Definition:

Let A be a square matrix. A Generalised inverse (G-inv) of A is any matrix A^- such that:

$$A A^- A = A$$

Properties! Let $G = A^- = (X^T X)^-$

(P1) G is also squared

(P2) G^T is also a G-inv of $A = X^T X$

(P3) Let $H_x := X G X^T$ projection matrix (hat matrix). projection onto column space of $\underline{X}$.

$\Rightarrow H_x \underline{X} = \underline{X}$

(P4) $H_x = X G X^T$ is invariant to the choice of G

(P5) $H_x = X G X^T$ is symmetric

new!

(P6) $\text{rank}(H_x) = \text{rank}(\underline{X} G \underline{X}^T) = \text{rank}(\underline{X})$

Now $\underline{y} - X\tilde{\beta} = \underline{y} - X(X^TX)^{-}X^T\underline{y} = \underline{y} - \underline{X}G\underline{X}^T\underline{y} = \underline{y} - H_x\underline{y} = \underbrace{(I-H_x)}_{\substack{\text{invariant to}\\ \text{choice of } G}}\underline{y}$

$\Rightarrow \quad SSE = (y - X\tilde{\beta})^T(y - X\tilde{\beta}) = \underline{y}^T(I-H_x)^T(I-H_x)\underline{y}$

But $(I-H_x)^T = I - H_x^T = I - H_x$ since H_x is symmetric by PS

Also $H_xH_x = \boxed{H_x\underline{X}}G\underline{X}^T = XGX^T = H_x$, $\quad$ H_x is idempotent

$\qquad\qquad\qquad \overset{\shortparallel}{\underline{X}}$

$\Rightarrow (I-H_x)(I-H_x) = I - H_x - H_x + H_xH_x = I - H_x - \cancel{H_x} + \cancel{H_x} = I - H_x$ $\quad \Rightarrow$ $I-H_x$ is also idempotent

$$\Rightarrow \boxed{SSE = \underline{y}^T(I-H_x)(I-H_x)\underline{y} = \underline{y}^T(I-H_x)\underline{y}}$$

and is invariant to the choice of G

ANOVA

$\beta = \begin{pmatrix} \mu \\ \tau_1 \\ \vdots \\ \tau_t \end{pmatrix}$ $X = \begin{pmatrix} \vdots & \cdots \end{pmatrix}$

CRD

Effects Model

$y_{ij} = \mu + \tau_i + \epsilon_{ij}$

$\underline{y} = X\beta + \underline{\epsilon}$

$SSE = (\underline{y} - X\tilde{\beta})^T(\underline{y} - X\tilde{\beta})$

$\tilde{\beta}$ solution to normal eq's

measure quality of fit

Reduced model

Test $H_0: \tau_1 = \tau_2 = \dots = \tau_t \Rightarrow y_{ij} = \mu + \epsilon_{ij}$

$\underline{y} = X_0\underline{\beta} + \underline{\epsilon}$ where $X_0 = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ $\beta = \mu$

$= \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}\mu + \epsilon$

$\hat{\mu} = \bar{y}$

$\Rightarrow SSE_0 = (\underline{y} - X_0\hat{\mu})^T(\underline{y} - X_0\hat{\mu}) = (\underline{y} - \underline{1}\hat{\mu})^T(\underline{y} - \underline{1}\hat{\mu}) = \sum_i\sum_j (y_{ij} - \hat{\mu})^2$

called SSTotal

$= (\underline{y} - \bar{y}\underline{1})^T(\underline{y} - \bar{y}\underline{1}) = (\underline{y} - \frac{1}{n}\underline{1}\underline{1}^T\underline{y})^T(\underline{y} - \frac{1}{n}\underline{1}\underline{1}^T\underline{y})$

$= \underline{y}^T(I - \frac{1}{n}\underline{1}\underline{1}^T)(I - \frac{1}{n}\underline{1}\underline{1}^T)\underline{y}$

Take $A = I - \frac{1}{n}\underline{1}\underline{1}^T$ clamp $\Rightarrow A^T = I - \frac{1}{n}\underline{1}\underline{1}^T$ symmetric $= \underline{y}^TAA\underline{y} = \underline{y}^TA\underline{y}$

$\Rightarrow AA = (I - \frac{1}{n}\underline{1}\underline{1}^T)(I - \frac{1}{n}\underline{1}\underline{1}^T) = I - \frac{1}{n}\underline{1}\underline{1}^T - \frac{1}{n}\underline{1}\underline{1}^T + \frac{1}{n^2}\underline{1}\underbrace{\underline{1}^T\underline{1}}_{n}\underline{1}^T = I - \frac{2}{n}\underline{1}\underline{1}^T + \frac{1}{n}\underline{1}\underline{1}^T$

$= I - \frac{1}{n}\underline{1}\underline{1}^T = A$

Idempotent

Result ① If A is idempotent then $\text{rank}(A) = \text{trace}(A)$ = sum diagonal elements.
② $H_x X = X \quad \Rightarrow \quad H_x \underline{1} = \underline{1}$ (since $\underline{1}$ is a column of $\underline{X}$!)

$$\text{rank}(I - \tfrac{1}{n}\underline{1}\underline{1}^T) = \text{trace}(I - \tfrac{1}{n}\underline{1}\underline{1}^T) = \text{trace}(I) - \tfrac{1}{n}\text{trace}(\underline{1}\underline{1}^T) = n - \tfrac{1}{n}n = n-1$$

$$\underline{1}\underline{1}^T = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ & & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

use $H_{\underline{1}} = \frac{1}{n}\underline{1}\underline{1}^T$

Always $SSE_0 > SSE$ (Fit is always worse if less parameters)

$\Rightarrow SSE_0 - SSE > 0$

$$= -\underline{y}^T(I - XGX^T)\underline{y} + \underline{y}^T(I - \tfrac{1}{n}\underline{1}\underline{1}^T)\underline{y} = \underline{y}^T \underbrace{\left(XGX^T - \tfrac{1}{n}\underline{1}\underline{1}^T\right)}_{H_x - H_{\underline{1}}} \underline{y}$$

$$B = \left(XGX^T - \tfrac{1}{n}\underline{1}\underline{1}^T\right)\left(XGX^T - \tfrac{1}{n}\underline{1}\underline{1}^T\right)$$

$$= XGX^T - \tfrac{1}{n}\underbrace{XGX^T\underline{1}}_{②\ \text{projection}}\underline{1}^T - \tfrac{1}{n}\underline{1}\underline{1}^T XGX^T + \tfrac{1}{n}\underline{1}\underline{1}\underline{1}^T$$

$$= XGX^T - \tfrac{1}{n}\underline{1}\underline{1}^T \cancel{- \tfrac{1}{n}\underline{1}\underline{1}^T} \cancel{+ \tfrac{1}{n}\underline{1}\underline{1}^T} = B \quad \text{idempotent}$$

$\text{rank}(B) = \text{trace}(B) = \text{trace}(H_x) - \frac{1}{n}\text{trace}(\underline{1}\underline{1}^T) =$

$\text{trace}(XX^TG) - 1 = \text{rank}(\underline{X}) - 1 = t-1$

P6

Theorem $\underline{y} \sim MVN(\underline{\mu}, \sigma^2 I)$

Define central χ^2

non central χ'^2

Warning

$\frac{\lambda}{2}$ in R is λ just

Then $\frac{\underline{y}^T A \underline{y}}{\sigma^2} \sim \chi'^2_{(r(A), \lambda)}$, $\lambda = \frac{1}{\sigma^2}\underline{\mu}^T A \underline{\mu}$

non-central χ^2

$\Leftrightarrow$ A idempotent

$\text{rank}(I - \frac{1}{n}\underline{1}\underline{1}^T) = n-1$

$\Rightarrow SSTotal = \frac{\underline{y}^T(I - \frac{1}{n}\underline{1}\underline{1}^T)\underline{y}}{\sigma^2} \sim \chi^2_{(n-1, 0)}$

here $\underline{\mu} = \mu \underline{1} \Rightarrow \lambda = \frac{1}{\sigma^2}\mu^2 \underline{1}^T(I - \frac{1}{n}\underline{1}\underline{1}^T)\underline{1}$

$\lambda = \frac{1}{\sigma^2}\mu^2 \left(\underbrace{\underline{1}^T\underline{1} - \frac{1}{n}n^2}_{n-n = 0}\right)$

$\text{rank}(I - XGX^T) = \text{rank}(I) - \text{rank}(XGX^T)$

$= n - t$

$\Rightarrow SSE = \frac{\underline{y}^T(I - XGX^T)\underline{y}}{\sigma^2} \sim \chi^2_{(n-t, 0)}$

here $\underline{\mu} = X\beta$

$\lambda = \frac{1}{\sigma^2}\beta^T X^T(I - XGX^T)X\beta$

$= \frac{1}{\sigma^2}\beta^T(X^TX - X^TXGX^TX)\beta$

$= \frac{1}{\sigma^2}\beta^T(\underbrace{X^TX - X^TX}_{0})\beta = 0$

$$SST = \frac{\underline{y}^T\left(X\Omega X^T - \frac{1}{n}\underline{1}\,\underline{1}^T\right)\underline{y}}{\sigma^2} \sim \chi^2_{(t-1,\ \lambda)}$$

$\text{rank}\left(X\Omega X^T - \frac{1}{n}\underline{1}\,\underline{1}^T\right) = t-1$

$\|$

$SS_{Total} - SSE$

$\|$

SSE_0

have

$\underline{\mu} = X\underline{\beta}$

note $\lambda = 0$ if H_0: true $\underline{\mu} = \mu\underline{1}$

$$\lambda = \frac{1}{2\sigma^2}\beta^T X^T\left(X\Omega X^T - \frac{1}{n}\underline{1}\,\underline{1}^T\right)X\beta$$

$$= \beta^T\left(X^T X\Omega X^T X - \frac{1}{n}X^T\underline{1}\,\underline{1}^T X\right)\beta$$

$$= \beta^T\left(X^T X - \frac{1}{n}X^T\underline{1}\,\underline{1}^T X\right)\beta$$

$$= \beta^T X^T\left(I - \frac{1}{n}\underline{1}\,\underline{1}^T\right)X\beta$$

$$= \underline{\mu}^T\left(I - \frac{1}{n}\underline{1}\,\underline{1}^T\right)\underline{\mu} = \underline{\mu}^T\underline{\mu} - n\bar{\mu}^2$$

$$\sum \mu_i^2 - n\bar{\mu}^2$$

$$\frac{\sum(\mu_i - \bar{\mu})^2}{2\sigma^2}$$

$$\underline{\mu} = \begin{pmatrix} \mu_1 \\ \mu_1 \\ \vdots \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_2 \\ \vdots \\ \mu_t \\ \vdots \\ \mu_t \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} r_1 \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} r_2 \\ \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} r_t \end{matrix}$$

note if balanced

$$\bar{\mu} = \frac{r\mu_1 + r\mu_2 \cdots r\mu_t}{tr} = \frac{\mu_1 + \cdots \mu_t}{t}$$

otherwise is weighted avg

$$\Rightarrow \lambda = \frac{1}{\sigma^2}\sum_{i=1}^{t} r_i(\mu_i - \bar{\mu})^2 \underset{\text{balanced}}{=} \frac{r}{\sigma^2}\sum_{i=1}^{t}(\mu_i - \bar{\mu})^2$$

ANOVA table

Source	df	Sum of Squares	Mean Squares	F
Treatment	$t-1$	SST	$MST = \frac{SST}{t-1}$	$F = \frac{MST}{MSE} = \frac{SST/t-1}{SSE/n-t}$
Error	$n-t$	SSE	$MSE = \frac{SSE}{n-t}$	
Total	$n-1$	SSTotal		

Under $H_0: \tau_1 = \tau_2 = \cdots = \tau_t$

central F

$$\left.\begin{array}{l} SST \sim \chi'^2_{(t-1,0)} \\ SSE \sim \chi'^2_{(n-t,0)} \end{array}\right\} \text{indep} \Rightarrow F = \frac{SST/t-1}{SSE/n-t} \sim F_{(t-1,\, n-t)}$$

(balanced design)

$$\lambda = \frac{r}{\sigma^2} \sum_{i=1}^{t} (\mu_i - \bar{\mu})^2$$

In general

non central F

non centrality parameter

$$\left.\begin{array}{l} SST \sim \chi^2_{(t-1,\lambda)} \\ SSE \sim \chi^2_{(n-t,0)} \end{array}\right\} \text{indep} \Rightarrow F = \frac{SST/t-1}{SSE/n-t} \sim F'_{(t-1,\, n-t,\, \lambda)}$$

ANOVA Factorial design

$y_{ijk} = \mu + \tau_i + \alpha_j + \gamma_{ij} + \epsilon_{ijk}$

$= (\underline{1}\ \underline{X}_1, \underline{X}_2, \underline{X}_{int})$

SSE $= \underline{y}^T (I - H_X) \underline{y}$ stays the same! just different $\underline{X}$ and $\Rightarrow$ different $H_{\underline{X}}$

Test null hypothesis $H_0: \gamma_{ij} = 0 \ \forall i, j$ (no interactions)

$\Rightarrow \underline{X}_0 = (1\ \underline{X}_1, \underline{X}_2) \Rightarrow SSE_0 = \underline{y}^T (I - H_{\underline{X}_0}) \underline{y}$

since clearly $H_{\underline{X}_0}$ is also idempotent $\Rightarrow I - H_{\underline{X}_0}$ idemp too!

Also $SSE_0 - SSE > 0$

$\Rightarrow SSE_0 - SSE = \underline{y}^T (H_X - H_{X_0}) \underline{y}$

" sum of squares due to the interactions

SS Int

$\text{rank}(H_X - H_0) = \text{trace}(H_X - H_{X_0})$

$= \text{trace}(H_X) - \text{trace}(H_{X_0}) = \text{rank}(X) - \text{rank}(X_0)$

$= 1 + (t-1) + (s-1) + (t-1)(s-1) - (1 + (t-1) + (s-1))$

$= (t-1)(s-1)$

Theorem

$\Rightarrow \quad \dfrac{\underline{y}^T (H_X - H_{X_0}) \underline{y}}{\sigma^2} \sim \chi^2_{((t-1)(s-1),\ \lambda)}$ where $\lambda = \frac{1}{\sigma^2} \underline{\mu}^T (H_X - H_{X_0}) \underline{\mu}$

under H_0 : $\gamma_{ij} = 0$

$\Rightarrow \underline{\mu} = X_0 \underline{\beta}_0$

$\Rightarrow \lambda = \frac{1}{\sigma^2} \underline{\beta}_0^T X_0^T (H_X - H_{X_0}) X_0 \underline{\beta}_0$

$= \frac{1}{\sigma^2} \underline{\beta}_0 X_0^T (\underbrace{H_X X_0}_{X_0} - \underbrace{H_{X_0} X_0}_{X_0}) \underline{\beta}_0$

$\Rightarrow \lambda = 0$

otherwise

$\lambda = \frac{1}{\sigma^2} \underline{\beta}^T X^T (H_X X - H_{X_0} X) \beta$

$= \frac{1}{\sigma^2} \underline{\beta}^T X^T (X - H_{X_0} X) \underline{\beta}$

$= \frac{1}{\sigma^2} \underline{\beta}^T X^T (I - H_{X_0}) X \underline{\beta}$

ANOVA Table for Factorial $H_0: \gamma_{ij} = 0 \ \forall ij$

Balanced design

Source	df	Sum of Squares	Mean Squares	F
interactions	$(t-1)(s-1)$	$SS\,Int$	$SSInt/(t-1)(s-1)$	$\dfrac{SSInt/(t-1)(s-1)}{SSE/st(r-1)}$
Error	$st(r-1)$	SSE	$SSE/st(r-1)$	

under H_0

$F \sim F_{((s-1)(t-1),\, st(r-1),\, 0)}$

in general

$F \sim F'_{((s-1)(t-1),\, st(r-1),\, \lambda)}$

λ previous page

$$n - t = str - (\cancel{1} + (t - \cancel{1}) + (s-1) + (t-1)(s-1))$$

$$\cancel{t} + \cancel{s} - \cancel{1} + t\cdot s - \cancel{t} - \cancel{s} + \cancel{1} \quad t\cdot s$$

$$= str - st = st(r-1)$$