HW3 Report

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1 Oscillatory Motion and Chaos.

1.1 max Ω_D value for small-angle (linear) approximation

The ODE has been defined as:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta - 2\gamma\frac{d\theta}{dt} + \alpha_D sin(\Omega_D t) \tag{1}$$

To solve the problem, we consider it as a complex differential equation,

$$\frac{d^2z}{dt^2} = -\frac{g}{l}z - 2\gamma\frac{dz}{dt} + \alpha_D exp(\Omega_D t) \tag{2}$$

where z is a complex number z(t)=x(t)+iy(t). Here x(t) and y(t) are both real functions and $\sqrt{-1} = \pm i$ and $\theta = Im(z)$

Since z obtains the equation:

$$z = Aexp(i\Omega_D t) \tag{3}$$

$$\frac{dz}{dt} = Ai\Omega_D e^{i\Omega_D t} \tag{4}$$

$$\frac{d^2z}{dt^2} = -A\Omega_D^2 e^{i\Omega_D t} \tag{5}$$

By plugging equations 3,4,5 into equation 2:

$$a_D e^{\Omega_D t} = Ae^{i\Omega_D t}(\frac{g}{l} - \Omega_D^2 + 2\gamma i\Omega_D) \tag{6}$$

Since the imaginary part and the non-imaginary part follow the rule:

$$a + ib = re^{i\theta} = (\sqrt{a^2 + b^2})e^{i\theta} \tag{7}$$

By plugging equation 7 into equation 6:

$$a_D e^{\Omega_D t} = Ae^{i\Omega_D t}\sqrt{(\frac{g}{l} - \Omega_D^2)^2 + (2\gamma i\Omega_D)^2} \tag{8}$$

Thus, we can obtain:

$$A = \frac{\alpha_D}{\sqrt{(\frac{g}{l} - \Omega_D^2)^2 + (2\gamma i \Omega_D)^2}} = \frac{\alpha_D}{\sqrt{(\frac{g}{l} - \Omega_D^2)^2 + 4\gamma^2\Omega_D^2}} \tag{9}$$

In order to find the maximum value of Ω_D, we need to find max value of A, which need to take the derivative of A and find the value when it is equal to 0. Since the top part is a number, we are focusing on the denominator part.

$$\frac{d}{d\Omega_D}((\frac{g}{l} - \Omega_D^2)^2 + 4\gamma^2\Omega_D^2)) = 0 \tag{10}$$

By applying the chain rule:

$$\frac{d}{d\Omega_D}((\frac{g}{l} - \Omega_D^2)^2 = \frac{d}{dg}g^2\frac{d}{d\Omega_D}(\frac{g}{l} - \Omega_D^2) = 4\Omega_D(\Omega_D^2 - \frac{g}{l}) \tag{11}$$

Plugging equation 11 back into equation 10:

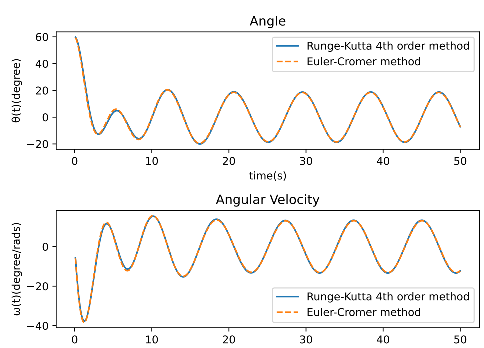
$$4\Omega_D(\Omega_D^2 - \frac{g}{l}) + 8\gamma^2\Omega_D = \Omega_D^2 - \frac{g}{l} + 2\gamma^2 = 0 \tag{12}$$

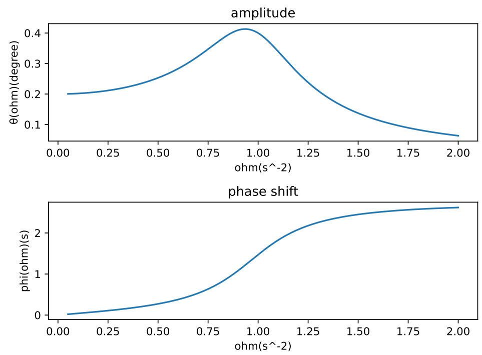
From which, we are able to obtain the Ω_D value:

$$\Omega_D = \sqrt{\frac{g}{l} - 2\gamma^2} = \sqrt{\frac{9.8m/s^2}{9.8m} - 2 * (0.25s^{-1})^2} \approx 0.71s^{-} \tag{13}$$

1.2 Euler-Cromer vs the Runge-Kutta 4th order

By comparing Euler-Cromer and Runge-Kutta 4th order methods in simulating the angle of the pendulum over time, we find that the difference between those two methods is super small. The difference in the $\theta(t)$ value over 500 points is 0.081841 and 0.055021 for $\omega(t)$. Even though the Euler-Cromer method has an error of $O(\Delta t^2)$ and Runge-Kutta 4th order method obtains an error of $O(\Delta t^5)$. There is no large difference in accuracy.



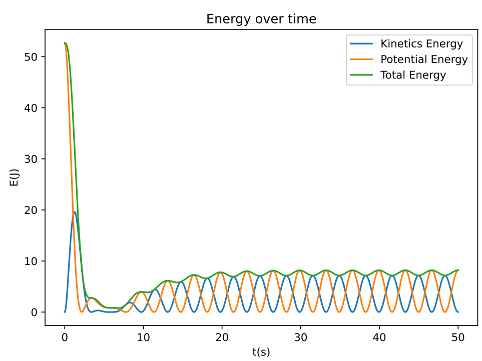


By further analyzing Runge-Kutta 4th order methods, we extract the amplitude $\theta_0(_D)$ and phase shift $\phi(_D)$ for at least 10 different driving frequencies, mapping out the resonance structure and plotting $\theta_0(_D)$ and $\phi(_D)$.

From there, we extract the full-width at half maximum of the amplitude graph to be 1.118 which is much larger than γ value (0.25). This is due to the driving force that makes the FWHM larger than expected.

1.3 energy calculation

To further analyze the pendulum system, we measure the kinetic, potential and total energy.

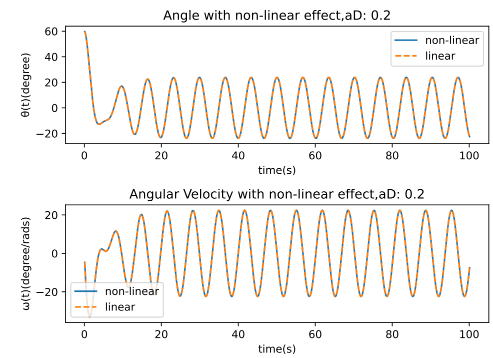


From the figure, we find that the PE and KE are fluctuating reversely, with the increment of PE would have the decrement of KE. This makes the total energy fluctuating in a stable range.

1.4 non-linear effects

By switching on non-linear effects by replacing θ with $sin(\theta)$ in the restoring force and plot and compare to your previous results for $\theta(t)$ and $\Omega(t)$ using Ω_D

close to resonance.



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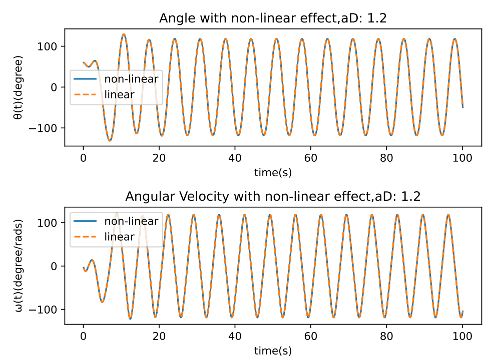
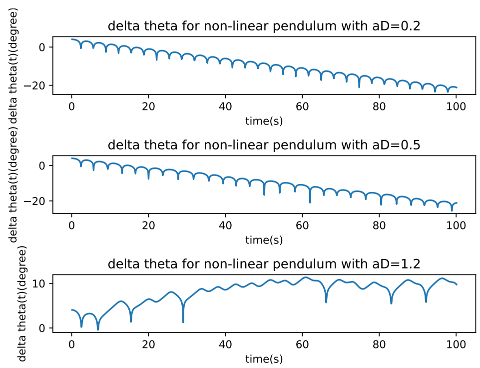


Since the Ω_D value we chose is close to the resonance, there are no large difference in terms of non-linear and linear graph.

1.5 Lyapunov exponent

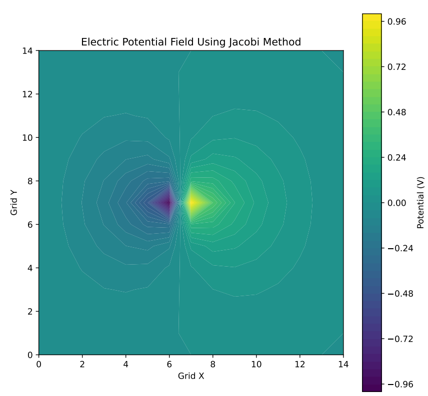
In this section, we use the non-linear pendulum with $\Omega_D = 0.666s^1$ and values of $a_D = 0.2$, 0.5 and 1.2 rad/s^2 to compute $|\Delta\theta(t)|$ for several trajectories with slightly different initial angle ($\Delta\theta$ is around 0.001 rad). From the graph, we find that the slope for aD = 0.2 and aD = 0.5 are negative, whereas the slope of aD= 1.2 is positive. This corresponds to the Lyapunov exponent of the system, $\lambda_{0.2} = -0.250, \lambda_{0.5} = -0.251, \lambda_{1.2} = 0.069$.



2 Poisson Equation

2.1 the equipotential lines

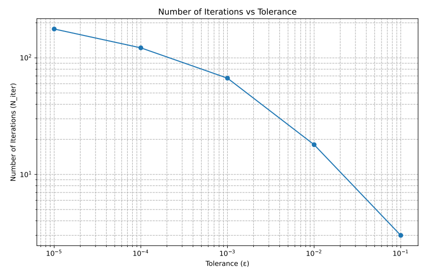
By using the Jacobi relaxation algorithm, we are able to plot the equipotential lines.



Compare to the literature, the general shape looks similar with one side positive and the other side negative. (https://en.wikipedia.org/wiki/Electric$_dipole_moment$

2.2 Iteration steps

The iteration steps have been plotted as below. The decrement of tolerence would leads to the increment of iteration, since there need to be more rounds in order to get the required tolerence. From the graph, I could see a N correlate to n^2 as mentioned in the third question.



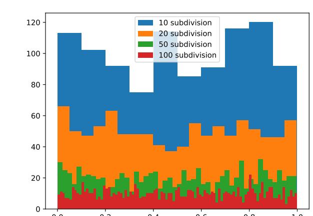
2.3 Simultaneous Over-Relaxation Method (not work out)

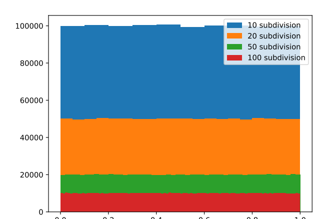
I have tried to work on this part but I am not able to figure out due to time constraints. I am able to print out the graph. However, I cannot get the graph converge with the smaller tolerance.

3 Random Numbers

3.1 random number generation

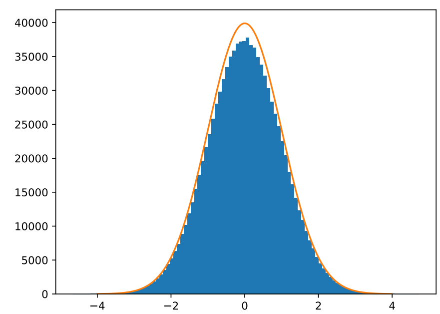
1000 and 1,000,000 random numbers have been generated accordingly. As we can see, 1,000,000 has a much equal amount comparing to 1000 random.





3.2 Gaussian distribution

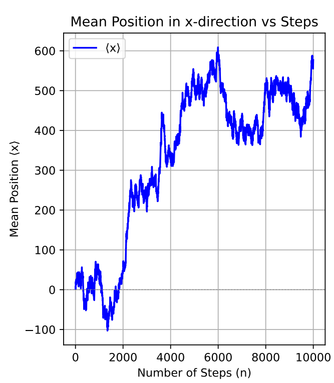
The distribution was plotted with the Box-Muller algorithm method. Comparedto the Gaussian ddistribution, thethe line shape and structuare are really similar.

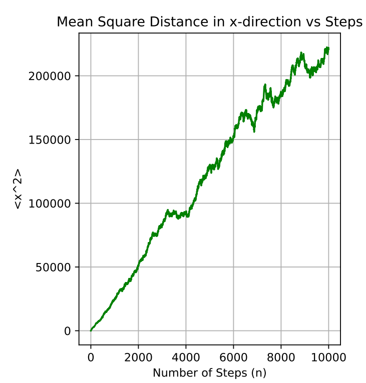


4 2D Random Walker

4.1 $< x_n >$ and $< x_n^2 >$ vs t

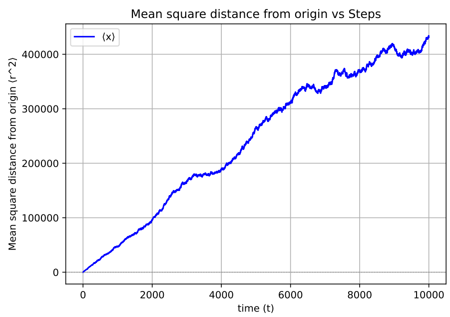
The results of $< x_n >$ and $< x_n^2 >$ vs t in the random walker are shown below. Compared with each other, $< x_n^2 >$ vs. t shows a smoother line.





4.2 diffusive motion

To show the motion is diffusive, we plug the mean square distance from the starting point $< r^2 > vst$. The diffusion constant is around 53.68.



5 Diffusion Equation

5.1 The spatial expectation value

To prove that $< x(t^2) > = \sigma(t)^2$, we first get the equation of $< x(t^2) >$ from the definition,

$$< x(t^2) >= \int x^2 \rho(x,t) dx \tag{14}$$

By pluging the $\rho(x,t)$ from given equation in the problem,

$$\rho(x,t) = \frac{1}{\sqrt{2\pi\sigma(t)^2}} exp(-\frac{x^2}{2\sigma(t)^2}) \tag{15}$$

The new equation would be,

$$< x(t^2) >= \int x^2 \frac{1}{\sqrt{2\pi\sigma(t)^2}} exp(-\frac{x^2}{2\sigma(t)^2}) dx = \frac{1}{\sqrt{2\pi\sigma(t)^2}} \int x^2 exp(-\frac{x^2}{2\sigma(t)^2}) dx \tag{16}$$

By using substitution $u = \frac{x}{\sqrt{2}\sigma(t)}$,

$$< x(t)^2 >= \frac{1}{\sqrt{2\pi\sigma(t)^2}} \sqrt{2}\sigma(t) 2\sigma(t)^2 \int u^2 exp(-u^2) du \tag{17}$$

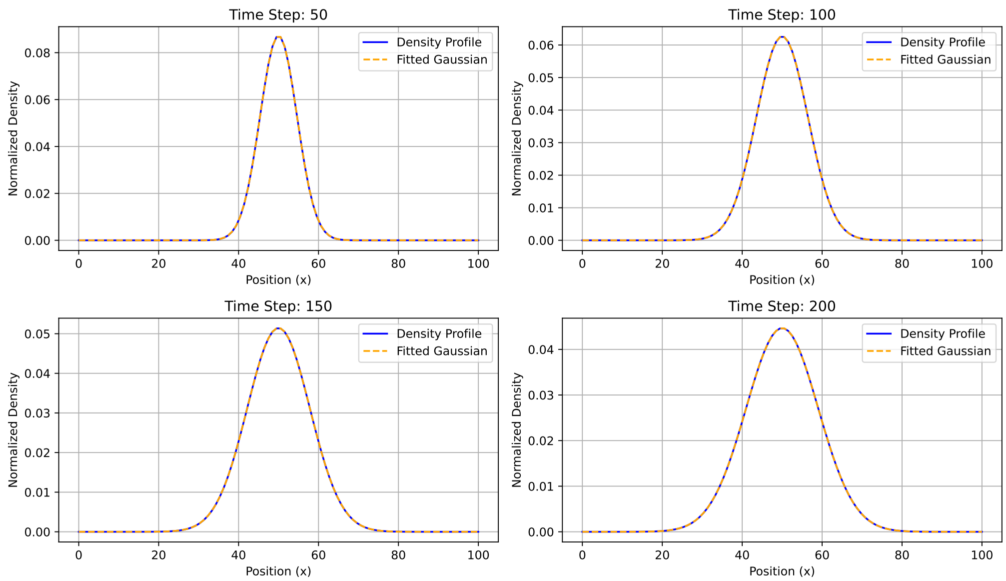
By changing into polar coordination, $\int u^2 exp(-u^2) du = \frac{1}{2}\sqrt{\pi}$

$$< x(t)^2 >= \frac{1}{\sqrt{2\pi\sigma(t)^2}} \sqrt{2}\sigma(t) 2\sigma(t)^2 \frac{1}{2}\sqrt{\pi} \tag{18}$$

By rewrite the equation, we are able to get $< x(t)^2 >= \sigma(t)^2$

5.2 1D diffusion equation

The 1D diffusion equation using the finite difference form with a diffusion constant D = 2. And compare it to normal distribution with $\sigma(t) = sqrt(2Dt)$



6 Contribution

Our group discusses questions with each other and compares our results and codes to try to understand each other.