Gauss Seidel & Gaussian Elimination

- Gaussian Elimination
  - ↳ A process where a matrix is made into this form : $\left[\begin{array}{ccc|c} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & z \end{array}\right]$

Example : $\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 9x_2 + 4x_3 = 9 \\ 8x_1 + x_2 + 2x_3 = 15 \end{cases} \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 9 & 4 & 9 \\ 8 & 1 & 2 & 15 \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 5 & -2 & 7 \\ 0 & -15 & -22 & 7 \end{array}\right] \xrightarrow{r_2/5} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & -15 & -22 & 7 \end{array}\right]$

$\xrightarrow[r_3 + 15r_2]{r_1 - 2r_2} \left[\begin{array}{ccc|c} 1 & 0 & 3.8 & -1.8 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & 0 & -28 & 28 \end{array}\right] \xrightarrow{r_3/-28} \left[\begin{array}{ccc|c} 1 & 0 & 3.8 & -1.8 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & 0 & 1 & -1 \end{array}\right] \xrightarrow{r_1 - 3.8r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & -0.4 & 1.4 \\ 0 & 0 & 1 & -1 \end{array}\right]$

$\xrightarrow{r_2 + 0.4r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right] \qquad \begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$ //

- Gauss Seidel
  - ↳ Figuring out if matrix is diagonally dominant or not along with iterations

Example : $\begin{cases} 9x + 2y + 3z = 23 \rightarrow |9| \geq |2| + |3| \\ 2x + 5y + z = 10 \rightarrow |5| \geq |2| + |1| \\ 2x + 3y + 9z = 16 \rightarrow |9| \geq |2| + |3| \end{cases}$ → conclusion : diagonally dominant

↳ Iteration

$x = \dfrac{23 - 2y - 3z}{9} \qquad y = \dfrac{10 - 2x - z}{5} \qquad z = \dfrac{16 - 2x - 3y}{9}$

Exact Values / Initial Values
: $x = 0$, $y = 0$, $z = 0$

1st
$x = 2.5556$ error: 100 % → 23/9
$y = 0.9778$ error : 100 % → 10-2(2.5556)/5
$z = 0.8840$ error: 100 % → 16-2(2.5556)-3(0.9778)/9

repeat steps until error < 1%

2nd
$x = 2.0436$ error : 25.0503 %
$y = 1.0058$ error : 2.7823 %
$z = 0.9884$ error : 10.5663 %

3rd
$x = 2.0026$ error : 2.0489 %
$y = 1.0013$ error : 0.4669 %
$z = 0.9990$ error : 1.0620 %

4th
$x = 2.0000$ error: 0.1271 %
$y = 1.0002$ error: 0.1150 %
$z = 0.9999$ error: 0.0937 %

if error doesn't go down, x, y, z is growing to infinity.

Error Formula

For 1st Iteration : $\left|\dfrac{x_1}{x_1}\right| \times 100\%$

For Next Iteration : $\left|\dfrac{\text{New} - \text{Old}}{\text{New}}\right| \times 100\%$

↳ From example question : $\left|\dfrac{2.5556 - 2.0436}{2.0436}\right| \times 100\%$

$= 25.0503\%$

Regression & Interpolation

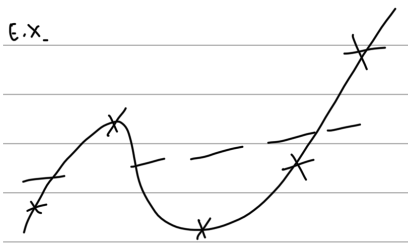
↳ Newton Interpolation : Finding one curve that passes through all points given.

↳ Regression : Finding one curve that passes through the mids of all points given

↳ Cubic Spline Interpolation : Slice a curve into parts and finding the respective cubic function.

NEWTON INTERPOLATION & REGRESSION METHOD

E.X.



$2 \le 2.3 \le 2.5 \rightarrow$ Take $(\overset{x_1}{2},\overset{y_1}{8})(\overset{x_2}{2.5},\overset{y_2}{12.5})$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12.5 - 8}{2.5 - 2} = \frac{4.5}{0.5} = 9$$

$y - y_1 = m(x - x_1)$

$y - 8 = 9(x - 2)$

$y = 9x - 10 \;/\; f(x) = 9x - 10$

$f(2.3) = 9(2.3) - 10$

$= 10.7$

x	y	∇y
2	8	•
2.5	12.5	9

$$\frac{12.5 - 8}{2.5 - 2} = 9$$

$y = 8 + 9(x - 2) = 9x - 10$

$y(2.3) = 9(2.3) - 10 = 10.7$ //

NEWTON'S POLYNOMIAL METHOD

E.X.

x	-4	3	-1	2	-5
y	27	6	6	3	38

Sample Question : a.) Find degree of Polynomial

b.) Find the polynomial

c.) Find $f(2.5)$

x	y	∇y	∇y^2	∇y^3	∇y^4
-4	27	•	•	•	.
3	6	-3	•	•	'
-1	6	-7	1	•	'
2	3	-4	1	0	'
-5	38	-11	1	0	'

$\frac{6-27}{3-(-4)} = -3$, $\frac{6-27}{-1-(-4)} = -7$, $\frac{3-27}{2-(-4)} = -4$

$\frac{38-27}{-5-(-4)} = -11$, $\frac{-7-(-3)}{-1-3} = 1$, $\frac{-4-(-3)}{2-3} = 1$

$\frac{-11-(-3)}{-5-3} = 1$, $\frac{1-1}{2-(-1)} = 0$, $\frac{1-1}{-5-(-1)} = 0$

a.) Non-zero stops at ∇y^2 deg = 2 //

b.) $y = 27 - 3(x-(-4)) + 1(x-(-4))(x-3)$

$= 27 - 3(x+4) + (x+4)(x-3)$

$= 27 - 3x - 12 + x^2 - 3x + 4x - 12$

$= x^2 - 2x + 3$ //

c.) $f(2.5) = (2.5)^2 - 2(2.5) + 3$

$= 4.25$ //

CUBIC SPLINE

$$k_{i-1} + 4k_i + k_{i+1} = \frac{6}{h^2}(y_{i-1} - 2y_i + y_{i+1})$$

$$f(x) = \frac{(x_i - x)^3}{6h} \cdot k_{i-1} + \frac{(x - x_{i-1})^3}{6h} \cdot k_i + \frac{x_i - x}{h} \cdot \left(y_{i-1} - \frac{h^2}{6} \cdot k_{i-1}\right) + \frac{(x - x_{i-1})}{h} \cdot \left(y_i - \frac{h^2}{6} \cdot k_i\right)$$

Taylor & McLaurin Series

MCLAURIN

$$f(x) = f(0) + \frac{f'(0)}{1!}(x)^1 + \frac{f''(0)}{2!}(x)^2 + \frac{f'''(0)}{3!}(x)^3 + \dots$$

TAYLOR

- sequence : ordered set of numbers $(T_1, T_2, T_3, \dots T_n)$
- Series : sum of all sequence up until n terms : $T_1 + T_2 + T_3 + \dots T_n$

Taylor Series $x \to a$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a)^1 + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

Decomposition

A = LU Decomposition

- L : Lower Triangular Matrix
- U : Upper Triangular Matrix

$$A: LU = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ +1 & 1 & 0 \\ +2 & -3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & -7 \end{bmatrix}}_{U}$$

E.X.

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 2x_1 + 9x_2 + 4x_3 = 9 \\ 8x_1 + x_2 + 2x_3 = 15 \end{cases} \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 2 & 9 & 4 & 9 \\ 8 & 1 & 2 & 15 \end{array}\right]$$

$AX = B \to LUX = B$

$$U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & -28 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & -3 & 1 \end{bmatrix}$$

Use elimination to obtain this

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & -3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & -28 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 9 \\ 15 \end{bmatrix}}_{B}$$

DOOLITTLE'S DECOMPOSITION

- $UX = Y$
- $LUX = B$
- $A = LU$
- $AX = B$
- $LY = B$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 8 & -3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} 1 \\ 9 \\ 15 \end{bmatrix}}_{B}$$

$y_1 = 1$

$2(1) + y_2 = 9 \to y_2 = 7$

$8(1) - 3(7) + y_3 = 15 \to y_3 = 28$

$Y = \{1, 9, 15\}$ or $\begin{cases} y_1 = 1 \\ y_2 = 9 \\ y_3 = 15 \end{cases}$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & -2 \\ 0 & 0 & -28 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 7 \\ 28 \end{bmatrix}}_{Y}$$

$-28x_3 = 28 \to x_3 = -1$

$5x_2 - 2(-1) = 7 \to x_2 = 1$

$x_1 + 2(1) - 1(3) = 1 \to x_1 = 2$

$X = \{2, 1, -1\}$ // or $\begin{cases} x_1 = 2 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$

Pivoting

- ↳ Using diagonally dominant (absolute value of each diagonal element should be dominant)
- ↳ The diagonal element must be greater than or equal to the sum of the other two values.

E.X. $\left[\begin{array}{ccc|c} -2 & 4 & -1 & 1 \\ 1 & -1 & 3 & 3 \\ -4 & 2 & -1 & -3 \end{array}\right]$ $\left.\begin{array}{l} |-2| < |4| + |-1| \\ |1| < |-1| + |3| \\ |-4| < |2| + |-1| \end{array}\right\}$ not diagonally dominant

r_{13} $\left[\begin{array}{ccc|c} \boxed{-4} & 2 & -1 & -3 \\ 1 & -1 & 3 & 3 \\ -2 & 4 & -1 & 1 \end{array}\right]$ $|-4| > |2| + |-1|$

r_{23} $\left[\begin{array}{ccc|c} \boxed{-4} & 2 & -1 & -3 \\ -2 & \boxed{4} & 1 & 1 \\ 1 & -1 & \boxed{3} & 3 \end{array}\right]$ $\left.\begin{array}{l} |-4| \geq |2| + |-1| \\ |4| \geq |-2| + |-1| \\ |3| \geq |1| + |-1| \end{array}\right\}$ diagonally dominant