Linear Equations

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Gauss Jordan

•) pivot must be non - 0

Scaled Row Pivoting + Gauss Jordan / LU Decomp.

1.) Get absolute first value of each row

$$\left(\begin{array}{ccc|c} \underline{2} & -2 & 6 & 16 \\ \underline{-2} & 4 & 3 & 0 \\ \underline{1} & -8 & 1 & 1 \end{array}\right) \begin{array}{l} \to |2| \\ \to |-2| \\ \to |1| \end{array}$$

2.) Get Scale of each row

Scale = absolute biggest value of each row

$$\left(\begin{array}{ccc|c} 2 & -2 & \underline{6} & 16 \\ -2 & \underline{4} & 3 & 0 \\ 1 & \underline{-8} & 1 & 1 \end{array}\right) \begin{array}{l} \to |6| \\ \to |4| \\ \to |-8| \end{array}$$

3.) Divide the results of (1) by (2)

$$\left\{ \frac{2}{6} ; \frac{2}{4} ; \frac{1}{8} \right\} \longrightarrow \left\{ \frac{1}{3} ; \frac{1}{2} ; \frac{1}{8} \right)$$

4.) Find the index of the biggest value

$$\left\{ \frac{1}{3} ; \boxed{\frac{1}{2}} ; \frac{1}{8} \right)$$

1 2 3

5.) Swap the index with the first row

1 <-> 2 row

$$\left(\begin{array}{ccc|c} 2 & -2 & 6 & 16 \\ -2 & 4 & 3 & 0 \\ 1 & -8 & 1 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} -2 & 4 & 3 & 0 \\ 2 & -2 & 6 & 16 \\ 1 & -8 & 1 & 1 \end{array}\right)$$

6.) Solve with gauss / LU decomposition

LU Decomposition

$A = L.U$

$Ax = B$

$LUX = B$

$UX = Y$

$LY = B$

Solve Y

Solve X with $UX = Y$

- Doesn't need to be diagonally dominant
- A Needs to be a square matrix

Gauss Seidel

- Iterative method
- Needs to be diagonally dominant
- Breaks if 1 value is too big/small

$$E = \frac{Value - PrevValue}{value} \times 100\%$$

$$\begin{pmatrix} 5 & -2 & 3 \\ -3 & 9 & 1 \\ 2 & -1 & -7 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

1.) Find the equation for x, y, and z

$$x = \frac{2y + 3z - 1}{5} \qquad y = \frac{3x - z + 2}{9} \qquad z = \frac{-2x + y + 3}{-7}$$

2.) Get initial values — biasanya dari soal

$\{0, 0, 0\}$

3.) Start iteration

it 1: $x = \frac{-1}{5} = -\frac{1}{5}$ $\boxed{\{-\frac{1}{5}, 0, 0\}}$ note: Use latest x, y, z

$y = \frac{3\cdot(-\frac{1}{5}) - 1}{5} = \frac{3}{25}$ $\boxed{\{-\frac{1}{5}, \frac{3}{25}, 0\}}$

$z = \frac{-2(-\frac{1}{5}) + (\frac{3}{25}) + 3}{7} \approx \frac{1}{2}$ $\boxed{\{-\frac{1}{5}, \frac{3}{25}, \frac{1}{2}\}}$

4.) Get errors for each value of x, y, z

$E_x = \frac{-\frac{1}{5} - 0}{-\frac{1}{5}} \cdot 100\% = \boxed{100\%}$

$E_y = \frac{\frac{3}{25} - 0}{\frac{3}{25}} \cdot 100\% = \boxed{100\%}$

$E_z = \ldots\ldots = \boxed{100\%}$

Stop if all errors are less than acceptance error

usually 12%

Regression & Interpolation

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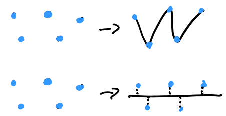
Regression vs Interpolation

Interpolation → hits all the dots

Regression → tries to estimate the next dot

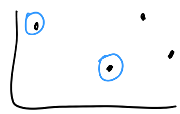


Interpolation
- Linear
- Newton
- Cubic spline

Regression Line Types
- linear
- exponential

Methods of Getting Regression Line — Least Square Regression

Linear Interpolation



example

find $x = 2,3$

points: $\{(2,8);(2,5, 12,5)\}$

1.) take 2 nearest dots

2.) get gradient

$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1}} = \frac{4,5}{0,5} = 9$$

3.) get the line

$$\boxed{y - y_1 = m(x - x_1)}$$

$y - 8 = 9 \cdot (x - 2)$

$y - 8 = 9x - 18$

$y = 9x - 10$

$f(2,3) = 9(2.3) - 10$

$= 10.7$

Newton Interpolation

Used to find a single function which interpolates all the points

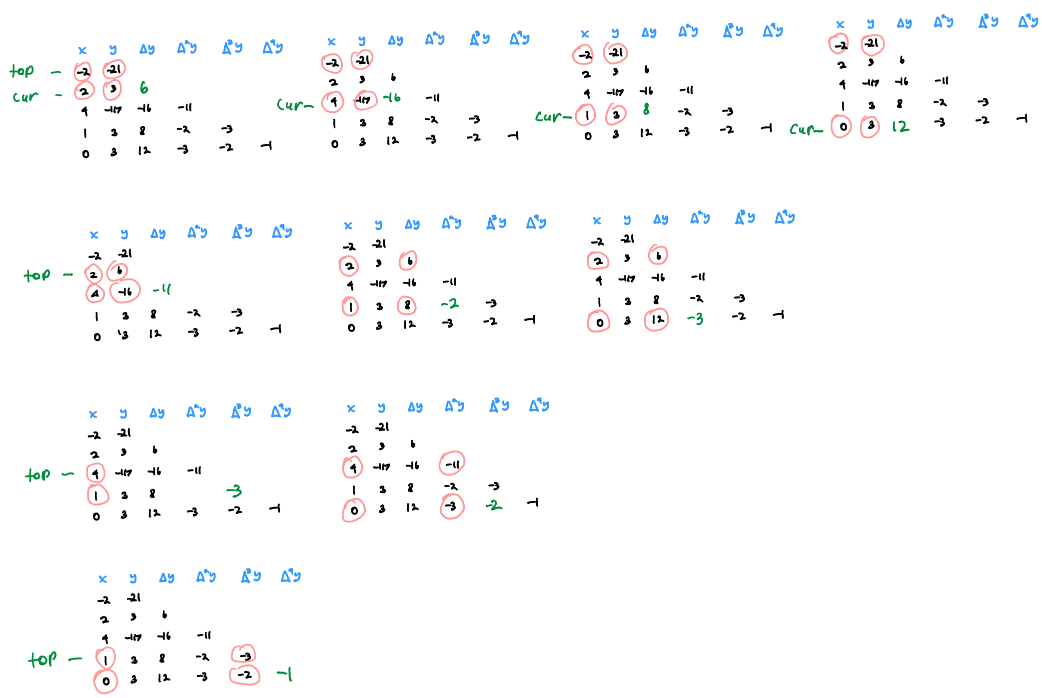
x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2	-21				
2	3	6			
4	-117	-16	-11		
1	3	8	-2	-3	
0	3	12	-3	-2	-1

→ Latest non-zero column is the degree

↓ 4th degree polynomial

1.) get all the values until $\Delta^n y = 0$

$$\boxed{\text{Value} = \frac{\text{cur Column } Y - \text{top Column } Y}{\text{cur Column } X - \text{top Column } X}}$$



2.) Get all the important values

for = ... $0 \quad 3 \quad 12 \quad -3 \quad -2 \quad -1$

2.) Get all the important values

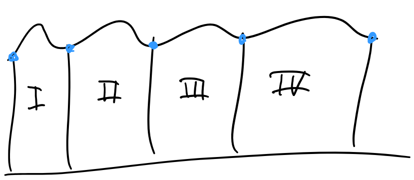
x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
-2 (x_1)	-21 (b_1)				
2 (x_2)	3	6 (b_2)			
4 (x_3)	-117	-16	-11 (b_3)		
1 (x_4)	3	8	-2	-3 (b_4)	
0 (x_5)	3	12	-3	-2	-1 (b_5)

3.) Insert Values to Formula

$$\boxed{f(x) = b_1 + b_2(x-x_1) + b_3(x-x_1)(x-x_2) + b_4(x-x_1)(x-x_2)(x-x_3)}$$

Cubic Spline Interpolation

- for interpolating n points
- Not a single function line newton's interpolation, but consists of multiple low degree functions (max degree = 3)
- X values must have the same interval $\boxed{\Delta x = h}$



$\boxed{\begin{array}{l} n = 5 \text{ points} \\ n-1 = 4 \text{ sub intervals} \end{array}}$

K_i = knot of index i $\quad \boxed{K_0 = 0 \quad K_{n-1} = 0}$

$$(1) \quad K_{i-1} + 4K_i + K_{i+1} = \frac{6}{h^2}\left(y_{i-1} - 2y_i + y_{i+1}\right)$$

$$(2) \quad f(x) = \frac{(x_i - x)^3}{6h} \cdot K_{i-1} + \frac{(x - x_{i-1})^3}{6h} K_i + \frac{(x_i - x)}{h} \cdot \left(y_{i-1} - \frac{h^6}{6} \cdot K_{i-1}\right) + \frac{(x - x_{i-1})}{h}\left(y_i - \frac{h^2}{6} K_i\right)$$

1.) Find all values of knots using formula (1)

- •) 5 points
- •) Find $K_0 - K_4$
- •) Substitute $i=1, i=2, i=3$ to (1) → gives 3 linear equations
- •) Solve all linear equations to get $K_0 - K_4$ → $K_0 = 0$

•) Substitute $i=1, i=2, i=3$ to (1) → gives 3 linear equations

•) Solve all linear equations to get $K_0 - K_4$ → $K_0 = 0$, $K_4 = 0$

2.) use the corresponding f(x) at i and i-1 and plug in the f(x)

Example

$x = 2.5$

index	0	1	2	3	4
x	1	2	3	4	5
y	0	1	0	1	2

x is between this interval → input K_1 & K_2 into (2) to get f(x)

↓

get the value of f(2.5)

Regression & Interpolation 2

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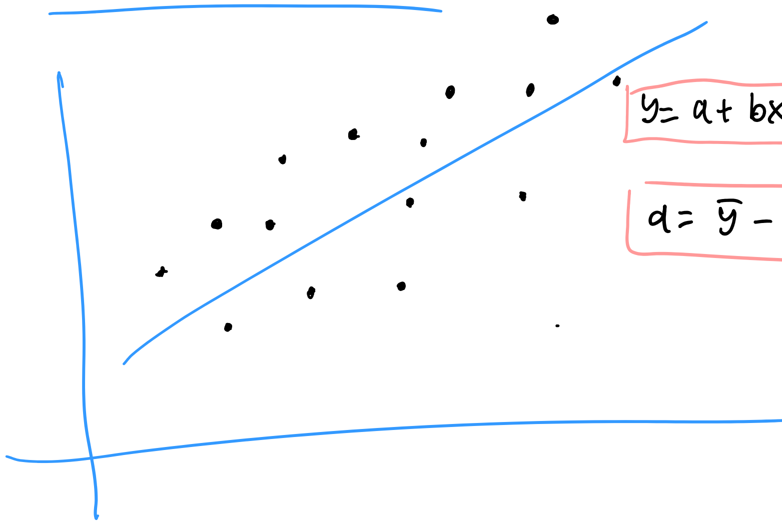
Linear Regression



$$\boxed{y = a + bx}$$

$$\boxed{a = \bar{y} - b\bar{x}}$$

$$\boxed{b = \frac{\Sigma(x.y) - \bar{x}.\Sigma y}{\Sigma x^2 - n.\bar{x}}}$$

Errors

$$\boxed{s = \Sigma(y - f(x))^2}$$

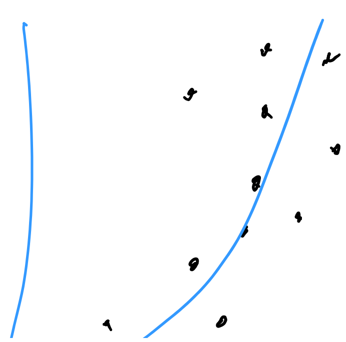
SSE

Diff between the actual y value and predicted y value

$$\boxed{\sigma = \sqrt{\frac{s}{n-2}}}$$

Exponential Regression

↳ no formula, so convert to linear regression first



$$\boxed{y = a.e^{bx}}$$

$$y = a.e^{bx}$$

$$\Downarrow$$

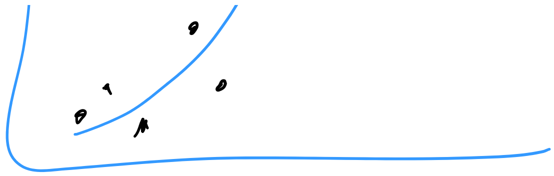
$$\ln(y) = \ln(a.e^{bx}) = \ln(a) + \ln(e^{b.x})$$

$$\underbrace{\ln y}_{} = \underbrace{\ln a}_{} + bx$$

$$\boxed{z = A + bx}$$

$$\boxed{\begin{aligned} z &= \ln y \\ A &= \ln a \end{aligned}}$$

$$b = \frac{\Sigma(x z) - \bar{x}.\Sigma z}{}$$



Steps:

1.) Find b and A

2.) find a from $\boxed{A = \ln a}$

3.) find the f(x)

$$b = \frac{\sum(xz) - \bar{x} \cdot \sum z}{\sum x^2 - n \cdot \bar{x}^2}$$

$$A = \bar{z} - b \cdot \bar{x}$$

Taylor Series & McLaurin Series

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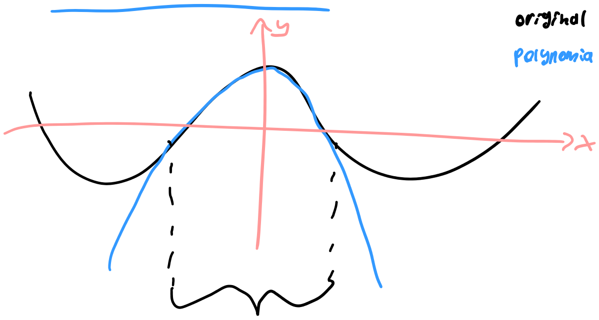
Taylor Series

original function

polynomial function generated by taylor series along $x=0$



taylor polynomial is only accurate along $x=0$

Note: the more terms you add, the more accurate the function is

Taylor Series along $x=a$

$$\boxed{f(x) = f(a) + f(a)' \cdot (x-a) + \frac{f(a)''}{2!}(x-a)^2 + \frac{f(a)'''}{3!}(x-a)^3 + \dots}$$

Remainder

$$\boxed{R_n = \int_a^x \frac{f^{(n+1)}(t)}{(n+1)!} \cdot (x-t)^{n+1} \cdot dt}$$

Errors
- Truncation error $\rightarrow$ error caused by not adding enough terms
- Rounding error

Estimating Truncation Error

- Denoted as $E_n(x)$

$$\boxed{|E_n(x)| \leq \frac{|f^{n+1}|}{(n+1)!} \cdot (x-a)^{n+1}}$$

$|E_n(x)| \leq$ the next term

$|E_n(x)| \lesssim$ the next term

Mc Laurin Series

— Taylor series but evaluated at $\boxed{x=0}$

$$\boxed{f(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} \cdot x^3 + \frac{f^4(0)}{4!} \cdot x^4}$$

Root of Equations

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Roots of Equation

1. $x^2 - 3x + 2 = 0 \rightarrow \dfrac{-b \pm \sqrt{-4ac}}{2a}$

2. $x = \cos x$

   $\left.\begin{matrix} y = x \\ y = \cos x \end{matrix}\right\}$ find the intersection $\rightarrow$ intersection at 0.74, root is at $x = 0.74$

3. Other methods $\rightarrow$
   - bisection
   - Newton Raphson
   - False Position
   - Secant - method
   - Fixed Point
   - Steffensen
   - Brent

   } iterative

in this course < bisection, Newton Raphson

criteria for stopping: error $\leq$ tolerance

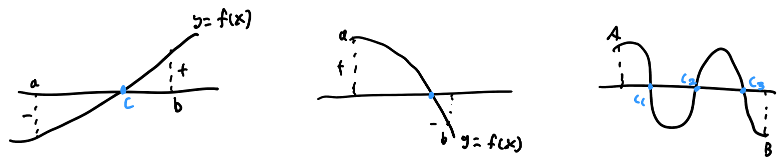
Bisection Method

$\left.\begin{matrix} y = f(x) \\ y = 0 \end{matrix}\right\}$ $f(x) = 0 \rightarrow$ find x (root)

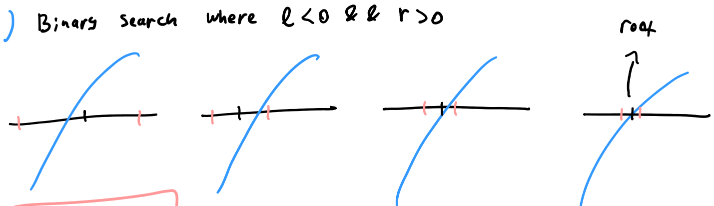
if $y = f(x)$ is continuous at $a < x < b$
and $f(a) \cdot f(b) < 0$
then
there's at least one root of x

Maximum Number of Evaluations

$$\boxed{n = \frac{\ln\left(\frac{\Delta x}{\varepsilon}\right)}{\ln 2}}$$ $\rightarrow$ Round n up



- One side must be (+) and the other needs to be (−)
- Binary search where $l < 0$ && $r > 0$



$$\boxed{m = \frac{l + r}{2}}$$

- target: $f(mid) = 0 \rightarrow$ stop when $f(mid) < \varepsilon$

•) target : $f(mid) = 0$ → stop when $f(mid) < \varepsilon$

Steps

1.) find the initial values of $f(l)$ and $f(r)$

2.) get the mid $\boxed{m = (l+r)/2}$ and $f(mid)$

3.) if $f(mid) < \varepsilon$, stop there. mid is the root

if $f(mid)$ is (+), change right = mid

if $f(mid)$ is (−), change left = mid

Example

$x - \cos(x) = 0$

Find a root between 0.7 and 0.8 , $\varepsilon = 0.0001$

How many function evaluations at most are there?

$f(x) = x - \cos(x) = 0$ $\quad 0.7 \leq x \leq 0.8$

$\varepsilon = 0.0001$

4 d.p.

Set Calculator to Radian Mode

not degree

---

x_0: $f(0.7) = -0.0648$

x_1: $f(0.8) = 0.1033$

$(0.7, 0.8) \rightarrow x_2 = 0.75 \rightarrow f(x) = 0.0183$ (+) → Change right

$(0.7, 0.75) \rightarrow x_3 = 0.725 \rightarrow f(x) = -0.0235$ (−) → change left

$(0.725, 0.75) \rightarrow x_4 = 0.7375 \rightarrow f(x) = -0.0027$ (−) → Change left

$(0.7375, 0.75) \rightarrow x_5 = 0.7438 \rightarrow f(x) = 0.0078$ (+) → Change right

$(0.7375, 0.7438) \rightarrow x_6 = 0.7406 \rightarrow f(x) = 0.0026$ (+) → Chang right

$(0.7375, 0.7406) \rightarrow x_7 = \boxed{0.7391} \rightarrow f(x) = 0.0000 \rightarrow 0.000 < \varepsilon$

Root

stop here

8 evaluations

Newton - Raphson

•) Find where $\boxed{y = 0}$

•) Starts with one point

•) Faster than Bisection but less accurate

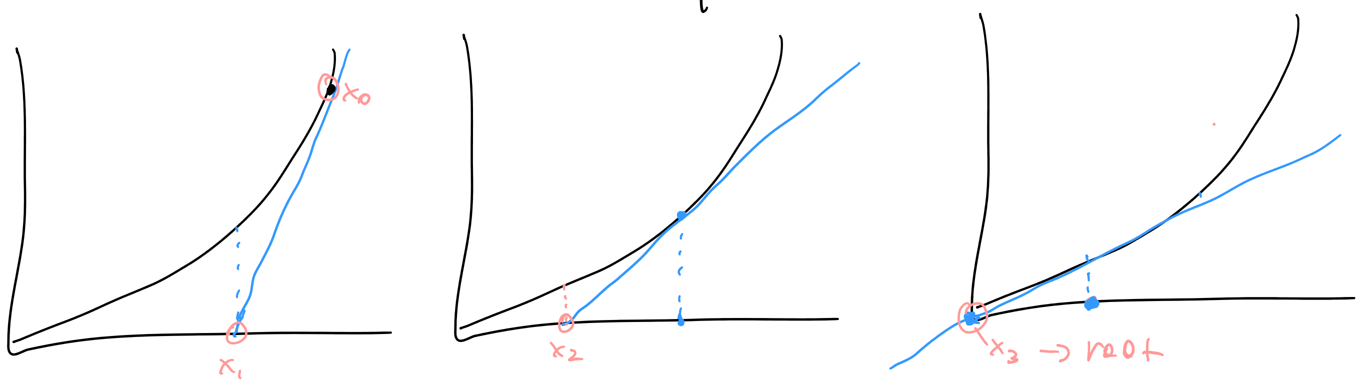
find the tangent line

→

find the intersection at $x = 0$

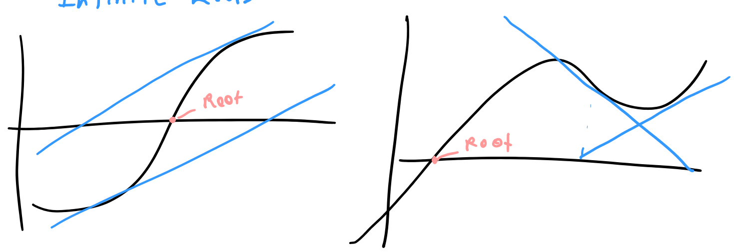
$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

•) Faster than Bisection but less accurate | at $x=0$



•) May fail sometimes (infinite loops)

Infinite Loops



Next X

$$\boxed{X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)}}$$

Truncation Error

$$\boxed{E_{n+1} = -\frac{f''(X_n)}{2f'(X_n)}(E_n)^2}$$

Steps

1.) Find values for X_{n+1}

2.) get truncation error for that X_n

3.) Stop if $|X_{n+1} - X_n| < \varepsilon$ or $|f(X_{n+1})| \leq \epsilon$
   ↳ X_{n+1} is the root

Example

$f(x) = x - \cos x = 0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = x - \cos x = 0$

$f'(x) = 1 + \sin x = 0$

$x_0 = 2$

$\varepsilon = 0.0001$ (4 d.p.)

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

$x_0 = 2 \rightarrow f(2) = 2.4161$

$f'(2) = 1.9093$

$x_1 = 0.7345 \rightarrow f(0.7345) = -0.0076$

$f'(0.7395) = 1.6702$

$x_3 = \boxed{0.7391} \rightarrow f(0.7391) = 0.0000 < 0.0001$

$f'(0.7391) = 1.6736$

stop

root

3 evaluations