

Investigation of Approximating π Using Probability

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Introduction

The idea of π was first introduced to me at a young age when learning the relationship between a circle's diameter and its perimeter. In fact, my class would hold competitions about who can recite the most digits of π . It was then that I took an interest in π , to the point where its historical component of discovery is just as fascinating compared to its mathematical component of innate complexity. In a video from PBS network's NOVA program series, "Mathematics is the queen of Sciences", mentioned that π has close relations to a river's length, the colors in a rainbow, and how middle C should sound. It shows up from how cells grow into spherical shapes to the brightness of a supernova. "It is like seeing π on a series of mountain peaks, poking out of a fog shrouded valley"[1]. Deeper explorations suggested a trace of π can be found in the field of probability, a field that seemed completely unrelated to the mathematical constant.

This essay aims to investigate the occurrence of π in the field of probability. The investigation consists of two parts. The first part about Buffon's needle experiment, the second part about a variation of Pólya's recurrence theorem. Both investigations would be conducted separately with different methods, and the results and analysis would be conducted collectively. The results will also be compared to see how efficiently they can approximate π .

Part One: Buffon's Needle Experiment

2.1 Background Information

Buffon's needle problem was a question first posed in 1733 by Georges-Louis Leclerc, Comte de Buffon [2]. The problem proposed was: suppose there is a floor made of parallel strips of wood, each of the same width. If a needle is dropped on the floor, what is the probability that the needle will land on a line between two strips [3]? The solution for the problem was presented to the problem in 1777 by himself, making it the earliest geometric probability to be solved [2].

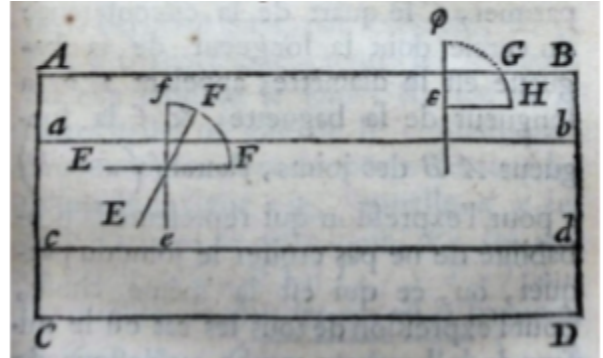


Figure 1: Buffon's needle experiment [5]

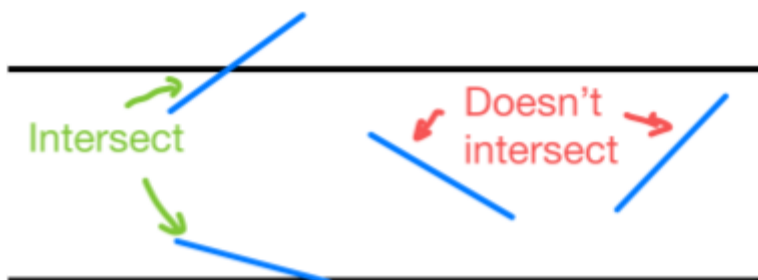
Buffon's solution to the problem can be used to design a Monte Carlo experiment, where repeated random sampling to obtain numerical results, to give insight to the approximation of π . This means that the experiment cannot guarantee 100% accuracy for the approximation of π , but rather the error of the approximation decreases with increasing numbers or samples.

2.2 Calculations

For the purpose of simplicity, this essay would only concern itself with the simplest case of Buffon's needle experiment, where

the length of the needle \leq the distance between the parallel lines. (Condition 1)

This implies that there are only two possible outcomes for each needle: to touch a line once, or not to touch any line at all. The aim of Buffon's experiment is thus to find $P(\text{needle intersects one of the lines})$.



To
for the outcome

Figure 2: Different orientations of the needle. Created using GoodNotes

formulate a model
of the needle,

some variables are defined:

- d : distance between the two parallel lines $0 \leq d$
- l : length of the needle. Given condition 1,
 $0 \leq l \leq d$
- x : distance between the midpoint of the needle to the nearest line. Given condition 1, $0 \leq x \leq d/2$
- θ : angle between the needle and the horizontal.

Without loss of generality assume θ is acute:

$$0 \leq \theta \leq \pi/2$$

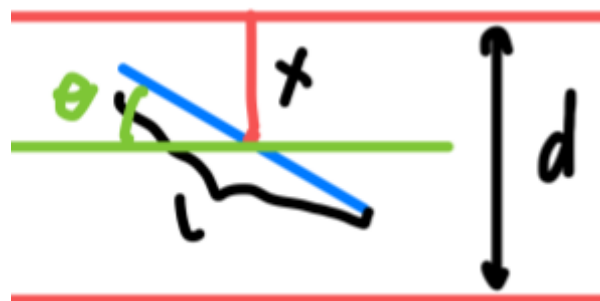


Figure 3: Representation of variables

It is here that another condition of the experiment is introduced:

There is no preferred orientation of the needle. (condition 2)

The needle can be dropped at any orientation with equal probability. There is equal probability for x in range to happen with each trial, and one outcome of a trial does not influence the outcome of another. In other words, the probability of x in each outcome is uniformly distributed in its range. The same applies for θ .

Now it is just the problem of associating a model to $P(\text{needle intersects one of the lines})$. To do this, the position of the needle is broken down. We observe that the needle intersects one line if and only if its vertical component of $l/2 \geq x$:

Lemma 1: for the needle to intersect, the condition $l/2 \cdot \sin\theta \geq x$ has to be guaranteed

Proof: In figure 4, the vertical components of $l/2$ are shown in yellow. The distance from the needle's midpoint to the nearest line, represented as x , is shown in black. An intersection occurs

when the vertical component $\geq x$.

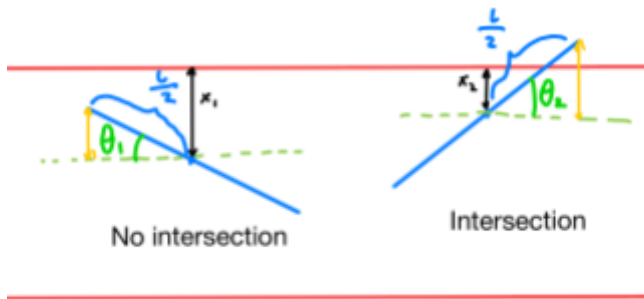


Figure 4: Vertical component and x

From figure 4, a triangle can be observed, constructed with the vertical component (yellow), $l/2$ segment (blue), and the horizontal (green). Trigonometry can be employed to obtain:

$$\text{vertical component of } l/2 = l/2 \cdot \sin\theta$$

As having the vertical component of $l/2 \geq x$ is both sufficient and necessary for the event of an intersection, an intersection can occur if and only if

$$l/2 \cdot \sin\theta \geq x$$

Thus proving Lemma 1.

From here there are two ways which can find out $P(\text{needle intersecting one of the lines})$. Each method would prove to show the same results.

Using graph: The problem can be shown by plotting a graph with D on the ordinate and $l/2 \cdot \sin\theta$ on the abscissa [14] (the graph is displayed with $0 \leq \theta \leq \pi$ to better demonstrate the relationship, if take θ as $0 \leq \theta \leq \pi/2$ the area of the sine function and the area of the rectangle would both be halved and does not violate the $P(x \leq l/2 \cdot \sin\theta)$ below):

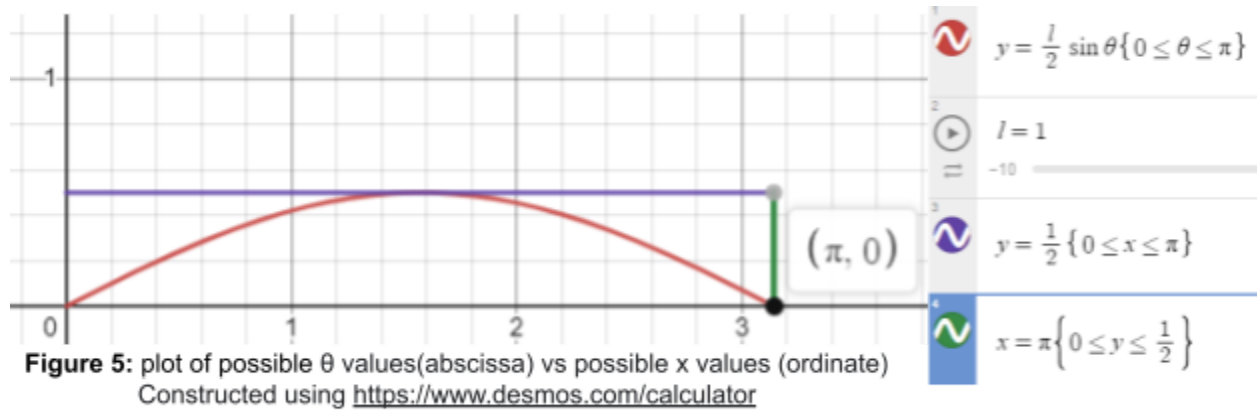


Figure five portrays all the possible values of θ and x (in the rectangle surrounded by the x -axis, y -axis, and the functions in purple and in green), with the length of the needle and the distance between consecutive lines assumed to be 1 unit (Note that in reality the value of $x = d/2$). Each point inside the rectangle represents a possible position/orientation of the needle when dropped.

Because $P(\text{needle intersects one of the lines}) \equiv P(x \leq l/2 \cdot \sin\theta)$, this is shown in the area between the x -axis and the function $y = l/2 \cdot \sin\theta$, as every point inside the region obeys $x \leq l/2 \cdot \sin\theta$ (the ordinate is smaller than the abscissa).

The area of the region can be found using integration:

$$A = \int_0^{\pi} \frac{l}{2} \sin\theta d\theta$$

$$A = \left[-\frac{l}{2} \cos\theta \right]_0^{\pi} = -\frac{l}{2}(-1) - \left[-\frac{l}{2}(1) \right] = l$$

The probability that the needle intersects one of the lines is equal to the area of the region divided by the total region of probabilities (the rectangle):

$$P(x \leq l/2 \cdot \sin\theta) = \frac{l}{\frac{\pi d}{2}} = \frac{2l}{\pi d}$$

Using joint PDF: Another way to find the probability of a needle intersecting one of the lines is to use the idea of joint probability density function. Assume the probability for x follows the probability density function (PDF) $f(x)$, θ follows the PDF of $f(\theta)$, then according to condition 2: $f(x)$ and $f(\theta)$ are constants.

Since the sum of the probabilities of a PDF equals 1, and the variable x has a domain of $[0, \frac{d}{2}]$

(i.e. $f(x)$ is only defined when $x \in [0, \frac{d}{2}]$), it is possible to deduce that:

$$\int_{-\infty}^{\infty} f(x) dx = \int_0^{d/2} f(x) dx = 1$$

Assume $f(x) = c$, where c is a constant:

$$\int_0^{d/2} c dx = 1$$

$$[cx]_0^{d/2} = 1$$

$$c = f(x) = \frac{2}{d}$$

Similarly,

$$\int_{-\infty}^{\infty} f(\theta) d\theta = \int_0^{\pi/2} f(\theta) d\theta = 1$$

$$f(\theta) = \frac{2}{\pi}$$

We now observe that the distance from the midpoint to the nearest line(x) and its angle with the horizontal(θ) are independent events, as they do not affect each other. Thus their joint PDF:

$$f(x, \theta) = P(x \underset{\text{and}}{\cap} \theta) = P(x)P(\theta) = \frac{2}{d} \cdot \frac{2}{\pi} = \frac{4}{d\pi}$$

The aim is to calculate $P(\text{needle intersects one of the lines})$, which from Lemma 1, is the equivalent of $P(x \leq l/2 \cdot \sin\theta)$. This probability concerns itself with two random variables. To calculate this probability multivariate integration is used. The joint PDF is integrated with respect to θ and x . θ can be any value between $\pi/2$ and 0. x , however, needs to satisfy the condition $x \leq l/2 \cdot \sin\theta$, so its upper and lower limits are $l/2 \cdot \sin\theta$ and 0 respectively.

$$P(x \leq l/2 \cdot \sin\theta) = \int_0^{\frac{\pi}{2}} \left(\int_0^{\frac{l}{2} \cdot \sin\theta} \frac{4}{d\pi} dx \right) d\theta$$

$$P(x \leq l/2 \cdot \sin\theta) = \int_0^{\frac{\pi}{2}} \left[\frac{4}{d\pi} x \right]_0^{\frac{l}{2} \cdot \sin\theta} d\theta$$

$$P(x \leq l/2 \cdot \sin\theta) = \int_0^{\frac{\pi}{2}} \frac{4}{d\pi} \cdot \frac{l}{2} \cdot \sin\theta d\theta$$

$$P(x \leq l/2 \cdot \sin\theta) = \left[\frac{4}{d\pi} \cdot \frac{l}{2} \cdot (-\cos\theta) \right]_0^{\frac{\pi}{2}}$$

$$P(x \leq l/2 \cdot \sin\theta) = \frac{2l}{d\pi}$$

Thus $P(\text{needle intersects one of the lines}) = \frac{2l}{d\pi}$

2.3 Experiment

It is possible to experimentally determine P (needle intersects one of the lines), or P , by altering

the equation $P = \frac{2l}{d\pi}$:

$$P = \frac{2l}{d\pi}$$

$$\pi = \frac{2l}{dP}$$

If the experiment is set up that the length of the needle = the distance between consecutive lines

($l = d$):

$$\pi = \frac{2}{P}$$

Theoretically, by determining the value of P , it is possible to obtain the value of π . However, this experiment is classified as a Monte Carlo experiment, where it relies on its inherent randomness to obtain numerical results [4]. It is impossible to predict a totally accurate value of π , but with more and more trials the degree of accuracy can increase. One might throw in millions of needle and still cannot get a good idea of the exact value of π .

In reality, as the experiment requires large quantities of trials, there is a shortage in both time, manpower, as well as lack of control over factors which can cause random errors for the experiment to be conducted in real life. A computer simulation based heavily on Lalithnarayan C [6] is conducted, taking various data points and plot a graph of the estimated π values (raw data in the appendix).

Below are the results of the experiment, constructed using <https://www.desmos.com/calculator>:

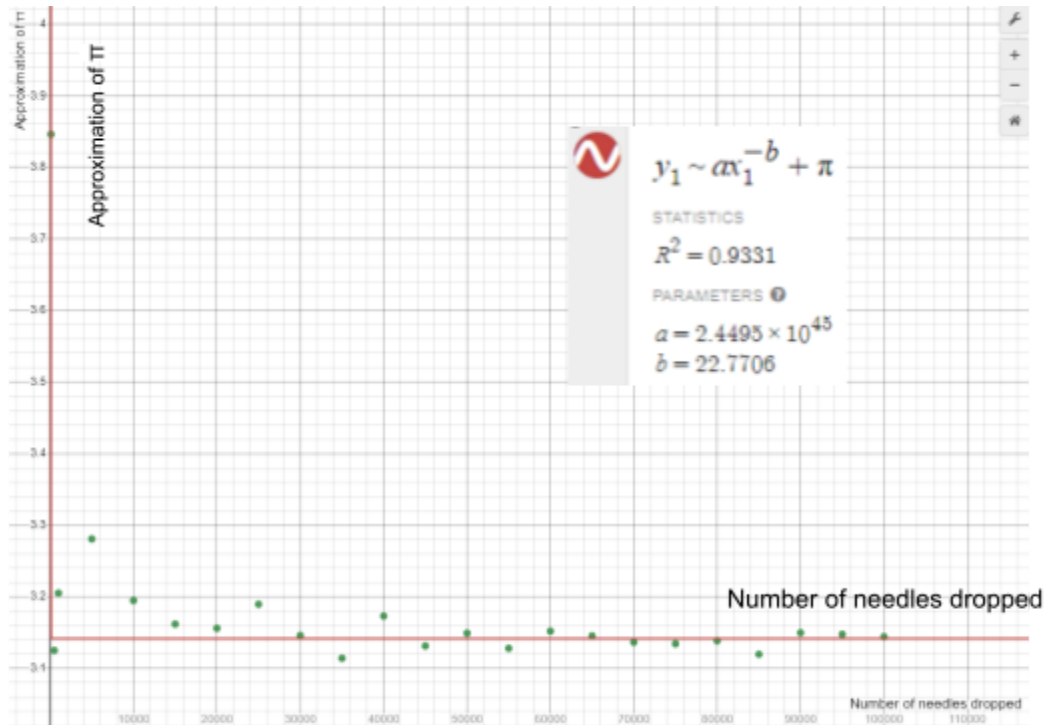


Figure 9: Graph for approximation of π of Buffon's needle experiment, with regression attempted

As seen from the graph, the results of the experiment portrayed a convergence to π . This is in accordance with expectations and further analysis can be conducted.

Part Two: Pólya's Recurrence Theorem

Another way to investigate the occurrence of π is modifying Polyá's recurrence theorem, another Monte Carlo experiment.

2.1 Background information

In the early 1900s, George Pólya was taking a walk in a park, when he noticed that he crossed the same couple quite often in that park, even though it seemed to him that they both were taking random walks [9]. He took great interest in this observation and

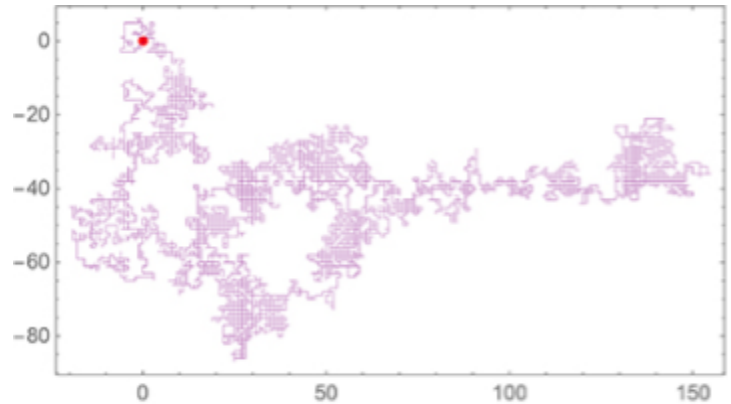


Figure 6: A random walk on a 2-D plane [10]

developed it into Pólya's recurrence theorem. Pólya's recurrence theorem states that a simple random walk on a d -dimensional lattice is recurrent for $d = 1, 2$ and transient for $d > 2$ [7]. A random walk is a path on a graph or lattice where each step is determined randomly [8]. He defined a random walk as recurrent if the walker passes through every single point on a lattice with probability one (i.e. the walker would always return to the origin), otherwise the walk is transient [8]. This means that for a walker in a 1-dimensional line or a 2-dimensional plane, the walker is always able to return to its starting position provided that he/she has taken ∞ steps. As mathematician Shizuo Kakutani famously remarked, "A drunk man will find his way home, but a drunk bird may get lost forever". For the case of simplicity, the d -dimensional lattice would be replaced with a 1-dimensional line in this recurrence experiment.

3.2 Calculations

Some variables have to be defined:

- n : number of steps taken by the walker.
- S_n : position of the walker after n steps.
- X_i : step taken at step number i .

Lemma 2: $S_n = 0$ occurs if and only if $n = 2k$ for some integer $k > 0$

Proof: In a 1-dimensional line, the walker has the option of taking a step towards the left or the right at random. Assume $S_0 = 0$, meaning the walker starts at position 0 along the number line.

Each step X_i taken can be one unit away from position S_{i-1} . Without loss of generality assume if the walker takes a step to the right, $X_i = +1$. If the walker takes a step leftwards, $X_i = -1$.

Therefore the position of the walker after n steps can be determined by:

$$S_n = \sum_{i=1}^n x_i$$

Since $X_i \in \{-1, +1\}$ (X_i is either positive one or negative one), X_i can only be an odd integer, so the parity of $S_n = n \cdot \text{odd}$. Basic mathematical operations dictates that

$$\text{odd} \cdot \text{even} = \text{even}$$

$$\text{odd} \cdot \text{odd} = \text{odd}$$

As S_n needs to be even (0 is an even integer), n must be even. Thus $n = \text{even}$ is both sufficient and necessary for $S_n = 0$, completing the lemma.

Basing on lemma 2, the problem transforms to: given that the walker takes $2n$ steps, find $P(S_{2n} = 0)$.

A peculiar observation is made by developing a stimulation for a population of random walkers and graphing a histogram based on their position S_{2n} on the number line:

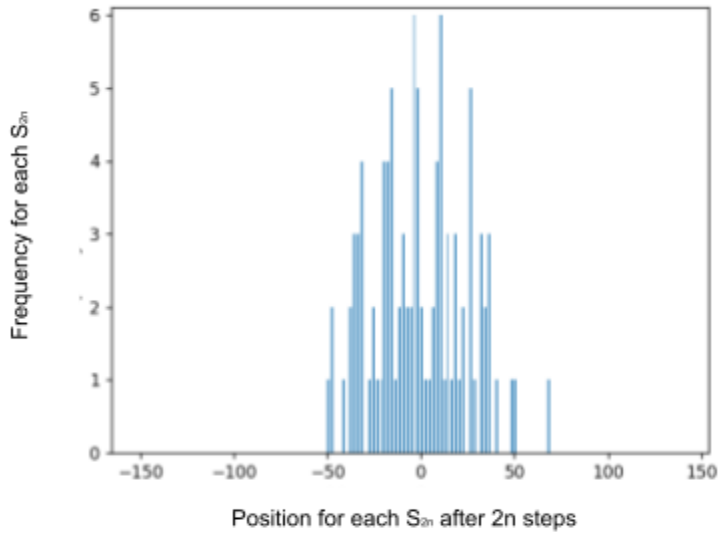


Figure 7: Graph of outcome of recurrence experiment with population 100, each sample taking a random walk of 500 steps

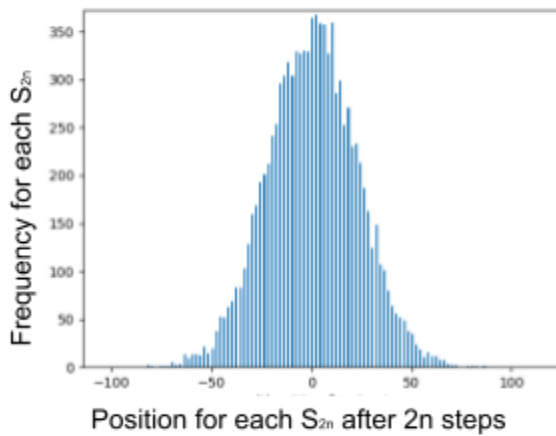


Figure 8: Graph of outcome of recurrence experiment with population 10000, each sample taking a random walk of 500 steps

As can be seen, for a certain number of $2n$, as the population increases, the graph converges more and more like a bell-shaped curve. The Central Limit Theorem dictates that a population

with a defined mean μ and standard deviation σ and take sufficiently large random samples from the population with replacement, then the distribution of the sample means will be approximately normally distributed [14]. Calculations for its mean and its standard deviation are thus conducted:

As $S_{2n} = \sum_{i=1}^{2n} x_i$, $E(S_{2n}) = E(\sum_{i=1}^{2n} x_i)$. Because each step x_i has equal probability of going left or right for some other step x_j (where $j \neq i$),

$$E(\sum_{i=1}^{2n} x_i) = E(2nx_i) = 2nE(x_i) [11]$$

For x_i it has an equal probability of going left or right. This means that $p = \frac{1}{2}$ for $x_i = -1$

(taking a step towards the left) and $p = \frac{1}{2}$ for $x_i = 1$ (taking a step towards the right). Thus:

$$E(x_i) = \frac{1}{2} \cdot (-1) + \frac{1}{2} \cdot 1 = 0$$

$$E(S_{2n}) = 2nE(x_i) = 0$$

The mean of S_n is 0, a constant. The standard deviation σ is calculated as the positive square root of its variance (σ^2). The variance can be calculated with the formula:

$$\sigma^2 = E(S_{2n}^2) - [E(S_{2n})]^2 = E(S_{2n}^2) [13]$$

$$E(S_{2n}^2) = E[(\sum_{i=1}^{2n} x_i)^2]$$

Expand the term $(\sum_{i=1}^{2n} x_i)^2 = (x_1 + x_2 + x_3 + \dots + x_{2n}) \cdot (x_1 + x_2 + x_3 + \dots + x_{2n})$

By multiplying and grouping same terms:

$$E\left(\sum_{i=1}^{2n} x_i\right)^2 = E\left[\left(\sum_{i=1}^{2n} x_i^2\right) + \sum_{i \neq j} x_i x_j\right]$$

According to the linearity of expectation:

$$E\left[\left(\sum_{i=1}^{2n} x_i^2\right) + \sum_{i \neq j} x_i x_j\right] = \sum_{i=1}^{2n} E(x_i^2) + \sum_{i \neq j} E(x_i x_j)$$

$$E(x_i^2) = \frac{1}{2} \cdot (-1)^2 + \frac{1}{2} \cdot 1^2 = 1$$

Linearity of Expectation :

For random expectations X,Y, no matter what the joint distribution is,
 $E(X + Y) = E(X) + E(Y)$ [16]

There are four different outcomes for $x_i x_j$:

$x_i = 1$	$x_i = 1$	$x_i = -1$	$x_i = -1$
$x_j = -1$	$x_j = 1$	$x_j = -1$	$x_j = 1$
$x_i x_j = -1$	$x_i x_j = 1$	$x_i x_j = 1$	$x_i x_j = -1$

With each situation sharing the same probability ($\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$)

$$\text{Thus } \sum_{i \neq j} E(x_i x_j) = 0$$

$$\sigma^2 = E\left[\left(\sum_{i=1}^{2n} x_i^2\right) + \sum_{i \neq j} x_i x_j\right] = \sum_{i=1}^{2n} 1 + 0 = 2n$$

The standard deviation $\sigma = \sqrt{2n}$, a constant value with a defined value of $2n$. This obeys the

Central Limit Theorem, so theoretically, the distribution of S_{2n} should converge towards a

normal distribution. However, as seen from Lemma 2, S_{2n} can only be even. This means that the

distribution of S_{2n} is not continuous (as all odd numbers have zero frequency). Instead, the

distribution can be modeled with a binomial distribution.

We observe that for $S_n = 0$, there must be an equal amount of steps to the left and steps to the right (as steps share equal length). Thus there must be n steps to the left and n steps to the right.

Both steps have the probability of $\frac{1}{2}$ to occur.

$$P(S_n = 0) = C_{2n}^n \left(\frac{1}{2}\right)^n \left(\frac{1}{2}\right)^{2n-n} = C_{2n}^n \left(\frac{1}{2}\right)^{2n}$$

Or it can be think of simply in a combinatorics perspective:

$$P(S_n = 0) = \frac{\text{outcomes with } S_n=0}{\text{total number of possible outcomes}}$$

The denominator can be thought as a combination of choosing n steps to the right/left out of $2n$ steps: C_{2n}^n

The numerator is all the possible trajectories of the walker when taking $2n$ steps. With each step the walker has two options (left or right), so the numerator simplifies to 2^{2n} .

Thus

$$P(S_{2n} = 0) = \frac{C_{2n}^n}{2^{2n}}$$

This expression at first glance seems unable to be simplified. Fortunately, there is a method named Stirling's approximation which can provide a great estimate of $n!$.

Stirling's approximation:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$$

Where \sim means that their ratio tends to 1 as n tends to infinity [12].

$$C_{2n}^n = \frac{(2n)!}{n! \cdot n!} = \frac{(2n)!}{(n!)^2} = \frac{\sqrt{4\pi n} \left[\frac{(2n)}{e} \right]^{2n}}{\left[\sqrt{2\pi n} \left(\frac{n}{e} \right)^n \right]^2}$$

$$C_{2n}^n = \frac{\sqrt{4\pi n} \cdot (2n)^{2n} \cdot e^{-2n}}{2\pi n \cdot n^{2n} \cdot e^{-2n}}$$

$$C_{2n}^n = \frac{2^{2n}}{\sqrt{\pi n}}$$

$$P(S_{2n} = 0) = \frac{C_{2n}^n}{2^{2n}} = \frac{\frac{2^{2n}}{\sqrt{\pi n}}}{2^{2n}} = \frac{1}{\sqrt{\pi n}}$$

Surprisingly π makes its appearance in this seemingly unrelated probability experiment. A way of experimentally deriving π is constructed.

3.3 Experiment

The equation obtained for the probability of returning to the starting point after $2n$ steps can be altered in the following manner:

$$P(S_{2n} = 0) = \frac{1}{\sqrt{\pi n}}$$

$$\sqrt{\pi n} = \frac{1}{P(S_n=0)}$$

$$\pi = \frac{1}{n[P(S_{2n}=0)]^2}$$

This equation links $P(S_{2n} = 0)$ with π with consideration of an arbitrary value n . Note that even though Pólya's recurrence theorem claimed that the number of steps approaches infinity,

such condition does not have to be met when trying to find the recurrence of π with the equation above. This is because as $2n$ approaches infinity, n approaches infinity also, and the probability

approaches asymptotically to $\frac{1}{\sqrt{n}}$.

$$P(S_{2n} = 0) = \frac{1}{\sqrt{\pi n}} \propto \frac{1}{\sqrt{n}}$$

A computer simulation of Pólya's recurrence experiment is created to obtain results and approximate π .

Below are the results of the experiment, constructed using <https://www.desmos.com/calculator>:

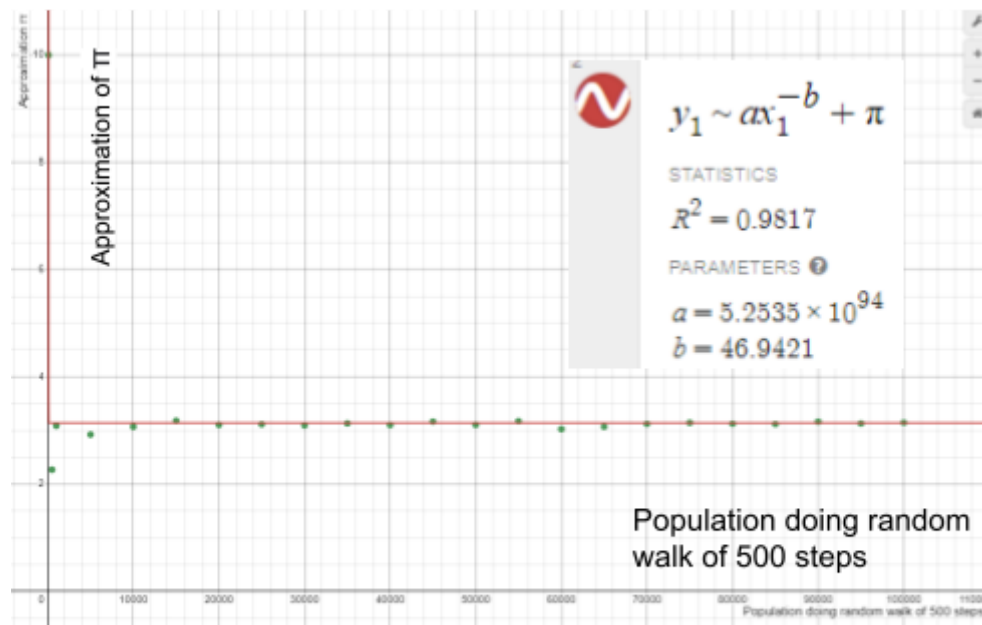


Figure 10: Graph for approximation of π of Pólya's recurrence experiment, With regression attempted

Similar to Buffon's needle experiment, The results for Pólya's recurrence experiment also demonstrates a convergence towards π . This is in accordance with expectations and further analysis can be conducted.

Analysis

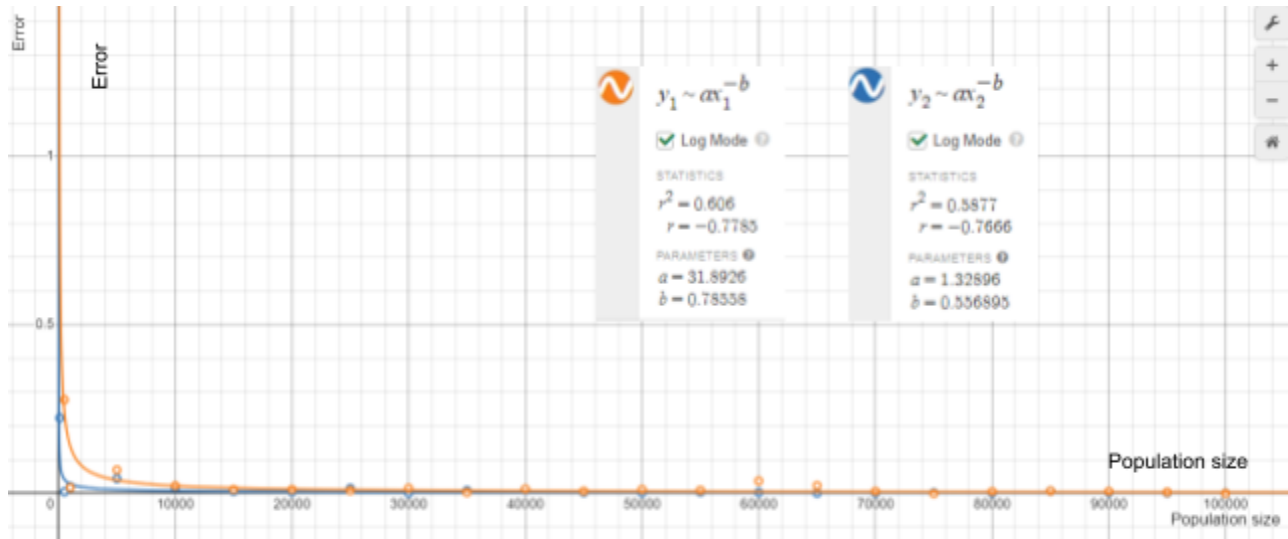


Figure 11: Error for both experiments (Buffon: Blue, Pólya: Orange), with Regression

(All of the above graphs are constructed using <https://www.desmos.com/calculator>)

The graphs in Figures 9 and 10 represent the approximation of π by both experiments. The regression applied is a power series function with a negative exponent and a vertical translation of π . This is because these experiments are Monte Carlo simulations where their inherent randomness is used to obtain numerical results. In theory, as the population increases, the effect of random error decreases, leading to a more accurate result. Yet the random error still persists, hence the results are only approximations. By this logic the regression function should have a horizontal asymptote of π as well as portraying a decaying-with-decreasing-rate relation:

$$\lim_{x \rightarrow \infty} R(x) = \pi$$

Note that this does not rule out a polynomial function in the form of:

$$P(x) = \pi + ax^{-1} + bx^{-2} + \dots + ix^{-j} \text{ for some positive integer } j. \text{ But for the sake of}$$

simplicity, we chose $R(x)$ to be in the form of $R(x) = ax^{-b} + \pi$ (an inverse power function) for some real number a and positive integer b .

Both regression equations showed to fit the data points well (with $R^2 = 0.9331, 0.9817$ respectively). The data points have successfully portrayed evidence of a decay relationship with the decay rate decreasing accompanied by increasing population size, which aligns with the hypothesis of the decreasing effect of random error.

Both graphs possess a very sharp turning point where the derivative of the function changes from a very large negative value to a very small negative value. This is the result of very large coefficients (a) which increases the y value dramatically when $x < 1$. The large b values also cause these sharp turning points, because as x increases after $x > 1$, the term ax^{-b} would quickly converge towards 0, making $R(x) \approx 0 + \pi = \pi$.

Note that $R(x) \neq \pi$ for any real x values. It just asymptotically converges towards π . The decay rate is upper bounded by $O(x^{-c})$ in this case which prevents it from crossing its horizontal asymptote. Thus the term “approximation” instead of “obtain”.

Figure 11 portrays the error of the experiments from π . The regression for such are also relatively accurate to the pattern of the data points as they both possess small R^2 values. Log mode is enabled on both functions, which linearizes the model by a logarithmic transformation [15] (seems appropriate to apply since the function is a power function of the form $(y = ax^{-b})$).

It can be observed that the pattern for error follows a similar decay relationship with the pattern for approximating π . The error decrease belongs to the error in the inherent randomness in the experiment (random error). As the population size increases, the effect of this error decreases.

By comparing both experiments, one interesting pattern to observe is that the error for Buffon's needle experiment is relatively smaller than Pólya's recurrence experiment. The one part that differs the two is the employment of Stirling's approximation in the latter. This opens up a limitation for Pólya's recurrence experiment: the degree of validity of Stirling's approximation. According to a paper by 蔡永裕 which experimented on the errors of Stirling's approximation[17], for $n!$, if $n = 500$ (employed in the calculations), the error of Stirling's approximation is around -1.66653×10^{-4} . This may seem insignificant, but when faced with data points with similar precision, the error can pose a significant effect.

Some other possible errors which can contribute to inaccuracies in both experiments include but not limited to:

Error description	Type of error	Possible solutions
Each data point is conducted with only one trial due to time concerns. There is uncertainty in each trial as each run of the program would yield slightly different results	Random	Conduct more trials for each data point and take their average value. Cannot eliminate this error but can decrease its effectiveness.
All the randomness necessary in the experiments are generated by the computer. This is classified as pseudorandom and each random value (the orientation of the needle in Buffon's needle experiment, the step of the person in Pólya's recurrence experiment) is generated using a pre-existing algorithm (values are purely computer generated). Hence it is not truly random.	Systematic	Employ an algorithm modeled on unpredictable physical phenomena. For example, the behavior of electrons in orbitals.

Conclusion

This paper aims to investigate the occurrence of π in Buffon's needle experiment and an experiment based on Pólya's recurrence theorem.

As shown by the analysis, both experiments contribute values similar to π , with the error getting progressively smaller as the population size is increased. The rate at which the error is decreasing slows down with the population size increasing at a constant rate. These conclusions help show that both experiments are able to approximate π to a given degree of accuracy. The word approximate is necessary in this context as it is impossible to obtain the exact value of π using these two experiments. Pólya's recurrence experiment employed Stirling's approximation which contributes to a marginally bigger error than Buffon's needle experiment. Nonetheless, they both follow the same trend of approximating π , and with a large population size the marginal error is negligible.

Overall, I believe the investigation was a success as I feel that I have a better grasp upon the marvelous property of π and how different mathematical concepts can be interlinked together.

Appendix

6.1 Programming used

All simulations are run on vivobook model M513U with the following specifications:

Device name	LAPTOP-5MF0JE7O	
Processor	AMD Ryzen 7 5700U with Radeon Graphics	1.80 GHz
Installed RAM	16.0 GB (15.4 GB usable)	
Device ID	C42A85A5-2CD8-4D22-9C82-0E752FD3193A	
Product ID	00325-97204-88164-AAOEM	
System type	64-bit operating system, x64-based processor	
Pen and touch	No pen or touch input is available for this display	
Edition	Windows 10 Home	
Version	21H2	
Installed on	2021/4/27	
OS build	19044.2364	
Experience	Windows Feature Experience Pack 120.2212.4190.0	

IDE used:

Name: PyCharm Community Edition
Version: 2022.3
Build: 223.7571.203

Buffon's Needle Experiment

```

import matplotlib.pyplot as plt
import random
import math
import numpy as np

NUMBER_OF_NEEDLES = 100000

class DefineNeedle:
    def __init__(self, x=None, y=None, theta=None, length=0.5):
        if x is None:
            x = random.uniform(0, 1)
        if y is None:
            y = random.uniform(0, 1)
        if theta is None:
            theta = random.uniform(0, math.pi)

        self.needle_coordinates = np.array([x, y])
        self.complex_representation = np.array(

            [length/2 * math.cos(theta), length/2*math.sin(theta)])
        self.end_points = np.array([np.add(self.needle_coordinates, -1*np.array(
            self.complex_representation)), np.add(self.needle_coordinates,
self.complex_representation)])

    def intersects_with_y(self, y):
        return self.end_points[0][1] < y and self.end_points[1][1] > y

class BuffonSimulation:
    def __init__(self):
        self.floor = []
        self.boards = 2
        self.list_of_needle_objects = []
        self.number_of_intersections = 0

        fig = plt.figure(figsize=(10, 10))
        self.buffon = plt.subplot()
        self.results_text = fig.text(
            0, 0, self.estimate_pi(), size=10)
        self.buffon.set_xlim(-0.1, 1.1)
        self.buffon.set_ylim(-0.1, 1.1)

    def plot_floor_boards(self):
        for j in range(self.boards):
            self.floor.append(0+j)
            self.buffon.hlines(

```

```

        y=self.floor[j], xmin=0, xmax=1, color='black', linestyle='--',
linewidth=2.0)

def toss_needles(self):
    needle_object = DefineNeedle()
    self.list_of_needle_objects.append(needle_object)
    x_coordinates = [needle_object.end_points[0]
                     [0], needle_object.end_points[1][0]]
    y_coordinates = [needle_object.end_points[0]
                     [1], needle_object.end_points[1][1]]

    for board in range(self.boards):
        if needle_object.intersects_with_y(self.floor[board]):
            self.number_of_intersections += 1
            self.buffon.plot(x_coordinates, y_coordinates,
                             color='green', linewidth=1)
        return
    self.buffon.plot(x_coordinates, y_coordinates,
                     color='red', linewidth=1)

def estimate_pi(self, needles_tossed=0):
    if self.number_of_intersections == 0:
        estimated_pi = 0
    else:
        estimated_pi = (needles_tossed) / \
            (1 * self.number_of_intersections)
    error = abs((math.pi - estimated_pi)/math.pi)*100
    print("Intersections:", str(self.number_of_intersections),
          "\nestedimated_pi:", str(estimated_pi),
          "\n error:", str(error/100))
    return (" Intersections:" + str(self.number_of_intersections) +
            "\n Total Needles: " + str(needles_tossed) +
            "\n Approximation of Pi: " + str(estimated_pi) +
            "\n Error: " + str(error) + "%")

def plot_needles(self):
    for needle in range(NUMBER_OF_NEEDLES):
        self.toss_needles()
        self.results_text.set_text(self.estimate_pi(needle+1))
        if (needle+1) % 200 == 0:
            plt.pause(1/200)
    plt.title("Estimation of Pi using Probability")

def plot(self):
    self.plot_floor_boards()
    self.plot_needles()
    plt.show()

```

```
simulation = BuffonSimulation()
simulation.plot()
```

The above code is based heavily on [4] with slight modifications.

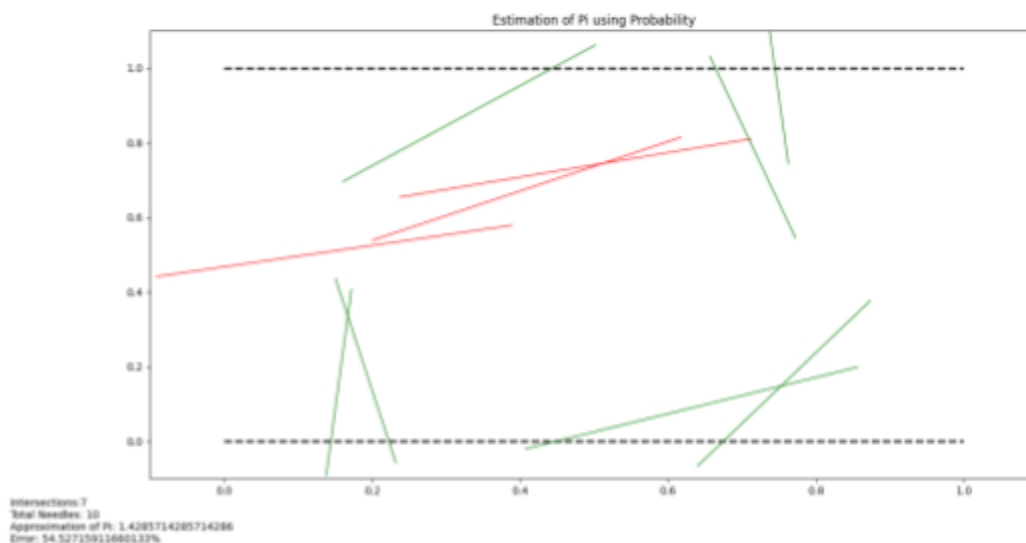


Figure 12: Image output of Buffon's needle experiment, when `NUMBER OF NEEDLES = 10`

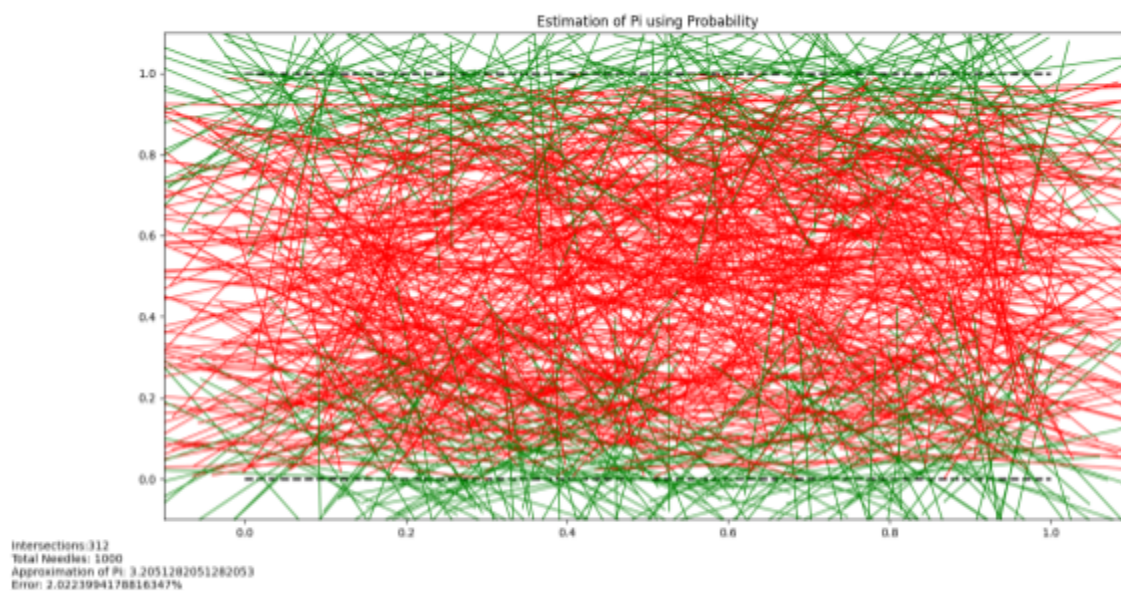


Figure 13: Image output of Buffon's needle experiment, when `NUMBER OF NEEDLES = 1000`

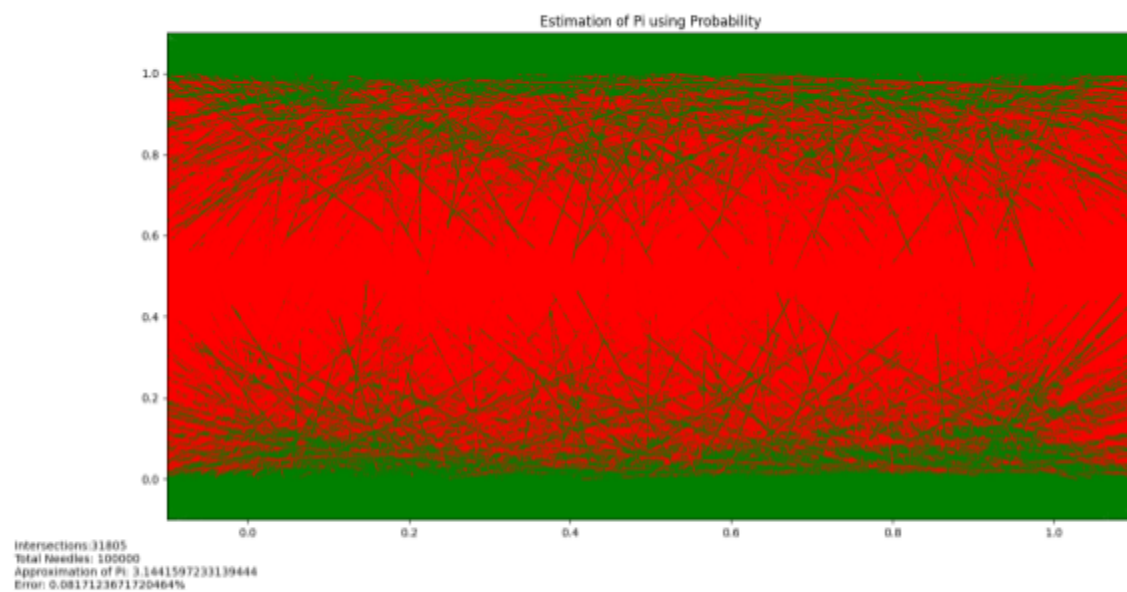


Figure 14: Image output of Buffon's needle experiment, when `NUMBER OF NEEDLES = 100000`

Pólya's Recurrence Experiment

```

import matplotlib.pyplot as plt
import random
import math
import numpy as np

collection = []
bins = []

def step_counter(i):
    sn = 0
    stepsList = [-1, 1]
    for j in range(i):
        x = random.choice(stepsList)
        sn = sn + x
    collection.append(sn)
    # if sn == 0:
    #     p += 1

def recurrence_experiment(i, j):
    for k in range(i):
        step_counter(j)

    count0 = (collection.count(0))
    print(count0)

    experimentpi = ((i/count0)**2) / (j/2)
    print(experimentpi)
    error = abs((math.pi - experimentpi) / math.pi)
    print(error)
    s = -1 * j
    for l in range(2 * j):
        bins.append(s)
        s += 1

recurrence_experiment(10000, 500) # i = population size, j = 2n
plt.hist(collection, bins=bins)
plt.xlabel("position S2n after 2n steps")
plt.ylabel("frequency for each S2n")
plt.show()

```

6.2 Raw data

(Data points highlighted in red are considered as outliers and are excluded from analysis.)

Raw data for Buffon's Needle Experiment

Needles tossed	Intersections	Approximation for π	Error
10	7	1.428571428571420000	0.545271591166013000
100	26	3.846153846153840000	0.224268793014579000
500	160	3.125000000000000000	0.005281605675654110
1000	312	3.205128205128200000	0.020223994178816300
5000	1524	3.280839895013120000	0.044323773568867100
10000	3130	3.194888178913730000	0.016964492599970300
15000	4744	3.161888701517700000	0.006460432705914830
20000	6337	3.156067539845350000	0.004607499396530450
25000	7838	3.189589180913490000	0.015277769149625700
30000	9536	3.145973154362410000	0.001394356702361690
35000	11239	3.114156063706730000	0.008733337802947450
40000	12606	3.173092178327780000	0.010026610134192200
45000	14371	3.131306102567670000	0.003274310885075390
50000	15877	3.149209548403350000	0.002424532921177400
55000	17583	3.128021384291640000	0.004319869185662920
60000	19036	3.151922672830420000	0.003288147248762340
65000	20666	3.145262750411300000	0.001168228101538470
70000	22318	3.136481763598880000	0.001626846811302600
75000	23928	3.134403209628880000	0.002288471088920910
80000	25489	3.138608811644230000	0.000949786390079894
85000	27246	3.119723996182960000	0.006961009850172190
90000	28574	3.149716525512700000	0.002585908747153370
95000	30182	3.147571400172280000	0.001903094144195770
100000	31805	3.144159723313940000	0.000817123671720464

Raw data for Pólya's Recurrence Experiment

population size		Occurrences for $S_{2n} = 0$	Approximation for π	Error
10		0	Null	Null
100		2	10.000000000000000	2.18309886183790
500		21	2.26757369614512	0.27820887486669
1000		36	3.08641975308641	0.01756207967966
5000		185	2.92184075967859	0.06994920033955
10000		361	3.06934415788706	0.02299741044409
15000		532	3.17994233704562	0.01220708337601
20000		718	3.10363823992675	0.01208126509325
25000		896	3.11403858418367	0.00877073269656
30000		1079	3.09214329335592	0.01574022022790
35000		1251	3.13098841470388	0.00337543407284
40000		1437	3.09932013507224	0.01345576055802
45000		1600	3.164062500000000	0.00715237425340
50000		1795	3.10363823992675	0.01208126509325
55000		1953	3.17234844760735	0.00978987329322
60000		2182	3.02449589634716	0.03727305547039
65000		2347	3.06803652379480	0.02341364330316
70000		2506	3.12100121719047	0.00655445777663
75000		2676	3.14202980152426	0.00013914850927
80000		2863	3.12318182932811	0.00586034737528
85000		3045	3.11690272621137	0.00785904797371
90000		3201	3.16208588752284	0.00652319896077
95000		3397	3.12835556213386	0.00421349707474
100000	2n = 500	3567	3.14379176969461	0.00070000039703

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Error in Stirling's approximation

17. 蔡永裕. 談 *stirling* 公式的改良. https://web.math.sinica.edu.tw/math_media/d204/20406.pdf.

表六. 計算 $n!$ 時, Stirling 公式與其改良式
分別之相對誤差

n	Stirling 公式	改良式
1	-7.78630E-02	-4.16219E-04
2	-4.04978E-02	-1.83817E-05
3	-2.72984E-02	-2.64893E-06
4	-2.05760E-02	-6.51314E-07
5	-1.65069E-02	-2.17155E-07
6	-1.37803E-02	-8.81200E-08
7	-1.18262E-02	-4.10132E-08
8	-1.03573E-02	-2.11183E-08
9	-9.21276E-03	-1.17508E-08
10	-8.29596E-03	-6.95219E-09
11	-7.54507E-03	-4.32298E-09
12	-6.91879E-03	-2.80103E-09
13	-6.38850E-03	-1.87879E-09
14	-5.93370E-03	-1.29793E-09
15	-5.53933E-03	-9.19761E-10
16	-5.19412E-03	-6.66388E-10
17	-4.88940E-03	-4.92317E-10
18	-4.61846E-03	-3.70051E-10
19	-4.37596E-03	-2.82468E-10
20	-4.15765E-03	-2.18619E-10
21	-3.96009E-03	-1.71327E-10
22	-3.78045E-03	-1.35791E-10
23	-3.61641E-03	-1.08746E-10
24	-3.46600E-03	-8.79178E-11
25	-3.32761E-03	-7.16885E-11
26	-3.19984E-03	-5.89342E-11
27	-3.08152E-03	-4.87976E-11
28	-2.97164E-03	-4.06906E-11
29	-2.86932E-03	-3.41431E-11
30	-2.77382E-03	-2.88256E-11
100	-8.32983E-04	-7.975 E-14
200	-4.16580E-04	-4.876 E-14
300	-2.77739E-04	-7.211 E-14
400	-2.08312E-04	8.530 E-14
500	-1.66653E-04	-7.188 E-14
600	-1.38879E-04	-1.126 E-13
700	-1.19041E-04	1.817 E-13
800	-1.04161E-04	-1.325 E-13
900	-9.25883E-05	-3.649 E-13
1000	-8.33299E-05	-2.971 E-13

Figure 15: When calculating $n!$, the relative error in Stirling's formula and its improved formula