]$$

$8\frac{1}{6}z = 16\frac{1}{3} \rightarrow z = 2$

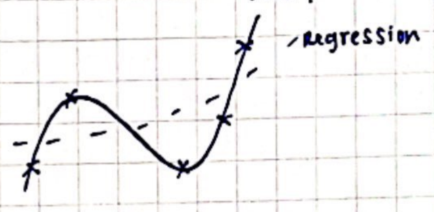
$-6y - 2\frac{1}{2}(2) = 1 \rightarrow y = -1$

$-2x + 4(-1) + 3(2) = 0$

$x = 1$

$\therefore \{(1, -1, 2)\}$

"Regression" & "Interpolation"



$2 \leq 2.3 \leq 2.5$ → Take $(2,8)$ $(2.5, 12.5)$

$x_1\ y_1 \quad x_2\ y_2$

$(2,8) \quad (2.5, 12.5)$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12.5 - 8}{2.5 - 2} = 9$

$y - y_1 = m(x - x_1)$

$y - 8 = 9(x - 2)$

$y - 8 = 9x - 18$

$y = 9x - 18 + 8$

$y = 9x - 10$

$y(2.3) = 9(2.3) - 10 = 10.7$

"Newton - Interpolation"

∇ = delta $\quad \delta$ = delta $\quad \Delta$ = delta

x	y	∇y
2	8	
2.5	12.5	9

$\frac{12.5 - 8}{2.5 - 2} = 9$

$y = 8 + 9(x - 2) = 9x - 10$

$y(2.3) = 9(2.3) - 10 = 10.7$

Newton Polynomial Method

EXAMPLE

Given

x	-4	3	-1	2	-5
y	27	6	6	3	38

a). Find the degree of the polynomial

b). Find the polynomial

c). Find $F(2.5)$

X	Y	ΔY	∇y^2	∇y^3	∇y^4
-4	27	.	.	.	
3	6	-3	.	.	
-1	6	-7	1	.	
2	3	-4	1	0	
-5	38	-11	1	0	

$\frac{6-27}{3-(-4)} = \frac{-21}{7} = -3$, $\frac{6-27}{-1-(-4)} = \frac{-21}{3} = -7$, $\frac{3-27}{2-(-4)} = \frac{-24}{6} = -4$

$\frac{38-27}{-5-(-4)} = \cancel{\frac{11}{7}}\ \frac{11}{-1} = -11$, $\frac{-7-(-3)}{-1-3} = \frac{-4}{-4} = 1$, $\frac{(-4)-(-3)}{2-3} = \frac{-1}{-1} = 1$

$\frac{-11-(-3)}{-5-3} = \frac{-8}{-8} = 1$, $\frac{1-1}{2-(-1)} = 0$, $\frac{1-1}{-5-(-1)} = 0$

a). NON-ZERO STOPS AT $\nabla y^2 \Rightarrow deg = 2$

b). $y = 27 - 3(x-(-4)) + 1(x-(-4))(x-3)$

$y = 27 - 3(x+4) + (x+4)(x-3) = 27 - 3x - 12 + x^2 + x - 12 = x^2 - 2x + 3$

c). $y(2.5) = (2.5)^2 - 2(2.5) + 3 = 4.25$

EXAMPLE

x	-2	3	-3	2	-4
y	-5	60	-36	19	-101

a). Find the degree of the polynomial

b). Find the polynomial

c). Find $f(2.5)$

X	Y	ΔY	∇y^2	∇y^3	∇y^4
-2	-5	.	.	.	.
3	60	13	.	.	.
-3	-36	31	-3	.	.
2	19	6	7	2	.
4	-101	48	-5	2	0

$\frac{60-(-5)}{3-(-2)} = \frac{65}{5} = 13$, $\frac{-36-(-5)}{-3-(-2)} = \frac{-31}{-1} = 31$, $\frac{19-(-5)}{2-(-2)} = \frac{24}{4} = 6$

$\frac{-101-(-5)}{-4-(-2)} = \frac{-96}{-2} = 48$, $\frac{31-13}{-3-3} = \frac{18}{-6} = -3$, $\frac{6-13}{2-13} = \frac{-7}{7} = 7$

$\frac{48-13}{-4-3} = \frac{35}{-7} = -5$, $\frac{7-(-3)}{2-(-3)} = \frac{10}{5} = 2$, $\frac{-5-(-3)}{-4-(-3)} = \frac{-2}{-1} = 2$

$\frac{2-2}{-4-2} = 0$

a). NON-ZERO STOPS AT $\nabla^3 y \longrightarrow deg = 3$

b). $y = -5 + 13(x-(-2)) - 3(x-(-2))(x-3) + 2(x-(-2))(x-3)(x-(-3))$

$y = -5 + 13(x+2) - 3(x+2)(x-3) + 2(x+2)(x-3)(x+3)$

$y = -5 + 13x + 26 - 3(x^2 - x - 6) + 2$

$(x+2)(x-3)(x+3)$

```
 1    2
 1   -3   x
-3   -6
  1    2    .
 --------------- +
  1   -1    6
  1    3
 --------------- x
  3   -3   18
  1   -1   -6    .
 --------------- +
  1    2   -9   -18
```

$y = -5 + 13x + 26 - 3(x^2 - x - 6) + 2(1x^3 + 2x^2 - 9x - 18)$

$y = 2x^3 + x^2 - 2x + 3$

c). $f(2.5) = y(2.5) = 2(2.5)^3 + (2.5)^2 - 2(2.5) + 3$

$= 35.5$

TASK 2

1).

$$\begin{cases} 9x + 2y + 3z = 23 \\ 2x + 5y + z = 10 \\ 2x + 3y + 9z = 16 \end{cases}$$

a). ① Equation → 9 > sum of 2+3 = Dominant <TRUE>

② Equation → 5 is not > sum of 2+1 = NOT Dominant <TRUE>

③ Equation → 9 is > sum of 2+3 = Diagonally Dominant <TRUE>

∴ system is Diagonally Dominant.

b). Gauss-Seidel

x = 0

y = 0 4 d.p.

z = 0

Iteration 1

ERROR %

$$x = \frac{23-2y-3z}{9} \qquad x = \frac{23(0)-2(0)-3(0)}{9} = 2.5556 \qquad e = \left|\frac{2.5556-0}{2.5556}\right| \times 100\% = 100\%$$

$$y = \frac{10-2x-z}{5} \qquad y = \frac{10-2(2.5556)-0}{5} = 0.9778 \qquad e = \left|\frac{0.9778-0}{0.9778}\right| \times 100\% = 100\%$$

$$z = \frac{16-2x-3y}{9} \qquad z = \frac{16-2(2.5556)-3(0.9778)}{9} = 0.884 \qquad e = \left|\frac{0.884-0}{0.884}\right| \times 100\% = 100\%$$

Iteration 2

ERROR %

$$x = \frac{23-2(0.9778)-3(0.884)}{9} = 2.0436 \qquad e = \left|\frac{2.0436-2.5556}{2.0436}\right| \times 100\% = 25.0503\%$$

$$y = \frac{10-2(2.0436)(0.884)}{5} = 1.0058 \qquad e = \left|\frac{1.0058-0.9778}{1.0058}\right| \times 100\% = \cancel{2.7623\%}\ 2.7838$$

$$z = \frac{16-2(2.0436)-3(1.0058)}{9} = 0.9884 \qquad e = \left|\frac{0.9884-0.884}{0.9884}\right| \times 100\% = 10.5625\%$$

Iteration 3

ERROR %

$$x = \frac{23-2(1.0058)-3(0.9884)}{9} = 2.0026 \qquad e = \left|\frac{2.0026-2.0436}{2.0026}\right| \times 100\% = 2.0473\%$$

$$y = \frac{10-2(2.0026)-(0.9884)}{5} = 1.0013 \qquad e = \left|\frac{1.0013-1.0058}{1.0013}\right| \times 100\% = 0.45\%$$

$$z = \frac{16-2(2.0026)-3(1.0013)}{9} = 0.999 \qquad e = \left|\frac{0.999-0.9884}{0.999}\right| \times 100\% = 1.0610\%$$

ERROR % ↓

Iteration 4

$$X = \frac{23-2(1.0013)-3(0.999)}{9} = 2$$

$$e = \left|\frac{2-2.0026}{2}\right| \times 100\% = 0.1300\%$$

$$y = \frac{10-2(2)-3(0.999)}{5} = 1.0002$$

$$e = \left|\frac{1.0002-1.0013}{1.0002}\right| \times 100\% = 0.11\%$$

$$Z = \frac{16-2(2)-3(1.0002)}{9} = 0.9999$$

$$e = \left|\frac{0.9999-0.999}{0.9999}\right| \times 100\% = 0.09\%$$

2.
$$\begin{cases} 0.1x + 2y + 3z = 5.2 \\ 2x + 0.5y + z = 5.5 \\ 2x + 3y + 0.2z = 7.2 \end{cases}$$

$$\left[\begin{array}{ccc|c} 0.1 & 2 & 3 & 5.2 \\ 2 & 0.5 & 1 & 5.5 \\ 2 & 3 & 0.2 & 7.2 \end{array}\right]$$

$$x = \frac{5.2-2y-3z}{0.1}$$

$$y = \frac{5.5-\cancel{0.5y}^{\,2y}-z}{0.5}$$

$$z = \frac{7.2-2x-3y}{0.2}$$

Initial Values!
x=0
y=0
z=0

▷ ITERATION NO.1

ERROR!

$$x = \frac{5.2-2(0)-3(0)}{0.1} = 52$$

$$e = \left|\frac{52-0}{52}\right| \times 100\% = 100\%$$

$$y = \frac{5.5-2(52)-0}{0.5} = -197$$

$$e = \left|\frac{-197-0}{-197}\right| \times 100\% = 100\%$$

$$z = \frac{7.2-2(52)-3(-197)}{0.2} = 2471$$

$$e = \left|\frac{2471-0}{2471}\right| \times 100\% = 100\%$$

▷ ITERATION NO.2

ERROR!

$$x = \frac{5.2-2(-197)-3(2471)}{0.1} = -7013.8$$

$$e = \left|\frac{-7013.8-52}{-7013.8}\right| \times 100\% = 100.7419\%$$

$$y = \frac{5.5-2(-7013.8)-2471}{0.5} = 1156.210$$

$$e = \left|\frac{1156.210-(-197)}{1156.210}\right| \times 100\% = 117.0389\%$$

$$z = \frac{7.2-2(-7013.8)-3(-1156.210)}{0.2} = 87517.15$$

$$e = \left|\frac{87517.15-2471}{87517.15}\right| \times 100\% = 97.1765\%$$

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- ITERATION 3

$$x = \frac{5.2 - 2(1156.210) - 3(87517.15)}{0.1} = -2648586.7$$

$$y = \frac{5.5 - 2(-2648586.7) - 87517.15}{0.5} = 10419323.5$$

$$z = \frac{7.2 - 2(2648586.7) - 3(10419323.5)}{0.2} = -182775683\cancel{5}.5$$

ERROR!

$$e = \left| \frac{-2648586.7 - (-7013.8)}{-2648586.7} \right| \times 100\% = 99.735\%$$

$$e = \left| \frac{10419323.5 - 1156.210}{10419323.5} \right| \times 100\% = 99.988\%$$

$$e = \left| \frac{-182775683.5 - 87517.15}{-182775683.5} \right| \times 100\% = 100.0478\%$$

- ITERATION 4

$$x = \frac{5.2 - 2(\cancel{1.04}\ 10419323.5) - 2(-182775683.5)}{0.1} = 3447127252$$

$$y = \frac{5.5 - 2(3447127252) - (-182775683.5)}{0.5} = -13422957630$$

$$z = \frac{7.2 - 2(\overset{3447127252}{\cancel{-13422957630}}) - 3(-13422957630)}{0.2} = 166873091966$$

ERROR!

$$e = \left| \frac{3447127252 - (-2648586.7)}{3447127252} \right| \times 100\% = 100.0768\%$$

$$e = \left| \frac{-13422957630 - 10419323.5}{-13422957630} \right| \times 100\% = 100.077\%$$

$$e = \left| \frac{166873091966 - (-182775683.5)}{166873091966} \right| \times 100\% = 100.1095\%$$

b). Conclusion!

Gauss-seidel method works when the matrix is considered to be diagonally dominant. :)

Newton Interpolation → Finding one curve that passes through all points given.
Regression → finding one curve that passes through "the mids" of all points given.

Example 4:

x	-2	2	4	1	0
y	-21	3	-117	3	3

a). Find the degree of the polynomial
b). Find the polynomial
c). Find $f(2.5)$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
-2	-21	•	•	•	•
2	3	6	•	•	•
4	-117	-16	-11	•	•
1	3	8	-2	-3	•
0	3	12	-3	-2	-1

$\frac{3-(-21)}{2-(-2)} = \frac{24}{4} = 6$, $\frac{-117-(-21)}{4-(-2)} = \frac{-96}{6} = -16$, $\frac{3-(-21)}{1-(-2)} = \frac{24}{3} = 8$, $\frac{3-(-21)}{0-(-2)} = \frac{24}{2} = 12$

$\frac{-16-6}{4-2} = \frac{-22}{2} = -11$, $\frac{8-6}{1-2} = \frac{2}{-1} = -2$, $\frac{12-6}{0-2} = \frac{6}{-2} = -3$, $\frac{-2-(-11)}{1-4} = \frac{9}{-3} = -3$

$\frac{-3-(-11)}{0-4} = \frac{8}{-4} = -2$, $\frac{-2-(-3)}{0-1} = \frac{1}{-1} = -1$

$\therefore$ a) deg = 4

b). $y = -21 + 6(x-(-2)) +$
$-11(x-(-2))(x-2) +$
$-3(x-(-2))(x-2)(x-4) +$
$-1(x-(-2))(x-2)(x-4)(x-1)$
$y = -x^4 + 2x^3 + x^2 - 2x + 3$

c). $f(2.5) = -(2.5)^4 + 2(2.5)^3 + (2.5)^2 - 2(2.5) + 3$
$= -3.5625$

KNOT	0	1	2	3	4
X	1	2	3	4	5
Y	0	1	0	1	0

minimum needed:
linear → 2 points
quadratic → 3 points
cubic → 4 points
quartic → 5 points

$x_0 = 1$, $x_1 = 2$, $x_2 = 3$, $x_3 = 4$, $x_4 = 5$
$y_0 = 0$, $y_1 = 1$, $y_2 = \cancel{1}\,0$, $y_3 = \cancel{0}\,1$, $y_4 = 0$

cubic spline interpolation → n subintervals whose length = h.

every subinterval → one cubic function

$i = 1, 2, \dots, n-1$

$k_0 = 0$, $k_n = 0$

$$k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1}) \text{ ---- (1)}$$

$$f(x) = \frac{(x_i - x)^3}{6h} k_{i-1} + \frac{(x - x_{i-1})^3}{6h} k_i + \frac{(x_i - x)}{h}\left[y_{i-1} - \frac{h^2}{6} k_{i-1}\right] +$$

$$+ \frac{(x - x_{i-1})}{h} \cdot \left[y_i - \frac{h^2}{6} k_i\right] \text{ ___ (2)}$$

Σ = sigma

σ = sigma = SD = Standard Deviation

$y = a + bx$

$$b = \frac{\Sigma xy - \bar{x}\Sigma y}{\Sigma x^2 - n\bar{x}^2}$$

$a = \bar{y} - b\bar{x}$

n = number of data

$$\sigma = \sqrt{\frac{S}{n-2}} = \sqrt{\frac{\Sigma(y - f(x))^2}{n-2}}$$

$\bar{x}$ = x bar = average of x

$\bar{y}$ = y bar = average of y

$a+b$	= a+b
$a-b$	= a-b
$a \times b$	= a*b
$\frac{a}{b}$	= a/b
a^2	= a^2
$\ln$	= ln()
Σx	= sum()
$\bar{x}$	= average ()
$\sqrt{x}$	= ~~SQRT(n)~~ SQRT(x)

$y = a \cdot e^{bx}$ → exponential

$\ln y = \ln(a \cdot e^{bx})$

$= \ln a + \ln e^{bx}$

$= \ln a + bx \cdot \ln e$

$\ln y = \ln a + bx$

$z = A + bx$ → linear

$$b = \frac{\Sigma xz - \bar{x}\Sigma z}{\Sigma x^2 - n \cdot \bar{x}^2}$$

$A = \bar{z} - b \cdot \bar{x}$

$\ln a = A$ → ${}^{e}\log a = A$

$a = e^A$

$a = \exp(A)$

Taylor Series

Sequence : ordered set of numbers ($T_1, T_2, T_3 \ldots T_n$).

Series : sum of all sequence up until n terms = $T_1 + T_2 + T_3 + \ldots T_n$.

sequence = $\{T_n\} = \{T_1, T_2, T_3, \ldots\}$

Series = $\sum_{n=1}^{\infty} T_n = T_1 + T_2 + T_3 + \ldots$

McLaurin Series $\rightarrow x = 0$

$$f(x) = f(0) + \frac{f'(0)}{1!}x^1 + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

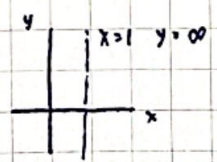
Taylor Series $\rightarrow x = a$.

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

$\therefore$ If $a = 0$ then, Taylor series becomes McLaurin series

$\{McL\} \subset \{Taylor\}$

$x = 1 \rightarrow y' = \infty$



$y = x^2 \sin x$

$y' = 2x \cdot \sin x + x^2 \cdot \cos x$

$y'' = 2\sin x + 2x\cos x + 2x\cos x + x^2(-\sin x)$

$y''' = ?$

$$y = x^2 \sin x = x^2\left[x - \frac{x^3}{3!} + \frac{x^5}{6!} - \frac{x^7}{7!} + \ldots\right]$$

$$= x^3 - \frac{x^5}{3!} + \frac{x^7}{5!} - \frac{x^9}{7!} + \ldots$$

$$y' = 3x^2 - \frac{5x^4}{3!} + \frac{7x^6}{5!} - \frac{9x^8}{7!} + \ldots$$

$$y'' = 6x - \frac{20x^3}{3!} + \frac{42x^5}{5!} - \frac{72x^7}{7!} + \ldots$$

1). $f(x) = \sin x$ put in Mc Lauren's Series (4 non zero terms).

$f(x) = \sin x \longrightarrow f(0) = \sin 0 = 0$
$f'(x) = \cos x \longrightarrow f'(0) = \cos 0 = 1$
$f''(x) = -\sin x \longrightarrow f''(0) = -\sin 0 = 0$
$f'''(x) = -\cos x \longrightarrow f'''(0) = -\cos 0 = -1$
$f^{(4)}(x) = \sin x \longrightarrow f^{(4)}(0) = \sin 0 = 0$
$f^{(5)}(0) = 1$
$f^{(6)}(0) = 0$
$f^{(7)}(0) = -1$
$f^{(8)}(0) = 0$
$f^{(9)}(0) = 1$

$$f(x) = \sin x = 0 + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 - \frac{1}{7!}x^7 + \dots$$

$$\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

2). $f(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

3). $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

4). $e^{3x} = 1 + \frac{3x}{1!} + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \frac{(3x)^4}{4!} + \dots$

5). $e^{-0.5x} = 1 + \frac{(-0.5x)}{1!} + \frac{(-0.5x)^2}{2!} + \frac{(-0.5x)^3}{3!} + \dots$

6). $e^{\ln \sin x} = \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

7). $x^3 e^x = x^3 \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)$

$= x^3 + \frac{x^4}{1!} + \frac{x^5}{2!} + \frac{x^6}{3!} + \dots$

8). $f(x) = \ln(x+1) \rightarrow$ Mc L. (4 non zero's terms)
$f(x) = \ln(x+1) \rightarrow f(0) = \ln 1 = 0$
$f'(x) = \frac{1}{x+1} = (x+1)^{-1} \longrightarrow f'(0) = 1^{-1} = 1$

$f''(x) = -1(x+1)^{-2} \rightarrow f''(0) = -1(1)^{-2} = -1 = -1!$
$f'''(x) = (-1)(-2)(x+1)^{-3} \longrightarrow f'''(0) = 2!(1)^{-3} = 2!$
$f^{(4)}(x) = (-1)(-2)(-3)(x+1)^{-4} \longrightarrow f^{(4)}(0) = -3!(1)^{-4} = -3!$

9). $f(x) = x^5 \ln(x+1) = x^5 \left[\frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots\right]$

$= \frac{x^6}{1} - \frac{x^7}{2} + \frac{x^8}{3} - \frac{x^9}{4} + \dots$

Taylor Series

→ A mathematical technique used in scientific computing to approximate a function as a polynomial series.

EX: $f(x) = \ln x$ Put in Taylor Series around $x = 1$
(4 non zero terms).

$f(x) = \ln x \rightarrow f(1) = \ln 1 = 0$

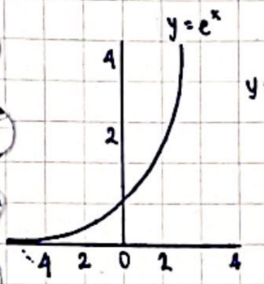
$f'(x) = \frac{1}{x} = x^{-1} \rightarrow f'(1) = 1^{-1} = 1$

$f''(x) = -1 \cdot x^{-2} \rightarrow f''(1) = -1(1)^{-2} = -1!$

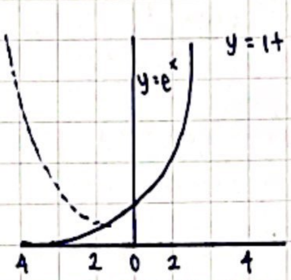
$f'''(x) = (-1)(-2)x^{-3} \rightarrow f'''(1) = 2!(1)^{-3} = 2!$

$f^{(4)}(x) = (-1)(-2)(-3)x^{-4} \rightarrow f^{(4)}(1) = -3!(1)^{-4} = -3!$

$$f(x) = 0 + \frac{1}{1!}(x-1) - \frac{1!}{2!}(x-1)^2 + \frac{2!}{3!}(x-1)^3 - \frac{3!}{4!}(x-1)^4 + \ldots$$



$y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}$



$y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

EX:

1. Given $f(x) = \ln(x)$
   a). Estimate $\ln(1.1)$ by 4^{th} order of Taylor series at $x = 1$.
   b). Find the maximum error.
   c). Find the actual error. (Answer NEXT PAGE).
   d). Compare B and C

$$f(x) = \ln x = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \underbrace{\ldots}_{\text{errors}}$$

$e^2 = 7.3891$

2). Approximate $\frac{1}{4.41}$ with Taylor Series (3 Non-zero terms). 4dp.

Actual value $\frac{1}{4.41} = 0.2267$

$\frac{1}{4.41} = \frac{1}{(2.1)^2} \longrightarrow f(x) = \frac{1}{x^2}$

$= \frac{1}{(2+0.1)^2} \longrightarrow$ around $x = 2$

$f(x) = x^{-2} \rightarrow f(2) = 2^{-2} = \frac{1}{4}$

$f'(x) = -2x^{-3} \rightarrow f'(2) = -2(2)^{-3}$

$f''(x) = (-2)(-3)x^{-4} \rightarrow f''(2) = 3!\,2^{-4}$

$f(x) = \frac{1}{x^2} = \frac{1}{4} - \frac{2(2)^{-3}}{1!}(x-2) + \frac{3!\,2^{-4}}{2!}(x-2)^2 + \ldots$

$f(x) = \frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 + \ldots$

$f(2.21) = \frac{1}{4} - \frac{1}{4}(2.1-2) + \frac{3}{16}(2.1-2)^2$

$= 0.226875$

≈ 0.2269

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Estimating Truncation Errors.

$F(x) = f_n(x) + E_n(x)$

$f_n(x)$ = Taylor series n terms

$E_n(x)$ = truncation error

$$E_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!} \cdot (x-a)^{n+1} \quad ; \quad a < z < x.$$

$$|E_n(x)| \leq \frac{M}{(n+1)!} \cdot (x-a)^{n+1} \quad ; \quad M = \text{max of } |f^{(n+1)}|$$

a). $\ln x = \frac{(x-1)}{1} - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$

$\ln 1.1 = \frac{(1.1-1)}{1} - \frac{(1.1-1)^2}{2} + \frac{(1.1-1)^3}{3} - \frac{(1.1-1)^4}{4}$

$= 0.09530833333$

b). $f(x) = \ln x \rightarrow$ at $x = 1$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -1!\, x^{-2}$

$f'''(x) = 2!\, x^{-3}$

$f^{(4)}(x) = -3! \cdot x^{-4}$

$\rightarrow f^{(5)}(x) = 4!\, x^{-5}$

$M = |4!\, x^{-5}| = 4!\, (1.1)^{-5}$

$|E_n(x)| \leq \frac{4!(1)}{5!} (1.1-1)^5 \leq \frac{(1)^5 (0.1)^5}{5} \leq \frac{0.00001}{5} = 0.000002$

c). Actual Error

$= |\text{actual value} - \text{approximation}|$

$= |\ln 1.1 - 0.09530833333|$

$= 0.00000184647 4325$

≈ 0.000002

d). They are nearly the same.

Task 5.

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Given $f(x) = \sqrt[3]{x}$. use 5 d.p.

a). Find Taylor series for $f(x)$ at $x = 8$ (4 Non-zero Terms).

b). What is the maximum error occured?

c). Approximate $\sqrt[3]{8.1}$ using part (a).

d). What is the actual error? compare (b) and (c).

a). $f(x) = \sqrt[3]{x}$

$\triangleright$ $x = 8$

$\therefore x^{\frac{1}{3}}$

$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \rightarrow 0.08333$

$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} \rightarrow -0.00694$

$f'''(x) = \frac{10}{27}x^{-\frac{8}{3}} \rightarrow 0.00144$

$f^{(4)}(x) = -\frac{80}{81}x^{-\frac{11}{3}} \rightarrow -0.00048$

$f(8) = 2$

$$f(x) = f(8) + \frac{f'(8)}{1!}(x-8) + \frac{f''(8)}{2!}(x-8)^2 + \frac{f'''(8)}{3!}(x-8)^3 + \dots$$

$$= 2 + \frac{0.08333}{1!}(x-8)^1 + \frac{-0.00694}{2!}(x-8)^2 + \frac{0.00144}{3!}(x-8)^3 + \frac{-0.00048}{4!}(x-8)^4$$

$$\rightarrow f(x) = 2 + 0.08333(x-8)^1 - \overset{0.00347}{\cancel{0.00698}}(x-8)^2 + 0.00024(x-8)^3 + \dots \cancel{-0.00002(x-8)^4}$$ "

b). $$|E_n(x)| \leq \frac{M}{(n+1)!} \cdot (x-a)^{n+1}$$

$M = |f^{(n+1)}(x)| = |f^{(4)}(x)|$

$= \left|-\frac{80}{81}x^{-\frac{11}{3}}\right|$

$= \left|-\frac{80}{81}(8)^{-\frac{11}{3}}\right|$

$= 0.00048$

$$\rightarrow |E_n(x)| \leq \frac{0.00048}{(3+1)!} \times (8.1-8)^{3+1} = 0.000000002$$

or 2×10^{-9} (max error)

c). $f(8.1) = 2 + 0.08333(x-8)^1 - 0.00347(x-8)^2 + 0.00024(x-8)^3 + \dots$

$= 2 + 0.08333(8.1-8) - 0.00347(8.1-8)^2 + 0.00024(8.1-8)^3 + \dots$

$= 2.00830$

d). $\sqrt[3]{8.1} = 2.00829$

≈ 2.00830

Actual Error : $2.00830 - 2.00830 = 0$

$\therefore$ Result of Taylor's series in inaccurate since 0 is < 0.000000002 or 2×10^{-9},

actual error < max error.

$f(x) = x^3 \sin x$

Put in McLaurin series (4 non-zero terms).

$f(x) = x^3 \sin x \longrightarrow f(0) = 0$

$f'(x) = 3x^2 \cdot \sin x + x^3 \cos x \longrightarrow f'(0) = 0$

$f''(x) = 6x \sin x + 3x^2 \cos x + 3x^2 \cos x + x^3(-\sin x) \longrightarrow f''(0) = 0$

↳ complicated !!!

$$f(x) : x^3\left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \right)$$

$$f(x) : \frac{x^4}{1!} - \frac{x^6}{3!} + \frac{x^8}{5!} - \frac{x^{10}}{7!} + \dots$$

$$\boxed{\begin{array}{l} e^x \\ \sin x \\ \cos x \end{array}}$$

Root Finding

$f(x) = 0 \longrightarrow$ find x_0 such that $f(x_0) = 0$

$x_0 =$ The root

$f(x) = x^2 - 3x + 2 = 0$ $\quad x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

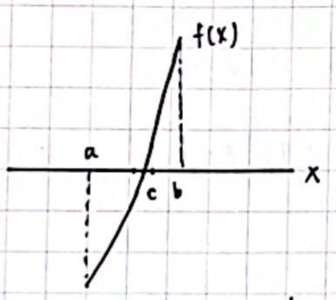
$$= \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2}$$

$$\left.\begin{array}{l} x_1 = \frac{3+1}{2} = 2 \\ x_2 = \frac{3-1}{2} = 1 \end{array}\right\} \text{Roots of } x^2 - 3x + 2 = 0$$

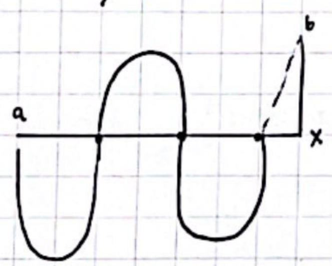
$x = \cos x \rightarrow$ NO Formula.

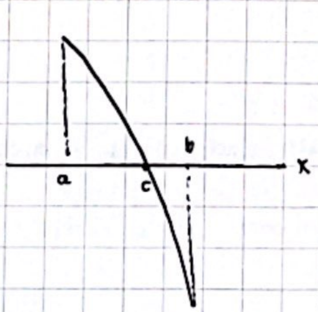
$f(x) = x - \cos x = 0$

Bisection Method



- $f(x)$ is continuous in $a < x < b$
- $f(a) * f(b) < 0 \rightarrow$ there is at least one.
- "c" such that $f(c) = 0$





Maximum number of evaluations in bisection algorithm is:

$$n = \frac{\ln(\Delta x / e)}{\ln 2}$$

Δx : Initial interval length
e : tolerance

* Bisection method is slow but sure.

Example:
$x = \cos(x)$
- Find a root between 0.7 and 0.8, with tolerance 0.0001 (4 dp)
- How many function evaluations at most are there?

Trigonometric Calculation → set the calculator to RAD mode !!!

Ans:

$f(x) = x - \cos x$

$$\left.\begin{array}{l} x_0 = 0.7 \rightarrow f(0.7) = -0.0648 = \ominus \rightarrow f(a) \\ [0.7, 0.8] \quad x_1 = 0.8 \rightarrow f(0.8) = 0.1033 = \oplus \rightarrow f(b) \end{array}\right\} f(a)\cdot f(b) < 0$$

$$x_2 = \frac{0.7+0.8}{2} = 0.75 \rightarrow f(0.75) = 0.0183 = \oplus$$

$$[0.7, 0.75] \rightarrow x_3 = \frac{0.7+0.75}{2} = 0.725 \rightarrow f(0.725) = -0.0235 = \ominus$$

$$[0.725, 0.75] \rightarrow x_4 = \frac{0.725+0.75}{2} = 0.7375 \rightarrow f(0.7375) = -0.0027 = \ominus$$

$$[0.7375, 0.75] \rightarrow x_5 = \frac{0.7375+0.75}{2} = 0.7438 \rightarrow f(0.7438) = 0.0078 = \oplus$$

$$[0.7375, 0.7438] \rightarrow x_6 = \frac{0.7375+0.7438}{2} = 0.7406 \rightarrow f(0.7406) = 0.0026 = \oplus$$

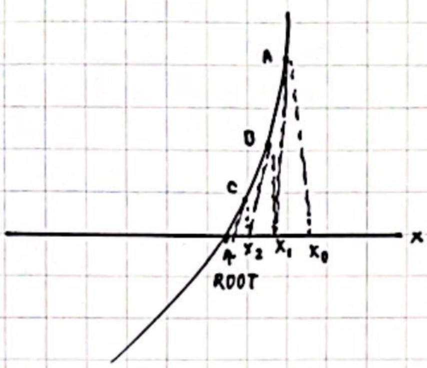
$$[0.7375, 0.7406] \rightarrow x_7 = \frac{0.7375+0.7406}{2} = 0.7391 \rightarrow f(0.7391) = 0.0000 \leq e$$

STOP!

∴ The root is 0.7391

Maximum evaluation $n = \dfrac{\ln\left(\frac{0.8-0.7}{0.0001}\right)}{\ln 2} = 9.96 \approx 10$ evaluations.

Newton - Raphson method



$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Pitfalls / Drawbacks

Ex! $x = \cos x$

$f(x) = x - \cos x = 0$

$f'(x) = 1 - (-\sin x) = 1 + \sin x$

$x_0 = 2$ $\quad f(2) = 2 - \cos 2 = 2.4161$

$f'(2) = 1 + \sin 2 = 1.9093$

$$x_1 = 2 - \frac{2.4161}{1.9093} = 0.7345$$

$\rightarrow f(0.7345) = 0.7345 - \cos(0.7345)$

$= -0.0076$

$f'(0.7345) = 1 + \sin(0.7345)$

$= 1.6702$

$$x_2 = 0.7345 - \frac{(-0.0076)}{1.6702} = 0.7391 \rightarrow f(0.7391) = 0.0000 \leq e = 0.0001$$

stop!

$\therefore$ The Root is 0.7391

$f(x) = x^3 - 3x^2 - x + 3 = 0$

$f'(x) = 3x^2 - 6x - 1$

$x_0 = -4 \rightarrow f(-4) = -64 - 48 + 4 + 3 = -105$

$f'(-4) = 48 + 24 - 1 = 71$

$$x_1 = -4 - \frac{(-105)}{71} = -2.5211 \rightarrow f(-2.5211) = -29.5716$$

$f'(-2.5211) = 33.1950$

$$x_2 = -2.5211 - \frac{(-29.5716)}{33.1950} = -1.6303 \rightarrow f(-1.6303) = -7.6762$$

$f'(-1.6303) = 16.7551$

$$x_3 : -1.6303 - \frac{(-7.6762)}{16.7551} = -1.1721$$

etc.... :)

TASK 6.

1.

a). $\ln y = \ln a + x \cdot \ln b$

$\ln y = \ln(a \cdot b^x)$

$\ln y = \ln a + \ln(b^x)$

$\ln y = \ln a + x^{\ln b}$

mean(lny) = 1.6336
mean(lnx) = 1.465

x	1	3	4	5	6
y	50	12.5	6.2	3.1	0.8

ln y →	3.912	2.525	1.823	1.181	−0.223
ln x →	0.000	1.099	1.386	1.609	1.792

cov → covariance

slope : cov(lny, lnx) ÷ Var(lnx)

intercept : mean (ln y) − slope × mean (lnx)

$\text{cov}(\ln y, \ln x) = \frac{1}{5} \times [(3.912 - 1.465)$

DRAFT of TASK 6

Numerical Differentiation

1. only for evenly spaces of 'x'
2. Function is unknown
3. There are 5 methods.

Taylor Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

x is replaced by x + h

a is replaced by x

$$f(x+h) = f(x) + \frac{f'(x)}{1!}(x+h-x) + \frac{f''(x)}{2!}(x+h-x)^2 + \frac{f'''(x)}{3!}(x+h-x)^3 + \dots$$

$$f(x+h) = f(x) + \frac{f'(x)}{1!}(h) + \frac{f''(x)}{2!}(h)^2 + \frac{f'''(x)}{3!}(h)^3 + \dots$$

(1) $f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \dots$

(2) $f(x-h) = f(x) - \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \dots$

(1) − (2) $\rightarrow$ $f(x+h) - f(x-h) = \frac{2f'(x)}{1}h + \frac{2f'''(x)}{3!}h^3 + \dots$

$f(x+h) - f(x-h) = 2f'(x)\cdot h$ (neglect the rest).

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

(1) + (2) $\rightarrow$ $f(x+h) + f(x-h) = 2f(x) + \frac{2f''(x)}{2!}h^2 + \dots$

$f(x+h) + f(x-h) = 2f(x) + f''(x)\cdot h^2$ (neglect the Rest).

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

① Central Difference Approximation $O(h^2)$

$$2hf'(x) = -1\cdot f(x-h) + 0\cdot f(x) + 1\cdot f(x+h)$$

$$\boxed{f'(x) = \frac{-f(x-h) + f(x+h)}{2h}}$$

$$h^2 f''(x) = 1\cdot f(x-h) - 2f(x) + 1\cdot f(x+h)$$

$$\boxed{f''(x) = \frac{f(x-h) - 2f(x) + f(x+h)}{h^2}}$$

②. First Forward Difference Approximation $(O(h))$

$$\boxed{f'(x) = \frac{-1\cdot f(x) + 1\cdot f(x+h)}{h}} \qquad \boxed{f''(x) = \frac{1\cdot f(x) - 2f(x+h) + 1\cdot f(x+2h)}{h^2}}$$

③. First Backward Difference Approximation $(O(h))$

$$\boxed{f'(x) = \frac{-1\cdot f(x-h) + 1\cdot f(x)}{h}} \qquad \boxed{f''(x) = \frac{1\cdot f(x-2h) - 2f(x-h) + 1\cdot f(x)}{h^2}}$$

④. Second Forward Difference Approximation $O(h^2)$

$$\boxed{f'(x) = \frac{-3f(x) + 4f(x+h) - 1f(x+2h)}{2h}} \qquad \boxed{f''(x) = \frac{2f(x) - 5f(x+h) + 4f(x+2h) - f(x+3)}{h^2}}$$

⑤. Second Backward Difference Approximation $O(h^2)$

$$\boxed{f'(x) = \frac{1\cdot f(x-2h) - 4f(x-h) + 3f(x)}{2h}} \qquad \boxed{f''(x) = \frac{-1\cdot f(x-3h) + 4f(x-2h) - 5f(x-h) + 2f(x)}{h^2}}$$

Past UTS S-comp

1. A known system of linear Equation:

$$\begin{cases} 2x - 3y + 21z = 22 \\ 21x + y - 3z = 40 \\ 3x + 21y - z = 26 \end{cases}$$

a). Change the order of the system so that the diagonal elements are dominant.
b). Use Gauss-Seidel iteration with initial value $\{(0,0,0)\}$, 3 iterations (4 dp) to find the solution for the system.
c). Are the results Convergent (explain).

Answers:

ⓐ.

$$\begin{aligned} 2x - 3y + 21z &= 22 \\ 21x + y - 3z &= 40 \\ 3x + 21y - z &= 26 \end{aligned} \quad \Bigg| \quad \begin{aligned} 21x + y - 3z &= 40 \\ 3x + 21y - z &= 26 \\ 2x - 3y + 21z &= 22 \end{aligned} \qquad \begin{aligned} |1| + |-3| &\le |21| \\ |3| + |-2| &\le |21| \\ |2| + |-3| &\le |21| \end{aligned}$$

$\Rightarrow$ Diagonally Dominant.

1st

$$x = \frac{40 - y + 3z}{21} \qquad x = \frac{40 - 0 + 3(0)}{21} = 1.9048$$

$$y = \frac{26 - 3x + z}{21} \qquad y = \frac{22 - 3(\ ) + 0}{21} = 0.9660$$

$$z = \frac{27 - 2x + 3y}{21} \qquad z = \frac{22 - 2(\ ) + 3(\ )}{21} = 1.0042$$

2nd (use Excel faster)

$x = 2.0022$
$y = 0.9999$
$z = 0.9998$

3rd (use excel)

$x = 2.0000$
$y = 1.0000$
$z = 1.0000$

2. Laboratory experiments are given in the table below.

Ni phase cair, a (gr/L)	2	2.5	3
Ni phase organik, g (gr/L)	8.57	10	12

Assume that a is the amount of 'Ni' in the liquid phase, and 'g' is the amount of 'Ni' in the organic phase. Quadratic interpolation is used to estimate the value of 'g', which is given by the following formula:

$$g = x_1 a^2 + x_2 a + x_3$$

a). Find three simultaneous equations based on the data given by the experimental results.

b). Use the Gauss-Seidel Elimination method to get the value of the x_1 and x_2 and x_3 and then estimate the amount of 'Ni' in the organic phase, if 2.3 g/L of 'Ni's available in the liquid phase.

c). Use the LU Decomposition method to get the values of x_1, x_2, x_3 and then estimate the amount of the 'Ni' in the organic phase, if 2.3 g/L of 'Ni' is available in the liquid phase.

ANSWER:

a). $g = x_1 a^2 + x_2 a + x_3$

$8.57 = x_1(2)^2 + x_2(2) + x_3 \rightarrow 4x_1 + 2x_2 + x_3 = 8.57$

$10 = x_1(2.5)^2 + x_2(2.5) + x_3 \rightarrow 6.25x_1 + 2.5x_2 + x_3 = 10$

$12 = x_1(3)^2 + x_2(3) + x_3 \rightarrow 9x_1 + 3x_2 + x_3 = 12$

b). $$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 8.57 \\ 6.25 & 2.5 & 1 & 10 \\ 9 & 3 & 1 & 12 \end{array}\right] \begin{array}{l} r_2 - \frac{6.25}{4} r_1 \\ r_3 - \frac{9}{4} r_1 \end{array} \sim \left[\begin{array}{ccc|c} 4 & 2 & 1 & 8.57 \\ 0 & -0.625 & -0.5625 & -3.3906 \\ 0 & -1.5 & -1.25 & -7.2825 \end{array}\right] r_3 - \frac{(-1.5)}{(-0.625)} r_2$$

$$\left[\begin{array}{ccc|c} 4 & 2 & 1 & 8.57 \\ 0 & -0.625 & -0.5625 & -3.3906 \\ 0 & 0 & 0.1 & 0.855 \end{array}\right]$$

$0.1 x_3 = 0.855$

$x_3 = 8.55$

$-0.625 x_2 - 0.5625(8.55) = -3.3906$

$x_2 = -0.37$

$4x_1 + 2(-0.37) + 1(8.55) = 8.57$

$x_1 = 0.19$

$g(2.3) = 0.19 \cdot (2.3)^2 + (-0.37)(2.3) + 8.55$

$= 1.0051 - 0.851 + 8.55$

$= 8.7041$ gr/l