Investigation of Approximating π Using Probability

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Introduction

The idea of π was first introduced to me at a young age when learning the relationship between a circle's diameter and its perimeter. In fact, my class would hold competitions about who can recite the most digits of π. It was then that I took an interest in π, to the point where its historical component of discovery is just as fascinating compared to its mathematical component of innate complexity. In a video from PBS network's NOVA program series, "Mathematics is the queen of Sciences", mentioned that π has close relations to a river's length, the colors in a rainbow, and how middle C should sound. It shows up from how cells grow into spherical shapes to the brightness of a supernova. "It is like seeing π on a series of mountain peaks, poking out of a fog shrouded valley"[1]. Deeper explorations suggested a trace of π can be found in the field of probability, a field that seemed completely unrelated to the mathematical constant.

This essay aims to investigate the occurrence of π in the field of probability. The investigation consists of two parts. The first part about Buffon's needle experiment, the second part about a variation of Pólya's recurrence theorem. Both investigations would be conducted separately with different methods, and the results and analysis would be conducted collectively. The results will also be compared to see how efficiently they can approximate π.

Part One: Buffon's Needle Experiment

2.1 Background Information

Buffon's needle problem was a question first posed in 1733 by Georges-Louis Leclerc, Comte de Buffon [2]. The problem proposed was: suppose there is a floor made of parallel strips of wood, each of the same width. If a needle is dropped on the floor, what is the probability that the needle will land on a line between two strips [3]? The solution for the problem was presented to the problem in 1777 by himself, making it the earliest geometric probability to be solved [2].

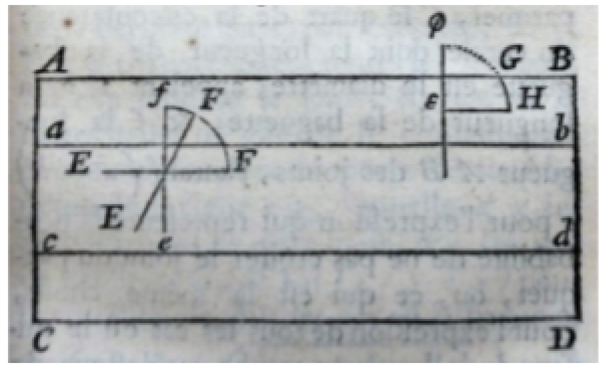

Figure 1: Buffon's needle experiment [5]

Buffon's solution to the problem can be used to design a Monte Carlo experiment, where repeated random sampling to obtain numerical results, to give insight to the approximation of π. This means that the experiment cannot guarantee 100% accuracy for the approximation of π, but rather the error of the approximation decreases with increasing numbers or samples.

2.2 Calculations

For the purpose of simplicity, this essay would only concern itself with the simplest case of Buffon's needle experiment, where

the length of the needle ≤ the distance between the parallel lines. (Condition 1)

This implies that there are only two possible outcomes for each needle: to touch a line once, or not to touch any line at all. The aim of Buffon's experiment is thus to find P(needle intersects one of the lines).

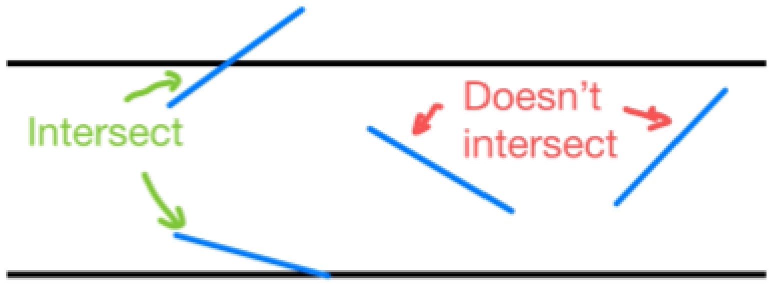

Figure 2: Different orientations of the needle. Created using GoodNotes

To formulate a model for the outcome of the needle, some variables are defined:

- d: distance between the two parallel lines $0 \leq d$
- ℓ: length of the needle. Given condition 1, $0 \leq \ell \leq d$
- x: distance between the midpoint of the needle to the nearest line. Given condition 1, $0 \leq x \leq d/2$
- θ: angle between the needle and the horizontal. Without loss of generality assume θ is acute: $0 \leq \theta \leq \pi/2$

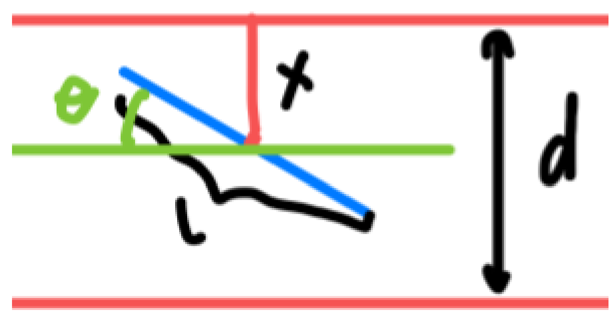

Figure 3: Representation of variables

It is here that another condition of the experiment is introduced:

There is no preferred orientation of the needle. (condition 2)

The needle can be dropped at any orientation with equal probability. There is equal probability for x in range to happen with each trial, and one outcome of a trial does not influence the outcome of another. In other words, the probability of x in each outcome is uniformly distributed in its range. The same applies for θ.

Now it is just the problem of associating a model to P(needle intersects one of the lines). To do this, the position of the needle is broken down. We observe that the needle intersects one line <u>if and only</u> if its vertical component of $l/2 \geq x$:

Lemma 1: for the needle to intersect, the condition $l/2 \cdot sin\theta \geq x$ *has to be guaranteed*

Proof: In figure 4, the vertical components of $l/2$ are shown in yellow. The distance from the needle's midpoint to the nearest line, represented as x, is shown in black. An intersection occurs when the vertical component $\geq$ x.

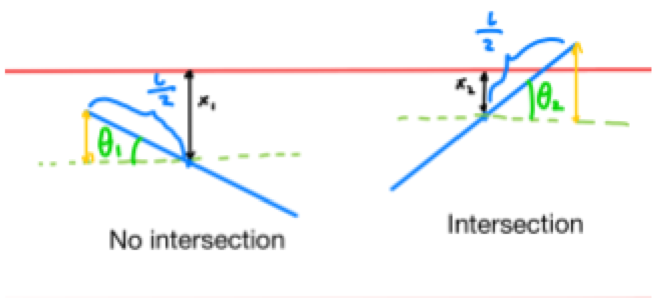

Figure 4: Vertical component and x

From figure 4, a triangle can be observed, constructed with the vertical component (yellow), $l/2$ segment (blue), and the horizontal (green). Trigonometry can be employed to obtain:

$$vertical\ component\ of\ l/2 = l/2 \cdot sin\theta$$

As having the vertical component of $l/2 \geq x$ is both sufficient and necessary for the event of an intersection, an intersection can occur if and only if

$$l/2 \cdot sin\theta \geq x$$

Thus proving Lemma 1.

From here there are two ways which can find out P(needle intersecting one of the lines). Each method would prove to show the same results.

Using graph: The problem can be shown by plotting a graph with D on the ordinate and $l/2 \cdot sin\theta$ on the abscissa [14] (the graph is displayed with $0 \le \theta \le \pi$ to better demonstrate the relationship, if take θ as $0 \le \theta \le \pi/2$ the area of the sine function and the area of the rectangle would both be halved and does not violate the $P(x \le l/2 \cdot sin\theta)$ below):

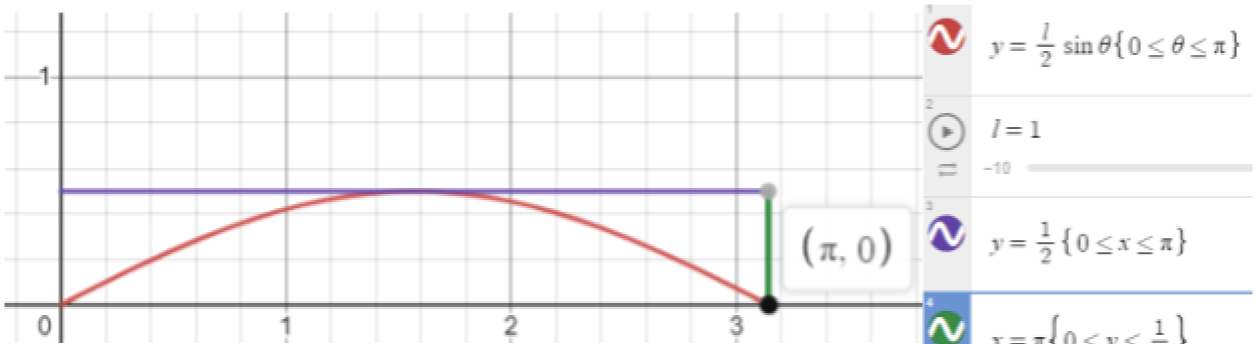

Figure 5: plot of possible θ values(abscissa) vs possible x values (ordinate)
Constructed using https://www.desmos.com/calculator

Figure five portrays all the possible values of θ and x (in the rectangle surrounded by the x-axis, y-axis, and the functions in purple and in green), with the length of the needle and the distance between consecutive lines assumed to be 1 unit (Note that in reality the value of x = d/2). Each point inside the rectangle represents a possible position/orientation of the needle when dropped.

Because P(needle intersects one of the lines) ≡ $P(x \le l/2 \cdot sin\theta)$, this is shown in the area between the x-axis and the function $y = l/2 \cdot sin\theta$, as every point inside the region obeys $x \le l/2 \cdot sin\theta$ (the ordinate is smaller than the abscissa).

The area of the region can be found using integration:

$$A = \int_0^{\pi} \frac{l}{2} sin\theta d\theta$$

$$A = \left[-\frac{l}{2}cos\theta\right]_0^{\pi} = -\frac{l}{2}(-1) - [-\frac{l}{2}(1)] = l$$

The probability that the needle intersects one of the lines is equal to the area of the region divided by the total region of probabilities (the rectangle):

$$P(x \le l/2 \cdot sin\theta) = \frac{l}{\frac{\pi d}{2}} = \frac{2l}{\pi d}$$

Using joint PDF: Another way to find the probability of a needle intersecting one of the lines is to use the idea of joint probability density function. Assume the probability for x follows the probability density function (PDF) $f(x)$, θ follows the PDF of $f(\theta)$, than according to condition 2: $f(x)$ and $f(\theta)$ are constants.

Since the sum of the probabilities of a PDF equals 1, and the variable x has a domain of $[0, \frac{d}{2}]$ (i.e. f(x) is only defined when $x \in [0, \frac{d}{2}]$, it is possible to deduce that:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{0}^{d/2} f(x)dx = 1$$

Assume f(x) = c, where c is a constant:

$$\int_{0}^{d/2} c\, dx = 1$$

$$[cx]_{0}^{d/2} = 1$$

$$c = f(x) = \frac{2}{d}$$

Similarly,

$$\int_{-\infty}^{\infty} f(\theta)dx = \int_{0}^{\pi/2} f(\theta)dx = 1$$

$$f(\theta) = \frac{2}{\pi}$$

We now observe that the distance from the midpoint to the nearest line(x) and its angle with the horizontal(θ) are independent events, as they do not affect each other. Thus their joint PDF:

$$f(x, \theta) = P(x \underset{and}{\cap} \theta) = P(x)P(\theta) = \frac{2}{d} \cdot \frac{2}{\pi} = \frac{4}{d\pi}$$

The aim is to calculate P(needle intersects one of the lines), which from Lemma 1, is the equivalent of P($x \le l/2 \cdot sin\theta$). This probability concerns itself with two random variables. To calculate this probability multivariate integration is used. The joint PDF is integrated with respect to θ and x. θ can be any value between π/2 and 0. x, however, needs to satisfy the condition $x \le l/2 \cdot sin\theta$, so its upper and lower limits are $l/2 \cdot sin\theta$ and 0 respectively.

$$P(x \le l/2 \cdot sin\theta) = \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{l}{2} \cdot sin\theta} \frac{4}{d\pi} dx \right) d\theta$$

$$P(x \le l/2 \cdot sin\theta) = \int_0^{\frac{\pi}{2}} \left[\frac{4}{d\pi} x \right]_0^{\frac{l}{2} \cdot sin\theta} d\theta$$

$$P(x \le l/2 \cdot sin\theta) = \int_0^{\frac{\pi}{2}} \frac{4}{d\pi} \cdot \frac{l}{2} \cdot sin\theta d\theta$$

$$P(x \le l/2 \cdot sin\theta) = \left[\frac{4}{d\pi} \cdot \frac{l}{2} \cdot (-\ cos\theta) \right]_0^{\frac{\pi}{2}}$$

$$P(x \le l/2 \cdot sin\theta) = \frac{2l}{d\pi}$$

Thus P(needle intersects one of the lines) = $\frac{2l}{d\pi}$

2.3 Experiment

It is possible to experimentally determine P(needle intersects one of the lines), or P, by altering the equation P = $\frac{2l}{d\pi}$:

$$P = \frac{2l}{d\pi}$$

$$\pi = \frac{2l}{dP}$$

If the experiment is set up that the length of the needle = the distance between consecutive lines (l = d):

$$\pi = \frac{2}{P}$$

Theoretically, by determining the value of P, it is possible to obtain the value of π. However, this experiment is classified as a Monte Carlo experiment, where it relies on its inherent randomness to obtain numerical results [4]. It is impossible to predict a totally accurate value of π, but with more and more trials the degree of accuracy can increase. One might throw in millions of needle and still cannot get a good idea of the exact value of π.

In reality, as the experiment requires large quantities of trials, there is a shortage in both time, manpower, as well as lack of control over factors which can cause random errors for the experiment to be conducted in real life. A computer simulation based heavily on Lalithnarayan C [6] is conducted, taking various data points and plot a graph of the estimated π values (raw data in the appendix).

Below are the results of the experiment, constructed using https://www.desmos.com/calculator:

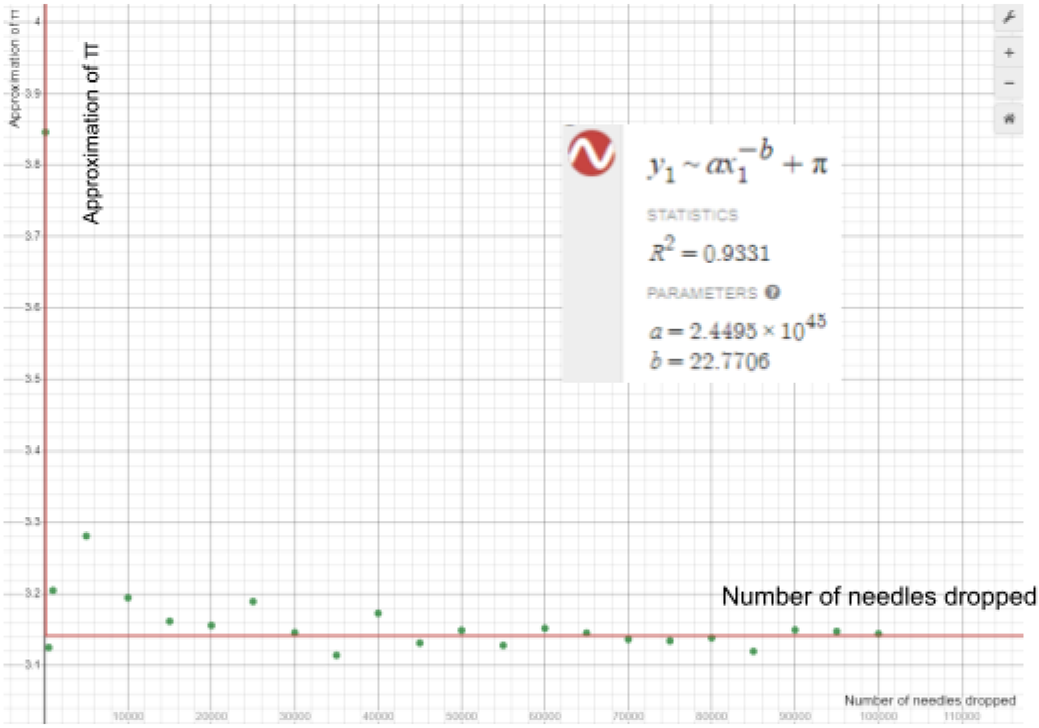

Figure 9: Graph for approximation of π of Buffon's needle experiment, with regression attempted

As seen from the graph, the results of the experiment portrayed a convergence to π. This is in accordance with expectations and further analysis can be conducted.

Part Two: Pólya's Recurrence Theorem

Another way to investigate the occurrence of π is modifying Polya's recurrence theorem, another Monte Carlo experiment.

2.1 Background information

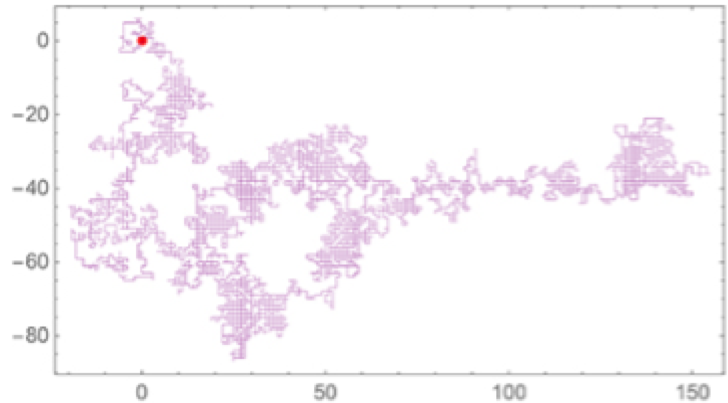

Figure 6: A random walk on a 2-D plane [10]

In the early 1900s, George Pólya was taking a walk in a park, when he noticed that he crossed the same couple quite often in that park, even though it seemed to him that they both were taking random walks [9]. He took great interest in this observation and developed it into Pólya's recurrence theorem. Pólya's recurrence theorem states that a simple random walk on a d-dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$ [7]. A random walk is a path on a graph or lattice where each step is determined randomly [8]. He defined a random walk as recurrent if the walker passes through every single point on a lattice with probability one (i.e. the walker would always return to the origin), otherwise the walk is transient [8]. This means that for a walker in a 1-dimensional line or a 2-dimensional plane, the walker is always able to return to its starting position provided that he/she has taken ∞ steps. As mathematician Shizuo Kakutani famously remarked, "A drunk man will find his way home, but a drunk bird may get lost forever". For the case of simplicity, the d-dimensional lattice would be replaced with a 1-dimensional line in this recurrence experiment.

3.2 Calculations

Some variables have to be defined:

- n: number of steps taken by the walker.
- S_n: position of the walker after n steps.
- X_i: step taken at step number i.

Lemma 2: $S_n = 0$ *occurs if and only if n = 2k for some integer k >0*

Proof: In a 1-dimensional line, the walker has the option of taking a step towards the left or the right at random. Assume $S_0 = 0$, meaning the walker starts at position 0 along the number line. Each step X_i taken can be one unit away from position S_{i-1}. Without loss of generality assume if the walker takes a step to the right, $X_i =+ 1$. If the walker takes a step leftwards, $X_i =- 1$.

Therefore the position of the walker after n steps can be determined by:

$$S_n = \sum_{i=1}^{n} x_i$$

Since $X_i \in \{- 1, + 1\}$ (X_i is either positive one or negative one), X_i can only be an odd integer, so the parity of $S_n = n \cdot odd$. Basic mathematical operations dictates that

$$odd \cdot even = even$$

$$odd \cdot odd = odd$$

As S_n needs to be even (0 is an even integer), n must be even. Thus n = even is both sufficient and necessary for $S_n = 0$, completing the lemma.

Basing on lemma 2, the problem transforms to: given that the walker takes 2n steps, find P($S_{2n} = 0$).

A peculiar observation is made by developing a stimulation for a population of random walkers and graphing a histogram based on their position S_{2n} on the number line:

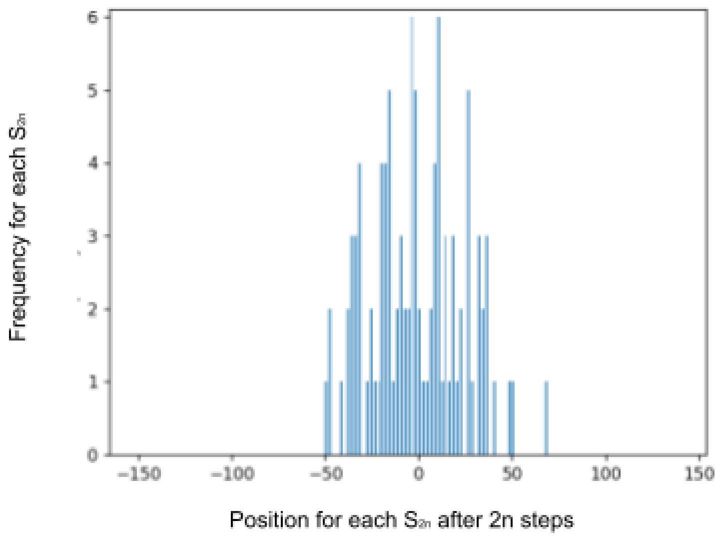

Figure 7: Graph of outcome of recurrence experiment with population 100, each sample talking a random walk of 500 steps

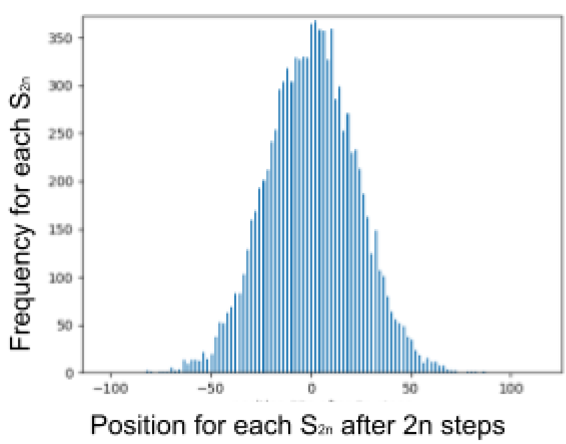

Figure 8: Graph of outcome of recurrence experiment with population 10000, each sample taking a random walk of 500 steps

As can be seen, for a certain number of 2n, as the population increases, the graph converges more and more like a bell-shaped curve. The Central Limit Theorem dictates that a population with a defined mean μ and standard deviation σ and take sufficiently large random samples from the population with replacement, then the distribution of the sample means will be approximately normally distributed [14]. Calculations for its mean and its standard deviation are thus conducted:

As $S_{2n} = \sum_{i=1}^{2n} x_i$, $E(S_{2n}) = E(\sum_{i=1}^{2n} x_i)$. Because each step x_i has equal probability of going left of right for some other step x_j $(where\ j \neq i)$,

$$E(\sum_{i=1}^{2n} x_i) = E(2nx_i) = 2nE(x_i)[11]$$

For x_i, it has an equal probability of going left or right. This means that $p = \frac{1}{2}$ for $x_i = -1$ (taking a step towards the left) and $p = \frac{1}{2}$ for $x_i = 1$ (taking a step towards the right). Thus:

$$E(x_i) = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$$

$$E(S_{2n}) = 2nE(x_i) = 0$$

The mean of S_n is 0, a constant. The standard deviation σ is calculated as the positive square root of its variance ($σ^2$). The variance can be calculated with the formula:

$$σ^2 = E(S_{2n}^{\ 2}) - \left[E(S_{2n})\right]^2 = E(S_{2n}^{\ 2})\ [13]$$

$$E(S_{2n}^{\ 2}) = E[(\sum_{i=1}^{2n} x_i)^2]$$

Expand the term $(\sum_{i=1}^{2n} x_i)^2 = (x_1 + x_2 + x_3 +\bullet\bullet\bullet+ x_{2n}) \cdot (x_1 + x_2 + x_3 +\bullet\bullet\bullet+ x_{2n})$

By multiplying and grouping same terms:

$$E(\sum_{i=1}^{2n} x_i)^2 = E[(\sum_{i=1}^{2n} x_i^2) + \sum_{i\neq j} x_i x_j)]$$

According to the linearity of expectation:

$$E[(\sum_{i=1}^{2n} x_i^2) + \sum_{i\neq j} x_i x_j)] = \sum_{i=1}^{2n} E(x_i^2) + \sum_{i\neq j} E(x_i x_j)$$

$$E(x_i^2) = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$$

Linearity of Expectation :

For random expectations X,Y, no matter what the joint distribution is,
$E(X + Y) = E(X) + E(Y)$[16]

There are four different outcomes for $x_i x_j$:

$x_i = 1$ $x_j = -1$ $x_i x_j = -1$	$x_i = 1$ $x_j = 1$ $x_i x_j = 1$	$x_i = -1$ $x_j = -1$ $x_i x_j = 1$	$x_i = -1$ $x_j = 1$ $x_i x_j = -1$

With each situation sharing the same probability ($\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$)

Thus $\sum_{i\neq j} E(x_i x_j) = 0$

$$\sigma^2 = E[(\sum_{i=1}^{2n} x_i^2) + \sum_{i\neq j} x_i x_j)] = \sum_{i=1}^{2n} 1 + 0 = 2n$$

The standard deviation $\sigma = \sqrt{2n}$, a constant value with a defined value of 2n. This obeys the Central Limit Theorem, so theoretically, the distribution of S_{2n} should converge towards a normal distribution. However, as seen from Lemma 2, S_{2n} can only be even. This means that the distribution of S_{2n} is not continuous (as all odd numbers have zero frequency). Instead, the distribution can be modeled with a binomial distribution.

We observe that for $S_n = 0$, there must be an equal amount of steps to the left and steps to the left (as steps share equal length). Thus there must be n steps to the left and n steps to the right. Both steps have the probability of $\frac{1}{2}$ to occur.

$$P(S_n = 0) = C_{2n}^{n}(\frac{1}{2})^{n}(\frac{1}{2})^{2n-n} = C_{2n}^{n}(\frac{1}{2})^{2n}$$

Or it can be think of simply in a combinatorics perspective:

$$P(S_n = 0) = \frac{outcomes\ with\ S_n = 0}{total\ number\ of\ possible\ outcomes}$$

The denominator can be thought as a combination of choosing n steps to the right/left out of 2n steps: C_{2n}^{n}

The numerator is all the possible trajectories of the walker when taking 2n steps. With each step the walker has two options (left or right), so the numerator simplifies to 2^{2n}.

Thus

$$P(S_{2n} = 0) = \frac{C_{2n}^{n}}{2^{2n}}$$

This expression at first glance seems unable to be simplified. Fortunately, there is a method named Stirling's approximation which can provide a great estimate of $n!$.

Stirling's approximation:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$$

Where ~ means that their ratio tends to 1 as n tends to infinity [12].

$$C^{n}_{2n} = \frac{(2n)!}{n! \cdot n!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{4\pi n}\left[\frac{(2n)}{e}\right]^{2n}}{\left[\sqrt{2\pi n}(\frac{n}{e})^{n}\right]^2}$$

$$C^{n}_{2n} = \frac{\sqrt{4\pi n} \cdot (2n)^{2n} \cdot e^{-2n}}{2\pi n \cdot n^{2n} \cdot e^{-2n}}$$

$$C^{n}_{2n} = \frac{2^{2n}}{\sqrt{\pi n}}$$

$$P(S_{2n} = 0) = \frac{C^{n}_{2n}}{2^{2n}} = \frac{\frac{2^{2n}}{\sqrt{\pi n}}}{2^{2n}} = \frac{1}{\sqrt{\pi n}}$$

Surprisingly π makes its appearance in this seemingly unrelated probability experiment. A way of experimentally deriving π is constructed.

3.3 Experiment

The equation obtained for the probability of returning to the starting point after 2n steps can be altered in the following manner:

$$P(S_{2n} = 0) = \frac{1}{\sqrt{\pi n}}$$

$$\sqrt{\pi n} = \frac{1}{P(S_{n}=0)}$$

$$\pi = \frac{1}{n\left[P(S_{2n}=0)\right]^2}$$

This equation links $P(S_{2n} = 0)$ with π with consideration of an arbitrary value n. Note that even though Pólya's recurrence theorem claimed that the number of steps approaches infinity, such condition does not have to be met when trying to find the recurrence of π with the equation above. This is because as 2n approaches infinity, n approaches infinity also, and the probability approaches asymptotically to $\frac{1}{\sqrt{n}}$.

$$P(S_{2n} = 0) = \frac{1}{\sqrt{\pi n}} \propto \frac{1}{\sqrt{n}}$$

A computer simulation of Pólya's recurrence experiment is created to obtain results and approximate π.

Below are the results of the experiment, constructed using https://www.desmos.com/calculator:

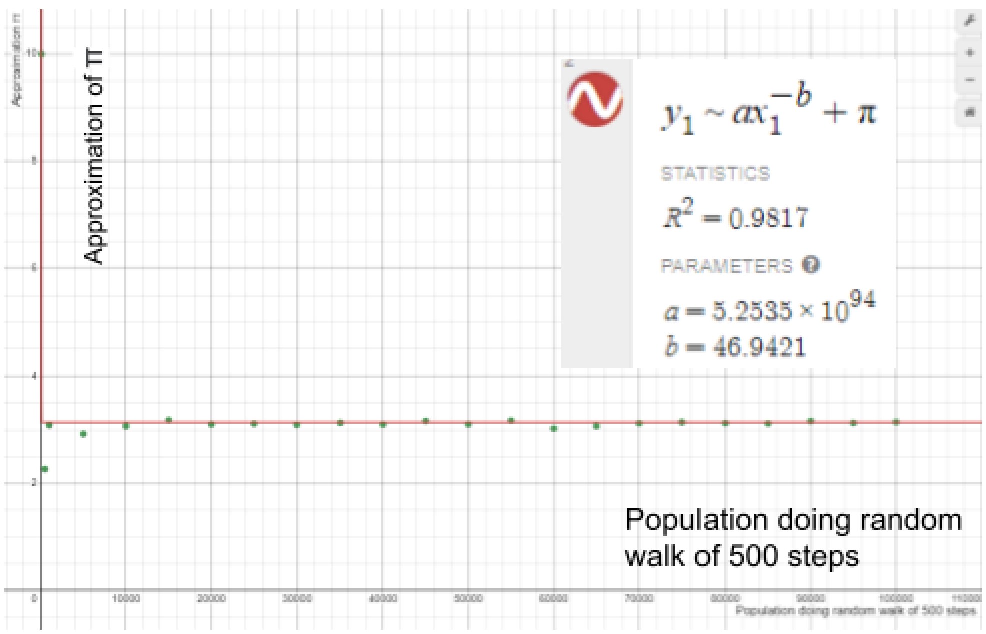

Figure 10: Graph for approximation of π of Pólya's recurrence experiment, With regression attempted

Similar to Buffon's needle experiment, The results for Pólya's recurrence experiment also demonstrates a convergence towards π. This is in accordance with expectations and further analysis can be conducted.

Analysis

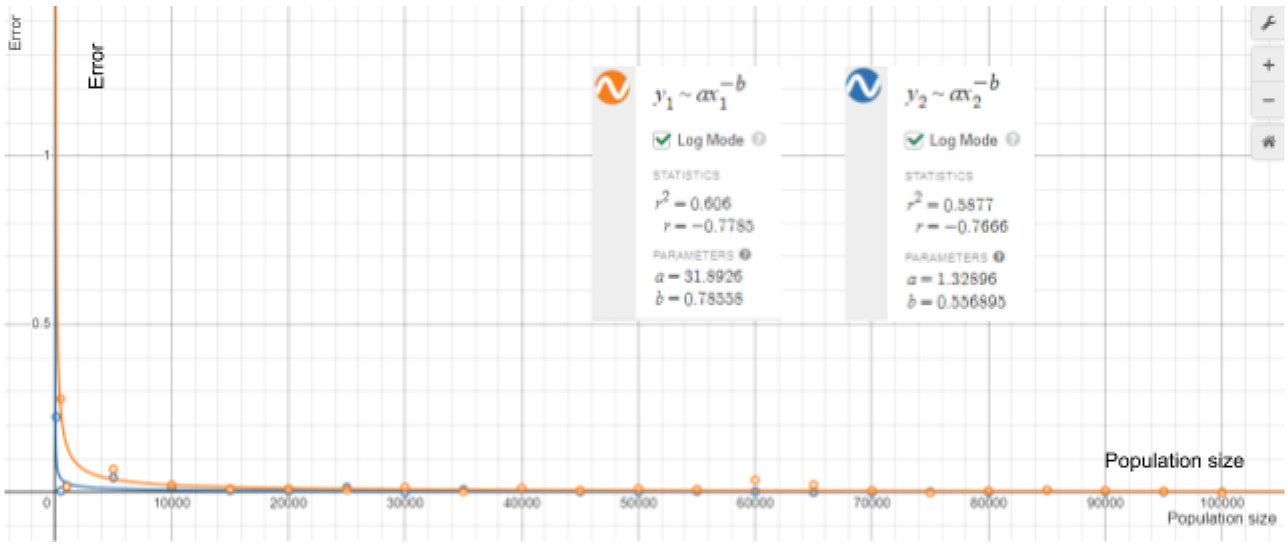

Figure 11: Error for both experiments (Buffon: Blue, Pólya: Orange), with Regression

(All of the above graphs are constructed using https://www.desmos.com/calculator)

The graphs in Figures 9 and 10 represent the approximation of π by both experiments. The regression applied is a power series function with a negative exponent and a vertical translation of π. This is because these experiments are Monte Carlo simulations where their inherent randomness is used to obtain numerical results. In theory, as the population increases, the effect of random error decreases, leading to a more accurate result. Yet the random error still persists, hence the results are only approximations. By this logic the regression function should have a horizontal asymptote of π as well as portraying a decaying-with-decreasing-rate relation:

$$\lim_{x \to \infty} R(x) = \pi$$

Note that this does not rule out a polynomial function in the form of:

$P(x) = \pi + ax^{-1} + bx^{-2} + \bullet\bullet\bullet + ix^{-j}$ for some positive integer j. But for the sake of simplicity, we chose $R(x)$ to be in the form of $R(x) = ax^{-b} + \pi$ (an inverse power function) for some real number a and positive integer b.

Both regression equations showed to fit the data points well (with $R^2 = 0.9331, 0.9817$ respectively. The data points have successfully portrayed evidence of a decay relationship with the decay rate decreasing accompanied by increasing population size, which aligns with the hypothesis of the decreasing effect of random error.

Both graphs possess a very sharp turning point where the derivative of the function changes from a very large negative value to a very small negative value. This is the result of very large coefficients (a) which increases the y value dramatically when $x < 1$. The large b values also cause these sharp turning points, because as x increases after $x > 1$, the term ax^{-b} would quickly converge towards 0, making $R(x) \approx 0 + \pi = \pi$.

Note that $R(x) \neq \pi$ for any real x values. It just asymptotically converges towards π. The decay rate is upper bounded by $O(x^{-c})$in this case which prevents it from crossing its horizontal asymptote. Thus the term “approximation” instead of “obtain”.

Figure 11 portrays the error of the experiments from π. The regression for such are also relatively accurate to the pattern of the data points as they both possess small R^2 values. Log mode is enabled on both functions, which linearizes the model by a logarithmic transformation [15] (seems appropriate to apply since the function is a power function of the form ($y = ax^{-b}$).

It can be observed that the pattern for error follows a similar decay relationship with the pattern for approximating π. The error decrease belongs to the error in the inherent randomness in the experiment (random error). As the population size increases, the effect of this error decreases.

By comparing both experiments, one interesting pattern to observe is that the error for Buffon's needle experiment is relatively smaller than Pólya's recurrence experiment. The one part that differs the two is the employment of Stirling's approximation in the latter. This opens up a limitation for Pólya's recurrence experiment: the degree of validity of Stirling's approximation. According to a paper by 蔡永裕 which experimented on the errors of Stirling's approximation[17], for n!, if n = 500 (employed in the calculations), the error of Stirling's approximation is around -1.66653×10^{-4}. This may seem insignificant, but when faced with data points with similar precision, the error can pose a significant effect.

Some other possible errors which can contribute to inaccuracies in both experiments include but not limited to:

Error description	Type of error	Possible solutions
Each data point is conducted with only one trial due to time concerns. There is uncertainty in each trial as each run of the program would yield slightly different results	Random	Conduct more trials for each data point and take their average value. Cannot eliminate this error but can decrease its effectiveness.
All the randomness necessary in the experiments are generated by the computer. This is classified as pseudorandom and each random value (the orientation of the needle in Buffon's needle experiment, the step of the person in Pólya's recurrence experiment) is generated using a pre-existing algorithm (values are purely computer generated). Hence it is not truly random.	Systematic	Employ an algorithm modeled on unpredictable physical phenomena. For example, the behavior of electrons in orbitals.

Conclusion

This paper aims to investigate the occurrence of π in Buffon's needle experiment and an experiment based on Pólya's recurrence theorem.

As shown by the analysis, both experiments contribute values similar to π, with the error getting progressively smaller as the population size is increased. The rate at which the error is decreasing slows down with the population size increasing at a constant rate. These conclusions help show that both experiments are able to approximate π to a given degree of accuracy. The word approximate is necessary in this context as it is impossible to obtain the exact value of π using these two experiments. Pólya's recurrence experiment employed Stirling's approximation which contributes to a marginally bigger error than Buffon's needle experiment. Nonetheless, they both follow the same trend of approximating π, and with a large population size the marginal error is negligible.

Overall, I believe the investigation was a success as I feel that I have a better grasp upon the marvelous property of π and how different mathematical concepts can be interlinked together.

Appendix

6.1 Environment used

All simulations are run on vivobook model M513U with the following specifications:

Device name	LAPTOP-5MF0JE7O
Processor	AMD Ryzen 7 5700U with Radeon Graphics 1.80 GHz
Installed RAM	16.0 GB (15.4 GB usable)
Device ID	C42A85A5-2CD8-4D22-9C82-0E752FD3193A
Product ID	00325-97204-88164-AAOEM
System type	64-bit operating system, x64-based processor
Pen and touch	No pen or touch input is available for this display
Edition	Windows 10 Home
Version	21H2
Installed on	2021/4/27
OS build	19044.2364
Experience	Windows Feature Experience Pack 120.2212.4190.0

IDE used:

Name: PyCharm Community Edition Version: 2022.3 Build: 223.7571.203

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<u>Error in Stirling's approximation</u>

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表六. 計算 n! 時, Stirling 公式與其改良式
分別之相對誤差

n	Stirling 公式	改良式
1	-7.78630E-02	-4.16219E-04
2	-4.04978E-02	-1.83817E-05
3	-2.72984E-02	-2.64893E-06
4	-2.05760E-02	-6.51314E-07
5	-1.65069E-02	-2.17155E-07
6	-1.37803E-02	-8.81200E-08
7	-1.18262E-02	-4.10132E-08
8	-1.03573E-02	-2.11183E-08
9	-9.21276E-03	-1.17508E-08
10	-8.29596E-03	-6.95219E-09
11	-7.54507E-03	-4.32298E-09
12	-6.91879E-03	-2.80103E-09
13	-6.38850E-03	-1.87879E-09
14	-5.93370E-03	-1.29793E-09
15	-5.53933E-03	-9.19761E-10
16	-5.19412E-03	-6.66388E-10
17	-4.88940E-03	-4.92317E-10
18	-4.61846E-03	-3.70051E-10
19	-4.37596E-03	-2.82468E-10
20	-4.15765E-03	-2.18619E-10
21	-3.96009E-03	-1.71327E-10
22	-3.78045E-03	-1.35791E-10
23	-3.61641E-03	-1.08746E-10
24	-3.46600E-03	-8.79178E-11
25	-3.32761E-03	-7.16885E-11
26	-3.19984E-03	-5.89342E-11
27	-3.08152E-03	-4.87976E-11
28	-2.97164E-03	-4.06906E-11
29	-2.86932E-03	-3.41431E-11
30	-2.77382E-03	-2.88256E-11
100	-8.32983E-04	-7.975 E-14
200	-4.16580E-04	-4.876 E-14
300	-2.77739E-04	-7.211 E-14
400	-2.08312E-04	8.530 E-14
500	-1.66653E-04	-7.188 E-14
600	-1.38879E-04	-1.126 E-13
700	-1.19041E-04	1.817 E-13
800	-1.04161E-04	-1.325 E-13
900	-9.25883E-05	-3.649 E-13
1000	-8.33299E-05	-2.971 E-13

Figure 12: When calculating n!, the relative error in Stirling's formula and ites improved formula