1 BL in PDE problem

1.1 2-D N-S equations

$$\begin{cases} U\dfrac{\partial U}{\partial X}+V\dfrac{\partial V}{\partial Y}=-\dfrac{1}{\rho}\dfrac{\partial P}{\partial X}+\upsilon(\dfrac{\partial^2 U}{\partial X^2}+\dfrac{1}{\delta^2}\dfrac{\partial^2 U}{\partial Y^2}) \\ \delta U\dfrac{\partial U}{\partial X}+\delta V\dfrac{\partial V}{\partial Y}=-\dfrac{1}{\delta\rho}\dfrac{\partial P}{\partial Y}+\upsilon(\delta^2\dfrac{\partial^2 U}{\partial X^2}+\dfrac{\partial^2 U}{\partial Y^2}) \\ \dfrac{\partial U}{\partial X}+\dfrac{\partial V}{\partial Y}=0 \end{cases}$$

By dominent balance of (1) $\rightarrow$ $\delta=\upsilon^2$ $\frac{\upsilon}{\delta^2}=Os(1)$

Remark: With dimension $\frac{\upsilon}{\delta^2}=Os(\frac{u_0^@}{L})$ L : typical length.

Meanwhile $\frac{\delta}{L}=Os(\frac{\upsilon^{\frac{1}{2}}}{u_0L^{\frac{1}{2}}})=Os(\frac{1}{Re^{\frac{1}{2}}})$ $\quad Re=\frac{u_0L}{\upsilon}$, RE is called Reynolds number, which is a big number($Re>>1$).

Leading order:

$$\begin{cases} U_0\dfrac{\partial U_0}{\partial X}+V_0\dfrac{\partial V_0}{\partial Y}=-\dfrac{1}{\rho}\dfrac{\partial P_0}{\partial X}+\dfrac{\partial^2 U_0}{\partial Y^2} \\ \dfrac{\partial P_0}{\partial Y}=0 \\ \dfrac{\partial U_0}{\partial X}+\dfrac{\partial V_0}{\partial Y}=0 \end{cases}$$

Assume no BL in pressure, $P=P(x)$.

$$\lim_{Y\to+\infty}=\lim_{y\to 0}P(x,y)=p_0$$

. So $P(X)=p_0=$ constant in BL.

(3) is similar to $\frac{\partial U}{\partial Y}=\frac{\partial V}{\partial X}\rightarrow\nabla x(U,V,0)=(0,0,\frac{\partial U}{\partial Y}-\frac{\partial V}{\partial X})=(0,0,0)$

Introduce a operator $\begin{pmatrix}\frac{\partial}{\partial x}\\ \frac{\partial}{\partial y}\\ \nabla\end{pmatrix}\rightarrow\begin{pmatrix}\frac{\partial}{\partial y}\\ -\frac{\partial}{\partial x}\\ \nabla^{\perp}\end{pmatrix}$

However, we take $U=\frac{\partial\psi}{\partial Y}$ $V=-\frac{\partial\psi}{\partial X}$ for some $\psi(X,Y)$, $(U,V)=\nabla^{\perp}\psi$, where ψ is called stream function.

$$(1)\Rightarrow\frac{\partial\psi}{\partial Y}\frac{\partial^2\psi}{\partial X\partial Y}-\frac{\partial\psi}{\partial X}\frac{\partial^2\psi}{\partial^2 Y}=\frac{\partial^3\psi}{\partial Y^3}$$

Use similaity method

to find self-similar sol. $\psi(X,Y)=\psi(\tilde{X},\tilde{Y})$.

Here we take $\bar{X}=\epsilon^a X$ $\bar{Y}=\epsilon^b Y$ $\bar{\psi}=\epsilon^c\psi$

$$\Rightarrow\epsilon^{-2c+2b+a}\frac{\partial\bar{\psi}}{\partial\bar{Y}}\frac{\partial^2\bar{\psi}}{\partial\bar{X}\partial\bar{Y}}-\epsilon^{-2c+2b+a}\frac{\partial\bar{\psi}}{\partial\bar{X}}\frac{\partial^2\bar{\psi}}{\partial^2\bar{Y}}=\epsilon^{-c+3b}\frac{\partial^3\bar{\psi}}{\partial\bar{Y}^3}$$

Only if $2c-2b-a=c-3b\Leftrightarrow a=b+c$, there exists self-similar sol.

This property is called scaling invariance.

$$\begin{aligned} \psi&=f(X,Y)\quad X\sim\epsilon^{-a}\\ \bar{\psi}&=f(\bar{X},\bar{Y})\quad\epsilon\sim X^{-\frac{1}{a}}\\ &=\frac{f(X,Y)}{\epsilon^{-c}}\\ &\sim\frac{f(X,Y)}{X^{c/a}}\\ \bar{Y}&=\frac{Y}{X^{b/a}} \end{aligned}$$

$$\psi = f(X,Y) = X^{\frac{a-b}{a}}\bar{\psi} = X^{\frac{a-b}{a}} f(1, \frac{Y}{X^{b/a}})$$

Denote that $f(y) = f(1,y)$

$$Z = \frac{Y}{X^{b/a}}$$
$$U = \frac{\partial \psi}{\partial Y} = X^{1-\frac{2b}{a}} f'(Z)$$

Assume that U does not depend on X explicitly $\Rightarrow 1 - \frac{2b}{a} = 0 \Rightarrow a = 2b$.

Then $U = f(Z) \quad \psi = X^{\frac{1}{2}} f(Z) \quad Z := \frac{Y}{2X^{\frac{1}{2}}}$ [the 2 in fraction is for computation convenience.]

$$\begin{aligned}
\text{Eq:} \quad & f''' + ff'' = 0 \\
& f(0) = 0 && \text{(comes from definition of potential,} \\
& && \text{up to a constant, choose constant to be zero.)} \\
& u|_{y=0} = 0 \\
& f'(+\infty) = u_0 && \text{(become matching} \lim_{Y\to+\infty} U = u_0)
\end{aligned}$$

Then the ode is easy to get numerical sol.

2 WKB theory

2.1 Introduction

$$\begin{cases} \epsilon y'' + y = 0 \\ y(0) = 0, y(1) = 0 \end{cases}$$

Exact sol. $y = \frac{\sin(x/\sqrt{\epsilon})}{\sin(1/\sqrt{\epsilon})}$

* solution varies rapidly throughout the whole domain

* No BL

2.2 WKB theory

An approximation method for such problems.

$$y = e^{S(x)} \Rightarrow \epsilon(s'' + (s')^2)e^{S(x)} + e^{S(x)} = 0$$

$$\begin{cases} S(x) \sim \dfrac{1}{\delta} S_0(x) + S_1(x) + \delta S_2(x) + \cdots, \ \epsilon \to 0. \\ \delta = \delta(\epsilon) \to 0 \text{ as } \epsilon \to 0 \end{cases}$$

Note that $\frac{1}{\delta} S_0(x)$ accounts for fast oscillation.

Consider the problem.

$$\epsilon y'' = Q(x) y \quad \epsilon > 0$$

$Q(x) \neq 0$ in the domain,

$S(x)$ satisfies $\epsilon(S'' + (S')^2) = Q(x)$

$$\Rightarrow \epsilon[\frac{1}{\delta^2}(S_0')^2 + \frac{1}{\delta}(S_0'' + 2S_0'S_1') + S_1'' + (S_1')^2 + 2S_0'S_2'] \sim Q(x)$$

Dominant balance $\Rightarrow \frac{\epsilon}{\delta^2} = Os(1) \Rightarrow \delta = \epsilon^{\frac{1}{2}}$
Leading order $O(1)$: $(S_0')^2 = Q(x)$Eikonal Eq

$$S_0 = \pm \int^x \sqrt{Q(t)}dt + A_0$$

$O(\epsilon^{\frac{1}{2}} : 2S_0'S_1' + S_0'' = 0)$

$$\begin{aligned} S_1' &= -\frac{S_0''}{2S_0'} \\ S_1 &= -\frac{1}{2}\log|S_0'(x)| + A_1 \\ &= -\frac{1}{4}\log|Q(x)| + A_1 \\ S(x) &\sim \frac{1}{\sqrt{\epsilon}}(\pm\int^x \sqrt{Q(t)}dt + A_0) \\ &\quad -\frac{1}{4}\log|Q(x)| + A_1 \end{aligned}$$

$$\Rightarrow y(x) \sim C_1 Q(x)^{-\frac{1}{4}} e^{\pm\int^x \sqrt{Q(t)}dt/\sqrt{\epsilon}}$$

General sol:

$$y(x) \sim C_1 Q(x)^{-\frac{1}{4}} e^{\int^x \sqrt{Q(t)}dt/\sqrt{\epsilon}} + C_2 Q(x)^{-\frac{1}{4}} e^{-\int^x \sqrt{Q(t)}dt/\sqrt{\epsilon}} \quad \epsilon \to 0$$

Up to leading order.
For the problem in the introduction

$$Q(X) \equiv -1 \qquad \int^x \sqrt{Q(t)}dt = ix$$

leading order WKB approximation is:

$$y(x) \sim C_1 e^{ix/\sqrt{\epsilon}} + C_2 e^{-ix/\sqrt{\epsilon}}$$

Remark: When $Q(x) = 0$ at some x, the above method is no longer working needs special treatment.

High order:

$$O(\epsilon): \qquad 2S_0'S_2' + S_1'' + (S_1')^2 = 0$$

$$O(\epsilon^{\frac{n}{2}}): \qquad 2S_0'S_n' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = 0$$

$$S_2 = \pm\int^x \left(\frac{Q''}{8Q^{3/2}} - \frac{5(Q')^2}{32Q^{5/2}}\right)\mathrm{d}t$$

$$S_3 = -\frac{Q''}{16Q^2} + \frac{5Q'^2}{64Q^3}$$

Remark:

- WKB theory works only for linear eq otherwise e^S cannot be canceled on both sides. Nonlinear eqns do not have such a simple eq for $S(x)$.
- Some linear boundary layer problems can also be solved by WKB theory.

2.3 More on the asymptotic expansions of $S(x)$ and $y(x)$

$$
\begin{aligned}
&S(x) \sim \frac{1}{\delta}S_0(x) + S_1(x) + \delta S_2(x) + \delta^2 S_3(x) + \cdots, \epsilon \to 0 \\
&S(x) - \frac{1}{\delta}S_0(x) \sim S_1(x), \epsilon \to 0 \\
&S(x) - [\frac{1}{\delta}S_0(x) + S_1(x)] \sim \delta S_2(x) \epsilon \to 0
\end{aligned}
$$

We want to find an approximation to $y(x) = e^{S(x)}$ st the relation error is small.

$$\lim_{\delta \to 0} \frac{y(x) - e^{\frac{1}{\delta}S_0(x)}}{y(x)} = \lim_{\delta \to 0} \frac{e^{S(x) - \frac{S_0(x)}{\delta}} - 1}{e^{S(x) - \frac{1}{\delta}S_0(x)}} = \lim_{\delta \to 0} \frac{e^{S_1(x)} - 1}{e^{S_1(x)}} \text{is not necessarily small.}$$

$$\lim_{\delta \to 0} \frac{y(x) - e^{\frac{1}{\delta}S_0(x) + S_1(x)}}{y(x)} = \lim_{\delta \to 0} \frac{e^{\delta S_2(X)} - 1}{e^{\delta S_2(x)}} = 0$$

$\Rightarrow e^{\frac{1}{\delta}S_0(x) + S_1(x)}$ is leading ordeer approximation of $y(x)$ uniform approxiamtion.

More precisely $\frac{y(x) - e^{\frac{1}{\delta}S_0(x) + S_1(x)}}{y(x)} \sim \delta S_2(x)$

Generally when $N \leq 1$

$$\frac{y(x) - e^{\frac{1}{\delta}\sum_{n=0}^{N} \delta^n S_N(x)}}{y(x)} \sim \delta^N S_{N+1}(x) \delta \to 0$$