Definition:

Let A be a square matrix. A Generalised inverse (G-inv) of A is any matrix A^- such that:

$$A A^- A = A$$

Properties: Let $G = A^- = (X^T X)^-$

(P1) G is also squared

(P2) G^T is also a G-inv of $A = X^T X$

(P3) Let $H_x := X G X^T$ projection matrix (hat matrix). projection onto column space of $\underline{X}$.

$\Rightarrow H_x \underline{X} = \underline{X}$

(P4) $H_x = X G X^T$ is invariant to the choice of G

(P5) $H_x = X G X^T$ is symmetric

new!

(P6) $\text{rank}(H_x) = \text{rank}(\underline{X} G \underline{X}^T) = \text{rank}(\underline{X})$

Now $\underline{y} - X\tilde{\beta} = \underline{y} - X(X^TX)^- X^T\underline{y} = \underline{y} - \underline{X}G\underline{X}^T\underline{y} = \underline{y} - H_x\underline{y} = (\underbrace{I-H_x}_{})\underline{y}$

invariant to choice of G

$\Rightarrow$ $SSE = (y - X\tilde{\beta})^T(y - X\tilde{\beta}) = \underline{y}^T(I-H_x)^T(I-H_x)\underline{y}$

But $(I-H_x)^T = I - H_x^T = I - H_x$ since H_x is symmetric by PS

Also $H_xH_x = \boxed{H_x \underset{=}{X}}GX^T = XGX^T = H_x$, H_x is idempotent

$\underset{=}{X}$

$\Rightarrow (I-H_x)(I-H_x) = I - H_x - H_x + H_xH_x = I - H_x - \cancel{H_x} + \cancel{H_x} = I - H_x$ $\Rightarrow$ $I-H_x$ is also idempotent

$$\boxed{\Rightarrow SSE = \underline{y}^T(I-H_x)(I-H_x)\underline{y} = \underline{y}^T(I-H_x)\underline{y}}$$

and is invariant to the choice of G

ANOVA $\quad \beta = \begin{pmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_t \end{pmatrix} \quad X = \begin{pmatrix} \vdots & \cdots \end{pmatrix}$

CRD

Effects Model $\quad y_{ij} = \mu + \tau_i + \epsilon_{ij}$

$\underline{y} = X\underline{\beta} + \underline{\epsilon} \qquad SSE = (\underline{y} - X\tilde{\beta})^T(\underline{y} - X\tilde{\beta})$ $\qquad \tilde{\beta}$ solution to normal eq's

measure quality of fit

Reduced model

Test $\quad H_0: \tau_1 = \tau_2 = \cdots = \tau_t \implies y_{ij} = \mu + \epsilon_{ij}$

$\underline{y} = X_0\underline{\beta} + \underline{\epsilon}$ where $X_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad \beta = \mu$

$= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\mu + \epsilon$

$\hat{\mu} = \bar{y}$

$\implies SSE_0 = (\underline{y} - X_0\hat{\mu})^T(\underline{y} - X_0\hat{\mu}) = (\underline{y} - \underline{1}\hat{\mu})^T(\underline{y} - \underline{1}\hat{\mu}) = \sum_i \sum_j (y_{ij} - \hat{\mu})^2$

called SSTotal

$= (\underline{y} - \bar{y}\underline{1})^T(\underline{y} - \bar{y}\underline{1}) = (\underline{y} - \frac{1}{n}\underline{1}\underline{1}^T\underline{y})^T(\underline{y} - \frac{1}{n}\underline{1}\underline{1}^T\underline{y})$

$= \underline{y}^T(I - \frac{1}{n}\underline{1}\underline{1}^T)(I - \frac{1}{n}\underline{1}\underline{1}^T)\underline{y}$

clamp

Take $\quad A = I - \frac{1}{n}\underline{1}\underline{1}^T \implies A^T = I - \frac{1}{n}\underline{1}\underline{1}^T$ symmetric $\qquad = \underline{y}^TAA\underline{y} = \underline{y}^TA\underline{y}$

$\implies AA = (I - \frac{1}{n}\underline{1}\underline{1}^T)(I - \frac{1}{n}\underline{1}\underline{1}^T) = I - \frac{1}{n}\underline{1}\underline{1}^T - \frac{1}{n}\underline{1}\underline{1}^T + \frac{1}{n^2}\underline{1}\,\underline{1}^T\underline{1}\,\underline{1}^T = I - \frac{2}{n}\underline{1}\underline{1}^T + \frac{1}{n}\underline{1}\underline{1}^T$

idempotent $\qquad n \qquad = I - \frac{1}{n}\underline{1}\underline{1}^T = A$

Result ① If A is idempotent then $\text{rank}(A) = \text{trace}(A)$ = sum diagonal elements.

② $H_x X = X$ ⟹ $H_x \mathbf{1} = \mathbf{1}$ (since $\mathbf{1}$ is a column of $\mathbb{X}$!)

$$\text{rank}(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T) = \text{trace}(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T) = \text{trace}(I) - \tfrac{1}{n}\text{trace}(\mathbf{1}\mathbf{1}^T) = n - \tfrac{1}{n}n = n-1$$

use $H_{\mathbf{1}} = \frac{1}{n}\mathbf{1}\mathbf{1}^T$

$$\mathbf{1}\mathbf{1}^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

Always $SSE_0 > SSE$ (Fit is always worse if less parameters)

$\Rightarrow SSE_0 - SSE > 0$

"

$$-\underline{y}^T(I - XGX^T)\underline{y} + \underline{y}^T(I - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)\underline{y} = \underline{y}^T \big(\underbrace{XGX^T - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T}_{H_x - H_{\mathbf{1}}}\big)\underline{y}$$

$$B = (XGX^T - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)(XGX^T - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T)$$

$$= XGX^T - \tfrac{1}{n}\underbrace{XGX^T\mathbf{1}}_{\text{② projection}}\mathbf{1}^T - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T XGX^T + \tfrac{1}{n}\mathbf{1}\mathbf{1}\mathbf{1}^T$$

$$= XGX^T - \tfrac{1}{n}\mathbf{1}\mathbf{1}^T - \cancel{\tfrac{1}{n}\mathbf{1}\mathbf{1}^T} + \cancel{\tfrac{1}{n}\mathbf{1}\mathbf{1}^T} = B \text{ idempotent}$$

$\text{rank}(B) = \text{trace}(B) = \text{trace}(H_x) - \frac{1}{n}\text{trace}(\mathbf{1}\mathbf{1}^T) =$ P6 $\text{trace}(XX^TG) - 1 = \text{rank}(\mathbb{X}) - 1 = t - 1$

Theorem $\underline{y} \sim MVN(\underline{\mu}, \sigma^2 I)$

Define central χ^2

non central χ'^2

Warning

$\frac{\lambda}{2}$ in R is λ just

Then $\frac{\underline{y}^T A \underline{y}}{\sigma^2} \sim \chi'^2(r(A), \lambda)$, $\lambda = \frac{1}{\sigma^2} \underline{\mu}^T A \underline{\mu}$

non-central χ^2

$\iff$ A idempotent

$\operatorname{rank}(I - \frac{1}{n}\underline{1}\underline{1}^T) = n-1$

$$\Rightarrow SSTotal = \frac{\underline{y}^T (I - \frac{1}{n}\underline{1}\underline{1}^T)\underline{y}}{\sigma^2} \sim \chi^2_{(n-1, 0)}$$

here $\underline{\mu} = \mu \underline{1} \Rightarrow \lambda = \frac{1}{\sigma^2}\mu^2 \underline{1}^T(I - \frac{1}{n}\underline{1}\underline{1}^T)\underline{1}$

$$\lambda = \frac{1}{\sigma^2}\mu^2 (\underbrace{\underline{1}^T\underline{1} - \frac{1}{n}n^2}_{n-n}) = 0$$

$$\Rightarrow SSE = \frac{\underline{y}^T (I - XGX^T)\underline{y}}{\sigma^2} \sim \chi^2_{(n-t, 0)}$$

$\operatorname{rank}(I - XGX^T) = \operatorname{rank}(I) - \operatorname{rank}(XGX^T)$

$= n - t$

here $\underline{\mu} = X\beta$

$$\lambda = \frac{1}{\sigma^2}\beta^T X^T (I - XGX^T) X \beta$$

$$= \frac{1}{\sigma^2}\beta^T (X^T X - X^T X G X^T X)\beta$$

$$= \frac{1}{\sigma^2}\beta^T (\underbrace{X^T X - X^T X}_{0})\beta = 0$$

$$SST = \frac{y^T\left(XGX^T - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)y}{\sigma^2} \sim \chi^2_{(t-1,\ \lambda)}$$

$$\mathrm{rank}\left(XGX^T - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right) = t-1$$

$\parallel$

$SS_{Total} - SSE$

$\parallel$

SSE_0

have

$\mu = X\beta$

note $\lambda = 0$ if H_0: true $\mu = \mu\mathbf{1}$

$$\lambda = \frac{1}{2\sigma^2}\beta^T X^T\left(XGX^T - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)X\beta$$

$$= \beta^T\left(X^TXGX^TX - \frac{1}{n}X^T\mathbf{1}\mathbf{1}^TX\right)\beta$$

$$= \beta^T\left(X^TX - \frac{1}{n}X^T\mathbf{1}\mathbf{1}^TX\right)\beta$$

$$= \beta^TX^T\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)X\beta$$

$$= \mu^T\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mu = \mu^T\mu - n\bar{\mu}^2$$

$$\Sigma\mu_i^2 - n\bar{\mu}^2$$

$$\frac{\Sigma(\mu_i - \bar{\mu})^2}{2\sigma^2}$$

otherwise is weighted avg

note if balanced

$$\bar{\mu} = \frac{r\mu_1 + r\mu_2 \cdots r\mu_t}{tr} = \frac{\mu_1 + \cdots \mu_t}{t}$$

$$\mu = \begin{pmatrix} \mu_1 \\ \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \\ \vdots \\ \mu_t \\ \vdots \\ \mu_t \end{pmatrix} \begin{matrix} \}\, r_1 \\ \}\, r_2 \\ \}\, r_t \end{matrix}$$

$$\Rightarrow \lambda = \frac{1}{\sigma^2}\sum_{i=1}^{t} r_i(\mu_i - \bar{\mu})^2 \underset{\text{balanced}}{=} \frac{r}{\sigma^2}\sum_{i=1}^{t}(\mu_i - \bar{\mu})^2$$

ANOVA table

Source	df	Sum of Squares	Mean Squares	F
Treatment	$t-1$	SST	$MST = \frac{SST}{t-1}$	$F = \frac{MST}{MSE} = \frac{SST/t-1}{SSE/n-t}$
Error	$n-t$	SSE	$MSE = \frac{SSE}{n-t}$	
Total	$n-1$	SSTotal		

Under $H_0: \tau_1 = \tau_2 = \cdots = \tau_t$

$$\left.\begin{array}{l} SST \sim \chi'^2_{(t-1,0)} \\ SSE \sim \chi'^2_{(n-t,0)} \end{array}\right\} \text{indep} \Rightarrow F = \frac{SST/t-1}{SSE/n-t} \sim F_{(t-1,\, n-t)}$$

central F

(balanced design)

$$\lambda = \frac{r}{\sigma^2} \sum_{i=1}^{t} (\mu_i - \bar{\mu})^2$$

In general

$$\left.\begin{array}{l} SST \sim \chi^2_{(t-1,\, \lambda)} \\ SSE \sim \chi^2_{(n-t,\, 0)} \end{array}\right\} \text{indep} \Rightarrow F = \frac{SST/t-1}{SSE/n-t} \sim F'_{(t-1,\, n-t,\, \lambda)}$$

non central F

non centrality parameter

ANOVA Factorial design $y_{ijk} = \mu + \tau_i + \alpha_j + \gamma_{ij} + \epsilon_{ijk}$

$= (\underline{1}\ \underline{X}_1, \underline{X}_2, \underline{X}_{int})$

$SSE = \underline{y}^T (I - H_X) \underline{y}$ stays the same! just different $\underline{X}$ and $\Rightarrow$ different $H_{\underline{X}}$

Test null hypothesis $H_0: \gamma_{ij} = 0 \ \forall i, j$ (no interactions)

$\Rightarrow \underline{X}_0 = (1\ \underline{X}_1, \underline{X}_2) \Rightarrow SSE_0 = \underline{y}^T (I - H_{\underline{X}_0}) \underline{y}$ since clearly $H_{\underline{X}_0}$ is also idempotent $\Rightarrow I - H_{\underline{X}_0}$ idemp too!

Also $SSE_0 - SSE \geq 0$

$\Rightarrow SSE_0 - SSE = \underline{y}^T (H_X - H_{X_0}) \underline{y}$

$\quad\quad$ ‖
sum of squares due to the interactions

SS_{Int}

$\text{rank}(H_X - H_0) = \text{trace}(H_X - H_{X_0})$

$= \text{trace}(H_X) - \text{trace}(H_{X_0}) = \text{rank}(X) - \text{rank}(X_0)$

$= 1 + (t-1) + (s-1) + (t-1)(s-1) - (1 + (t-1) + (s-1))$

$= (t-1)(s-1)$

Theorem

$$\Rightarrow \frac{\underline{y}^T (H_X - H_{X_0}) \underline{y}}{\sigma^2} \sim \chi^2_{((t-1)(s-1),\, \lambda)}$$

where $\lambda = \frac{1}{\sigma^2} \underline{\mu}^T (H_X - H_{X_0}) \underline{\mu}$

under H_0 : $\gamma_{ij} = 0$

$$\Rightarrow \underline{\mu} = \underline{X}_0 \underline{\beta}_0$$

$$\Rightarrow \lambda = \frac{1}{\sigma^2} \underline{\beta}_0^T \underline{X}_0^T (H_X - H_{X_0}) \underline{X}_0 \underline{\beta}_0$$

$$= \frac{1}{\sigma^2} \underline{\beta}_0 \underline{X}_0^T (\underbrace{H_X \underline{X}_0}_{\underline{X}_0} - \underbrace{H_{X_0} \underline{X}_0}_{\underline{X}_0}) \underline{\beta}_0$$

$$\Rightarrow \lambda = 0$$

otherwise

$$\lambda = \frac{1}{\sigma^2} \underline{\beta}^T \underline{X}^T (H_X X - H_{X_0} X) \beta$$

$$= \frac{1}{\sigma^2} \underline{\beta}^T \underline{X}^T (\underline{X} - H_{X_0} X) \underline{\beta}$$

$$= \frac{1}{\sigma^2} \underline{\beta}^T \underline{X}^T (I - H_{X_0}) X \underline{\beta}$$

ANOVA Table for Factorial $H_0: \gamma_{ij} = 0 \ \forall ij$

Balanced design

Source	df	Sum of Squares	Mean Squares	F
interactions	$(t-1)(s-1)$	SS Int	$SSInt/(t-1)(s-1)$	$\dfrac{SSInt/(t-1)(s-1)}{SSE/st(r-1)}$
Error	$st(r-1)$	SSE	$SSE/st(r-1)$	

under H_0

$F \sim F_{((s-1)(t-1),\, st(r-1),\, 0)}$

in general

$F \sim F'_{((s-1)(t-1),\, st(r-1),\, \lambda)}$

λ previous page

$$n-t = str - \left(\cancel{1} + (t-\cancel{1}) + (s-1) + (t-1)(s-1)\right)$$

$$\cancel{t} + \cancel{s} - \cancel{1} + t\cdot s - \cancel{t} - \cancel{s} + \cancel{1} \quad t\cdot s$$

$$= str - st = st(r-1)$$