Research on general unified normalization computing for the gravitational potential tensor of arbitrary orders

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***Abstract*—A simple and consolidated structure, based on the general expression of Cunningham, is proposed to compute gravity potential and its derivatives up to any order. A novel normalization factor is introduced to synthesize various normalization forms into a unified one. Spherical harmonic series in a rectangular structure of all orders are constructed herein for the evaluation of arbitrary order derivatives, whose coefficients are irrelevant to the position information, possible for preprocessing and pre-storage as a high order tensor. The series remain to be the only terms needed for evaluation in the process by the chain role, greatly improving the computing efficiency and accuracy compared to the traditional ones. Simulations are conducted to analyze the practicality and validity of our structure. The accuracy and efficiency of various normalization algorithms are verified with comparison. Results indicate that the proposed expression has achieved its generality and unification. It is promising for potential use in fields with a similar expression of gravity potential.**

Keywords-gravity potential; arbitrary order tensor; normalization; numerical analysis; general recursion method

I. INTRODUCTION

The gravitational potential and its derivatives are widely used in various aspects, such as Astrodynamics, Geodesy, Geophysics [1], Archaeology [2] and so forth. In particular, the first derivative of the non-spherical part of the potential is the main source of satellite's perturbative forces. Second order derivatives appear in the precise orbit determination to evaluate the state transition matrix. In archaeological studying, the second and the third order derivatives are useful for revealing some closed peculiarities of different Physical-Archaeological models [2]. Higher order derivatives of the gravity field are frequently applied in gravity exploration and gravity gradient torque determination.

Heiskanen and Moritz [3] put forward the fundamental expression of planetary gravity potential for the first time in terms of the non-normalized quantities. Later, various expressions for the first order and second order derivatives were presented with respect to x, y, z in the Spherical coordinates [4-6]. It needs to be indicated that such expressions have unavoidable singularities at the poles. Hotine [7] constructed the series of the standard solid spherical harmonics for the expressions of the potential and its first two derivatives in the Cartesian coordinates to circumvent the singularity problems. The expression was suitable for differentiation with respect to x, y, z . However, the terms of each expression for the potential and its derivatives depended on changing Legendre functions, not in general attractive for numerical implementation. Cunningham [8] derived a general non-normalized formula representing the derivatives of the gravitational potential up to any order as a linear combination of the expressions associated with the Legendre polynomials in the Cartesian frames. Based on Cunningham's work, Metris et al [9] and Montenbruck [10] gave some improvements to simplify the expressions of the gravitational potential and its derivatives and optimize the computing performance. Petrovskaya [11] constructed a common structure of the spherical harmonic series for the potential derivatives of arbitrary orders in the geocentric reference frame. In the series, the coefficients of a certain order derivative were expressed as a linear function of preceding order coefficients, suitable for numerical implementation.

To avoid computing overflow, various normalization factors were introduced. Heiskanen and Moritz [3] defined a full normalization factor which is the most popular one widely applied in various algorithms and schemes for the evaluation of potential and its derivatives. A brand new normalization factor was presented by Belikov [12] of the associated Legendre functions in the procedure of an efficient algorithm. Lin [13] introduced a new normalization factor and summarized a fast computation method in four satellite laser ranging stations of China in Shanghai, Changchun, Wuhan and Beijing in practice. In this paper, the aforementioned three normalizations are taken into consideration and comparison in terms of precision and time efficiency.

Abad [1] implemented a new method to use an iterative scheme to simultaneously evaluate all the desired derivatives instead of the explicit expression. Sun [14] derived the concrete full normalized potential derivatives of the first and second orders based on numerical differentiation. It has to be noted that the numerical method avoids the complex symbolic expressions, however, usually introduces round-off errors and leads poor numerical results. Also, it is quite suitable for low order calculations, since the numerical forms of a high derivative is really complicated for the general expression deduction of arbitrary orders.

Though plenty of studies have been carried out for the efficient procedures, the evaluation of gravitational potential and its derivatives of arbitrary orders remained to be extremely complicated. Different methods with associated normalization factors were developed, cumbersome to take various forms into consideration. So it is necessary to implement a simple and unified structure to calculate the gravitational potential and its derivatives of arbitrary orders under various normalization cases.

The present paper is organized as follows. In Section II, the fundamental work of Cunningham's and Montenbruck's for the gravitational potential and its first derivatives is presented. A rectangular form dividing the expressions of gravity potential and it derivatives into two parts as horizontal terms and vertical ones is adopted, which can be extended later for the evaluation of high order derivatives. Section III is devoted to introduce the novel unified normalization factor, which produces a general procedure to taking all kinds of normalization forms into consideration. The coefficients of the Spherical harmonic series for arbitrary orders are constructed and the general and unified form of potential derivatives up to any order is given. In Section IV, different normalization methods are simulated with the help of the proposed general and unified form, and corresponding characteristics are studied and discussed. Section V presents the conclusions of the paper.

II. General Evaluation of Gravitational Potential and Its First Derivatives

In this section, the general expressions of the gravitational potential and its first derivatives are simply introduced. The aforementioned method of Cunningham's and Montenbruck's to calculate the gravity field and its first derivatives tensors are given in a rectangular form for subsequent extensions.

The gravitational potential can be presented with regard to non-normalized quantities as

$$U=\frac{GM_\oplus}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{R_\oplus^n}{r^n}P_{nm}(\sin\phi)\times\left[C_{nm}\cos(m\lambda)+S_{nm}\sin(m\lambda)\right] \quad (1)$$

where G is the gravitational constant, $R_\oplus$, $M_\oplus$ the equatorial radius and mass of the Earth respectively, r the geocentric distance of a point, (λ,ϕ) the latitude and the longitude measured eastward from the meridian of Greenwich, C_{nm} and S_{nm} Earth's spherical harmonic coefficients, and $P_{nm}(x)$ Legendre polynomials.

However, it is too cumbersome to derive a rigorous analytical expression of the derivatives of the gravitational potential from (1). And a method was presented by Cunningham, for the efficient computing of the forces and their derivatives, which was effective, however, rather obscure. Montenbruck put forward an equivalent expression, easy for understanding and program implementation.

Define the horizontal term and the vertical term of the spherical harmonics as

$$\begin{cases} V_{nm}=\left(\dfrac{R_\oplus}{r}\right)^{n+1}P_{nm}(\sin\phi)\cos(m\lambda) \\ W_{nm}=\left(\dfrac{R_\oplus}{r}\right)^{n+1}P_{nm}(\sin\phi)\sin(m\lambda) \end{cases} \quad (2)$$

Thus

$$U=\frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(C_{nm}V_{nm}+S_{nm}W_{nm}\right) \quad (3)$$

where

$$\begin{cases} V_{mm}=(2m-1)\left(\dfrac{xR_\oplus}{r^2}V_{m-1,m-1}-\dfrac{yR_\oplus}{r^2}W_{m-1,m-1}\right) \\ W_{mm}=(2m-1)\left(\dfrac{xR_\oplus}{r^2}W_{m-1,m-1}+\dfrac{yR_\oplus}{r^2}V_{m-1,m-1}\right) \end{cases} \quad (4.1)$$

$$\begin{cases} V_{m+1,m}=(2m+1)\dfrac{zR_\oplus}{r^2}V_{m,m} \\ W_{m+1,m}=(2m+1)\dfrac{zR_\oplus}{r^2}W_{m,m} \end{cases} \quad (4.2)$$

$$\begin{cases} V_{nm}=\dfrac{2n-1}{n-m}\dfrac{zR_\oplus}{r^2}V_{n-1,m}-\dfrac{n+m-1}{n-m}\dfrac{R_\oplus^{\,2}}{r^2}V_{n-2,m} \\ W_{nm}=\dfrac{2n-1}{n-m}\dfrac{zR_\oplus}{r^2}W_{n-1,m}-\dfrac{n+m-1}{n-m}\dfrac{R_\oplus^{\,2}}{r^2}W_{n-2,m} \end{cases} \quad (4.3)$$

whose initial values are

$$\begin{cases} V_{00}=\dfrac{R_\oplus}{r},W_{00}=0 \\ V_{10}=\dfrac{zR_\oplus}{r^2}V_{00},W_{10}=0 \end{cases} \quad (5)$$

The recursive computations of V_{nm} and W_{nm} are not based on the Spherical coordinates parameters (r,λ,ϕ) but artesian coordinates instead, fundamentally eliminating the singularity at the poles when ϕ=90°.

According to (3), the accelerations of gravitational potential are

$$\begin{aligned} g_x&=\frac{\partial U}{\partial x}=\frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(C_{nm}\frac{\partial V_{nm}}{\partial x}+S_{nm}\frac{\partial W_{nm}}{\partial x}\right) \\ g_y&=\frac{\partial U}{\partial y}=\frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(C_{nm}\frac{\partial V_{nm}}{\partial y}+S_{nm}\frac{\partial W_{nm}}{\partial y}\right) \\ g_z&=\frac{\partial U}{\partial z}=\frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(C_{nm}\frac{\partial V_{nm}}{\partial z}+S_{nm}\frac{\partial W_{nm}}{\partial z}\right) \end{aligned} \quad (6)$$

Hereinto $\dfrac{\partial V_{nm}}{\partial x},\dfrac{\partial V_{nm}}{\partial y},\dfrac{\partial V_{nm}}{\partial z},\dfrac{\partial W_{nm}}{\partial x},\dfrac{\partial W_{nm}}{\partial y},\dfrac{\partial W_{nm}}{\partial z}$ can be derived from Cunningham's study.

Define

$$\boldsymbol{V}_{nm}(x,y,z)=\frac{P_{nm}(\sin\varphi)(\cos m\lambda+\boldsymbol{i}\sin m\lambda)}{r^{n+1}} \quad \boldsymbol{i}=\sqrt{-1} \quad (7)$$

where

$$x=r\cos\varphi\cos\lambda$$
$$y=r\cos\varphi\sin\lambda$$
$$z=r\sin\varphi$$

Cunningham's general formula indicates that the arbitrary-order derivatives of $\boldsymbol{V}_{nm}(x,y,z)$ with respect to

Cartesian coordinates x, y, z are liner combination of $\{\boldsymbol{V}_{nm}(x,y,z)\}$, whose concrete expressions are

$$\frac{\partial \boldsymbol{V}_{nm}^{\alpha+\beta+\gamma}}{\partial x^{\alpha}\partial y^{\beta}\partial z^{\gamma}} = \boldsymbol{i}^{\beta}\sum_{k=0}^{\alpha+\beta}\frac{(-1)^{\alpha+\gamma-k}}{2^{\alpha+\beta}}\frac{(n-m+\gamma+2k)!}{(n-m)!}C_{\alpha,\beta,k}\boldsymbol{V}_{n+\alpha+\beta+\gamma,m+\alpha+\beta-2k} \tag{8.1}$$

where

$$C_{\alpha,\beta,k} = \sum_{i=\max(0,k-\alpha)}^{\min(\beta,k)}(-1)^{i}\binom{\alpha}{k-i}\binom{\beta}{i}$$

Let

$$\bar{\boldsymbol{V}}_{nm}(x,y,z) = V_{nm}(x,y,z) + iW_{nm}(x,y,z)$$

The relationship between the $\bar{\boldsymbol{V}}_{nm}(x,y,z)$ in our paper and the $\boldsymbol{V}_{nm}(x,y,z)$ in Cunningham's definition is

$$\bar{\boldsymbol{V}}_{nm}(x,y,z) = R_{\oplus}^{n+1}\boldsymbol{V}_{nm}(x,y,z)$$

Then (8) can be rewritten in terms of $\bar{\boldsymbol{V}}_{nm}(x,y,z)$ as

$$R_{\oplus}^{\alpha+\beta+\gamma}\frac{\partial \bar{\boldsymbol{V}}_{nm}^{\alpha+\beta+\gamma}}{\partial x^{\alpha}\partial y^{\beta}\partial z^{\gamma}} = \boldsymbol{i}^{\beta}\sum_{k=0}^{\alpha+\beta}\frac{(-1)^{\alpha+\gamma-k}}{2^{\alpha+\beta}}\frac{(n-m+\gamma+2k)!}{(n-m)!}C_{\alpha,\beta,k}\bar{\boldsymbol{V}}_{n+\alpha+\beta+\gamma,m+\alpha+\beta-2k} \tag{8.2}$$

The first derivatives of $\bar{\boldsymbol{V}}_{nm}$ with respect to x, y, z are decomposed into real and imaginary parts.

For $m = 0$:

$$\begin{cases} \dfrac{\partial V_{n,0}}{\partial x} = -\dfrac{V_{n+1,1}}{R_{\oplus}} \\ \dfrac{\partial V_{n,0}}{\partial y} = -\dfrac{W_{n+1,1}}{R_{\oplus}} \\ \dfrac{\partial V_{n,0}}{\partial z} = -\dfrac{(n+1)V_{n+1,0}}{R_{\oplus}} \end{cases} \tag{9}$$

For $m > 0$:

$$\begin{cases} \dfrac{\partial V_{nm}}{\partial x} = -\dfrac{1}{2R_{\oplus}}\left[V_{n+1,m+1} - (n-m+1)(n-m+2)V_{n+1,m-1}\right] \\ \dfrac{\partial V_{nm}}{\partial y} = -\dfrac{1}{2R_{\oplus}}\left[W_{n+1,m+1} + (n-m+1)(n-m+2)W_{n+1,m-1}\right] \\ \dfrac{\partial V_{nm}}{\partial z} = -\dfrac{n-m+1}{R_{\oplus}}V_{n+1,m} \end{cases} \tag{10}$$

$$\begin{cases} \dfrac{\partial W_{nm}}{\partial x} = -\dfrac{1}{2R_{\oplus}}\left[W_{n+1,m+1} - (n-m+1)(n-m+2)W_{n+1,m-1}\right] \\ \dfrac{\partial W_{nm}}{\partial y} = \dfrac{1}{2R_{\oplus}}\left[V_{n+1,m+1} + (n-m+1)(n-m+2)V_{n+1,m-1}\right] \\ \dfrac{\partial W_{nm}}{\partial z} = -\dfrac{n-m+1}{R_{\oplus}}W_{n+1,m} \end{cases} \tag{11}$$

Substituting (9), (10) and (11) into (6), the gravitational accelerations in the Earth centered fixed coordinate frame are obtained

$$g_x = \sum_{nm}\frac{\partial U_{nm}}{\partial x},\ g_y = \sum_{nm}\frac{\partial U_{nm}}{\partial y},\ g_z = \sum_{nm}\frac{\partial U_{nm}}{\partial z} \tag{12}$$

where

$$\frac{\partial U_{nm}}{\partial x} \overset{m=0}{=} \frac{GM_{\oplus}}{R_{\oplus}^{2}}\left\{-C_{n0}V_{n+1,1}\right\}$$

$$\overset{m>0}{=} \frac{GM_{\oplus}}{R_{\oplus}^{2}}\frac{1}{2}\left\{-\left(C_{nm}V_{n+1,m+1} + S_{nm}W_{n+1,m+1}\right) + (n-m+2)(n-m+1)\left(C_{nm}V_{n+1,m-1} + S_{nm}W_{n+1,m-1}\right)\right\}$$

$$\frac{\partial U_{nm}}{\partial y} \overset{m=0}{=} \frac{GM_{\oplus}}{R_{\oplus}^{2}}\left\{-C_{n0}W_{n+1,1}\right\}$$

$$\overset{m>0}{=} \frac{GM_{\oplus}}{R_{\oplus}^{2}}\frac{1}{2}\left\{-\left(C_{nm}W_{n+1,m+1} - S_{nm}V_{n+1,m+1}\right) - (n-m+2)(n-m+1)\left(C_{nm}W_{n+1,m-1} - S_{nm}V_{n+1,m-1}\right)\right\}$$

$$\frac{\partial U_{nm}}{\partial z} = \frac{GM_{\oplus}}{R_{\oplus}^{2}}\left\{-(n-m+1)\left(C_{nm}V_{n+1,m} + S_{nm}W_{n+1,m}\right)\right\}$$

Equation (12) was demonstrated by Montenbruck, and it is very convenient to calculate the gravitational accelerations in the Earth centered fixed coordinate frame according to this expression. All calculations are related to the position information (x, y, z) in the Earth centered fixed coordinate frame with a simple form, no singularities and of high efficiency in implementation.

III. CONSTRUCTION OF THE UNIFIED NORMALIZATION STRUCTURE FOR THE GRAVITY DERIVATIVES OF ARBITRARY ORDERS

As discussed above, different normalization methods are proposed for efficient processing to some extent, which leads to complex and intricate expressions in respective method that brings considerably inconvenience for programming. To unify all these different normalization cases, a novel general normalized expression of the gravitational potential and its derivatives of arbitrary orders are presented in accordance to Montenbruck's form in the Cartesian reference frame, which composites various normalization structures into a unified one. Coefficients of spherical harmonics series are constructed by means of chain roles and can be pre-processed and pre-stored as multi-dimensional tensors, requiring considerably less time in the procedure for the potential and derivatives computing.

A. Construction of the unified normalization structure for the potential (zero-order tensor) and its acceleration (first-order tensor)

Taking the potential and its first derivatives into consideration, a unified normalization factor is introduced. The unified structure is suitable for various normalization cases of the potential and its first derivatives are constructed with a high order potential.

Cunningham's expressions are derived from (1) that is based on Legendre polynomial series of potential model, and an unavoidable problem for this algorithm is that, as degree and order increases, there will be overflow in the computation of V_{nm} and W_{nm}. As a consequence, many amelioration methods were put forward with respective normalization factors. Herein, we take the most commonly used three normalization forms into consideration listed as Tab.I.

TABLE I. LIST OF THREE NORMALIZATION FORMS

	full normalization	Belikov's normalization	Lin's normalization
β_{nm}	$\sqrt{\dfrac{(n+m)!}{(2n+1)(n-m)!(2-\delta_{m0})}}$	$\dfrac{(n+m)!}{2^n n!}$	$\dfrac{(2n+1)!}{(2-\delta_{mo})2^n n!(n-m)!}$

Each normalization form has their own characteristics, and of each one the derived computing forms of gravitational potential (zero-order tensor) and its accelerations (first order tensor) are different, as well as the algorithms for programming. In considering that it is an extremely cumbersome task with great inconvenience for programming, a new normalization factor of Legendre polynomials is introduced in this subsection, which integrates various normalization forms of the gravitational expressions into a unified one, adaptive to different normalization cases.

All kinds of normalized Legendre polynomials satisfy the following equations

$$\bar{P}_{nm}(x)=P_{nm}(x)/\beta_{nm} \tag{13}$$

The relationship between the corresponding normalized spherical harmonic coefficients $\bar{C}_{nm},\bar{S}_{nm}$ and the non-normalized ones C_{nm},S_{nm} are

$$\bar{C}_{nm}=\beta_{nm}C_{nm}\,,\ \bar{S}_{nm}=\beta_{nm}S_{nm} \tag{14}$$

According to Montenbruck's method, let

$$\begin{cases}\bar{V}_{nm}=\left(\dfrac{R_\oplus}{r}\right)^{n+1}\bar{P}_{nm}(\sin\phi)\cos(m\lambda)\\ \bar{W}_{nm}=\left(\dfrac{R_\oplus}{r}\right)^{n+1}\bar{P}_{nm}(\sin\phi)\sin(m\lambda)\end{cases} \tag{15}$$

Then the normalized gravitational potential becomes

$$U=\frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\bar{C}_{nm}\bar{V}_{nm}+\bar{S}_{nm}\bar{W}_{nm}\right) \tag{16}$$

where

$$\begin{aligned}V_{nm}(x)&=\beta_{nm}\bar{V}_{nm}(x)\\ W_{nm}(x)&=\beta_{nm}\bar{W}_{nm}(x)\end{aligned} \tag{17}$$

Define a unified normalization factor as

$$\Pi_{nm}^{pq}=\frac{\beta_{pq}}{\beta_{nm}} \tag{18}$$

The chain roles of $\bar{V}_{nm}$ and $\bar{W}_{nm}$ then become

$$\begin{cases}\bar{V}_{mm}=(2m-1)\Pi_{mm}^{m-1,m-1}\left(\dfrac{xR_\oplus}{r^2}\bar{V}_{m-1,m-1}-\dfrac{yR_\oplus}{r^2}\bar{W}_{m-1,m-1}\right)\\ \bar{W}_{mm}=(2m-1)\Pi_{mm}^{m-1,m-1}\left(\dfrac{xR_\oplus}{r^2}\bar{W}_{m-1,m-1}-\dfrac{yR_\oplus}{r^2}\bar{V}_{m-1,m-1}\right)\end{cases} \tag{19.1}$$

$$\begin{cases}\bar{V}_{m+1,m}=(2m+1)\Pi_{m+1,m}^{mm}\dfrac{zR_\oplus}{r^2}\bar{V}_{m,m}\\ \bar{W}_{m+1,m}=(2m+1)\Pi_{m+1,m}^{mm}\dfrac{zR_\oplus}{r^2}\bar{W}_{m,m}\end{cases} \tag{19.2}$$

$$\begin{cases}\bar{V}_{nm}=\dfrac{2n-1}{n-m}\Pi_{nm}^{n-1,m}\dfrac{zR_\oplus}{r^2}\bar{V}_{\mathrm{n}-1,m}-\dfrac{n+m-1}{n-m}\Pi_{nm}^{n-2,m}\dfrac{R_\oplus^{\ 2}}{r^2}\bar{V}_{\mathrm{n}-2,m}\\ \bar{W}_{nm}=\dfrac{2n-1}{n-m}\Pi_{nm}^{n-1,m}\dfrac{zR_\oplus}{r^2}\bar{W}_{\mathrm{n}-1,m}-\dfrac{n+m-1}{n-m}\Pi_{nm}^{n-2,m}\dfrac{R_\oplus^{\ 2}}{r^2}\bar{W}_{\mathrm{n}-2,m}\end{cases} \tag{19.3}$$

For different normalization methods, the corresponding $\left\{\Pi_{mm}^{m-1,m-1},\ \Pi_{m+1,m}^{mm},\ \Pi_{nm}^{n-1,m},\ \Pi_{nm}^{n-2,m}\right\}$ differ from each other. Refer to Tab. II for specific expressions.

TABLE II. THE UNIFIED NORMALIZATION FACTOR IN POTENTIAL CALCULATION UNDER DIFFERENT CASES

	full normalization	Belikov's normalization	Metris's normalization
β_{nm}	$\sqrt{\dfrac{(n+m)!}{(2n+1)(n-m)!(2-\delta_{m0})}}$	$\dfrac{(n+m)!}{2^n n!}$	$\dfrac{(2n+1)!}{(2-\delta_{mo})2^n n!(n-m)!}$
Π_{nm}^{pq}	$\sqrt{\dfrac{2-\delta_{m0}}{2-\delta_{q0}}\dfrac{2n+1}{2p+1}\dfrac{(n-m)!}{(p-q)!}\dfrac{(p+q)!}{(n+m)!}}$	$2^{n-p}\dfrac{n!}{p!}\dfrac{(p+q)!}{(n+m)!}$	$2^{n-p}\dfrac{2-\delta_{m0}}{2-\delta_{q0}}\dfrac{n!}{p!}\dfrac{(2p+1)!}{(2n+1)!}\dfrac{(n-m)!}{(p-q)!}$
$\Pi_{mm}^{m-1,m-1}$	$\dfrac{1}{2m-1}\sqrt{\dfrac{2m+1}{2m}}(1+\delta_{m,1})$	$\dfrac{1}{2m-1}$	$\dfrac{1+\delta_{m,1}}{2m+1}$
$\Pi_{m+1,m}^{mm}$	$\dfrac{\sqrt{2m+3}}{2m+1}$	$\dfrac{2m+2}{2m+1}$	$\dfrac{1}{(2m+3)}$
$\Pi_{nm}^{n-1,m}$	$\sqrt{\dfrac{n-m}{n+m}\dfrac{2n+1}{2n-1}}$	$\dfrac{2n}{n+m}$	$\dfrac{n-m}{2n+1}$
$\Pi_{nm}^{n-2,m}$	$\sqrt{\dfrac{(2n+1)(n-m)(n-m-1)}{(2n-3)(n+m)(n+m-1)}}$	$\dfrac{4n(n-1)}{(n+m)(n+m-1)}$	$\dfrac{(n-m)(n-m-1)}{(2n+1)(2n-1)}$

Also, the initial conditions of $\bar{V}_{nm},\bar{W}_{nm}$ vary from each other for different normalization methods.

TABLE III. THE INITIAL VALUES OF $\bar{V}_{nm},\bar{W}_{nm}$ UNDER DIFFERENT CASES

	full normalization	Belikov's normalization	Metris's normalization
$\bar{V}_{0,0}$	$\dfrac{R}{r}$	$\dfrac{R}{r}$	$\dfrac{R}{r}$
$\bar{V}_{1,0}$	$\sqrt{3}\dfrac{zR}{r^2}\dfrac{R}{r}$	$2\dfrac{zR}{r^2}\dfrac{R}{r}$	$\dfrac{1}{3}\dfrac{zR}{r^2}\dfrac{R}{r}$
$\bar{W}_{0,0}$	0	0	0
$\bar{W}_{1,0}$	0	0	0

The normalized gravitational potential accelerations presented by $\bar{V}_{nm},\bar{W}_{nm}$ are

$$g_x = \sum_{nm} \frac{\partial U_{nm}}{\partial x},\ \ g_y = \sum_{nm} \frac{\partial U_{nm}}{\partial y},\ \ g_z = \sum_{nm} \frac{\partial U_{nm}}{\partial z} \tag{20}$$

where

$$\begin{aligned}
\frac{\partial U_{nm}}{\partial x} &\overset{m=0}{=} \frac{GM_\oplus}{R_\oplus^2}\left\{-\bar{C}_{n0}\Pi_{n,0}^{n+1,1}\bar{V}_{n+1,1}\right\} \\
&\overset{m>0}{=} \frac{GM_\oplus}{R_\oplus^2}\frac{1}{2}\left\{-\Pi_{nm}^{n+1,m+1}\left(\bar{C}_{nm}\bar{V}_{n+1,m+1}-\bar{S}_{nm}\bar{W}_{n+1,m+1}\right)\right. \\
&\qquad \left.+\left(n-m+2\right)\left(n-m+1\right)\Pi_{nm}^{n+1,m-1}\left(\bar{C}_{nm}\bar{V}_{n+1,m-1}+\bar{S}_{nm}\bar{W}_{n+1,m-1}\right)\right\} \\
\frac{\partial U_{nm}}{\partial y} &\overset{m=0}{=} \frac{GM_\oplus}{R_\oplus^2}\left\{-\bar{C}_{n0}\Pi_{n,0}^{n+1,1}\bar{W}_{n+1,1}\right\} \\
&\overset{m>0}{=} \frac{GM_\oplus}{R_\oplus^2}\frac{1}{2}\left\{\Pi_{nm}^{n+1,m+1}\left(-\bar{C}_{nm}\bar{W}_{n+1,m+1}+\bar{S}_{nm}\bar{V}_{n+1,m+1}\right)\right. \\
&\qquad \left.+\left(n-m+2\right)\left(n-m+1\right)\Pi_{nm}^{n+1,m-1}\left(-\bar{C}_{nm}\bar{W}_{n+1,m-1}+\bar{S}_{nm}\bar{V}_{n+1,m-1}\right)\right\}
\end{aligned}$$

$$\frac{\partial U_{nm}}{\partial z} = \frac{GM_\oplus}{R_\oplus^2}\left\{\left(n-m+1\right)\Pi_{nm}^{n+1,m}\left(-\bar{C}_{nm}\bar{V}_{n+1,m}-\bar{S}_{nm}\bar{W}_{n+1,m}\right)\right\}$$

The specific expressions of the corresponding $\left\{\Pi_{n,0}^{n+1,1},\ \Pi_{nm}^{n+1,m+1},\ \Pi_{nm}^{n+1,m-1},\ \Pi_{nm}^{n+1,m}\right\}$ are listed in Tab.IV.

TABLE IV. THE UNIFIED NORMALIZATION FACTOR IN ACCELERATION CALCULATION UNDER DIFFERENT CASES

	full normalization	*Belikov's normalization*	*Metris's normalization*
β_{nm}	$\sqrt{\frac{(n+m)!}{(2n+1)(n-m)!(2-\delta_{m0})}}$	$\frac{(n+m)!}{2^n n!}$	$\frac{(2n+1)!}{(2\text{-}\delta_{mo})2^n n!(n-m)!}$
Π_{nm}^{pq}	$\sqrt{\frac{2-\delta_{m0}}{2-\delta_{q0}}\frac{2n+1}{2p+1}\frac{(n-m)!}{(p-q)!}\frac{(p+q)!}{(n+m)!}}$	$2^{n-p}\frac{n!}{p!}\frac{(p+q)!}{(n+m)!}$	$2^{n-p}\frac{2\text{-}\delta_{mo}}{2\text{-}\delta_{qo}}\frac{n!}{p!}\frac{(2p+1)!}{(2n+1)!}\frac{(n-m)!}{(p-q)!}$
$\Pi_{n,0}^{n+1,1}$	$\sqrt{\frac{2n+1}{2n+3}\frac{(n+2)(n+1)}{2}}$	$\frac{n+2}{2}$	$\frac{2n+3}{2}$
$\Pi_{nm}^{n+1,m+1}$	$\sqrt{\frac{2n+1}{2n+3}\frac{(n+m+2)(n+m+1)}{1+\delta_{m0}}}$	$\frac{(n+m+2)(n+m+1)}{2(n+1)}$	$\frac{2n+3}{1+\delta_{m0}}$
$\Pi_{nm}^{n+1,m-1}$	$\sqrt{\frac{2n+1}{2n+3}\frac{1+\delta_{m,1}}{(n-m+2)(n-m+1)}}$	$\frac{1}{2(n+1)}$	$\frac{(1+\delta_{m,1})(2n+3)}{(n-m+2)(n-m+1)}$
$\Pi_{nm}^{n+1,m}$	$\sqrt{\frac{2n+1}{2n+3}\frac{n+m+1}{n-m+1}}$	$\frac{n+m+1}{2(n+1)}$	$\frac{2n+3}{n-m+1}$

Substituting the unified normalization factor in (16) (20) with corresponding concrete values under different normalization methods, the results are verified as the same as those presented in other papers [8, 12, 13]. Since our method is based on the general expression of Cunningham's, it maintains all the advantages for accurate and efficient computing. The unified structure is very succinct, providing a basic template form to calculate arbitrary Legendre polynomials under any normalization cases. Hence, as long as the following two series of unified normalization factors $\left\{\Pi_{mm}^{m-1,m-1},\ \Pi_{m+1,m}^{mm},\ \Pi_{nm}^{n-1,m},\ \Pi_{nm}^{n-2,m}\right\}$ and $\left\{\Pi_{n,0}^{n+1,1},\ \Pi_{nm}^{n+1,m+1},\ \Pi_{nm}^{n+1,m-1},\ \Pi_{nm}^{n+1,m}\right\}$ are given, the corresponding algorithms for the various normalization evaluation of potential and its first derivatives can be realized.

B. Construction of the unified normalization structure for the gravity derivatives of arbitrary orders

Firstly, let us consider the potential acceleration in the x direction with regard to normalized quantities.

$$g_x = \sum_{n=2}^{N}\sum_{m=0}^{\min(n,M)} \frac{\partial U_{nm}}{\partial x}$$

$$\begin{aligned}
\frac{\partial U_{nm}}{\partial x} &\overset{m=0}{=} \frac{GM_\oplus}{R_\oplus^2}\left\{-\bar{C}_{n0}\Pi_{n,0}^{n+1,1}\bar{V}_{n+1,1}\right\} \\
&\overset{m>0}{=} \frac{GM_\oplus}{R_\oplus^2}\frac{1}{2}\left\{-\Pi_{nm}^{n+1,m+1}\left(\bar{C}_{nm}\bar{V}_{n+1,m+1}-\bar{S}_{nm}\bar{W}_{n+1,m+1}\right)\right. \\
&\qquad \left.+\left(n-m+2\right)\left(n-m+1\right)\Pi_{nm}^{n+1,m-1}\left(\bar{C}_{nm}\bar{V}_{n+1,m-1}+\bar{S}_{nm}\bar{W}_{n+1,m-1}\right)\right\}
\end{aligned} \tag{21}$$

Expressions (21) are described perspective from $\bar{C}_{nm}\ \bar{S}_{nm}$, superficially, the subscripts of the expressions n,m are in accordance with $\bar{C}_{nm}\ \bar{S}_{nm}$'s serial numbers. The serial numbers of the corresponding $\bar{V}_{nm}\ \bar{W}_{nm}$ to $\bar{C}_{nm}\ \bar{S}_{nm}$ are $n{+}1, m-1$ and $n{+}1, m{+}1$, separately, as showed in Tab.V.

TABLE V. CONVERSION BETWEEN $\bar{C}_{nm}\ \bar{S}_{nm}$ AND $\bar{V}_{nm}\ \bar{W}_{nm}$ ON SERIAL NUMBERS

	$m-2$	$m-1$	m	$m+1$	$m+2$	$m+3$	
$n-1$							$n'-2$
n	$\bar{C},\bar{S}$		$\bar{C},\bar{S}$		$\bar{C},\bar{S}$		$n'-1$
$n+1$		$\bar{V},\bar{W}$		$\bar{V},\bar{W}$		$\bar{V},\bar{W}$	n'
$n+2$							$n'+1$
	$m'-3$	$m'-2$	$m'-1$	m'	$m'+1$	$m'+2$	

It needs to be noted that n', m' are new series numbers from the standpoint of $\bar{V}_{nm}\ \bar{W}_{nm}$. If we consider the aforementioned situations perspective from $\bar{V}_{nm}\ \bar{W}_{nm}$ conversely, the serial numbers of the multiplied gravity coefficient $\bar{C}_{nm}\ \bar{S}_{nm}$ then become $n'-1, m'-1$ and $n'-1, m'{+}1$. The merge of the corresponding $\bar{C}_{nm}\ \bar{S}_{nm}$ is presented as $\bar{C}_{nm}^{x},\ \bar{S}_{nm}^{x}$, and a new form is presented to describe $\bar{V}_{nm}\ \bar{W}_{nm}$'s contributions to the gravity gradient,

$$H_{nm}^{x} = \frac{GM_\oplus}{R_\oplus^2}\left(\bar{C}_{nm}^{x}\bar{V}_{nm}+\bar{S}_{nm}^{x}\bar{W}_{nm}\right) \tag{22.1}$$

$\bar{C}_{nm}^{x},\ \bar{S}_{nm}^{x}$ are normalized spherical harmonic series coefficients of the first-order tensor. From (21), the chain roles to calculate $\bar{C}_{nm}^{x},\ \bar{S}_{nm}^{x}$ are obtained

$$\begin{cases}\bar{C}_{n,0}^{x} = \dfrac{1}{2}n\left(n+1\right)\bar{C}_{n-1,1}\Pi_{n-1,0}^{n,0} \\ \bar{S}_{n,0}^{x} = 0\end{cases}$$

$$\begin{cases}\bar{C}_{n,1}^{x} = \dfrac{1}{2}\left[\left(n-1\right)\left(n-2\right)\bar{C}_{n-1,2}\Pi_{n-1,2}^{n,1} - 2\bar{C}_{n-1,0}\Pi_{n-1,0}^{n,1}\right] \\ \bar{S}_{n,1}^{x} = \dfrac{1}{2}\left(n-1\right)\left(n-2\right)\bar{S}_{n-1,2}\end{cases}$$

$$\begin{cases}\bar{C}_{ns}^{x}=-\dfrac{1}{2}\bar{C}_{n-1,s-1}\Pi_{n-1,s-1}^{ns}\\ \bar{S}_{ns}^{x}=-\dfrac{1}{2}\bar{S}_{n-1,s-1}\Pi_{n-1,s-1}^{ns}\end{cases}\min(n-1,M+1)\le s\le\min(n,M+1)$$

$$\begin{cases}\bar{C}_{nm}^{x}=\dfrac{1}{2}\left[(n-m)(n-m-1)\bar{C}_{n-1,m+1}\Pi_{n-1,m+1}^{nm}-\bar{C}_{n-1,m-1}\Pi_{n-1,m-1}^{nm}\right]\\ \bar{S}_{nm}^{x}=\dfrac{1}{2}\left[(n-m)(n-m-1)\bar{S}_{n-1,m+1}\Pi_{n-1,m+1}^{nm}-\bar{S}_{n-1,m-1}\Pi_{n-1,m-1}^{nm}\right]\end{cases}$$

$$3\le n\le N+1\quad 2\le m<\min(n-1,M+1)\qquad(22.2)$$

where the degree and order of H_{nm}^{x} are $n=3,4,...,N+1, m=0,1,...,n$ respectively.

The case when $C_{n,0}\equiv 0$ and $S_{n,0}\equiv 0$ are considered in (22). The degrees and orders of V_{nm}, W_{nm} in the calculation of the first-order derivatives are $n=3,4,\cdots N+1;\ m=0,1,\cdots n$, respectively. Summing up all the items of H_{nm}^{x}, it is obtained that the first derivative of U respect to x is

$$g_x=\frac{\partial U}{\partial x}=\sum_{n=3}^{N}\sum_{m=0}^{\min(n,M+1)}H_{nm}^{x}=\frac{GM_{\oplus}}{R_{\oplus}^{2}}\sum_{n=3}^{N+1}\sum_{m=0}^{\min(n,M+1)}\left(\bar{C}_{nm}^{x}\bar{V}_{nm}+\bar{S}_{nm}^{x}\bar{W}_{nm}\right)\qquad(23)$$

The similar form is obtained for the first derivative of U respect to y

$$g_y=\frac{\partial U}{\partial y}=\frac{GM_{\oplus}}{R_{\oplus}^{2}}\sum_{n=3}^{N+1}\sum_{m=0}^{\min(n,M+1)}\left(\bar{C}_{nm}^{y}\bar{V}_{nm}+\bar{S}_{nm}^{y}\bar{W}_{nm}\right)\qquad(24)$$

where

$$\begin{cases}\bar{C}_{n,0}^{y}=\dfrac{1}{2}n(n-1)\bar{S}_{n-1,1}\Pi_{n-1,1}^{n,0}\\ \bar{S}_{n,0}^{y}=0\end{cases}$$

$$\begin{cases}\bar{C}_{n1}^{y}=\dfrac{1}{2}(n-1)(n-2)\bar{S}_{n-1,2}\Pi_{n-1,2}^{n,1}\\ \bar{S}_{n1}^{y}=-\dfrac{1}{2}\left[(n-1)(n-2)\bar{C}_{n-1,2}\Pi_{n-1,2}^{n,1}+2\bar{C}_{n-1,0}\Pi_{n-1,0}^{n,1}\right]\end{cases}$$

$$\begin{cases}\bar{C}_{ns}^{y}=\dfrac{1}{2}\bar{S}_{n-1,s-1}\Pi_{n-1,s-1}^{ns}\\ \bar{S}_{ns}^{y}=-\dfrac{1}{2}\bar{C}_{n-1,s-1}\Pi_{n-1,s-1}^{ns}\end{cases}\min(n-1,M+1)\le s\le\min(n,M+1)$$

$$\begin{cases}C_{nm}^{y}=\dfrac{1}{2}\left[(n-m)(n-m-1)S_{n-1,m+1}\Pi_{n-1,m+1}^{nm}+S_{n-1,m-1}\Pi_{n-1,m-1}^{nm}\right]\\ S_{nm}^{y}=-\dfrac{1}{2}\left[(n-m)(n-m-1)C_{n-1,m+1}\Pi_{n-1,m+1}^{nm}+C_{n-1,m-1}\Pi_{n-1,m-1}^{nm}\right]\end{cases}$$

$$3\le n\le N+1\quad 2\le m<\min(n-1,M+1)\qquad(25)$$

The form for the first derivative of U respect to z is slightly different from others. We have the following expression

$$\frac{\partial U_{nm}}{\partial z}=\frac{GM_{\oplus}}{R_{\oplus}^{2}}\left\{(n-m+1)\left(-\bar{C}_{nm}\bar{V}_{n+1,m}-\bar{S}_{nm}\bar{W}_{n+1,m}\right)\right\}\ (26)$$

From the perspective of $\bar{C}_{nm}$, $\bar{S}_{nm}$, if their serial numbers are n,m, then the corresponding subscripts of $\bar{V}_{nm}$, $\bar{W}_{nm}$ are $n+1,m$. On the contrary, from the perspective of $\bar{V}_{nm}$, $\bar{W}_{nm}$, if their serial numbers are n',m', the corresponding serial numbers of $\bar{C}_{nm}$, $\bar{S}_{nm}$ are $n'-1,m'$, with only one associated item. So the change of subscripts only appear on the degree n. The similar form of (23), (24) with respect to z is then obtained

$$g_z=\frac{GM_{\oplus}}{R_{\oplus}^{2}}\sum_{n=3}^{N+1}\sum_{m=0}^{\min(n-1,M)}\left(\bar{C}_{nm}^{z}\bar{V}_{nm}+\bar{S}_{nm}^{z}\bar{W}_{nm}\right)\qquad(27)$$

where

$$\bar{C}_{nm}^{z}=-(n-m)\bar{C}_{n-1,m}\Pi_{n-1,m}^{nm}$$
$$\bar{S}_{nm}^{z}=-(n-m)\bar{S}_{n-1,m}\Pi_{n-1,m}^{nm}$$
$$3\le n\le N+1\quad 0\le m<\min(n,M+1)\quad(28)$$

To get an expression completely consistent with (23), (24), the cases when $\bar{C}_{nn}^{z}=0$ $\bar{S}_{nn}^{z}=0$ $m=\min(n,M+1)$ are complemented in (27), then the following expression is derived:

$$g_z=\frac{\partial U}{\partial z}=\frac{GM_{\oplus}}{R_{\oplus}^{2}}\sum_{n=3}^{N+1}\sum_{m=0}^{\min(n,M+1)}\left(\bar{C}_{nm}^{z}\bar{V}_{nm}+\bar{S}_{nm}^{z}\bar{W}_{nm}\right)\quad(29)$$

Though (23), (24)and(29)are merely transformation forms of (20), they are all equivalent to (6) for the first-order derivatives of gravitational potential model , indicating that the first order derivative of gravitational potential model $\frac{\partial U}{\partial x},\frac{\partial U}{\partial y},\frac{\partial U}{\partial z}$ share the similar structure with that of the zero-order of U. The differences exist in the spherical harmonics coefficients. For the former one, the spherical harmonics coefficients are $\bar{C}_{nm}^{x}$, $\bar{S}_{nm}^{x}$, while for the latter one are $\bar{C}_{nm}$, $\bar{S}_{nm}$. Also, the negative order of reference radius $R_{\oplus}$ increases, and the index of the summation with respect to n is changed from $\sum_{n=2}^{N}$ to $\sum_{n=3}^{N+1}$. Further, by replacing $\bar{C}_{nm}$, $\bar{S}_{nm}$ with the first-degree spherical harmonics coefficients $\bar{C}_{nm}^{x}$, $\bar{S}_{nm}^{x}$ and taking advantage of (22), (25) and (28), it is not difficult to obtain the spherical harmonics coefficients $\bar{C}_{nm}^{x^2},\bar{S}_{nm}^{x^2},\bar{C}_{nm}^{xy},\bar{S}_{nm}^{xy},\bar{C}_{nm}^{xz},\bar{S}_{nm}^{xz}$ of the second-degree derivatives. Based on the first-order tensor showed in (23), (24) and (29), the further derivatives of $\frac{\partial U}{\partial x},\frac{\partial U}{\partial y},\frac{\partial U}{\partial z}$ respect to x, namely, $\frac{\partial^2 U}{\partial x^2}=\frac{\partial}{\partial x}\frac{\partial U}{\partial x}$, $\frac{\partial^2 U}{\partial x\partial y}=\frac{\partial}{\partial y}\frac{\partial U}{\partial x}$, $\frac{\partial^2 U}{\partial x\partial z}=\frac{\partial}{\partial z}\frac{\partial U}{\partial x}$ can be obtained. Similarly, $\frac{\partial^2 U}{\partial y\partial z}=\frac{\partial}{\partial y}\frac{\partial U}{\partial z}$, $\frac{\partial^2 U}{\partial z^2}=\frac{\partial}{\partial z}\frac{\partial U}{\partial z}$ can be implemented as the structure of those in (23). Making use of recursion relations, the following expression in a general form for the unified normalization computing for gravitational potential derivatives of arbitrary orders is derived:

$$\frac{\partial U^{\alpha+\beta+\gamma}}{\partial x^{\alpha}\partial y^{\beta}\partial z^{\gamma}}=\frac{GM_{\oplus}}{R_{\oplus}^{\alpha+\beta+\gamma+1}}\sum_{n=2+\alpha+\beta+\gamma}^{N+\alpha+\beta+\gamma}\sum_{m=0}^{\min(n,M+\alpha+\beta+\gamma)}\left(\bar{C}_{nm}^{x^{\alpha}y^{\beta}z^{\gamma}}\bar{V}_{nm}+\bar{S}_{nm}^{x^{\alpha}y^{\beta}z^{\gamma}}\bar{W}_{nm}\right)\qquad(30)$$

The chain roles of $\bar{V}_{nm},\bar{W}_{nm}$ are as the same as that presented in (19). Based on (22), (25) and (28), the recurrence relations of spherical harmonic coefficients $\bar{C}_{nm}^{x^\alpha y^\beta z^\gamma}$ $\bar{S}_{nm}^{x^\alpha y^\beta z^\gamma}$ of degree $\alpha+\beta+\gamma$ can be derived, as well as that of degree $\alpha+\beta+\gamma+1$. The obtained chain roles are divided into three cases with respect to x, y, z respectively.

The case when the summation degree is $\alpha+\beta+\gamma+1$ and the partial derivative degree respect to x is $\alpha+1$

$$\begin{cases}\bar{C}_{n,0}^{x^{\alpha+1}y^\beta z^\gamma} = \frac{1}{2}n(n+1)\bar{C}_{n-1,1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,1}^{n,0}\\ \bar{S}_{n,0}^{x^{\alpha+1}y^\beta z^\gamma} = 0\end{cases}$$
$$\begin{cases}\bar{C}_{n,1}^{x^{\alpha+1}y^\beta z^\gamma} = \frac{1}{2}\left[(n-1)(n-2)\bar{C}_{n-1,2}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,2}^{n,1} - 2\bar{C}_{n-1,0}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,0}^{n,1}\right]\\ \bar{S}_{n,1}^{x^{\alpha+1}y^\beta z^\gamma} = \frac{1}{2}(n-1)(n-2)\bar{S}_{n-1,2}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,2}^{n,1}\end{cases}$$
$$\begin{cases}\bar{C}_{ns}^{x^{\alpha+1}y^\beta z^\gamma} = -\frac{1}{2}\bar{C}_{n-1,s-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,s-1}^{ns}\\ \bar{S}_{ns}^{x^{\alpha+1}y^\beta z^\gamma} = \frac{1}{2}\bar{S}_{n-1,s-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,s-1}^{ns}\end{cases}$$
$$\min(n-1, M+\alpha+\beta+\gamma) \le s \le \min(n, M+\alpha+\beta+\gamma)$$
$$\begin{cases}\bar{C}_{nm}^{x^{\alpha+1}y^\beta z^\gamma} = \frac{1}{2}\left[(n-m)(n-m-1)\bar{C}_{n-1,m+1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m+1}^{nm} - \bar{C}_{n-1,m-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m-1}^{nm}\right]\\ \bar{S}_{nm}^{x^{\alpha+1}y^\beta z^\gamma} = \frac{1}{2}\left[(n-m)(n-m-1)\bar{S}_{n-1,m+1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m+1}^{nm} - \bar{S}_{n-1,m-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m-1}^{nm}\right]\end{cases}$$
$$2+\alpha+\beta+\gamma \le n \le N+\alpha+\beta+\gamma \quad 2 \le m < \min(n-1, M+\alpha+\beta+\gamma) \tag{31}$$

The case when the summation degree is $\alpha+\beta+\gamma+1$ and the partial derivative degree respect to y is $\beta+1$.

$$\begin{cases}\bar{C}_{n,0}^{x^\alpha y^{\beta+1} z^\gamma} = \frac{1}{2}n(n+1)\bar{S}_{n-1,1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,1}^{n,0}\\ \bar{S}_{n,0}^{x^\alpha y^{\beta+1} z^\gamma} = 0\end{cases}$$
$$\begin{cases}\bar{C}_{n,1}^{x^\alpha y^{\beta+1} z^\gamma} = \frac{1}{2}(n-1)(n-2)\bar{S}_{n-1,2}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,2}^{n,1}\\ \bar{S}_{n,1}^{x^\alpha y^{\beta+1} z^\gamma} = -\frac{1}{2}\left[(n-1)(n-2)\bar{C}_{n-1,2}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,2}^{n,1} + 2\bar{C}_{n-1,0}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,0}^{n,1}\right]\end{cases}$$
$$\begin{cases}\bar{C}_{ns}^{x^\alpha y^{\beta+1} z^\gamma} = \frac{1}{2}\bar{S}_{n-1,s-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,s-1}^{ns}\\ \bar{S}_{ns}^{x^\alpha y^{\beta+1} z^\gamma} = -\frac{1}{2}\bar{C}_{n-1,s-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,s-1}^{ns}\end{cases} \quad s = \min(n, M+\alpha+\beta+\gamma)$$
$$\begin{cases}\bar{C}_{nm}^{x^\alpha y^{\beta+1} z^\gamma} = \frac{1}{2}\left[(n-m)(n-m-1)\bar{S}_{n-1,m+1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m+1}^{nm} + \bar{S}_{n-1,m-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m-1}^{nm}\right]\\ \bar{S}_{nm}^{x^\alpha y^{\beta+1} z^\gamma} = -\frac{1}{2}\left[(n-m)(n-m-1)\bar{C}_{n-1,m+1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m+1}^{nm} + \bar{C}_{n-1,m-1}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m-1}^{nm}\right]\end{cases}$$
$$2+\alpha+\beta+\gamma \le n \le N+\alpha+\beta+\gamma \quad 2 \le m < \min(n, M+\alpha+\beta+\gamma) \tag{32}$$

The case when the summation degree is $\alpha+\beta+\gamma+1$ and the partial derivative degree respect to z is $\gamma+1$.

$$\begin{cases}\bar{C}_{ns}^{x^\alpha y^\beta z^{\gamma+1}} = 0\\ \bar{S}_{ns}^{x^\alpha y^\beta z^{\gamma+1}} = 0\end{cases} \quad s = \min(n, M+\alpha+\beta+\gamma)$$
$$\begin{cases}\bar{C}_{nm}^{x^\alpha y^\beta z^{\gamma+1}} = -(n-m)\bar{C}_{n-1,m}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m}^{nm}\\ \bar{S}_{nm}^{x^\alpha y^\beta z^{\gamma+1}} = -(n-m)\bar{S}_{n-1,m}^{x^\alpha y^\beta z^\gamma}\Pi_{n-1,m}^{nm}\end{cases}$$
$$2+\alpha+\beta+\gamma \le n \le N+\alpha+\beta+\gamma \quad 0 \le m < \min(n, M+\alpha+\beta+\gamma) \tag{33}$$

The unified normalization factor Π_{nm}^{pq} is introduced to satisfy the general cases. The respective concrete forms of Π_{nm}^{pq} can be derived in terms of different definitions of normalization factor β_{nm}. For original Legendre polynomials, $\beta_{nm} \equiv 1$. Other conditions can be referred to Tab.VI.

TABLE VI. THE UNIFIED NORMALIZATION FACTORS IN ACCELERATION CALCULATION UNDER VARIOUS CASES

	full normalization	*Belikov's normalization*	*Metris's normalization*
β_{nm}	$\sqrt{\frac{(n+m)!}{(2n+1)(n-m)!(2-\delta_{m0})}}$	$\frac{(n+m)!}{2^n n!}$	$\frac{(2n+1)!}{(2-\delta_{mo})2^n n!(n-m)!}$
Π_{nm}^{pq}	$\sqrt{\frac{2-\delta_{m0}}{2-\delta_{q0}}\frac{2n+1}{2p+1}\frac{(n-m)!}{(p-q)!}\frac{(p+q)!}{(n+m)!}}$	$2^{n-p}\frac{n!}{p!}\frac{(p+q)!}{(n+m)!}$	$2^{n-p}\frac{2-\delta_{m0}}{2-\delta_{q0}}\frac{n!}{p!}\frac{(2p+1)!}{(2n+1)!}\frac{(n-m)!}{(p-q)!}$
$\Pi_{n-1,0}^{n,1}$	$\sqrt{\frac{2n-1}{2n+1}\frac{n(n+1)}{2}}$	$\frac{n+1}{2}$	$\frac{(2n+1)}{2}$
$\Pi_{n-1,m-1}^{nm}$	$\sqrt{\frac{2n-1}{2n+1}\frac{(n+m)(n+m-1)}{1+\delta_{m,1}}}$	$\frac{(n+m)(n+m-1)}{2n}$	$\frac{2n+3}{1+\delta_{m,1}}$
$\Pi_{n-1,m+1}^{nm}$	$\sqrt{\frac{2n-1}{2n+1}\frac{1+\delta_{m,0}}{(n-m)(n-m-1)}}$	$\frac{1}{2n}$	$\frac{(1+\delta_{m,0})(2n+1)}{(n-m)(n-m-1)}$
$\Pi_{n-1,m}^{nm}$	$\sqrt{\frac{2n-1}{2n+1}\frac{n+m}{n-m}}$	$\frac{n+m}{2n}$	$\frac{2n+1}{n-m}$

IV. SIMULATION AND ANALYSIS

Simulations of aforementioned three normalization methods are implemented in this section coded in C++ language. The validity and practicality of the unified normalization structure for the gravity derivatives of arbitrary orders is demonstrated to some extent by Laplace equations of low orders. The accuracy and efficiency among various normalization methods are taken into consideration to be compared and discussed.

The satellite orbital data in this procedure adopts the published position information based on two-body model. The algorithms start at 2017-01-01.00:00 with initial parameters $\vec{r} = (6678136.776010, 0, -1729.45633)$ (m), $\vec{v} = (1.987279, -1389.552202, 7672.856442)$ (m/s), and the latitude ranging from -80°~80°. Duration of 90 days is considered with a sampling step of 30 seconds and a total of 259200 sample points. The Earth gravity model WGS84 of 70 × 70 is adopted for the initial spherical harmonic coefficients. The computations are performed on PC with Intel(R) Core(TM) i7-4790K CPU, 4.00GHz processor, 8GB 1867MHz DDR3 RAM, and Windows 10 operating system of 64 bits.

Since the published spherical harmonic coefficients are in the form of full normalization, they need to be converted to other normalization forms for corresponding tensor computing.

According to (14), the relations between different normalization factors are

$$\frac{\bar{c}_{nm}}{\bar{\beta}_{nm}} = \frac{c_{nm}}{\beta_{nm}}, \frac{\bar{s}_{nm}}{\bar{\beta}_{nm}} = \frac{s_{nm}}{\beta_{nm}} \tag{34}$$

Then

$$c_{nm} = \frac{\beta_{nm}}{\bar{\beta}_{nm}}\bar{c}_{nm}, s_{nm} = \frac{\beta_{nm}}{\bar{\beta}_{nm}}\bar{s}_{nm}$$

where $\frac{\beta_{nm}}{\bar{\beta}_{nm}}$ is the conversion factor, $\bar{c}_{nm}$ $\bar{s}_{nm}$ the published fully normalized spherical harmonic coefficients and c_{nm} s_{nm} the desired spherical harmonic coefficients.

The conversion factors under different normalization cases are listed as Tab.VII.

TABLE VII. CONVERSION FACTORS BETWEEN FULL NORMALIZATION FACTOR AND OTHERS

	β_{nm}	$\beta_{nm}/\bar{\beta}_{nm}$
Full normalization $\bar{\beta}_{nm}$	$\sqrt{\frac{(n+m)!}{(2n+1)(n-m)!(2-\delta_{m0})}}$	1
Belikov's normalization $\tilde{\beta}_{nm}$	$\frac{(n+m)!}{2^n n!}$	$\frac{\sqrt{(2-\delta_{m0})(2n+1)}}{2^n\sqrt{(n-m)!(n+m)!}}$
Metris's normalization ε_{nm}	$\frac{(2n+1)!}{(2\text{-}\delta_{mo})2^n n!(n-m)!}$	$\frac{(2n+1)(2n)!!}{\sqrt{(2-\delta_{m0})(n-m)!(n+m)!}}$

A. Correctness verification of the general and unified form for arbitrary order derivatives

The validation for the correctness of the proposed form is implemented in this subsection. Specifically, we compare the summations of the diagonal components of corresponding second-order tensors. The closer the results are to zero, the better it satisfies the Laplace equations. The results are shown in Fig.1.

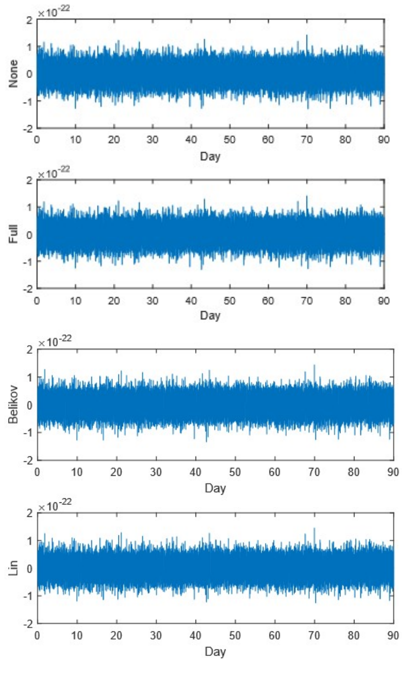

Figure 1. Comparison of Laplace equation results under various normalization cases.

It can be seen that the differences between non-normalized equations and normalized ones are quite modest. Hereinto the magnitude of diagonal components of corresponding second-order tensors is 10^{-9}, while that of summations is 10^{-22}. With the permission of the error, the computing results under four cases all satisfy the Laplace equations. The validity of the results proves indirectly that the unified form presented in this paper is correct.

In accordance to the verification of high-order tensor components of the gravitational potential, the correctness of the proposed form is demonstrated further.

Based on the Laplace equations of the gravitational potential $U_{xx}+U_{yy}+U_{zz}=0$, we obtain the following expression with respect to x,y,z in further derivation

$$U_{xxx^\alpha y^\beta z^\gamma}+U_{yyx^\alpha y^\beta z^\gamma}+U_{zzx^\alpha y^\beta z^\gamma}=0 \tag{35}$$

where α,β,γ are positive integers.

For tensors of degree $N(N\geq 2)$, there are $\frac{(N-1)N}{2}$ equations of (35). Taking the first five order tensor as instance, there are $\sum_{N=2}^{5}\frac{(N-1)N}{2}=20$ equations for each sampling point. The higher of the degree, the smaller the value of the tensors. So relative values of (35) $\frac{U_{xxx^\alpha y^\beta z^\gamma}+U_{yyx^\alpha y^\beta z^\gamma}+U_{zzx^\alpha y^\beta z^\gamma}}{\left|U_{xxx^\alpha y^\beta z^\gamma}\right|+\left|U_{yyx^\alpha y^\beta z^\gamma}\right|+\left|U_{zzx^\alpha y^\beta z^\gamma}\right|}$ is considered rather than $U_{xxx^\alpha y^\beta z^\gamma}+U_{yyx^\alpha y^\beta z^\gamma}+U_{zzx^\alpha y^\beta z^\gamma}$.

The computing results of relative values of the summation for each sampling point are listed in a row, consisting of $259200\times 20=5184000$ data information in the duration of 90 days.

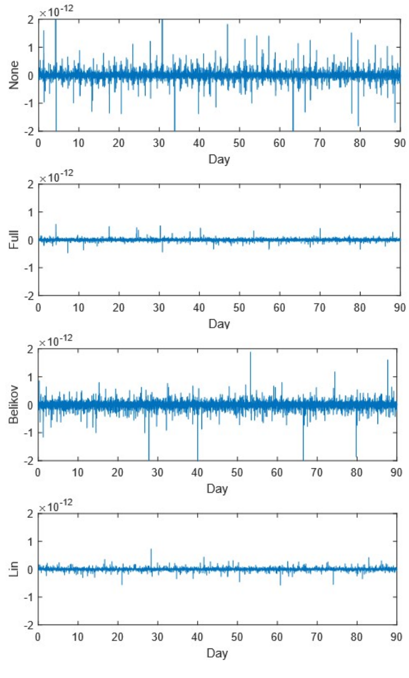

Figure 2. Relative value summation for the first five order derivatives under various cases.

Note that for high-order tensors of the gravity field, the general and unified form proposed in the paper still meets the requirement of Laplace equations in the range of allowable errors. It can be concluded that higher computing accuracy can be achieved with the help of normalization. Among these three normalization methods, Belikov's normalization form shows the worst results, while the Performance of full normalization and Lin's are evenly matched.

B. Relative precision comparison among various normalization methods

In this subsection, based on the proposed unified form in the paper, programming is easily implemented to compare the precision among various normalization cases.

Since there is not such a truth-value for the gravitational potential and its derivative tensors [15], Relative Precision (RP) is considered in the form as

$$RP = \left| \frac{Value(A) - Value(B)}{Value(B)} \right| \tag{36}$$

where $Value(B)$ is the fiducial value calculated by full normalization method, $Value(A)$ is the reference value of other normalization results.

The relative precision of the gravitational potential and its tensors of low-order derivatives is shown in Fig. 3−Fig. 12.

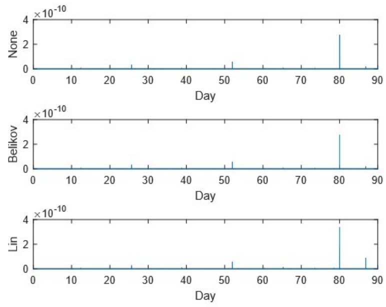

Figure 3. Relative precision comparison of the gravitational potential.

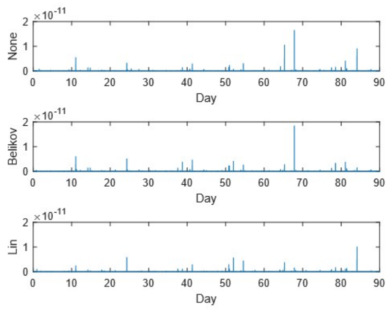

Figure 4. Relative precision comparison of U_x

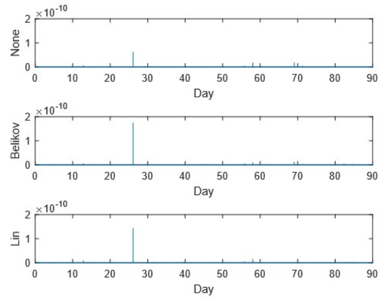

Figure 5. Relative precision comparison of U_y

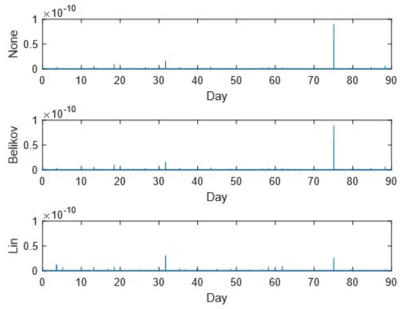

Figure 6. Relative precision comparison of U_z

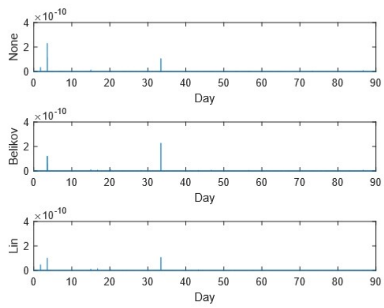

Figure 7. Relative precision comparison of U_{xx}

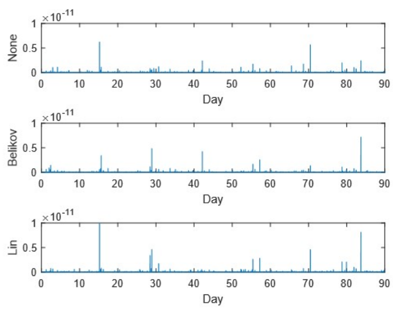

Figure 8. Relative precision comparison of U_{xy}

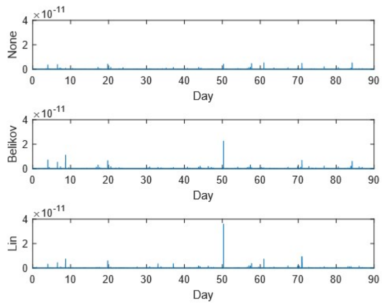

Figure 9. Relative precision comparison of U_{xz}

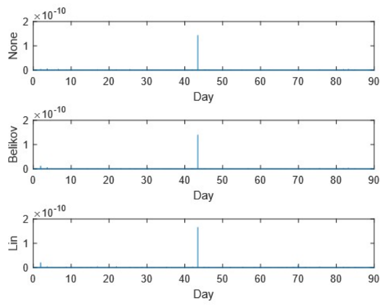

Figure 10. Relative precision comparison of U_{yy}

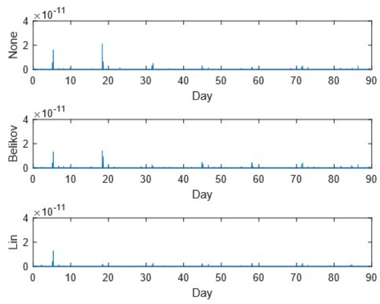

Figure 11. Relative precision comparison of U_{yz}

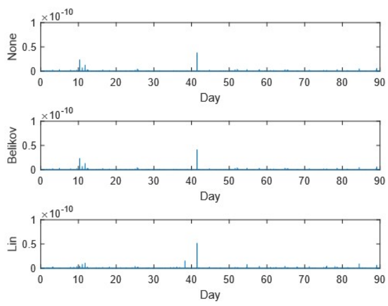

Figure 12. Relative precision comparison of U_{zz}

The results indicate that the relative precision between non-normalized values and other two normalized ones with respect to the full normalized one is quite small. Each normalization form has a similar accuracy.

C. Computing efficiency comparison among various normalization methods

The consuming time for the computation of the first five-order tensors under aforementioned conditions is counted. The transforming time for parameters of WGS84 model among various normalization forms is considered, while the orbit predicting time is eliminated.

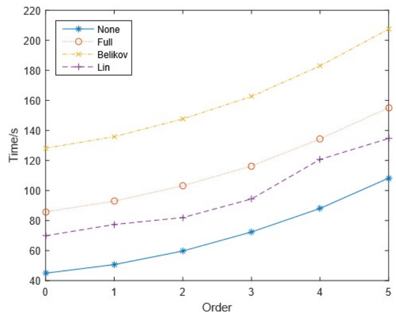

Figure 13. Computing time comparison among various normalization methods.

The consuming time of the method proposed in this paper has a near quadratic function relationship with orders in general. The non-normalized form is the most efficient, better than Lin's and full normalization one successively. Belikov's method behaves the worst, consuming approximately twice and triple time of that with Lin's method and non-normalized one respectively.

V. Conclusions

A general and unified form as a template to compute various normalized gravity potential tensor of arbitrary orders is derived in this paper with a simple and standard form. A unified normalization factor is introduced to process varieties of normalization situations. The concrete forms of Π_{nm}^{pq}, derived from corresponding definitions of β_{nm}, have four basic ones in the algorithms of this paper as $\Pi_{nm}^{n-1,m}\Pi_{nm}^{n-2,m}\Pi_{nm}^{n-1,m-1}\Pi_{nm}^{n-1,m+1}$. Other forms of Π_{nm}^{pq} are either the deformation results of changing the serial numbers or the simplification structure under some special conditions. The general form of normalization achieves a unified programming for various conditions, especially suitable for programing implementation of object-oriented programming languages, such as c + + and Java.

The series of spherical harmonic coefficients of the corresponding tensor $\left\{C_{nm}^{x^\alpha y^\beta z^\gamma},\ S_{nm}^{x^\alpha y^\beta z^\gamma}\right\}$ are constructed in this paper. $\left\{C_{nm}^{x^\alpha y^\beta z^\gamma},\ S_{nm}^{x^\alpha y^\beta z^\gamma}\right\}$ are merely related to initial data $\left\{C_{nm},\ S_{nm}\right\}$, degree n and order m and no longer position information (x, y, z), available for preprocessing and pre-storage. In practice, it is of high efficiency compared to the traditional expressions since $\{V_{nm},\ W_{nm}\}$ are the only terms needed to be evaluated according to recurrence relations. $\{V_{nm},\ W_{nm}\}$ can be directly evaluated by the projected position vector information (x, y, z) in the Cartesian Earth-fixed reference frame, getting rid of singularity problems and position transformation to planetocentric coordinates and improving the computing efficiency to some extent.

The simulation results indicate that our general and unified form is convenient and reliable for programming implementation and comparison among various normalization cases. The precision of the three normalization forms is quite near, while the computing efficiency vary considerably. The non-normalized form consumes the least time, and Belikov's method is the most inefficient.

The method proposed in our paper can be extended and applied to any other computing cases in a similar form as gravitational potential to obtain derivative tensors up to any order, such as geomagnetic field and bodily tide.

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