(Same as Lecture ~~6~~4 slides)

ANOVA table (For $H_0: \tau_1 = \dots = \tau_t$) (CRD)

$SST = SSE_0 - SSE$

Source	df	Sum of Squares	Mean Squares	F
Treatment	$t-1$	SST	$MST = \frac{SST}{t-1}$	$F = \frac{MST}{MSE} = \frac{SST/t-1}{SSE/n-t}$
Error	$n-t$	SSE	$MSE = \frac{SSE}{n-t}$	Recall $t-1 = \text{rank}(H_X - H_{X_0})$ $n-t = \text{rank}(I - H_X)$
Total	$n-1$	$SSTotal$		

Under $H_0: \tau_1 = \tau_2 = \dots = \tau_t$ (Recall model $y_{ij} = \mu + \tau_i + \epsilon_{ij}$)

$$\left.\begin{array}{l} SST \sim \chi'^2_{(t-1,0)} \\ SSE \sim \chi'^2_{(n-t,0)} \end{array}\right\} \text{indep} \Rightarrow F = \frac{SST/t-1}{SSE/n-t} \sim \underbrace{F_{(t-1,\, n-t)}}_{\text{central } F}$$

(balanced design)

$$\lambda = \frac{r}{\sigma^2} \sum_{i=1}^{t} (\mu_i - \bar{\mu})^2$$

In general

$$\left.\begin{array}{l} SST \sim \chi'^2_{(t-1,\lambda)} \\ SSE \sim \chi'^2_{(n-t,0)} \end{array}\right\} \text{indep} \Rightarrow F = \frac{SST/t-1}{SSE/n-t} \sim F'_{(t-1,\, n-t,\, \lambda)}$$

non central F

non centrality parameter

Determining the number of replicates (In a CRD)

Assume design is balanced!

Idea: Focus on hypothesis testing $H_0: \tau_1 = \dots = \tau_t = 0$ vs $H_a: \tau_i \neq 0$ for some i

We have test statistic $F = \dfrac{SST/t-1}{SSE/(n-t)}$

For a given α we reject H_0 if $F_{obs} > F_c$ where F_c = critical value

such that $P(F > F_c \mid H_0 \text{ true}) = \alpha$ and we know $F \sim F_{(t-1, n-t, 0)}$ central F distribution

(in R: 'qf' function) (under H_0)

Recall:

Type I Error: Reject H_0 when H_0 true!

Prob(type I Error) $= \alpha$ (controlled)

Type II Error: Not Reject H_0 when H_0 False!

Since test statistic is fixed then

P(type II Error) depends on 'r' and r, σ^2 and $\tau_1, \dots, \tau_t$ (since null hypothesis is not true!)

So we know that when H_0 is not true (H_0^c)

$\Rightarrow F \sim F_{(t-1,\, n-t,\, \lambda)}$ → non-central F where $\lambda = \frac{r}{\sigma^2}\sum_{i=1}^{t}(\tau_i - \bar{\tau})^2$, $\lambda > 0$

where $\lambda = \frac{\underline{\mu}^T(\underline{I} - H_{\underline{1}})\underline{\mu}}{\sigma^2}$ with $\underline{\mu} = X\underline{\beta}$, $\underline{\beta} = \begin{pmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_t \end{pmatrix}$

$$X = \begin{array}{c} \begin{matrix} \mu & \tau_1 & \tau_2 & \cdots & \tau_t \end{matrix} \\ \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 0 & & 0 \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 0 & 0 & & 1 \end{pmatrix} \\ \underline{1} \end{array}$$

Recall that $\mu_i = \mu + \tau_i$, $i = 1, 2, \ldots t$

$$\Rightarrow \underline{\mu} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \\ \vdots \\ \mu_t \\ \vdots \\ \mu_t \end{pmatrix} \quad H_{\underline{1}} = \frac{1}{n}\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \quad n = rt \quad \Rightarrow H_{\underline{1}}\underline{\mu} = \bar{\mu}\,\underline{1}$$

Balanced design

and $\bar{\mu} = \frac{r\mu_1 + r\mu_2 + \cdots + r\mu_t}{r \cdot t} = \frac{\mu_1 + \mu_2 + \cdots + \mu_t}{t} =$

$$= \frac{\mu + \tau_1 + \mu + \tau_2 + \cdots + \mu + \tau_t}{t} = \mu + \bar{\tau}$$

$$\Rightarrow \underline{\mu}^T\underline{\mu} = r\mu_1^2 + r\mu_2^2 + \cdots + r\mu_t^2 = r\sum_{i=1}\mu_i^2$$

$$\Rightarrow \underline{\mu}^T(\underline{I} - H_{\underline{1}})\underline{\mu} = \underline{\mu}^T\underline{\mu} - \underline{\mu}^T H_{\underline{1}}\underline{\mu} = \underline{\mu}^T\underline{\mu} - n\bar{\mu}\,\underline{\mu}^T\underline{1} = \underline{\mu}^T\underline{\mu} - n\bar{\mu}^2 = r\sum_{i=1}^{t}\mu_i^2 - rt\bar{\mu}^2 = r\left(\sum_{i=1}^{t}\mu_i^2 - t\bar{\mu}^2\right)$$

$$= r\sum_{i=1}^{t}(\mu_i - \bar{\mu})^2 = r\sum_{i=1}^{t}(\mu + \tau_i - \mu - \bar{\tau})^2 = \sum_{i=1}^{t}(\tau_i - \bar{\tau})^2$$

$$\Rightarrow \lambda = \frac{r \cdot \sum_{i=1}^{n}(\tau_i - \bar{\tau})^2}{\sigma^2} = \frac{r\sum_{i=1}^{n}(\mu_i - \bar{\mu})^2}{\sigma^2}$$

$$\text{Prob}(\text{type II Error}) = P(\text{not Reject } H_0 \mid H_0^c) = P(F \leq F_c \mid H_0^c)$$

we would like this to be large!

$$\text{Power}(r, \sigma^2, \underline{\tau}) = 1 - P(\text{type II Error}) = 1 - P(F \leq F_c \mid H_0^c) = P(F > F_c \mid H_0^c) = \underbrace{P(\text{Reject } H_0 \mid H_0 \text{ not true})}$$

upper tail probability of a non-central F distribution

in Lab 6 we showed it was an increasing function of λ.

MAIN IDEA

⊖ Fix σ^2 and $\underline{\tau}$

⊖ Target $\boxed{\text{Power}(r, \sigma^2, \underline{\tau}) = P_0}$ (∗) (0.9 say)

⊖ Determine r such that (∗) is true

Notes

⊕ σ^2 is determined by experience or a pilot experiment.

⊕ $\underline{\tau}$ will be fixed in a worst case scenario (next page)

The hardest situation to detect is when:

(A) The effects of two of the factor levels (say first and last) differ by Δ

(B) All other effect are equal and midway

e.g.

(A)

$$\mu+\tau_1 = k + \Delta/2$$
$$\mu+\tau_t = \cancel{k+\Delta/2}\ K - \frac{\Delta}{2}$$

(B)

$$\mu+\tau_2 = \mu+\tau_3 = \cdots = \mu+\tau_{t-1} = k$$

for some constant k (irrelevant)

$$\Rightarrow \lambda = \frac{r}{\sigma^2}\sum_{i=1}^{t}(\mu_i-\bar{\mu})^2 = \frac{r\Delta^2}{2\sigma^2} \Rightarrow P_0 = \frac{r\Delta^2}{2\sigma^2} \Leftrightarrow \boxed{r = \frac{2P_0\sigma^2}{\Delta^2}}$$

$$P_0 = \text{powern}\left(r, \sigma^2, \frac{r\Delta^2}{2\sigma^2}\right)$$

ANOVA Factorial design

$y_{ijk} = \mu + \tau_i + \alpha_j + \gamma_{ij} + \epsilon_{ijk}$

(Same as Lecture ~~4~~ 5 slides)

$= (\underline{1}\ \underline{X}_1, \underline{X}_2, \underline{X}_{int})$

$SSE = \underline{y}^T (I - H_X)\underline{y}$ stays the same! Just different $\underline{X}$ and $\Rightarrow$ different $H_{\underline{X}}$

Test null hypothesis $H_0: \gamma_{ij} = 0 \ \forall i, j$ (no interactions)

$\Rightarrow \underline{X}_0 = (1\ \underbrace{\underline{X}_1}_{\text{Effect of } \tau_i}, \underbrace{\underline{X}_2}_{\text{Effect of } \alpha_j}) \Rightarrow SSE_0 = \underline{y}^T (I - H_{\underline{X}_0})\underline{y}$

since clearly $H_{\underline{X}_0}$ is also idempotent $\Rightarrow I - H_{\underline{X}_0}$ idemp too!

Also $SSE_0 - SSE \geq 0$

$\Rightarrow \boxed{SSE_0 - SSE} = \underline{y}^T (H_X - H_{X_0})\underline{y}$

$\parallel$ sum of squares due to the interactions

$\boxed{SSInt}$

$\text{rank}(H_X - H_0) = \text{trace}(H_X - H_{X_0})$

$= \text{trace}(H_X) - \text{trace}(H_{X_0}) = \text{rank}(X) - \text{rank}(X_0)$

$= 1 + (t-1) + (s-1) + (t-1)(s-1) - (1 + (t-1) + (s-1))$

$= (t-1)(s-1)$

(Same as Lecture ~~8~~ 9 slides)

Theorem $\Rightarrow$

$$\frac{\underline{y}^T (H_X - H_{X_0}) \underline{y}}{\sigma^2} \sim \chi^2_{((t-1)(s-1),\ \lambda)}$$

where $\lambda = \frac{1}{\sigma^2} \underline{\mu}^T (H_X - H_{X_0}) \underline{\mu}$

under H_0 : $\gamma_{ij} = 0$

$$\Rightarrow \underline{\mu}_0 = X_0 \underline{\beta}_0$$

$$\Rightarrow \lambda = \frac{1}{\sigma^2} \underline{\beta}_0^T X_0^T (H_X - H_{X_0}) X_0 \underline{\beta}_0$$

$$= \frac{1}{\sigma^2} \beta_0 X_0^T (\underbrace{H_X X_0}_{X_0} - \underbrace{H_{X_0} X_0}_{X_0}) \underline{\beta}_0$$

$$\Rightarrow \lambda = 0$$

since X_0 is a subset of X!

otherwise

$$\lambda = \frac{1}{\sigma^2} \underline{\beta}^T \underline{X}^T (H_X \underline{X} - H_{X_0} \underline{X}) \beta$$

$$= \frac{1}{\sigma^2} \underline{\beta}^T \underline{X}^T (\underline{X} - H_{X_0} X) \underline{\beta}$$

$$= \frac{1}{\sigma^2} \underline{\beta}^T \underline{X}^T (I - H_{X_0}) X \underline{\beta}$$

$\boxed{\text{note: } H_{X_0} X \neq X \text{ or } \neq X_0}$

ANOVA Table for Factorial $H_0: \gamma_{ij} = 0 \ \forall ij$ (Same as Lecture ~~&~~ slides)

Balanced design

Source	df	Sum of Squares	Mean Squares	F
interactions	$(t-1)(s-1)$	SS Int	$SSInt/(t-1)(s-1)$	$\dfrac{SSInt/(t-1)(s-1)}{SSE/st(r-1)}$
Error	$st(r-1)$	SSE	$SSE/st(r-1)$	

$$n - t = str - (\cancel{1} + (t - \cancel{1}) + (s-1) + (t-1)(s-1))$$

$$\cancel{t} + \cancel{s} - \cancel{1} + t \cdot s - \cancel{t} - \cancel{s} + \cancel{1} \quad t \cdot s$$

$$= str - st = st(r-1)$$

Under H_0:

$$F \sim F_{((s-1)(t-1),\ st(r-1),\ 0)}$$

In general:

$$F \sim F'_{((s-1)(t-1),\ st(r-1),\ \lambda)}$$

λ in previous page!

Blocking

- **Objective:** Reduce the variance of the experimental error (σ^2) and increase the power for detecting treatment factor effects so that results **generalise to whole population**
- Choose the experimental units for a study to be as homogeneous as possible. Sometimes difficult!
- Heterogeneous experimental units are grouped into homogeneous subgroups before they are randomly assigned to treatment factor levels
- The act of grouping the experimental units together in homogeneous groups is called blocking.
- In a **randomized block design**, a group of heterogeneous experimental units is used so that the conclusions can be more general

Examples of blocking

- Plots of land in agricultural experiments are usually **blocked by proximity** because plots in close proximity normally have similar soil characteristics
- When experimental units are animals, the **grouping (blocking) of genetically similar animals**, such as littermates, often reduces variability within groups
- When experimental units are trials, or points in time where treatments will be applied, they are often **blocked by time** since many lurking variables may change over time and trials in close temporal proximity are more alike

Randomized complete block design (RCB) with one treatment factor

- Treatment factor has t levels
- b blocks (or subgroups of homogeneous experimental units)
- Each block contains exactly t experimental units for a total of $t \times b$ experimental units
- The t experimental units within each block are as similar as possible
- The groups of experimental units vary enough from block to block to allow general conclusions to be drawn
- The randomization of experimental units to treatment factor levels is performed within each block.

Comparison betwwen CRD and RCB $t = 3$, $b = 4$

```
levels<-c("level 1","level 2","level 3")
fac <- levels %>% rep(each = 4) %>% sample(12) %>% factor()
blocks <- factor( rep(c("block 1", "block 2", "block 3", "block 4"), each=3))
CRD <- data.frame( units=1:12,block=blocks,treatmentCRD=fac)
block1 <- sample(levels,3); block2 <- sample(levels,3)
block3 <- sample(levels,3); block4 <- sample(levels,3)
t<-c(block1,block2,block3,block4) %>% factor()
RCB<-data.frame(block = blocks, treatmentRCB = t)
cbind(CRD,RCB)
   units   block treatmentCRD   block treatmentRCB
1      1 block 1      level 1 block 1      level 1
2      2 block 1      level 1 block 1      level 2
3      3 block 1      level 3 block 1      level 3
4      4 block 2      level 3 block 2      level 3
5      5 block 2      level 1 block 2      level 1
6      6 block 2      level 3 block 2      level 2
7      7 block 3      level 3 block 3      level 1
8      8 block 3      level 2 block 3      level 3
9      9 block 3      level 2 block 3      level 2
10    10 block 4      level 2 block 4      level 3
11    11 block 4      level 1 block 4      level 2
12    12 block 4      level 2 block 4      level 1
```

Statistical model

$$y_{ij} = \mu + b_i + \tau_j + \epsilon_{ij}$$

where

- b_i is the effect of block $i \in \{1, \ldots, b\}$
- $\tau_1, \ldots, \tau_t$ are the treatment effects
- All $\{\epsilon_{ij}\}$ are $N(0, \sigma^2)$ and independent
- Note: There is no interaction between block and treatment
- Only $t \times b$ experimental units, there would be zero degrees of freedom for the error term ssE if a block by treatment interaction term were included!

ANOVA for Randomised complete block design

Model $\quad y_{ij} = \mu + b_i + \tau_j + \varepsilon_{ij}$

$\Rightarrow$ in matrix form $\quad \underline{\beta} = \begin{pmatrix} \mu \\ b_1 \\ \vdots \\ b_b \\ \tau_1 \\ \vdots \\ \tau_t \end{pmatrix}$, $\quad \underline{X} = \left(\begin{array}{c|cccc|cccc} 1 & 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 1 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline 1 & 0 & 1 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ \hline \vdots & \vdots & \vdots & & \vdots & & & & \\ \hline 1 & 0 & 0 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 1 \end{array} \right)$

$$\underline{y} = X\underline{\beta} + \underline{\varepsilon}$$

$\underline{1} \qquad U \qquad V$

$X_{ob} := (\underline{1}, V)$, $X_{or} = (\underline{1}, U)$

$H_0^{(\gamma)}: b_i = 0 \ \forall i$ vs $H_a: b_i \neq 0$ for some i

Under H_0: $\quad y_{ij} = \mu + \tau_j + \varepsilon_{ij}$

$\Rightarrow$ in matrix form $\quad \underline{y} = X_{ob}\,\underline{\alpha} + \underline{\varepsilon} \qquad \underline{\alpha} = \begin{pmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_t \end{pmatrix}$

$H_0^{(\tau)}: \tau_i = 0 \ \forall i$ vs $H_a: \tau_i \neq 0$ some i

under H_0 $\quad y_{ij} = \mu + b_i + \varepsilon_{ij}$

$\Rightarrow \quad \underline{y} = X_{or}\,\underline{\gamma} + \underline{\varepsilon}, \qquad \underline{\gamma} = \begin{pmatrix} \mu \\ b_1 \\ \vdots \\ b_b \end{pmatrix}$

Now, we can define

$$SSE = \underline{y}^T(I - H_X)\underline{y}$$

, $\text{rank}(I - H_X) = n - \text{rank}(X)$
$= t \cdot b - (\cancel{1} + t - \cancel{1} + b - 1)$
$= t \cdot b - t - b + 1 = t(b-1) - (b-1)$
$= (t-1)(b-1)$

$\Rightarrow \dfrac{SSE}{\sigma^2} \sim \chi^2_{((t-1)(b-1), 0)}$

$SSBlocks = SSE_{(H_0^{(b)})} - SSE$
$\underline{y}^T(H_X - H_{X_{ob}})\underline{y}$

, $\text{rank}(H_X - H_{X_{ob}}) = \text{rank}(X) - \text{rank}(X_{ob})$
$= 1 + t - 1 + b - 1 - (1 + t - 1) = b - 1$

$\Rightarrow \dfrac{SSBlocks}{\sigma^2} \sim \chi^2_{(b-1, \lambda_b)}$

$\lambda_b = \underline{\mu}^T(I - H_{X_{ob}})\underline{\mu}/\sigma^2$

$SST = SSE_{(H_0^{(\tau)})} - SSE$
$= \underline{y}^T(H_X - H_{X_{o\tau}})\underline{y}$

, $\text{rank}(H_X - H_{X_{o\tau}}) =$
$= 1 + t - 1 - b - 1 - (1 + b - 1) = t - 1$

$\Rightarrow \dfrac{SST}{\sigma^2} \sim \chi^2_{(t-1, \lambda_\tau)}$

$\lambda_t = \underline{\mu}^T(I - H_{X_{o\tau}})\underline{\mu}/\sigma^2$

ANOVA Table For RCB

Source	df	Sum of Squares	Mean Squares	F ratio
Blocks	$b-1$	SS_{Blocks}	$\frac{SS_{Blocks}}{b-1}$	
Treatments	$t-1$	SST	$\frac{SST}{t-1}$	$\frac{SST/t-1}{SSE/(t-1)(b-1)}$
Error	$(t-1)(b-1)$	SSE	$\frac{SSE}{(t-1)(b-1)}$	

$$F = \frac{SST/t-1}{SSE/(t-1)(b-1)} \sim F_{(t-1,\ (t-1)(b-1),\ \lambda)}$$