# Fourier Transform

## Euler's formula with introductory group theory

### Group theory

https://www.youtube.com/watch?v=mvmuCPvRoWQ

In abstract algebra, **group theory** studies the algebraic structures known as groups (symmetry groups).

Any group / collection of these symmetric actions. And group theory studies the associations between pairs of actions and the single action that is equivalent to applying one after the other.

### Addictive group of real numbers

The group operation of applying one action followed by another in the **addictive** group is like **slide** the origin in the real number line or complex plate

### Mulitiplicative group

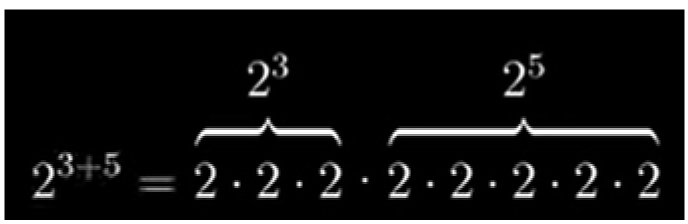
The group operation of applying one action followed by another in multiplication group is like **streching and squishing** the real number line or the complex plate.

Note: The multiplication action associated with i (at the imaginary line) is 90 deg rotation

Thus all the complex number eg. 3+2i is generated from the slide the origin first and then rotate it.

### Exponent:

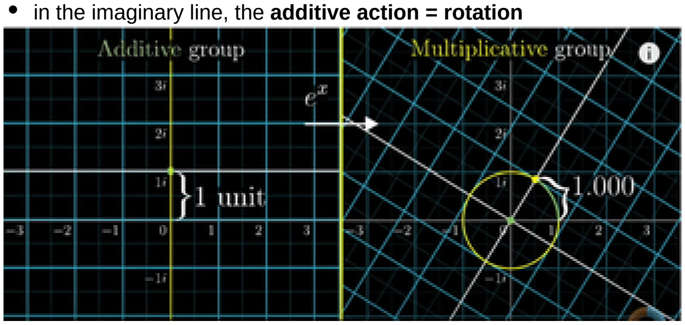
2^(x + y) = 2^x + 2^y



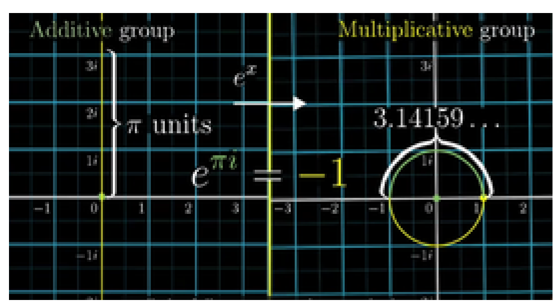

This property of exponent maps the addictive group into multiplication group.

Hence with exponent, the addictive action in the additive group = multiplication group.

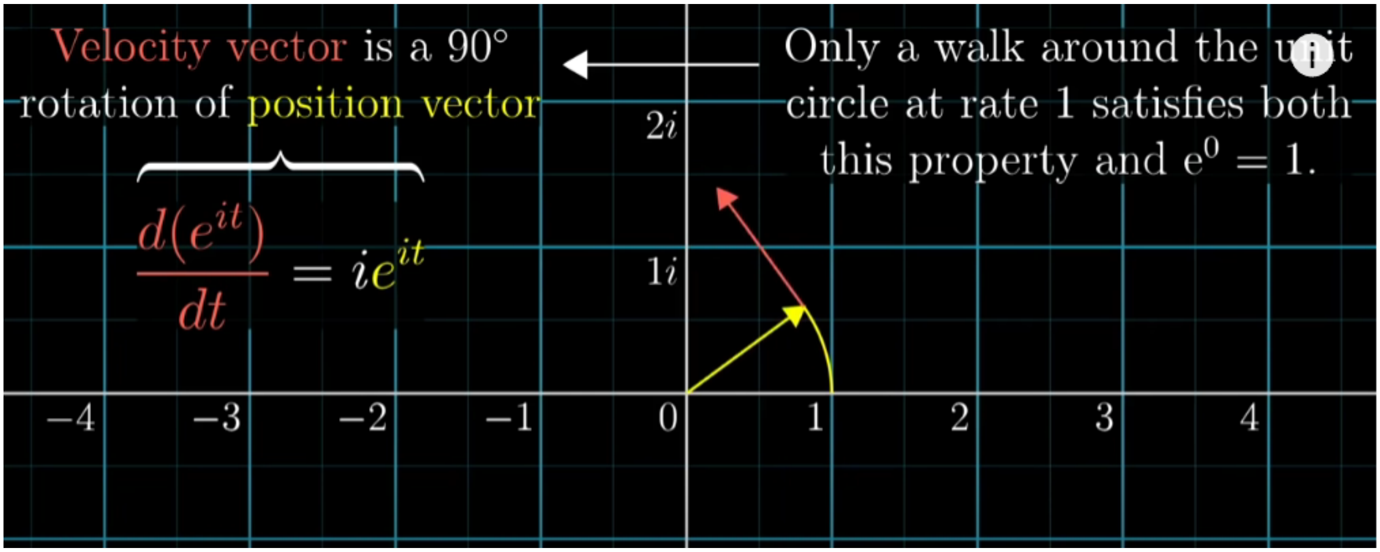
- in the real number line, the **additive action = slide the origin**
- in the imaginary line, the **additive action = rotation**




## Why e could as the base?




What we want is the distance in the imagary line (one unit i ) is equal to the distance( 1 radian) along the rotation line. Since radian for one circle is 2 pie, the one full rotation corresponding to 2pie i(only e as the base achieve that). -->if choose e as base,additive 2pie in the additive group corresponding to the full rotation.




How Fourier Transform works (recipe )?

**Fourier series are infinite series that represent periodic functions in terms of cosines and sines**.

using the equality sign, we write

(5) $$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

and call (5) the **Fourier series** of $f(x)$. We shall prove that in this case the coefficients of (5) are the so-called **Fourier coefficients** of $f(x)$, given by the **Euler formulas**

(6)

(0) $$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

(a) $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \qquad n = 1, 2, \cdots$$

(b) $$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \qquad n = 1, 2, \cdots.$$

q

**Periodic Rectangular Wave (Fig. 260)**

Find the Fourier coefficients of the periodic function $f(x)$ in Fig. 260. The formula is

(7) $$f(x) = \begin{cases} -k & \text{if} \quad -\pi < x < 0 \\ k & \text{if} \quad 0 < x < \pi \end{cases} \qquad \text{and} \qquad f(x + 2\pi) = f(x).$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\sin nx\,dx + \int_{0}^{\pi} k\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[k\frac{\cos nx}{n}\Big|_{-\pi}^{0} - k\frac{\cos nx}{n}\Big|_{0}^{\pi}\right].$$

Since $\cos(-\alpha) = \cos\alpha$ and $\cos 0 = 1$, this yields

$$b_n = \frac{k}{n\pi}[\cos 0 - \cos(-n\pi) - \cos n\pi + \cos 0] = \frac{2k}{n\pi}(1 - \cos n\pi).$$

Now, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, etc.; in general,

$$\cos n\pi = \begin{cases} -1 & \text{for odd } n, \\ 1 & \text{for even } n, \end{cases} \quad \text{and thus} \quad 1 - \cos n\pi = \begin{cases} 2 & \text{for odd } n, \\ 0 & \text{for even } n. \end{cases}$$

Hence the Fourier coefficients $b_n$ of our function are

$$b_1 = \frac{4k}{\pi}, \quad b_2 = 0, \quad b_3 = \frac{4k}{3\pi}, \quad b_4 = 0, \quad b_5 = \frac{4k}{5\pi}, \cdots.$$

Q1. How are continuous functions (cosines and sines) able to represent a given discontinuous function?

A ?https://math.stackexchange.com/questions/1358485/what-does-it-mean-when-two-functions-are-orthogonal-why-is-it-important#:~:text=the%20functions%20sin(n%CF%80,the%20same%20goes%20for%20Cosine).

How does the quality of the approximation increase if you take more and more terms of the series?

A ?

Q: Why are the approximating functions, called the partial sums of the series, in this example always zero at 0 and pie?

A: All the series is zero at 0 and pie

Q: Why is the factor 1/n (obtained in the integration) important?

A: help to converge

Linear algebra

Suppose the basis vectors are actually orthonormal. The $\boldsymbol{a}$'s become $\boldsymbol{q}$'s. Then $A^{\mathrm{T}}A$ simplifies to $Q^{\mathrm{T}}Q = I$. Look at the improvements in $\widehat{x}$ and $\boldsymbol{p}$ and $P$. Instead of $Q^{\mathrm{T}}Q$ we print a blank for the identity matrix:

$$\underline{\qquad}\,\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b} \quad \text{and} \quad \boldsymbol{p} = Q\widehat{x} \quad \text{and} \quad P = Q\,\underline{\qquad}\,Q^{\mathrm{T}}. \tag{4}$$

***The least squares solution of $Q\boldsymbol{x} = \boldsymbol{b}$ is $\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b}$. The projection matrix is $P = QQ^{\mathrm{T}}$.***

There are no matrices to invert. This is the point of an orthonormal basis. The best $\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b}$ just has dot products of $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ with $\boldsymbol{b}$. We have $n$ 1-dimensional projections! The "coupling matrix" or "correlation matrix" $A^{\mathrm{T}}A$ is now $Q^{\mathrm{T}}Q = I$. There is no coupling. When $A$ is $Q$, with orthonormal columns, here is $\boldsymbol{p} = Q\widehat{x} = QQ^{\mathrm{T}}\boldsymbol{b}$:

***Projection onto q's***

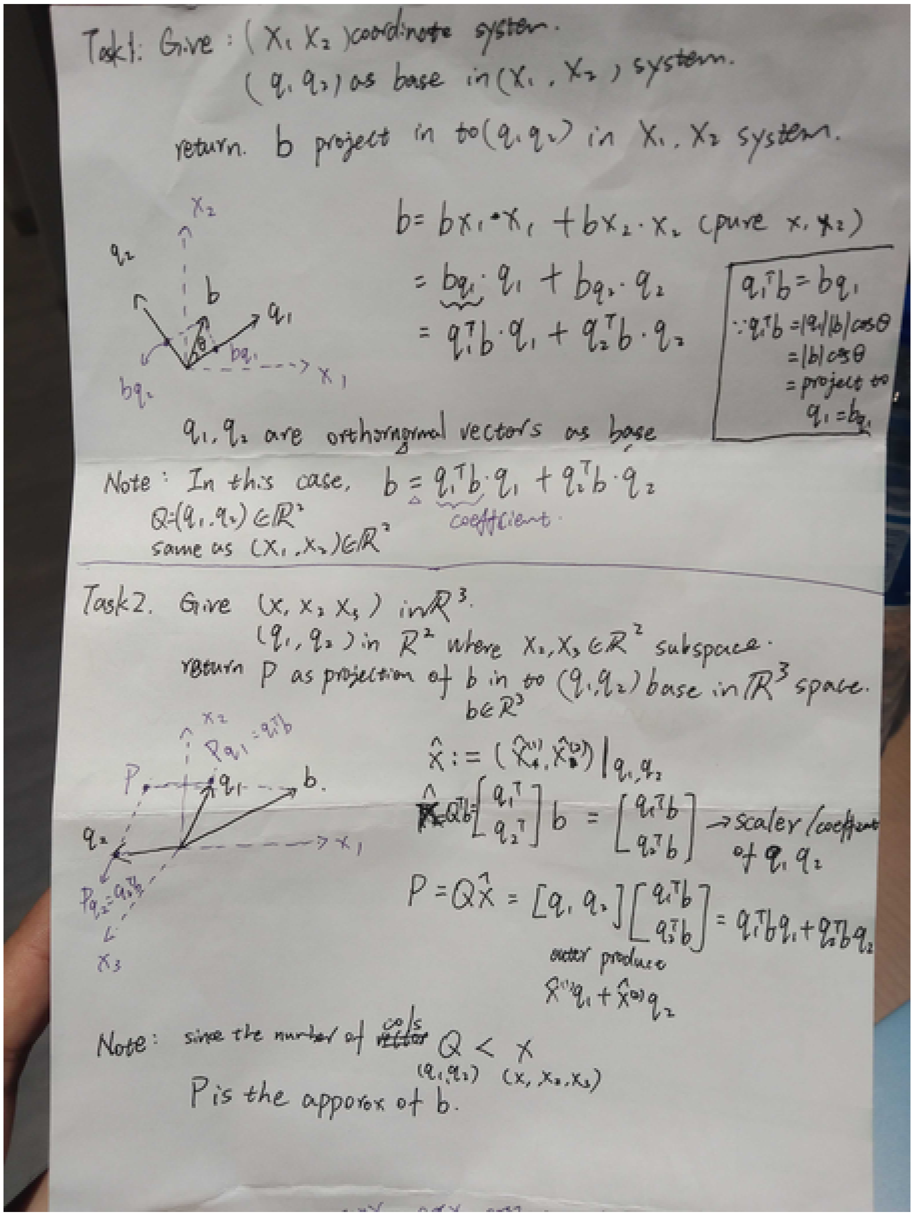
$$\boldsymbol{p} = \begin{bmatrix} | & & | \\ \boldsymbol{q}_1 & \cdots & \boldsymbol{q}_n \\ | & & | \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b} \\ \vdots \\ \boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b} \end{bmatrix} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b}). \tag{5}$$

**Important case:** When $Q$ is square and $m = n$, the subspace is the whole space. Then $Q^{\mathrm{T}} = Q^{-1}$ and $\widehat{x} = Q^{\mathrm{T}}\boldsymbol{b}$ is the same as $\boldsymbol{x} = Q^{-1}\boldsymbol{b}$. The solution is exact! The projection of $\boldsymbol{b}$ onto the whole space is $\boldsymbol{b}$ itself. In this case $P = QQ^{\mathrm{T}} = I$.

You may think that projection onto the whole space is not worth mentioning. But when $\boldsymbol{p} = \boldsymbol{b}$, our formula assembles $\boldsymbol{b}$ out of its 1-dimensional projections. If $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_n$ is an orthonormal basis for the whole space, so $Q$ is square, then every $\boldsymbol{b} = QQ^{\mathrm{T}}\boldsymbol{b}$ is the sum of its components along the $\boldsymbol{q}$'s:

$$\boldsymbol{b} = \boldsymbol{q}_1(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{b}) + \boldsymbol{q}_2(\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{b}) + \cdots + \boldsymbol{q}_n(\boldsymbol{q}_n^{\mathrm{T}}\boldsymbol{b}). \tag{6}$$

That is $QQ^{\mathrm{T}} = I$. It is the foundation of Fourier series and all the great "transforms" of applied mathematics. They break vectors or functions into perpendicular pieces. Then by adding the pieces, the inverse transform puts the function back together.



Task1: Give : ( $X_1$ $X_2$ ) coordinate system.

( $q_1$ $q_2$ ) as base in ( $X_1$ , $X_2$ ) system.

return. b project in to ( $q_1$ $q_2$ ) in $X_1$, $X_2$ system.

$b = bx_1 \cdot x_1 + bx_2 \cdot x_2$ (pure $x_1$ $x_2$)

$= bq_1 \cdot q_1 + bq_2 \cdot q_2$

$= q_1^T b \cdot q_1 + q_2^T b \cdot q_2$

$\boxed{q_1^T b = bq_1 \quad \because q_1^T b = |q_1||b|\cos\theta = |b|\cos\theta = \text{project to } q_1 = bq_1}$

$q_1$, $q_2$ are orthornormal vectors as base

Note : In this case, $b = q_1^T b \cdot q_1 + q_2^T b \cdot q_2$ coefficient.

$Q = (q_1, q_2) \in \mathbb{R}^2$

same as $(X_1, X_2) \in \mathbb{R}^2$

Task2. Give ( $x_1$ $x_2$ $x_3$ ) in $\mathbb{R}^3$.

( $q_1$ , $q_2$ ) in $\mathbb{R}^2$ where $x_2, x_3 \in \mathbb{R}^2$ subspace.

return P as projection of b in to ( $q_1$ $q_2$ ) base in $\mathbb{R}^3$ space.

$b \in \mathbb{R}^3$

$\hat{x} := (\hat{x}^{(1)}, \hat{x}^{(2)}) \,|_{q_1, q_2}$

$\hat{x} = Q^T b = \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} b = \begin{bmatrix} q_1^T b \\ q_2^T b \end{bmatrix}$ → scaler / coefficient of $q_1$ $q_2$

$P = Q\hat{x} = [q_1 \; q_2] \begin{bmatrix} q_1^T b \\ q_2^T b \end{bmatrix} = q_1^T b q_1 + q_2^T b q_2$

outer produce

$\hat{x}^{(1)} q_1 + \hat{x}^{(2)} q_2$

Note : since the number of cols of Q < X
( $q_1$ , $q_2$ ) ( $x_1$ , $x_2$ , $x_3$ )

P is the apporox of b.

What is the difference of Fourier serise and Fourier transfrom?

Case1 Periodic function

- continues function: the sum of Fourier serise will be finite (cld achieve accurate represent like task one)
- discontinue function: the sum .. will be infinite. when implement, always approximition

Case2 Aperiodic function

- use intergral instead of sum, here we call the intergral form of repersent as *Fourier Transfrom*