MA50260 Statistical Modelling

Lecture 16: Mixed Effects Models - Nested and Crossed Designs

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March 29, 2024

Mixed Effect Models

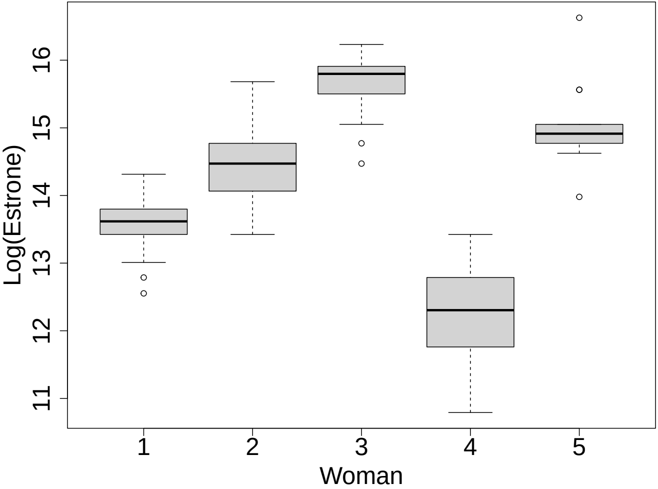
A mixed effects (normal) linear model is written as

$$\mathbf{Y} = \mathbf{X}\underline{\beta} + \mathbf{Z}\mathbf{b} + \underline{\epsilon}$$

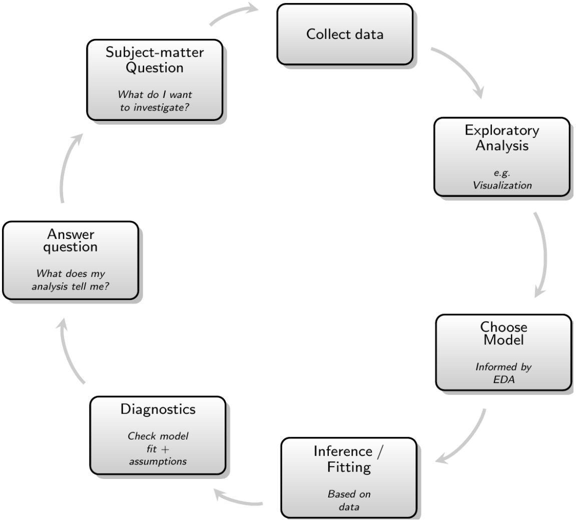
where

- $\mathbf{Y}$ is the vector of responses
- $\mathbf{X}$ is the design matrix
- $\underline{\beta}$ are the regression coefficients
- $\mathbf{Z}$ is the matrix of covariates associated to the q (unknown) random effects $\mathbf{b}$.
- $\underline{\epsilon}$ is the vector of residuals

Example - Analysis of Estrone Levels



Summary of Last Lecture



Nested and Crossed Effects

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How can we extend the introduced linear mixed model?

We generally distinguish between two types of random effects:

- **Nested effects:** levels of one factor are **completely contained** within the levels of another factor
- **Crossed effects:** levels of a factor vary across the levels of another

Example - Setup (I)

Consider a study with

- N machines
- P types of moulds that can be used on any machine
- n components being produced for each mould and each machine

Example - Setup (II)

Let $Y_{i,j,s}$ be the measurement for the s-th component from the j-th mould for the i-th machine.

$\Rightarrow$ $P \times n$ measurements for each machine.

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We are interested in the influence of machine and mould on the measurements:

- treat machine effect as random effect
- treat mould effect as random effect

Example - Nested Random Effects

Suppose that a different set of moulds is used on each machine.

We then model

$$Y_{i,j,s} = \mu + b_{1,i} + b_{12,ij} + \epsilon_{i,j,s},$$

where

- $b_{1,i} \sim \mathrm{Normal}\left(0, \sigma_M^2\right)$
- $b_{12,ij} \sim \mathrm{Normal}\left(0, \sigma_P^2\right)$
- $\epsilon_{i,j,s} \sim \mathrm{Normal}\left(0, \sigma_\epsilon^2\right)$

Note, the mould factor appears **only** within a particular level of the machine factor.

Example - Crossed Random Effects

Assume that the same moulds are used on each machine.

This is modelled as

$$Y_{i,j,s} = \mu + b_{1,i} + b_{2,j} + \epsilon_{i,j,s},$$

where

- $b_{1,i} \sim \mathrm{Normal}\left(0, \sigma_M^2\right)$
- $b_{2,j} \sim \mathrm{Normal}\left(0, \sigma_P^2\right)$
- $\epsilon_{i,j,s} \sim \mathrm{Normal}\left(0, \sigma_\epsilon^2\right)$

Note, mould effects are no longer specific to machines, but each mould effect b_j remains the same for all machine effects b_i.

Exercise 1

Measurements of NO_X across Bath at 4:00, 10:00, 16:00 and 22:00.

We have to consider

- Location
- Time of day

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Nested or crossed design?

Exercise 2

Delivery time for pizza service.

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- Variation across companies
- Variation across delivery drivers
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General Notation

For nested models, we can write

$$\mathbf{Y} = \mathbf{X}\underline{\beta} + \mathbf{Z}_1\mathbf{b}_1 + \mathbf{Z}_{12}\mathbf{b}_{12} + \underline{\epsilon},$$

where we use the double subscript to indicate that the random effects are nested.

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On the other hand, for crossed random effects, we can write

$$\mathbf{Y} = \mathbf{X}\underline{\beta} + \mathbf{Z}_1\mathbf{b}_1 + \mathbf{Z}_2\mathbf{b}_2 + \underline{\epsilon},$$

where $\mathbf{b}_1$ and $\mathbf{b}_2$ are **crossed random effects**.

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We can also write these models in the general form

$$\mathbf{Y} = \mathbf{X}\underline{\beta} + \mathbf{Z}\mathbf{b} + \underline{\epsilon}.$$

The Story so far (I)

In the past weeks, we considered a range of models, including:

- Normal linear regression
- Logistic regression (Binomial GLM)
- Poisson regression
- Gamma/Exponential regression
- Ordinal regression
- Linear mixed models

The Story so far (II)

Linear models

$$Y_i \sim \mathrm{Normal}(\mathbf{x}_i^{\mathrm{T}}\underline{\beta}, \sigma^2)$$

$$\mathbf{Y} = \mathbf{X}\underline{\beta} + \underline{\epsilon}$$

Generalized linear models

$$\mathbb{E}(Y_i) = g^{-1}\left(\mathbf{x}_i^{\mathrm{T}}\underline{\beta}\right)$$

Y_i from exponential family

Linear mixed models

$$Y_{i,j} \sim \mathrm{Normal}\left(\mathbf{x}_i^{\mathrm{T}}\underline{\beta}, \sigma_b^2 + \sigma_\epsilon^2\right)$$

$$\mathbf{Y} = \mathbf{X}\underline{\beta} + \mathbf{Z}\mathbf{b} + \underline{\epsilon}$$

The Story so far (III)

Linear models
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Linear mixed models
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Generalized linear mixed models

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1. Size of a car insurance claim

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4. Student performance across units

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5. Wingspan of an albatross

   $\rightarrow$ Normal linear regression